MODE STRUCTURE OF A NOISELESS PHASE-SENSITIVE IMAGE AMPLIFIER

by

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"In restless dreams I walked alone
Narrow streets of cobblestone
'Neath the halo of a street lamp
I turned my collar to the cold and damp
When my eyes were stabbed by the flash of a neon light
That split the night
And touched the sound of silence”

-'The Sound of Silence’ by Simon & Garfunkel.

Dedicated to my mother Kamala, and father Annamalai
for providing me everything
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ABSTRACT

MODE STRUCTURE OF A NOISELESS PHASE-SENSITIVE IMAGE AMPLIFIER

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Optical parametric amplifiers (OPAs) configured as phase-sensitive amplifiers (PSAs) can be used for noiseless optical image amplification, generation of non-classical states of light, and, in particular, for multimode squeezed light generation. For all these applications and for effective use of the quantum properties of the multimode PSA, we need to know its independently squeezed (or amplified) eigenmodes.

First we present the quantum theory of a spatially multimode traveling-wave PSA pumped by a high-power pump beam with arbitrary spatial profile. To quantitatively identify the Green’s function of the PSA, we have developed a semi-analytical coupled-mode-theory of the PSA using 2D Hermite-Gaussian mode expansions of the signal- and pump-beam spatial distributions.

By using Green’s functions of the classical OPA, we calculate the normally-ordered quadrature correlators at its output, which provide complete quantum description of the phase-sensitive OPA and enable determination of its independently squeezed eigenmodes. We find the number of the supported eigenmodes and their
shapes for a spatially broadband frequency-degenerate optical parametric amplifier with elliptical Gaussian pump.

We conclude by discussing our recent extensions of the coupled-mode-theory to study higher-order pump modes, compact representation of the PSA eigenmodes (especially convenient for experiments), and effect of non-zero phase mismatch on the PSA eigenmodes. We expect that the results from our model can be used for optimum mode matching in phase-sensitive image amplification and multimode squeezing generation.
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CHAPTER 1

INTRODUCTION

1.1 Quantum Limited Measurements

Noise determines system performance beyond the Rayleigh limit [1]. Qualitatively we may understand this phenomena in a double-slit system, where we want to resolve the image in the far-field. We can do this by detecting diffraction pattern of double-slit image in the Rayleigh-limited (diffraction limited) optical system. Here the effectiveness of the spatial-resolution depends also on the quantum-efficiency of the detector, in addition to the optics. In this classical imaging situation, the resolution of a diffraction limited optical system is limited by the detector shot noise.

Quantum optics proposals for experiments have shown the improvement in optical interferometry [1], and experiments in communication systems [2], and imaging systems [3],[4] beyond the shot noise limited measurement by generating novel quantum states of light, or by using parametric-amplifiers.

Light is described in Quantum Optics, as a harmonic oscillator with energy eigenstates given by photon numbers [5]. A single-mode quantum state of the photon is associated with a spatial mode, and its state given by the energy distribution of the harmonic oscillator states. Analogous to a mechanical oscillator, the quantum state of light has position and momentum quadrature operators in the harmonic oscillator description, and their commutation relations form the distinction between the classical and quantum descriptions of light. The quantum harmonic oscillator description of light adds half a photon energy to the ground state of every mode, which is a manifestation of the vacuum fluctuations. The inability of simultaneous measurement of
all components of electric and magnetic fields of the photon mode, is a consequence of the Heisenberg uncertainty relations between the quadrature components of the photonic mode [6].

In the minimum uncertainty state such as coherent state of laser light, the quantum noise manifests itself as fluctuations (variances) of Heisenberg-limited non-commuting pairs of variables, like the photon number-phase, or momentum-position quadratures. Squeezing refers to reducing fluctuations on any one of the quadratures (with corresponding anti-squeezing on other quadrature due to Heisenberg uncertainty) compared to the fluctuations of the corresponding coherent state. Squeezed states of light, have been shown to improve the sensing and measurement beyond the standard quantum limit set by the minimum uncertainty states.

Results of research into squeezed state generation of light has led to potential improvements in sensing and measurement systems beyond the standard quantum limit, which forms the science of Quantum Metrology. Squeezed states of light can help reach the quantum limit (and possibly improve over this) in interferometric measurements like the Laser-Interferometer Gravitational Wave Observatory project (LIGO) [7], which aims to detect gravitational waves.

1.2 Quantum Metrology and Quantum Imaging

The field of Quantum Metrology, seeks to use squeezed states of light to minimize the quantum fluctuations and obtain the least noisy measurement possible. This kind of experimental ability is unprecedented and is expected to be useful in extreme physics applications such as synchronizing optical atomic clocks, detecting gravitational waves, developing the standard references for fundamental quantities, and performing other sensitive measurements. Quantum Imaging forms a part of the larger science of Quantum Metrology. Quantum Imaging aims to use the non-
classical/quantum properties of photon to image in situations beyond the Rayleigh limit $0.6\lambda/D$ of angular separation between resolvable elements (super resolution), when the signal to noise ratio (SNR) is very low through a variety of protocols. Some of Quantum Imaging methods [4], include the joint detection of a weak phase object imaged by one of the entangled signal-idler pair beams in optical parametric amplifier, the correlation detection of any object using entangled or thermal beams by ghost imaging (GI), as well as phase-sensitive amplification followed by squeezed vacuum injection (SVI) and homodyne detection.

Optical phase-sensitive amplification offers potential for imaging beyond the standard quantum limit by preserving the SNR at input and output, i.e 0 dB noise figure (NF).

1.3 Conventional Image Amplifiers
We describe the electronic and optical methods for image amplification, in this section. Specifically these systems are phase-insensitive and have a limiting NF=2 (3 dB), due to the amplified-spontaneous-emission (ASE).

In an electronic image amplifier, the light from image is incident on a photocathode generates electrons, which are multiplied and focused onto a phosphorus screen to be converted back into visible light at a higher intensity. Such electronic image amplifier uses electronic amplification after an optical detector, leading to several sources of classical noise like spontaneous emission from multichannel plates, and multiplication noise in case of avalanche diodes, Johnson (thermal) and readout noise associated with the photodetectors.

Conventional signal amplifiers uses a gain medium under population inversion to add extra photons in the signal mode to get an amplified output; however spontaneous
emission adds noise equivalent to one half of a noisy photon at the input of each amplified mode.

Several nonlinear optical effects have been tested as possible candidates for image amplification, based on multicore fiber [8], photorefractive effect [9], stimulated Brillouin scattering (SBS) [10], stimulated Raman scattering (SRS) [11] and optical parametric amplification (OPA) [12]. Typically, image amplification is obtained by multimode gain media either through stimulated emission or by transferring energy into the spatial modes of the weak image-bearing signal beam from a strong pump beam through a nonlinear effect under appropriate phase-matching and mode-coupling conditions. OPA [13],[14], uses a high frequency pump to amplify any signal with frequency arbitrarily lower than that of the strong pump by the second order nonlinear interaction in a non-centrosymmetric crystal when a pump photon is destroyed and a photon is created each at the signal and idler frequencies respectively under energy and momentum conservation rules. Parametric amplification is in this sense a converse of the sum-frequency generation (SFG). In this sense, nonlinear image amplifiers are similar to the cross-field microwave amplifiers where a electron emitted from a thermionic cathode in a vacuum tube is slowed down by periodic magnetic fields whereby it exchanges power with the incoming radio-frequency (RF) field. Only amplifiers based on nonlinear effect can provide phase-sensitive amplification (with 0 dB NF) whereas all other systems are restricted to the phase insensitive operation incurring 3 dB degradation in the SNR of a shot-noise limited input signal. For laser-light (coherent-state $|\alpha\rangle$) $\text{SNR} = \langle \hat{n} \rangle = |\alpha|^2$, due to its Poissonian probability distribution of the photon number.
1.4 Degenerate OPA as Noiseless Amplifier

Degenerate OPA (DOPA) was originally used to generate squeezed states of light, and later it was demonstrated to be a noiseless amplifier in fiber optical systems [2], bulk optical setup [14], and as a noiseless image amplifier [12]. The important difference between the two is the latter situation demands a spatially multimode squeezed vacuum noise generation under tight constraints of the PSA phase matching spatial bandwidth. Noiseless image amplification is a harder problem to solve compared to the noiseless fiber amplification.

As discussed previously the 3 dB NF in optically amplified systems place a threshold on the smallest amount of light detected. This can become important in several applications where we do not have control over the received light, and we expect the availability of low noise or noiseless image amplifiers to potentially enable doing science in low-light regimes never accessible before. Envisioned applications of noiseless image amplification include,

1. Biological imaging from thick tissue samples, imaging from turbid media
2. Resolving faint images in astronomy, with high fidelity resulting in smaller collection times, and better images at good SNR
3. Free space optical communications
4. Hi-fidelity LADAR, remote sensing & target identification, ocean imaging etc

1.5 Quantum Image Enhancer - Quantum Laser RADAR

A Quantum Imaging system can image at very low SNR, for example, by using an entangled photon pair generator such as spontaneous-parametric down converter (SPDC) crystal used in a correlated imaging setup [15], or a degenerate optical parametric amplifier (DOPA) configured as a noiseless phase-sensitive amplifier (PSA) [12].
Figure 1.1. Figure illustrates effect of, (top-row) degradation of spatial resolution with a non-unity quantum efficiency detector, and (bot-row) recovery of the spatial resolution with a phase-sensitive pre-amplifier.

As a pre-amplifier to compensate for the non-unity detector quantum efficiency, the PSA is used to amplify the signal before detection, so that deconvolution algorithms (which are highly sensitive to SNR) in post-processing can possibly recover more information from the signal. Specifically Ref. [16], proposed a Quantum Laser RADAR, where finite-bandwidth PSA would be used to provide gain at 0 dB NF to compensate for detector inefficiency, and also to inject squeezed vacuum into the high-spatial frequencies which are cutoff due to the receiver aperture pupil, thereby improving resolution. We illustrate in Fig. 1.1, the effect of a non-unity ($\eta < 1$) quantum-efficiency (QE) detector on spatial resolution, with and without a phase-sensitive pre-amplifier.

Theoretical studies of a pre-amplifier in a LADAR system with an infinite-bandwidth PSA (i.e. undistorted PSA output from an extremely short crystal) were
carried out [17], and their Monte-Carlo modeling proved gains over the baseline system in the form of an improved sub-Rayleigh resolution of the targets in the far-field of the LADAR for moderate PSA gain (0-15 dB). Using the results from their study [17], we show in Fig. 1.2 the connection between PSA gains, improvement of SNR and the angular resolution of a 2-point targets. However, the results need to be qualified, since they have assumed undistorted operation of the PSA, which is impractical as we show in next section.

![Figure 1.2](image.png)

**Figure 1.2.** We make connection between, (top-row) SNR increase with the PSA gain, and (bot-row) improvement in angular resolution (over the Rayleigh limit) as function of SNR gains.

### 1.6 Modeling PSA at High Gains

Practical PSA are based on finite-length nonlinear crystals (from several mm to several cm), which leads to the phenomenon of gain-induced-diffraction where the
non-uniformly amplified signal beam diffracts faster, mixing with higher-order signal modes as it propagates over the length of the crystal [12]. Previously the theoretical models of [18],[19],[20] allowed one to calculate the gain and squeezing obtained from such a traveling-wave PSA, but with restrictions on the pump symmetry, or crystal length.

In this thesis, we develop a method for finding the number and shapes of the independently squeezed or amplified modes of a spatially-broadband, travelling-wave, frequency- and polarization-degenerate optical parametric amplifier in the general case of an elliptical Gaussian pump. The obtained results show that for tightly focused pump only one mode is squeezed, and this mode has a Gaussian TEM\(_{00}\) shape. For larger pump spot sizes that support multiple modes, the shapes of the most-amplified modes are close to Hermite- or Laguerre-Gaussian profiles. These results can be used to generate matched local oscillators for detecting high amounts of squeezing, and to design parametric image amplifiers that introduce minimal distortion.

Such a realistic PSA model allows the studies of practical LADAR system, and gives quantifiable results as demonstrated later in [21]. We show that an outcome of our theory is a powerful generalization of the traveling-wave PSA allowing us to solve problems of local-oscillator selection, multimode squeezed state generation, and eigenmodes in noiseless image amplification.

1.7 Organization of Thesis

This thesis is arranged in the following order. In Chapter 2, we present the background on quantum squeezing and explain how squeezing along with phase-sensitive amplification can be applied to improve target range and resolution in the form of a quantum image enhancer before a photodetector in LADAR system.
We next show theoretically the procedure to calculate the independently correlated eigenmodes of a PSA, and derive the normally ordered correlator which completely determines all the quantum properties of the PSA. The quantum theory of a spatially multimode traveling-wave phase-sensitive optical parametric amplifier (OPA) pumped by a high-power pump beam with arbitrary spatial profile, is presented in Chapter 3. We develop for the first time, a semi-analytical coupled-mode-theory of the PSA field evolution, and identify its independent (i.e. uncorrelated) spatial eigenmodes using a Green’s-function correlator approach, which is presented in Chapter 4. We discuss results of PSA eigenmode calculation, for various pump power and spot sizes with various pump-mode shapes and phase-matching conditions, in Chapter 5. In Chapter 6, we conclude by summarizing the contributions of this thesis, and outlining potential future directions of research for both experiments and theoretical analysis.
CHAPTER 2
THEORY OF PHASE-SENSITIVE AMPLIFICATION

Quantum-optical description of systems with amplification involves coupling with a vacuum reservoir due to the conditions on the preservation of the commutators between the Bosonic input and output field operators. This is the source of noise, which is described in semi-classical models by adding half-photon of vacuum noise to each input mode to account for spontaneous emission noise.

2.1 Ideal Single-Mode Linear Amplification

In this section we give the single-mode operator input-output transformation relations of an ideal amplifier in a Quantum Optics framework [12], [22]. These relations depend on the amplifier being a phase-sensitive amplifier (PSA) or phase-insensitive amplifier (PIA). These operator relations enable us to study the quantum properties of the amplifier in the Heisenberg picture, and help us evaluate the mean and variance of (among others) the quadrature operators $\hat{q}_+ = \left(\frac{\hat{a} + \hat{a}^\dagger}{2}\right)$, $\hat{q}_- = \left(\frac{\hat{a} - \hat{a}^\dagger}{2i}\right)$, and number operators $\hat{n} = \hat{a}^\dagger \hat{a}$, assuming either the input vacuum state $|0\rangle$ or the input coherent state $|\alpha\rangle$.

For PIA the output annihilation operator $\hat{b}$ is related to the input annihilation operator $\hat{a}$ and input vacuum operator $\hat{\nu}$ by,

$$\hat{b} = \mu \hat{a} + \nu \hat{\nu}^\dagger$$

(2.1)
Figure 2.1. Quadrature representation of the input (dotted, empty circle) and output (solid and dotted, filled circle) coherent-state before and after amplification by a PIA, where both quadratures of input are amplified equally, and with a larger uncertainty area. The minimum additional noise (with NF= 3 dB) with output state uncertainty area shown by inner circle, while larger NF amplification is shown by the outer shaded circle (of a larger uncertainty area).

$$\sigma_{\text{out}}^2 = (2G-1)\sigma_{\text{in}}^2$$

Figure 2.2. Quadrature representation of the input (dotted, empty circle) and output (solid, filled ellipse) coherent-state before and after amplification by a PSA. The PSA output state has a minimum additional noise (with NF = 0 dB) with same uncertainty area.

$$\sigma_X^2 = G\sigma_{\text{in}}^2 \quad \sigma_Y^2 = G^{-1}\sigma_{\text{in}}^2$$
and for the PSA the relations are,

\[ \hat{b} = \mu \hat{a} + \nu \hat{a}^\dagger \]  \hspace{1cm} (2.2)\]

where in both cases \(|\mu|^2 - |\nu|^2 = 1\) (typically \(\mu = \cosh(\gamma L)\) and \(\nu = \sinh(\gamma L)\)). These transformation relations (Bogoliubov transform) have this particular form because they have to preserve the Bosonic commutator relations between the corresponding input and output field operators; i.e., \([\hat{a}, \hat{a}^\dagger] = 1\), and \([\hat{b}, \hat{b}^\dagger] = 1\). Using this framework we define the signal-to-noise ratio (SNR) at input and output as,

\[ \hat{n}_a = \hat{a}^\dagger \hat{a}, \quad \text{SNR}_{\text{in}} = \frac{\langle \hat{n}_a \rangle^2}{\langle \Delta \hat{n}_a^2 \rangle} \] \hspace{1cm} (2.3)\]

\[ \hat{n}_b = \hat{b}^\dagger \hat{b}, \quad \text{SNR}_{\text{out}} = \frac{\langle \hat{n}_b \rangle^2}{\langle \Delta \hat{n}_b^2 \rangle} \] \hspace{1cm} (2.4)\]

which leads to the definition of the noise-figure (NF) as ratio of SNR between input and output,

\[ \text{NF} = \frac{\text{SNR}_{\text{in}}}{\text{SNR}_{\text{out}}}, \] \hspace{1cm} (2.5)\]

and the amplifier gain as the ratio of the mean photon number at output to that of the input,

\[ G = \frac{\langle \hat{n}_b \rangle}{\langle \hat{n}_a \rangle}. \] \hspace{1cm} (2.6)\]
Using the above relations for a PIA, we find the mean photon number (evaluated for a coherent state $|\alpha\rangle$) at the output, gain and variance as,

\[
\langle \hat{n}_a \rangle = |\alpha|^2,
\]
\[
\langle \hat{n}_b \rangle = \gamma|\alpha|^2 + \gamma - 1, \quad \text{where} \quad \gamma = |\mu|^2,
\]
\[
G = \gamma \quad \text{for} \quad |\alpha|^2 \gg 1,
\]
\[
\langle \Delta \hat{n}_b^2 \rangle = G(2G - 1)|\alpha|^2 + G(G - 1).
\]

With these relations it's a simple matter to write the NF for the PIA from the SNR relations as, $\text{NF} = 2 - 1/G$, in the high-gain limit.

Similarly for the PSA, we find the mean photon number (evaluated for a coherent state $|\alpha\rangle$) at the output, gain and variance as,

\[
\langle \hat{n}_a \rangle = |\alpha|^2,
\]
\[
\langle \hat{n}_b \rangle = \gamma|\alpha|^2 + (\gamma - 1)(|\alpha|^2 + 1) + 2\sqrt{\gamma(\gamma - 1)}|\alpha|^2 \cos(\theta), \quad \text{where} \quad \gamma = |\mu|^2,
\]
\[
\theta = \arg(\nu) - \arg(\mu) - 2\arg(\alpha)
\]
\[
G = 2\gamma - 1 + 2\sqrt{\gamma(\gamma - 1)} \quad \text{for} \quad |\alpha|^2 \gg 1, \theta = 0,
\]
\[
\langle \Delta \hat{n}_b^2 \rangle = G^2|\alpha|^2 + 2\gamma(\gamma - 1), \quad \text{for} \quad \theta = 0.
\]

Using the above relations find the NF of the PSA as, $\text{NF} = 1$, in the high-gain limit, which is a factor of 2 less than corresponding NF of PIA. This is primarily due to the fundamental quantum mechanical considerations of commutator involved in the PSA.
2.1.1 Effect of Lossy Photodetectors

We write the field operator relations for a lossy photodetector; i.e., where the quantum-efficiency $\eta < 1$. The quantum optical model for such a detector includes the vacuum coupling at the input through the operator $\hat{c}$, with signal $\hat{b}$,

$$\hat{d} = \sqrt{\eta} \hat{b} + \sqrt{1 - \eta} \hat{c}.$$  \hspace{1cm} (2.9)

A direct-detector measures the number operator $\hat{n}_d = \hat{d}^\dagger \hat{d}$ as $\langle \hat{n}_d \rangle$, with the fluctuation in the photo-current signal of $\langle (\Delta \hat{n}_d)^2 \rangle$. Then we can write the total NF of such a pre-amplifier detector system, with a pre-amplifier (PSA or PIA) of a corresponding NF$_{amp}$ as,

$$NF_\eta = NF_{amp} + \frac{1 - \eta}{\eta G}.$$ \hspace{1cm} (2.10)

Note that the SNR for quadrature operators $\hat{q}_\pm$, $\frac{\langle \hat{q}_\pm \rangle^2}{\langle (\Delta \hat{q}_\pm)^2 \rangle}$, is preserved by the PSA for $\theta = 0$ without any assumptions on magnitude of $|\alpha|^2$ and gain.

2.2 Optical Parametric Amplification

In practice, the PIA or PSA can be realized by the non-degenerate or degenerate OPA configurations respectively. A pioneering experiment of the OPA configured for image amplification in the PSA (and PIA) modes was demonstrated in [12]. In this 3-wave mixing process a high-energy photon in pump mode parametrically adds 1 photon into signal and 1 photon into idler modes under energy and momentum matching (phase-matching) conditions.
The operator relations for the OPA in a general (non-degenerate mode) can be written as,

\[ \hat{b}_s = \mu \hat{a}_s + \nu \hat{a}_i^\dagger, \quad (2.11) \]
\[ \hat{b}_i = \mu \hat{a}_i + \nu \hat{a}_s^\dagger, \quad (2.12) \]

where the annihilation operators are given by \( \hat{a}_{i,s} \) at the input, and by \( \hat{b}_{i,s} \) at the output. When only \( \hat{a}_s \) is excited at the input the OPA works as a PIA, and when both \( \hat{a}_s \) and \( \hat{a}_i \) are excited, it works as a PSA. Specifically, to operate in the PSA mode requires exciting the input mode \((\hat{a}_i + \hat{a}_s)/\sqrt{2}\) and detecting the output mode \((\hat{b}_i + \hat{b}_s)/\sqrt{2}\). A suitable mode transformation of the two-mode equations allows it to be written as two single-mode PSA systems.

For a plane-wave pump with \( \vec{q} = 0 \) spatial frequency in the OPA, the multimode image amplification has known transformation relations (unlike the elliptical Gaussian pump studied in this thesis), in terms of the spatial frequencies at \( \vec{q} \) and \( -\vec{q} \) of the signal and idler,

\[ \hat{b}_s(\vec{q}) = \mu \hat{a}_s(\vec{q}) + \nu \hat{a}_i^\dagger(\vec{-q}), \quad (2.13) \]
\[ \hat{b}_i(\vec{q}) = \mu \hat{a}_i(\vec{q}) + \nu \hat{a}_s^\dagger(\vec{-q}). \quad (2.14) \]

The quantum properties of the signal and idler beam from PIA in low gain regime, show identical correlations and they are entangled. Such entangled beams can be used to image weak phase objects below the shot noise limit, through a correlation measurement [15]. In the degenerate OPA (DOPA), the signal and idler of equal frequencies are generated by the pump photon, and this configuration of the OPA forms a phase-sensitive amplifier when excited with a signal at input, or a broadband
squeezed source when no input signal (idler) is present. Phase-sensitive amplifiers are however immune to the degradation in SNR: when they amplify the in phase quadrature, they also deamplify the out of phase quadrature by the same amount, because of which even the output amplified states continue to have the same area of the Heisenberg uncertainty ellipse, unlike the PIA which amplifies both quadratures leading to increased uncertainty area and degraded SNR. When operated without an input signal the PSA acts on the input zero point vacuum fluctuations creating squeezed vacuum at the output. Thus phase sensitive amplifiers are capable of providing 0 dB NF and preserve input SNR at the output.

2.3 Multimode PSA Applications to LADAR

The multimode PSA generates quantum states of light, whose properties can be used to enhance the angular and spatial resolution of optical remote sensors.

A quantum-enhanced image receiver for Laser Detection and Ranging (LADAR) that can offer a significant improvement in resolution over conventional laser sensors [16]. Our analysis and design is key to realizing the parametric-amplifier-based quantum image enhancer for the recovery of the information lost by both attenuation of high spatial frequencies of the image (due to free-space propagation acting as a low-pass filter) and inefficiency of the detector array (introducing extra noise from the vacuum modes responsible for loss). The quantum-enhanced LADAR research has potential to allow future laser-enabled cameras to see remote objects more clearly and have a much smaller threshold of operation.

