GENERALIZED INVERSE SCATTERING TRANSFORM FOR THE NONLINEAR SCHRÖDINGER EQUATION

by

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Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

May 2008

ACKNOWLEDGEMENTS

I would like to thank all those who helped me complete the work for my Ph.D. degree. I would first like to thank everyone in the Mathematics Department at the University of Texas at Arlington along with my committee members. I would also like to thank the University for the support through the GAANN fellowship from the US Department of Education awarded to the University of Texas at Arlington Mathematics Department.

Specifically I would like to thank Tuncay Aktosun for his guidance and encouragement in the research and formation of this paper. I would also like to thank Gaik Ambartsoumian and Cornelis van der Mee for all of their help and advice with my paper. Then of course I want to thank Ian Martines for his help and constant support. Being able to deal with me through all of the hard times and the good is not the easiest thing to do, but I could not have done it without him. Finally, I would like to thank my parents Debbie and Ron and my sister Christina for their love and support over the years. Without them I would not be the person I am today.

April 15, 2008

ABSTRACT

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The nonlinear Schrödinger (NLS) equation describes wave propagation in optical fibers, and it is one of the most well-known nonlinear partial differential equations. In 1972 Zakharov and Shabat introduced a powerful method (known as the inverse scattering transform) to solve the initial-value problem for the NLS equation. Due to mathematical and technical difficulties, this method has been available mainly in the case where the multiplicity of each bound state is one. In our research we remove that restriction and generalize the inverse scattering transform for the NLS equation to the case where the multiplicity of each bound state is arbitrarily chosen.

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CHAPTER 1

INTRODUCTION

In this paper we consider the focusing nonlinear Schrödinger equation (NLS)

$$iu_t + u_{xx} + 2|u|^2 u = 0, \qquad x, t \in \mathbb{R}$$

where the subscripts denote the partial derivatives with respect to appropriate independent variables. We generalize the corresponding inverse scattering transformation in the presence of bound states which may have multiplicities greater than one. The NLS equation has a number of physical applications such as waves in optical fibers [19] and surface waves in deep waters [18]. We consider the initial value problem associated with the NLS equation, namely given $u(x, 0)$ finding $u(x, t)$. To do this we consider the inverse scattering transform [20], the steps of which are outlined in the following diagram:

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{direct scattering}} & \{R(\lambda,0), \{\lambda_j\}, \{c_{js}(0)\}\} \\ {\scriptstyle\text{solution to NLS}}\big\downarrow & & \big\downarrow{\scriptstyle\text{time evolution}} \\ u(x,t) & \xleftarrow[\text{inverse scattering}]{} & \{R(\lambda,t), \{\lambda_j\}, \{c_{js}(t)\}\} \end{array}$$

Here, $R(\lambda, t)$ is the reflection coefficient, $\{\lambda_j\}$ is the set of bound state poles and $\{c_{js}(t)\}$ is the set of bound state norming constants. These three sets of data are collectively called the scattering data. These terms will be considered in greater detail in Chapters 2 and 3. The inverse scattering transform helps to solve for solutions to the NLS equation by associating the initial condition $u(x, 0)$ with the initial scattering data. We then consider the time evolution of the scattering data from $t = 0$ to any positive time t. Once this is accomplished we are able to use the scattering data for all t to recover for the desired function $u(x, t)$. We accomplish the generalization of the inverse scattering transform by deriving an explicit compact formula for the norming constants c_{js} along with their respective time evolution in the presence of nonsimple bound states, namely when each bound state λ_j has a set of bound state norming constants associated with it.

In 1972 Zakharov and Shabat introduced [20] the inverse scattering transform for the NLS equation and provided explicit formulas for the time evolution of the reflection coefficient as well as the norming constants when the bound states are all simple. The bound states correspond to the poles of the transmission coefficient in the upper half complex plane $\mathbb{C}^+$ and the algebraic multiplicity of each bound state is the same as the multiplicity of the corresponding pole. The number of norming constants for each bound state is equal to the algebraic multiplicity of that bound state. In the literature the analysis of the inverse scattering transform with bound states of multiplicity greater than one has mainly been avoided due to technical complications [7, 9, 10, 11, 12, 14, 15]. For example, Zakharov and Shabat tried to deal with nonsimple bound states by coalescing two distinct simple bound states into one and they illustrated this by a concrete example [20]. However, as pointed out by Olmedilla in [17], Zakharov and Shabat's "limiting process gives the appropriate value . . . but their final result for the potential is mistaken." Namely, the construction of the kernel of the integral equation was correct; however, the final result for the solution to the NLS equation was incorrect. It seems as if [17] is the only reference in which a systematic method has been sought to determine the time evolution of the norming constants corresponding to nonsimple bound states. Olmedilla was able to find some formulas for the time evolution of the norming constants for a bound state with multiplicity two or three, but he added [17], "in an actual calculation it is very complex to exceed four or five." Using the symbolic computer software REDUCE he was able to reach a multiplicity of nine, but his formulas were too complicated to generalize to a bound state of any multiplicity.

We begin in Chapter 1 with the discussion of direct scattering and the definitions of the scattering coefficients and their relevant properties. In Chapter 2 we consider the case when the transmission coefficient has a set of bound state poles each having order one, and we derive the kernel of the associated Marchenko integral equation. Although the analysis in that chapter is outlined in [20], it is helpful to review the problem in this case and all of its details and then extend it to the case with poles of higher multiplicity. In Chapter 3 we extend the analysis of the Marchenko integral equation to the case of a set of bound state poles, each with arbitrary order. In that chapter we also discuss the dependency constants as well as the bound state norming constants in the presence of poles of multiple order. In Chapter 4 we consider the second step of inverse scattering transform method, namely the time evolution of the scattering data in the presence of bound state poles of multiple order. By determining the time evolution of the dependency constants and exploiting the linear relationship between the norming and dependency constants we determine the time evolution of the norming constants. We then formulate the Marchenko integral equation in the case of bound states of multiple order. In Chapter 5 we show the time evolution of the scattering data in the presence of bound states of multiple order. In Chapter 6 we briefly review the scalar Marchenko integral equation associated with the Zakhaov-Shabat system when the bound states may have multiplicities greater than one.

CHAPTER 2

DIRECT SCATTERING FOR SIMPLE BOUND STATE POLES

In this chapter we review the inverse scattering transform for the focusing NLS equation. Even though everything in this section is known, we nevertheless provide some details to establish our notation and to help the reader. To solve the initial value problem for the NLS equation using the inverse scattering transform we first consider the Zakharov-Shabat system [20]. The inverse scattering transform associates [20] the NLS equation with the Zakharov-Shabat system

$$\begin{cases} \xi' = -i\lambda\xi + u(x,t)\eta, \\ \eta' = i\lambda\eta - u^*(x,t)\xi, \end{cases}$$

where the prime denotes the x-derivative, the asterisk denotes the complex conjugate, λ is the complex-valued spectral parameter, u is a complex-valued integrable function of x for each t, and ξ and η are functions of x and t. Although we are interested in the generalization of the inverse scattering transform for the NLS equation, in this paper we will review the more general case [1, 2, 16]

$$\begin{cases} \xi' = -i\lambda\xi + q(x,t)\eta, \\ \eta' = i\lambda\eta + r(x,t)\xi, \end{cases} \tag{2.1}$$

where q and r are complex-valued integrable potentials. In this case, given a particular relationship between $q(x,t)$ and $r(x,t)$ this method can be used for other integrable nonlinear partial differential equations. There exist two linearly independent vector solutions to (2.1), $\varphi(\lambda,x,t)$ and $\psi(\lambda,x,t)$, known as the Jost solutions [20], which are uniquely determined by imposing the asymptotic conditions

$$\varphi(\lambda,x,t)=\begin{bmatrix}1\\0\end{bmatrix}e^{-i\lambda x}+o(1),\qquad \overline{\varphi}(\lambda,x,t)=\begin{bmatrix}0\\1\end{bmatrix}e^{i\lambda x}+o(1),\tag{2.2}$$

as $x\to-\infty$, and

$$\psi(\lambda,x,t)=\begin{bmatrix}0\\1\end{bmatrix}e^{i\lambda x}+o(1),\qquad \overline{\psi}(\lambda,x,t)=\begin{bmatrix}1\\0\end{bmatrix}e^{-i\lambda x}+o(1),\tag{2.3}$$

as $x\to+\infty$, where an overbar indicates quantities that can be extended analytically to the lower half complex λ-plane in $\mathbb{C}^-$. For simplicity, we may drop the arguments and write φ for $\varphi(\lambda,x,t)$, ψ for $\psi(\lambda,x,t)$, $\overline{\varphi}$ for $\overline{\varphi}(\lambda,x,t)$, and $\overline{\psi}$ for $\overline{\psi}(\lambda,x,t)$.

2.1 Scattering Coefficients

The asymptotic behavior of the Jost solutions at the opposite ends of the x-axis will help us define the scattering coefficients. Since the potentials q and r appearing in (2.1) decay in some sense as $x\to\pm\infty$, it follows that for certain constants α, $\overline{\alpha}$, β, $\overline{\beta}$, γ, $\overline{\gamma}$, ϵ, and $\overline{\epsilon}$ we have

$$\psi=\begin{bmatrix}\alpha e^{-i\lambda x}\\ \beta e^{i\lambda x}\end{bmatrix}+o(1),\qquad \overline{\psi}=\begin{bmatrix}\overline{\alpha}e^{-i\lambda x}\\ \overline{\beta}e^{i\lambda x}\end{bmatrix}+o(1),\tag{2.4}$$

as $x\to-\infty$, and

$$\varphi=\begin{bmatrix}\gamma e^{-i\lambda x}\\ \varepsilon e^{i\lambda x}\end{bmatrix}+o(1),\qquad \overline{\varphi}=\begin{bmatrix}\overline{\gamma}e^{-i\lambda x}\\ \overline{\varepsilon}e^{i\lambda x}\end{bmatrix}+o(1),\tag{2.5}$$

as $x\to+\infty$. We will relate the coefficients appearing in (2.4) and (2.5) to the scattering coefficients, namely, the right reflection coefficient $R(\lambda,t)$, the left reflection coefficient $L(\lambda,t)$, the transmission coefficient from the right $T_r(\lambda,t)$, the transmission coefficient from the left $T_l(\lambda,t)$, the right reflection coefficient $\overline{R}(\lambda,t)$, the left reflection coefficient $\overline{L}(\lambda,t)$, the transmission coefficient from the right $\overline{T}_r(\lambda,t)$ and the transmission coefficient from the left $\overline{T}_l(\lambda,t)$. Again, for simplicity we drop the arguments of such functions.

The coefficients in (2.4) and (2.5) are related to the reflection coefficients and transmission coefficients as follows:

$$\alpha = \tfrac{L}{T}, \qquad \overline{\alpha} = \tfrac{1}{\overline{T}}, \qquad \beta = \tfrac{1}{T}, \qquad \overline{\beta} = \tfrac{\overline{L}}{\overline{T}},$$
$$\gamma = \tfrac{1}{T}, \qquad \overline{\gamma} = \tfrac{\overline{R}}{\overline{T}}, \qquad \epsilon = \tfrac{R}{T}, \qquad \overline{\epsilon} = \tfrac{1}{\overline{T}}.$$

The multiplication of the transmission coefficient from the right, T_r, with the Jost solution φ in (2.5) gives us the asymptotics

$$T_r\varphi = \begin{cases} \begin{bmatrix} T_r\gamma e^{-i\lambda x} \\ T_r\varepsilon e^{i\lambda x} \end{bmatrix} + o(1)\,, & x \to +\infty, \\ \begin{bmatrix} T_r e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1)\,, & x \to -\infty. \end{cases} \tag{2.6}$$

We consider the analogy from quantum mechanics and wave propagation to help us with the physical interpretation of (2.6). We can interpret $e^{i\lambda x}$ as a wave traveling in the positive x-direction and $e^{-i\lambda x}$ as a wave traveling in the negative x-direction by imagining a time factor $e^{-i\omega t}$. Choosing $T_r\gamma = 1$, we can interpret (2.6) as a unit amplitude wave is being sent from $x = +\infty$. Then T_r, the coefficient of $e^{-i\lambda x}$ as $x = -\infty$, becomes a transmission coefficient from the right and $T_r\epsilon$ becomes a reflection coefficient from the right, i.e. $T_r\epsilon = R$. Therefore, using this analogy we want $T_r\gamma$ from (2.6) to equal

1, which implies $\varepsilon = \frac{R}{T_r}$. Similarly, if we multiply the Jost solution ψ in (2.4) by the transmission coefficient from the left, T_l, we find

$$T_l\psi = \begin{cases} \begin{bmatrix} T_l\alpha e^{-i\lambda x} \\ T_l\beta e^{i\lambda x} \end{bmatrix} + o(1)\,, \quad x \to -\infty, \\ \begin{bmatrix} 0 \\ T_l e^{i\lambda x} \end{bmatrix} + o(1)\,, \quad x \to +\infty. \end{cases}$$

Again from the analogy of wave propagation we want $T_l\beta = 1$, which in turn implies $\alpha = \frac{L}{T_l}$.

Let $[f, g] := \begin{vmatrix} f_1 & g_1 \\ f_2 & g_2 \end{vmatrix}$ denote the Wronskian for the vector valued functions $f := \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ and $g := \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$. It can be shown that if the vectors f and g are solutions to (2.1), then their Wronskian is independent of x. Using $[\varphi, \psi]_{x=+\infty} = [\varphi, \psi]_{x=-\infty}$, from (2.4) and (2.5) we obtain $T_r = T_l$ because

$$[\varphi, \psi]_{x=+\infty} = \begin{vmatrix} \gamma e^{-i\lambda x} & 0 \\ \varepsilon e^{i\lambda x} & e^{i\lambda x} \end{vmatrix} = \gamma = \frac{1}{T_r},$$

$$[\varphi, \psi]_{x=-\infty} = \begin{vmatrix} e^{-i\lambda x} & \alpha e^{-i\lambda x} \\ 0 & \beta e^{i\lambda x} \end{vmatrix} = \beta = \frac{1}{T_l}. \tag{2.7}$$

Similarly, consider the multiplication of the transmission coefficient from the right, $\overline{T}_r$, with the Jost solution $\overline{\varphi}$ in (2.5):

$$\overline{T}_r\overline{\varphi} = \begin{cases} \begin{bmatrix} \overline{T}_r\overline{\gamma} e^{-i\lambda x} \\ \overline{T}_r\overline{\varepsilon} e^{i\lambda x} \end{bmatrix} + o(1)\,, \quad x \to +\infty, \\ \begin{bmatrix} 0 \\ \overline{T}_r e^{i\lambda x} \end{bmatrix} + o(1)\,, \quad x \to -\infty. \end{cases}$$

Similarly we want $\overline{\varepsilon}\overline{T}_r = 1$, which in turn implies that $\overline{\gamma} = \frac{\overline{R}}{\overline{T}_r}$. We can then consider the multiplication of the transmission coefficient from the left, $\overline{T}_l$, with the Jost solution $\overline{\psi}$ in (2.4):

$$\overline{T}_l\overline{\psi} = \begin{cases} \begin{bmatrix} \overline{T}_l\overline{\alpha}e^{-i\lambda x} \\ \overline{T}_l\overline{\beta}e^{i\lambda x} \end{bmatrix} + o(1)\,, & x \to -\infty, \\ \begin{bmatrix} \overline{T}_l e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1)\,, & x \to +\infty. \end{cases}$$

Using the analogy from wave propagation we want $\overline{\alpha}\overline{T}_l = 1$, which in turn implies that $\alpha = \frac{\overline{R}}{\overline{T}_l}$. Again we know $[\overline{\varphi}, \overline{\psi}]_{x=+\infty} = [\overline{\varphi}, \overline{\psi}]_{x=-\infty}$, and hence

$$[\overline{\varphi}, \overline{\psi}]_{x=+\infty} = \begin{vmatrix} \overline{\gamma}e^{-i\lambda x} & e^{-i\lambda x} \\ \overline{\varepsilon}e^{i\lambda x} & 0 \end{vmatrix} = -\overline{\varepsilon} = -\frac{1}{\overline{T}_r},$$

$$[\overline{\varphi}, \overline{\psi}]_{x=-\infty} = \begin{vmatrix} 0 & \overline{\alpha}e^{-i\lambda x} \\ e^{i\lambda x} & \overline{\beta}e^{i\lambda x} \end{vmatrix} = -\overline{\alpha} = -\frac{1}{\overline{T}_l}. \tag{2.8}$$

Notice that in the above argument we have seen that $T_r = T_l$ and $\overline{T}_r = \overline{T}_l$. We will simply call them the transmission coefficients T and $\overline{T}$, respectively. We can then rewrite the asymptotic behaviors of the Jost solutions in terms of the scattering coefficients as

$$\varphi = \begin{cases} \begin{bmatrix} \frac{1}{T}e^{-i\lambda x} \\ \frac{R}{T}e^{i\lambda x} \end{bmatrix} + o(1)\,, & x \to +\infty, \\ \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1)\,, & x \to -\infty, \end{cases}$$

$$\overline{\varphi} = \begin{cases} \begin{bmatrix} \frac{\overline{R}}{\overline{T}} e^{-i\lambda x} \\ \frac{1}{\overline{T}} e^{i\lambda x} \end{bmatrix} + o(1)\,, \quad x \to +\infty, \\ \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1)\,, \quad x \to -\infty, \end{cases}$$

$$\psi = \begin{cases} \begin{bmatrix} \frac{L}{T} e^{-i\lambda x} \\ \frac{1}{T} e^{i\lambda x} \end{bmatrix} + o(1)\,, \quad x \to -\infty, \\ \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1)\,, \quad x \to +\infty, \end{cases}$$

$$\overline{\psi} = \begin{cases} \begin{bmatrix} \frac{1}{\overline{T}} e^{-i\lambda x} \\ \frac{\overline{L}}{\overline{T}} e^{i\lambda x} \end{bmatrix} + o(1)\,, \quad x \to -\infty, \\ \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1)\,, \quad x \to +\infty. \end{cases}$$

We can exploit certain Wronskians of the Jost solutions to derive various properties of the scattering coefficients. Consider

$$[\varphi, \overline{\psi}]_{x=+\infty} = \begin{vmatrix} \frac{1}{T} e^{-i\lambda x} & e^{-i\lambda x} \\ \frac{R}{T} e^{i\lambda x} & 0 \end{vmatrix} = -\frac{R}{T},$$

$$[\varphi, \overline{\psi}]_{x=-\infty} = \begin{vmatrix} e^{-i\lambda x} & \frac{1}{T} e^{-i\lambda x} \\ 0 & \frac{\overline{L}}{\overline{T}} e^{i\lambda x} \end{vmatrix} = \frac{\overline{L}}{\overline{T}}.$$

We then have for $\lambda \in \mathbb{R}$

$$-\frac{R}{T} = \frac{\overline{L}}{\overline{T}}.$$

From the x-independence of the Wronskian we have

$$[\varphi, \overline{\varphi}]_{x=+\infty} = \begin{vmatrix} \frac{1}{T} e^{-i\lambda x} & \frac{\overline{R}}{\overline{T}} e^{-i\lambda x} \\ \frac{R}{T} e^{i\lambda x} & \frac{1}{\overline{T}} e^{i\lambda x} \end{vmatrix} = \frac{1}{T\overline{T}} - \frac{R\overline{R}}{T\overline{T}},$$

$$[\varphi, \overline{\varphi}]_{x=-\infty} = \begin{vmatrix} e^{-i\lambda x} & 0 \\ 0 & e^{i\lambda x} \end{vmatrix} = 1.$$

Therefore, we have for $\lambda \in \mathbb{R}$

$$T\overline{T} + R\overline{R} = 1.$$

Again from the x-independence of the Wronskian we set

$$[\psi, \overline{\psi}]_{x=+\infty} = \begin{vmatrix} 0 & e^{-i\lambda x} \\ e^{i\lambda x} & 0 \end{vmatrix} = 1,$$

$$[\psi, \overline{\psi}]_{x=-\infty} = \begin{vmatrix} \frac{L}{T}e^{-i\lambda x} & \frac{1}{\overline{T}}e^{-i\lambda x} \\ \frac{1}{T}e^{i\lambda x} & \frac{\overline{L}}{\overline{T}}e^{i\lambda x} \end{vmatrix} = \frac{L\overline{L}}{T\overline{T}} - \frac{1}{T\overline{T}}.$$

Then for any $\lambda \in \mathbb{R}$

$$T\overline{T} + L\overline{L} = 1.$$

Finally, from the x-independence of the Wronskian we obtain

$$[\psi, \overline{\varphi}]_{x=+\infty} = \begin{vmatrix} 0 & \frac{\overline{R}}{\overline{T}}e^{-i\lambda x} \\ e^{i\lambda x} & \frac{1}{\overline{T}}e^{i\lambda x} \end{vmatrix} = -\frac{\overline{R}}{\overline{T}},$$

$$[\psi, \overline{\varphi}]_{x=-\infty} = \begin{vmatrix} \frac{L}{T}e^{-i\lambda x} & 0 \\ \frac{1}{T}e^{i\lambda x} & e^{i\lambda x} \end{vmatrix} = \frac{L}{T}.$$

Therefore, for any $\lambda \in \mathbb{R}$

$$-\frac{\overline{R}}{\overline{T}} = \frac{L}{T}.$$

We would now like to find scalar coefficients A, B, C, and, D such that

$$\varphi = A\overline{\psi} + B\psi,$$

$$\overline{\varphi} = C\overline{\psi} + D\psi,$$

or equivalently

$$\begin{bmatrix} \varphi \\ \overline{\varphi} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \overline{\psi} \\ \psi \end{bmatrix}. \tag{2.9}$$

The relationship in (2.9) follows from the fact that any two linearly independent vector valued solutions to (2.1) form a basis to express any other vector solution to (2.1).

Proposition 2.1.1 *The coefficients in (2.9) are related to the scattering coefficients as*

$$A = \frac{1}{T}, \qquad B = -\frac{\overline{L}}{\overline{\overline{T}}}, \qquad C = \frac{\overline{R}}{\overline{\overline{T}}}, \qquad D = \frac{1}{T}.$$

Proof Exploiting Wronskian relations in (2.9) we get

$$\begin{aligned} [\varphi, \psi] &= A[\overline{\psi}, \psi] + B[\psi, \psi], \\ \frac{1}{T} &= A. \end{aligned}$$

Therefore, we have $A = \frac{1}{T}$. Similarly using

$$[\varphi, \overline{\psi}] = [A\overline{\psi} + B\psi, \overline{\psi}],$$

and evaluating the Wronskians as $x \to +\infty$ and as $x \to -\infty$, we obtain $B = \frac{R}{T} = -\frac{\overline{L}}{\overline{\overline{T}}}$. Again, using

$$[\psi, \overline{\varphi}] = C[\psi, \overline{\psi}] + D[\psi, \psi],$$

and evaluating the Wronskians as $x \to +\infty$ and as $x \to -\infty$ we get $C = \frac{\overline{R}}{\overline{\overline{T}}} = -\frac{L}{T}$. Finally, using

$$[\overline{\psi}, \overline{\varphi}] = C[\overline{\psi}, \overline{\psi}] + D[\overline{\psi}, \psi],$$

we get $D = \frac{1}{T}$. We can now put all of these together and write (2.9) as

$$\begin{aligned} \begin{bmatrix} \varphi \\ \overline{\varphi} \end{bmatrix} &= \begin{bmatrix} \frac{1}{T} & -\frac{\overline{L}}{\overline{\overline{T}}} \\ \frac{\overline{R}}{\overline{\overline{T}}} & \frac{1}{\overline{\overline{T}}} \end{bmatrix} \begin{bmatrix} \overline{\psi} \\ \psi \end{bmatrix}, \\ \begin{bmatrix} \varphi \\ \overline{\varphi} \end{bmatrix} &= \begin{bmatrix} \frac{1}{T} & \frac{R}{T} \\ -\frac{L}{T} & \frac{1}{T} \end{bmatrix} \begin{bmatrix} \overline{\psi} \\ \psi \end{bmatrix}, \end{aligned} \tag{2.10}$$

which completes the proof. ∎

2.2 Bound States

Let us now introduce the dependency constant γ_j and relate it to the residue of T at the bound state λ_j. From (2.7) we know

$$[\varphi, \psi] = \frac{1}{T}.$$

Therefore, if T has a pole at some $\lambda_j \in \mathbb{C}^+$, the Jost solutions φ and ψ are linearly dependent at that λ_j value. Thus there exists γ_j such that

$$\varphi(\lambda_j, x, t) = \gamma_j \psi(\lambda_j, x, t).$$

For simplicity we will drop the arguments x and t and use $\varphi(\lambda_j)$ for $\varphi(\lambda_j, x, t)$, $\psi(\lambda_j)$ for $\psi(\lambda_j, x, t)$, $\overline{\varphi}(\overline{\lambda}_j)$ for $\overline{\varphi}(\overline{\lambda}_j, x, t)$, and $\overline{\psi}(\overline{\lambda}_j)$ for $\overline{\psi}(\overline{\lambda}_j, x, t)$. From (2.8) we know that

$$[\overline{\varphi}, \overline{\psi}] = \frac{1}{\overline{T}}.$$

Therefore, if $\overline{T}$ has a pole at $\overline{\lambda}_j$ in the upper half complex λ-plane, the Jost solutions $\overline{\varphi}$ and $\overline{\psi}$ become linearly dependent at that $\overline{\lambda}_j$ value. Thus there exists a value $\overline{\gamma_j}$ such that

$$\overline{\varphi}(\overline{\lambda}_j) = \overline{\gamma_j} \overline{\psi}(\overline{\lambda}_j).$$

We call γ_j and $\overline{\gamma_j}$ the dependency constants. Similarly we call the poles of T and $\overline{T}$, λ_j and $\overline{\lambda}_j$, respectively, bound states poles or simply bound states.

Theorem 2.2.1 *The dependency constant γ_j and the residues of T at λ_j are related as*

$$i\gamma_j Res(T, \lambda_j) = \frac{-1}{2\int_{-\infty}^{\infty} \psi_1(\lambda_j, s)\psi_2(\lambda_j, s)ds},$$

where ψ_1 and ψ_2 denote the first and second components of the Jost solution ψ.

Proof When T has a simple pole at λ_j, consider the expansion of T about λ_j as

$$T(\lambda) = \frac{1}{t_j} \frac{1}{(\lambda - \lambda_j)} + O(1)\,, \quad \lambda \to \lambda_j.$$

Then $t_j = \dfrac{1}{\mathrm{Res}(T,\lambda_j)}$. Since λ_j is a simple pole we have

$$\mathrm{Res}(T,\lambda_j) = \frac{1}{\dfrac{\partial}{\partial\lambda}\left(\dfrac{1}{T(\lambda)}\right)\Big|_{\lambda=\lambda_j}},$$

which can be expressed as

$$\begin{aligned}\frac{1}{\mathrm{Res}(T,\lambda_j)} &= \frac{\partial}{\partial\lambda}\left(\frac{1}{T(\lambda)}\right)\Big|_{\lambda=\lambda_j} = \frac{\partial}{\partial\lambda}[\varphi,\psi]\Big|_{\lambda=\lambda_j} \\ &= \begin{vmatrix}\dot\varphi_1 & \psi_1\\ \dot\varphi_2 & \psi_2\end{vmatrix}_{\lambda=\lambda_j} + \begin{vmatrix}\varphi_1 & \dot\psi_1\\ \varphi_2 & \dot\psi_2\end{vmatrix}_{\lambda=\lambda_j}.\end{aligned}$$

Note that an overdot indicates the λ-derivative. Let us define

$$Q_1(\lambda,x,t) := \frac{d}{dx}\begin{vmatrix}\dot\varphi_1 & \psi_1\\ \dot\varphi_2 & \psi_2\end{vmatrix}, \qquad Q_2(\lambda,x,t) := \frac{d}{dx}\begin{vmatrix}\varphi_1 & \dot\psi_1\\ \varphi_2 & \dot\psi_2\end{vmatrix}.$$

Now consider

$$Q_1(\lambda,x,t) = \frac{d}{dx}\begin{vmatrix}\dot\varphi_1 & \psi_1\\ \dot\varphi_2 & \psi_2\end{vmatrix} = \begin{vmatrix}\dot\varphi_1' & \psi_1\\ \dot\varphi_2' & \psi_2\end{vmatrix} + \begin{vmatrix}\dot\varphi_1 & \psi_1'\\ \dot\varphi_2 & \psi_2'\end{vmatrix},$$

where we recall that a prime denotes the x-derivative. From the λ-derivative of (2.1) we get

$$\begin{cases}\dot\xi' = -i\xi - i\lambda\dot\xi + q\dot\eta,\\ \dot\eta' = i\eta + i\lambda\dot\eta + r\dot\xi.\end{cases} \tag{2.11}$$

Using (2.11) we can rewrite $Q_1(\lambda,x,t)$ with only the λ-derivative as

$$\begin{aligned}Q_1(\lambda,x,t) &= \begin{vmatrix}-i\varphi_1 - i\lambda\dot\varphi_1 + q\dot\varphi_2 & \psi_1\\ i\varphi_2 + i\lambda\dot\varphi_2 + r\dot\varphi_1 & \psi_2\end{vmatrix} + \begin{vmatrix}\dot\varphi_1 & -i\lambda\psi_1 + q\psi_2\\ \dot\varphi_2 & i\lambda\psi_2 + r\psi_1\end{vmatrix}\\ &= -i(\varphi_1\psi_2 + \varphi_2\psi_1).\end{aligned} \tag{2.12}$$

Similarly,

$$\begin{aligned} Q_2 &= \frac{d}{dx}\begin{vmatrix} \varphi_1 & \dot{\psi}_1 \\ \varphi_2 & \dot{\psi}_1 \end{vmatrix} = \begin{vmatrix} \varphi_1' & \dot{\psi}_1 \\ \varphi_2' & \dot{\psi}_2 \end{vmatrix} + \begin{vmatrix} \varphi_1 & \dot{\psi}_1' \\ \varphi_2 & \dot{\psi}_2' \end{vmatrix} \\ &= i(\varphi_1\psi_2 + \varphi_2\psi_1). \end{aligned} \tag{2.13}$$

Now, consider $Q_1(\lambda, x, t)$ and $Q_2(\lambda, x, t)$ evaluated at λ_j by letting $x \to \pm\infty$:

$$Q_1(\lambda_j, x, t) = -Q_2(\lambda_j, x, t) = -i[\varphi_1(\lambda_j)\psi_2(\lambda_j) + \varphi_2(\lambda_j)\psi_1(\lambda_j)]$$

$$= -i[\gamma_j\psi_1(\lambda_j)\psi_2(\lambda_j) + \gamma_j\psi_2(\lambda_j)\psi_1(\lambda_j)]$$

$$= 2i\gamma_j\psi_1(\lambda_j)\psi_2(\lambda_j),$$

where we have used the fact that $\frac{1}{T(\lambda_j)} = 0$. Also, notice that $Q_1(\lambda_j, x, t)$ and $Q_2(\lambda_j, x, t)$ vanish as $x \to -\infty$, since $\psi_2(\lambda_j) \to 0$ as $x \to -\infty$. Now consider

$$\begin{vmatrix} \dot{\varphi}_1(\lambda_j) & \psi_1(\lambda_j) \\ \dot{\varphi}_2(\lambda_j) & \psi_2(\lambda_j) \end{vmatrix} = \int_{+\infty}^{x} Q_1(\lambda_j, s, t)ds = -2i\gamma_j \int_{+\infty}^{x} \psi_1(\lambda_j, s, t)\psi_2(\lambda_j, s, t)ds,$$

$$\begin{vmatrix} \varphi_1(\lambda_j) & \dot{\psi}_1(\lambda_j) \\ \varphi_2(\lambda_j) & \dot{\psi}_2(\lambda_j) \end{vmatrix} = \int_{-\infty}^{x} Q_2(\lambda_j, s, t)ds = 2i\gamma_j \int_{-\infty}^{x} \psi_1(\lambda_j, s, t)\psi_2(\lambda_j, s, t)ds.$$

We then have the following expression

$$\left(\begin{vmatrix} \dot{\varphi}_1(\lambda_j) & \psi_1(\lambda_j) \\ \dot{\varphi}_2(\lambda_j) & \psi_2(\lambda_j) \end{vmatrix} + \begin{vmatrix} \varphi_1(\lambda_j) & \dot{\psi}_1(\lambda_j) \\ \varphi_2(\lambda_j) & \dot{\psi}_2(\lambda_j) \end{vmatrix} \right) = -2i\gamma_j \int_{-\infty}^{\infty} \psi_1(\lambda_j, s, t)\psi_2(\lambda_j, s, t)ds.$$

Therefore, if λ_j is a simple pole then

$$\frac{1}{\mathrm{Res}(T, \lambda_j)} = -2i\gamma_j \int_{-\infty}^{\infty} \psi_1(\lambda_j, s, t)\psi_2(\lambda_j, s, t)ds,$$

which completes the proof. ∎

Through a similar procedure, if $\overline{\lambda}_j$ is a simple pole of $\overline{T}$, we obtain

$$\frac{1}{\text{Res}(\overline{T}, \overline{\lambda}_j)} = 2i\overline{\gamma}_j \int_{-\infty}^{\infty} \overline{\psi}_1(\overline{\lambda}_j, s, t)\overline{\psi}_2(\overline{\lambda}_j, s, t)ds.$$

We define the bound state norming constants c_j and $\overline{c}_j$ in terms of the dependency constants γ_j and $\overline{\gamma_j}$, respectively, as

$$c_j := i\gamma_j \text{Res}(T, \lambda_j) = \frac{-1}{2\int_{-\infty}^{\infty} \psi_1(\lambda_j, s)\psi_2(\lambda_j, s)ds}, \tag{2.14}$$

$$\overline{c}_j := i\overline{\gamma}_j \text{Res}(\overline{T}, \overline{\lambda}_j) = \frac{1}{2\int_{-\infty}^{\infty} \overline{\psi}_1(\overline{\lambda}_j, s)\overline{\psi}_2(\overline{\lambda}_j, s)ds}. \tag{2.15}$$

We now have a representation of the norming constants in terms of the dependency constants and the residues of the transmission coefficients in the presence of simple poles. One of our goals in this thesis is to analyze the relationship among the dependency constants, norming constants, and residues when the bound states are no longer simple.

CHAPTER 3

MARCHENKO INTEGRAL EQUATION FOR SIMPLE BOUND STATE POLES

In this chapter we review the steps involved in deriving the Marchenko integral equation when T has simple poles at values λ_j in $\mathbb{C}^+$ and $\overline{T}$ has simple poles at $\overline{\lambda}_j$ in $\mathbb{C}^-$. We then review how the solution to the Marchenko integral equation yields the solution $u(x,t)$ to the NLS equation.

3.1 Marchenko Integral Equation Associated with $\mathbb{C}^+$

We first consider the case where the transmission coefficient, T, has the set of simple bound state poles $\{\lambda_j\}_{j=1}^N$ in $\mathbb{C}^+$.

Theorem 3.1.1 *The Marchenko integral equation related to (2.1) is given by*

$$0 = \overline{K}(x,y,t) + \Omega(x+y,t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_{-\infty}^{\infty} \Omega(s+y,t)K(x,s,t)ds, \quad y > x,$$

where $\Omega(z,t) := \hat{R}(z,t) + \sum_{j=1}^{N} c_j(t)e^{i\lambda_j z}$.

Proof From (2.10)

$$T\varphi = \overline{\psi} + R\psi. \tag{3.1}$$

Let us use $\overline{\mathbb{C}^+} := \mathbb{C}^+ \bigcup \mathbb{R}$ and $\overline{\mathbb{C}^-} := \mathbb{C}^- \bigcup \mathbb{R}$. It is known [1, 2, 8, 16, 20] that as $\lambda \to \infty$

$$\overline{\psi} = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty \text{ in } \overline{\mathbb{C}^-},$$

$$\psi = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty \text{ in } \overline{\mathbb{C}^+}.$$

Rewriting (3.1) as

$$\varphi + (T-1)\varphi = \overline{\psi} - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + R\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + R\left(\psi - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}\right),$$

we get

$$\varphi - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + (T-1)\varphi = \left(\overline{\psi} - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right) + R\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + R\left(\psi - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}\right).$$

If we apply the integral operator $\displaystyle\int_{-\infty}^{\infty} e^{i\lambda y}\frac{d\lambda}{2\pi}$ for $y > x$ in the above equation, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\varphi - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right) e^{i\lambda y}\frac{d\lambda}{2\pi} + \int_{-\infty}^{\infty}(T-1)\varphi e^{i\lambda y}\frac{d\lambda}{2\pi} &= \int_{-\infty}^{\infty} R\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} e^{i\lambda y}\frac{d\lambda}{2\pi} \\ + \int_{-\infty}^{\infty}\left(\overline{\psi} - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right) e^{i\lambda y}\frac{d\lambda}{2\pi} &+ \int_{-\infty}^{\infty} R\left(\psi - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}\right) e^{i\lambda y}\frac{d\lambda}{2\pi}.\end{aligned} \tag{3.2}$$

We now consider each term in (3.2) individually. It is known [1, 2, 8, 16, 20] that $\left(\varphi - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right)$ is analytic in $\mathbb{C}^+$ with respect to λ, is continuous in $\overline{\mathbb{C}^+}$ with respect to λ, and behaves like $O(\frac{1}{\lambda})$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. Therefore,

$$\int_{-\infty}^{\infty}\left(\varphi - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right) e^{i\lambda y}\frac{d\lambda}{2\pi} = 0, \quad y > x.$$

For the next term we consider the residues of T at each pole λ_j for $j = 1, 2, \ldots, N$. Since these poles are in $\mathbb{C}^+$ the integral term can be written as

$$\int_{-\infty}^{\infty}(T-1)\varphi e^{i\lambda y}\frac{d\lambda}{2\pi} = i\sum_{j=1}^{N}\operatorname{Res}(T, \lambda_j)\varphi(\lambda_j)e^{i\lambda_j y}.$$

Recall that $\varphi(\lambda_j) = \gamma_j\psi(\lambda_j)$, where γ_j is the dependency constant for λ_j. Then, the corresponding integral term can be written as

$$\int_{-\infty}^{\infty}(T-1)\varphi e^{i\lambda y}\frac{d\lambda}{2\pi} = i\sum_{j=1}^{N}\operatorname{Res}(T, \lambda_j)\gamma_j\psi(\lambda_j)e^{i\lambda_j y}. \tag{3.3}$$

We now define

$$K(x,y,t) := \int_{-\infty}^{\infty} \left(\psi - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} \right) e^{-i\lambda y} \frac{d\lambda}{2\pi}. \tag{3.4}$$

Then

$$\int_{-\infty}^{\infty} K(x,y,t) e^{i\lambda y} dy + \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} = \psi(\lambda, x, t). \tag{3.5}$$

By substituting (3.5) into (3.3) we find

$$\int_{-\infty}^{\infty} (T-1)\varphi e^{i\lambda y} \frac{d\lambda}{2\pi} = i \sum_{j=1}^{N} \gamma_j \mathrm{Res}(T, \lambda_j) \begin{bmatrix} 0 \\ e^{i\lambda_j (x+y)} \end{bmatrix}$$

$$+ i \sum_{j=1}^{N} \gamma_j \mathrm{Res}(T, \lambda_j) \int_{-\infty}^{\infty} K(x,s,t) e^{i\lambda_j (s+y)} ds.$$

We can replace the second term on the right hand side of (3.2) by $\overline{K}(x,y,t)$, where we have defined

$$\overline{K}(x,y,t) := \int_{-\infty}^{\infty} \left(\overline{\psi} - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} \right) e^{i\lambda y} \, \frac{d\lambda}{2\pi}. \tag{3.6}$$

Similarly we can replace the third term of (3.2) by $\hat{R}(x+y,t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where we have defined

$$\hat{R}(y,t) := \int_{-\infty}^{\infty} R(\lambda, t) e^{i\lambda y} \frac{d\lambda}{2\pi}. \tag{3.7}$$

For the fifth term of (3.2), we can consider a rearrangement of (3.7) along with (3.6) to obtain

$$\int_{-\infty}^{\infty} R \left(\psi - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} \right) e^{i\lambda y} \frac{d\lambda}{2\pi} = \int_{-\infty}^{\infty} \hat{R}(s+y,t) K(x,s,t) ds.$$

Hence the Fourier transform of (3.1) for $y > x$ yields

$$0 = \sum_{j=1}^{N} \gamma_j \mathrm{Res}(T, \lambda_j) \begin{bmatrix} 0 \\ e^{i\lambda_j (x+y)} \end{bmatrix} + i \sum_{j=1}^{N} \gamma_j \mathrm{Res}(T, \lambda_j) \int_{-\infty}^{\infty} K(x,s,t) e^{i\lambda_j (s+y)} ds$$

$$-\overline{K}(x,y,t) - \hat{R}(x+y,t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \int_{-\infty}^{\infty} \hat{R}(s+y,t)K(x,s,t)ds,$$

or equivalently

$$0 = \overline{K}(x,y,t) + \left(\hat{R}(x+y,t) - i\sum_{j=1}^{N}\gamma_j \mathrm{Res}(T,\lambda_j)e^{i\lambda_j(x+y)}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$+ \int_{-\infty}^{\infty} K(x,s,t)\left(\hat{R}(s+y,t) - i\sum_{j=1}^{N}\gamma_j \mathrm{Res}(T,\lambda_j)e^{i\lambda_j(s+y)}\right)ds.$$

Now define

$$\Omega(z,t) := \hat{R}(z,t) - i\sum_{j=1}^{N}\gamma_j \mathrm{Res}(T,\lambda_j)e^{i\lambda_j z}.$$

Thus, using(2.14) we see that the kernel of the Marchenko integral equation can also be expressed as

$$\Omega(z,t) := \hat{R}(z,t) + \sum_{j=1}^{N} c_j e^{i\lambda_j z},$$

with the understanding that c_j depends on t. Therefore, the Marchenko integral equation can be written as

$$0 = \overline{K}(x,y,t) + \Omega(x+y,t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_{-\infty}^{\infty} \Omega(s+y,t)K(x,s,t)ds, \quad y > x,$$

completing the proof. ∎

3.2 Marchenko Integral Equation Associated with $\mathbb{C}^-$

Similarly, we consider the case where the transmission coefficient $\overline{T}$ has the set of simple bound state poles $\{\overline{\lambda}_j\}_{j=1}^{\overline{N}}$ in $\mathbb{C}^-$.

Theorem 3.2.1 *The Marchenko integral equation associated with $\mathbb{C}^-$ and corresponding to the system of ordinary differential equations in (2.1) is given by*

$$0 = K(x,y,t) + \overline{\Omega}(x+y,t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_{-\infty}^{\infty} \overline{\Omega}(s+y,t)\overline{K}(x,s,t)ds, \quad x > y,$$

where $\overline{\Omega}(z,t) := \hat{\overline{R}}(z,t) + \sum_{j=1}^{\overline{N}} \overline{c}_j(t) e^{-i\overline{\lambda}_j z}$.

Proof Recall again from (2.10)

$$\overline{T}\overline{\varphi} = \psi + \overline{R\psi}. \tag{3.8}$$

It is known [1, 2, 8, 16, 20] that

$$\overline{\psi} = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty \text{ in } \overline{\mathbb{C}^-},$$

$$\psi = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + O\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty \text{ in } \overline{\mathbb{C}^+}.$$

From (3.9) we get

$$\overline{\varphi} + (\overline{T} - 1)\overline{\varphi} = \psi - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + \overline{R} \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + \overline{R}\left(\overline{\psi} - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right), \tag{3.9}$$

or equivalently by rearranging terms we obtain

$$\overline{\varphi} - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + (\overline{T} - 1)\overline{\varphi} = \left(\psi - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}\right) + \overline{R} \begin{bmatrix} e^{i\lambda x} \\ 0 \end{bmatrix} + \overline{R}\left(\overline{\psi} - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right).$$

If we apply the integral operator $\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-i\lambda y}$ for $x > y$ on the above equation, we get

$$\begin{gathered}
\int_{-\infty}^{\infty} \left(\overline{\varphi} - \begin{bmatrix} 0 \\ e^{-\lambda x} \end{bmatrix}\right) e^{-i\lambda y} \frac{d\lambda}{2\pi} + \int_{-\infty}^{\infty} (\overline{T} - 1)\overline{\varphi} e^{-i\lambda y} \frac{d\lambda}{2\pi} = \\
\int_{-\infty}^{\infty} \left(\psi - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}\right) e^{-i\lambda y} \frac{d\lambda}{2\pi} + \int_{-\infty}^{\infty} \overline{R} \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} e^{-i\lambda y} \frac{d\lambda}{2\pi} \\
+ \int_{-\infty}^{\infty} \overline{R} \left(\overline{\psi} - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right) e^{-i\lambda y} \frac{d\lambda}{2\pi}.
\end{gathered} \tag{3.10}$$

We now consider each term in the above equation individually. It is known [1, 2, 8, 16, 20] that $\overline{\varphi} - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}$ is analytic in $\mathbb{C}^-$ with respect to λ and continuous in the $\overline{\mathbb{C}^-}$ with respect to λ and behaves like $O(\frac{1}{\lambda})$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$. Therefore,

$$\int_{-\infty}^{\infty} \left(\overline{\varphi} - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} \right) e^{-i\lambda y} \frac{d\lambda}{2\pi} = 0, \quad x > y.$$

For the next term we have to consider the residues of $\overline{T}$ at each of its simple poles $\overline{\lambda}_j$ for $j = 1, 2, \ldots, \overline{N}$. Since these poles are located in $\mathbb{C}^-$ the integral term is evaluated as

$$\int_{-\infty}^{\infty} (\overline{T} - 1)\overline{\varphi} e^{-i\lambda y} \frac{d\lambda}{2\pi} = i \sum_{j=1}^{\overline{N}} \operatorname{Res}(\overline{T}, \overline{\lambda}_j) \overline{\varphi}(\overline{\lambda}_j) e^{-i\overline{\lambda}_j y}.$$

From Chapter 2 we know that $\overline{\varphi}(\overline{\lambda}_j) = \overline{\gamma_j}\overline{\psi}(\overline{\lambda}_j)$, where $\overline{\gamma_j}$ is the dependency constant for $\overline{\lambda}_j$. Thus

$$\int_{-\infty}^{\infty} (\overline{T} - 1)\overline{\varphi} e^{-i\lambda y} \frac{d\lambda}{2\pi} = i \sum_{j=1}^{\overline{N}} \operatorname{Res}(\overline{T}, \overline{\lambda}_j) \overline{\gamma_j}\overline{\psi}(\overline{\lambda}_j) e^{-i\overline{\lambda}_j y}. \tag{3.11}$$

From (3.4) we see that

$$\int_{-\infty}^{\infty} \overline{K}(x, y, t) e^{-i\lambda y} dy + \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} = \overline{\psi}(\lambda, x, t). \tag{3.12}$$

Then by substituting (3.12) into (3.11) we find

$$\int_{-\infty}^{\infty} (\overline{T} - 1)\overline{\varphi} e^{-i\lambda y} \frac{d\lambda}{2\pi} = i \sum_{j=1}^{\overline{N}} \overline{\gamma_j} \operatorname{Res}(\overline{T}, \overline{\lambda}_j) e^{-i\overline{\lambda}_j (x+y)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$+ i \sum_{j=1}^{\overline{N}} \overline{\gamma_j} \operatorname{Res}(\overline{T}, \overline{\lambda}_j) \int_{-\infty}^{\infty} \overline{K}(x, s, t) e^{-i\overline{\lambda}_j (s+y)} ds.$$

We can replace the third term of (3.10) by $K(x, y, t)$ because of (3.4). Similarly we can replace the fourth term of (3.10) by $\hat{\overline{R}}(x + y, t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where we have defined

$$\hat{\overline{R}}(y, t) := \int_{-\infty}^{\infty} \overline{R}(\lambda, t) e^{-i\lambda y} \frac{d\lambda}{2\pi}. \tag{3.13}$$

For the fifth term of (3.10) we can consider a rearrangement of (3.13) along with (3.4) to obtain

$$\int_{-\infty}^{\infty} \overline{R}\left(\overline{\psi} - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right) e^{-i\lambda y}\frac{d\lambda}{2\pi} = \int_{-\infty}^{\infty} \hat{\overline{R}}(s+y,t)\overline{K}(x,s,t)ds.$$

Hence, (3.8) can be rewritten as

$$0 = -\sum_{j=1}^{\overline{N}} \overline{\gamma_j}\mathrm{Res}(\overline{T},\overline{\lambda}_j)e^{-i\overline{\lambda}_j(x+y)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - i\sum_{j=1}^{\overline{N}} \overline{\gamma_j}\mathrm{Res}(\overline{T},\overline{\lambda}_j)\int_{-\infty}^{\infty} \overline{K}(x,s,t)e^{-i\overline{\lambda}_j(s+y)}ds$$

$$+K(x,y,t) - \hat{\overline{R}}(x+y,t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_{-\infty}^{\infty} \hat{\overline{R}}(s+y,t)\overline{K}(x,s,t)ds,$$

or equivalently

$$0 = K(x,y,t) + \left(\hat{\overline{R}}(x+y,t) - i\sum_{j=1}^{\overline{N}} \overline{\gamma_j}\mathrm{Res}(\overline{T},\overline{\lambda}_j)e^{-i\overline{\lambda}_j(x+y)}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$+\int_{-\infty}^{\infty} \overline{K}(x,s,t)\left(\hat{\overline{R}}(s+y,t) - i\sum_{j=1}^{\overline{N}} \overline{\gamma_j}\mathrm{Res}(\overline{T},\overline{\lambda}_j)e^{-i\overline{\lambda}_j(s+y)}\right).$$

Now define

$$\overline{\Omega}(z,t) := \hat{\overline{R}}(z,t) - i\sum_{j=1}^{\overline{N}} \overline{\gamma_j}\mathrm{Res}(\overline{T},\overline{\lambda}_j)e^{-i\overline{\lambda}_j z}.$$

Then, the Marchenko integral equation can be written as

$$0 = K(x,y,t) + \overline{\Omega}(x+y,t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_{-\infty}^{\infty} \overline{\Omega}(s+y,t)\overline{K}(x,s,t)ds, \quad x > y.$$

From (2.15) we see that the kernel $\overline{\Gamma}(z,t)$ can also be written as

$$\overline{\Omega}(z,t) = \hat{\overline{R}}(z,t) + \sum_{j=1}^{\overline{N}} \overline{c}(t)_j e^{-i\overline{\lambda}_j z}.$$

Thus the Marchenko integral equation can be written as

$$0 = K(x,y,t) + \overline{\Omega}(x+y)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_{-\infty}^{\infty} \overline{\Omega}(s+y)\overline{K}(x,s,t)ds, \quad \text{for } x > y,$$

and hence the proof is complete. $\blacksquare$

It is known [1, 2, 8, 16, 20] that $\left(\overline{\psi} - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right)$ is analytic in $\mathbb{C}^-$ and continuous in $\overline{\mathbb{C}^-}$ and behaves as $O(\frac{1}{\lambda})$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$. Therefore, $\overline{K}(x,y,t) = 0$ if $x > y$. Similarly, $K(x,y,t) = 0$ if $y > x$. Therefore, we can write the two vector-valued Marchenko integral equations as

$$\begin{cases} 0 = \overline{K}(x,y,t) + \Omega(x+y,t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \displaystyle\int_{x}^{\infty} \Omega(s+y,t)K(x,s,t)ds, \quad y > x, \\ 0 = K(x,y,t) + \overline{\Omega}(x+y,t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \displaystyle\int_{-\infty}^{x} \overline{\Omega}(s+y,t)\overline{K}(x,s,t)ds, \quad x > y. \end{cases}$$

CHAPTER 4

EXTENSION OF DIRECT SCATTERING FOR BOUND STATE POLES WITH HIGHER MULTIPLICITY

The theory of the inverse scattering transform in the presence of simple bound states has been thoroughly studied. However, a complete satisfactory theory does not exist for a bound state pole with higher multiplicity. To remedy this we revisit the derivation of the Marchenko integral equations. We derive the relationship between the norming constants and the dependency constants in the presence of bound states of multiple order. The motivation behind our method came from the analysis in [5].