In a linear system imaging, the field distribution in the image plane is the convolution of the field in the object plane with the point spread function (PSF) of the propagation and imaging device. It is the PSF of the imaging system that places the classical resolution limits (Rayleigh Limit) on the image.
The SNR of the image is calculated at any pixel in the image as the ratio of the mean square photon number to the variance in the photon number. For Poissonian distribution, the variance is the square root of the mean, which is the origin of the shot noise in the photodetector.

LADAR target resolution and useful range is limited by the minimum SNR at which the detection algorithms can reliably perform. When quantum image enhancer (QIE) is used before the photodetector, it can significantly improve the SNR [16]. Essential component of such a system is squeezed vacuum injection (SVI) in all those spatial frequencies cutoff by the soft aperture in front of the detector and followed by the spatially multimode PSA, with the homodyne detection at the CCD. SVI has the effect of reducing the quantum fluctuations in the highest spatial frequencies, while the PSA is tuned to amplify the information carrying quadrature of the signal.

2.4 Conclusion
In order to realize a practical multimode-PSA in a Quantum Image Enhancer like in [17] but with finite spatial-bandwidth, we need to identify the shape of the images which are best amplified with zero noise-figure, without coupling to other modes. This problem is quite hard since the practical PSA with elliptical Gaussian pump creates a strong gain-induced diffraction in the amplified signal beam, eventually causing distortion in the signal modes [19],[20]. We tackle this problem by developing the quantum theory of PSA correlators and a procedure to calculate the eigenmodes of the PSA in Chapter 3. The actual implementation of the procedure itself requires a knowledge of the PSA Greens function by numerically solving the coupled-mode equations, which will be developed in Chapter 4.
CHAPTER 3
QUANTUM THEORY OF PSA

3.1 Introduction

We present the quantum theory of a spatially-multimode traveling-wave phase-sensitive optical parametric amplifier (OPA) pumped by a beam with arbitrary spatial profile. By using Greens functions of the classical OPA, we derive the normally-ordered quadrature correlators at the OPA output, which provide complete quantum description of the phase-sensitive OPA and enable determination of its independently-squeezed eigenmodes. Two analytically treatable examples of plane-wave pump and infinite spatial bandwidth of the crystal are discussed in detail. The following work was published as journal article [23].

3.2 Quantum Theory of PSA

Spatially-broadband optical parametric amplifiers (OPAs) are important for the generation of correlated modes for quantum information processing as well as for noiseless amplification of images, with recent work nicely summarized in [24, 25]. The latter application requires phase-sensitive OPAs with strongly focused pump beams in either traveling-wave [12, 26] or self-imaging-cavity [27] configurations. While the classical traveling-wave OPA with plane-wave pump has a well known analytical solution [28, 29] that is straightforwardly extendable to the quantum case, the inhomogeneous (i.e. spatially-varying) pump case requires numerical modeling even for the classical signal [30], unless the nonlinear medium of the OPA is very short. Although computationally efficient numerical methods based on Hermite-Gaussian and
Laguerre-Gaussian mode expansions have been developed for both cavity-based [18] and traveling-wave [20] OPAs for some pumping configurations, determination of the complete quantum properties of the OPAs from numerical modeling have remained a serious challenge.

In this work, we provide a general framework relating the complete quantum description of the traveling-wave phase-sensitive OPA with arbitrary pump to Greens function of the underlying classical propagation equation, which is obtainable by numerical or (in rare instances) analytical methods. By diagonalizing the derived quantum correlators, we show that a set of independently-squeezed orthogonal modes (eigenmodes) of the OPA can be obtained, in analogy to the Karhunen-Loeve expansion for classical random processes. Such eigenmodes of the traveling-wave OPA are also related to the supermodes of a self-imaging-cavity-based OPA studied in [31]. The analysis of the present paper serves as a basis for our determination of the actual OPA eigenmodes via numerically-obtained Greens functions in Hermite-Gaussian representation [32].

3.3 Definitions from Classical Free-Space Propagation

A detailed theory of parametric amplification of multimode fields is summarized in a recently published book [24]. Here, we concentrate on OPA equations in paraxial approximation with undepleted pump. Assuming that the signal, idler, and pump fields are polarized, we look for solutions in the form

\[ e(\vec{r}, t) = E(\vec{\rho}, z) e^{i(kz - \omega t)} + c.c, \]  

(3.1)
where $E(\vec{\rho}, z)$ is a slowly-varying field envelope, $\vec{\rho}$ is a transverse vector with coordinates $(x, y)$, and the intensity is given by

$$I(\vec{\rho}, z) = 2\epsilon_0 nc|E(\vec{\rho}, z)|^2.$$  \hfill (3.2)

In the presence of a strong pump $E_p(\vec{\rho}, z)$ at frequency $\omega_p$, signal electric field $E_s(\vec{\rho}, z)$ at frequency $\omega_s$ is coupled to the idler electric field at frequency $E_i(\vec{\rho}, z)$ at frequency $\omega_i = \omega_p - \omega_s$ through the following equation:

$$\frac{\partial E_s}{\partial z} = \frac{i}{2k_s} \nabla^2 E_s + \frac{i2\omega_s d_{\text{eff}}}{n_s c} E_p E_i^* e^{i\Delta k z},$$  \hfill (3.3)

where $\Delta k = k_p - k_s - k_i$ is the wavevector mismatch, $d_{\text{eff}}$ is the effective nonlinear coefficient accounting for the field polarizations and crystal orientation, and the equation for the idler beam is obtained by interchanging subscripts $s$ and $i$ in Equation (3.3). Equation (3.3) describes the traveling-wave OPA in paraxial approximation with a pump of arbitrary spatial profile. Let us introduce the spatial-frequency ($\vec{q}$) domain through the Fourier transform

$$\tilde{E}(\vec{q}, z) = \int E(\vec{\rho}, z)e^{-i\vec{q}\vec{\rho}} d\vec{\rho}$$  \hfill (3.4)

and the inverse Fourier transform

$$E(\vec{\rho}, z) = \int \tilde{E}(\vec{q}, z)e^{-i\vec{q}\vec{\rho}} \frac{d\vec{q}}{(2\pi)^2}.$$  \hfill (3.5)
In the absence of the pump \((E_p = 0)\), Equation (3.3) is reduced to the paraxial Helmholtz equation, whose solution in the Fourier domain is given by

\[
\tilde{E}(\vec{q}, z) = \tilde{E}(\vec{q}, 0) \exp \left(-\frac{i q^2 z}{2k}\right),
\]

(3.6)

which translates into the following spatial-domain solutions (Fresnel integrals):

\[
E(\vec{\rho}, z) = \frac{k}{2\pi iz} \int E(\vec{\rho}', 0) \exp \left[\frac{ik|\vec{\rho} - \vec{\rho}'|^2}{2z}\right] d\vec{\rho}', \quad \text{(2D)}
\]

(3.7)

\[
E(x, z) = \sqrt{\frac{k}{2\pi z}} e^{-i\pi/4} \int E(x', 0) \exp \left[\frac{ik(x - x')^2}{2z}\right] dx', \quad \text{(1D)}
\]

(3.8)

where the kernels of integrals in Equations (3.7) and (3.8) are, respectively, the 2D and 1D Greens functions

\[
G(\vec{\rho}, \vec{\rho}', z) = \frac{k}{2\pi iz} \exp \left[\frac{ik|\vec{\rho} - \vec{\rho}'|^2}{2z}\right], \quad \text{(2D)}
\]

(3.9)

\[
G(x, x', z) = \sqrt{\frac{k}{2\pi z}} e^{-i\pi/4} \exp \left[\frac{ik(x - x')^2}{2z}\right], \quad \text{(1D)}
\]

(3.10)

i.e. the solutions of the paraxial Helmholtz equation satisfying the initial conditions

\[
G(\vec{\rho}, \vec{\rho}', 0) = \delta(\vec{\rho} - \vec{\rho}'), \quad \text{(2D)}
\]

(3.11)

\[
G(x, x', 0) = \delta(x - x'). \quad \text{(1D)}
\]

(3.12)
3.4 Degenerate OPA

Assuming the signal and idler beam to have the same frequency and polarization, we can drop the $s$ and $i$ subscripts. We can express the signal field in terms of two real-valued quadratures,

$$ E(\vec{\rho}, z) = X(\vec{\rho}, z) + iY(\vec{\rho}, z) $$  \hspace{1cm} (3.13) 

so that the solution is given by

$$ E(\vec{\rho}, z) = \int \left[ G_x(\vec{\rho}, \vec{\rho}', z)X(\vec{\rho}', 0) + iG_y(\vec{\rho}, \vec{\rho}', z)Y(\vec{\rho}', 0) \right] d\vec{\rho}', $$  \hspace{1cm} (3.14) 

where $G_x(\vec{\rho}, \vec{\rho}', z)$ and $iG_y(\vec{\rho}, \vec{\rho}', z)$ are the Greens functions of Equation (3.3), i.e. its solutions with initial conditions

$$ G_x(\vec{\rho}, \vec{\rho}', 0) = \delta(\vec{\rho} - \vec{\rho}'), $$  \hspace{1cm} (3.15) 

$$ iG_y(\vec{\rho}, \vec{\rho}', 0) = i\delta(\vec{\rho} - \vec{\rho}'). $$

The solution (3.14) can be re-written in vector form as

$$ \mathbf{E}(\vec{\rho}, z) = \int \mathbf{G}(\vec{\rho}, \vec{\rho}', z)\mathbf{E}(\vec{\rho}', 0) d\vec{\rho}', $$  \hspace{1cm} (3.16) 

where

$$ \mathbf{E}(\vec{\rho}, z) = \begin{bmatrix} X(\vec{\rho}, z) \\ Y(\vec{\rho}, z) \end{bmatrix}, $$  \hspace{1cm} (3.17)
and the elements of real matrix $G$ in Equation (3.18) are related to $G_x$ and $G_y$ of Equation (3.14) as follows:

$$G_x(\vec{\rho}, \vec{\rho}', z) = C_x(\vec{\rho}, \vec{\rho}', z) + iS_x(\vec{\rho}, \vec{\rho}', z), \quad (3.19)$$

$$G_y(\vec{\rho}, \vec{\rho}', z) = S_y(\vec{\rho}, \vec{\rho}', z) - iC_y(\vec{\rho}, \vec{\rho}', z).$$

Note that Equation (3.16) can also be written in the spatial-frequency domain as

$$\tilde{\mathbf{E}}(\vec{q}, z) = \int \mathbf{\tilde{G}}(\vec{q}, -\vec{q}', z) \mathbf{\tilde{E}}(\vec{q}', 0) \frac{dq'}{(2\pi)^2}, \quad (3.20)$$

where

$$\mathbf{\tilde{E}}(\vec{q}, z) = \begin{bmatrix} \tilde{X}(\vec{q}, z) \\ \tilde{Y}(\vec{q}, z) \end{bmatrix} = \begin{bmatrix} \frac{\tilde{E}(\vec{q}, z) + \tilde{E}^*(\vec{-q}, z)}{2} \\ \frac{\tilde{E}(\vec{q}, z) - \tilde{E}^*(\vec{-q}, z)}{2i} \end{bmatrix}$$

(3.21)

is the Fourier transform of electric field in equation (3.17), and $\mathbf{\tilde{G}}(\vec{q}, \vec{q}', z)$ is the Fourier transform of the Greens function (3.18) with respect to both $\vec{\rho}$ and $\vec{\rho}'$.

3.5 Quantum description of the Degenerate OPA

Since Equation (3.3) is linear, it (as well as all the other formulae above) also holds in the quantum case by assuming the signal electric field to be an operator. It is
convenient to normalize this operator so as to produce the following commutation relations:

\[
\begin{align*}
\left[ E(\vec{\rho}, z), E^\dagger(\vec{\rho}', z) \right] &= \delta(\vec{\rho} - \vec{\rho}') \\
\left[ E(\vec{\rho}, z), E(\vec{\rho}', z) \right] &= 0.
\end{align*}
\] (3.22)

Preservation of the commutators (3.22) during the field evolution in the OPA requires that

\[
\begin{align*}
\int \left[ C_x(\vec{\rho}, \vec{\rho}', \vec{\rho}'', z)C_y(\vec{\rho}, \vec{\rho}', \vec{\rho}'', z) - C_y(\vec{\rho}, \vec{\rho}'', \vec{\rho}', z)C_x(\vec{\rho}, \vec{\rho}'', z) \right] d\vec{\rho}'' &= 0, \\
\int \left[ S_x(\vec{\rho}, \vec{\rho}'', z)S_y(\vec{\rho}', \vec{\rho}'', z) - S_y(\vec{\rho}, \vec{\rho}'', z)S_x(\vec{\rho}', \vec{\rho}'', z) \right] d\vec{\rho}'' &= 0, \\
\int \left[ C_x(\vec{\rho}, \vec{\rho}', \vec{\rho}'', z)S_y(\vec{\rho}, \vec{\rho}', \vec{\rho}'', z) - C_y(\vec{\rho}, \vec{\rho}'', \vec{\rho}', z)S_x(\vec{\rho}, \vec{\rho}'', z) \right] d\vec{\rho}'' &= \delta(\vec{\rho} - \vec{\rho}'), \\
\int \left[ S_y(\vec{\rho}, \vec{\rho}'', z)C_x(\vec{\rho}, \vec{\rho}', \vec{\rho}'', z) - S_x(\vec{\rho}, \vec{\rho}'', z)C_y(\vec{\rho}, \vec{\rho}', \vec{\rho}'', z) \right] d\vec{\rho}'' &= \delta(\vec{\rho} - \vec{\rho}').
\end{align*}
\] (3.23)

which can be expressed in the matrix form as

\[
\int \mathbf{G}(\vec{\rho}, \vec{\rho}'', z)\mathbf{J}^T \mathbf{G}^T(\vec{\rho}', \vec{\rho}'', z)\mathbf{J} d\vec{\rho}'' = \begin{bmatrix} \delta(\vec{\rho} - \vec{\rho}') & 0 \\ 0 & \delta(\vec{\rho} - \vec{\rho}') \end{bmatrix}, \tag{3.24}
\]

where

\[
\int \mathbf{J}^T \mathbf{G}^T(\vec{\rho}, \vec{\rho}', z)\mathbf{J} d\vec{\rho}'' = \begin{bmatrix} S_y(\vec{\rho}, \vec{\rho}', z) & -C_y(\vec{\rho}, \vec{\rho}', z) \\ -S_x(\vec{\rho}, \vec{\rho}', z) & C_x(\vec{\rho}, \vec{\rho}', z) \end{bmatrix}. \tag{3.25}
\]
Equation (3.24) means that the transformation in Equation (3.16) is symplectic, i.e. it preserves an antisymmetric matrix

\[
J = \begin{bmatrix}
  0 & 1 \\
-1 & 0
\end{bmatrix}.
\] (3.26)

Assuming the state at the input of the OPA to be vacuum (coherent state is treated the same way by separating the classical mean field from the vacuum fluctuations), we can completely describe the quantum properties of the output light by a correlation matrix

\[
R(\vec{\rho}, \vec{\rho}', z) = 4 \times \begin{bmatrix}
  \langle X(\vec{\rho}, z)X(\vec{\rho}', z) \rangle & \langle X(\vec{\rho}, z)Y(\vec{\rho}', z) \rangle \\
  \langle Y(\vec{\rho}, z)X(\vec{\rho}', z) \rangle & \langle Y(\vec{\rho}, z)Y(\vec{\rho}', z) \rangle
\end{bmatrix},
\] (3.27)

whose value at the OPA input is

\[
R(\vec{\rho}, \vec{\rho}', 0) = \delta(\vec{\rho} - \vec{\rho}') \times \begin{bmatrix}
  1 & i \\
-i & 1
\end{bmatrix}
\] (3.28)
The elements of the correlation matrix in Equation (3.27), to be referred to as the correlators, are given by

\[\langle X(\vec{\rho}, z)X(\vec{\rho}', z) \rangle = \frac{1}{4} \int \left[ C_x(\vec{\rho}, \vec{\rho}', z)C_x(\vec{\rho}', \vec{\rho}', z) + C_y(\vec{\rho}, \vec{\rho}', z)C_y(\vec{\rho}', \vec{\rho}', z) \right] d\vec{\rho}', \]
\[\langle Y(\vec{\rho}, z)Y(\vec{\rho}', z) \rangle = \frac{1}{4} \int \left[ S_x(\vec{\rho}, \vec{\rho}', z)S_x(\vec{\rho}', \vec{\rho}', z) + S_y(\vec{\rho}, \vec{\rho}', z)S_y(\vec{\rho}', \vec{\rho}', z) \right] d\vec{\rho}', \]
\[\langle X(\vec{\rho}, z)Y(\vec{\rho}', z) \rangle = \frac{i}{4} \delta(\vec{\rho} - \vec{\rho}') + \langle X(\vec{\rho}, z)Y(\vec{\rho}', z) \rangle_N, \]
\[\langle Y(\vec{\rho}, z)X(\vec{\rho}', z) \rangle = -\frac{i}{4} \delta(\vec{\rho} - \vec{\rho}') + \langle Y(\vec{\rho}, z)X(\vec{\rho}', z) \rangle_N, \]
\[\langle X(\vec{\rho}, z)Y(\vec{\rho}', z) \rangle_N = \frac{1}{4} \int \left[ C_x(\vec{\rho}, \vec{\rho}', z)S_x(\vec{\rho}', \vec{\rho}', z) + C_y(\vec{\rho}, \vec{\rho}', z)S_y(\vec{\rho}', \vec{\rho}', z) \right] d\vec{\rho}', \]
\[\langle Y(\vec{\rho}, z)X(\vec{\rho}', z) \rangle_N = \frac{1}{4} \int \left[ S_x(\vec{\rho}, \vec{\rho}', z)C_x(\vec{\rho}', \vec{\rho}', z) + S_y(\vec{\rho}, \vec{\rho}', z)C_y(\vec{\rho}', \vec{\rho}', z) \right] d\vec{\rho}', \]

where the subscript \( N \) denotes a normally-ordered correlator. Eliminating the delta-function from the off diagonal elements of Equation (3.27) yields a related correlation matrix

\[\mathbf{R}'(\vec{\rho}, \vec{\rho}', z) = 4 \times \begin{bmatrix} \langle X(\vec{\rho}, z)X(\vec{\rho}', z) \rangle & \langle X(\vec{\rho}, z)Y(\vec{\rho}', z) \rangle_N \\ \langle Y(\vec{\rho}, z)X(\vec{\rho}', z) \rangle_N & \langle Y(\vec{\rho}, z)Y(\vec{\rho}', z) \rangle \end{bmatrix} \] (3.30)

with initial conditions

\[\mathbf{R}'(\vec{\rho}, \vec{\rho}', 0) = \delta(\vec{\rho} - \vec{\rho}') \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \] (3.31)

One can also see that \( \mathbf{R}'(\vec{\rho}, \vec{\rho}', z) = \mathbf{R}'(\vec{\rho}, \vec{\rho}', 0) + \mathbf{R}'_N(\vec{\rho}, \vec{\rho}', z) \), where \( \mathbf{R}'_N(\vec{\rho}, \vec{\rho}', z) \) has zero initial condition and is given by Equation (3.30) with the diagonal-element
correlators replaced by the corresponding normally-ordered correlators. It is easy to show that

\[ R'(\vec{\rho}, \vec{\rho}', z) = \int G(\vec{\rho}, \vec{\rho}'', z)G^T(\vec{\rho}', \vec{\rho}''', z)d\vec{\rho}'' \] (3.32)

In the spatial-frequency domain, this correlation matrix is given by

\[ R'(\vec{q}, \vec{q}', z) = \int R'(\vec{\rho}, \vec{\rho}', z)e^{-i\vec{q}\cdot\vec{\rho}}e^{-i\vec{q}'\cdot\vec{\rho}'}d\vec{\rho}d\vec{\rho}' \]

\[ = \int \tilde{G}(\vec{q}, \vec{q}'', z)\tilde{G}^T(\vec{q}', -\vec{q}'', z)\frac{d\vec{q}''}{(2\pi)^2}. \] (3.33)

The Green’s functions in Equation (3.14) can, in general, be found numerically (analytical solutions are known for the case of a plane-wave pump and the case of a very short crystal with inhomogeneous pump), which enables the evaluation of the correlators in Equations (3.27), (3.29), (3.30), (3.32), (3.33).

3.6 Modes of Maximum Squeezing and Anti-Squeezing

Let us project the amplified light onto a mode

\[ E_{\text{LO}} = \begin{bmatrix} X_{\text{LO}}(\vec{\rho}) \\ Y_{\text{LO}}(\vec{\rho}) \end{bmatrix}, \] (3.34)

called the local oscillator (LO) mode. The detected squeezing factor (measured noise normalized by the standard quantum limit) is given by

\[ \lambda(z) = \frac{\int \int E_{\text{LO}}^T(\vec{\rho})R(\vec{\rho}, \vec{\rho}', z)E_{\text{LO}}(\vec{\rho}')d\vec{\rho}d\vec{\rho}'}{\int E_{\text{LO}}(\vec{\rho})E_{\text{LO}}(\vec{\rho})d\vec{\rho}} \] (3.35)
where the kernel $R(\vec{\rho}, \vec{\rho}', z)$ obtained from Equation (3.27) is Hermitian, i.e.

$$R^\dagger(\vec{\rho}, \vec{\rho}', z) = R(\vec{\rho}, \vec{\rho}', z)$$ (3.36)

Equation (3.35) would still hold if we replace the kernel $R(\vec{\rho}, \vec{\rho}', z)$ by $R'(\vec{\rho}, \vec{\rho}', z)$ which is also Hermitian, but is more convenient for numerical evaluations. The projection (3.35) onto the LO mode can also be done in the spatial-frequency domain:

$$\begin{align*}
\lambda(z) &= \frac{\int \int \tilde{E}^*_{LO}(\vec{q}) \tilde{R}^{\prime*}(\vec{q}, \vec{q}', z) \tilde{E}_{LO}(\vec{q}') \frac{d\vec{q}d\vec{q}'}{(2\pi)^2}}{\int \tilde{E}^*_{LO}(\vec{q}) \tilde{E}_{LO}(\vec{q}) \frac{d\vec{q}}{(2\pi)^2}} \\
&= \frac{\int \int \tilde{E}^*_{LO}(\vec{q}) \tilde{R}^{\prime*}(\vec{q}, \vec{q}', z) \tilde{E}_{LO}(\vec{q}') \frac{d\vec{q}d\vec{q}'}{(2\pi)^2}}{\int \tilde{E}^*_{LO}(\vec{q}) \tilde{E}_{LO}(\vec{q}) \frac{d\vec{q}}{(2\pi)^2}}
\end{align*}$$ (3.37)

where

$$\tilde{E}^*_{LO}(\vec{q}) = \tilde{E}_{LO}(-\vec{q})$$ (3.38)

and

$$\tilde{R}^{\prime*}(\vec{q}, \vec{q}', z) = \tilde{R}^{\prime*}(-\vec{q}, -\vec{q}', z).$$ (3.39)

Once we obtain the correlation matrix $R(\vec{\rho}, \vec{\rho}', z)$ or $R'(\vec{\rho}, \vec{\rho}', z)$ from the Greens functions, we can find the extrema of the functional (3.35) (known as the Rayleigh quotient) as eigensolutions of the Fredholm integral equation

$$\int R'(\vec{\rho}, \vec{\rho}', z) E_{LO}^\lambda(\vec{\rho}') d\vec{\rho}' = \lambda(z) E_{LO}^\lambda(\vec{\rho}),$$ (3.40)
which defines a complete set of orthogonal uncorrelated modes \( \mathbf{E}_{\text{LO}}^\lambda(\vec{\rho}) \), classified by their squeezing factors \( \lambda \). This procedure is a quantum analog of the standard Karhunen-Loève expansion. If

\[
\mathbf{E}_{\text{LO}}^\lambda(\vec{\rho}) = \begin{bmatrix}
X_{\text{LO}}(\vec{\rho}) \\
Y_{\text{LO}}(\vec{\rho})
\end{bmatrix}
\]  
(3.41)

is an eigenmode of Equation (3.40) with eigenvalue \( \lambda < 1 \) (squeezed quadrature), then

\[
\mathbf{E}_{\text{LO}}^{1/\lambda}(\vec{\rho}) = \mathbf{J} \mathbf{E}_{\text{LO}}^\lambda(\vec{\rho}) = \begin{bmatrix}
Y_{\text{LO}}(\vec{\rho}) \\
- X_{\text{LO}}(\vec{\rho})
\end{bmatrix}
\]  
(3.42)

is also an eigenmode of Equation (3.40) with eigenvalue \( 1/\lambda > 1 \) (anti-squeezed quadrature). The mode \( \mathbf{E}_{\text{LO}}^{1/\lambda}(\vec{\rho}) \) is a \(-\pi/2\)-shifted version of \( \mathbf{E}_{\text{LO}}^\lambda(\vec{\rho}) \). Note that in the spatial-frequency domain, Equation (3.40) takes the form

\[
\int \mathbf{R}'(\vec{q}, -\vec{q}', z) \mathbf{E}_{\text{LO}}^\lambda(\vec{q}') \frac{d\vec{q}'}{(2\pi)^2} = \lambda(z) \mathbf{E}_{\text{LO}}^\lambda(\vec{q}').
\]  
(3.43)

Thus, by solving Equation (3.40) one can obtain the shapes of the independently-squeezed modes, and their spectrum of squeezing/anti-squeezing, for an arbitrary pump profile. This will completely answer the questions about the effective number of amplified modes and their spatial profiles. Possibilities for additional analysis include (a) iteration of the pump profile to maximize the number of well-squeezed modes (i.e. spatial bandwidth of squeezing) and (b) finding a unitary transformation that maps these modes to known modes, e.g. Hermite-Gaussian modes or plane waves (in the latter case, \( \lambda \) becomes the spatial squeezing spectrum).
It is worth noting that the procedure for diagonalizing the case of multimode squeezing into independently-squeezed modes was originally developed in a general form in [33, 34] for the case of an arbitrary quadratic Hamiltonian. The theory for optimal (matched) LO with Equations (3.35) and (3.40) was introduced in [35] and later applied to the calculation of soliton squeezing in [36, 37]. Similar decomposition techniques were recently reviewed in [38] in the context of parametric interactions in optical fibers.