4.1 Marchenko Integral Equation Associated with $\mathbb{C}^+$

We consider the case where the function T has N poles, λ_j, in $\mathbb{C}^+$ each with order n_j. The Marchenko integral equation will only differ from the previous case for simple poles in the term

$$\int_{-\infty}^{\infty} (T-1)\varphi e^{i\lambda y} \frac{d\lambda}{2\pi} \tag{4.1}$$

since we can no longer simply use the residue of T in evaluating this term.

Theorem 4.1.1 *The Marchenko integral equation associated with the system in (2.1) is given by*

$$0 = \overline{K}(x,y,t) + \Omega(x+y,t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_{-\infty}^{\infty} K(x,z,t)\Omega(z+y,t)dz, \quad y > x,$$

in the presence of a set of bound states $\{\lambda_j\}_{j=1}^N$ *each of multiple order* n_j *with kernel*

$$\Omega(z,t) := \hat{R}(z,t) + \sum_{j=1}^{N} \sum_{m=0}^{n_j-1} \frac{c_{jm}(t)}{m!} z^m e^{i\lambda_j z}.$$

Proof Consider the expansions about λ_j of the three functions in the integrand of (4.1):

$$T(\lambda)-1=\frac{t_{jn_j}}{(\lambda-\lambda_j)^{n_j}}+\frac{t_{j(n_j-1)}}{(\lambda-\lambda_j)^{n_j-1}}+\ldots+\frac{t_{j1}}{(\lambda-\lambda_j)}+\ldots,$$

$$e^{i\lambda y}=e^{i\lambda_j y}\left[1+iy(\lambda-\lambda_j)+\frac{(iy)^2}{2!}(\lambda-\lambda_j)^2+\ldots+\frac{(iy)^{n_j-1}}{(n_j-1)!}(\lambda-\lambda_j)^{n_j-1}+\ldots\right],$$

$$\varphi(\lambda)=\varphi(\lambda_j)+\dot{\varphi}(\lambda_j)(\lambda-\lambda_j)+\frac{\ddot{\varphi}(\lambda_j)}{2!}(\lambda-\lambda_j)+\ldots+\frac{\varphi^{(n_j-1)}(\lambda_j)}{(n_j-1)!}(\lambda-\lambda_j)^{n_j-1}+\ldots. \quad (4.2)$$

Notice that the coefficients t_{js} in the above expansion can be obtained as

$$t_{js}=\frac{1}{(n_j-s)!}\frac{d^{n_j-s}}{d\lambda^{n_j-s}}\left(\lambda-\lambda_j\right)^{n_j}T(\lambda)\bigg|_{\lambda=\lambda_j},$$

where $j=1,2,\ldots,N$ and $s=1,2,\ldots,n_j$. Recall also that we drop the arguments of x and t for simplicity. When the integrand in (4.1) is expanded about λ_j, the only term that contributes to the integral is the coefficients of the term $\frac{1}{(\lambda-\lambda_j)}$, and the contributions from the remaining terms are nil. Since there are N poles of order n_j each, respectively, we get

$$\begin{aligned}\int_{-\infty}^{\infty}(T-1)\varphi e^{i\lambda y}\frac{d\lambda}{2\pi}=\sum_{j=1}^{N} ie^{i\lambda_j y}&\left[t_{j1}\varphi(\lambda_j)+t_{j2}(\varphi(\lambda_j)(iy)+\dot{\varphi}(\lambda_j))\right.\\ &+t_{j3}\left(\varphi(\lambda_j)\frac{(iy)^2}{2!}+\dot{\varphi}(\lambda_j)(iy)+\frac{\ddot{\varphi}(\lambda_j)}{2!}\right)\\ &+t_{j4}\left(\varphi(\lambda_j)\frac{(iy)^3}{3!}+\dot{\varphi}(\lambda_j)\frac{(iy)^2}{2!}+\frac{\ddot{\varphi}(\lambda_j)}{2!}(iy)+\frac{\varphi^{(3)}(\lambda_j)}{3!}\right)+\ldots\\ &\left.+\,t_{jn_j}\left(\varphi(\lambda_j)\frac{(iy)^{n_j-1}}{(n_j-1)!}+\dot{\varphi}(\lambda_j)\frac{(iy)^{n_j-2}}{(n_j-2)!}+\ldots+\frac{\varphi^{(n_j-1)}(\lambda_j)}{(n_j-1)!}\right)\right],\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\int_{-\infty}^{\infty}(T-1)\varphi e^{i\lambda y}\frac{d\lambda}{2\pi}=\sum_{j=1}^{N} ie^{i\lambda_j y}&\left[\varphi(\lambda_j)\left(t_{j1}+t_{j2}(iy)+t_{j3}\frac{(iy)^2}{2!}+\ldots+t_{n_j}\frac{(iy)^{n_j-1}}{(n_j-1)!}\right)\right.\\ &+\dot{\varphi}(\lambda_j)\left(t_{j2}+t_{j3}(iy)+t_{j4}\frac{(iy)^2}{2!}+\ldots+t_{n_j}\frac{(iy)^{n_j-2}}{(n_j-2)!}\right)\\ &+\frac{\ddot{\varphi}(\lambda_j)}{2!}\left(t_{j3}+t_{j4}(iy)+t_{j5}\frac{(iy)^2}{2!}+\ldots+t_{n_j}\frac{(iy)^{n_j-3}}{(n_j-3)!}\right)\\ &\left.+\ldots+\frac{\varphi^{n_j-1}(\lambda_j)}{(n_j-1)!}(t_{jn_j})\right].\end{aligned} \quad (4.3)$$

We can rewrite the right hand sides of (4.3) as a matrix product as

$$\int_{-\infty}^{\infty}(T-1)\varphi(\lambda,x,t)e^{i\lambda y}\frac{d\lambda}{2\pi}=\sum_{j=1}^{N}ie^{i\lambda_j y}\Phi_j F_j T_j Y_j,$$

where the matrices Φ_j, F_j, T_j, and Y_j are defined as

$$\Phi_j := \begin{bmatrix} \varphi(\lambda_j) & \dot{\varphi}(\lambda_j) & \dots & \varphi^{(n_j-1)}(\lambda_j) \end{bmatrix}, \quad F_j := \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{1!} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{(n_j-1)!} \end{bmatrix},$$

$$T_j := \begin{bmatrix} \frac{t_{j1}}{0!} & \frac{t_{j2}}{1!} & \dots & \frac{t_{j(n_j-1)}}{(n_j-2)!} & \frac{t_{jn_j}}{(n_j-1)!} \\ \frac{t_{j2}}{0!} & \frac{t_{j3}}{1!} & \dots & \frac{t_{jn_j}}{(n_j-2)!} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{t_{jn_j}}{0!} & 0 & \dots & 0 & 0 \end{bmatrix}, \qquad Y_j := \begin{bmatrix} 1 \\ (iy)^1 \\ \vdots \\ (iy)^{n_j-1} \end{bmatrix}.$$

As in the case of simple poles we would like to express the functions $\varphi(\lambda_j,x,t)$, $\dot{\varphi}(\lambda_j,x,t)$, $\dots$, $\varphi^{n_j-1}(\lambda_j,x,t)$ in terms of the functions $\psi(\lambda_j,x,t)$, $\dot{\psi}(\lambda_j,x,t)$, $\dots$, $\psi^{n_j-1}(\lambda_j,x,t)$. To find such representations we consider the expansion of $\frac{1}{T}$ about λ_j. Let $a:=\frac{1}{T}$. Then

$$a(\lambda)=a(\lambda_j)+\dot{a}(\lambda_j)(\lambda-\lambda_j)+\frac{\ddot{a}(\lambda_j)}{2!}(\lambda-\lambda_j)^2+\dots+\frac{a^{(n_j)}(\lambda_j)}{(n_j)!}(\lambda-\lambda_j)^{n_j}+\dots. \quad (4.4)$$

Since λ_j is a pole of order n_j for T, it is a zero of order n_j for $a(\lambda)$. Therefore, we have

$$a(\lambda_j)=\dot{a}(\lambda_j)=\dots=a^{(n_j-1)}(\lambda_j)=0. \quad (4.5)$$

We know from Chapter 2 that $a=[\varphi,\psi]$. Expanding the Jost solutions φ and ψ about λ_j we obtain

$$\varphi=\varphi(\lambda_j)+\dot{\varphi}(\lambda_j)(\lambda-\lambda_j)+\frac{\ddot{\varphi}(\lambda_j)}{2!}(\lambda-\lambda_j)^2+\dots,$$

$$\psi = \psi(\lambda_j) + \dot{\psi}(\lambda_j)(\lambda - \lambda_j) + \frac{\ddot{\psi}(\lambda_j)}{2!}(\lambda - \lambda_j)^2 + \ldots.$$

Thus,

$$a(\lambda) = [\varphi, \psi] = \begin{vmatrix} \varphi_1(\lambda_j) + \dot{\varphi}_1(\lambda_j)(\lambda - \lambda_j) + \ldots & \psi_1(\lambda_j) + \dot{\psi}_1(\lambda_j)(\lambda - \lambda_j) + \ldots \\ \varphi_2(\lambda_j) + \dot{\varphi}_2(\lambda_j)(\lambda - \lambda_j) + \ldots & \psi_2(\lambda_j) + \dot{\psi}_2(\lambda_j)(\lambda - \lambda_j) + \ldots \end{vmatrix}$$

$$\begin{aligned} &= [\varphi_1(\lambda_j)\psi_2(\lambda_j) - \varphi_2(\lambda_j)\psi_1(\lambda_j)] + (\lambda - \lambda_j)[\varphi_1(\lambda_j)\dot{\psi}_2(\lambda_j) + \dot{\varphi}_1(\lambda_j)\psi_2(\lambda_j) \\ &\quad -\varphi_2(\lambda_j)\dot{\psi}_1(\lambda_j) - \psi_1(\lambda_j)\dot{\varphi}(\lambda_j)] + (\lambda - \lambda_j)^2[\varphi_1(\lambda_j)\frac{\ddot{\psi}(\lambda_j)}{2!} + \psi_2(\lambda_j)\frac{\ddot{\varphi}_1(\lambda_j)}{2!} \\ &\quad +\dot{\varphi}_1(\lambda_j)\dot{\psi}_2(\lambda_j) - \psi_1(\lambda_j)\frac{\ddot{\varphi}(\lambda_j)}{2!} - \varphi_2(\lambda_j)\frac{\ddot{\psi}_1(\lambda_j)}{2!} - \dot{\psi}_1(\lambda_j)\dot{\varphi}(\lambda_j)] + \ldots. \end{aligned}$$

Comparing the above expansion with (4.4) and using (4.5) we get the following relations:

$$\begin{aligned} 0 &= a(\lambda_j) = \varphi_1(\lambda_j)\psi_2(\lambda_j) - \varphi_2(\lambda_j)\psi_1(\lambda_j), \\ 0 &= \dot{a}(\lambda_j) = \varphi_1(\lambda_j)\dot{\psi}_2(\lambda_j) + \dot{\varphi}_1(\lambda_j)\psi_2(\lambda_j) - \varphi_2(\lambda_j)\dot{\psi}_1(\lambda_j) - \psi_1(\lambda_j)\dot{\varphi}(\lambda_j), \\ 0 &= \ddot{a}(\lambda_j) = \varphi_1(\lambda_j)\tfrac{\ddot{\psi}(\lambda_j)}{2!} + \psi_2(\lambda_j)\tfrac{\ddot{\varphi}_1(\lambda_j)}{2!} + \dot{\varphi}_1(\lambda_j)\dot{\psi}_2(\lambda_j) - \psi_1(\lambda_j)\tfrac{\ddot{\varphi}(\lambda_j)}{2!}, \\ &\qquad -\varphi_2(\lambda_j)\tfrac{\ddot{\psi}_1(\lambda_j)}{2!} - \dot{\psi}_1(\lambda_j)\dot{\varphi}(\lambda_j), \\ &\vdots \\ 0 &= a^{(n_j-1)}(\lambda_j) = \sum_{l=0}^{n_j-1} \binom{n_j - 1}{l}[\varphi^{(n_j-1-l)}(\lambda_j), \psi^{(l)}(\lambda_j)]. \end{aligned} \tag{4.6}$$

The first relation involving $a(\lambda_j)$ in (4.6) implies

$$\begin{vmatrix} \varphi_1(\lambda_j) & \psi_2(\lambda_j) \\ \varphi_2(\lambda_j) & \psi_2(\lambda_j) \end{vmatrix} = 0,$$

where we use the subscripts 1 and 2 to indicate the first and second components of the Jost solutions. Thus $\varphi(\lambda_j)$ and $\psi(\lambda_j)$ are linearly dependent. This means that there exists γ_{j0} that may only depend on t, but not on x such that

$$\varphi(\lambda_j) = \gamma_{j0}\psi(\lambda_j). \tag{4.7}$$

By substituting (4.7) into the second relation in (4.6) involving $\dot{a}(\lambda_j)$, we obtain

$$\gamma_{j0}\psi_1(\lambda_j)\dot{\psi}_2(\lambda_j) + \dot{\varphi}_1(\lambda_j)\psi_2(\lambda_j) - \gamma_{j0}\psi_2(\lambda_j)\dot{\psi}_1(\lambda_j)\psi_1(\lambda_j)\dot{\varphi}_1(\lambda_j) = 0,$$

which can be written as

$$\begin{vmatrix} \dot{\varphi}_1(\lambda_j) - \gamma_{j0}\dot{\psi}_1(\lambda_j) & \psi_1(\lambda_j) \\ \dot{\varphi}_2(\lambda_j) - \gamma_{j0}\dot{\psi}_2(\lambda_j) & \psi_2(\lambda_j) \end{vmatrix} = 0. \tag{4.8}$$

From (4.8) we see that $\dot{\varphi}(\lambda_j) - \gamma_{j0}\dot{\psi}(\lambda_j)$ and $\psi(\lambda_j)$ are linearly dependent. Therefore, there exists γ_{j1} that may depend on t, but not on x, such that

$$\dot{\varphi}(\lambda_j) - \gamma_{j0}\psi(\lambda_j) = \gamma_{j1}\psi(\lambda_j),$$

or equivalently

$$\dot{\varphi}(\lambda_j) = \gamma_{j0}\psi(\lambda_j) + \gamma_{j1}\psi(\lambda_j). \tag{4.9}$$

Again substituting (4.7) and (4.9) into the third expression in (4.6) we have

$$\begin{gathered} 0 = \gamma_{j0}\psi_1(\lambda_j)\tfrac{\ddot{\psi}_2(\lambda_j)}{2!} + \psi_2(\lambda_j)\tfrac{\ddot{\varphi}_1(\lambda_j)}{2!} + \gamma_{j0}\psi_1(\lambda_j)\dot{\psi}_2(\lambda_j) + \gamma_{j2}\psi_1(\lambda_j)\dot{\psi}_2(\lambda_j) \\ -\psi_2\tfrac{\ddot{\varphi}_2(\lambda_j)}{2!} - \gamma_{j0}\psi_2(\lambda_j)\tfrac{\ddot{\psi}_1(\lambda_j)}{2!} - \dot{\psi}(\lambda_j)\gamma_{j0}\dot{\psi}_2(\lambda_j) - \dot{\psi}_1(\lambda_j)\gamma_{j1}\psi_2(\lambda_j), \end{gathered}$$

which can be written as

$$\begin{vmatrix} \ddot{\varphi}_1(\lambda_j) - 2\gamma_{j1}\dot{\psi}_1(\lambda_j) - \gamma_{j0}\ddot{\psi}_1(\lambda_j) & \psi_1(\lambda_j) \\ \ddot{\varphi}_2(\lambda_j) - 2\gamma_{j1}\dot{\psi}_2(\lambda_j) - \gamma_{j0}\ddot{\psi}_2(\lambda_j) & \psi_2(\lambda_j) \end{vmatrix} = 0. \tag{4.10}$$

From (4.10) we see that $\ddot{\varphi}(\lambda_j) - 2\gamma_{j1}\dot{\psi}(\lambda_j) - \gamma_{j0}\ddot{\psi}(\lambda_j)$ and $\psi(\lambda_j)$ are linearly dependent. Therefore, there exists γ_{j2} that may be dependent on t but not on x such that

$$\ddot{\varphi}(\lambda_j) - 2\gamma_{j1}\dot{\psi}(\lambda_j) - \gamma_{j0}\ddot{\psi}(\lambda_j) = \gamma_{j2}\psi(\lambda_j), \tag{4.11}$$

or equivalently

$$\ddot{\varphi}(\lambda_j) = \gamma_{j0}\ddot{\psi}(\lambda_j) + 2\gamma_{j1}\dot{\psi}(\lambda_j) + \gamma_{j2}\psi(\lambda_j).$$

Now we want prove that there is a similar representation for $\varphi^{(n_j-1)}(\lambda_j)$ in terms of $\psi(\lambda_j)$, $\dot{\psi}(\lambda_j)$, ..., $\psi^{n_j-1}(\lambda_j)$. Recall we know for any $n = 0, 1, \dots n_j - 1$ that

$$0 = a^{(n)}(\lambda_j) = \sum_{l=0}^{n} \binom{n}{l} [\varphi^{(n-l)}(\lambda_j), \psi^{(l)}(\lambda_j)], \tag{4.12}$$

which can be written as

$$\begin{aligned}
&[\varphi^{(n)}(\lambda_j), \psi^{(0)}(\lambda_j)] + \tbinom{n}{1}\gamma_{j(n-1)}[\psi^{(0)}(\lambda_j), \psi^{(1)}(\lambda_j)] + \ldots + \tbinom{n}{n}\gamma_{j0}[\psi^{(0)}(\lambda_j), \psi^{(n)}(\lambda_j)] \\
&+\tbinom{n}{1}\tbinom{n-1}{n-2}\gamma_{j(n_j-2)}[\psi^{(1)}(\lambda_j), \psi^{(1)}(\lambda_j)] + \ldots + \tbinom{n}{n-1}\gamma_{j0}[\psi^{(1)}(\lambda_j), \psi^{(n-1)}(\lambda_j)] \\
&+\tbinom{n}{1}\tbinom{n-1}{n-3}\gamma_{j(n_j-3)}[\psi^{(2)}(\lambda_j), \psi^{(1)}(\lambda_j)] + \tbinom{n}{2}\tbinom{n-2}{n-4}\gamma_{j(n_j-4)}[\psi^{(2)}(\lambda_j), \psi^{(2)}(\lambda_j)] + \ldots \\
&+\tbinom{n}{1}\tbinom{n-1}{n-4}\gamma_{j(n_j-4)}[\psi^{(3)}(\lambda_j), \psi^{(1)}(\lambda_j)] + \tbinom{n}{2}\tbinom{n-2}{n-5}\gamma_{j(n_j-5)}[\psi^{(3)}(\lambda_j), \psi^{(2)}(\lambda_j)] + \ldots \qquad . \\
&+\tbinom{n}{1}\tbinom{n-1}{n-5}\gamma_{j(n_j-5)}[\psi^{(4)}(\lambda_j), \psi^{(1)}(\lambda_j)] + \tbinom{n}{2}\tbinom{n-2}{n-6}\gamma_{j(n_j-6)}[\psi^{(4)}(\lambda_j), \psi^{(2)}(\lambda_j)] + \ldots \\
&\vdots \\
&+\tbinom{n}{1}\tbinom{n-1}{0}\gamma_{j0}[\psi^{(n-1)}(\lambda_j), \psi^{(1)}(\lambda_j)].
\end{aligned} \tag{4.13}$$

The expansion in (4.13) implies the presence of a matrix structure. We know that $[\psi^{(k)}, \psi^{(k)}] = 0$ for any k and $[\psi^{(k)}, \psi^{(j)}] = -[\psi^{(j)}, \psi^{(k)}]$; therefore, if we can show the constants of similar terms are the same, then all of the values excluding the first row of (4.13) will be zero. Consider the constants associated with $\gamma_{j(n-3)}$; for these to cancel out we must have

$$\binom{n}{1}\binom{n-1}{n-3} = \binom{n}{2}\binom{n-2}{n-3},$$

which is certainly true and equivalent to

$$\frac{n!}{1!(n-1)!}\frac{(n-1)!}{(n-3)!2!} = \frac{n!}{2!(n-2)!}\frac{(n-2)!}{(n-3)!1!}. \tag{4.14}$$

We can see that the equality in (4.14) in fact holds, and hence the terms containing $\gamma_{j(n_j-3)}$ will cancel out. Consider the terms associated with $\gamma_{j(n_j-4)}$; for these to cancel out we must have

$$\binom{n}{1}\binom{n-1}{n-4} = \binom{n}{3}\binom{n-3}{n-4},$$

which is equivalent to

$$\frac{n!}{1!(n-1)!}\frac{(n-1)!}{(n-4)!3!}=\frac{n!}{3!(n-3)!}\frac{(n-3)!}{(n-4)!1!}. \tag{4.15}$$

We can see that (4.15) holds, and hence the terms containing $\gamma_{j(n_j-4)}$ will cancel out. Consider the constants associated with $\gamma_{j(n_j-5)}$, for these to cancel out we must have

$$\binom{n}{2}\binom{n-2}{n-5}=\binom{n}{3}\binom{n-3}{n-5} \quad \text{and} \quad \binom{n}{1}\binom{n-1}{n-5}=\binom{n}{4}\binom{n-4}{n-5},$$

which are equivalent to

$$\begin{aligned}\frac{n!}{2!(n-2)!}\frac{(n-2)!}{(n-5)!3!}&=\frac{n!}{3!(n-3)!}\frac{(n-3)!}{(n-5)!2!},\\ \frac{n!}{1!(n-1)!}\frac{(n-1)!}{(n-5)!4!}&=\frac{n!}{4!(n-4)!}\frac{(n-4)!}{(n-5)!1!}.\end{aligned} \tag{4.16}$$

We can see again that (4.16) holds, and hence the terms containing $\gamma_{j(n_j-5)}$ in (4.13) cancel out. Therefore, for all but the first row of (4.13) to cancel out we need in general

$$\binom{n}{k}\binom{n-k}{n-p}=\binom{n}{p-k}\binom{n-p+k}{n-p},$$

which is equivalent to

$$\frac{n!}{k!(n-k)!}\frac{(n-k)!}{(n-p)!(p-k)!}=\frac{n!}{(p-k)!(n-p+k)!}\frac{(n-p+k)!}{(n-p)!k!}. \tag{4.17}$$

We can see (4.17) in fact holds, and hence the corresponding terms in (4.13) cancel out. Thus, we have

$$0=[\varphi^{(n)}(\lambda_j),\psi^{(0)}(\lambda_j)]+\binom{n}{1}\gamma_{j(n_j-1)}[\psi^{(0)}(\lambda_j),\psi^{(1)}(\lambda_j)]+\ldots+\binom{n}{n}\gamma_{j0}[\psi^{(0)}(\lambda_j),\psi^{(n)}(\lambda_j)],$$

or equivalently

$$0=\left[\varphi^{(n)}(\lambda_j)-\binom{n}{1}\gamma_{j(n_j-1)}\psi^{(1)}(\lambda_j)-\binom{n}{2}\gamma_{j(n_j-2)}\psi^{(2)}(\lambda_j)-\ldots-\binom{n}{n}\gamma_{j0}\psi^{(n)}(\lambda_j),\psi^{(0)}(\lambda_j)\right].$$

Therefore, the zero value of the Wronskian above implies that there exists $\gamma_{j(n+1)}$ such that

$$\varphi^{(n)}(\lambda_j)-\binom{n}{1}\gamma_{j(n_j-1)}\psi^{(1)}(\lambda_j)-\binom{n}{2}\gamma_{j(n_j-2)}\psi^{(2)}(\lambda_j)-\ldots-\binom{n}{n}\gamma_{j0}\psi^{(n)}(\lambda_j)=\gamma_{j(n_j+1)}\psi^{(0)}(\lambda_j),$$

or equivalently

$$\varphi^{(n)}(\lambda_j) = \binom{n}{1}\gamma_{j(n_j-1)}\psi^{(1)}(\lambda_j) - \binom{n}{2}\gamma_{j(n_j-2)}\psi^{(2)}(\lambda_j) + \ldots + \binom{n}{n}\gamma_{j0}\psi^{(n)}(\lambda_j) + \gamma_{j(n_j+1)}\psi^{(0)}(\lambda_j).$$

Therefore, we can say for $l = 0, \ldots, n_j - 1$ we have

$$\varphi^{(l)}(\lambda_j) = \sum_{k=0}^{l} \binom{l}{k} \gamma_{j(l-k)}\psi^{(k)}(\lambda_j), \tag{4.18}$$

which can be expressed in matrix form as

$$\hat{\Phi}_j = \Psi_j \Gamma_j, \tag{4.19}$$

where the matrices $\hat{\Phi}_j$, Γ_j, and Ψ_j are defined as

$$\hat{\Phi}_j := \begin{bmatrix} \varphi(\lambda_j) & \dot{\varphi}(\lambda_j) & \ldots & \varphi^{(n_j-1)}(\lambda_j) \end{bmatrix},$$

$$\Psi_j := \begin{bmatrix} \psi(\lambda_j) & \dot{\psi}(\lambda_j) & \ldots & \psi^{(n_j-1)}(\lambda_j) \end{bmatrix},$$

$$\Gamma_j := \begin{bmatrix} \gamma_{j0} & \gamma_{j1} & \gamma_{j2} & \cdots & \gamma_{j(n_j-1)} \\ 0 & \binom{1}{1}\gamma_{j0} & \binom{2}{1}\gamma_{j1} & \cdots & \binom{n_j-1}{1}\gamma_{j(n_j-2)} \\ 0 & 0 & \binom{2}{2}\gamma_{j0} & \cdots & \binom{n_j-1}{2}\gamma_{j(n_j-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \binom{n_j-1}{n_j-1}\gamma_{j0} \end{bmatrix}. \tag{4.20}$$

Using (4.19) in (4.4) we obtain

$$\int_{-\infty}^{\infty} (T-1)\varphi e^{i\lambda y} \frac{d\lambda}{2\pi} = \sum_{j=1}^{N} ie^{i\lambda_j y}\Psi_j \Gamma_j F_j T_j Y_j, \tag{4.21}$$

where we have defined

$$F_j := \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{1!} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{(n_j-1)!} \end{bmatrix}, \quad Y_j := \begin{bmatrix} 1 \\ (iy)^1 \\ \vdots \\ (iy)^{n_j-1} \end{bmatrix},$$

$$T_j := \begin{bmatrix} \frac{t_{j1}}{0!} & \frac{t_{j2}}{1!} & \dots & \frac{t_{j(n_j-1)}}{(n_j-2)!} & \frac{t_{jn_j}}{(n_j-1)!} \\ \frac{t_{j2}}{0!} & \frac{t_{j3}}{1!} & \dots & \frac{t_{jn_j}}{(n_j-2)!} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{t_{jn_j}}{0!} & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{4.22}$$

From (3.4) we have

$$\psi(\lambda) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + \int_{-\infty}^{\infty} K(x,z,t)e^{i\lambda z}dz. \tag{4.23}$$

If we take the λ-derivative of (4.23) we find that for any n

$$\psi^{(n)}(\lambda_j) = \begin{bmatrix} 0 \\ (ix)^n e^{i\lambda_j x} \end{bmatrix} + \int_{-\infty}^{\infty} K(x,z,t)e^{i\lambda_j z}(iz)^n dz.$$

Now we can express Ψ_j given in (4.20) as

$$\Psi_j = \begin{bmatrix} 0 \\ e^{i\lambda_j x} \end{bmatrix} X_j + \int_{-\infty}^{\infty} K(x,z,t)e^{i\lambda_j z}Z_j dz,$$

where the matrices X_j and Z_j are defined as

$$\begin{aligned} X_j &:= \begin{bmatrix} 1 & ix & \dots & (ix)^{n_j-1} \end{bmatrix}, \\ Z_j &:= \begin{bmatrix} 1 & iz & \dots & (iz)^{n_j-1} \end{bmatrix}. \end{aligned} \tag{4.24}$$

Using (4.24) we write (4.23) as

$$\int_{-\infty}^{\infty}(T-1)\varphi e^{i\lambda y}\frac{d\lambda}{2\pi} = \sum_{j=1}^{N} ie^{i\lambda_j(x+y)}X_j\Gamma_jF_jT_jY_j\begin{bmatrix}0\\1\end{bmatrix} + \sum_{j=1}^{N} i\int_{-\infty}^{\infty}dzK(x,z,t)e^{i\lambda_j(x+z)}Z_j\Gamma_jF_jT_jY_j.$$

Thus, as in the case of simple poles we form the Marchenko integral equation

$$0 = \overline{K}(x,y,t) + \left(\hat{R}(x+y,t) - \sum_{j=1}^{N} ie^{i\lambda_j(x+y)}X_j\Gamma_jF_jT_jY_j\begin{bmatrix}0\\1\end{bmatrix}\right) + \int_{-\infty}^{\infty}dzK(x,z,t)\left(\hat{R}(z+y,t) - \sum_{j=1}^{N} ie^{i\lambda_j(x+z)}Z_j\Gamma_jF_jT_jY_j\right). \tag{4.25}$$

Let us use $\Gamma_jF_jT_j =: C_j$, where the matrix C_j is defined as

$$C_j := \begin{bmatrix} \sum_{k=0}^{n_j-1}\frac{\gamma_{jk}}{k!}\frac{t_{j(k+1)}}{0!} & \sum_{k=0}^{n_j-2}\frac{\gamma_{jk}}{k!}\frac{t_{j(k+2)}}{1!} & \cdots & \sum_{k=0}^{0}\frac{\gamma_{jk}}{k!}\frac{t_{j(k+n_j)}}{(n_j-1)!} \\ \sum_{k=0}^{n_j-2}\frac{\binom{k+1}{1}\gamma_{jk}}{(k+1)!}\frac{t_{j(k+2)}}{0!} & \sum_{k=0}^{n_j-3}\frac{\binom{k+1}{1}\gamma_{jk}}{(k+1)!}\frac{t_{j(k+3)}}{1!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=0}^{0}\frac{\binom{k+n_j-1}{n_j-1}\gamma_{jk}}{(k+n_j-1)!}\frac{t_{j(k+n_j)}}{0!} & 0 & \cdots & 0 \end{bmatrix},$$

which simplifies to

$$C_j = \begin{bmatrix} \sum_{k=0}^{n_j-1} \frac{\gamma_{jk}}{k!} t_{j(k+1)} & \sum_{k=0}^{n_j-2} \frac{\gamma_{jk}}{k!} t_{j(k+2)} & \sum_{k=0}^{n_j-3} \frac{\gamma_{jk}}{k!} t_{j(k+3)} & \cdots & \sum_{k=0}^{0} \frac{\gamma_{jk}}{k!} t_{j(k+n_j)} \\ \sum_{k=0}^{n_j-2} \frac{\gamma_{jk}}{k!} t_{j(k+2)} & \binom{2}{1} \sum_{k=0}^{n_j-3} \frac{\gamma_{jk}}{k!} t_{j(k+3)} & \binom{3}{1} \sum_{k=0}^{n_j-4} \frac{\gamma_{jk}}{k!} t_{j(k+4)} & \cdots & 0 \\ \sum_{k=0}^{n_j-3} \frac{\gamma_{jk}}{k!} t_{j(k+3)} & \binom{3}{2} \sum_{k=0}^{n_j-4} \frac{\gamma_{jk}}{k!} t_{j(k+4)} & \binom{4}{2} \sum_{k=0}^{n_j-5} \frac{\gamma_{jk}}{k!} t_{j(k+5)} & \cdots & 0 \\ \sum_{k=0}^{n_j-4} \frac{\gamma_{jk}}{k!} t_{j(k+4)} & \binom{4}{3} \sum_{k=0}^{n_j-5} \frac{\gamma_{jk}}{k!} t_{j(k+5)} & \binom{5}{3} \sum_{k=0}^{n_j-6} \frac{\gamma_{jk}}{k!} t_{j(k+6)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{k=0}^{0} \frac{\gamma_{jk}}{k!} t_{j(k+n_j)} & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{4.26}$$

The Marchenko integral equation in (4.25) can then be written as

$$\begin{aligned} 0 = \overline{K}(x,y,t) + \left(\hat{R}(x+y) - \sum_{j=1}^{N} i e^{i\lambda_j (x+y)} X_j C_j Y_j \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ + \int_{-\infty}^{\infty} dz K(x,z,t) \left(\hat{R}(z+y) - \sum_{j=1}^{N} i e^{i\lambda_j (x+z)} Z_j C_j Y_j \right). \end{aligned}$$

Now consider the expansions of the matrices $X_j C_j Y_j$ and $Z_j C_j Y_j$:

$$\begin{aligned} X_j C_j Y_j = [c_{j0}(ix)^0 + c_{j1}(ix)^1 + \ldots + c_{j(n_j-1)}(ix)^{n_j-1} +](iy)^0 \\ +[c_{j1}(ix)^0 + c_{j2}(ix)^1 \tbinom{2}{1} + \ldots + c_{j(n_j-1)}(ix)^{n_j-2} \tbinom{n_j-1}{n_j-2}](iy)^1 \\ +[c_{j2}(ix)^0 + c_{j3}(ix)^1 \tbinom{3}{1} + \ldots + c_{j(n_j-1)}(ix)^{n_j-3} \tbinom{n_j-1}{n_j-3}](iy)^2 \\ + \ldots + [c_{j(n_j-1)}(ix)^0 \tbinom{n_j-1}{n_j-1}](iy)^{n_j-1}, \end{aligned} \tag{4.27}$$

$$\begin{aligned} Z_j C_j Y_j = [c_{j0}(iz)^0 + c_{j1}(iz)^1 + \ldots + c_{j(n_j-1)}(iz)^{n_j-1} +](iy)^0 \\ +[c_{j1}(iz)^0 + c_{j2}(iz)^1 \tbinom{2}{1} + \ldots + c_{j(n_j-1)}(iz)^{n_j-2} \tbinom{n_j-1}{n_j-2}](iy)^1 \\ +[c_{j2}(iz)^0 + c_{j3}(iz)^1 \tbinom{3}{1} + \ldots + c_{j(n_j-1)}(iz)^{n_j-3} \tbinom{n_j-1}{n_j-3}](iy)^2 \\ + \ldots + [c_{j(n_j-1)}(iz)^0 \tbinom{n_j-1}{n_j-1}](iy)^{n_j-1}. \end{aligned} \tag{4.28}$$

We would like to write the kernel of the Marchenko integral equation in (4.25) in a form similar to that of simple poles. We can reorder the terms of (4.27) and (4.28) to obtain

$$\begin{aligned}X_jC_jY_j = c_{j0}[(ix)^0(iy)^0] + c_{j1}[(ix)^1(iy)^0 + (ix)^0(iy)^1]\\
+c_{j2}[(ix)^2(iy)^0 + \tbinom{2}{1}(ix)^1(iy)^1 + (ix)^0(iy)^2]\\
+c_{j3}[(ix)^3(iy)^0 + \tbinom{3}{2}(ix)^2(iy)^1 + \tbinom{3}{1}(ix)^1(iy)^2 + (ix)^0(iy)^3] + \ldots\\
+c_{j(n_j-1)}[(ix)^{n_j-1}(iy)0 + \tbinom{n_j-1}{n_j-2}(ix)^{n_j-2}(iy)^1 + \tbinom{n_j-1}{n_j-3}(ix)^{n_j-3}(iy)^2 + \ldots + (ix)^0(iy)^{n_j-1}],\end{aligned}$$

$$\begin{aligned}Z_jC_jY_j = c_{j0}[(iz)^0(iy)^0] + c_{j1}[(iz)^1(iy)^0 + (iz)^0(iy)^1]\\
+c_{j2}[(iz)^2(iy)^0 + \tbinom{2}{1}(iz)^1(iy)^1 + (iz)^0(iy)^2]\\
+c_{j3}[(iz)^3(iy)^0 + \tbinom{3}{2}(iz)^2(iy)^1 + \tbinom{3}{1}(iz)^1(iy)^2 + (iz)^0(iy)^3] + \ldots\\
+c_{j(n_j-1)}[(iz)^{n_j-1}(iy)0 + \tbinom{n_j-1}{n_j-2}(iz)^{n_j-2}(iy)^1 + \tbinom{n_j-1}{n_j-3}(iz)^{n_j-3}(iy)^2 + \ldots + (iz)^0(iy)^{n_j-1}],\end{aligned}$$

which can be written as

$$X_jC_jY_j = -i\sum_{m=0}^{n_j-1}\frac{c_{jm}}{m!}(x+y)^m,$$

$$Z_jC_jY_j = -i\sum_{m=0}^{n_j-1}\frac{c_{jm}}{m!}(z+y)^m,$$

where we have defined

$$c_{jm} := \sum_{k=0}^{n_j-m-1}\frac{\gamma_{jk}}{k!}i^{m+1}t_{j(m+k+1)}. \qquad (4.29)$$

Note that c_{jm} is a function of t because γ_{jk} depends on t. Now we can rewrite our Marchenko integral equation in (4.25) as

$$\begin{aligned}0 = \overline{K}(x,y,t) + \left(\hat{R}(x+y) - i\sum_{j=1}^{N} ie^{i\lambda_j(x+y)}\sum_{m=0}^{n_j-1}\frac{c_{jm}}{m!}(x+y)^m\begin{bmatrix}0\\1\end{bmatrix}\right)\\
+\int_{-\infty}^{\infty} dzK(x,z,t)\left(\hat{R}(z+y) - i\sum_{j=1}^{N} ie^{i\lambda_j(x+z)}\sum_{m=0}^{n_j-1}\frac{c_{jm}}{m!}(z+y)^m\right).\end{aligned}$$

Finally, letting

$$\Omega(y,t) := \hat{R}(y,t) + \sum_{j=1}^{N}\sum_{m=0}^{n_j-1}\frac{c_{jm}(t)}{m!}y^m e^{i\lambda_j y},$$

we see that the Marchenko integral equation can be written as

$$0 = \overline{K}(x,y,t) + \Omega(x+y,t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_{-\infty}^{\infty} K(x,z,t)\Omega(z+y,t)dz, \quad y > x,$$

which has the same form as in the case of the simple poles. ∎

4.2 Marchenko Integral Equation Associated with $\mathbb{C}^-$

We now consider when the function $\overline{T}$ has $\overline{N}$ poles located at $\lambda = \overline{\lambda}_j$ in $\mathbb{C}^-$ each having order $\overline{n}_j$. The Marchenko integral equation will again differ from the previous case of simple poles in the term $\int_{-\infty}^{\infty} (\overline{T}-1)\overline{\varphi}e^{-i\lambda y}\frac{d\lambda}{2\pi}$. This is because of the evaluation of this integral requires more than a simple residue evaluation, since we cannot simply use the residue of $\overline{T}$ alone.

Theorem 4.2.1 *The Marchenko integral equation for the system in (2.1) associated with $\mathbb{C}^-$ is*

$$0 = K(x,y,t) + \overline{\Omega}(x+y)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_{-\infty}^{\infty} \overline{K}(x,z,t)\overline{\Omega}(z+y)dz, \quad x > y,$$

in the presence of a set of bound states $\{\overline{\lambda}_j\}_{j=1}^{\overline{N}}$ each of order $\overline{n}_j$ with kernel

$$\overline{\Omega}(s,t) := \hat{\overline{R}}(s,t) + \sum_{j=1}^{\overline{N}} \sum_{m=0}^{\overline{n}_j - 1} \frac{\bar{c}_{jm}}{m!} s^m e^{-i\overline{\lambda}_j s}.$$

Proof Consider the expansions about $\overline{\lambda}_j$:

$$\overline{T} - 1 = \frac{\bar{t}_{j\overline{n}_j}}{(\lambda - \overline{\lambda}_j)^{\overline{n}_j}} + \frac{\bar{t}_{j(\overline{n}_j-1)}}{(\lambda - \overline{\lambda}_j)^{\overline{n}_j-1}} + \ldots + \frac{\bar{t}_{j1}}{(\lambda - \overline{\lambda}_j)} + \ldots,$$

$$e^{-i\lambda y} = e^{-i\overline{\lambda}_j y}\left[1 - iy(\lambda - \overline{\lambda}_j) + \frac{(-iy)^2}{2!}(\lambda - \overline{\lambda}_j)^2 + \ldots + \frac{(-iy)^{\overline{n}_j-1}}{(\overline{n}_j-1)!}(\lambda - \overline{\lambda}_j)^{\overline{n}_j-1} + \ldots\right],$$

$$\overline{\varphi} = \overline{\varphi}(\overline{\lambda}_j) + \dot{\overline{\varphi}}(\overline{\lambda}_j)(\lambda - \overline{\lambda}_j) + \frac{\ddot{\overline{\varphi}}(\overline{\lambda}_j)}{2!}(\lambda - \overline{\lambda}_j) + \ldots + \frac{\overline{\varphi}^{(\overline{n}_j-1)}(\overline{\lambda}_j)}{(\overline{n}_j-1)!}(\lambda - \overline{\lambda}_j)^{\overline{n}_j-1} + \ldots. \quad (4.30)$$

Notice that we have

$$\bar{t}_{js} = \frac{1}{(\overline{n}_j - s)!}\frac{d^{\overline{n}_j - s}}{d\lambda^{\overline{n}_j - s}}\overline{T}\left(\lambda - \overline{\lambda}_j\right)|_{\lambda=\overline{\lambda}_j}.$$

When we multiply the terms of (4.30) and integrate, the only term contributing to the integral will be the term containing $\frac{1}{(\lambda - \overline{\lambda}_j)}$, and all others will contribute zero to the integral. Since there are $\overline{N}$ poles with respective orders $\overline{n}_j$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}(\overline{T}-1)\overline{\varphi}e^{-i\lambda y}\frac{d\lambda}{2\pi} &= \sum_{j=1}^{\overline{N}} ie^{-i\overline{\lambda}_j y}\left[\bar{t}_{j1}\overline{\varphi}(\overline{\lambda}_j) + \bar{t}_{j2}(\overline{\varphi}(\overline{\lambda}_j)(-iy) + \dot{\overline{\varphi}}(\overline{\lambda}_j))\right.\\
&+\bar{t}_{j3}\left(\overline{\varphi}(\overline{\lambda}_j)\frac{(-iy)^2}{2!} + \dot{\overline{\varphi}}(\overline{\lambda}_j)(-iy) + \frac{\ddot{\overline{\varphi}}(\overline{\lambda}_j)}{2!}\right)\\
&+\bar{t}_{j4}\left(\overline{\varphi}(\overline{\lambda}_j)\frac{(-iy)^3}{3!} + \dot{\overline{\varphi}}(\overline{\lambda}_j)\frac{(-iy)^2}{2!} + \frac{\ddot{\overline{\varphi}}(\overline{\lambda}_j)}{2!}(-iy) + \frac{\overline{\varphi}^{(3)}(\overline{\lambda}_j)}{3!}\right) + \ldots\\
&\left.+\,\bar{t}_{j\overline{n}_j}\left(\overline{\varphi}(\overline{\lambda}_j)\frac{(-iy)^{\overline{n}_j-1}}{(\overline{n}_j-1)!} + \dot{\overline{\varphi}}(\overline{\lambda}_j)\frac{(-iy)^{\overline{n}_j-2}}{(\overline{n}_j-2)!} + \ldots + \frac{\overline{\varphi}^{(\overline{n}_j-1)}(\overline{\lambda}_j)}{(\overline{n}_j-1)!}\right)\right],\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&= \sum_{j=1}^{\overline{N}} ie^{-i\overline{\lambda}_j y}\left[\overline{\varphi}(\overline{\lambda}_j)\left(\bar{t}_{j1} + \bar{t}_{j2}(-iy) + \bar{t}_{j3}\frac{(-iy)^2}{2!} + \ldots + \bar{t}_{j\overline{n}_j}\frac{(-iy)^{\overline{n}_j-1}}{(\overline{n}_j-1)!}\right)\right.\\
&\quad+\dot{\overline{\varphi}}(\overline{\lambda}_j)\left(\bar{t}_{j2} + \bar{t}_{j3}(-iy) + \bar{t}_{j4}\frac{(-iy)^2}{2!} + \ldots + \bar{t}_{j\overline{n}_j}\frac{(-iy)^{\overline{n}_j-2}}{(\overline{n}_j-2)!}\right)\\
&\quad+\frac{\ddot{\overline{\varphi}}(\overline{\lambda}_j)}{2!}\left(\bar{t}_{j3} + \bar{t}_{j4}(-iy) + \bar{t}_{j5}\frac{(-iy)^2}{2!} + \ldots + \bar{t}_{j\overline{n}_j}\frac{(-iy)^{\overline{n}_j-3}}{(\overline{n}_j-3)!}\right)\\
&\quad\left.+\ldots + \frac{\overline{\varphi}^{\overline{n}_j-1}(\overline{\lambda}_j)}{(\overline{n}_j-1)!}(\bar{t}_{j\overline{n}_j})\right].\end{aligned}$$

The integral term can now be written in matrix form as

$$\int_{-\infty}^{\infty}(\overline{T}-1)\overline{\varphi}e^{-i\lambda y}\frac{d\lambda}{2\pi} = \sum_{j=1}^{\overline{N}} ie^{-i\lambda_j y}\overline{\Phi}_j\overline{F}_j\overline{T}_j\overline{Y}_j, \tag{4.31}$$

where the matrices $\overline{\Phi}_j$, $\overline{F}_j$, $\overline{T}_j$, and $\overline{Y}_j$ are defined as

$$\overline{\Phi}_j := \begin{bmatrix} \overline{\varphi} & \dot{\overline{\varphi}} & \ldots & \overline{\varphi}^{(\overline{n}_j-1)} \end{bmatrix}, \qquad \overline{F}_j := \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \ldots & 0 \\ 0 & \frac{1}{1!} & 0 & \ldots & 0 \\ 0 & 0 & \frac{1}{2!} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{1}{(n_j-1)!} \end{bmatrix}, \tag{4.32}$$

$$\overline{T}_j := \begin{bmatrix} \frac{\bar{t}_{j1}}{0!} & \frac{\bar{t}_{j2}}{1!} & \ldots & \frac{\bar{t}_{j\overline{n}_j-1}}{(\overline{n}_j-2)!} & \frac{\bar{t}_{j\overline{n}_j}}{(\overline{n}_j-1)!} \\ \frac{\bar{t}_{j2}}{0!} & \frac{\bar{t}_{j3}}{1!} & \ldots & \frac{\bar{t}_{j\overline{n}_j}}{(\overline{n}_j-2)!} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\bar{t}_{j\overline{n}_j}}{0!} & 0 & \ldots & \ldots & 0 \end{bmatrix}, \qquad \overline{Y}_j := \begin{bmatrix} 1 \\ (iy)^1 \\ \vdots \\ (iy)^{\overline{n}_j-1} \end{bmatrix}.$$

As in the case of simple poles we would like to have approximate representations for $\overline{\varphi}(\overline{\lambda}_j)$ and its derivatives in terms of $\overline{\psi}(\overline{\lambda}_j)$. To find such representations we must consider the expansion of $\frac{1}{\overline{T}}$ about $\overline{\lambda}_j$. Define $\overline{a} := \frac{1}{\overline{T}}$ and observe that

$$\overline{a}(\lambda) = \overline{a}(\overline{\lambda}_j) + \dot{\overline{a}}(\overline{\lambda}_j)(\lambda - \overline{\lambda}_j) + \frac{\ddot{\overline{a}}(\overline{\lambda}_j)}{2!}(\lambda - \overline{\lambda}_j)^2 + \ldots + \frac{\overline{a}^{(\overline{n}_j)}(\overline{\lambda}_j)}{(\overline{n}_j)!}(\lambda - \overline{\lambda}_j)^{\overline{n}_j} + \ldots. \tag{4.33}$$

Since $\overline{\lambda}_j$ is a pole of order $\overline{n}_j$ of $\overline{T}$ it is a zero of order $\overline{n}_j$ of $\overline{a}(\lambda)$. Therefore,

$$\overline{a}(\overline{\lambda}_j) = \dot{\overline{a}}(\overline{\lambda}_j) = \ldots = \overline{a}^{(\overline{n}_j-1)}(\overline{\lambda}_j) = 0. \tag{4.34}$$

We know from page 8 that $\overline{a} = [\overline{\varphi}, \overline{\psi}]$. Consider the expansions of $\overline{\varphi}$ and $\overline{\psi}$ about $\overline{\lambda}_j$:

$$\overline{\varphi}(\lambda_j) = \overline{\varphi}(\overline{\lambda}_j) + \dot{\overline{\varphi}}(\overline{\lambda}_j)(\lambda - \overline{\lambda}_j) + \frac{\ddot{\overline{\varphi}}(\overline{\lambda}_j)}{2!}(\lambda - \overline{\lambda}_j)^2 + \ldots,$$

$$\overline{\psi}(\lambda_j) = \overline{\psi}(\overline{\lambda}_j) + \dot{\overline{\psi}}(\overline{\lambda}_j)(\lambda - \overline{\lambda}_j) + \frac{\ddot{\overline{\psi}}(\overline{\lambda}_j)}{2!}(\lambda - \overline{\lambda}_j)^2 + \ldots.$$

Thus,

$$\overline{a}(\lambda) = [\overline{\varphi}, \overline{\psi}] = \begin{vmatrix} \overline{\varphi}_1(\overline{\lambda}_j) + \dot{\overline{\varphi}}_1(\overline{\lambda}_j)(\lambda - \overline{\lambda}_j) + \ldots & \overline{\psi}_1(\overline{\lambda}_j) + \dot{\overline{\psi}}_1(\overline{\lambda}_j)(\lambda - \overline{\lambda}_j) + \ldots \\ \overline{\varphi}_2(\overline{\lambda}_j) + \dot{\overline{\varphi}}_2(\overline{\lambda}_j)(\lambda - \overline{\lambda}_j) + \ldots & \overline{\psi}_2(\overline{\lambda}_j) + \dot{\overline{\psi}}_2(\overline{\lambda}_j)(\lambda - \overline{\lambda}_j) + \ldots \end{vmatrix}$$

$$= [\overline{\varphi}_1(\overline{\lambda}_j)\overline{\psi}_2(\overline{\lambda}_j) - \overline{\varphi}_2(\overline{\lambda}_j)\overline{\psi}_1(\overline{\lambda}_j)] + (\lambda - \overline{\lambda}_j)[\overline{\varphi}_1(\overline{\lambda}_j)\dot{\overline{\psi}}_2(\overline{\lambda}_j) + \dot{\overline{\varphi}}_1(\overline{\lambda}_j)\overline{\psi}_2(\overline{\lambda}_j)$$
$$-\overline{\varphi}_2(\overline{\lambda}_j)\dot{\overline{\psi}}_1(\overline{\lambda}_j) - \overline{\psi}_1(\overline{\lambda}_j)\dot{\overline{\varphi}}_2(\overline{\lambda}_j)] + (\lambda - \overline{\lambda}_j)^2[\overline{\varphi}_1(\overline{\lambda}_j)\frac{\ddot{\overline{\psi}}_2(\overline{\lambda}_j)}{2!} + \overline{\psi}_2(\overline{\lambda}_j)\frac{\ddot{\overline{\varphi}}_1(\overline{\lambda}_j)}{2!}$$
$$+\dot{\overline{\varphi}}_1(\overline{\lambda}_j)\dot{\overline{\psi}}_2(\overline{\lambda}_j) - \overline{\psi}_1(\overline{\lambda}_j)\frac{\ddot{\overline{\varphi}}_2(\overline{\lambda}_j)}{2!} - \overline{\varphi}_2(\overline{\lambda}_j)\frac{\ddot{\overline{\psi}}_1(\overline{\lambda}_j)}{2!} - \dot{\overline{\psi}}_1(\overline{\lambda}_j)\dot{\overline{\varphi}}_2(\overline{\lambda}_j)] + \dots,$$

where we recall that the subscripts 1 and 2 indicate the first and second components of the Jost solutions. Now if we compare this expansion of $\overline{a}(\lambda)$ to (4.33) and use (4.34) we have the following equalities:

$$\begin{aligned}
0 &= \overline{a}(\overline{\lambda}_j) = \overline{\varphi}_1(\overline{\lambda}_j)\overline{\psi}_2(\overline{\lambda}_j) - \overline{\varphi}_2(\overline{\lambda}_j)\overline{\psi}_1(\overline{\lambda}_j),\\
0 &= \dot{\overline{a}}(\overline{\lambda}_j) = \overline{\varphi}_1(\overline{\lambda}_j)\dot{\overline{\psi}}_2(\overline{\lambda}_j) + \dot{\overline{\varphi}}_1(\overline{\lambda}_j)\overline{\psi}_2(\overline{\lambda}_j) - \overline{\varphi}_2(\overline{\lambda}_j)\dot{\overline{\psi}}_1(\overline{\lambda}_j) - \overline{\psi}_1(\overline{\lambda}_j)\dot{\overline{\varphi}}_2(\overline{\lambda}_j),\\
0 &= \ddot{\overline{a}}(\overline{\lambda}_j) = \overline{\varphi}_1(\overline{\lambda}_j)\tfrac{\ddot{\overline{\psi}}_2(\overline{\lambda}_j)}{2!} + \overline{\psi}_2(\overline{\lambda}_j)\tfrac{\ddot{\overline{\varphi}}_1(\overline{\lambda}_j)}{2!} + \dot{\overline{\varphi}}_1(\overline{\lambda}_j)\dot{\overline{\psi}}_2(\overline{\lambda}_j) - \overline{\psi}_1(\overline{\lambda}_j)\tfrac{\ddot{\overline{\varphi}}_2(\overline{\lambda}_j)}{2!}\\
&\qquad -\overline{\varphi}_2(\overline{\lambda}_j)\tfrac{\ddot{\overline{\psi}}_1(\overline{\lambda}_j)}{2!} - \dot{\overline{\psi}}_1(\overline{\lambda}_j)\dot{\overline{\varphi}}_2(\overline{\lambda}_j),\\
&\vdots\\
0 &= \overline{a}^{(\overline{n}_j-1)}(\overline{\lambda}_j) = \sum_{l=0}^{\overline{n}_j-1}\binom{\overline{n}_j-1}{l}[\overline{\varphi}^{(\overline{n}_j-1-l)}(\overline{\lambda}_j), \overline{\psi}^{(l)}(\overline{\lambda}_j)].
\end{aligned} \tag{4.35}$$