3.7 Analytical Solution 1: Plane-wave Pump

In the presence of a plane-wave pump $E_p(\vec{\rho}, z) = |E_p|e^{i\theta_p} = \text{const}$, Equation (3.3) still preserves the shift-invariance of the original paraxial Helmholtz equation and is, therefore, easily solved in the Fourier domain [28]:

$$\tilde{E}_s(q, z) = \mu(q, z)\tilde{E}_s(q, 0) + \nu(q, z)\tilde{E}_i^*(-q, 0),$$  

(3.44)

where the coefficients of the input/output transformation are

$$\mu(q, z) = \left[\cosh(\gamma z) - \frac{i \Delta k_{\text{eff}}}{2\gamma} \sinh(\gamma z)\right] \times \exp\left(\frac{i \Delta k_{\text{eff}}}{2} z\right) \times \exp\left(-i \frac{q^2}{2k_s} z\right),$$

$$\nu(q, z) = \frac{\kappa_s}{\gamma} \sinh(\gamma z) \times \exp\left(\frac{i \Delta k_{\text{eff}}}{2} z\right) \times \exp\left(-i \frac{q^2}{2k_s} z\right),$$

(3.45)

$q = |\vec{q}|$, the effective wavevector mismatch is

$$\Delta k_{\text{eff}} = k_p - k_s - k_i + \frac{q^2}{2} \left(\frac{1}{k_s} + \frac{1}{k_i}\right)$$

$$= \Delta k + \frac{q^2}{2} \left(\frac{1}{k_s} + \frac{1}{k_i}\right)$$

(3.46)
and the parametric gain coefficient is

\[
\gamma = \sqrt{\kappa^2 - \Delta k_{\text{eff}}^2/4} \text{ with }
\]

\[
\kappa^2 = \kappa_s \kappa_i^* = \frac{\omega_s \omega_i d_{\text{eff}}^2 I_p}{2 \varepsilon_0 n_s n_i c^3}, \quad \text{and}
\]

\[
\kappa_s = \frac{\omega_s d_{\text{eff}}}{n_s c} |E_p| e^{i\theta_p}
\]

Note that, in the degenerate case, the product of the exponentials in Equation (3.45), containing the effective wavevector mismatch and diffraction phase terms, becomes simply \(\exp (i \Delta k z/2)\).

From this point on, we will assume wavelength and polarization-degenerate signal and idler waves, which allows us to drop the \(s\) and \(i\) subscripts. Greens functions in the plane-wave pump case are given by

\[
G_x(\vec{\rho}, \vec{\rho}', z) = \int e^{i\vec{q} \cdot (\vec{\rho} - \vec{\rho}')} [\mu(q, z) + \nu(q, z)] \frac{d\vec{q}}{(2\pi)^2}
\]

\[
= G_x(|\vec{\rho} - \vec{\rho}'|, z),
\]

\[
G_y(\vec{\rho}, \vec{\rho}', z) = \int e^{i\vec{q} \cdot (\vec{\rho} - \vec{\rho}')} [\mu(q, z) - \nu(q, z)] \frac{d\vec{q}}{(2\pi)^2}
\]

\[
= G_y(|\vec{\rho} - \vec{\rho}'|, z)
\]

where the \(\mu\) and \(\nu\) coefficients are taken from the Equation (3.45). In the spatial-frequency domain, the matrix form of the Green’s function is given by

\[
\tilde{G}(\vec{q}, \vec{q}', z) = (2\pi)^2 \delta(\vec{q} + \vec{q}') \mathbf{M}(q, z)
\]
where

\[
M(q, z) = \begin{bmatrix}
\text{Re}(\mu + \nu) & -\text{Im}(\mu - \nu) \\
\text{Im}(\mu + \nu) & \text{Re}(\mu - \nu)
\end{bmatrix}
\] (3.49)

that is

\[
\tilde{E}(\vec{q}, z) = M(q, z)\tilde{E}(\vec{q}, 0).
\] (3.50)

The correlation-matrix kernel \( R'(\vec{q}, \vec{q}', z) \) is, therefore, given by

\[
R'(\vec{q}, \vec{q}', z) = (2\pi)^2 \delta(\vec{q} + \vec{q}') M(q, z) M^T(q, z)
\]
\[
= (2\pi)^2 \delta(\vec{q} + \vec{q}')\begin{bmatrix}
|\mu + \nu|^2 & 2\text{Im}(\mu\nu) \\
2\text{Im}(\mu\nu) & |\mu - \nu|^2
\end{bmatrix}
\] (3.51)

the squeezing/anti-squeezing factor is determined by

\[
\lambda(z) = \frac{\int \tilde{E}^T_{\text{LO}}(\vec{q}) M(q, z) M^T(q, z) \tilde{E}^*_{\text{LO}}(\vec{q}) d\vec{q}}{\int \tilde{E}^T_{\text{LO}}(\vec{q}) \tilde{E}_{\text{LO}}(\vec{q}) d\vec{q}}
\] (3.52)

and the independently squeezed modes are the eigenvectors of the matrix \( MM^T \):

\[
M(q, z) M^T(q, z) \tilde{E}^\lambda_{\text{LO}}(\vec{q}) = \lambda(z) \tilde{E}^\lambda_{\text{LO}}(\vec{q}).
\] (3.53)

One can easily see that the modes \( \tilde{E}_{\text{LO}}(\vec{q}) \) corresponding to different spatial frequencies are squeezed independently. For each spatial frequency, there are two eigenvalues,

\[
\lambda_1 = (|\mu| + |\nu|)^2
\] (3.54)
\[
\lambda_2 = (|\mu| - |\nu|)^2
\] (3.55)
with corresponding eigenvectors given by

\[
\bar{E}_{\lambda_1}^\lambda (\vec{q}) = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix},
\]

(3.56)

\[
\bar{E}_{\lambda_2}^\lambda (\vec{q}) = \begin{bmatrix} \sin(\varphi) \\ -\cos(\varphi) \end{bmatrix},
\]

(3.57)

where

\[
\varphi = \frac{\arg(\mu) + \arg(\nu)}{2},
\]

(3.58)

and the second eigenvector represents the electric field of the first shifted by \(-\pi/2\).

Note that the phase (3.58) is the eigenmodes phase at the output of the PSA. It is different from the optimum input phase

\[
\theta = -\frac{\arg(\mu) - \arg(\nu)}{2}
\]

(3.59)

that ensures maximum amplification. In other words, light entering the PSA with optimum input phase (3.59) will emerge from the PSA with output phase (3.58). Similarly, input light phase shifted by \(-\pi/2\) from the phase (3.59) will emerge with \(-\pi/2\) shift from phase (3.58). The eigenvalues of Equations (3.55) and (3.55) are related as \(\lambda_1 = 1/\lambda_2\), which is a consequence of the symplectic transformation (3.16) leading to the condition \(|\mu|^2 - |\nu|^2 = 1\) for the Bogoliubov transformation (3.44).
Figure 3.1. Diagonal and off-diagonal elements of the correlation matrix $R_N'(\xi, L)$ of Equation (3.60) (where $\xi = \vec{\rho} - \vec{\rho}'$ and $\xi = |\vec{\xi}|$) without (a) and with (b) a phase plate in the Fourier plane. Parameter values for the OPA are the same as those in [12, 29]: crystal length $L = 5.21$ mm, $\Delta k = 0$, $\kappa L = 0.88$ i.e. a phase-sensitive gain $(|\mu| + |\nu|)^2 = 5.8$ for $q = 0$, signal wavelength $\lambda_s = 1064$ nm, refractive index $n_s = 1.78$, and pump phase $\theta_p = -\pi/2$.

From Equation (3.51), one can also obtain the normally-ordered correlator in the spatial ($\vec{\rho}$)-domain as

$$R'(\vec{\rho}, \vec{\rho}'; z) = \int_0^\infty M(q, z)J_0(q|\vec{\rho} - \vec{\rho}'|)q dq/(2\pi)$$

$$= R'(|\vec{\rho} - \vec{\rho}'|, 0) + R'_N(|\vec{\rho} - \vec{\rho}'|, z)$$

(3.60)

where $J_0$ is the Bessel function and the initial correlator value is given by Equation (3.31). The four matrix elements of the correlator $R'_N(|\vec{\rho} - \vec{\rho}'|, L)$ are plotted as functions of $\xi = |\vec{\rho} - \vec{\rho}'|$ in Figure 3.1(a) and as functions of $\xi = \vec{\rho} - \vec{\rho}' = (\xi_x, \xi_y)$ in Figure 3.2(a) for a nonlinear crystal with parameter values similar to those in [12, 29] and $\theta_p = -\pi/2$. The presence of the off-diagonal elements indicates that the quantum field emerging from the OPA does not have a flat phase front. Such distortion can be corrected by inserting a Fourier-plane phase plate to rotate the phases of the spatial-frequency components of the output field by $-\phi(q)$ (for small parametric gains this
Figure 3.2. The diagonal and off-diagonal elements of the correlation matrix $R'_N(\xi, L)$ of Figure 3.1 shown in the $(\xi_x, \xi_y)$-plane without (a) and with (b) a phase plate in the Fourier plane.

This transformation is equivalent to multiplication of Equation (3.50) by the matrix

$$S = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix}, \quad \text{(3.61)}$$

which transforms the correlation matrix of Equation (3.51) into

$$S\tilde{R}'(\vec{q}, \vec{q}', z)S^T = (2\pi)^2 \delta(\vec{q} + \vec{q}') \begin{bmatrix} (|\mu| + |\nu|)^2 & 0 \\ 0 & (|\mu| - |\nu|)^2 \end{bmatrix}, \quad \text{(3.62)}$$

with the corresponding spatial-domain normally ordered counterpart shown in Figures 3.1(b) and Figure 3.2(b). The singularity at $\vec{\rho} = \vec{\rho}'$ is due to rectification of the sinc.
function in the spatial-frequency domain, which occurs when taking the absolute value of $\nu$ [29].

3.8 Analytical Solution 2: Short Crystal with Inhomogeneous Pump

For sufficiently short crystals, the diffraction term in Equation (3.3) can be neglected, and the resulting equation takes the following form:

$$\frac{\partial E_s}{\partial z} = \frac{i\omega_s d_{\text{eff}}}{n_s c} E_p E_i^* e^{i\Delta k z}, \quad (3.63)$$

or

$$\frac{\partial E_s}{\partial z} = i\kappa_s E_i^* e^{i\Delta k z}, \quad (3.64)$$

where $\kappa_s = 2\omega_s d_{\text{eff}} E_p / (n_s c)$, in general, is a complex parameter that depends on the coordinate $\vec{\rho}$ (if the pump is inhomogeneous). One can then introduce the parameters $\mu$ and $\nu$ as

$$\mu(\vec{\rho},z) = \left( \cosh(\gamma z) - \frac{i\Delta k}{2\gamma} \sinh(\gamma z) \right) \times \exp \left( i \frac{\Delta k}{2} z \right),$$

$$\nu(\vec{\rho},z) = i\frac{\kappa_s}{\gamma} \sinh(\gamma z) \times \exp \left( i \frac{\Delta k}{2} z \right),$$

(3.65)

where

$$\gamma(\vec{\rho}) = \sqrt{\kappa^2(\vec{\rho}) - \Delta k^2/4},$$

$$\kappa^2 = \kappa_s \kappa_i^* = \frac{\omega_s \omega_i d_{\text{eff}}^2 I_p(\vec{\rho})}{2\varepsilon_0 n_s n_i n_p c^2},$$

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so that the solution takes the form of point-by-point (pixel-by-pixel) field amplification:

\[
E_s(\vec{\rho}, z) = \mu(\vec{\rho}, z)E_s(\vec{\rho}, 0) + \nu(\vec{\rho}, z)E_s^*(\vec{\rho}, 0).
\] (3.66)

In the degenerate case, we have

\[
G_x(\vec{\rho}, \vec{\rho}', z) = [\mu(\vec{\rho}, z) + \nu(\vec{\rho}, z)] \delta(\vec{\rho} - \vec{\rho}'), \\
G_y(\vec{\rho}, \vec{\rho}', z) = [\mu(\vec{\rho}, z) - \nu(\vec{\rho}, z)] \delta(\vec{\rho} - \vec{\rho}'),
\] (3.67)

and

\[
\mathbf{G}(\vec{\rho}, \vec{\rho}', z) = \mathbf{M}(\vec{\rho}, z) \times \delta(\vec{\rho} - \vec{\rho}'),
\] (3.68)

where

\[
\mathbf{M}(\vec{\rho}, z) = \begin{bmatrix}
\text{Re}(\mu + \nu) & -\text{Im}(\mu - \nu) \\
\text{Im}(\mu + \nu) & \text{Re}(\mu - \nu)
\end{bmatrix}.
\] (3.69)

From Equations (3.68) and (3.69) we can see that the situation is very similar to the plane-wave pump case, but, instead of the spatial-frequency domain, the same input-output relations take place in the image domain. Namely,

\[
\mathbf{E}(\vec{\rho}, z) = \mathbf{M}(\vec{\rho}, z)\mathbf{E}(\vec{\rho}, 0)
\] (3.70)
and

\[ R'(\vec{\rho}, \vec{\rho}', z) = \delta(\vec{\rho} - \vec{\rho}') \times M(\vec{\rho}, z)M^T(\vec{\rho}, z) \]

\[ = \delta(\vec{\rho} - \vec{\rho}') \begin{bmatrix} |\mu + \nu^*|^2 & 2\text{Im}(\mu\nu) \\ 2\text{Im}(\mu\nu) & |\mu - \nu^*|^2 \end{bmatrix}, \tag{3.71} \]

so that the independently-squeezed modes are the eigenvectors of the matrix \( MM^T \):

\[ M(\vec{\rho}, z)M^T(\vec{\rho}, z)E_{\lambda Lo}^\lambda(\vec{\rho}) = \lambda(z)E_{\lambda Lo}^\lambda(\vec{\rho}). \tag{3.72} \]

One can easily see that the modes \( E_{\lambda Lo}^\lambda(\vec{\rho}) \) corresponding to different pixels of the image are squeezed independently. For each pixel, there are two eigenvalues,

\[ \lambda_1 = (|\mu| + |\nu|)^2, \tag{3.73} \]

\[ \lambda_2 = (|\mu| - |\nu|)^2, \tag{3.74} \]

with corresponding eigenvectors

\[ E_{\lambda Lo}^{\lambda_1}(\vec{\rho}) = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix}, \tag{3.75} \]

\[ E_{\lambda Lo}^{\lambda_2}(\vec{\rho}) = \begin{bmatrix} \sin(\varphi) \\ -\cos(\varphi) \end{bmatrix}, \tag{3.76} \]

where

\[ \varphi = \frac{\text{arg}(\mu) + \text{arg}(\nu)}{2}, \tag{3.77} \]
and the second eigenvector represents the electric field of the first shifted by $-\pi/2$. Similar to our discussion above for the plane-wave pump case, we note that the phase (3.77) is the eigenmodes phase at the output of the PSA. It is different from the optimum input phase

$$\theta = -\frac{\arg(\mu) - \arg(\nu)}{2}$$

(3.78)

that ensures maximum amplification. In other words, light entering the PSA with optimum input phase (3.78) will emerge from the PSA with output phase (3.77). Similarly, input light with phase shifted by $-\pi/2$ from the phase (3.78) will emerge with $-\pi/2$ shift from phase (3.77). The eigenvalues of Equations (3.73) and (3.74) are related as $\lambda_1 = 1/\lambda_2$, which is a consequence of the symplectic transformation (3.16) leading to the condition $|\mu|^2 - |\nu|^2 = 1$ for the Bogoliubov transformation (3.66).

3.9 Summary

We have presented a methodology for the complete quantum description of a traveling-wave phase-sensitive optical parametric amplifier (OPA) in terms of its normally-ordered quadrature correlators that are obtainable from the numerically or analytically solvable Greens function of the classical OPA propagation equation. This approach applies to the case of a pump beam with arbitrary spatial profile and enables determination of the independently-squeezed orthogonal eigenmodes of the OPA. This methodology serves as a basis for our study of the modes of the OPA with an elliptical Gaussian pump beam, which is presented in Chapter 4 and 5.
CHAPTER 4
COUPLED-MODE THEORY OF PSA

4.1 Introduction

We present for the first-time a semi-analytical theory of PSA to calculate its independently squeezed modes, for the 2D case without restrictions on symmetry of the signal or pump shape. We achieve this in two steps, by developing the HG-mode theory of PSA to calculate its Green’s function of the PSA and then by calculating the quantum correlators in terms of the numerical Green’s function based on results in Chapter 3.

4.2 Coupled-Mode Theory for PSA Spatial Modes

We present the coupled-mode-theory of PSA in this section, and calculate its independently squeezed modes. We also calculate squeezing, gain of the PSA in this method using the knowledge of Green’s functions.

Our model has a distinct advantage of studying PSA evolution for various higher-order pump profiles (possibly non-Gaussian), and extends the previous 1D models to 2D, by eliminating the cylindrical symmetry requirement in the transverse plane in the earlier works of La Porta, Slusher [19], and Köprülü and O. Aytür [20]. In this sense, we extend logically the Laguerre-Gaussian coupled-mode theory of the PSA under a Gaussian beam pump [20], to the 2D case without restrictions on symmetry of the signal or pump shape, to compute squeezing and gain under homodyne detection.
4.2.1 Calculation Overview

Derivation of the coupled-mode equations for the Type-I (Type-II OPA can be treated as two independent Type-I OPA [3] by appropriate basis transformations of the field polarizations) degenerate Optical Parametric Amplifier (OPA), is presented here. We solve the degenerate OPA problem by writing transverse mode representations of the signal, and pump beams in terms of eigenmodes of paraxial Helmholtz equation, a popular method used earlier in [20], and integrate the resulting coupled-mode equations. Extending previous cavity-based OPO work [31], we replaced the Laguerre-Gaussian (LG) mode with the Hermite-Gaussian (HG) mode, with the signal-idler degeneracy. The similar Laguerre-Gaussian (LG) mode coupling equations are also developed.

The transverse signal, idler and pump fields of PSA can represented in the Hermite-Gaussian (HG), or Laguerre-Gaussian (LG) bases, both of which form the transverse modes in freespace. Next we substitute these field expansions into the nonlinear-Helmholtz equation for OPA in the undepleted pump approximation, and reduce the PDE to a system of ODE's by using the fact that HG-modes (or LG-modes) are eigenfunctions of the paraxial-Helmholtz equation. Following this simplification we project the equations onto the orthonormal HG-modes which yield the system of coupled ODEs. Same process is carried out for the LG modes as well.

This procedure reduces the OPA coupled PDE (Partial Differential Equation) to a system of coupled ODE (Ordinary Differential Equation) Initial Value Problems (IVP). It also converts the problem of studying the field evolution of the signal and pump modes into the problem of studying their mode coefficient evolution across the nonlinear crystal. Also this procedure will allow us to calculate the Green’s function of the OPA, which is equivalent to propagating a spatially narrow Gaussian beam in
the transverse plane approximating the delta function \( \delta(\vec{p} - \vec{p}') \) numerically, for every point on the input plane of OPA.

4.3 Free-space Modes

Free-space modes unlike their counterparts in the waveguide, or resonators do not have a dispersion relation for higher-order mode numbers with the frequency of light beam; i.e., their wavenumbers are independent of the mode numbers \((m, n)\) or \((p, l)\). Free-space modes generally have a property of preserving their shapes, but with a scaling term dependent on the axial propagation distance which accounts for its diffraction properties.

We use the notation \( \exp(+i(kz - \omega t)) \) for a \(+z\) propagating wave to obtain the paraxial Helmholtz equation in the time-harmonic form, to represent the free-space modes (LG or HG). The higher-order Gaussian modes LG, and HG are defined with a transverse spot-size (at any point on the axis of propagation \(+z\)) at the \(1/e\) intensity radius point (some texts use spot size at \(1/e^2\) intensity radius).

The higher-order modes are also normalized to have unity energy across any transverse plane, in addition to the orthogonality property of the modes. Next, we present the notations of the free-space higher-order orthogonal modes the LG, HG, and astigmatic-HG modes as we use them for our calculations.

4.3.1 Laguerre-Gaussian Modes

The Laguerre-Gaussian mode (LG) are eigenfunctions of the paraxial Helmholtz equation. The normalized LG-mode with radial index \(p \geq 0\), and azimuthal index \(l \in \mathbb{Z}\),

\[
\int_0^\infty \int_0^{2\pi} |\varphi_{p,l}|^2 r dr d\theta = 1
\] (4.1)
with the waist size defined at the 1/e intensity $a(z)$, is given by

$$\varphi_{p,l}(r, \theta, z) = \sqrt{\frac{p! \pi p + |l|)!}{\pi(p + |l|)!}} \exp\left(-i(2p + |l| + 1)\psi(z)\right) \left(\frac{r}{a(z)}\right)^{|l|} I_{|l|}^{|l|} \left(\frac{r^2}{a^2(z)}\right) \times \exp\left(+i \frac{kr^2}{2R(z)} + il\theta - \frac{r^2}{2a^2(z)}\right).$$  \hspace{1cm} (4.2)

4.3.2 Hermite-Gaussian Modes

Hermite-Gaussian modes acquire only a phase shift $\exp(+ikz)$ and the beam-shape is scaled by the spot-size $a(z)$ on free-space propagation, due to their eigenfunction property [39]. It is well known that, Hermite-Gaussian function is an eigenfunction of Laplacian operator, (in any number of dimensions), also an eigenfunction of the Fourier transform operator, and satisfies a 3-term recurrence relations among other properties. We modify the 1D Hermite-Gaussian mode of order $n$ from [39], for the beam waist defined as the radius of the 1/e intensity spot while keeping the same Rayleigh length for 1D HG mode by,

$$\Phi_n(x, z) = \sqrt{\frac{1}{\pi}} \left[\frac{\exp(-i((2n + 1)\psi(z))/2)}{\sqrt{2^n n!a(z)}}\right] H_n\left(\frac{x}{a(z)}\right) \exp\left(+i \frac{kr^2}{2R(z)} - \frac{x^2}{2a^2(z)}\right).$$  \hspace{1cm} (4.3)

with the normalization

$$\int_{-\infty}^{+\infty} |\Phi_m(x, z)|^2 dx = 1,$$  \hspace{1cm} (4.4)
Figure 4.1. The map shows an LG-mode $\varphi_{p,l}$ with fundamental beam at $200 \times 200 \mu m^2$ (a) intensity and (b) phase distribution evaluated at $z = 0$ for several values of $p$ radial index, and $l$ azimuthal index, both going from $0 - 2$. The radial index $p$ lies horizontally, while azimuthal index $l$ lies vertically downward.
where the beam parameters $\psi(z)$, $a(z)$, and $R(z)$, are the parametric axial functions for the axial Gouy phase shift, modified beam spot size, and wavefront curvature given by,

$$\psi(z) = \arctan \left( \frac{z}{z_R} \right), \quad R(z) = z \left[ 1 + \left( \frac{z_R}{z} \right)^2 \right]$$

$$z_R = \frac{2\pi a_0^2}{\lambda}, \quad a(z) = a_0 \sqrt{1 + \left( \frac{z}{z_R} \right)^2}.$$ 

(4.5)

(4.6)

The product of the beam spot size and the mode order $a(z) \times \sqrt{n}$ is an easy way to find the spatial extent of a 1D HG-mode $\Phi_n(x)$.

The 2D Hermite-Gaussian function is defined as

$$\Phi_{m,n}(r, z) = \Phi_m(x, z)\Phi_n(y, z), \text{ where } r = \hat{x}x + \hat{y}y,$$

(4.7)

$$\Phi_{m,n}(r, z) = \sqrt{\frac{1}{\pi}} \exp \left( -i \left[ m + n + 1 \right] \psi(z) \right) \frac{1}{\sqrt{2^m 2^n m! n! a^2(z)}} \left( \frac{x}{a(z)} \right)^m \left( \frac{y}{a(z)} \right)^n H_m \left( \frac{x}{a(z)} \right) H_n \left( \frac{y}{a(z)} \right) \times \exp \left( -\frac{k (x^2 + y^2)}{2 R(z)} - \frac{(x^2 + y^2)}{2 a^2(z)} \right),$$

(4.8)

and normalized as

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\Phi_{m,n}(x, y, z)|^2 dx dy = 1,$$

(4.9)

with the property of being solutions of the paraxial Helmholtz equation

$$\nabla^2 \Phi_{m,n} + i k \frac{\partial \Phi_{m,n}}{\partial z} = 0.$$

(4.10)
Figure 4.2. The map shows the absolute value squared of the HG-mode $|\Phi_{m,n}|^2$ evaluated at $z = 0$, for several values of $m$ horizontal mode index, and $n$ vertical mode index, both going from $0 \rightarrow 3$. The fundamental HG-mode waist was set at $200 \times 200\mu m^2$. 
4.3.3 Elliptical HG Mode

We can define an elliptical HG-mode where the X, Y transverse modes have a different beam spot-size $a_{0x}, a_{0y}$ resulting in a different axial Rayleigh length for each mode $z_{Rx}$ and $z_{Ry}$ respectively. This separation also leads to the different Gouy phase shifts for the X, Y transverse modes $\psi_x(z), \psi_y(z)$ respectively. The beam curvature and spot-size at any axial distance $z$ is now different for each transverse mode and is represented by attaching subscripts to the $R(z)$ and $a(z)$ corresponding to the respective modes. Orthonormality properties follow simply just like in previous discussion due to the separability of the astigmatic mode into its X, and Y components.

The elliptical 2D Hermite-Gaussian function is defined as,

$$
\Phi_{m,n}(\vec{r}, z) = \sqrt{\frac{1}{\pi}} \exp\left(-i \left([m + 1/2] \psi_x(z) + [n + 1/2] \psi_y(z)\right)\right) \frac{1}{\sqrt{2^m m! n! a_x(z) a_y(z)}} \times H_m\left(\frac{x}{a_x(z)}\right) H_n\left(\frac{y}{a_y(z)}\right) \times \exp\left(\frac{-k x^2}{2R_x(z)} + \frac{-k y^2}{2R_y(z)} - \frac{x^2}{2a_x^2(z)} - \frac{y^2}{2a_y^2(z)}\right). \quad (4.11)
$$

4.4 Integral Transformations in LG and HG Modes

Any finite image $f(x, y)$ in the X-Y plane can be uniquely represented in the free-space LG modes $\varphi_{p, l}(r, \theta, z)$ or HG modes $\Phi_{m,n}(x, y, z)$, which satisfy completeness and orthogonality in the $L^2$ space. It is the LG or HG mode analogue of the frequently used angular-spectrum representation, in Fourier-Optics. In this section we implicitly assume the dependence of both basis modes on $k, a_0$ and $Z_R$, as it simplifies the
Figure 4.3. The map shows the absolute value squared of the elliptical HG-mode $|\Phi_{m,n}|^2$ evaluated at $z = 0$, for several values of $m$ horizontal mode index, and $n$ vertical mode index, both going from $0 - 3$. Fundamental elliptical HG-mode waist was set at $200 \times 100 \mu m^2$. 
representations of the bi-orthogonal integral transformations. More precisely, one can represent \( f(x, y) \) using the complex valued mode coefficients \( a_{m,n} \) or \( b_{p,l} \) as,

\[
f(x, y) = \sum_{m,n} a_{m,n} \Phi_{m,n}(x, y, z) = \sum_{p,l} b_{p,l} \varphi_{p,l}(r, \theta, z)
\]

(4.12)

where,

\[
a_{m,n} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) \Phi_{m,n}^*(x, y, z) \, dx \, dy,
\]

and

\[
b_{p,l} = \int_{0}^{2\pi} \int_{0}^{+\infty} f(r \cos \theta, r \sin \theta) \varphi_{p,l}^*(r, \theta, z) \, r \, dr \, d\theta.
\]

(4.13)

(4.14)

The image \( f(x, y) \) can be represented in a particular basis \( \Phi_{m,n} \) or \( \varphi_{p,l} \) in a finite number of modes by proper choice of fundamental Gaussian spot-size \( a_0 \). The appropriate number of modes in a particular representation of the image can be defined according to the minimum mean square error

\[
\sum_{x} \sum_{y} \left| f(x, y) - \sum_{m,n} a_{m,n} \Phi_{m,n}(x, y, z) \right|^2
\]

(4.15)

and

\[
\sum_{x} \sum_{y} \left| f(x, y) - \sum_{p,l} b_{p,l} \varphi_{p,l}(r, \theta, z) \right|^2.
\]

(4.16)
4.5 Derivation of PSA Coupled-Mode Theory

Using the complete orthogonal basis of HG or LG modes functions $\phi_{q,m}(\vec{\rho}, z, k, a_0)$ with the property

$$\int_{\rho} \phi_{q,m}(\vec{\rho}, z, k, a_0) \phi_{q,m}^*(\vec{\rho}, z, k, a_0) d\vec{\rho} = \delta_{q,q'} \delta_{m,m'}$$  \hspace{1cm} (4.17)

enables us to write a decomposition of the transverse electric field profiles for the pump, signal and idler as follows

$$E_p(\vec{r}, z) = \sum_{q,m} \exp(i\theta_p) \sqrt{\frac{P_0}{2\varepsilon_0 n_p c}} C_{q,m}(z) \phi_{q,m}(\vec{r}, z, k_p, a_0), \text{ pump field}$$  \hspace{1cm} (4.18)

$$E_s(\vec{r}, z) = \sum_{p,l} \frac{D_{p,l}(z)}{\sqrt{2\varepsilon_0 n_s c}} \phi_{p,l}(\vec{r}, z, k_s, \sqrt{2}a_0), \text{ signal field}$$  \hspace{1cm} (4.19)

$$E_d(\vec{r}, z) = \sum_{r,n} \frac{B_{r,n}(z)}{\sqrt{2\varepsilon_0 n_s c}} \phi_{r,n}(\vec{r}, z, k_d, \sqrt{2}a_0), \text{ idler field}$$  \hspace{1cm} (4.20)

with $B$, $C$, and $D$ are the mode expansion coefficients of their respective fields. For the case of free-space propagation these coefficients are constants. Here $\phi$ can either be $\Phi_{m,n}$ HG-mode or $\varphi_{p,l}$ LG-mode. Due to nonlinearity in the OPA, mode coupling manifests itself as $z$ – dependent coefficients.

4.5.1 Coupled-Mode Theory for OPA

Our development uses the undepleted pump approximation, i.e $\partial E_p(z)/\partial z = 0$; the most important consequence of this assumption is to reduce the evolution equation from a system of 3 coupled PDE (signal, pump, and idler) to 2 coupled PDE (signal, and idler). The field equation model for the OPA following [23] is,

$$\frac{\partial E_s}{\partial z} = \frac{i}{2k_s} \nabla^2 E_s + \frac{i\omega_s \times 2d_{\text{eff}}}{n_s c} E_p E_d^* e^{i\Delta k z}.$$  \hspace{1cm} (4.21)
Here $\Delta k = k_p - k_s - k_d$, the phase mismatch for the OPA. We have similar equation for the idler evolution under the substitution $s \to d$, and $d \to s$. Since we expect to only study the PSA we no longer follow the idler evolution, as our interests lie in DOPA operation (signal and idler are the fields at the same frequency).

Rewriting the above equation as,

$$\nabla^2 \rho E_s + i2k_s \frac{\partial E_s}{\partial z} = -\frac{2k_s\omega_s \times 2d_{\text{eff}}}{n_s c} E_p E_d^* e^{i\Delta k z}$$

and on replacing the mode expressions for the signal, pump and idler into above we obtain,

$$-\sum_{p,l} D_{p,l}(z) \left( \nabla^2 \Phi_{p,l}(\vec{r}, z) + i2k_s \frac{\partial \Phi_{p,l}(\vec{r}, z)}{\partial z} \right) + i2k_s \Phi_{p,l}(\vec{r}, z) \frac{dD_{p,l}(z)}{dz} =$$

$$-\frac{2k_s\omega_s d_{\text{eff}}}{n_s c} \exp(i\theta_p) \sqrt{\frac{P_0}{2\varepsilon_0 n_p c}} \sum_{q,r} \left( \Phi_{q,m}(\vec{r}, z) \Phi_{r,n}^*(\vec{r}, z) C_{q,m}(z) B_{r,n}^*(z) e^{i\Delta k z} \right)$$

where the paraxial Helmholtz equation terms simplifies to zero, operating on the HG-mode functions.

$$-\frac{2k_s\omega_s \times 2d_{\text{eff}}}{n_s c} \exp(i\theta_p) \sqrt{\frac{P_0}{2\varepsilon_0 n_p c}} \sum_{q,r} \left( \Phi_{q,m}(\vec{r}, z) \Phi_{r,n}^*(\vec{r}, z) C_{q,m}(z) B_{r,n}^*(z) e^{i\Delta k z} \right)$$

$$= -\sum_{p,l} i2k_s \Phi_{p,l}(\vec{r}, z) \frac{dD_{p,l}(z)}{dz}$$

Projecting this equation onto a specific HG-mode $\Phi_{p,l}$ and using the orthonormality of the HG-modes, we get
\[
\frac{dD_{p,l}(z)}{dz} = \omega_s \times 2d_{\text{eff}} \exp (i\theta_p) \sqrt{\frac{P_0}{2\epsilon_0 n_p n_s^2 c^3}} e^{i\Delta k z} \sum_{q,r}^{m,n} (\Lambda_{p,q,r}^{l,m,n} C_{q,m}(z) D_{r,n}^*(z)) \tag{4.25}
\]

where the coupling integral is given by

\[
\Lambda_{p,q,r}^{l,m,n} = \int \Phi_{p,l}^*(\vec{r},z) \Phi_{r,n}^*(\vec{r},z) \Phi_{q,m}(\vec{r},z) d\vec{r}. \tag{4.26}
\]

Specifically for a degenerate signal and idler in a PSA, pumped by the fundamental-mode pump with \( C_{0,0} = 1 \) this equation simplifies to

\[
\frac{dD_{p,l}(z)}{dz} = \omega_s \times 2d_{\text{eff}} \exp (i\theta_p) \sqrt{\frac{P_0}{2\epsilon_0 n_p n_s^2 c^3}} e^{i\Delta k z} \sum_{q,r}^{m,n} (\Lambda_{p,r}^{l,n}(z) D_{r,n}^*(z)), \tag{4.27}
\]

where \( \Lambda_{p,r}^{l,n}(z) = \Lambda_{p,q=0,r}^{l,m=0,n}(z) \).

This system of evolution equations shows how various mode coefficients evolve in the longitudinal direction. Specifically, these equations show the phenomenon of mode mixing, where signal mode \((p,l)\) has contributions from all the idler modes \((r,n)\), pump modes \((q,m)\) mediated by the coupling integral \(\Lambda_{p,q,r}^{l,m,n}\) which might have one or more selection rules. The physics of gain-induced diffraction with quadrature rotation and mode mixing is described in this system.

The form of the above equations remains same for both LG and HG modes, as they have a similar phase dependence on \(z\) and \(\omega\) variables when applied on the paraxial Helmholtz equation.

In the following sections we derive the closed-form expressions for the overlap integrals presented in our derivation for various combinations of pump mode, signal-pump spot size ratio \(a_s/a_p = \sqrt{2}\) (fixed) or \(a_s/a_p = f_s\) (general case). We note that in all the different situations the closed-form overlap-integrals describe the coupled-
Table 4.1. Closed form overlap integrals, for the HG and elliptical HG beam modes.

<table>
<thead>
<tr>
<th>HG Mode Overlap Integral</th>
<th>Elliptical HG-Mode Overlap Integral</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Overlap Integral Form</strong></td>
<td><strong>Overlap Integral Form</strong></td>
</tr>
<tr>
<td>( \Lambda_{p,r}^{l,n} (z) = D_{p,r}^{x} (z) D_{l,n}^{y} (z) / \sqrt{\pi a_{0}^{2}} )</td>
<td>( \Lambda_{p,r}^{l,n} (z) = D_{p,r}^{x} (z) D_{l,n}^{y} (z) / \sqrt{\pi a_{0}^{2}} )</td>
</tr>
<tr>
<td><strong>Closed Form</strong></td>
<td><strong>Closed Form</strong></td>
</tr>
<tr>
<td>( D_{p,r}^{x} (z) = \frac{\exp(+i[p+r+1/2] \psi(z)) (-1)^{p+r} (p+r-1)!!}{\sqrt{1+(z/z_{R})^{2}} \sqrt{2^{p+r+1} p!}} )</td>
<td>( D_{p,r}^{x} (z) = \frac{\exp(+i[p+r+1/2] \psi(z)) (-1)^{p+r} (p+r-1)!!}{\sqrt{1+(z/z_{R})^{2}} \sqrt{2^{p+r+1} p!}} )</td>
</tr>
<tr>
<td><strong>Closed Form z-dependence</strong></td>
<td><strong>Closed Form z-dependence</strong></td>
</tr>
<tr>
<td>( \Lambda_{p,r}^{l,n} (z) = D_{p,r}^{x} (z=0) \times D_{l,n}^{y} (z=0) \times \frac{\exp(+i[p+r+l+n+1] \psi(z))}{\sqrt{1+(z/z_{R})^{2}} \sqrt{1+(z/z_{R})^{2}}} )</td>
<td>( \Lambda_{p,r}^{l,n} (z) = D_{p,r}^{x} (z=0) \times D_{l,n}^{y} (z=0) \times \frac{\exp(+i[p+r+l+n+1] \psi(z))}{\sqrt{1+(z/z_{R})^{2}} \sqrt{1+(z/z_{R})^{2}}} )</td>
</tr>
<tr>
<td><strong>Selection Rule</strong></td>
<td><strong>Selection Rule</strong></td>
</tr>
<tr>
<td>( p + r ) is even, and ( l + n ) is even</td>
<td>( p + r ) is even, and ( l + n ) is even</td>
</tr>
</tbody>
</table>
Table 4.2. Closed form overlap integrals, for the LG beam modes.

<table>
<thead>
<tr>
<th>LG Mode Overlap Integral</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Overlap Integral Form</td>
<td>$\Lambda_{l,n}^{p,r}(z) = \Lambda_{p,r}^{l}(z)$</td>
</tr>
<tr>
<td>Closed Form</td>
<td>$\Lambda_{p,r}^{l}(z) = \frac{1}{\sqrt{\pi a(z)}} \frac{(p+r+</td>
</tr>
<tr>
<td>Closed Form z-dependence</td>
<td>$\Lambda_{p,r}^{l}(z) = \frac{\Lambda_{p,r}^{l}(z=0)}{\sqrt{1+(z/z_R)^2}} \exp (\mp i [2(r+p+</td>
</tr>
<tr>
<td>Selection Rule</td>
<td>$l = -n$</td>
</tr>
</tbody>
</table>
mode-theory of the same Eq.4.25, with appropriate selection rules and mode indices for choice of LG or HG basis.

4.6 Coupling Integral for Fixed $\sqrt{2}$-scale Signal, Pump spot size

Analytical closed form expression for coupling integral can speed up the calculations by significantly, as we eliminate the computationally intensive numerical evaluation of the orthogonal polynomials, and instead evaluating a (relatively) simple expression.

In this section we present analytical expressions for the HG mode coupling integral, and the LG mode coupling integral. The simplification of the coupling integral is made possible primarily by the choice of signal beam waist $a_s = \sqrt{2}a_p$ in terms of the pump waist $a_p$, so that the two wavefronts overlap over entire length of the crystal, with $\Delta k = 0$, perfect phase matching. The results of this section are summarized in Tables 4.1, 4.2.

In the following results (since signal and pump have fixed scale of $\sqrt{2}$), the pump spot size if same for both $X, Y$ dimensions are represented by $a$, and for elliptical cases by $a_x$ and $a_y$ respectively. For general scale results the spot sizes have appropriate subscripts $a_s$ for signal, and $a_p$ for pump spot sizes.

4.6.1 Coupling Integral for HG modes

In the case of a Gaussian pump TEM$_{00}$ we have the fundamental HG mode with $q = 0, m = 0$ and allow us to study the coupling integral at $z$, $\Lambda_{l,m}^{l,m} (z) = \Lambda_{p,q=0,r}^{l,m=0,n} (z)$. We work out the form of this overlap integral, in detail in the Appendix A.1. We write the overlap integral as a product,

$$\Lambda_{p,r}^{l,n}(z) = \frac{1}{\sqrt{\pi a_0}} D_{p,r}(z) D_{l,n}(z), \quad (4.28)$$
where the factor $D_{p,r}(z)$ in the overlap integral form has a closed form analytical expression obtained as

$$D_{p,r}(z) = \frac{\exp \left( +i \left[ p + r + 1/2 \right] \psi(z) \right) \left( -1 \right)^{\frac{p-r}{2}} (p + r - 1)!!}{\sqrt{1 + (z/z_R)^2} \sqrt{2^{p+r+1}p!r!}}. \quad (4.29)$$

We note the selection rule for $D_{p,r}(z)$, is non-zero for even $(p+r)$, and 0 for odd values; total selection rule becomes $\Lambda_{p,r}^{l,n}$, is non-zero for $\text{mod}(p + r, 2) + \text{mod}(l + n, 2) = 0$.

Since the overlap integral has axial phase factors which do not depend on the transverse coordinates, we can write it as a product of overlap integral value at the waist plane $z = 0$ (where there is no axial phase contribution) and the axial phase factor

$$\Lambda_{p,r}^{l,n}(z) = \Lambda_{p,r}^{l,n}(z = 0) \frac{\exp \left( +i \left[ p + r + l + n + 1 \right] \psi(z) \right)}{\sqrt{1 + (z/z_R)^2}}. \quad (4.30)$$

4.6.2 Coupling Integral for Elliptical HG Modes

The elliptical HG modes also factor similarly as standard HG modes, into 2 terms corresponding to each transverse mode. Again this overlap integral is worked out in Appendix A.2 as,

$$\Lambda_{p,r}^{l,n}(z) = \frac{1}{\sqrt{\pi a_{0x} a_{0y}}} D_{p,r}^{x}(z) D_{l,n}^{y}(z) \quad (4.31)$$

where $D_{p,r}^{x}(z)$ is

$$D_{p,r}^{x}(z) = \frac{\exp \left( +i \left[ p + r + 1/2 \right] \psi_x(z) \right) \left( -1 \right)^{\frac{p-r}{2}} (p + r - 1)!!}{\sqrt{1 + (z/z_{Rx})^2} \sqrt{2^{p+r+1}p!r!}}. \quad (4.32)$$
Figure 4.4. The map shows the absolute value of the 1D HG-mode overlap integral $D_{m,m'}$ evaluated at $z = 0$, as function of the signal, and idler indices $m, m'$ respectively.

We note the selection rule for $D_{p,r}^x(z)$, is non-zero for even $(p+r)$, and 0 for odd values; total selection rule becomes $\Lambda_{l,n}^{p,r}$, is non-zero for $\text{mod}(p + r, 2) + \text{mod}(l + n, 2) = 0$.

Now we can rewrite the overlap integral along any point on axial direction $z$, in terms of its value at the focal plane $z = 0$

$$\Lambda_{p,r}^{l,n}(z) = \frac{\Lambda_{p,r}^{l,n}(z = 0)}{\sqrt{1 + (z/z_{Rx})^2} \sqrt{1 + (z/z_{Ry})^2}} \times \exp (+i \left[ p + r + 1/2 \right] \psi_x(z)) \times \exp (+i \left[ l + n + 1/2 \right] \psi_y(z)). \quad (4.33)$$

We note the self-consistency between the different cases by using $a_x = a_y$, then the coupling integral for elliptical modes Eq. 4.33, becomes same as Eq. 4.30, the coupling integral for the standard HG modes.
4.6.3 Coupling Integral for LG Modes

Even though the LG modes the coupling integral has the same form as HG modes in Eq. 4.26 with LG-modes instead, it has a very different closed form expression, and selection rule. We work out the form of this overlap integral, in detail in the Appendix A.3 as, (we note the selection rule for $\Lambda_{l,n}^{p,r}$, is non-zero only for $l = -n$, leading us to write $\Lambda_{p,r}^{l,n} = \Lambda_{p,r}^{l,n}$)

$$\Lambda_{p,r}^{l,n}(z) = \frac{1}{\sqrt{\pi a(z)}} \frac{(p + r + |l|)!}{\sqrt{z}!} \exp \left( + i \left[ 2(r + p + |l|) + 1 \right] \psi(z) \right).$$  

This completes the analytic expression for the LG-mode overlap integral, and allows quick solution of the mode evolution equations of OPA.

\[\text{Figure 4.5. The map shows the absolute value of the LG-mode overlap integral} \sqrt{\pi a(z)}\Lambda_{p,p'}^{l}(z) \text{ evaluated at } z = 0, \text{ for signal and idler modes } p, p' \text{ respectively, with a fixed azimuthal index } l = 0.\]

Similar to the HG-mode case, we can write the general overlap integral as a product of the overlap integral value at focus plane $z = 0$ and an axial phase factor.
\[ \Lambda_{p,r}^{[l]}(z) = \frac{\Lambda_{p,r}^{[l]}(z = 0)}{\sqrt{1 + (z/z_R)^2}} \exp (+i[2(r + p + |l|) + 1] \psi(z)) . \] (4.35)

4.6.4 Coupling Integral for Elliptic Higher-Order HG\(_{q,m}\) Pump mode

The previous calculations are concerned only with the 0-order pump mode in the PSA amplification process, while sufficient for the experimental situations, we would like to explore the effect of a higher-order pump on PSA eigenmodes and its effects on the gain and spatial bandwidth of the PSA. To study a higher-order pump in a PSA, we carry out a overlap-integral calculation similar to 4.26, but with a higher-order pump mode \((q, m) \neq (0, 0)\), motivated by search for new degrees of freedom to control spatial bandwidth of PSA, and its passband. The details of the calculation are presented in A.4. We calculate the overlap of pump mode \(m\) with signal and idler modes \(l\) and \(n\).