The first equality in (4.35) implies

$$\begin{vmatrix} \overline{\varphi}_1(\overline{\lambda}_j) & \overline{\psi}_1(\overline{\lambda}_j) \\ \overline{\varphi}_2(\overline{\lambda}_j) & \overline{\psi}_2(\overline{\lambda}_j) \end{vmatrix} = 0,$$

which indicates that $\overline{\varphi}(\overline{\lambda}_j)$ and $\overline{\psi}(\overline{\lambda}_j)$ are linearly dependent. Then there exists $\overline{\gamma}_{j0}$ such that

$$\overline{\varphi}(\overline{\lambda}_j) = \overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j). \tag{4.36}$$

By substituting (4.36) into the second equation in (4.35) we obtain

$$\overline{\gamma}_{j0}\overline{\psi}_1(\overline{\lambda}_j)\dot{\overline{\psi}}_2(\overline{\lambda}_j) + \dot{\overline{\varphi}}_1(\overline{\lambda}_j)\overline{\psi}_2(\overline{\lambda}_j) - \overline{\gamma}_{j0}\overline{\psi}_2(\overline{\lambda}_j)\dot{\overline{\psi}}_1(\overline{\lambda}_j) - \overline{\psi}_1(\overline{\lambda}_j)\dot{\overline{\varphi}}_2(\overline{\lambda}_j) = 0,$$

which can be written as

$$\begin{vmatrix} \dot{\overline{\varphi}}_1(\overline{\lambda}_j) - \overline{\gamma}_{j0}\dot{\overline{\psi}}_1(\overline{\lambda}_j) & \overline{\psi}_1(\overline{\lambda}_j) \\ \dot{\overline{\varphi}}_2(\overline{\lambda}_j) - \overline{\gamma}_{j0}\dot{\overline{\psi}}_2(\overline{\lambda}_j) & \overline{\psi}_2(\overline{\lambda}_j) \end{vmatrix} = 0. \tag{4.37}$$

Now (4.37) implies that $\dot{\overline{\varphi}}(\overline{\lambda}_j) - \overline{\gamma}_{j0}\dot{\overline{\psi}}(\overline{\lambda}_j)$ and $\overline{\psi}(\overline{\lambda}_j)$ are linearly dependent. Therefore, there exists $\overline{\gamma}_{j1}$ such that

$$\dot{\overline{\varphi}}(\overline{\lambda}_j) = \overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) + \overline{\gamma}_{j1}\overline{\psi}(\overline{\lambda}_j). \tag{4.38}$$

Again substituting (4.36) and (4.38) into the third equation in (4.35) we get

$$\begin{aligned} 0 = \overline{\gamma}_{j0}\overline{\psi}_1(\overline{\lambda}_j)\tfrac{\ddot{\overline{\psi}}_2(\overline{\lambda}_j)}{2!} + \overline{\psi}_2(\overline{\lambda}_j)\tfrac{\ddot{\overline{\varphi}}_1(\overline{\lambda}_j)}{2!} + \overline{\gamma}_{j0}\dot{\overline{\psi}}_1(\overline{\lambda}_j)\dot{\overline{\psi}}_2(\overline{\lambda}_j) + \overline{\gamma}_{j1}\overline{\psi}_1(\overline{\lambda}_j)\dot{\overline{\psi}}_2(\overline{\lambda}_j) \\ -\overline{\psi}_1\tfrac{\ddot{\overline{\varphi}}_2(\overline{\lambda}_j)}{2!} - \overline{\gamma}_{j0}\overline{\psi}_2(\overline{\lambda}_j)\tfrac{\ddot{\overline{\psi}}_1(\overline{\lambda}_j)}{2!} - \overline{\gamma}_{j0}\dot{\overline{\psi}}(\overline{\lambda}_j)\dot{\overline{\psi}}_2(\overline{\lambda}_j) - \overline{\gamma}_{j1}\dot{\overline{\psi}}_1(\overline{\lambda}_j)\overline{\psi}_2(\lambda_j), \end{aligned}$$

which can be written as

$$\begin{vmatrix} \ddot{\overline{\varphi}}_1(\overline{\lambda}_j) - 2\overline{\gamma}_{j1}\dot{\overline{\psi}}_1(\overline{\lambda}_j) - \overline{\gamma}_{j0}\ddot{\overline{\psi}}_1(\overline{\lambda}_j) & \overline{\psi}_1(\overline{\lambda}_j) \\ \ddot{\overline{\varphi}}_2(\overline{\lambda}_j) - 2\overline{\gamma}_{j1}\dot{\overline{\psi}}_2(\overline{\lambda}_j) - \overline{\gamma}_{j0}\ddot{\overline{\psi}}_2(\overline{\lambda}_j) & \overline{\psi}_2(\overline{\lambda}_j) \end{vmatrix} = 0. \tag{4.39}$$

From (4.39) we see that, $\ddot{\overline{\varphi}}(\overline{\lambda}_j) - 2\overline{\gamma}_{j1}\dot{\overline{\psi}}(\overline{\lambda}_j) - \overline{\gamma}_{j0}\ddot{\overline{\psi}}(\overline{\lambda}_j)$ and $\overline{\psi}(\overline{\lambda}_j)$ are linearly dependent. Therefore, there exists $\overline{\gamma}_{j2}$ such that

$$\ddot{\overline{\varphi}}(\overline{\lambda}_j) = \overline{\gamma}_{j0}\ddot{\overline{\psi}}(\overline{\lambda}_j) + 2\overline{\gamma}_{j1}\dot{\overline{\psi}}(\overline{\lambda}_j) + \overline{\gamma}_{j2}\overline{\psi}(\overline{\lambda}_j). \tag{4.40}$$

Now we wish to prove that there is a similar representation for $\overline{\varphi}^{(\overline{n}_j-1)}(\overline{\lambda}_j)$. Recall that for any $n = 0, 1, \dots, \overline{n}_j - 1$ we have

$$0 = \overline{a}^{(n)}(\overline{\lambda}_j) = \sum_{l=0}^{n} \binom{n}{l} [\overline{\varphi}^{(n-l)}(\overline{\lambda}_j), \overline{\psi}^{(l)}(\overline{\lambda}_j)]. \tag{4.41}$$

As in the previous section, (4.41) can be written as

$$0 = [\overline{\varphi}^{(n)}(\overline{\lambda}_j), \overline{\psi}^{(0)}(\overline{\lambda}_j)] + \binom{n}{1}\overline{\gamma}_{j(n-1)}[\overline{\psi}^{(0)}(\overline{\lambda}_j), \overline{\psi}^{(1)}(\overline{\lambda}_j)] + \ldots + \binom{n}{n}\overline{\gamma}_{j0}[\overline{\psi}^{(0)}(\overline{\lambda}_j), \overline{\psi}^{(n)}(\overline{\lambda}_j)],$$

or equivalently

$$0 = \left[\overline{\varphi}^{(n)}(\overline{\lambda}_j) - \binom{n}{1}\overline{\gamma}_{j(n-1)}\overline{\psi}^{(1)}(\overline{\lambda}_j) - \binom{n}{2}\overline{\gamma}_{j(n-2)}\overline{\psi}^{(2)}(\overline{\lambda}_j) - \ldots - \binom{n}{n}\overline{\gamma}_{j0}\overline{\psi}^{(n)}(\overline{\lambda}_j), \overline{\psi}^{(0)}(\overline{\lambda}_j)\right].$$

Therefore, there exists $\overline{\gamma}_{j(n+1)}$ such that

$$\overline{\varphi}^{(n)}(\overline{\lambda}_j) - \binom{n}{1}\overline{\gamma}_{j(n-1)}\overline{\psi}^{(1)}(\overline{\lambda}_j) - \binom{n}{2}\overline{\gamma}_{j(n-2)}\overline{\psi}^{(2)}(\overline{\lambda}_j) - \ldots - \binom{n}{n}\overline{\gamma}_{j0}\overline{\psi}^{(n)}(\overline{\lambda}_j) = \overline{\gamma}_{j(n+1)}\overline{\psi}^{(0)}(\overline{\lambda}_j),$$

or equivalently

$$\overline{\varphi}^{(n)}(\overline{\lambda}_j) = \binom{n}{1}\overline{\gamma}_{j(n-1)}\overline{\psi}^{(1)}(\overline{\lambda}_j) - \binom{n}{2}\overline{\gamma}_{j(n-2)}\overline{\psi}^{(2)}(\overline{\lambda}_j) + \ldots + \binom{n}{n}\overline{\gamma}_{j0}\overline{\psi}^{(n)}(\overline{\lambda}_j) + \overline{\gamma}_{j(n+1)}\overline{\psi}^{(0)}(\overline{\lambda}_j).$$

Therefore, we see that for $l = 0, \ldots, \overline{n}_j - 1$ we have

$$\overline{\varphi}^{(l)}(\overline{\lambda}_j) = \sum_{k=0}^{l} \binom{l}{k}\overline{\gamma}_{j(l-k)}\overline{\psi}^{(k)}(\overline{\lambda}_j). \tag{4.42}$$

The above expression can be written in matrix form as

$$\overline{\Phi}_j = \overline{\Psi}_j \overline{\Gamma}_j, \tag{4.43}$$

where the matrices $\overline{\Phi}_j$, $\overline{\Psi}_j$, and $\overline{\Gamma}_j$ are defined as

$$\overline{\Phi}_j := \begin{bmatrix} \overline{\varphi}(\overline{\lambda}_j) & \dot{\overline{\varphi}}(\overline{\lambda}_j) & \ldots & \overline{\varphi}^{(\overline{n}_j-1)}(\overline{\lambda}_j) \end{bmatrix},$$

$$\overline{\Psi}_j := \begin{bmatrix} \overline{\psi}(\overline{\lambda}_j) & \dot{\overline{\psi}}(\overline{\lambda}_j) & \ldots & \overline{\psi}^{(\overline{n}_j-1)}(\overline{\lambda}_j) \end{bmatrix}, \tag{4.44}$$

$$\overline{\Gamma}_j := \begin{bmatrix} \overline{\gamma}_{j0} & \overline{\gamma}_{ji} & \cdots & \overline{\gamma}_{j(\overline{n}_j-1)} \\ 0 & \binom{1}{1}\overline{\gamma}_{j0} & \cdots & \binom{\overline{n}_j-1}{1}\overline{\gamma}_{j(\overline{n}_j-2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \binom{\overline{n}_j-1}{\overline{n}_j-1}\overline{\gamma}_{j0} \end{bmatrix}.$$

Substituting (4.43) into (4.31) we get

$$\int_{-\infty}^{\infty} (\overline{T}-1)\overline{\varphi}e^{-i\lambda y}\frac{d\lambda}{2\pi} = \sum_{j=1}^{\overline{N}} ie^{-i\overline{\lambda}_j y}\overline{\Psi}_j\overline{\Gamma}_j\overline{F}_j\overline{T}_j\overline{Y}_j. \tag{4.45}$$

From (3.4) we get

$$\overline{\psi}(\overline{\lambda}_j) = \begin{bmatrix} e^{-i\lambda_j x} \\ 0 \end{bmatrix} + \int_{-\infty}^{\infty} \overline{K}(x,z,t)e^{-i\overline{\lambda}_j z}dz.$$

From the λ-derivative of (3.4) we obtain for $n = 0, 1, 2, \ldots$

$$\overline{\psi}^{(n)}(\overline{\lambda}_j) = \begin{bmatrix} (-ix)^n e^{-i\lambda_j x} \\ 0 \end{bmatrix} + \int_{-\infty}^{\infty} \overline{K}(x,z,t)e^{-i\overline{\lambda}_j z}(-iz)^n dz.$$

We can express $\overline{\Psi}_j$ as

$$\overline{\Psi}_j = \begin{bmatrix} e^{-i\lambda_j x} \\ 0 \end{bmatrix} \overline{X}_j + \int_{-\infty}^{\infty} \overline{K}(x,z,t) e^{-i\bar{\lambda}_j z} \overline{Z}_j dz,$$

where $\overline{X}_j$ and $\overline{Z}_j$ are defined as

$$\overline{X}_j := \begin{bmatrix} 1 & (-ix) & \dots & (-ix)^{\overline{n}_j - 1} \end{bmatrix},$$

$$\overline{Z}_j := \begin{bmatrix} 1 & (-iz) & \dots & (-iz)^{\overline{n}_j - 1} \end{bmatrix}.$$

We can then write the integral in (4.45) as

$$\int_{-\infty}^{\infty} (\overline{T} - 1)\overline{\varphi} e^{-i\lambda y} \frac{d\lambda}{2\pi} = \sum_{j=1}^{\overline{N}} i e^{-i\bar{\lambda}_j (x+y)} \overline{X}_j \overline{\Gamma}_j \overline{T}_j \overline{Y}_j \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$+ \sum_{j=1}^{\overline{N}} i \int_{-\infty}^{\infty} \overline{K}(x,z,t) e^{-i\bar{\lambda}_j (x+z)} \overline{Z}_j \overline{\Gamma}_j \overline{F}_j \overline{T}_j \overline{Y}_j dz,$$

where $\overline{F}_j$, $\overline{T}_j$, and $\overline{Y}_j$ are defined in (4.32) and $\overline{\Gamma}_j$ is defined in (4.44). As in the case of simple poles we can now form the Marchenko integral equation

$$0 = K(x,y,t) + \left(\hat{\overline{R}}(x+y,t) - \sum_{j=1}^{\overline{N}} i e^{-i\bar{\lambda}_j (x+y)} \overline{X}_j \overline{\Gamma}_j \overline{F}_j \overline{T}_j \overline{Y}_j \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$
$$+ \int_{-\infty}^{\infty} \overline{K}(x,z,t) \left(\hat{\overline{R}}(z+y,t) - \sum_{j=1}^{\overline{N}} i e^{-i\lambda_j (x+z)} \overline{Z}_j \overline{\Gamma}_j \overline{F}_j \overline{T}_j \overline{Y}_j \right) dz.$$

We now let $\overline{\Gamma}_j \overline{F}_j \overline{T}_j =: \overline{C}_j$, where we have

$$\overline{C}_j := \begin{bmatrix} \displaystyle\sum_{k=0}^{\overline{n}_j - 1} \frac{\overline{\gamma}_k}{k!} \frac{\bar{t}_{k+1}}{0!} & \displaystyle\sum_{k=0}^{\overline{n}_j - 2} \frac{\overline{\gamma}_{jk}}{k!} \frac{\bar{t}_{j(k+2)}}{1!} & \dots & \displaystyle\sum_{k=0}^{0} \frac{\overline{\gamma}_{jk}}{k!} \frac{\bar{t}_{j(k+\overline{n}_j)}}{(\overline{n}_j - 1)!} \\ \displaystyle\sum_{k=0}^{\overline{n}_j - 2} \frac{\binom{k+1}{1}\overline{\gamma}_{jk}}{(k+1)!} \frac{\bar{t}_{j(k+2)}}{0!} & \displaystyle\sum_{k=0}^{\overline{n}_j - 3} \frac{\binom{k+1}{1}\overline{\gamma}_{jk}}{(k+1)!} \frac{\bar{t}_{j(k+3)}}{1!} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \displaystyle\sum_{k=0}^{0} \frac{\binom{k+n_j-1}{n_j-1}\overline{\gamma}_{jk}}{(k+\overline{n}_j - 1)!} \frac{\bar{t}_{j(k+\overline{n}_j)}}{0!} & 0 & \dots & 0 \end{bmatrix},$$

which can also be written as

$$\overline{C}_j = \begin{bmatrix} \sum_{k=0}^{\overline{n}_j-1} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+1)} & \sum_{k=0}^{\overline{n}_j-2} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+2)} & \sum_{k=0}^{\overline{n}_j-3} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+3)} & \dots & \sum_{k=0}^{0} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+n_j)} \\ \sum_{k=0}^{\overline{n}_j-2} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+2)} & \binom{2}{1}\sum_{k=0}^{\overline{n}_j-3} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+3)} & \binom{3}{1}\sum_{k=0}^{\overline{n}_j-4} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+4)} & \dots & 0 \\ \sum_{k=0}^{\overline{n}_j-3} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+3)} & \binom{3}{2}\sum_{k=0}^{\overline{n}_j-4} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+4)} & \binom{4}{2}\sum_{k=0}^{\overline{n}_j-5} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+5)} & \dots & 0 \\ \sum_{k=0}^{\overline{n}_j-4} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+4)} & \binom{4}{3}\sum_{k=0}^{\overline{n}_j-5} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+5)} & \binom{5}{3}\sum_{k=0}^{\overline{n}_j-6} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+6)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{k=0}^{0} \frac{\overline{\gamma}_{jk}}{k!}\overline{t}_{j(k+\overline{n}_j)} & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Thus, we can write the Marchenko integral equation as

$$\begin{aligned} 0 = K(x,y,t) + \left(\hat{\overline{R}}(x+y) - \sum_{j=1}^{\overline{N}} i e^{-i\overline{\lambda}_j(x+y)} \overline{X}_j \overline{C}_j \overline{Y}_j \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ + \int_{-\infty}^{\infty} dz \overline{K}(x,z,t) \left(\hat{\overline{R}}(z+y) - \sum_{j=1}^{\overline{N}} i e^{-i\overline{\lambda}_j(x+z)} \overline{Z}_j \overline{C}_j \overline{Y}_j \right). \end{aligned}$$

Consider the expansions of the matrices $\overline{X}_j\overline{C}_j\overline{Y}_j$ and $\overline{Z}_j\overline{C}_j\overline{Y}_j$:

$$\begin{aligned} \overline{X}_j\overline{C}_j\overline{Y}_j &= [\overline{c}_{j0}(ix)^0 + \overline{c}_{j1}(ix)^1 + \ldots + \overline{c}_{j,\overline{n}_j-1}(ix)^{\overline{n}_j-1} +](iy)^0 \\ &+ [\overline{c}_{j1}(ix)^0 + \overline{c}_{j2}(ix)^1 \binom{2}{1} + \ldots + \overline{c}_{j,\overline{n}_j-1}(ix)^{\overline{n}_j-2} \binom{\overline{n}_j-1}{\overline{n}_j-2}](iy)^1 \\ &+ [\overline{c}_{j2}(ix)^0 + \overline{c}_{j3}(ix)^1 \binom{3}{1} + \ldots + \overline{c}_{j,\overline{n}_j-1}(ix)^{\overline{n}_j-3} \binom{\overline{n}_j-1}{\overline{n}_j-3}](iy)^2 \\ &+ \ldots + [\overline{c}_{j,\overline{n}_j-1}(ix)^0 \binom{\overline{n}_j-1}{\overline{n}_j-1}](iy)^{\overline{n}_j-1}, \end{aligned} \tag{4.46}$$

$$\begin{aligned}\overline{Z}_j\overline{C}_j\overline{Y}_j &= [\overline{c}_{j0}(iz)^0+\overline{c}_{j1}(iz)^1+\ldots+\overline{c}_{j,\overline{n}_j-1}(iz)^{\overline{n}_j-1}+](iy)^0 \qquad (4.47)\\ &+ [\overline{c}_{j1}(iz)^0+\overline{c}_{j2}(iz)^1\binom{2}{1}+\ldots+\overline{c}_{j,\overline{n}_j-1}(iz)^{\overline{n}_j-2}\binom{\overline{n}_j-1}{\overline{n}_j-2}](iy)^1\\ &+ [\overline{c}_{j2}(iz)^0+\overline{c}_{j3}(iz)^1\binom{3}{1}+\ldots+\overline{c}_{j,\overline{n}_j-1}(iz)^{\overline{n}_j-3}\binom{\overline{n}_j-1}{\overline{n}_j-3}](iy)^2\\ &+ \ldots+[\overline{c}_{j,\overline{n}_j-1}(iz)^0\binom{\overline{n}_j-1}{\overline{n}_j-1}](iy)^{\overline{n}_j-1}. \qquad (4.48)\end{aligned}$$

We would like to write the kernel of the Marchenko integral equation in a form similar to the case of simple poles. We can reorder the terms in (4.48) to obtain

$$\begin{gathered}\overline{X}_j\overline{C}_j\overline{Y}_j=\overline{c}_{j0}[(ix)^0(iy)^0]+\overline{c}_{j1}[(ix)^1(iy)^0+(ix)^0(iy)^1]\\ +\overline{c}_{j2}[(ix)^2(iy)^0+\tbinom{2}{1}(ix)^1(iy)^1+(ix)^0(iy)^2]\\ +\overline{c}_{j3}[(ix)^3(iy)^0+\tbinom{3}{2}(ix)^2(iy)^1+\tbinom{3}{1}(ix)^1(iy)^2+(ix)^0(iy)^3]+\ldots\\ +\overline{c}_{j(\overline{n}_j-1)}[(ix)^{\overline{n}_j-1}(iy)0+\tbinom{\overline{n}_j-1}{\overline{n}_j-2}(ix)^{\overline{n}_j-2}(iy)^1+\tbinom{\overline{n}_j-1}{\overline{n}_j-3}(ix)^{\overline{n}_j-3}(iy)^2+\ldots+(ix)^0(iy)^{\overline{n}_j-1}],\end{gathered}$$

$$\begin{gathered}\overline{Z}_j\overline{C}_j\overline{Y}_j=\overline{c}_{j0}[(iz)^0(iy)^0]+c_{j1}[(iz)^1(iy)^0+(iz)^0(iy)^1]\\ +\overline{c}_{j2}[(iz)^2(iy)^0+\tbinom{2}{1}(iz)^1(iy)^1+(iz)^0(iy)^2]\\ +\overline{c}_{j3}[(iz)^3(iy)^0+\tbinom{3}{2}(iz)^2(iy)^1+\tbinom{3}{1}(iz)^1(iy)^2+(iz)^0(iy)^3]+\ldots\\ +\overline{c}_{j(\overline{n}_j-1)}[(iz)^{\overline{n}_j-1}(iy)^0+\tbinom{\overline{n}_j-1}{\overline{n}_j-2}(iz)^{\overline{n}_j-2}(iy)^1+\tbinom{\overline{n}_j-1}{\overline{n}_j-3}(iz)^{\overline{n}_j-3}(iy)^2+\ldots+(iz)^0(iy)^{\overline{n}_j-1}],\end{gathered}$$

which can be written as

$$\overline{X}_j\overline{C}_j\overline{Y}_j=-i\sum_{m=0}^{\overline{n}_j-1}\frac{\overline{c}_{jm}}{m!}(x+y)^m,$$

$$\overline{Z}_j\overline{C}_j\overline{Y}_j=-i\sum_{m=0}^{\overline{n}_j-1}\frac{\overline{c}_{jm}}{m!}(z+y)^m,$$

where $\overline{c}_{jm}$ is defined as

$$\overline{c}_{jm}:=\sum_{k=0}^{\overline{n}_j-m-1}\frac{\overline{\gamma}_{jk}}{k!}i^{m+1}\overline{t}_{j(m+k+1)}. \qquad (4.49)$$

We can rewrite our integral equation as

$$0 = K(x,y,t) + \left(\hat{\overline{R}}(x+y,t) + \sum_{j=1}^{\overline{N}} e^{-i\overline{\lambda}_j(x+y)} \sum_{m=0}^{\overline{n}_j-1} \frac{\overline{c}_{jm}}{m!}(x+y)^m \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$
$$+ \int_{-\infty}^{\infty} dz \overline{K}(x,z,t) \left(\hat{\overline{R}}(z+y,t) + \sum_{j=1}^{\overline{N}} e^{-i\overline{\lambda}_j(x+z)} \sum_{m=0}^{\overline{n}_j-1} \frac{\overline{c}_{jm}}{m!}(z+y)^m \right).$$

Thus, it is appropriate to write the kernel of the Marchenko integral equation as

$$\overline{\Omega}(y,t) := \hat{\overline{R}}(y,t) + \sum_{j=1}^{\overline{N}} \sum_{m=0}^{\overline{n}_j-1} \frac{\overline{c}_{jm}(t)}{m!} y^m e^{-i\overline{\lambda}_j y}.$$

Now for $x > y$, the Marchenko integral equation associated with $\mathbb{C}^-$ can be written as

$$0 = K(x,y,t) + \overline{\Omega}(x+y,t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_{-\infty}^{\infty} \overline{K}(x,z,t)\overline{\Omega}(z+y,t)dz$$

resembling the case of simple poles. ∎

Thus, we have accomplished our goal for writing the two Marchenko integral equations in a simple form. Summarizing, we can write the system of Marchenko integral equations as

$$\begin{cases} 0 = \overline{K}(x,y,t) + \Omega(x+y,t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \displaystyle\int_x^{\infty} K(x,s,t)\Omega(s+y,t)ds, \quad y > x, \\ 0 = K(x,y,t) + \overline{\Omega}(x+y,t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \displaystyle\int_{-\infty}^{x} \overline{K}(x,s,t)\overline{\Omega}(s+y,t)ds, \quad x > y. \end{cases} \tag{4.50}$$

CHAPTER 5

TIME EVOLUTION OF THE SCATTERING DATA ASSOCIATED WITH BOUND STATE POLES WITH HIGHER MULTIPLICITY

We now consider the time evolution of the scattering data in the presence of bound states of multiple orders. Since the reflection coefficients are not dependent on the bound states, the time evolution for them is the same as in the case of simple bound state poles. Even though the time evolution of the reflection coefficients is known [20], we include a derivation of their time evolution for the convince of the reader. In this chapter we first derive the time evolution of the dependency constants. We then exploit the linear relationship between the norming constants and the dependency constants and hence obtain the time evolution of the bound state norming constants in the presence of bound state poles of any multiplicity.

5.1 Lax Pair

We begin by reviewing the Lax pair associated with the NLS equation. Rewrite the system in (2.1) as

$$\frac{d}{dx}\Phi = -i\lambda J\Phi + Q\Phi, \tag{5.1}$$

where Φ, J, and Q are defined as

$$\Phi := \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \qquad J := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad Q := \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix}.$$

Notice that $J^2 = I$, where I denotes the 2×2 identity matrix. We can rewrite (5.1) as

$$iJ\partial_x\Phi = \lambda\Phi + iJQ\Phi,$$

or equivalently as

$$\left(iJ\partial_x - iJQ\right)\Phi = \lambda\Phi.$$

It is known [20] that the Lax pair operators $\mathcal{L}$ and $\mathcal{A}$ defined as

$$\mathcal{L} := iJ\partial_x - iJQ,$$

$$\mathcal{A} := 2iJ\frac{\partial^2}{\partial x^2} + \begin{bmatrix} 0 & -2iq \\ 2ir & 0 \end{bmatrix} + \begin{bmatrix} -iqr & -iq_x \\ ir_x & iqr \end{bmatrix},$$

form the Lax pair associated with (2.1). In other words

$$0 = \mathcal{L}_t + \mathcal{L}\mathcal{A} - \mathcal{A}\mathcal{L}, \tag{5.2}$$

yields the coupled system of nonlinear equations

$$\begin{cases} iq_t + q_{xx} - 2q^2 r = 0 \\ ir_t - r_{xx} + 2r^2 q = 0. \end{cases} \tag{5.3}$$

If we let $r = -q^*$, then the two equations in (5.3) both reduce to the NLS equation

$$iq_t + q_{xx} + 2|q|^2 q = 0.$$

5.2 Time Evolution of Scattering Data Associated with $\mathbb{C}^+$

With the help of the Lax pair we will consider the time evolution of the scattering data. It is known [1, 2, 3, 4, 8, 16, 20] that $\psi_t - \mathcal{A}\psi$ is a solution to $\mathcal{L}\psi = \lambda\psi$, i.e.

$$\mathcal{L}\left(\psi_t - \mathcal{A}\psi\right) = \lambda\left(\psi_t - \mathcal{A}\psi\right).$$

This is equivalent to saying that $\psi_t - \mathcal{A}\psi$ can be written as a linear combination of the Jost solutions ψ and φ as

$$\psi_t - \mathcal{A}\psi = c_1\psi + c_2\varphi. \tag{5.4}$$

Using the asymptotics of the Jost solutions appearing on pages 8 and 9, let us determine c_1 and c_2. As $x \to \pm\infty$, we know that q, r, q_x, $r_x \to 0$ at each fixed t. Therefore, $\mathcal{A} \to 2iJ\partial_x^2$ as $x \to \pm\infty$. As $x \to +\infty$, from (5.4) we get

$$\partial_t \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x^2 \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{T} e^{-i\lambda x} \\ \frac{R}{T} e^{i\lambda x} \end{bmatrix},$$

which yields

$$\begin{bmatrix} 0 \\ -2i\lambda^2 e^{i\lambda x} \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{T} e^{-i\lambda x} \\ \frac{R}{T} e^{i\lambda x} \end{bmatrix}.$$

Hence, we have $c_1 = -2i\lambda^2$ and $c_2 = 0$. Thus, we see that the Jost solution ψ evolves according to

$$\psi_t - \mathcal{A}\psi = -2i\lambda^2 \psi. \tag{5.5}$$

Similarly we know [1, 2, 3, 4, 8, 13, 16, 20] that $\varphi_t - \mathcal{A}\varphi$ is also a solution to $\mathcal{L}\varphi = \lambda\varphi$, and hence

$$\mathcal{L}\left(\varphi_t - \mathcal{A}\varphi\right) = \lambda\left(\varphi_t - \mathcal{A}\varphi\right).$$

Consequently, $\varphi_t - \mathcal{A}\varphi$ can be written as a linear combination of the Jost solutions ψ and φ as

$$\varphi_t - \mathcal{A}\varphi = c_3\psi + c_4\varphi. \tag{5.6}$$

With the help of the asymptotics on pages 8 and 9, as $x \to -\infty$ from (5.6) we obtain

$$\partial_t \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} - 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x^2 \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} = c_3 \begin{bmatrix} \frac{L}{T} e^{-i\lambda x} \\ \frac{R}{T} e^{i\lambda x} \end{bmatrix} + c_4 \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix},$$

which simplifies to

$$\begin{bmatrix} 2i\lambda^2 e^{-i\lambda x} \\ 0 \end{bmatrix} = c_3 \begin{bmatrix} \frac{L}{T} e^{-i\lambda x} \\ \frac{R}{T} e^{i\lambda x} \end{bmatrix} + c_4 \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}.$$

Therefore, we have $c_3 = 0$ and $c_4 = 2i\lambda^2$. We then have the following time evolution for the Jost solution φ:

$$\varphi_t - \mathcal{A}\varphi = 2i\lambda^2\varphi. \tag{5.7}$$

The time evolution of the scattering coefficients $L(\lambda, t)$, $R(\lambda, t)$, and $T(\lambda, t)$ are already known:

$$L(\lambda, t) = L(\lambda, 0)e^{-4i\lambda^2 t}, \quad R(\lambda, t) = R(\lambda, 0)e^{4i\lambda^2 t}, \quad T(\lambda, t) = T(\lambda, 0). \tag{5.8}$$

Note that the evolutions in (5.8) can be obtained from the asymptotics of (5.5) and (5.7) as $x \to \pm\infty$ as follows. As $x \to -\infty$ from (5.5) we obtain

$$\partial_t \begin{bmatrix} \frac{L}{T}e^{-i\lambda x} \\ \frac{1}{T}e^{i\lambda x} \end{bmatrix} - 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x^2 \begin{bmatrix} \frac{L}{T}e^{-i\lambda x} \\ \frac{1}{T}e^{i\lambda x} \end{bmatrix} = -2i\lambda^2 \begin{bmatrix} \frac{L}{T}e^{-i\lambda x} \\ \frac{1}{T}e^{i\lambda x} \end{bmatrix}. \tag{5.9}$$

We know that $\lambda_t = 0$, and hence (5.9) reduces to

$$\begin{bmatrix} \left(\frac{L}{T}\right)_t e^{-i\lambda x} \\ \left(\frac{1}{T}\right)_t e^{i\lambda x} \end{bmatrix} + \begin{bmatrix} 2i\lambda^2 \frac{L}{T}e^{-i\lambda x} \\ -2i\lambda^2 \frac{1}{T}e^{i\lambda x} \end{bmatrix} = \begin{bmatrix} -2i\lambda^2 \frac{L}{T}e^{-i\lambda x} \\ -2i\lambda^2 \frac{1}{T}e^{i\lambda x} \end{bmatrix},$$

which yields

$$\left(\frac{L}{T}\right)_t e^{-i\lambda x} + 2i\lambda^2 \frac{L}{T}e^{-i\lambda x} = -2i\lambda^2 \frac{L}{T}e^{-i\lambda x}, \tag{5.10}$$

$$\left(\frac{1}{T}\right)_t e^{i\lambda x} - 21\lambda^2 \frac{1}{T}e^{i\lambda x} = -2i\lambda^2 \frac{1}{T}e^{i\lambda x}. \tag{5.11}$$

From (5.11) we have

$$\left(\frac{1}{T}\right)_t = 0,$$

which implies

$$T_t = 0.$$

From (5.10) we get

$$\left(\frac{L}{T}\right)_t = -4i\lambda^2 \frac{L}{T},$$

which implies

$$L_t = -4i\lambda^2 L.$$

Therefore, due to the fact that $T_t = 0$ we have

$$L(\lambda,t) = L(\lambda,0)e^{-4i\lambda^2 t}.$$

Similarly, as $x \to +\infty$ from (5.7) we get

$$\partial_t \begin{bmatrix} \frac{1}{T}e^{-i\lambda x} \\ \frac{R}{T}e^{i\lambda x} \end{bmatrix} - 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x^2 \begin{bmatrix} \frac{1}{T}e^{-i\lambda x} \\ \frac{R}{T}e^{i\lambda x} \end{bmatrix} = 2i\lambda^2 \begin{bmatrix} \frac{1}{T}e^{-i\lambda x} \\ \frac{R}{T}e^{i\lambda x} \end{bmatrix}. \tag{5.12}$$

Again we know that $\lambda_t = 0$, and hence (5.12) reduces to

$$\begin{bmatrix} \left(\frac{1}{T}\right)_t e^{-i\lambda x} \\ \left(\frac{R}{T}\right)_t e^{i\lambda x} \end{bmatrix} + \begin{bmatrix} 2i\lambda^2 \frac{1}{T}e^{-i\lambda x} \\ -2i\lambda^2 \frac{R}{T}e^{i\lambda x} \end{bmatrix} = \begin{bmatrix} 2i\lambda^2 \frac{1}{T}e^{-i\lambda x} \\ 2i\lambda^2 \frac{R}{T}e^{i\lambda x} \end{bmatrix}.$$

Thus, we get

$$\left(\frac{1}{T}\right)_t = 0, \tag{5.13}$$

$$\left(\frac{R}{T}\right)_t = 4i\lambda^2 \frac{R}{T}. \tag{5.14}$$

Notice that (5.13) again shows that $T_t = 0$. We then consider (5.14), which reduces to

$$R_t = 4i\lambda^2 R.$$

Therefore, we obtain the time evolution of $R(\lambda,t)$ as

$$R(\lambda,t) = R(\lambda,0)e^{4i\lambda^2 t}.$$

Notice that since $T(\lambda,t)$ is independent of time, this implies that the coefficients t_{jm} and the matrix T_j appearing in (4.22) are also independent of time.

Let us now consider the time evolution of the dependency constants γ_{jm}.

Theorem 5.2.1 *The dependency constants γ_{jk} satisfy the ordinary differential equations*

$$\frac{d\gamma_{jm}}{dt} = 4i\lambda^2\gamma_{jm} + 8im\lambda\gamma_{j(m-1)} + 4im(m-1)\gamma_{j(m-2)}.$$

Proof The proof proceeds by induction. We first consider the base case, $\varphi(\lambda_j) = \gamma_{j0}\psi(\lambda_j)$. Its time derivative gives us

$$\varphi_t(\lambda_j) = \gamma_{j0}\psi_t(\lambda_j) + (\gamma_{j0})_t\psi(\lambda_j). \tag{5.15}$$

Using (5.5) we can write (5.15) as

$$2i\lambda_j^2\varphi(\lambda_j) + \mathcal{A}\varphi(\lambda_j) = \gamma_{j0}\psi_t(\lambda_j) + (\gamma_{j0})_t\psi(\lambda_j). \tag{5.16}$$

With the help of (5.7) we can write (5.16) as

$$2i\lambda_j^2\gamma_{j0}\psi(\lambda_j) + \mathcal{A}\gamma_{j0}\psi(\lambda_j) = \gamma_{j0}\mathcal{A}\psi(\lambda_j) - 2i\lambda_j^2\gamma_{j0}\psi(\lambda_j) + (\gamma_{j0})_t\psi(\lambda_j). \tag{5.17}$$

Note that (5.17) implies

$$\frac{d\gamma_{j0}}{dt} = 4i\lambda_j^2\gamma_{j0}. \tag{5.18}$$

From the time derivative of (4.9) we find

$$\dot{\varphi}_t(\lambda_j) = \gamma_{j0}\dot{\psi}_t(\lambda_j) + (\gamma_{j0})_t\dot{\psi}(\lambda_j) + \gamma_{j1}\psi_t(\lambda_j) + (\gamma_{j1})_t\psi(\lambda_j). \tag{5.19}$$

Consider the λ-derivatives of (5.5) and (5.7):

$$\begin{aligned} \dot{\varphi}_t - \mathcal{A}\dot{\varphi} &= 2i\lambda^2\dot{\varphi} + 4i\lambda\varphi, \\ \dot{\psi}_t - \mathcal{A}\dot{\psi} &= -2i\lambda^2\dot{\psi} - 4i\lambda\psi. \end{aligned} \tag{5.20}$$

Using (5.5), (5.7), (5.18), and (5.20) we find that (5.19) can be written as

$$\begin{aligned} (\gamma_{j1})_t\psi(\lambda_j) = -2i\lambda_j^2\gamma_{j0}\dot{\psi}(\lambda_j) + 2i\lambda_j^2\gamma_{j1}\psi(\lambda_j) + 4i\lambda_j\gamma_{j0} - \mathcal{A}\gamma_{j1}\psi(\lambda_j) - 4i\lambda_j^2\gamma_{j0}\psi(\lambda_j) \\ +\mathcal{A}\gamma_{j0}\psi(\lambda_j) + 2i\lambda_j^2\gamma_{j0}\psi(\lambda_j) + 4i\lambda_j\gamma_{j0}\psi(\lambda_j) + 2i\lambda_j^2\gamma_{j1}\psi(\lambda_j) + \mathcal{A}\gamma_{j0}\psi(\lambda_j) + \mathcal{A}\gamma_{j1}\psi(\lambda_j), \end{aligned}$$

which simplifies to

$$(\gamma_{j1})_t = 8i\lambda_j\gamma_{j0} + 4i\lambda_j^2\gamma_{j1},$$

or equivalently

$$\frac{d\gamma_{j1}}{dt} = 4i\lambda_j^2\gamma_{j1} + 8i\lambda_j\gamma_{j0}. \tag{5.21}$$

From the λ-derivative of (5.20) we get

$$\begin{aligned} \ddot{\varphi}_t - \mathcal{A}\ddot{\varphi} &= 2i\lambda^2\ddot{\varphi} + 8i\lambda\dot{\varphi} + 4i\varphi, \\ \ddot{\psi}_t - \mathcal{A}\ddot{\psi} &= -2i\lambda^2\ddot{\psi} - 8i\lambda\dot{\psi} - 4i\psi. \end{aligned} \tag{5.22}$$

From the time derivative of (4.11), with the help of (5.5), (5.7), and (5.18) – (5.22) we get

$$\begin{aligned} (\gamma_{j2})_t\psi(\lambda_j) = 2i\lambda_j^2\gamma_{j0}\ddot{\psi}(\lambda_j) + 4i\lambda_j^2\gamma_{j1}\dot{\psi}(\lambda_j) + 2i\lambda_j^2\gamma_{j2}\psi(\lambda_j) + 8i\lambda_j\gamma_{j0}\dot{\psi}(\lambda_j) + 8i\lambda_j\gamma_{j1}\psi(\lambda_j) \\ +4i\gamma_{j0}\psi(\lambda_j) + 2i\lambda_j^2\gamma_{j0}\ddot{\psi}(\lambda_j) + 8i\lambda_j\gamma_{j0}\dot{\psi}(\lambda_j) + 4i\gamma_{j0}\psi(\lambda_j) - 4i\lambda_j\gamma_{j0}\ddot{\psi}(\lambda_j) + 4i\lambda_j^2\gamma_{j1}\dot{\psi}(\lambda_j) \\ +8i\lambda_j\gamma_{j1}\psi(\lambda_j) - 16i\lambda_j\gamma_{j0}\dot{\psi}(\lambda_j) - 8i\lambda_j^2\gamma_{j1}\dot{\psi}(\lambda_j) + 2i\lambda_j^2\gamma_{j2}\psi(\lambda_j), \end{aligned}$$

which yields

$$(\gamma_{j2})_t = 4i\lambda_j^2\gamma_{j2} + 16i\lambda_j\gamma_{j1} + 8i\gamma_{j0}.$$

For the inductive argument we will write $(\gamma_{j2})_t$ in the following way

$$\frac{d\gamma_{j2}}{dt} = 4i\lambda_j^2\gamma_{j2} + (2)8i\lambda_j\gamma_{j1} + (2\cdot 1)4i\gamma_{j0}, \tag{5.23}$$

which suggests the recursive formula

$$\frac{d\gamma_{jm}}{dt} = 4i\lambda_j^2\gamma_{jm} + 8im\lambda_j\gamma_{j(m-1)} + 4im(m+1)\gamma_{j(m-2)}. \tag{5.24}$$

Let us assume that the above recursive formula is true for $m = k$. To show that the recursive formula is true for $m = k+1$ we must first obtain an equivalent expression for $\sum_{m=0}^{l}\binom{l}{m}\frac{d\gamma_{j(l-m)}}{dt}\psi^{(m)}(\lambda_j)$. The λ-derivatives of (5.7) for $m = 0, 1, \ldots, n_j - 1$ are given by

$$\varphi_t^{(m)} - \mathcal{A}\varphi^{(m)} = 2i\lambda^2\varphi^{(m)} + \binom{m}{1}4i\lambda\varphi^{(m-1)} + \binom{m}{2}4i\varphi^{(m-2)}. \tag{5.25}$$

Similarly, the higher order λ-derivatives of (5.5) for $m = 0, 1, \ldots, n_j - 1$ are given by

$$\psi_t^{(m)} - \mathcal{A}\psi^{(m)} = -2i\lambda^2\psi^{(m)} - \binom{m}{1}4i\lambda\psi^{(m-1)} - \binom{m}{2}4i\psi^{(m-2)}. \tag{5.26}$$

If we take the time derivative of (4.18) we obtain

$$\frac{\partial \varphi^{(l)}}{\partial t}(\lambda_j) = \sum_{k=0}^{l} \binom{l}{k} \gamma_{j(l-k)} \frac{\partial \psi^{(k)}}{\partial t}(\lambda_j) + \sum_{k=0}^{l} \binom{l}{k} \frac{d\gamma_{j(l-k)}}{dt} \psi^{(k)}(\lambda_j). \tag{5.27}$$

From (4.18) and (5.27) we get

$$\begin{aligned} \frac{\partial \varphi^{(l)}}{\partial t}(\lambda_j) - \mathcal{A}\varphi^{(l)}(\lambda_j, x) &= \sum_{k=0}^{l} \binom{l}{k} \gamma_{j(l-k)} \frac{\partial \psi^{(k)}}{\partial t}(\lambda_j) \\ + \sum_{k=0}^{l} \binom{l}{k} \frac{d\gamma_{j(l-k)}}{dt} \psi^{(k)}(\lambda_j) &- \mathcal{A} \sum_{k=0}^{l} \binom{l}{k} \gamma_{j(l-k)} \psi^{(k)}(\lambda_j). \end{aligned} \tag{5.28}$$

Using (4.18), (5.25), (5.26), and (5.28) we obtain

$$\begin{aligned} \sum_{k=0}^{l} \binom{l}{k} \frac{d\gamma_{j(l-k)}}{dt} \psi^{(k)}(\lambda_j) &= 4i\lambda_j^2 \sum_{k=0}^{l} \binom{l}{k} \gamma_{j(l-k)} \psi^{(k)}(\lambda_j) \\ &+ 4i\lambda_j \sum_{k=0}^{l-1} \left[l\binom{l-1}{k} + (l-k)\binom{l}{k} \right] \gamma_{j(l-k)} \psi^{(k-1)}(\lambda_j) \\ &+ 2i \sum_{k=0}^{l-2} \left[\binom{l-2}{k} l(l-1) + (l-k)(l-k-1)\binom{l}{k} \right] \gamma_{j(l-k)} \psi^{(k-2)}(\lambda_j). \end{aligned} \tag{5.29}$$

Note that we have

$$\begin{aligned} l\binom{l-1}{k} + (l-k)\binom{l}{k} &= \frac{2\, l!}{k!(l-k-1)!}, \\ \binom{l-2}{k} l(l-1) + (l-k)(l-k-1)\binom{l}{k} &= \frac{2\, l!}{k!(l-k-2)!}. \end{aligned} \tag{5.30}$$

Using (5.30) we can rewrite (5.29) as

$$\begin{aligned} \sum_{k=0}^{l} \binom{l}{k} \frac{d\gamma_{j(l-k)}}{dt} \psi^{(k)}(\lambda_j) &= 4i\lambda_j^2 \sum_{k=0}^{l} \binom{l}{k} \gamma_{j(l-k)} \psi^{(k)}(\lambda_j) \\ &+ 4i\lambda_j \sum_{k=0}^{l-1} \frac{2\, l!}{k!(l-k-1)!} \gamma_{j(l-k)} \psi^{(k-1)}(\lambda_j) \\ &+ 2i \sum_{k=0}^{l-2} \frac{2\, l!}{k!(l-k-2)!} \gamma_{j(l-k)} \psi^{(k-2)}(\lambda_j). \end{aligned} \tag{5.31}$$

We will use (5.31) to prove our recursive formula (5.24) for $m = k+1$. Multiplying both sides of (5.24) with $m = k$ by $\binom{k+1}{s}\psi^{(k+1-s)}(\lambda_j, x)$, we obtain

$$\begin{aligned}\binom{k+1}{s}\frac{d\gamma_{jk}}{dt}\psi^{(k+1-s)} &= \binom{k+1}{s}4i\lambda_j^2\gamma_{jk}\psi^{(k+1-s)}(\lambda_j)\\ +\binom{k+1}{s}8ik\lambda_j\gamma_{j(k-1)}\psi^{(k+1-s)}(\lambda_j) &+ \binom{k+1}{s}4ik(k-1)\gamma_{j(k-2)}\psi^{(k+1-s)}(\lambda_j).\end{aligned} \tag{5.32}$$

Applying $\sum_{s=0}^{k}$ to both sides of (5.32), we get

$$\begin{aligned}\sum_{s=0}^{k}\binom{k+1}{s}\frac{d\gamma_{jk}}{dt}\psi^{(k+1-s)}(\lambda_j) &= \sum_{s=0}^{k}\binom{k+1}{s}4i\lambda_j^2\gamma_{jk}\psi^{(k+1-s)}(\lambda_j)\\ &+\sum_{s=0}^{k}\binom{k+1}{s}8ik\lambda_j\gamma_{j(k-1)}\psi^{(k+1-s)}(\lambda_j)\\ &+\sum_{s=0}^{k}\binom{k+1}{s}4ik(k-1)\gamma_{j(k-2)}\psi^{(k+1-s)}(\lambda_j).\end{aligned} \tag{5.33}$$

Let us rewrite (5.31) as

$$\begin{aligned}\sum_{s=0}^{k+1}\binom{k+1}{s}\frac{d\gamma_{js}}{dt}\psi^{(k+1-s)}(\lambda_j) &= 4i\lambda_j^2\sum_{s=0}^{k+1}\binom{k+1}{s}\gamma_{js}\psi^{(k-s+1)}(\lambda_j)\\ +4i\lambda_j\sum_{s=0}^{k}\frac{2(k+1)!}{s!(k-s)!}\gamma_{js}\psi^{(k-s)}(\lambda_j) &+ 2i\sum_{s=0}^{k-1}\frac{2(k+1)!}{s!(k-s-1)!}\gamma_{js}\psi^{(k-s-1)}(\lambda_j),\end{aligned}$$

which can also be written as

$$\begin{aligned}\frac{d\gamma_{j(k+1)}}{dt}\psi^{(0)}(\lambda_j) + \sum_{s=0}^{k}\binom{k+1}{s}\frac{d\gamma_{js}}{dt}\psi^{(k+1-s)}(\lambda_j) &=\\ 4i\lambda_j^2\sum_{s=0}^{k+1}\binom{k+1}{s}\gamma_{js}\psi^{(k-s+1)}(\lambda_j) + 4i\lambda_j\sum_{s=0}^{k}\frac{2(k+1)!}{s!(k-s)!}\gamma_{js}\psi^{(k-s)}(\lambda_j)&\\ +2i\sum_{s=0}^{k-1}\frac{2(k+1)!}{s!(k-s-1)!}\gamma_{js}\psi^{(k-s-1)}(\lambda_j).&\end{aligned} \tag{5.34}$$