The selection rule of the overlap integral is 0 for odd combinations of \((m + n + l)\), and it exists for even combinations of the \((m + n + l)\) only. While the overlap integral itself takes the form,

\[ \Lambda_{l,m,n}(z = 0) = \frac{\sqrt{l!} \; n!}{\pi^{7/4} \sqrt{a_0 y^{2m+1} m!}} \] (4.36)

\[ \times \sum_{t=0}^{\min(l,n)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(-k + r) \Gamma(k - m + r) \Gamma(-l - n + r + 2t)}{t! (l - t)! (n - t)!} \],

where \(r = (1 + l + n + m - 2t)/2\).
Figure 4.6. The map shows the absolute value of the LG-mode overlap integral $\sqrt{\pi a(z)} A_{l, p, p'}^{|l|}(z)$ evaluated at $z = 0$, as function of the signal $p$, and idler $p'$ radial mode index, for several azimuthal mode index values $l = 0, \pm 1, \ldots, \pm 5$ in figures (a)-(f). Overlap integral shown here, is calculated by excluding the pump spot size scaling factors for generality.
For $z \neq 0$ the overlap integral just picks up a complex phase and is given by,

$$\Lambda_{l,m,n}(z) = \frac{\sqrt{l!} \, n! \exp[+i \left(l + n - m + 1/2\right) \psi_y(z)]}{\pi^{7/4} \sqrt{a_y(z)} 2^{m+1} m!} \times \sum_{t=0}^{\min(l,n)} \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(-k+r) \Gamma(k-m+r) \Gamma(-l-n+r+2t)}{t! \, (l-t)! \, (n-t)!}.$$  \hspace{1cm} (4.37)

Further we write,

$$\Lambda_{l,m,n}(z) = \exp[+i \left(l + n - m + 1/2\right) \psi_y(z)] \Lambda_{l,m,n}(z = 0),$$  \hspace{1cm} (4.38)

using which we calculate the complete overlap integral for higher-order pump $\Lambda_{p,q,r}(z)$ in the fixed scale spot size ($\sqrt{2}$) case is the product of overlap integrals along each transverse direction,

$$\Lambda_{p,q,r}^{l,m,n}(z) = \Lambda_{p,q,r}(z) \times \Lambda_{l,m,n}(z).$$  \hspace{1cm} (4.39)

The magnitude of the overlap integral is shown in Fig. 4.7 for several higher-order pump modes.

4.6.5 Coupling Integral for Higher-Order LG\textsubscript{q,m} Pump mode

In this section we compute the overlap integral, for various signal, and idler modes, with a higher-order Laguerre-Gaussian (LG) mode pump, which defines the strength of the 3-wave nonlinear interaction. Details of the calculations are found in A.5.

We use the pump mode $(q, m)$, signal $(p, l)$ and idler $(r, n)$ in the LG basis. We write the final overlap integral, including the selection rule as (non-zero for only
Figure 4.7. Overlap integral $|\Lambda_{m,p,m'}(z = 0)|/\sqrt{\pi a_0}$, between signal modes $m$ and $m'$ for a higher-order pump modes ($p = 0 - 9$, indicated on top of maps) preferably couples to signal modes numbers away from 0-mode. The bandwidth of the non-zero elements [spread in $(m - m')$ direction] increases with pump order $p$. 
m = l + n, where azimuthal mode index for signal and idler must add up to the pump),

\[ \Lambda_{p,q,r}^{l,m,l+n,n}(z = 0) = \sqrt{\frac{1}{2^{l+|n|+|n|+2}} \frac{p! q! r!}{(p+|l|)! (q+|m|)! (r+|n|)!}} \times \sum_{w_0=0}^{r} \sum_{w_1=0}^{p} \sum_{w_2=0}^{q} \left[ \frac{(-1)^{w_0+w_1+w_2}}{w_0! w_1! w_2!} \left( \frac{r+|n|}{r-w_0} \right) \left( \frac{p+|l|}{p-w_1} \right) \left( \frac{q+|m|}{q-w_2} \right) \right. \\
\left. \times \frac{\Gamma \left( \frac{|m|+|l|+|n|}{2} + w_0 + w_1 + w_2 + 1 \right)}{2^{w_0+w_1}} \right]. \quad (4.40) \]

with the complete overlap integral, given by,

\[ \Lambda_{p,q,r}^{l,m,l+n,n}(z) = \frac{\Lambda_{p,q,r}^{l,m,l+n,n}(z = 0)}{\sqrt{1 + (z/Z_R)^2}} \times \exp \left( +i \left[ 2 \left( r + p - q + \frac{|l| + |n| - |m| + 1}{2} \right) \right] \psi(z) \right). \quad (4.41) \]

### 4.6.6 Coupling Integral for Arbitrary Pump Profile

The solution of the higher-order pump overlap integral Eq.4.39, allows us to study PSA action under arbitrary pump profile. We can use non-trivial pump fields, composed of linear superposition of higher-order HG-modes. The overlap integral in such cases would be a weighted (same weights of HG-mode in pump) sum of the overlap integrals of the type discussed in Eq.4.39. For a general pump field represented in the HG-basis as

\[ E_p(\vec{r}, z) = \sum_{q,m} \eta_{q,m}(z) \phi_{q,m}(\vec{r}, z, k_p, a_{0y}), \quad (4.42) \]
we have the overlap integral represented over the pump mode coefficients for its respective order,

\[
\mathcal{V}_{l,n}^{\prime} \left( z \right) = \sum_{q,m} \eta_{q,m} \Lambda_{l,m,n}^{\prime} \left( z \right),
\]

with \( \Lambda_{l,m,n}^{\prime} \left( z \right) \) same as in higher-order pump case. Thus we can solve the coupled-mode equations to arrive at the eigenmodes of PSA, for the arbitrary pump case in what is a powerful generalization of the PSA model.

Unlimited resolution can be obtained from a plane-wave pumped OPA, under its spatial bandwidth. This can be calculated or approximated by a pump beam which is a superposition of the several HG-modes in this theory. This can be used to check the scaling of the exponential eigenvalue spectrum (for an elliptic Gaussian beam) to the analytically predicted parabolic eigenvalue spectrum (for plane wave pump).

The ability to treat any arbitrary pump mode, enables us to answer several questions on optimal pump shaping for a particular class of signal images, than can be obtained by simple pump shapes. Such a procedure has potential to form basis of an adaptive pump mode generation in practical implementations.

4.7 Coupling Integral for General Scaled Signal, Pump spot size

Previously we have calculated the elliptical overlap integral for the signal, pump and idler of the degenerate optical parametric amplifier (DOPA) when the signal spot size \( a_s \) is scaled by \( \sqrt{2} \) times the pump spot \( a_p \). One important limitation of the current solver (at a fixed scale \( \sqrt{2} \) between the signal & pump spot size) is the scaling in number of basis functions quadratically with the pump spot size. Subsequently the extensive numerical eigenmode calculations indicated this Hermite-Gaussian (HG)
basis was not compact, i.e required several hundred modes in one dimension, for accurate solution of moderate pump spot sizes (equivalent in area upto $200 \times 200 \mu m^2$).

Studying larger spot sizes would be impossible in this context. Our guess that a different signal spot size (than $a_s = \sqrt{2} a_p$) would provide a more compact representation proved to be correct from our calculations of mode overlap of fundamental eigenmode of the PSA with a different scale signal spot size $a_s = f_s a_p$, typically with $f_s < 1$. The problem with any scale factor $f_s$ different from $\sqrt{2}$ is a complication of the overlap integral calculation, since the phase curvatures of signal and pump donot cancel out over the crystal leaving a and $z$-dependent complex exponentials in the integrals making for a harder numerical evaluation; however we hope even with the added numerical inconvenience, this signal scaling provides us with a compact low-dimensional representation of eigenmodes, allowing us to tackle larger pump spot sizes. We can also optimize the memory requirements of the solver by moving to a general scaled spot size, by choosing an increased complexity of evaluating coupling integral.

We perform this calculation in hope to enable studying much larger pump spot area (larger than $200 \times 200 \mu m^2$) as we expect to stay within memory limits in the improved solver, with probably longer computation times. In addition we would also have a compact representations of correlators and Green’s functions which can be studied in a less resource intensive (moderate memory and CPU) computer.

It is interesting to note that in previous studies [20], researchers speculated the possibility of obtaining a larger degree of deamplification or squeezing when the signal-pump spot size ratio was different from the canonical $\sqrt{2}$ used in the calculations then [19],[20].
4.7.1 Coupling Integral for TEM$_{00}$ Pump, with General Scaled spot size

In this section we present the closed form expression requiring some pre-calculations, for quick evaluation of the overlap integrals, and discuss some numerical issues. This approach can scale up to 80 or so modes in each spatial dimension before yielding to numerical overflow issues. We show in detail in Appendix A.6, that the overlap integral for the general scaled spot size can be written as follows. The complete overlap integral is given by,

$$\Lambda_{p,r}^{l,n}(z) = \frac{\exp \left( -i \left[ \psi_{px}(z)/2 - (p + r + 1)\psi_{sx}(z) \right] \right)}{\sqrt{\pi a_{0px} a_{0py} \sqrt{1 + (z/Z_{Rpx})^2} \sqrt{1 + (z/Z_{Rpy})^2}}} \times \frac{1}{\sqrt{\pi} \sqrt{2^{p+r} p! r!}} \sum_{i=0,2,4,...}^{i \leq p+r} c_i \times (\xi_x(z))^{-(1+i)/2} \Gamma((i+1)/2)$$

$$\times \frac{1}{\sqrt{\pi}} \frac{\exp \left( -i \left[ \psi_{py}(z)/2 - (l + n + 1)\psi_{sy}(z) \right] \right)}{\sqrt{2^{l+n} l! n!}} \times \left[ \sum_{k=0,2,4,...}^{k \leq l+n} d_k \times (\xi_y(z))^{-(1+k)/2} \Gamma((k+1)/2) \right],$$

where $c_i$ are the discrete convolution coefficients representing product $H_p(x)H_r(x)$, and $d_k$ are the discrete convolution coefficients representing the product $H_l(y)H_n(y)$ respectively. Note that in the general scale case, the signal and pump spot sizes are elliptical, and related by their respective scale factors as $a_{0sx} = f_{sx} a_{0px}$, and $a_{0sy} = f_{sy} a_{0py}$; also the signal and pump have different Rayleigh distances $Z_{Rpx} \neq Z_{Rsx}$, or $Z_{Rpy} \neq Z_{Rsy}$, unlike the $\sqrt{2}$ scale case. We note the selection rule for $\Lambda_{p,r}^{l,n}$ is non-zero for $\text{mod}(p + r, 2) + \text{mod}(l + n, 2) = 0$.  

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4.7.2 Coupling Integral under Higher-Order HG\(_{q,m}\) Pump Mode, with General Scaled spot size

We carry out a similar calculation for the overlap integral, in the general scaled spot size. Details of derivation are presented at A.7. The complete overlap integral is given by,

\[
\Lambda_{l,m,n}^{p,q,r}(z) = \frac{\exp\left(-i\left((q + 1)\psi_{px}(z)/2 - (p + r + 1)\psi_{sx}(z)\right)\right)}{\sqrt{\pi}a_{0px}a_{0py}} \sqrt{1 + (z/Z_{Rpx})^2} \frac{\sqrt{\pi}}{\sqrt{2^{p+q+r}p!q!r!}} a_{sx}^2(z) \times \left[ \sum_{i=0,2,4,...} c_i(z) \times (\xi_x(z))^{-(1+i)/2} \Gamma((i + 1)/2) \right] \times \frac{1}{\sqrt{\pi}} \frac{\exp\left(-i\left((m + 1)\psi_{py}(z)/2 - (l + n + 1)\psi_{sy}(z)\right)\right)}{\sqrt{1 + (z/Z_{Rpy})^2} \sqrt{2^{l+m+n}l!m!n!}} a_{sy}^2(z) \times \left[ \sum_{k=0,2,4,...} d_k(z) \times (\xi_y(z))^{-(1+k)/2} \Gamma((k + 1)/2) \right],
\]

where \(c_i(z)\) and \(d_k(z)\) are the discrete convolution coefficients representing products of Hermite polynomials in the X, Y directions. Numerical evaluation strategies for this overlap integrals follows from a mix of analytical formula as shown here, and pre-computed coefficients to sidestep overflows and numerical artifacts. We note the selection rule for \(\Lambda_{l,m,n}^{p,q,r}\), is non-zero for \(\mod(p + q + r, 2) + \mod(l + m + n, 2) = 0\).

4.8 Relationships between the Overlap Integrals

First we note the strength of the interactions between the fundamental modes of the signal, pump and idler, remain the same in both the LG and the HG representations; i.e. \(\sqrt{\pi}a(z)\Lambda_{0,0}^{0,0}(z = 0) = \sqrt{\pi}a(z)\Lambda_{0,0}^{0,0}(z = 0) = |D_{0,0}(z = 0)|^2 = 1/2\).

We use Fig. 4.8, for simplified presentation of the relations between the various overlap integrals derived in the previous sections, classified based on pump order and the ratio of the signal to pump spot size. In the case of HG-mode overlap integrals,
the general scale spot size forms reduce to the fixed scale for choice of scale $f_s = \sqrt{2}$, and the higher-order mode forms reduce to the fundamental pump mode case for $q = 0, m = 0$. For the LG-mode overlap integrals (we have only derived the fixed scale forms) the higher-order forms for choice of $q = 0, m = 0$, reduce to the fundamental pump mode case. These relationships show the reduceability of the formulas, and also their self-consistency.

Figure 4.8. We show the relationships among (a) HG-overlap integrals in the fixed ($f_s = \sqrt{2}$), and general $f_s$ case for fundamental and higher-order pump beams; (b) LG-overlap integrals in the fixed scale for fundamental and higher-order pump beams. Arrows indicate the transformations required to reduce a more general formula to the specialized one.

4.9 Compact Representation of PSA Eigenmodes

Although our original approach has provided a simple and efficient way of finding the PSA Greens function, it also had a serious drawback: the choice of the signals HG expansion basis required the use of a large number (proportional to the square of pump-beam area) of HG modes, which both demanded large computing resources
and complicated the interpretation of the results. In this section, we report a better HG basis (for elliptical pump case; or Laguerre-Gaussian, or LG, basis for circular pump case), which provides far more compact representation of the PSA modes. This compact basis differs from the original HG basis by the choice of the beam waist: instead of the waist $2^{1/2}$ larger than the pump waist, it has a waist ($a_{sz'} = \sqrt{a_{pz}/q_{px}}$), that is roughly equal to the geometric average of the pump waist ($a_{pz}$), and the inverse spatial bandwidth ($q_{px} = \sqrt{k_p/L}$) of the PSA.

We identify this low-dimensional subspace for the PSA signal eigenmodes, by solving the optimization problem of the overlap integral between the $HG_{m,n}$ mode (similar procedure can be used with LG modes also) and the eigenmode $\Phi_{eig}$ to identify the spot size which characterizes the compact basis

$$\arg \max_{a_{0x} \times a_{0y}} \int \rho HG_{m,n}(\rho, a_{0x}, a_{0y}, z = 0)\Phi_{eig}^*(\rho)d\rho. \quad (4.46)$$

The resulting compact basis reduces the requirement on the number of HG or LG modes needed for solving the PSA propagation equations. We find the basis size grows linearly rather than quadratically with pump-beam area (as with $\sqrt{2}$ scale case), which drastically reduces the memory needs and the computation time. Even more importantly, in the new basis the PSA eigenmodes can be approximated by just a handful of HG or LG modes, making it easy to interpret the findings and implement them experimentally for local-oscillator (LO) generation in traveling wave squeezing experiments.
4.10 Overlap Integral with relative Signal-Pump $z$-offset

Our coupled-mode-theory framework for the phase-sensitive amplifier (PSA), provides several degrees of freedom to maximize the gain for given crystal, pump power, and spot-size in a PSA. One additional degree of freedom in PSA was experimentally found to be the relative $z$-offset (axial offset), between the focus of the signal and pump (see Fig. 4.9), which has been demonstrated in previous studies [20], to provide a larger deamplification or squeezing than the focused pump and signal at the crystal center. Tuning the $z$-offset, shows further improved performance compared to the co-focused signal and pump configuration. We can study this situation by computing

![Diagram of signal and pump beam configuration with $\Delta z_{\text{off}}$](image)

Figure 4.9. Signal, and Pump beam configuration of the PSA, illustrating positive valued $\Delta z_{\text{off}}$, for pump focused after the signal, from the point $z = 0$ of the signal waist.

the new overlap integral, similar in form to the overlap integral with a general scaled waist (signal spot size is not necessarily $\sqrt{2} \times$ pump spot size) Eq.4.45, and Eq.4.44. We modify this existing formula using the new parameter $\Delta z_{\text{off}}$ (positive valued for pump focused after the signal, negative when preceding the signal), which accounts for the offset of pump beam waist, from the point $z = 0$ of the signal beam waist.
Basically we find that, the overlap integral remains same as in the case of the general scale beam waist, with only the following transformations

\[ a_{px}(z) \rightarrow a_{px}(z - \Delta z_{off}), \]
\[ \psi_{px}(z) \rightarrow \psi_{px}(z - \Delta z_{off}), \]  \hspace{1cm} (4.47)
\[ R_{px}(z) \rightarrow R_{px}(z - \Delta z_{off}). \]

In the interest of brevity, we donot write the effect of transformations on the closed forms equations (corresponding to the general scaled waist case), presented in the previous sections. We end this section, by noting the overlap integral for a higher-order pump mode, with signal and idler beams in the z-offset case may be obtained in a similar manner to the method used here, by following the transformations in Eq. 4.47.

4.11 Quantum Operator Relations for Multimode PSA

Using the gain \( g_n \) and shape of the eigenmodes of the PSA at input \( \Phi_n(\rho) \) and out \( \phi_n(\rho) \), we can write the annihilation operator relations for the multimode PSA [21],[40]. Let input and output electric field be decomposed over their respective operators and the corresponding eigenmodes,

\[ \hat{E}_{\text{in}}(\rho) = \sum_n \hat{a}_n^{\text{in}} \Phi_n(\rho), \]  \hspace{1cm} (4.48)
\[ \hat{E}_{\text{out}}(\rho) = \sum_n \hat{a}_n^{\text{out}} \phi_n(\rho). \]  \hspace{1cm} (4.49)

Now we can write the input-output operator relations as,

\[ \hat{a}_{\text{out}} = \sqrt{g_n} \hat{a}_{\text{in}} + \sqrt{g_n - 1} \left( \hat{a}_{\text{in}} \right)^\dagger, \]  \hspace{1cm} (4.50)
and use the orthogonality of the input-output eigenmodes to write the corresponding
field operator relations to be,

\[ \hat{E}_{\text{out}}(\vec{\rho}) = \int \left[ \mu(\vec{\rho}, \vec{\rho}') \hat{E}_{\text{in}}(\vec{\rho}') + \nu(\vec{\rho}, \vec{\rho}') \hat{E}_{\text{in}}(\vec{\rho}')^\dagger \right] d\vec{\rho}', \]

where,

\[ \mu(\vec{\rho}, \vec{\rho}') = \sum_n \sqrt{g_n} \phi_n(\vec{\rho}) \Phi_n^*(\vec{\rho}'), \]

\[ \nu(\vec{\rho}, \vec{\rho}') = \sum_n \sqrt{g_n - 1} \phi_n(\vec{\rho}) \Phi_n(\vec{\rho}). \]

4.12 Coupled-Mode Theory - Matrix Formulation

Here the system of equations for each of the orthogonal mode coefficients \( D_{p,l} \) of
signal. The mode coupling coefficients \( \Lambda_{p,q,r}^{l,m,n} \) are the overlap integral values of the
pump mode \((q, m)\) to the signal mode \((p, l)\) and idler mode \((r, n)\) respectively.

\[ \frac{dD_{p,l}(z)}{dz} = i\kappa e^{i(\theta_p)} e^{i\Delta k z} \sum_{q,r}^{m,n} \left( \Lambda_{p,q,r}^{l,m,n} C_{q,m}(z) B_{r,n}^*(z) \right) \]

(4.52)

where \( \kappa = \omega_s \times 2d_{\text{eff}} \sqrt{P_0} / \sqrt{2\varepsilon_0 n_p \mu_0 c^2}. \)

4.12.1 Degenerate OPA Coupled-Mode Equations

Two more equations just like the one above exist for the pump and idler mode evolutions as a function of axial distance. But under the undepleted pump approximation, the pump mode coefficients \( C_{q,m}(z) \) remain constants, independent of variation along \( z \). This reduces the system from systems of 3 coupled ODEs to 2 coupled ODEs.

If we also further assume the degenerate co-polarized signal and idler beams we
end up with \( E_s = E_d^* \) and the mode evolution equation reduces to a single system of
coupled ODEs in mode unknowns \( D_{q,m}(z) \),

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\[
\frac{dD_{p,l}(z)}{dz} = i Ke^{(\theta_p)} e^{i\Delta kz} \sum_{q,r} (\Lambda_{p,q,r}^{l,m,n} C_{q,m} D_{r,n}^*(z))
\] (4.53)

This equation can be numerically solved using an Initial Value Problem (IVP) ODE solver, through an integrator routine like popular order 4 Runge-Kutta (RK-4) method.

### 4.12.2 Fundamental Mode Pump beam

For a pump beam in fundamental mode \(HG_{00}\) (or \(LG_{00}\)), we can use the overlap integrals \(\Lambda_{p,q,r}^{l,m,n}(z)\) instead of \(\Lambda_{p,q,r}^{l,m,n}(z)\) where \(q = 0, m = 0\), resulting in the equation

\[
\frac{dD_{p,l}(z)}{dz} = i Ke^{(\theta_p)} e^{i\Delta kz} \sum_{r} (\Lambda_{p,r}^{l,n}(z) D_{r,n}^*(z)).
\] (4.54)

In this section we develop the dimension reduction of the equations assuming HG-mode coefficients. A similar procedure for applying these results with the LG-mode coefficients is explained towards the end. We use the undepleted pump approximation in fundamental mode and set \(C_{0,0}(z) = 1\) for all \(z\). We convert the 4D equation into a 2D equation, by the index substitution

\[
(p, l) \rightarrow i = p \times M_y + l
\] (4.55)

\[
(r, n) \rightarrow j = r \times M_y + n
\] (4.56)

\[
\Lambda_{p,q,r}^{l,m,n}(z) \rightarrow \Lambda_{i,j}(z)
\] (4.57)

\[
(M_x - 1, M_y - 1) \rightarrow M_e = M_x \times M_y - 1
\] (4.58)
Figure 4.10. Schematic of 4D matrix rearrangement into 2D matrix for calculation of Transfer matrix or Green’s function matrix.

where \( l, n \in [0, M_y - 1] \), \( p, r \in [0, M_x - 1] \), and \( M_x \) & \( M_y \) are the number of mode coefficients used in the transverse \( X, Y \) directions respectively; schematically shown in Fig. 4.10.