Now substituting (5.33) into the second term of (5.34) and rearranging terms we get

$$\begin{aligned}\frac{d\gamma_{j(k+1)}}{dt}\psi^{(0)}(\lambda_j) &= 4i\lambda_j^2\sum_{s=0}^{k+1}\binom{k+1}{s}\gamma_{js}\psi^{(k-s+1)}(\lambda_j) + 4i\lambda_j\sum_{s=0}^{k}\frac{2(k+1)!}{s!(k-s)!}\gamma_{js}\psi^{(k-s)}(\lambda_j)\\ &+2i\sum_{s=0}^{k-1}\frac{2(k+1)!}{s!(k-s-1)!}\gamma_{js}\psi^{(k-s-1)}(\lambda_j) - \sum_{s=0}^{k}\binom{k+1}{s}4i\lambda_j^2\gamma_{jk}\psi^{(k+1-s)}(\lambda_j)\\ &-\sum_{s=0}^{k}\binom{k+1}{s}8ik\lambda_j\gamma_{j(k-1)}\psi^{(k+1-s)}(\lambda_j) - \sum_{s=0}^{k}\binom{k+1}{s}4ik(k-1)\gamma_{j(k-2)}\psi^{(k+1-s)}(\lambda_j),\end{aligned}$$

or equivalently

$$\begin{aligned}\frac{d\gamma_{j(k+1)}}{dt}\psi^{(0)}(\lambda_j) &= 4i\lambda_j^2\left[\sum_{s=0}^{k+1}\binom{k+1}{s}\gamma_{js}\psi^{(k-s+1)}(\lambda_j) - \sum_{s=0}^{k}\binom{k+1}{s}\gamma_{jk}\psi^{(k+1-s)}(\lambda_j)\right]\\ &+8i\lambda_j\left[\sum_{s=0}^{k}\frac{2(k+1)!}{s!(k-s)!}\gamma_{js}\psi^{(k-s)}(\lambda_j) - \sum_{s=0}^{k}\binom{k+1}{s}k\gamma_{j(k-1)}\psi^{(k+1-s)}(\lambda_j)\right]\\ &+4i\left[\sum_{s=0}^{k-1}\frac{(k+1)!}{s!(k-s-1)!}\gamma_{js}\psi^{(k-s-1)}(\lambda_j) - \sum_{s=0}^{k}\binom{k+1}{s}k(k-1)\gamma_{j(k-2)}\psi^{(k+1-s)}(\lambda_j)\right].\end{aligned}$$

This last equality can be simplified to

$$\begin{aligned}\frac{d\gamma_{j(k+1)}}{dt}\psi^{(0)}(\lambda_j) &= 4i\lambda_j^2\left[\gamma_{j(k+1)}\psi^{(0)}(\lambda_j)\right] + 8i\lambda_j\left[(k+1)\gamma_{jk}\psi^{(0)}(\lambda_j)\right]\\ &+4i\left[(k+1)k\gamma_{j(k-1)}\psi^{(0)}(\lambda_j)\right],\end{aligned}$$

which yields

$$\frac{d\gamma_{j(k+1)}}{dt} = 4i\lambda_j^2\gamma_{j(k+1)} + 8i\lambda_j(k+1)\gamma_{jk} + 4ik(k+1)\gamma_{j(k-1)}. \tag{5.35}$$

Thus, the time evolution of the dependency constants γ_{js} has been established for any $k = 0, 1, \ldots, n_j - 1$. ∎

Next we establish the time evolution of the norming constants from $t = 0$ to an arbitrary time. For this purpose we introduce the $n_j \times n_j$ matrix A_j as

$$A_j := \begin{bmatrix} -i\lambda_j & -1 & 0 & \dots & 0 & 0 \\ 0 & -i\lambda_j & -1 & \dots & 0 & 0 \\ 0 & 0 & -i\lambda_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -i\lambda_j & -1 \\ 0 & 0 & 0 & \dots & 0 & -i\lambda_j \end{bmatrix}. \tag{5.36}$$

Theorem 5.2.2 *The time evolution of the norming constants c_{jk} is governed by*

$$\begin{bmatrix} c_{j(n_j-1)}(t) & c_{j(n_j-2)}(t) & \dots & c_{j0}(t) \end{bmatrix} = \begin{bmatrix} c_{j(n_j-1)}(0) & c_{j(n_j-2)}(0) & \dots & c_{j0}(0) \end{bmatrix} e^{-4iA_j^2 t}. \tag{5.37}$$

Proof Recall that the norming constants and the dependency constants are related to each other by (4.29), which can be written in matrix form as

$$\widehat{C}_j = \widehat{\Gamma}_j \widehat{P}_j, \tag{5.38}$$

where $\widehat{C}_j$, $\widehat{\Gamma}_j$, and $\widehat{P}_j$ are defined as

$$\widehat{C}_j := \begin{bmatrix} c_{j(n_j-1)} & \dots & c_{j0} \end{bmatrix}, \qquad \widehat{\Gamma}_j := \begin{bmatrix} \gamma_{j(n_j-1)} & \dots & \gamma_{j0} \end{bmatrix},$$

$$\widehat{P}_j := \begin{bmatrix} 0 & 0 & \dots & 0 & \frac{t_{jn_j} i^{-1}}{(n_j-1)!} \\ 0 & 0 & \dots & \frac{t_{jn_j} i^{0}}{(n_j-2)!} & \frac{t_{j(n_j-1)} i^{-1}}{(n_j-2)!} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & \frac{t_{jn_j} i^{n_j-3}}{1!} & \dots & \frac{t_{j3} i^{0}}{1!} & \frac{t_{j2} i^{-1}}{1!} \\ \frac{t_{jn_j} i^{n_j-2}}{0!} & \frac{t_{j(n_j-1)} i^{n_j-3}}{0!} & \dots & \frac{t_{j2} i^{0}}{0!} & \frac{t_{j1} i^{-1}}{0!} \end{bmatrix}, \tag{5.39}$$

and the entries of $\widehat{C}_j$ and $\widehat{\Gamma}_j$ are dependent on t. Since t_{jk} is independent of time, the time derivative of (5.38) yield

$$\frac{d\widehat{C}_j}{dt} = \frac{d\widehat{\Gamma}_j}{dt}\widehat{P}_j, \tag{5.40}$$

which can also be written as

$$\begin{bmatrix} \frac{dc_{j(n_j-1)}}{dt} & \frac{dc_{j(n_j-2)}}{dt} & \dots & \frac{dc_{j0}}{dt} \end{bmatrix} = \begin{bmatrix} \frac{d\gamma_{j(n_j-1)}}{dt} & \frac{d\gamma_{j(n_j-2)}}{dt} & \dots & \frac{d\gamma_{j0}}{dt} \end{bmatrix} \widehat{P}_j. \tag{5.41}$$

Using (5.35) in (5.41) we obtain

$$\begin{bmatrix} \frac{d\gamma_{j(n_j-1)}}{dt} & \frac{d\gamma_{j(n_j-2)}}{dt} & \dots & \frac{d\gamma_{j0}}{dt} \end{bmatrix} = \begin{bmatrix} \gamma_{j(n_j-1)} & \gamma_{j(n_j-2)} & \dots & \gamma_{j0} \end{bmatrix} L_j, \tag{5.42}$$

where we have defined

$$L_j := \begin{bmatrix} 4i\lambda_j^2 & 0 & 0 & \dots & 0 \\ (n_j-1)8i\lambda_j & 4i\lambda_j^2 & 0 & \dots & 0 \\ (n_j-2)(n_j-1)4i & (n_j-2)8i\lambda_j & 4i\lambda_j^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 4i\lambda_j^2 \end{bmatrix}. \tag{5.43}$$

Let us write (5.42) as

$$\frac{d\widehat{\Gamma}_j}{dt} = \widehat{\Gamma}_j L_j. \tag{5.44}$$

Substituting (5.44) into (5.40) we obtain

$$\frac{d\widehat{C}_j}{dt} = \widehat{\Gamma}_j L_j \widehat{P}_j. \tag{5.45}$$

Since P_j is congruent to a triangular matrix with nonzero diagonal entries, we know that it is invertible. Thus we can write (5.45) in the equivalent form

$$\frac{d\widehat{C}_j}{dt} = \widehat{\Gamma}_j(\widehat{P}_j\widehat{P}_j^{-1})L_j\widehat{P}_j. \tag{5.46}$$

From (5.38) we see this is equivalent to

$$\frac{d\widehat{C}_j}{dt} = \widehat{C}_j\widehat{P}_j^{-1}L_j\widehat{P}_j,$$

which has the unique solution

$$\widehat{C}_j(t) = \widehat{C}_j(0)e^{(\widehat{P}_j^{-1}L_j\widehat{P}_j)t}.$$

With the help of (5.39) and (5.43) we obtain

$$\widehat{P}_j^{-1}L_j\widehat{P}_j = -4iA_j^2,$$

where A_j is the matrix defined in (5.36). Thus, (5.37) is proved. ∎

5.3 Time Evolution of Scattering Data Associated with $\mathbb{C}^-$

Again, since $\mathcal{L}$ and $\mathcal{A}$ form a Lax pair, we know that $\overline{\psi}_t - \mathcal{A}\overline{\psi}$ is also a solution to $\mathcal{L}\overline{\psi} = \lambda\overline{\psi}$. Then $\overline{\psi}_t - \mathcal{A}\overline{\psi}$ can be written as a linear combination of the Jost solutions $\overline{\psi}$ and $\overline{\varphi}$ as

$$\overline{\psi}_t - \mathcal{A}\overline{\psi} = c_1\overline{\psi} + c_2\overline{\varphi}, \tag{5.47}$$

where c_1 and c_2 are some coefficients independent of x, which can be determined from the asymptotics as $x \to \pm\infty$. It is already known [1, 2, 3, 4, 8, 13, 16, 20] that the time evolution of $\overline{\psi}$ is given by

$$\overline{\psi}_t - \mathcal{A}\overline{\psi} = 2i\lambda^2\overline{\psi},$$

and this can be established by letting $x \to \pm\infty$ in (5.47). As $x \to \pm\infty$, we know that q, r, q_x, $r_x \to 0$, and hence $\mathcal{A} \to 2iJ\partial_x^2$ as $x \to \pm\infty$. Now consider $\overline{\psi}_t - \mathcal{A}\overline{\psi} = c_1\overline{\psi} + c_2\overline{\varphi}$ as $x \to +\infty$ and use the asymptotics on pages 8 and 9 we get

$$\partial_t \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} - 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x^2 \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \frac{\overline{R}}{\overline{T}}e^{-i\lambda x} \\ \frac{1}{\overline{T}}e^{i\lambda x} \end{bmatrix},$$

from which we conclude that

$$\begin{bmatrix} 2i\lambda^2 e^{-i\lambda x} \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \frac{\overline{R}}{\overline{T}}e^{-i\lambda x} \\ \frac{}{\overline{T}}e^{i\lambda x} \end{bmatrix}.$$

Therefore, we have $c_1 = 2i\lambda^2$ and $c_2 = 0$, and the time evolution of $\overline{\psi}$ is governed by

$$\overline{\psi}_t - \mathcal{A}\overline{\psi} = 2i\lambda^2\overline{\psi}. \tag{5.48}$$

Similarly, we can derive the time evolution of $\overline{\varphi}$. Since $\overline{\varphi}_t - \mathcal{A}\overline{\varphi}$ is also a solution to $\mathcal{L}\overline{\varphi} = \lambda\overline{\varphi}$ we have

$$\overline{\varphi}_t - \mathcal{A}\overline{\varphi} = c_3\overline{\psi} + c_4\overline{\varphi},$$

for some coefficients c_3 and c_4 not depending on x. Letting $x \to -\infty$ and using the asymptotics on pages 8 and 9, from the above relationship we obtain

$$\partial_t \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} - 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x^2 \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} = c_3 \begin{bmatrix} \frac{1}{\overline{T}} e^{-i\lambda x} \\ \frac{\overline{L}}{\overline{T}} e^{i\lambda x} \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix},$$

which yields

$$\begin{bmatrix} 2i\lambda^2 e^{-i\lambda x} \\ 0 \end{bmatrix} = c_3 \begin{bmatrix} \frac{\overline{L}}{\overline{T}} e^{-i\lambda x} \\ \frac{\overline{R}}{\overline{T}} e^{i\lambda x} \end{bmatrix} + c_4 \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}.$$

Therefore, we have $c_3 = 0$ and $c_4 = -2i\lambda^2$, and the evolution of $\overline{\varphi}$ is governed by

$$\overline{\varphi}_t - \mathcal{A}\overline{\varphi} = -2i\lambda^2\overline{\varphi}. \tag{5.49}$$

The time evolution of the scattering coefficients $\overline{R}$, $\overline{L}$, and $\overline{T}$ is already known [1, 2, 8, 16] and can easily be obtained from the asymptotics of (5.48) and (5.49) as follows. Letting $x \to -\infty$ in (5.48) we get

$$\partial_t \begin{bmatrix} \frac{1}{\overline{T}} e^{-i\lambda x} \\ \frac{\overline{L}}{\overline{T}} e^{i\lambda x} \end{bmatrix} - 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x^2 \begin{bmatrix} \frac{1}{\overline{T}} e^{-i\lambda x} \\ \frac{\overline{L}}{\overline{T}} e^{i\lambda x} \end{bmatrix} = -2i\lambda^2 \begin{bmatrix} \frac{1}{\overline{T}} e^{-i\lambda x} \\ \frac{\overline{L}}{\overline{T}} e^{i\lambda x} \end{bmatrix}. \tag{5.50}$$

Since $\lambda_t = 0$, (5.50) gives us

$$\begin{bmatrix} \left(\frac{1}{\overline{T}}\right)_t e^{-i\lambda x} \\ \left(\frac{\overline{L}}{\overline{T}}\right)_t e^{i\lambda x} \end{bmatrix} + \begin{bmatrix} 2i\lambda^2 \frac{1}{\overline{T}} e^{-i\lambda x} \\ -2i\lambda^2 \frac{\overline{L}}{\overline{T}} e^{i\lambda x} \end{bmatrix} = \begin{bmatrix} 2i\lambda^2 \frac{1}{\overline{T}} e^{-i\lambda x} \\ 2i\lambda^2 \frac{\overline{L}}{\overline{T}} e^{i\lambda x} \end{bmatrix},$$

which yields

$$\left(\frac{\overline{L}}{\overline{\overline{T}}}\right)_t e^{i\lambda x} - 2i\lambda^2 \frac{\overline{L}}{\overline{\overline{T}}} e^{i\lambda x} = 2i\lambda^2 \frac{\overline{L}}{\overline{\overline{T}}} e^{i\lambda x},$$

$$\left(\frac{1}{\overline{\overline{T}}}\right)_t e^{-i\lambda x} + 2i\lambda^2 \frac{1}{\overline{\overline{T}}} e^{-i\lambda x} = 2i\lambda^2 \frac{1}{\overline{\overline{T}}} e^{-i\lambda x}.$$

From (5.51) and (5.51) we obtain

$$\left(\frac{\overline{L}}{\overline{\overline{T}}}\right)_t = 4i\lambda^2 \frac{\overline{L}}{\overline{\overline{T}}}, \quad \left(\frac{1}{\overline{\overline{T}}}\right)_t = 0,$$

which are equivalent to

$$\overline{L}_t = 4i\lambda^2 \overline{L}, \quad \overline{T}_t = 0.$$

Therefore, we obtain

$$\overline{L}(\lambda, t) = \overline{L}(\lambda, 0) e^{4i\lambda^2 t}, \quad \overline{T}(\lambda, t) = \overline{T}(\lambda, 0).$$

Thus $\overline{T}$ is independent of time. Letting $x \to +\infty$ in (5.49) we get

$$\partial_t \begin{bmatrix} \frac{\overline{R}}{\overline{\overline{T}}} e^{-i\lambda x} \\ \frac{1}{\overline{\overline{T}}} e^{i\lambda x} \end{bmatrix} - 2i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \partial_x^2 \begin{bmatrix} \frac{\overline{R}}{\overline{\overline{T}}} e^{-i\lambda x} \\ \frac{1}{\overline{\overline{T}}} e^{i\lambda x} \end{bmatrix} = 2i\lambda^2 \begin{bmatrix} \frac{\overline{R}}{\overline{\overline{T}}} e^{-i\lambda x} \\ \frac{1}{\overline{\overline{T}}} e^{i\lambda x} \end{bmatrix}. \tag{5.51}$$

Since $\lambda_t = 0$, from (5.51) we get

$$\begin{bmatrix} \left(\frac{\overline{R}}{\overline{\overline{T}}}\right)_t e^{-i\lambda x} \\ \left(\frac{1}{\overline{\overline{T}}}\right)_t e^{i\lambda x} \end{bmatrix} + \begin{bmatrix} 2i\lambda^2 \frac{\overline{R}}{\overline{\overline{T}}} e^{-i\lambda x} \\ -2i\lambda^2 \frac{\overline{R}}{\overline{\overline{T}}} e^{i\lambda x} \end{bmatrix} = \begin{bmatrix} -2i\lambda^2 \frac{\overline{R}}{\overline{\overline{T}}} e^{-i\lambda x} \\ -2i\lambda^2 \frac{1}{\overline{\overline{T}}} e^{i\lambda x} \end{bmatrix},$$

which yields

$$\left(\frac{1}{\overline{\overline{T}}}\right)_t = 0, \tag{5.52}$$

$$\left(\frac{\overline{R}}{\overline{\overline{T}}}\right)_t = -4i\lambda^2 \frac{\overline{R}}{\overline{\overline{T}}}. \tag{5.53}$$

Thus, we again gt $\overline{T}_t = 0$ and

$$\overline{R}(\lambda, t) = \overline{R}(\lambda, 0) e^{-4i\lambda^2 t}.$$

Notice that since $\overline{T}$ is independent of time, the coefficients $\bar{t}_{jm}$ appearing in (4.49) are also independent of time. Let us now consider the time evolution of the dependency constants $\overline{\gamma}_{jm}$.

Theorem 5.3.1 *The time evolution of the dependency constants* $\overline{\gamma}_{jm}$ *is governed by the ordinary differential equation*

$$\frac{d\overline{\gamma}_{jm}}{dt} = -4i\lambda^2\overline{\gamma}_{jm} - 8im\lambda\overline{\gamma}_{j(m-1)} - 4im(m-1)\overline{\gamma}_{j(m-2)}. \tag{5.54}$$

Proof We will again use induction to prove our theorem. Taking the time derivative of (4.36) we obtain

$$\overline{\varphi}_t(\overline{\lambda}_j) = \overline{\gamma}_{j0}\overline{\psi}_t(\overline{\lambda}_j) + (\overline{\gamma}_{j0})_t\overline{\psi}(\overline{\lambda}_j). \tag{5.55}$$

Using (5.49), we can write (5.55) as

$$-2i\overline{\lambda}_j^2\overline{\varphi}(\overline{\lambda}_j) + \mathcal{A}\overline{\varphi}(\overline{\lambda}_j) = \overline{\gamma}_{j0}\overline{\psi}_t(\overline{\lambda}_j) + (\overline{\gamma}_{j0})_t\overline{\psi}(\overline{\lambda}_j). \tag{5.56}$$

Using (4.36) and (5.48) in (5.56) we get

$$-2i\overline{\lambda}_j^2\overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) + \mathcal{A}\overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) = \overline{\gamma}_{j0}\mathcal{A}\overline{\psi}(\overline{\lambda}_j) - 2i\overline{\lambda}_j^2\overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) + (\overline{\gamma}_{j0})_t\overline{\psi}(\overline{\lambda}_j). \tag{5.57}$$

From (5.57) we obtain

$$\frac{d\overline{\gamma}_{j0}}{dt} = -4i\overline{\lambda}_j^2\overline{\gamma}_{j0}. \tag{5.58}$$

Taking the time derivative of (4.38) we get

$$\dot{\overline{\varphi}}_t(\overline{\lambda}_j) = \overline{\gamma}_{j0}\dot{\overline{\psi}}_t(\overline{\lambda}_j) + (\overline{\gamma}_{j0})_t\dot{\overline{\psi}}(\overline{\lambda}_j) + \overline{\gamma}_{j1}\overline{\psi}_t(\overline{\lambda}_j) + (\overline{\gamma}_{j1})_t\overline{\psi}(\overline{\lambda}_j). \tag{5.59}$$

The λ-derivatives of (5.48) and (5.49) give us

$$\begin{aligned} \dot{\overline{\varphi}}_t - \mathcal{A}\dot{\overline{\varphi}} &= -2i\lambda^2\dot{\overline{\varphi}} - 4i\lambda\overline{\varphi}, \\ \dot{\overline{\psi}}_t - \mathcal{A}\dot{\overline{\psi}} &= 2i\lambda^2\dot{\overline{\psi}} + 4i\lambda\overline{\psi}. \end{aligned} \tag{5.60}$$

Using (5.48), (5.49), (5.58), and (5.60) in (5.59) we obtain

$$
\begin{aligned}
(\overline{\gamma}_{j1})_t\overline{\psi}(\overline{\lambda}_j) &= +2i\overline{\lambda}_j^2\overline{\gamma}_{j0}\dot{\overline{\psi}}(\overline{\lambda}_j) - 2i\overline{\lambda}_j^2\overline{\gamma}_{j1}\overline{\psi}(\overline{\lambda}_j) - 4i\overline{\lambda}_j\overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) - \mathcal{A}\overline{\gamma}_{j1}\overline{\psi}(\overline{\lambda}_j) + 4i\overline{\lambda}_j^2\overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) \\
&-\mathcal{A}\overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) - 2i\overline{\lambda}_j^2\overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) - 4i\overline{\lambda}_j\overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) - 2i\overline{\lambda}_j^2\overline{\gamma}_{j1}\psi(\overline{\lambda}_j) + \mathcal{A}\overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) + \mathcal{A}\overline{\gamma}_{j1}\overline{\psi}(\overline{\lambda}_j),
\end{aligned}
$$

which simplifies to

$$(\overline{\gamma}_{j1})_t = -8i\overline{\lambda}_j\overline{\gamma}_{j0} - 4i\overline{\lambda}_j^2\overline{\gamma}_{j1},$$

or equivalently

$$\frac{d\overline{\gamma}_{j1}}{dt} = -4i\overline{\lambda}_j^2\overline{\gamma}_{j1} - 8i\overline{\lambda}_j\overline{\gamma}_{j0}. \tag{5.61}$$

From the λ-derivative of (5.60) we get

$$
\begin{aligned}
\ddot{\overline{\varphi}}_t - \mathcal{A}\ddot{\overline{\varphi}} &= -2i\lambda^2\ddot{\overline{\varphi}} - 8i\lambda\dot{\overline{\varphi}} - 4i\overline{\varphi}, \\
\ddot{\overline{\psi}}_t - \mathcal{A}\ddot{\overline{\psi}} &= 2i\lambda^2\ddot{\overline{\psi}} + 8i\lambda\dot{\overline{\psi}} + 4i\overline{\psi}.
\end{aligned} \tag{5.62}
$$

Using (5.48), (5.49), (5.56), (5.58), (5.61), and (5.62) we find that (4.40) can be written as

$$
\begin{aligned}
(\overline{\gamma}_{j2})_t\overline{\psi}(\overline{\lambda}_j) &= -2i\overline{\lambda}_j^2\overline{\gamma}_{j0}\ddot{\overline{\psi}}(\overline{\lambda}_j) - 4i\overline{\lambda}_j^2\overline{\gamma}_{j1}\dot{\overline{\psi}}(\overline{\lambda}_j) - 2i\overline{\lambda}_j^2\overline{\gamma}_{j2}\overline{\psi}(\overline{\lambda}_j) - 8i\overline{\lambda}_j\overline{\gamma}_{j0}\dot{\overline{\psi}}(\overline{\lambda}_j) - 8i\overline{\lambda}_j\overline{\gamma}_{j1}\overline{\psi}(\overline{\lambda}_j) \\
&-4i\overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) - 2i\overline{\lambda}_j^2\overline{\gamma}_{j0}\ddot{\overline{\psi}}(\overline{\lambda}_j) - 8i\overline{\lambda}_j\overline{\gamma}_{j0}\dot{\overline{\psi}}(\overline{\lambda}_j) - 4i\overline{\gamma}_{j0}\overline{\psi}(\overline{\lambda}_j) + 4i\overline{\lambda}_j\overline{\gamma}_{j0}\ddot{\overline{\psi}}(\overline{\lambda}_j) - 4i\overline{\lambda}_j^2\overline{\gamma}_{j1}\dot{\overline{\psi}}(\overline{\lambda}_j) \\
&-8i\overline{\lambda}_j\overline{\gamma}_{j1}\overline{\psi}(\overline{\lambda}_j) + 16i\overline{\lambda}_j\overline{\gamma}_{j0}\dot{\overline{\psi}}(\overline{\lambda}_j) + 8i\overline{\lambda}_j^2\overline{\gamma}_{j1}\dot{\overline{\psi}}(\overline{\lambda}_j) - 2i\overline{\lambda}_j^2\overline{\gamma}_{j2}\overline{\psi}(\overline{\lambda}_j),
\end{aligned}
$$

which reduces to

$$(\overline{\gamma}_{j2})_t = -4i\overline{\lambda}_j^2\overline{\gamma}_{j2} - 16i\overline{\lambda}_j\overline{\gamma}_{j1} - 8i\overline{\gamma}_{j0},$$

or equivalently

$$\frac{d\overline{\gamma}_{j2}}{dt} = -4i\overline{\lambda}_j^2\overline{\gamma}_{j2} - (2)8i\overline{\lambda}_j\overline{\gamma}_{j1} - (2\cdot 1)4i\overline{\gamma}_{j0}. \tag{5.63}$$

Therefore, (5.63) satisfies the recursive formula for $m = 2$. Now assume the recursive formula is true for $m = k$, i.e.

$$\frac{d\overline{\gamma}_{jk}}{dt} = -4i\overline{\lambda}_j^2\overline{\gamma}jk - 8ik\overline{\lambda}_j\overline{\gamma}_{j(k-1)} - 4ik(k+1)\overline{\gamma}_{j(k-2)}. \tag{5.64}$$

To show that the recursive formula is also true for $m = k+1$, we must first find an equivalent expression for $\sum_{k=0}^{l}\binom{l}{k}\frac{d\overline{\gamma}_{j(l-k)}}{dt}\overline{\psi}^{(k)}(\overline{\lambda}_j)$. The λ-derivatives of (5.49) for $m = 0, 1, \ldots, \overline{n}_j - 1$ yield

$$\overline{\varphi}_t^{(m)} - \mathcal{A}\overline{\varphi}^{(m)} = -2i\lambda^2\overline{\varphi}^{(m)} - \binom{m}{1}4i\lambda\overline{\varphi}^{(m-1)} - \binom{m}{2}4i\overline{\varphi}^{(m-2)}. \tag{5.65}$$