This leads to rewriting Eq. 4.54, as

\[
\begin{bmatrix}
\frac{d}{dz} D_0(z) \\
\vdots \\
\frac{d}{dz} D_u(z) \\
\frac{d}{dz} D_{Me}(z)
\end{bmatrix} = k_1 \begin{bmatrix}
\Lambda_{0,0}(z) & \ldots & \Lambda_{0,M_x-1}(z) & \ldots & \Lambda_{0,Me}(z) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\Lambda_{u,0}(z) & \ldots & \Lambda_{u,M_x-1}(z) & \ldots & \Lambda_{u,Me}(z) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\Lambda_{Me,0}(z) & \ldots & \Lambda_{Me,M_x-1}(z) & \ldots & \Lambda_{Me,Me}(z)
\end{bmatrix} \begin{bmatrix}
D_0(z) \\
\vdots \\
D_u(z) \\
\vdots \\
D_{Me}(z)
\end{bmatrix}^*
\]

\( (4.59) \)

the equivalent matrix equation for the 4D system of linear homogeneous ODEs.

where \( k_1 \) is the corresponding constant from Eq. 4.59,

\[
k_1 = \imath k e^{(\theta_p)} e^{+i\Delta kz}.
\]
The dimension of the unknown vector $D$ is $[M_e \times 1]$, and transfer matrix $\Lambda$ at point $dz$ has dimension $[M_e \times M_e]$. The equivalent problem now looks like

$$\frac{dD(z)}{dz} = k_1 \Lambda(z) D^*(z). \quad (4.61)$$

An identical equation can be developed for LG mode overlap integral coefficients. For mode indices are appropriately identified due to $2M_L + 1$ azimuthal modes running from $l, n \in [-M_L, \ldots 0 \ldots + M_L]$, and $M_r$ radial modes running from $p, r \in [0, \ldots M_r - 1]$, which lead to defining reduced dimension index $i$, and $M_e$ as follows:

$$(p, l) \rightarrow i = p \times (2M_L + 1) + (l + M_L) \quad (4.62)$$

$$(r, n) \rightarrow j = r \times (2M_L + 1) + (n + M_L) \quad (4.63)$$

$$(M_r - 1, M_L) \rightarrow M_e = M_r \times (2M_L + 1) - 1. \quad (4.64)$$

Finally applying the transformations $M_x \rightarrow M_r$ and $M_y \rightarrow (2M_L + 1)$ to the reduced dimension Eq. 4.59, we arrive at the complete matrix equation for LG mode coefficients

$$\begin{bmatrix}
D_0(z) \\
\vdots \\
D_u(z) \\
\vdots \\
D_{M_e}(z)
\end{bmatrix}
\begin{bmatrix}
\Lambda_{0,0}(z) & \ldots & \Lambda_{0,M_L}(z) & \ldots & \Lambda_{0,M_e}(z) \\
\vdots & \ddots & \ldots & \ddots & \vdots \\
\Lambda_{u,0}(z) & \ldots & \Lambda_{u,M_L}(z) & \ldots & \Lambda_{u,M_e}(z) \\
\vdots & \ddots & \ldots & \ddots & \vdots \\
\Lambda_{M_e,0}(z) & \ldots & \Lambda_{M_e,M_L}(z) & \ldots & \Lambda_{M_e,M_e}(z)
\end{bmatrix}
\begin{bmatrix}
D_0(z) \\
\vdots \\
D_u(z) \\
\vdots \\
D_{M_e}(z)
\end{bmatrix}^* = k_1 \begin{bmatrix}
D_0(z) \\
\vdots \\
D_u(z) \\
\vdots \\
D_{M_e}(z)
\end{bmatrix} \quad (4.65)$$
4.12.3 Green’s Function Calculation

We can write the matrix evolution equations of the DOPA, in the form of quadratures. We do this by expressing the mode coefficients of the signal and idler in their real and imaginary quadrature format

\[ D_{p,l}(z) = X_{p,l}(z) + iY_{p,l}(z) \]  \hspace{1cm} (4.66)

\[ D_{p,l}(z) = \begin{bmatrix} X_{p,l}(z) \\ Y_{p,l}(z) \end{bmatrix}. \]  \hspace{1cm} (4.67)

Clearly, by solving for the Green’s function as explained in detail C.2, we are able to write the transfer function of the PSA of length L, as a linear transformation by the Greens matrix G

\[ D_{m,n}(L/2) = \sum_{m,n} G_{m,n,m',n'}(L)D_{m',n'}(-L/2). \]  \hspace{1cm} (4.68)

4.13 Numerical Methods for Nonlinear Paraxial Helmholtz Equation

The nonlinear paraxial Helmholtz wave equation can be solved using the fast Fourier transform beam propagation method (FFT-BPM), or by integrating the system of linear differential equations arising by orthogonal eigenfunction expansions [41].

Spectral methods using the FFT have very high accuracy increasing exponentially in the number of Fourier modes used in the numerical calculation. One significant limitation of the FFT-BPM method is that we cannot use it to study the Green’s function of the PSA due to the inability to propagate a spatial delta function which tends to fill the Fourier space causing aliasing in the calculation. However, this method is very accurate to study the signal amplification by an inhomogeneous
pump beam in a PSA [30]. Our classical PSA solver using FFT-BPM is documented in Appendix B.

However, it was found the finite difference beam propagation method (FD-BPM) where the diffraction is solved by the Crank-Nicolson method does not work for this problem since their errors accumulated in the finite different step causing the nonlinear propagation step to become unstable. FD-BPM works quite well to solve the associated problem of the nonlinear Schrödinger equation (NLSE) for soliton propagation in an optical fiber.

4.14 Limitations of PSA Theoretical Model

Even though our theory is quite general, we find that it could cover a few more situations, which are experimentally useful. We discuss the limits of our theory in this section.

4.14.1 Pump-Signal Phase Fluctuations

Theory does not take into account pump phase fluctuations or phase drift, which is an important experimental effect that can affect the outcome in squeezing experiments due to limitations on phase locking of pump [20]. Closely related issue is our treatment of the image amplification as a continuous-wave (CW) problem (ignoring the temporal modes), instead of taking into account the quantitative nature of envelope of pulsed pump, and image beam signals and their overlap in time. Current treatment of the temporal modes, assumes a flat-top, CW-nature or delta-function in time, and perfect overlap in time of the signal and image beams; this facet of the theory is also motivated by experimental control on the envelope and timing of these beams.
4.14.2 Theoretical Methods

Our current solution method of integrating the PSA equations, even in a compact basis format, suffers from the scaling issues. One possible solution method to enable studying large pump spot sizes, is the usage of the Fast Multipole Method (FMM) [42]. Contentious issue, is whether if we can use the idea of FMM, and its variants to actually speedup the solutions of the coupled-ODE system, by approximating the interaction strength (overlap integral) to span over only few modes (i.e for larger values of mode difference, $|m - n|$, it vanishes). We speculate that it may not possible to obtain a correct solution by this approximation, because the assumptions of the FMM which help make those approximations donot really apply to our problem of PSA mode mixing dynamics. The PSA modes undergo strong mode mixing, and the coupling falls off slowly with the neighboring mode index, which make the system difficult to organize hierarchically to study their mode evolution; in other words the mode indices do not group into naturally ordered blocks of strong and weak coupling based on strength of interaction, which make it difficult to directly apply the FMM to this dense mode coupling matrix. However, if one could apply this method, then we could study using FMM the eigenmodes of the PSA for very large pump spot sizes.

4.15 PSA Amplifier Configurations

To amplify an image using this theoretical model, we suggest the following configuration; the traveling wave image to be amplified at the signal wavelength, is focused to fall into the center of the PSA (typically a $\chi^{(2)}$ crystal like PPKTP or BBO). The pump beams are made coincident with the signal beam focus using a dichroic mirror.

We note that the same configuration was previously used in pioneering experiments of [12], and theoretical basis of this configuration was supported by estimates of
Figure 4.11. Configuration of degenerate traveling-wave phase-sensitive amplifier, with signal and pump (at twice signal frequency) beams focused at the center of the $\chi^{(2)}$ crystal, for optimum amplification.

spatial-bandwidth under phase-matching in [29], since it helps achieve phase-matching across the large number of spatial frequencies.

4.16 Conclusion

In this thesis we solve the nonlinear wave equation by a semi-analytical method of integrating the system of linear differential equations arising from orthogonal eigenfunction expansions. We use the order-4 Rüng-Kutta (RK-4) as the integration method of choice to propagate the signal mode coefficients at the input of the PSA to its output signal mode amplified coefficients. This method allows the calculation of the Green’s functions of the PSA by exciting each mode in each quadrature individually at the input and saving the corresponding output over all modes. The Green’s functions calculated in this way are used to calculate the correlator matrix and through this the independently squeezed eigenmodes. We also note in a Fourier
imaging setup (where the signal is focused at the center of the nonlinear crystal and
the wavefronts of the signal and pump overlap within the crystal), we can calculate
the image amplification by the decomposing the signal field at the input crystal face
over the signal HG-modes, which are used in the PSA propagation routine, and then
we can reconstruct the amplified signal field at the output from the amplified signal
HG-modes we have calculated.

This semi-analytical method provides greater accuracy compared to the purely
numerical split-step Fourier method solutions, albeit at a greater computational cost
of the projection and coupling coefficient calculations.

We note that optical volume holograms (based on planar lightwave circuit or
photorefractive materials) can be used to create unitary transforms to mode convert
the eigenmodes of the PSA into the standard HG-modes and vice-versa, to amplify
images within the limited spatial bandwidth of PSA.
CHAPTER 5
RESULTS OF PSA SPATIAL EIGENMODES CALCULATIONS

In this chapter we present the results of our PSA eigenmode calculations for many
different situations. We discuss the results of our theory, how the concepts of compact
basis, higher order pump modes, and phase-mismatch can be used as new degrees
of freedom to tailor the number, gain and shape of PSA eigenmodes. We discuss
some of the qualitative aspects of the traveling wave optical parametric amplification
process based on our results. Finally we touch upon the limits of our theory both
computational, and theoretical restrictions in our model. The computational issues
involved in solving the PSA coupled ODE system, and estimates for numerical solver
parameters are explained in Appendix. C.

5.1 Independently Squeezed Modes of PSA

Using the theory of the PSA modes developed in Ch.3, and Ch.4, we can calculate the
Green’s function of PSA for the particular choice of crystal, pump mode, spot size,
and pump power. By diagonalizing the quantum correlators of PSA Ch.3 (which
can be calculated only with the knowledge of Green’s function), we find the inde-
pendently squeezed signal modes of PSA. By calculating the correlator matrix from
the Green’s function and diagonalization, we obtain the eigenmode (independently
squeezed modes) shapes and the degree of squeezing. In our discussion of squeezing
we assume the squeezed vacuum generated by the travelling wave PSA, is detected in
a balanced homodyne measurement using a matched local oscillator beam, and with
perfect mode-matching efficiency.
The eigenvalues of the correlator matrix, obtained by diagonalization, is called the squeezing spectrum, and roughly has a sigmoid-shape (see Fig.C.2), where the lower and upper halves are exactly inverse of each other, $\lambda_1 = 1/\lambda_2$; i.e for every squeezed mode there is an anti-squeezed mode with eigenvectors having the $90^\circ$ rotated quadratures. The eigenvalue spectrum starts at the most squeezed mode and gradually rolls off to a unity-gain modes, before climbing up a slope of the high gain (anti-squeezed) modes. The most important information we can obtain from the eigenmode spectrum, apart from the gains/squeezing values of each individual mode, is the number of squeezed modes, at a given pump parameters.

The eigenvalues are related to the power gain or deamplification for each mode excited at the input with 0-phase or $90^\circ$-phase shifts respectively, by the following relations: $\lambda_1 = (|\mu| + |\nu|)^2$ for gain, and $\lambda_2 = (|\mu| - |\nu|)^2$ for deamplification.

The eigenvector corresponding to the squeezed mode, contains the eigenmode shape, as mode coefficients in the signal HG basis (computational basis). We use a simple mode field expansion by inverse HG transformation, to visualized the eigenmode shape from the mode coefficients.

5.2 Eigenmodes of PSA with 0-order Pump

5.2.1 Parameters for the PSA Modeling

We chose parameters of PSA crystal length $L = 2\text{cm}$, with effective nonlinearity at $d_{\text{eff}} = 4.35 \text{ pm/V}$, at signal and pump wavelength of $\lambda_s = 1560\text{nm}$ and $\lambda_p = 780\text{nm}$ respectively, with pump phase at $\theta_p = -\pi/2$, and pump power in 10 kW range. We also assume the perfect phase-matching with $\Delta k = 0$. The results in this section were presented in the conference talk [43].

First we study the eigenmode spectrum for pump-beam spot sizes $a_{0px} \times a_{0py}$: $25 \times 25\mu m^2$, $100 \times 25\mu m^2$, $100 \times 50\mu m^2$, and $100 \times 100\mu m^2$, and later extend it to cases
of 200 × 200µm², 400 × 100µm² and 800 × 50µm². For each pump spot size, we chose the pump power in such a way that the fundamental eigenmode (most squeezed) has a gain of 15 (or a squeezing of 0.066). For the calculation we need at least 32 × 32 HG modes in X-Y spatial directions for 100 × 100µm² spot size and lesser. For larger spot size (in a given dimension), our rule of thumb is to use 4× larger mode number for a 2× increase in the pump spot size, along that dimension; i.e for 200 × 200µm² we use 128 × 128 modes along X-Y directions to calculate the eigenmode spectrum, which is 4× the 32 × 32 modes used to study a pump spot of 100 × 100µm².

We note that for a larger pump power at the same pump spot size, we are able to bring more modes into the squeezing bandwidth, while improving on the largest achievable squeezing for the fundamental eigenmode. Eigenmode shapes are mostly the same for a larger powers. We find that PSA eigenmodes shapes do not change with crystal type, since changes in the effective nonlinearity $d_{\text{eff}}$, are equivalent to the power scaling, carried out by the effective parametric gain $\kappa$.

The eigenmode squeezing spectrum (gain) for the spot sizes 25 × 25µm², 100 × 25µm², 100 × 50µm², and 100 × 100µm², and later extend it to cases of 200 × 200µm², 400 × 100µm² and 800 × 50µm² are shown in Fig. 5.1. The black dashed line in the figure shows the 3 dB point from the peak gain. As expected, we see the number of modes within 3 dB of peak gain goes up, with the pump spot area in a linear way. We can also identify that a purely circular pump spot size, shows a highly degenerate eigenmode spectrum as in Fig. 5.1. The staggered nature of the spectrum where two or more eigenmodes have the same gain is an indication of the degeneracy. Breakdown of the eigenmode spectrum degeneracy, was found to be related to the ellipticity of the pump spot size, for example in Fig. 5.2.

Next we plot the spatial distribution of the intensity of the electric field, of the first several eigenmodes from the diagonalization procedure we carried out earlier.
Figure 5.1. Eigenmode squeezing spectrum for elliptical pump & circular pump spot sizes, where power is chosen for the eigenmode-0 gain to be normalized to $\approx 15$. The black dashed line in the figure shows the 3 dB point from the peak gain. We can see an increase in the number of squeezed modes with the pump spot area.

Figure 5.2. The staggered nature of the spectrum where two or more eigenmodes have the same gain is an indication of the degeneracy. Eigenmode squeezing spectrum for elliptical pump, shows the non-degenerate spectrum as compared to degenerate spectrum of the circular pump spot size.
First several eigenmode shapes for pump spot sizes $25 \times 25\mu m^2$, $100 \times 25\mu m^2$, $100 \times 50\mu m^2$, and $100 \times 100\mu m^2$ are shown in Fig. 5.3, for pump spot size $200 \times 200\mu m^2$ shown in Fig. 5.4, for $400 \times 100\mu m^2$ shown in Fig. 5.5, and for $800 \times 50\mu m^2$ shown in Fig. 5.6. We clearly note the trend in the eigenmode shapes, and the influence of the inhomogeneous pump shape; elliptic pump modes support eigenmodes along the larger dimension discriminating against the smaller area pump, whereas circular pump modes support circularly-symmetric eigenmodes as a consequence of the degenerate eigenvalue spectrum.

Shapes of the fundamental eigenmode for pump spot size at $25 \times 25\mu m^2$, $100 \times 25\mu m^2$, $100 \times 50\mu m^2$, and $100 \times 100\mu m^2$, are shown in HG basis in Fig. 5.7(A), and Fig. 5.7 (B), yielding 35%, 50%, 61% and 97% overlap with TEM$_{00}$ mode for corresponding spot-sizes. We plot in Fig. 5.8, a 3D representation of mode power distributions for the smallest pump spot sizes ($25 \times 25\mu m^2$, $100 \times 25\mu m^2$, $100 \times 50\mu m^2$, and $100 \times 100\mu m^2$). Number of squeezed modes increases with spot size Fig. 5.7 (B), when pump power is chosen to match the gain of the fundamental mode with spot size $100 \times 100\mu m^2$. These pump powers are not linear in beam area Fig. 5.7 (C). Eigenmode shapes in XY domain are found in Fig. 5.3.
Figure 5.3. First 4, 6, 9 and 15 spatial (X-Y) profiles of (most amplified / most squeezed) eigenmodes of the PSA (all drawn to same scale) for 4 different pump spot sizes $25 \times 25 \mu m^2, 100 \times 25 \mu m^2, 100 \times 50 \mu m^2, 100 \times 100 \mu m^2$ respectively. Pump powers are adjusted to produce the same PSA gain of $\approx 15$ for the first mode (#0). Higher eigenmodes involve superpositions of several HG modes in computational basis.
Figure 5.4. First 15, spatial eigenmode shapes for large pump spot sizes at 200 × 200μm².
Figure 5.5. First 15, spatial eigenmode shapes for large pump spot sizes at $400 \times 100 \mu m^2$. 
Figure 5.6. First 17, spatial eigenmode shapes for large pump spot sizes at $800 \times 50 \mu m^2$. 

$800 \times 50 \mu m^2, P_0=21.5 kW$
Figure 5.7. (a) $|A_{mn}|^2$ for mode #0. (b) Eigenvalue (gain and squeezing) spectra. (c) Blue circles: pump power needed for PSA gain of 15 for mode #0, vs. pump beam area $a_{0px} \times a_{0py}$. Red triangles: PSA gain of mode #0 for the four cases if the peak intensity in all cases is equal to that for $100 \times 100 \mu m^2$. Blue dashed line: linear power dependence (guide for eyes). Even though mode #0 is single-lobe in all 4 cases, it is far from being close to the fundamental TEM$_{00}$ mode [as can be seen in (a)], except for the $25 \times 25 \mu m^2$ case.

Figure 5.8. Even though Modes #0 in all the 4 cases have single-lobe shape, they are far from being Gaussian TEM$_{00}$ in $\sqrt{2}$-calculation basis, except in $25 \times 25 \mu m^2$ beam spot size, which is 97% TEM$_{00}$. 

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Figure 5.9. We show the mode representation of the fundamental eigenmode of PSA for several different pump spot sizes. The (top) first row, shows their spatial distribution while second row shows their mode representation in calculation $\sqrt{2}$ signal basis, and in the compact a HG-mode (third row) or a LG-mode (fourth row) parameterized by different signal spot size by $f_s \times$, (different from $\sqrt{2} \times$) the pump spot size.
Figure 5.10. Eigenmode shapes for pump spot size $200 \times 200 \mu m^2$, and their mode content in (top-row, first) spatial domain, (second row) calculation HG-basis, (third row) compact LG-basis based on eigenmode #0, (bottom row, fourth) compact LG-basis based on eigenmode #5.
Figure 5.11. Eigenmode shapes for pump spot size $800 \times 50 \mu m^2$, and their mode content in spatial domain, and compact HG-basis.
5.3 Compact Representation of PSA Eigenmodes

Using our theory for calculating the compact low-dimensional subspace to represent the PSA eigenmodes in Sec 4.9, we identify quantitatively the compact basis for several different situations. The following results were first presented at a conference talk [44], and also at a poster session [45].

The HG expansion coefficients $|A_{m,n}|^2$, for the eigenmodes of the most elliptical ($800 \times 50 \mu m^2$) of these pump beams, are shown in the original basis (Fig. 5.9) and the compact basis (Fig. 5.11), along with their spatial profiles. The eigenmodes #0 and #5 have overlap of 98.4% with TEM$_{00}$ and $\geq 90\%$ with TEM$_{50}$, respectively. The modes within 3 dB from the maximum gain (gain of mode #0) are well represented by either one or, at most, two HG modes, whereas the eigenmodes outside of the -3 dB bandwidth require a handful of HG modes for representation. The overlap of mode #0 with TEM$_{00}$ can be improved to 99% by tweaking (to $113 \times 38.4 \mu m^2$) the waist of the expansion basis for best overlap with mode #0, at the expense of slightly worse overlap with higher-order eigenmodes. We calculate the compact basis for signal eigenmodes of $400 \times 100 \mu m^2$ pump spot size, through the overlap integral calculations visualized in Fig. 5.12.

The case of a circular pump is illustrated in Fig. 5.4, showing the spatial profiles of few of the first 14 modes; some of the modes are double-degenerate with respect to azimuthal rotation), as well as their representations in the original HG basis [see Fig. 5.10 (second row)] and in the compact LG basis [see Fig. 5.10 (bottom row)]. It is easy to see that, unlike the original basis, the compact basis enables representation of each of the 14 most prominent eigenmodes by a superposition of 4 or fewer LG modes. The signal spot size was chosen for the maximum 96.5% overlap of mode #5 with LG$_{10}$, also yielding 98% overlap of mode #0 with LG$_{00}$. The overlap of mode #0
Figure 5.12. Overlap integral of the TEM$_{00}$ mode with the signal eigenmode #0 of PSA for a 400 $\times$ 100$\mu m^2$ pump spot size at 16.25 kW for several spot sizes, the best of which is the compact signal basis of size 81.6 $\times$ 48.7$\mu m^2$ with 99.2% overlap.

Figure 5.13. Scaling of compact basis signal spot size in xy-dimensions with the corresponding pump spot sizes, while the calculation basis with slope $2^{1/2}$ (whereas compact mode spot size data have a slope $f_{sx}$ or $f_{sy}$) is shown for reference.
with LG\textsubscript{00} can be improved to 99.4% by slightly adjusting the waist of the expansion basis (to \(62 \times 62 \mu m^2\)), which reduces the mode \#5 overlap with LG\textsubscript{10} to 89.9%.

Naturally PSA eigenmodes are represented in a basis contingent on the pump shape; circular pump PSA eigenmodes find good representation in the LG-basis, whereas the elliptical pump PSA eigenmodes are well represented in the HG-basis. While these are soft restrictions, since using a non-compact basis only has the effect of a larger computational load.

While there is a big difference in the representation between the original basis and the new compact basis, moderate changes in the basis waist size around the optimum do not reduce the compactness. This is illustrated for example in Fig. 5.24, where the compact basis single-mode Gaussian has \(> 90\%\) overlap with fundamental eigenmode, for over \(6 \mu m\) around the optimum beam waist at \(34.6 \mu m\). This means that the approximately optimal waist size can be computed as a geometric average of the pump waist and the inverse spatial bandwidth before solving the PSA equation. We show the scaling of the compact basis beam waist with the calculation basis in Fig. 5.13. The equation is then expanded over the optimum basis and the coupled-mode equations are efficiently solved owing to drastically reduced number of the required expansion modes.

To summarize, we have found the eigenmodes of a spatially-broadband PSA and expressed them as superpositions of a very small number of HG or LG modes. This procedure is important for both amplification (boosting image power before detection) and squeezing (suppressing the quantum noise in many spatial modes) applications of the PSA, where it would help, respectively, in minimizing the amplified image distortions and in providing matched local oscillator for maximum squeezing detection.
5.4 Eigenmodes of PSA under Higher-Order Pump

Here we investigate qualitative relations between PSA eigenmodes obtained by pumping with higher order HG modes at perfect phase matching ($\Delta k = 0$), and nonzero ($\Delta k \neq 0$) phase mismatch case under a HG 0-order pump using our numerical solver. This would have consequences for the achievable spatial-bandwidth of the PSA, by using phase-mismatch as a parameter dial for moving the PSA pass-band around to mode space only achievable with a higher order pump.

In the second situation (pumping by a higher-order mode), the coupled-mode theory of the PSA is exactly the same as that for an elliptical fundamental Gaussian pump, but with mode-coupling coefficients determined by a new overlap integral of the signal mode $m$, idler mode $n$, and pump mode $l$, was derived earlier in Eq. 4.38. The complete overlap integral is the product of overlap integrals along each transverse direction (i.e. $X$ and $Y$) as in Eq. 4.39.

Analysis of the structure of Eq. 4.38, tells us that the pump of higher-order $l$, spreads the coupling away from the main diagonal $m = n$, i.e., facilitates the coupling between the modes of significantly different orders, clearly seen in Fig.4.7. Thus, one can expect the shift of the gain from HG$_{00}$ signal modes to higher-order modes.

Using Eq. 4.39, also allows us to generalize the model to study the PSA behavior under an arbitrary pump profile: by mode decomposition of the general pump mode over the HG basis and calculating the respective overlap integrals of each HG component of the pump beam with all signal and idler HG modes, we can use the formalism to obtain the PSA eigenmodes, as shown in Eq. 4.43. This, for example, can potentially help in finding the optimal pump shape for a particular class of signal images.

The procedure was carried out for pump spot size 200 × 200$\mu m^2$, pumped by a HG$_{22}$ mode beam at 16.25kW pump power. The mode shapes shown in Fig. 5.14,
Figure 5.14. First 17 eigenmodes, calculated for a pump spot size $200 \times 200\mu m^2$, pumped by a HG$_{22}$ mode beam at 16.25kW pump power.
Figure 5.15. First 20 eigenmodes, calculated for a pump spot size $400 \times 100\mu m^2$, pumped by a HG$_{40}$ mode beam at 17.25kW pump power.
clearly lies far away from the eigenmode shapes (compare with Fig. 5.4) corresponding to the fundamental-pump beam PSA at a similar power. The first 20 eigenmodes of an elliptic pump of HG_{40} mode, 17.25kW power and spot size 400 × 100µm², is shown in Fig. 5.15, demonstrating the advantages of using the higher-order pump beam; discussion on this issue continues in the following section.

5.5 Eigenmodes of PSA with Phase Mismatch $\Delta k \neq 0$

It is well known that phase-mismatch provides an additional degree of freedom in operating the parametric oscillators and amplifiers, to study quantum entanglement and frequency metrology [46].

In the following section, we study the PSA under HG_{00} pump-mode pumping, but with a non-zero phase mismatch. Since the theory is able to handle situations with the phase-mismatch $\Delta k \neq 0$, we are have explored the results of these parameter values. All our current results are for perfect phase-matching with $\Delta k = 0$. The results in this section were presented at a conference talk [47].

Our results of eigenmode calculations with effective non-zero phase-mismatch are calculated for two different cases; for pump spot size 100 × 100µm², at 5kW pump power the modes are shown in Fig. 5.16, whereas for a larger pump spot size 200 × 200µm², at 16kW pump power the modes are shown in Fig. 5.17. Comparing these eigenmode shapes visually with the corresponding modes calculated for perfect phase-matching like in Fig.5.3 and Fig. 5.4, shows the movement of the eigenmodes with the largest gain toward the higher order mode shapes in the non-zero phase-mismatched case.

Let us understand what happens in this situation. Non-zero phase mismatch $\Delta k = k_p - k_s - k_i$ (subscripts for pump, signal and idler wave numbers), in the simplest case of a plane-wave pump, moves the maximum gain from low spatial frequencies.
Figure 5.16. First 20 eigenmodes, calculated for a pump spot size $100 \times 100 \mu m^2$, at 5kW pump power and phase-mismatch of $\Delta k = -220m^{-1}$.
Figure 5.17. First 20 eigenmodes, calculated for a pump spot size $200 \times 200 \mu m^2$, at 16kW pump power and phase-mismatch of $\Delta k = -220 m^{-1}$. 
(low-pass-filtering behavior of a PSA with $\Delta k = 0$) to higher spatial frequencies (bandpass-filtering behavior of a PSA with $\Delta k \neq 0$). This only happens for $\Delta k < 0$, because negative phase mismatch compensates the diffractive term in the effective phase mismatch $\Delta k_{\text{eff}}(q) = \Delta k + q^2/k_s$ (where $q$, is spatial frequency of interest), making $\Delta k_{\text{eff}}(q) = 0$, at some high spatial frequency $q$ of interest. It is reasonable to expect that, with a Gaussian pump, a similar shift of the highest gain from the fundamental $HG_{00}$ signal mode toward higher-order HG modes (i.e. modes with larger presence of high spatial frequencies) will take place.

5.6 Comparison of $\Delta k \neq 0$ and Higher-Order Pump Eigenmodes

We study the eigenmodes for (i) phase mismatched situation (e.g. obtained by tilting the crystal from the phase-matched angle) for $\Delta k = -220 m^{-1}$ and several different $HG_{00}$ pump spot sizes and powers, and (ii) phase-matched situation ($\Delta k = 0$) with elliptical ($HG_{40}$ mode with $400 \times 100 \mu m^2$ spot size, 17.25 kW power) or circular ($HG_{22}$ mode with $200 \times 200 \mu m^2$ spot size, 16.25 kW power) Gaussian pump.

Figure 5.18 shows eigenvalue spectra for various studied cases. One can see that, for the same power and pump spot size, phase mismatch (or higher-order pumping) significantly reduces the highest gain, but distributes it more evenly among the modes. This indicates that a more meaningful comparison should be done between the phase-mismatched case for one pump size and phase-matched case for a bigger pump size, chosen so that the highest-mode gains are similar in both cases: e.g., mismatched $100 \times 100 \mu m^2$ case vs. matched $200 \times 200 \mu m^2$ one have spectra with comparable decay rates. Similar comparison with a larger pump size can be carried for the case (ii) as well, which confirms that the number of amplified modes (width of the spectrum) is determined by the pump power rather than by pumps shape or the presence or absence of the phase matching. The shapes of the eigenmodes are shown in Fig. 5.19(a)(d),
proving that both phase mismatch (b),(d) and higher-order pumping (a),(c) shift the gain toward higher-order modes (i.e. HG_{00}-like signal mode is no longer #0, except (d), where highest-gain #0 mode is HG_{00}-like, but has significant content of non-HG_{00} modes as well).

To summarize, we have found the eigenmodes of a spatially-broadband PSA pumped by higher-order HG pump modes with Δk = 0, and by HG_{00} pump mode with Δk ≠ 0. We have shown the similarity of the two situations in their ability to tune the passband of the PSA toward eigenmodes with higher-spatial-frequency content (i.e. higher HG-order content). This procedure is important for tuning the PSA into the higher-spatial-frequency range, previously inaccessible due to the limited spatial-bandwidth of PSA centered around the zero-spatial-frequency.
Figure 5.19. We compare different spatial profiles of several eigenmodes (mode number indicated on the image) (corresponding to Fig. 5.18) for 4 different situations of non-zero phase-mismatch and higher order pumping in order (a)-(d) for (a) $400 \times 100 \mu m^2$ spot size, 17.25 kW, HG$_{40}$ pump mode at $\Delta k = 0$; (b) $800 \times 50 \mu m^2$ spot size, 21.5 kW, HG$_{00}$ pump mode at $\Delta k = -220 m^{-1}$; (c) $200 \times 200 \mu m^2$ spot size, 16.25 kW, $\Delta k = 0$ with HG$_{22}$ pump mode; and (d) $200 \times 200 \mu m^2$ spot size, 16 kW, $\Delta k = -220 m^{-1}$ with HG$_{00}$ pump mode.
5.7 Resolution Properties of PSA with general Pump mode and Phase matching

We discuss the resolution of an amplified image in PSA by exploring the effects of a fundamental mode pump with and without phase-mismatch, higher-order mode pump at perfect phase-matching on the PSA. We find the effect of phase-matching and pump mode shape on image amplification through a Monte Carlo numerical modeling technique using our previously calculated eigenmode shapes.

In all the following results we model the PSA amplification of images using the signal eigenmodes for a $200 \times 200 \mu m^2$ pump spot size with TEM_{00} or HG_{22} pump mode at $\Delta k = 0$ or a fundamental pump mode with $\Delta k = -220 m^{-1}$. We include effect of quantum noise at the input, and due to the non-unity quantum efficiency at the photo-detector in our modeling. Optimum spatial-filtering around the signal spatial-bandwidth at the photo-detector is also included in our modeling.

First the effect of pre-amplifier PSA with a $\eta = 15\%$ photo-detector on image resolution is demonstrated dramatically in Fig. 5.20, as compared to the baseline case. We can visually trace the shape of the letters in the photo-detection with a pre-amplifier which would not be possible in the baseline situation, demonstrating the resolution improvement due to a spatially multimode PSA, and the effect of noiseless image amplification.

Next we show the comparative effect of spatial characteristics of different PSA’s by considering signal images with the checkerboard, and bulls eye patterns. It is clear from Figs.5.21,5.22,5.23, that amplification achieved by the presence of higher-order pump mode, or non-zero phase mismatch, as compared to fundamental pump mode PSA, can significantly increase the contrast in these two images. The quantitative movement of PSA signal eigenmodes toward the higher spatial-frequencies for a non-zero phase mismatch or a higher-order pump mode seen in Section. 5.5, is matched well by the qualitative improvement in high resolution image amplification.
Figure 5.20. Image resolution of letters ‘3’, ‘A’, and ‘N’ with $N = 200$ or $1000$ photons, in a lossy photo-detector with and without a pre-amplifier PSA, is shown for two different detector quantum efficiencies $\eta = 75\%$ or $15\%$. Clearly the presence of a pre-amplifier PSA, decides the resolution of the image in the case of the more lossy photo-detector ($N = 200$, $\eta = 15\%$) which would not be possible otherwise, while providing brighter images in the higher quantum efficiency ($\eta = 75\%$) case. The PSA has gain 15 on eigenmode #0, with pump power $P_0 = 16.25kW$, TEM$_{00}$ mode of spot size $200 \times 200 \mu m^2$. 
Figure 5.21. Image amplification with fundamental Gaussian pump of spot size $200 \times 200\mu m^2$ at pump power 16.25kW (adjusted to produce a PSA gain of $\approx 15$ for the first mode #0), showing signal before (a) and after amplification (b).
Figure 5.22. Image amplification with HG$_{22}$ Gaussian pump of spot size $200 \times 200 \mu m^2$ at pump power 16.25kW (adjusted to produce a PSA gain of $\approx 15$ for the first mode #0), showing signal before (a) and after amplification (b).
Figure 5.23. Image amplification with fundamental Gaussian pump of spot size $200 \times 200 \mu m^2$ and $\Delta k = -220 m^{-1}$ at pump power $16.25 kW$ (adjusted to produce a PSA gain of $\approx 15$ for the first mode $\neq 0$), showing signal before (a) and after amplification (b).
5.8 Experimental Verification of PSA Eigenmodes

Recently our collaborators at the Northwestern University in Chicago, have demonstrated the fundamental eigenmode of the PSA [49]. They achieve this by, generating and verifying its gain-deamplification properties as function of the Gaussian-beam waist size (we calculated a Gaussian mode to be coincident upto 96% with the fundamental eigenmode). Sure signature of a PSA eigenmode, is that they undergo highest gain or deamplification, only depending on choice of signal-pump phase offset.

The charts from their experiment [49], which illustrate the overlap of the fundamental Gaussian mode with the fundamental eigenmode, and the experimental result of gain/deamplification as function of beam waist.

From Fig. 5.25, Fig. 5.24, we see the coincidence of peak gain of 11.4 dB, and deamplification -7.9 dB at the $1/e$ Gaussian waist of 34.6 $\mu$m, pickout the generation and existence of the PSA eigenmode.

Theoretical predictions were used to confirm results from yet another experiment on cascaded PSA’s [50].

5.9 Qualitative Discussion of PSA Eigenmode Formation

From the shape and profile of PSA eigenmodes calculated for the several pump power, area, mode and crystal parameters in this chapter, we deduce a qualitative idea of what goes on in the PSA. Our calculations have shown how a single mode excited at PSA input can undergo gain induced diffraction and couple to many other modes with significant gain, by the time they reach the crystal output. This leads to a complex view of the PSA amplification process. However, if instead of exciting just one mode at the input, we excite a specific combination of modes, the PSA dynamics make sure for a specific crystal parameters, the same modes are reproduced at output, with a gain. This can be thought of as a opaque process where power transfer from the pump
Figure 5.24. Overlap of fundamental eigenmode with a circular Gaussian mode, with $1/e$ waist of 34.6 $\mu$m, upto 96%.

Figure 5.25. Eigenmodes of PSA undergo highest gain or deamplification; these experiments show, (top-row) gain of PSA as a function of Gaussian beam $1/e$ waist, and (bot-row) same for deamplification. We can see that at the beam waist size of 34.6 $\mu$m, the signal Gaussian beam undergoes maximum amplification or deamplification in the PSA (at the same beam-waist, shown by the green line), which is the experimental signature of an eigenmode, in agreement with modeling.
keeps mode amplitudes growing throughout the length of a crystal and scattering by gain-induced diffraction is confined with the eigenmode set, while the physics of mode-coupling and quadrature rotations will allow these few mode combinations to be reproduced in the same proportion as the input - and hence called eigenmodes. Such eigenmodes undergo gain and deamplification without mixing with each other, forming a complete orthonormal basis and also a simpler picture of the PSA dynamics.
CHAPTER 6
CONCLUSION AND FUTURE WORK

We expect that our work will be a starting point for future analysis of other multimode PSA’s, and will help in experimental demonstration of PSA-based multimode squeezing or image amplification in advanced Quantum Imaging and Metrology [24]. Quantum Information and Communication protocols [51], can also benefit by using PSA’s due to their noiseless properties. While some of this work has been published as conference articles for the coupled-mode theory [43], compact basis representations [44], [45], and the effect of higher-order mode pump or phase-mismatch [47], the journal articles on the same topics are also under preparation, whereas the article on the quantum theory of PSA has been published [23].

6.1 Contributions
In this thesis we have shown the procedure to calculate the independently amplified signal eigenmodes of a PSA, their number and shapes, by developing a fully quantum theory of the PSA, and a semi-analytical coupled-mode-theory of the PSA. We also quantitatively identify eigenmodes of the PSA for various pump powers and $1/e$ spot sizes, by using numerical calculations. Knowledge of the number and spatial distribution of eigenmodes would enable building mode converters for use in the phase-sensitive image amplification or multimode squeezing generation, or as a part of the LADAR system. We have quantified the performance of PSA for fundamental, and higher-order pump modes, both with and without phase-mismatch. When not used for image amplification, the same eigenmodes from our work represent the
multimode-squeezed light generated by a DOPA, and can be used a source of Quantum Images [4].

On a historical note, we have advanced on some of the intractable problems identified in decade-old work; we consider these problems solved or accessible to exploration, based on the theory developments in this thesis. Specifically, researchers speculated on the possibility of obtaining a larger degree of deamplification or squeezing for the signal-pump spot size ratio different from the canonical $\sqrt{2}$, or on the signal-pump focusing off the crystal center [19],[20]; our theory work with general scaled waist $a_s = f_s a_p$ and $z$-offset overlap integrals, which makes these situations easy to explore. The difficulty of generating matched local-oscillator beams (now possible with SLM) [3], [19], and shaping the pump mode to obtain largest squeezing in a fundamental Gaussian mode was highlighted in [19], [48], both of which are solved in our calculation of the eigenmodes (perfect local-oscillator beam is the conjugate of the squeezed eigenmode to be measured), and the coupled-mode theory for the arbitrary pump mode respectively.

In the larger framework of the Quantum-Sensor Program (QSP) which sponsored this work, our models have been used to design experiments and system simulations. Our model for the PSA eigenmodes was used in the Monte-Carlo simulations of LADAR system with a finite-spatial-bandwidth PSA pre-amplifier with homodyne detection [21]. Recently, our collaborators at the Northwestern University, have demonstrated the fundamental eigenmode of the PSA [49]. A recent experiment demonstrated a coherent LADAR system [52], using our modeling input for PSA parameters and phase-locking schemes for the amplified quadrature.

We also like to highlight the capabilities of our PSA coupled-mode-theory for signal, idler, and pump modes, which allows to study PSA in very general terms to treat the following situations (some of which are illustrated in Fig. 6.1),
Figure 6.1. PSA coupled-mode-theory variants arranged according to fixed/general-scale spot size and the zero/higher-order pump modes.

1. two choices of computation basis (LG or HG),
2. either a $\sqrt{2}$-scale factor between signal and pump spot size or general scaled $a_s = f_s a_p$, mode solver (in increasing order of complexity, and generality)
3. zero-order or higher-order (elliptical) Gaussian pump requirements,
4. superpositions of LG or HG modes for pump beams,
5. $z$-axis axial offset of signal and pump focusing situation,
6. Non-zero $\Delta k$ phase-mismatch

forming a comprehensive theoretical framework to study the PSA.

6.2 Future Work

In this section, we give a sense of the possibilities for our theoretical program, and make few proposals for theoretical study, as well as experiments which are possible primarily due to the PSA theory and analysis developed in this thesis.

We give a skeletal picture of our proposed experiments, either for applications, or verification of the theory, or both. This is a preliminary list with a sketchy protocol, as conducting these experiments is beyond the scope of this thesis work.
6.2.1 Experiments to Verify Squeezing and Higher PSA Eigenmodes

Our Northwestern University collaborators - Prof. Kumar’s group, continuing from their verification of the lowest order eigenmode [49], are planning future experiments expected to verify the higher order eigenmodes as well. In particular, an experiment is planned in which various PSA eigenmodes are synthesized using a spatial-light modulator (SLM), and their amplification and deamplification gains are measured and compared with the theoretical predictions. A second experiment on squeezing will use our theory to generate and detect the mode with the maximum squeezing of quantum noise.

6.2.2 Optimization of Image Amplification using PSA

We envision a method to optimize the PSA amplification of specific image or classes of images using our coupled-mode-theory of PSA for an arbitrary pump mode. We can achieve the optimum amplification of the image by tuning pump mode profile (optimizing the weights of various higher-order pump mode coefficients involved in the superposition sum of HG-modes forming the pump profile) while observing the PSA image amplification in a closed loop system. For example, one can use evolutionary algorithms in a real-time system. A schematic setup of the optimized PSA of images using multimode pump profile is shown in the Fig. 6.2.

6.2.3 Experimental determination of PSA Green’s functions

Based on new protocols developed in the area of light propagation in disordered media, we propose to measure PSA Green’s functions directly. Ref. [53] successfully demonstrated propagation a large fraction of light through a disordered by effectively determining the eigenmodes of the media during the experiment. As in [53], we can experimentally determine the Green’s functions of PSA. We can use neutral density
Figure 6.2. Proposed architecture for optimized PSA operation using a multimode pump profile in a closed loop configuration.
filters with spatial-light-modulator (SLM), to shape the input beams and learn the output beam amplitudes and phase by interferometry. This will provide the point-wise PSA Green’s function input-output pairs.

6.3 Open Problems

We identify some of the open problems which could use the techniques developed in this thesis, and be solved theoretically.

6.3.1 Multimode Waveguide Image Amplifier

We propose to study theoretically a multimode nonlinear waveguide image amplifiers using a new coupled-mode theory similar to the one developed in this thesis, but accounting for the spatial-mode mixing among the frequency degenerate waveguide modes. This model of a phase-insensitive amplifier is expected to describe the spatial-mode effects of a semiconductor laser device [54], modified to work as an image amplifier or a waveguide with parabolic index which naturally has HG modes. Similar studies of multimode-interference have been carried out in case of waveguides [55],[56] and silicon Raman image amplifiers [57]. We propose to study both degenerate and non-degenerate OPA with the spatial mode mixing and gain-induced-diffraction of the signal-idler spatial modes along with the self-focusing nature of the waveguide.

6.3.2 $\chi^{(3)}$ based Multimode Noiseless Image Amplifiers

Recently $\chi^{(3)}$ based multi-mode noiseless image amplifiers have been demonstrated [58]. However, there seems to be no method to predict the eigenmodes of the system. We believe that this question can be explored by developing an equivalent theory for the Four-Wave-Mixing (FWM) based parametric amplification, similar to the $\chi^{(2)}$ media theory developed here in this thesis.
6.3.3 Chaos in PSA

Future work can look at the chaos and stability of the OPA as function of pump powers, spot size, and phase mismatch, by studying the PSA evolution in the total coupled signal-pump-idler equations instead of the undepleted approximation (unlike current work), even if it is mostly of academic interest. We can probe the deterministic chaotic dynamics of the OPA, in the pump depletion regime. It is expected that one can find unique pump, signal and idler power dependencies along the crystal evolution, and possible Quantum Image pattern formation mechanisms similar to the ones identified earlier [59].

6.4 Conclusion

The theory developments presented in this thesis, enable building $\chi^{(2)}$-based phase-sensitive image amplifiers and multimode squeezers with desired performance, and known limitations for engineering and science applications, such as Quantum Imaging, Quantum Metrology, Nonlinear Optics, etc. The ultimate possibility is to have a spatially- and spectrally- broadband image amplifier, i.e. PSA which can amplify images in several colors enabling us to build better cameras for optical telescopes, and amplifiers for gravitational-wave astronomy telescopes. We believe our work forms a small step towards this goal.
APPENDIX A

SIMPLIFYING OVERLAP INTEGRAL FOR HG AND LG MODES
A.1 HG-mode Overlap Integral for 0-order Pump

In the case of a Gaussian pump TEM$_{00}$ we have the fundamental HG mode with $q = 0$, $m = 0$ and allow us to study the coupling integral at $z$, $C_{p,r}^{l,n}(z) = \Lambda_{p,q=0,r}^{l,m=0,n}(z)$.

$$\Phi_{0,0}(\vec{r}, z, k_p, a_0) = \frac{\exp(-i\psi(z))}{\sqrt{\pi}a(z)} \exp\left(+ik_p\frac{(x^2 + y^2)}{2R(z)}\right) \exp\left(-\frac{(x^2 + y^2)}{2a^2(z)}\right) \quad (A.1)$$

$$\Phi_{p,l}(\vec{r}, z, k_s, \sqrt{2}a_0) = \frac{1}{\sqrt{2\pi}} \frac{\exp(-i[p + l + 1/2] \psi(z))}{\sqrt{2^{p+l}l!n!a^2(z)}} H_p\left(\frac{x}{\sqrt{2}a(z)}\right) H_l\left(\frac{y}{\sqrt{2}a(z)}\right) \times \exp\left(+ik_s\frac{(x^2 + y^2)}{2R(z)}\right) \exp\left(-\frac{(x^2 + y^2)}{4a^2(z)}\right) \quad (A.2)$$

$$C_{p,r}^{l,n}(z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi_{0,0}(\vec{r}, z, k_p, a_0)\Phi^*_{r,n}(\vec{r}, z, k_s, \sqrt{2}a_0)$$
$$\times \Phi^*_{p,l}(\vec{r}, z, k_s, \sqrt{2}a_0)dx \ dy \quad (A.3)$$

which can be separated after factoring in $k_p = 2k_s$. Both independent factors in the integral are similar and can be represented by taking

$$D_{l,n}(z) = \frac{1}{\sqrt{2\pi}} \frac{\exp(+i[l + n + 1/2] \psi(z))}{\sqrt{2^{l+n}l!n!a^2(z)}} \times \int_{-\infty}^{+\infty} H_l\left(\frac{y}{\sqrt{2}a(z)}\right) H_n\left(\frac{y}{\sqrt{2}a(z)}\right) \exp\left(-\frac{y^2}{a^2(z)}\right) dy \quad (A.4)$$
from which we can write the overlap integral as a product

\[ C_{p,r}^{l,n}(z) = \frac{1}{\sqrt{\pi a(z)}} D_{p,r}(z) D_{l,n}(z). \]  

(A.5)

The factor \( D_{p,r}(z) \) in the overlap integral form has a closed-form analytical expression obtained from [60], under substitution \( t = \frac{x}{\sqrt{2a(z)}} \). The simplified form of the factor \( D_{p,r}(z) \) follows

\[
D_{p,r}(z) = \begin{cases} 
\exp(\pm i[p+r+1/2]\psi(z)) \frac{(-1)^{p-r}(p+r-1)!!}{\sqrt{2^{p+r+1}p!}} & \text{if } p + r \text{ is even} \\
0 & \text{otherwise}
\end{cases}
\]  

(A.6)

Since the overlap integral has axial phase factors which do not depend on the transverse coordinates, we can write it as a product of overlap integral value at the waist plane \( z = 0 \) (where there is no axial phase contribution) and the axial phase factor.

\[
C_{p,r}^{l,n}(z) = \frac{C_{p,r}^{l,n}(z = 0)}{\sqrt{1 + (z/z_R)^2}} \exp(\pm i [p + r + l + n + 1] \psi(z))
\]  

(A.7)

A.2 Elliptical HG-mode Overlap Integral for 0-order Pump

The elliptical HG modes also factor similarly to standard HG modes and separate into 2 terms corresponding to each transverse mode. The overlap integral is

\[
C_{p,r}^{l,n}(z) = \frac{1}{\sqrt{\pi a_x(z)a_y(z)}} D_{p,r}^x(z) D_{l,n}^y(z),
\]  

(A.8)
where

\[
D_{l,n}^y(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\pm i \left[ l + n + 1/2 \right] \psi_y(z) \right) \left( \frac{2^l 2^n l! n! a_y^2(z)}{\sqrt{2} \pi} \right) \times \int_{-\infty}^{\infty} H_l \left( \frac{y}{\sqrt{2} a_y(z)} \right) H_n \left( \frac{y}{\sqrt{2} a_y(z)} \right) \exp \left( -\frac{y^2}{a_y^2(z)} \right) \, dy, \quad (A.9)
\]

and \( D_{p,r}^x(z) \) is obtained from above under subscript replacement \( y \to x \). The simplification of this integral is

\[
D_{p,r}^x(z) = \begin{cases} 
\exp(\pm i \left[ p + r + 1/2 \right] \psi_x(z)) \left( -1 \right)^{\frac{p-r}{2}} \frac{\psi_x(z)}{\sqrt{2}^{p+r+1} p! r!} & \text{if } p + r \text{ is even} \\
0 & \text{otherwise.} 
\end{cases} \quad (A.10)
\]

Now we can rewrite the overlap integral along any point on axial direction \( z \), in terms of its value at the focal plane \( z = 0 \):

\[
C_{l,n}^{l,n}(z) = \frac{C_{p,r}^{l,n}(z = 0)}{\sqrt{1 + (z/z_{Rx})^2}, \sqrt{1 + (z/z_{Ry})^2}} \times \exp(\pm i \left[ p + r + 1/2 \right] \psi_x(z)) \exp(\pm i \left[ l + n + 1/2 \right] \psi_y(z)). \quad (A.11)
\]

It is interesting to note that when \( a_{0x} = a_{0y} \), the coupling integral for elliptical modes Eq. A.11, becomes same as Eq. A.7, the coupling integral for the standard HG modes.