Similarly, the λ-derivatives of (5.48) give us

$$\overline{\psi}_t^{(m)} - \mathcal{A}\overline{\psi}^{(m)} = +2i\lambda^2\overline{\psi}^{(m)} + \binom{m}{1}4i\lambda\overline{\psi}^{(m-1)} + \binom{m}{2}4i\overline{\psi}^{(m-2)}. \tag{5.66}$$

From the time derivative of (4.42) we find

$$\frac{d\overline{\varphi}^{(l)}}{dt}(\overline{\lambda}_j) = \sum_{k=0}^{l}\binom{l}{k}\overline{\gamma}_{j(l-k)}\frac{d\overline{\psi}^{(k)}}{dt}(\overline{\lambda}_j) + \sum_{k=0}^{l}\binom{l}{k}\frac{d\overline{\gamma}_{j(l-k)}}{dt}\overline{\psi}^{(k)}(\overline{\lambda}_j). \tag{5.67}$$

With the help of (4.42) and (5.67) we obtain

$$\begin{aligned}\frac{d\overline{\varphi}^{(l)}}{dt}(\overline{\lambda}_j) - \mathcal{A}\overline{\varphi}^{(l)}(\overline{\lambda}_j) &= \sum_{k=0}^{l}\binom{l}{k}\overline{\gamma}_{j(l-k)}\frac{d\overline{\psi}^{(k)}}{dt}(\overline{\lambda}_j)\\ +\sum_{k=0}^{l}\binom{l}{k}\frac{d\overline{\gamma}_{j(l-k)}}{dt}\overline{\psi}^{(k)}(\overline{\lambda}_j) &- \mathcal{A}\sum_{k=0}^{l}\binom{l}{k}\overline{\gamma}_{j(l-k)}\overline{\psi}^{(k)}(\overline{\lambda}_j).\end{aligned} \tag{5.68}$$

From (5.65), (5.66), (5.67), and (5.68) we have

$$\begin{aligned}\sum_{k=0}^{l}\binom{l}{k}\frac{d\overline{\gamma}_{j(l-k)}}{dt}\overline{\psi}^{(k)}(\overline{\lambda}_j) &= -4i\overline{\lambda}_j^2\sum_{k=0}^{l}\binom{l}{k}\overline{\gamma}_{j(l-k)}\overline{\psi}^{(k)}(\overline{\lambda}_j)\\ &-4i\overline{\lambda}_j\sum_{k=0}^{l-1}\left[l\binom{l-1}{k} + (l-k)\binom{l}{k}\right]\overline{\gamma}_{j(l-k)}\overline{\psi}^{(k-1)}(\overline{\lambda}_j)\\ -2i\sum_{k=0}^{l-2}&\left[\binom{l-2}{k}l(l-1) + (l-k)(l-k-1)\binom{l}{k}\right]\overline{\gamma}_{j(l-k)}\overline{\psi}^{(k-2)}(\overline{\lambda}_j).\end{aligned} \tag{5.69}$$

Using (5.30) we can rewrite (5.69) as

$$\begin{aligned}\sum_{k=0}^{l}\binom{l}{k}\frac{d\overline{\gamma}_{j(l-k)}}{dt}\overline{\psi}^{(k)}(\overline{\lambda}_j) &= -4i\overline{\lambda}_j^2\sum_{k=0}^{l}\binom{l}{k}\overline{\gamma}_{j(l-k)}\overline{\psi}^{(k)}(\overline{\lambda}_j)\\ &-4i\overline{\lambda}_j\sum_{k=0}^{l-1}\frac{2\,l!}{k!(l-k-1)!}\overline{\gamma}_{j(l-k)}\overline{\psi}^{(k-1)}(\overline{\lambda}_j)\\ &-2i\sum_{k=0}^{l-2}\frac{2\,l!}{k!(l-k-2)!}\overline{\gamma}_{j(l-k)}\overline{\psi}^{(k-2)}(\overline{\lambda}_j).\end{aligned} \tag{5.70}$$

We will use (5.64) to prove the recursive formula for $m = k+1$. Multiplying both sides of (5.70) by $\binom{k+1}{s}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j)$, we obtain

$$\begin{aligned}\binom{k+1}{s}\frac{d\overline{\gamma}_{jk}}{dt}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j) = -\binom{k+1}{s}4i\overline{\lambda}_j^2\overline{\gamma}_{jk}\overline{\psi}^{k+1-s}(\overline{\lambda}_j)\\ -\binom{k+1}{s}8ik\overline{\lambda}_j\overline{\gamma}_{j(k-1)}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j) - \binom{k+1}{s}4ik(k-1)\overline{\gamma}_{j(k-2)}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j).\end{aligned} \tag{5.71}$$

Applying the summation $\sum_{s=0}^{k}$ to both sides of (5.71), we get

$$\begin{aligned}\sum_{s=0}^{k}\binom{k+1}{s}\frac{d\overline{\gamma}_{jk}}{dt}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j) &= -\sum_{s=0}^{k}\binom{k+1}{s}4i\overline{\lambda}_j^2\overline{\gamma}_{jk}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j)\\ &-\sum_{s=0}^{k}\binom{k+1}{s}8ik\overline{\lambda}_j\overline{\gamma}_{j(k-1)}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j)\\ &-\sum_{s=0}^{k}\binom{k+1}{s}4ik(k-1)\overline{\gamma}_{j(k-2)}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j).\end{aligned} \tag{5.72}$$

From (5.70) we have

$$\begin{aligned}\sum_{s=0}^{k+1}\binom{k+1}{s}\frac{d\overline{\gamma}_{js}}{dt}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j) = -4i\overline{\lambda}_j^2\sum_{s=0}^{k+1}\binom{k+1}{s}\overline{\gamma}_{js}\overline{\psi}^{(k-s+1)}(\overline{\lambda}_j)\\ -4i\overline{\lambda}_j\sum_{s=0}^{k}\frac{2(k+1)!}{s!(k-s)!}\overline{\gamma}_{js}\psi^{(k-s)}(\overline{\lambda}_j) - 2i\sum_{s=0}^{k-1}\frac{2(k+1)!}{s!(k-s-1)!}\overline{\gamma}_{js}\overline{\psi}^{(k-s-1)}(\overline{\lambda}_j),\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\frac{d\gamma_{j(k+1)}}{dt}\psi^{(0)}(\overline{\lambda}_j) + \sum_{s=0}^{k}\binom{k+1}{s}\frac{d\gamma_{js}}{dt}\psi^{(k+1-s)}(\overline{\lambda}_j)\\ = 4i\lambda_j^2\sum_{s=0}^{k+1}\binom{k+1}{s}\gamma_{js}\psi^{(k-s+1)}(\overline{\lambda}_j) + 4i\lambda_j\sum_{s=0}^{k}\frac{2(k+1)!}{s!(k-s)!}\gamma_{js}\psi^{(k-s)}(\overline{\lambda}_j)\\ +2i\sum_{s=0}^{k-1}\frac{2(k+1)!}{s!(k-s-1)!}\gamma_{js}\psi^{(k-s-1)}(\overline{\lambda}_j).\end{aligned} \tag{5.73}$$

By substituting (5.72) into the second term of (5.73) and rearranging terms, we get

$$\frac{d\overline{\gamma}_{j(k+1)}}{dt}\overline{\psi}^{(0)}(\overline{\lambda}_j) = -4i\overline{\lambda}_j^2 \sum_{s=0}^{k+1} \binom{k+1}{s} \overline{\gamma}_{js}\overline{\psi}^{(k-s+1)}(\overline{\lambda}_j) - 4i\overline{\lambda}_j \sum_{s=0}^{k} \frac{2(k+1)!}{s!(k-s)!}\overline{\gamma}_{js}\overline{\psi}^{(k-s)}(\overline{\lambda}_j)$$
$$-2i\sum_{s=0}^{k-1} \frac{2(k+1)!}{s!(k-s-1)!}\overline{\gamma}_{js}\overline{\psi}^{(k-s-1)}(\overline{\lambda}_j) + \sum_{s=0}^{k}\binom{k+1}{s} 4i\overline{\lambda}_j^2\overline{\gamma}_{jk}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j)$$
$$+\sum_{s=0}^{k}\binom{k+1}{s} 8ik\overline{\lambda}_j\overline{\gamma}_{j(k-1)}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j) + \sum_{s=0}^{k}\binom{k+1}{s} 4ik(k-1)\overline{\gamma}_{j(k-2)}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j),$$

or equivalently

$$\frac{d\overline{\gamma}_{j(k+1)}}{dt}\overline{\psi}^{(0)}(\overline{\lambda}_j) = -4i\overline{\lambda}_j^2 \left[\sum_{s=0}^{k+1} \binom{k+1}{s} \overline{\gamma}_{js}\overline{\psi}^{(k-s+1)}(\overline{\lambda}_j) - \sum_{s=0}^{k}\binom{k+1}{s}\overline{\gamma}_{jk}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j)\right]$$
$$-8i\overline{\lambda}_j\left[\sum_{s=0}^{k} \frac{2(k+1)!}{s!(k-s)!}\overline{\gamma}_{js}\overline{\psi}^{(k-s)}(\overline{\lambda}_j) - \sum_{s=0}^{k}\binom{k+1}{s} k\overline{\gamma}_{j(k-1)}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j)\right]$$
$$-4i\left[\sum_{s=0}^{k-1} \frac{(k+1)!}{s!(k-s-1)!}\overline{\gamma}_{js}\overline{\psi}^{(k-s-1)}(\overline{\lambda}_j) - \sum_{s=0}^{k}\binom{k+1}{s} k(k-1)\overline{\gamma}_{j(k-2)}\overline{\psi}^{(k+1-s)}(\overline{\lambda}_j)\right].$$

This last expression can be simplified to

$$\frac{d\overline{\gamma}_{j(k+1)}}{dt}\overline{\psi}^{(0)}(\overline{\lambda}_j) = -4i\overline{\lambda}_j^2\left[\overline{\gamma}_{j(k+1)}\overline{\psi}^{(0)}(\overline{\lambda}_j)\right] - 8i\overline{\lambda}_j\left[(k+1)\overline{\gamma}_{jk}\overline{\psi}^{(0)}(\overline{\lambda}_j)\right]$$
$$-4i\left[(k+1)k\overline{\gamma}_{j(k-1)}\overline{\psi}^{(0)}(\overline{\lambda}_j)\right],$$

or equivalently

$$\frac{d\overline{\gamma}_{j(k+1)}}{dt} = -4i\overline{\lambda}_j^2\overline{\gamma}_{j(k+1)} - 8i\lambda_j(k+1)\overline{\gamma}_{jk} - 4i(k+1)k\overline{\gamma}_{j(k-1)}.$$

Thus, we have shown that (5.64) holds for all $k = 0, 1, \ldots, \overline{n}_j$, and hence the proof of our theorem is complete. ∎

Next we consider the time evolution of the bound state norming constants from $t = 0$ to an arbitrary time. For this purpose let us define the $n_j \times n_j$ matrix $\overline{A}_j$ as

$$\overline{A}_j := \begin{bmatrix} -i\overline{\lambda}_j & -1 & 0 & \dots & 0 \\ 0 & -i\overline{\lambda}_j & -1 & \dots & 0 \\ 0 & 0 & -i\overline{\lambda}_j & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -i\overline{\lambda}_j \end{bmatrix}. \tag{5.74}$$

Theorem 5.3.2 *The time evolution of the norming constants $\overline{c}_{jm}$ is governed by*

$$\begin{bmatrix} \overline{c}_{j(\overline{n}_j-1)}(t) & \overline{c}_{j(\overline{n}_j-2)}(t) & \dots & \overline{c}_{j0}(t) \end{bmatrix} = \begin{bmatrix} \overline{c}_{j(\overline{n}_j-1)}(0) & \overline{c}_{j(\overline{n}_j-2)}(0) & \dots & \overline{c}_{j0}(0) \end{bmatrix} e^{4i\overline{A}_j^2 t}. \tag{5.75}$$

Proof Recall that the norming constants $\overline{c}_{jm}$ and the dependency constants $\overline{\gamma}_{jm}$ are related to each other trough (4.49), which we write in the matrix form as

$$\widehat{\overline{C}}_j = \widehat{\overline{\Gamma}}_j \widehat{\overline{P}}_j, \tag{5.76}$$

where the matrices $\widehat{\overline{C}}_j$, $\widehat{\overline{\Gamma}}_j$, and $\widehat{\overline{P}}_j$ are defined as

$$\widehat{\overline{C}}_j := \begin{bmatrix} \overline{c}_{j(\overline{n}_j-1)} & \dots & \overline{c}_{j0} \end{bmatrix},$$

$$\widehat{\overline{\Gamma}}_j := \begin{bmatrix} \overline{\gamma}_{j(\overline{n}_j-1)} & \dots & \overline{\gamma}_{j0} \end{bmatrix},$$

$$\widehat{\overline{P}}_j := \begin{bmatrix} 0 & 0 & \dots & 0 & \frac{\overline{t}_{j\overline{n}_j} i^{-1}}{(\overline{n}_j-1)!} \\ 0 & 0 & \dots & \frac{\overline{t}_{j\overline{n}_j} i^{0}}{(\overline{n}_j-2)!} & \frac{\overline{t}_{j(\overline{n}_j-1)} i^{-1}}{(\overline{n}_j-2)!} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & \frac{\overline{t}_{j\overline{n}_j} i^{\overline{n}_j-3}}{1!} & \dots & \frac{\overline{t}_{j3} i^{0}}{1!} & \frac{\overline{t}_{j2} i^{-1}}{1!} \\ \frac{\overline{t}_{j\overline{n}_j} i^{\overline{n}_j-2}}{0!} & \frac{\overline{t}_{j(\overline{n}_j-1)} i^{\overline{n}_j-3}}{0!} & \dots & \frac{\overline{t}_{j2} i^{0}}{0!} & \frac{\overline{t}_{j1} i^{-1}}{0!} \end{bmatrix}. \tag{5.77}$$

Now since $\overline{t}_{jk}$ is independent of time from (5.76) we get

$$\frac{d\widehat{\overline{C}}_j}{dt} = \frac{d\widehat{\overline{\Gamma}}_j}{dt} \widehat{\overline{P}}_j, \tag{5.78}$$

or equivalently

$$\left[\begin{array}{cccc} \frac{d\overline{c}_{j(\overline{n}_j-1)}}{dt} & \frac{d\overline{c}_{j(\overline{n}_j-2)}}{dt} & \dots & \frac{d\overline{c}_{j0}}{dt} \end{array} \right] = \left[\begin{array}{cccc} \frac{d\overline{\gamma}_{j(\overline{n}_j-1)}}{dt} & \frac{d\overline{\gamma}_{j(\overline{n}_j-2)}}{dt} & \dots & \frac{d\overline{\gamma}_{j0}}{dt} \end{array} \right] \widehat{\overline{P}}_j.$$

Define the $n_j \times n_j$ matrix $\overline{L}$ as

$$\overline{L}_j := \begin{bmatrix} -4i\overline{\lambda}_j^2 & 0 & 0 & \dots & 0 \\ -(\overline{n}_j-1)8i\overline{\lambda}_j & -4i\overline{\lambda}_j^2 & 0 & \dots & 0 \\ -(\overline{n}_j-2)(\overline{n}_j-1)4i & -(\overline{n}_j-2)8i\overline{\lambda}_j & -4i\overline{\lambda}_j^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -4i\overline{\lambda}_j^2 \end{bmatrix}. \tag{5.79}$$

We can write (5.54) as

$$\frac{d\widehat{\overline{\Gamma}}_j}{dt} = \widehat{\overline{\Gamma}}_j \, \overline{L}_j. \tag{5.80}$$

Using (5.80) in (5.78) we get

$$\frac{d\widehat{\overline{C}}_j}{dt} = \widehat{\overline{\Gamma}}_j \, \overline{L}_j \, \widehat{\overline{P}}_j. \tag{5.81}$$

Now since $\overline{P}_j$ is congruent to a triangular matrix with nonzero diagonal entries, it is invertible. Writing (5.81) in the equivalent form as

$$\frac{d\widehat{\overline{C}}_j}{dt} = \widehat{\overline{\Gamma}}_j (\widehat{\overline{P}}_j \, \widehat{\overline{P}}_j^{-1}) \overline{L}_j \, \widehat{\overline{P}}_j, \tag{5.82}$$

with the help of (5.76) we obtain

$$\frac{d\widehat{\overline{C}}_j}{dt} = \widehat{\overline{C}}_j \, \widehat{\overline{P}}_j^{-1} \overline{L}_j \, \widehat{\overline{P}}_j,$$

which has the unique solution

$$\widehat{\overline{C}}_j(t) = \widehat{\overline{C}}_j(0) e^{(\widehat{\overline{P}}_j^{-1} \overline{L}_j \, \widehat{\overline{P}}_j)t}.$$

With the help of (5.74), (5.77), and (5.79) we obtain

$$\widehat{\overline{P}}_j^{-1} \overline{L}_j \, \widehat{\overline{P}}_j = 4i\overline{A}_j^2.$$

Therefore, the theorem is proved. ∎

Moreover, a generalization of the formula in (5.37) for a set of bound states $\{\lambda_j\}_{j=1}^{N}$ each of multiplicity n_j can be expressed as

$$C(t) = C(0)e^{-4iA^2t}, \tag{5.83}$$

where C and A are defined as

$$C(t) := \begin{bmatrix} C_1(t) & C_2(t) & \dots & C_N(t) \end{bmatrix}, \tag{5.84}$$

$$A := \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_N \end{bmatrix}, \tag{5.85}$$

where the matrix A_j is defined in (5.36) and C_j defined as

$$C_j := \begin{bmatrix} c_{j(n_j-1)}(t) & c_{j(n_j-2)}(t) & \dots & c_{j0}(t) \end{bmatrix}.$$

Similarly, the analog of the formula in (5.37) for a set of bound states $\{\overline{\lambda}_j\}_{j=1}^{\overline{N}}$ each of multiplicity $\overline{n}_j$ is given by

$$\overline{C}(t) = \overline{C}(0)e^{4i\overline{A}^2t},$$

where $\overline{C}$ and $\overline{A}$ are defined as

$$\overline{C}(t) := \begin{bmatrix} \overline{C}_1(t) & \overline{C}_2(t) & \dots & \overline{C}_{\overline{N}}(t) \end{bmatrix},$$

$$\overline{A} := \begin{bmatrix} \overline{A}_1 & 0 & \dots & 0 \\ 0 & \overline{A}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \overline{A}_{\overline{N}} \end{bmatrix},$$

where the matrix $\overline{A}_j$ is defined in (5.74) and $\overline{C}_j$ defined as

$$\overline{C}_j := \begin{bmatrix} \overline{c}_{j(\overline{n}_j-1)}(t) & \overline{c}_{j(\overline{n}_j-2)}(t) & \dots & \overline{c}_{j0}(t) \end{bmatrix}.$$

CHAPTER 6

INVERSE SCATTERING TRANSFORM FOR THE NLS EQUATION

As mentioned in Chapter 5, the first-order system in (2.1) with $r = -q^*$ reduces to the Zakharov-Shabat system, which is associated with the NLS equation. In that case, the Jost solutions $\overline{\psi}$ and $\overline{\varphi}$, the scattering coefficients $\overline{T}$, $\overline{R}$, and $\overline{L}$, the bound state dependency constants $\overline{\gamma}_{jm}$, and the bound state norming constants $\overline{c}_{jm}$ can all be expressed in terms of the corresponding quantities ψ, φ, T, R, L, γ_{jm}, and c_{jm}. In this chapter we review such relationships. The connecting relationships for the Jost solutions and the scattering coefficients are already known [1, 2, 16] in the case of simple bound states and they remain unchanged also in the case of poles of higher multiplicities.

When $r(x,t) = -q^*(x,t)$ we get

$$\overline{\psi}(\lambda, x, t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \psi^*(\lambda^*, x, t). \tag{6.1}$$

Note that (6.1) can be verified by showing that both $\psi(\lambda, x, t)$ and $\overline{\psi}(\lambda, x, t)$ satisfy the following Zakharov-Shabat system and the corresponding asymptotics:

$$\overline{\psi'} = \begin{bmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{bmatrix} \overline{\psi} + \begin{bmatrix} 0 & q \\ -q^* & 0 \end{bmatrix} \overline{\psi}. \tag{6.2}$$

Substituting λ^* for λ in (6.2) we have

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \psi'^*(\lambda^*) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i\lambda^* & 0 \\ 0 & i\lambda^* \end{bmatrix} \psi^*(\lambda^*)$$
$$+ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & q(x) \\ -q^*(x) & 0 \end{bmatrix} \psi^*(\lambda^*).$$

However

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

and hence

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \psi'^*(\lambda^*) = &- \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i\lambda^* & 0 \\ 0 & -i\lambda^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \psi^*(\lambda^*) \\ &- \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & q(x) \\ -q^*(x) & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \psi^*(\lambda^*) \\ = &\begin{bmatrix} -i\lambda^* & 0 \\ 0 & i\lambda^* \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \psi^*(\lambda^*) + \begin{bmatrix} 0 & q(x) \\ -q^*(x) & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \psi^*(\lambda^*). \end{aligned}$$

From pages 8 and 9 we know that ψ and $\overline{\psi}$ satisfy the asymptotic conditions as $x \to +\infty$

$$\psi \to \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad \overline{\psi} \to \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1),$$

and hence, as $x \to +\infty$, we have

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \psi^*(\lambda^*) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (e^{-i\lambda^* x})^* \end{bmatrix} + o(1) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1).$$

Thus, (6.1) is established. In a similar way it can be shown that

$$\overline{\varphi}(\lambda, x, t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \varphi^*(\lambda^*, x, t). \tag{6.3}$$

By comparing the asymptotics as $x \to -\infty$ in (6.1) and as $x \to +\infty$ in (6.3), we obtain when $r = -q^*$ the relationships among the scattering coefficients as

$$\overline{T}(\lambda, t) = T^*(\lambda^*, t), \qquad \overline{R}(\lambda, t) = -R^*(\lambda^*, t), \qquad \overline{L}(\lambda) = -L^*(\lambda^*, t). \tag{6.4}$$

With the help of (6.1), (6.3), and (6.4) one can obtain the following result [7]:

Theorem 6.0.3 *When $r = -q^*$, we have*

1. $\overline{\lambda}_j = \lambda_j^*$
2. *algebraic multiplicity of* $\overline{\lambda}_j$ = *algebraic multiplicity of* λ_j^*
3. $\overline{\gamma}_{jm} = \gamma_{jm}^*$
4. $\overline{c}_{jm} = -c_{jm}^*$
5. $\overline{\Omega}(z,t) = -\Omega^*(z,t)$
6. $\overline{K}(x,y,t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} K^*(x,y,t).$

As a consequence of Theorem 6.0.3, from the two vector-valued Marchenko equations given in (4.50), one derives the scalar Marchenko equation

$$K(x,y,t) = \Omega^*(x+y,t) - \int_x^\infty dz \int_x^\infty ds\, K(x,s,t)\, \Omega(s+z,t)\, \Omega^*(z+y,t),$$

where the scalar kernel is [7]

$$\Omega(z,t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\lambda,t)e^{i\lambda z}d\lambda + \sum_{j=1}^{N}\sum_{s=0}^{n_j-1} c_{js}(t)\frac{z^s}{s!}e^{i\lambda_j z}.$$

Then, the solution $u(x,t)$ to the NLS equation can be found by solving the scalar Marchenko integral equation for the scalar function $K(x,y,t)$ and by using

$$u(x,t) = -2K(x,x,t).$$

CHAPTER 7

CONCLUSION

When Zakharov and Shabat introduced the inverse scattering transform for the NLS equation in 1972 [20], they assumed the bound states were all simple. They tried to deal with bound states of multiplicity two by coalescing two simple poles into one. Even though the idea of coalescing two or more poles into one is theoretically feasible, the practical implementation of coalescing is not that easy, as evident from the concrete example that Zakharov and Shabat provided with a computational error [17, 20]. As pointed out by Olmedilla [17], Zakharov and Shabat's "limiting process gives the appropriate value ... but their final result for the potential is mistaken." The error in [20] certainly does not diminish the importance of the work by Zakharov and Shabat, but it indicates that the process of dealing with bound states of higher multiplicity is not easy even when the multiplicity is two. Olmedilla derived [17] some formulas to deal with bound states of multiplicities two and three, but he also added that "in actual calculation it is very complex to exceed four of five." Using the symbolic computer software REDUCE he was able to reach a multiplicity of nine, but his formulas were too complicated to generalize to a bound state of any multiplicity.

In our thesis we have provided a complete generalization of the inverse scattering transform with bound states of any multiplicities. We have accomplished our goal by deriving the time evolution for the norming constants c_{jm} when there are N bound states each having multiplicity n_j, respectively. Our elegant formula given in (5.83)

$$C(t) = C(0)e^{-4iA^2t},$$

where C is the $1\times(Nn_j)$ matrix given in (5.84) and A is the $(Nn_j)\times(Nn_j)$ matrix given in (5.85), provides the generalization of the evolution formula

$$c_j(t) = c_j(0)e^{-4i\lambda_j t},$$

which holds only when there is a simple bound state at λ_j.

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BIOGRAPHICAL STATEMENT

Theresa Busse grew up in Elk Grove Village, IL, a suburb of Chicago. Her parents instilled the importance of college from a very young age, and she took that very seriously. She then went off to college not too far away from home at Lewis University in Romeville, IL. There she majored in mathematics and physics with a minor in secondary education. She had planned to teach high school math and physics. As graduation drew closer, she did not feel she was done with learning and so she applied to graduate schools to both mathematics and physics. She was accepted to a number of universities, but Texas Tech University gave her a unique opportunity and an assistantship where she was guaranteed teaching at the undergraduate level. So she made the big move from Illinois to Texas. She was able to teach a wide range of undergraduate courses while taking graduate courses and preparing her research for a master's thesis. While at Texas Tech University she had the pleasure of working with Marianna Shubov and could not have asked for anyone better. Then it was off to Arlington, TX to work on her Ph.D. at the University of Texas at Arlington. She was again able to teach at the undergraduate level while she was working on her Ph.D. degree. She was there for a semester before her current adviser, Tuncay Aktosun, arrived at the university. With a common interest in partial differential equations, they started working on a research topic almost immediately. Then three short years later Theresa was defending her dissertation.