A.3 LG-mode Overlap Integral for 0-order Pump

For the LG modes the coupling integral has the same form as HG modes in Eq. 4.26, but only using LG-modes:

\[
\Lambda_{l,m,n}^{p,q,r} = \int \varphi_{p,l}(\vec{r}, z) \varphi_{r,m}^{\ast}(\vec{r}, z) \varphi_{q,m}(\vec{r}, z) d\vec{r}. \quad (A.12)
\]
When the pump is chosen to be cylindrically symmetric, \( q = 0, m = 0 \), then the selection rule becomes that \( m = l + n \) for a non-zero overlap integral, and under this condition leads to a requirement \( l = -n \):

\[
C_{p,r}^l(z) = \Lambda_{p,q=0,r}^{l,m=0,-l}(z)
\]

\[
= \int \varphi_{p,l}^*(\vec{r}, z, k_s, \sqrt{2}a_0) \varphi_{r,-l}^*(\vec{r}, z, k_s, \sqrt{2}a_0) \varphi_{0,0}(\vec{r}, z, k_p, a_0) d\vec{r}; \tag{A.13}
\]

Now the form of the pump, signal and idler factoring in \( C_{p,r}^l(z) \) are

\[
\varphi_{0,0}(r, \theta, z, k_p, a_0) = \frac{1}{a(z)\sqrt{\pi}} \exp \left( +i \left( \frac{k_p r^2}{2R(z)} - \psi(z) \right) \right) \exp \left( -\frac{r^2}{2a^2(z)} \right) \tag{A.14}
\]

\[
\varphi_{p,l}(r, \theta, z, k_s, \sqrt{2}a_0) = \sqrt{\frac{p!}{\sqrt{\pi}}} \frac{\exp (-i(2p + |l| + 1)\psi(z))}{2 \pi (p + |l|)!} \frac{1}{a(z)} \times \left( \frac{r}{\sqrt{2a(z)}} \right)^{|l|} \frac{L_p^{|l|}}{L_r^{|l|}} \exp \left( +i \frac{k_s r^2}{2R(z)} + il\theta - \frac{r^2}{4a^2(z)} \right). \tag{A.15}
\]

Evaluating further, after cancellation of the azimuthal phase factors we have

\[
C_{p,r}^l(z) = \frac{1}{\sqrt{\pi a(z)}} \sqrt{\frac{r|p|!}{(r + |l|)!(p + |l|)!}} \exp \left( +i \left[ 2(r + p + |l|) + 1 \right] \psi(z) \right)
\]

\[
\int_0^{+\infty} \left[ \left( \frac{r^2}{2a^2(z)} \right)^{|l|} L_p^{|l|} \left( \frac{r^2}{2a^2(z)} \right) L_r^{|l|} \left( \frac{r^2}{2a^2(z)} \right) \exp \left( -2 \left( \frac{r^2}{2a^2(z)} \right) \right) \right] d \left( \frac{r^2}{2a^2(z)} \right). \tag{A.16}
\]
The above equation resembles the standard integral form on replacing \( t = \frac{r^2}{2a^2(z)} \), which allows us to reduce \( C_{p,r}^l(z) \) as,

\[
C_{p,r}^l(z) = \begin{cases} 
\frac{1}{\sqrt{\pi}} \frac{1}{a(z)} \frac{(p+r+|l|)!}{\sqrt{p! \; r!(p+|l|)!(r+|l|)!}} \exp\left(\frac{+i[2(r+p+|l|)+1]\psi(z)}{2p+r+|l|+1}\right) & \text{if } l = -n \\
0 & \text{otherwise}
\end{cases}
\] (A.17)

Similar to the HG-mode case, we can write the general overlap integral as a product of the overlap integral value at focus plane \( z = 0 \) and an axial phase factor

\[
C_{p,r}^l(z) = \frac{C_{p,r}^l(z = 0)}{\sqrt{1 + (z/z_R)^2}} \exp\left(\frac{+i[2(r+p+|l|)+1]\psi(z)}{2p+r+|l|+1}\right). \tag{A.18}
\]

A.4 HG-mode Overlap Integral for Higher-Order Pump

The overlap integrals between the spatial modes of the pump, signal and idler define the strength of the 3-wave interaction in the nonlinear process.

Lugiato et-al [18] have shown generalized overlap integral between arbitrary order modes of the pump, signal and idler in the Laguerre-Gaussian (LG) basis. In our previous work, we have shown the similar overlap integral calculations for fundamental pump mode in the Hermite-Gaussian (HG) basis. This section documents the extension of the overlap integral to the general order pump mode overlap with arbitrary mode order of signal and idler.

Motivation of studying the higher order pump excitations is to allow model numerically many complicated settings of DOPA. The coupling integral in 2D factors as a product of the coupling integrals in each dimension, which makes the study of
1D integral sufficiently general. The 1D coupling integral at position $z = 0$ is given by

$$
\Lambda_{l,m,n}(z = 0) = \int \Phi^*_l(y, \sqrt{2}a_{0y}, \lambda_p) \Phi^*_n(y, \sqrt{2}a_{0y}, \lambda_s) \Phi_m(y, a_{0y}, \lambda_p) dy,
$$

(A.19)

where $\lambda_p = \lambda_s/2$, and the wavefronts are phase matched by choice of $a_{0ys} = \sqrt{2}a_{0y}$. Mode indices $m, l, n$ represent the pump, signal, and idler fields with the corresponding HG modes $\Phi$. Simplifying the above equation we get

$$
\Lambda_{l,m,n}(z = 0) = k_1 \int_{-\infty}^{\infty} \frac{\exp \left( -\frac{y^2}{2a_{0y}^2} \right) H_l \left( \frac{y}{\sqrt{2}a_{0y}} \right) H_n \left( \frac{y}{\sqrt{2}a_{0y}} \right) H_m \left( \frac{y}{a_{0y}} \right)}{\pi^{3/4} a_{0y}^{3/2} 2^{(m+n+l+1)/2} \sqrt{m! n! l!}} dy.
$$

(A.20)

Upon converting to a standard form using $t = \frac{y}{\sqrt{2}a_{0y}}$ we write

$$
\Lambda_{l,m,n}(z = 0) = k_1 \int_{-\infty}^{\infty} H_l(t) H_n(t) H_m(t\sqrt{2}) \exp \left( -2t^2 \right) dt,
$$

(A.21)

where

$$
k_1 = \sqrt{2}a_{0y} \left( \pi^{3/4} a_{0y}^{3/2} 2^{(i+n+m+1)/2} \sqrt{i! n! m!} \right)^{-1}.
$$

(A.22)

Using the following identity to express the Hermite polynomial of scaled argument in terms of products of Hermite polynomials of different orders,

$$
2^{n/2} H_n \left( \sqrt{2y} \right) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(y) H_k(y),
$$

(A.23)
and the Feldheims identity relating product of 2 Hermite polynomials to a series sum of several Hermite polynomials of different orders,

$$H_m(y) H_n(y) = m! n! \sum_{t=0}^{\min(m,n)} \frac{2^t H_{m+n-2t}(y)}{t! (m-t)! (n-t)!}, \quad (A.24)$$

we obtain a form of overlap integral which has a analytic closed form expression. The closed form integral product of three Hermite polynomials of different orders provided in [60], exists when \(l+n+m\) is even, with \(l+n \geq m\), and \(s = (l+n+m+1)/2\):

$$\int_{-\infty}^{\infty} \exp \left(-2y^2\right) H_l(y) H_n(y) H_m(y) dy = \frac{1}{\pi} 2^{(l+n+m-1)/2} \Gamma(s-l) \Gamma(s-n) \Gamma(s-m), \quad (A.25)$$

and overlap integral is 0 for other odd combinations of \(l, n, m\). After some algebra, these results reduce the overlap integral to

$$\Lambda_{l,m,n}(z = 0) = \frac{\sqrt{2} a_{0y} l! n!}{\pi^{3/4} a_{0y}^{3/2} 2^{(l+n+2m+1)/2} \sqrt{l! n! m!}} \times \sum_{t=0}^{\min(l,n)} \sum_{k=0}^{m} \binom{m}{k} \frac{2^t J_{mln}^{lk}}{t! (l-t)! (n-t)!} \quad (A.26)$$

where \(r = (1+l+n+m-2t)/2\) and

$$J_{mln}^{lk} = \frac{2^{(l+n+m-2t-1)/2}}{\pi} \Gamma(-k+r) \Gamma(k-m+r) \Gamma(-l-n+r+2t), \quad (A.27)$$

when \((l+n+m-2t)\) is even and 0, otherwise. Also the odd-parity of Hermite polynomials forces \(\Lambda_{l,m,n}\) to vanish when \(l+n+m\) is odd. Since \(l+n+m\) is always
even for non-zero $\Lambda_{l,m,n}$, and $2t$ is also even, $J_{\text{min}}^{nk}$ remains non-zero in the summation. Simplifying further, we get,

$$\Lambda_{l,m,n}(z = 0) = \frac{\sqrt{l! n!}}{\pi^{7/4} a_0 y^{2m+1} m!} \left( \min(l,n) \sum_{t=0}^{m} \sum_{k=0}^{m-k} \binom{m}{k} \frac{\Gamma(-k+r)\Gamma(k-m+r)\Gamma(-l-n+r+2t)}{t! (l-t)! (n-t)!} \right), \tag{A.28}$$

where $r = (1 + l + n + m - 2t)/2$. For $z \neq 0$ the overlap integral just picks up a complex phase and is given by

$$\Lambda_{l,m,n}(z) = \frac{\sqrt{l! n!}}{\pi^{7/4} a_0 (z) y^{2m+1} m!} \exp \left[ +i (l + n - m + 1/2) \psi_y(z) \right] \left( \min(l,n) \sum_{t=0}^{m} \sum_{k=0}^{m-k} \binom{m}{k} \frac{\Gamma(-k+r)\Gamma(k-m+r)\Gamma(-l-n+r+2t)}{t! (l-t)! (n-t)!} \right). \tag{A.29}$$

Further,

$$\Lambda_{l,m,n}^2 = \frac{\exp \left[ +i (l + n - m + 1/2) \psi_y(z) \right] \Lambda_{l,m,n}(z = 0)}{\sqrt{1 + (z/Z_{Ry})^2}} \Lambda_{l,m,n}(z = 0), \tag{A.30}$$

which gives the total overlap integral for higher order pump in the fixed scale spot size ($\sqrt{2}$) case to be

$$C_{l,m,n}^{p,q,r}(z) = \Lambda_{p,q,r}(z) \times \Lambda_{l,m,n}(z). \tag{A.31}$$

### A.5 LG-mode Overlap Integral for Higher-Order Pump

In this section we compute the overlap integral, for signal, and idler, with a higher order Laguerre-Gaussian (LG) mode pump. The overlap integrals between
the spatial modes of the pump, signal and idler define the strength of the 3-wave interaction in the nonlinear process.

We calculate the higher order general LG pump mode overlap integrals (even though [18] gives higher order LG pump, but with restriction to case of azimuthal index 0, with a flat phase) with fully general phase-structured pump beam. This might have potential applications in a non uniform phase-structured pump along with amplitude modulations in the transverse pump profile. Moreover a phase-structured pump can have a compact representation in the higher order generalized LG mode compared to say Hermite-Gaussian basis, with complex coefficients.

We use the pump mode \((q, m)\), signal \((p, l)\) and idler \((r, n)\) in the LG basis. The notations of the LG mode follows from previous forms.

\[
\varphi_{q,m}(r, \theta, z, k_p, a_0) = \frac{1}{a(z)} \sqrt{\frac{q!}{\pi(q + |m|)!}} \exp \left[ -i(2q + |m|) \psi(z) \right] \times \left( \frac{r}{a(z)} \right)^{|m|} L_q^{|m|} \left( \frac{r^2}{a^2(z)} \right) \times \exp \left[ -i k_p r^2 \frac{1}{2R(z)} + i m \theta - \frac{r^2}{2a^2(z)} \right], \quad (A.32)
\]

\[
\varphi_{p,l}(r, \theta, z, k_s, \sqrt{2a_0}) = \frac{1}{a(z)} \sqrt{\frac{p!}{2\pi(p + |l|)!}} \exp \left[ -i(2p + |l|) \psi(z) \right] \times \left( \frac{r}{\sqrt{2a(z)}} \right)^{|l|} L_p^{|l|} \left( \frac{r^2}{2a^2(z)} \right) \times \exp \left[ -i k_s r^2 \frac{1}{2R(z)} + i l \theta - \frac{r^2}{4a^2(z)} \right], \quad (A.33)
\]

\[
\varphi_{r,n}(r, \theta, z, k_s, \sqrt{2a_0}) = \frac{1}{a(z)} \sqrt{\frac{r!}{2\pi(r + |n|)!}} \exp \left[ -i(2r + |n|) \psi(z) \right] \times \left( \frac{r}{\sqrt{2a(z)}} \right)^{|n|} L_r^{|n|} \left( \frac{r^2}{2a^2(z)} \right) \times \exp \left[ -i k_s r^2 \frac{1}{2R(z)} + i n \theta - \frac{r^2}{4a^2(z)} \right]. \quad (A.34)
\]
The overlap integral $C_{p,q,r}^{l,m,n}(z)$ is calculated by evaluating

$$C_{p,q,r}^{l,m,n}(z) = \int_0^{2\pi} \left[ \int_0^{2\pi} \phi_{p,l}^*(r,\theta,z) \phi_{r,n}^*(r,\theta,z) \phi_{q,m}(r,\theta,z) r dr \right] d\theta. \quad (A.35)$$

We find the selection rule, for non-zero overlap to be limiting the azimuthal mode index of the three beams as $m = l + n$. Carrying out the simplifications we arrive at the intermediate result at plane $z = 0$,

$$C_{p,q,r}^{l,m,l+n,n}(z = 0) = \left[ \sqrt{\frac{2|l|+|n|}{\pi a_0^2}} \frac{p! q! r!}{(p + |l|)! (q + |m|)! (r + |n|)!} \right]$$

$$\times \int_0^\infty \left[ \exp(-2t) t^{|l|} L_p^{|l|}(t) L_r^{|n|}(t) L_q^{|m|}(2t) \right] dt. \quad (A.36)$$

We simplify this integral expressing the Laguerre polynomials as power series, using
the Rodrigues formula successively and then use the Gamma function closed forms [60],

\[
L_\alpha^\beta(x) = \sum_{m=0}^{\infty} \left[ (-1)^m \binom{n + \alpha}{n - m} \frac{x^m}{m!} \right],
\]

(A.37)

\[
\Gamma(x) = \int_0^\infty \exp(-t) \, t^{x-1} \, dt,
\]

(A.38)

which leads to the closed form,

\[
C_{p,q,r}^{l,n,n}(z = 0) = \frac{1}{\sqrt{2|l|+|n|+2} \sqrt{\pi a_0}} \sqrt{\frac{p! \, q! \, r!}{(p + |l|)(q + |m|)(r + |n|)!}}
\times \sum_{w_0=0}^r \sum_{w_1=0}^p \sum_{w_2=0}^q \frac{(-1)^{w_0+w_1+w_2}}{w_0! \, w_1! \, w_2!} \left( \binom{r + |n|}{w_0+w_1+w_2} \binom{r - w_0}{p-w_1} \binom{q + |m|}{q-w_2} \Gamma\left( \frac{|l|+|n|+|m|}{2} + w_0 + w_1 + w_2 + 1 \right) \right). \tag{A.39}
\]

The complete overlap integral, is given by

\[
C_{p,q,r}^{l,n,n}(z) = \frac{C_{p,q,r}^{l,n,n}(z = 0)}{\sqrt{1 + (z/Z_R)^2}} \times \exp\left( +i \left[ 2(r + p - q + (|l| + |n| - |m| + 1)/2) \right] \psi(z) \right). \tag{A.40}
\]

For value of \(q = 0, m = 0\), it reduces to the case of fundamental pump overlap integral (previously derived) as,

\[
C_{p,r}^{l,0,0}(z) = \frac{1}{\sqrt{\pi a(z)}} \frac{(p + r + |l|)!}{\sqrt{p! \, r!(p + |l|)!(r + |l|)!}} \frac{\exp\left( +i \left[ 2(r + p + |l|) + 1 \right] \psi(z) \right)}{2^{p+r+|l|+1}} \tag{A.41}
\]

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A.6 HG-mode Overlap Integral for General Scaled Beam Waist and 0-order Pump 

For choice of scaling signal spot size \( a_s = f_s a_p \), typically with \( f_s < 1 \) (for elliptical HG beams \( x,y \) subscripts attached to \( a_p, a_s, f_s \) respectively) the overlap integrals need to be treated in the following way to achieve closed form expressions. Due to the general scale \( f_s \) we have a nonzero \( z \)-dependent complex phase (vanishes for \( f_s = \sqrt{2} \)). Since no closed form solutions exist for this case, we tackle this situation by pre-calculating the discrete convolution of the scaled Hermite polynomials (equivalent to product of two Hermite polynomials). Even if we cannot eliminate \( z \)-dependence of the overlap integral, using this method we can at least replace their numerical evaluation with an exact analytical formula.

For the coupling integral in the case of a Gaussian pump TEM\(_{00} \) we have the fundamental HG mode with \( q = 0, m = 0 \) and allow us to study the coupling integral at \( z \), \( C_{l,n}^{p,r}(z) = \Lambda_{l,m=0,n}^{p,q=0,r}(z) \). We assume the signal spot sizes along X, Y directions are scaled by \( f_{sx} \) and \( f_{sy} \) over the corresponding pump spot sizes.

\[
C_{p,r}^{l,n}(z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi_{0,0}(\vec{r}, z, k_p, a_{px}, a_{py}) \Phi_{r,n}^{*}(\vec{r}, z, k_s, f_{sx} a_{px}, f_{sy} a_{py}) \\
\times \Phi_{p,l}(\vec{r}, z, k_s, f_{sx} a_{px}, f_{sy} a_{py}) \ dx \ dy, \quad (A.42)
\]

which can be separated after factoring in \( k_p = 2k_s \) and using resolvent nature of the HG-modes as separate products of X, Y dependent variables with indices \( (p, r) \) and \( (l, n) \) respectively

\[
C_{p,r}^{l,n}(z) = D_{p,r}(z)D_{l,n}(z). \quad (A.43)
\]

Unlike \( \sqrt{2} \) scaled case earlier, the general scaled overlap integral has axial phase factors longitudinal distance, and cannot be written as a product of overlap integral
value at the focal plane \( z = 0 \) (where there is no axial phase contribution) and the axial phase factor.

Both the independent factors in the integral are similar and can be represented by (for \((l, n)\) replace \(x\) with \(y\))

\[
D_{p,r}(z) = \frac{1}{\pi^{3/4}} \frac{\exp \left( \frac{-i \left[ \psi_{px}(z)/2 - \left( p + r + 1 \right) \psi_{sx}(z) \right]}{\sqrt{2^p 2^r p! r! a_{sx}^2(z) a_{px}(z)}} \right)}{\sqrt{2^p 2^r p! r! a_{sx}^2(z) a_{px}(z)}} \int_{-\infty}^{+\infty} \left[ \frac{x}{f_{sx} a_{px}(z)} \right] \frac{x}{f_{sx} a_{px}(z)} \exp \left( \frac{+i k_p a^2}{2} \left( \frac{1}{R_{px}(z)} - \frac{1}{R_{sx}(z)} \right) \right) \times \exp \left( -\frac{x^2}{2 a_{px}(z)^2} - \frac{x^2}{a_{sx}(z)^2} \right) \right] dx. \tag{A.44}
\]

The discrete convolution of Hermite polynomials of order \( p, l \) represents their product, through the \( p + l + 1 \) coefficients starting from \( c_0 \) to \( c_{p+l} \)

\[
H_p(x) H_l(x) = \sum_{i=0}^{p+l} c_i x^i. \tag{A.45}
\]

Using this fact we can evaluate the following integral which forms the RHS of Eq. A.44,

\[
\int_{-\infty}^{+\infty} H_p(x) H_r(x) \exp \left( -x^2 \xi_x(z) \right) dx, \tag{A.46}
\]

where \( \xi_x(z) = \delta_x(z) - i \gamma_x(z) \), \( \delta_x(z) = a_{sx}(z)^2 / 2 a_{px}(z) \) + 1, \( \gamma_x(z) = \frac{k_p a_{sx}^2(z)}{2} \left( \frac{1}{R_{px}(z)} - \frac{1}{R_{sx}(z)} \right) \). \tag{A.47}

We note that \( \xi_y(z) \) is also defined identically with \( y \)-dependent spot size and curvature.
Replacing Eq. A.45 into Eq. A.46, we can express the integral of sums, as a sum of integrals, each of which is of the form

\[ \int_{-\infty}^{+\infty} c_i x^i \exp\left(-x^2 \xi_x(z)\right) \, dx, \]  

(A.48)

where \( c_i \) is the i-th coefficient of discrete convolution. These new integrals have closed forms which are scaled by the pre-calculated coefficient \( c_i \), as long as \( \Re(\xi_x(z)) > 0 \) or \( \delta_x(z) > 0 \),

\[ \int_{-\infty}^{+\infty} x^i \exp\left(-x^2 \xi_x(z)\right) \, dx = \begin{cases} (\xi_x(z))^{-(i+1)/2} \Gamma\left(\frac{i+1}{2}\right) & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases} \]  

(A.49)

We can write the factor \( D_{p,r}(z) \) in the overlap integral form as a closed form analytical expression

\[ D_{p,r}(z) = \frac{1}{\pi^{3/4}} \frac{\exp\left(-\frac{1}{2} [\psi_{px}(z)/2 - (p + r + 1) \psi_{sx}(z)]\right)}{\sqrt{2^{p+r} p! r! a_{0px} \sqrt{1 + (z/Z_{Rpx})^2}}} \left[ \sum_{i=0,2,4,...}^{i \leq p+r} c_i \times (\xi_x(z))^{-(1+i)/2} \Gamma\left(\frac{i + 1}{2}\right) \right], \]  

(A.50)

where \( c_i \) are the discrete convolution coefficients of \( H_p(x)H_r(x) \). A similar equation can be obtained for \( D_{t,n} \) by substitution.
The complete overlap integral is given by,

\[
C_{l,n}^{p,r}(z) = \frac{\exp(-i [\psi_{px}(z)/2 - (p + r + 1)\psi_{sx}(z)])}{\sqrt{\pi a_{0px} a_{0py}} \sqrt{1 + (z/Z_{Rpx})^2} \sqrt{1 + (z/Z_{Rpy})^2} \sqrt{\pi} \sqrt{2^p 2^r p! r!}} \times \left[ \sum_{i=0,2,4,...}^{i \leq p+r} c_i \times (\xi_x(z))^{-(1+i)/2} \Gamma((i + 1)/2) \right] \times \frac{1}{\sqrt{\pi}} \frac{\exp(-i [\psi_{py}(z)/2 - (l + n + 1)\psi_{sy}(z)])}{\sqrt{2^l 2^n l! n!}} \times \left[ \sum_{k=0,2,4,...}^{k \leq l+n} d_k \times (\xi_y(z))^{-(1+k)/2} \Gamma((k + 1)/2) \right],
\]  

(A.51)

where \(c_i\) are the discrete convolution coefficients representing product \(H_p(x)H_r(x)\), and \(d_k\) are the discrete convolution coefficients representing the product \(H_l(y)H_n(y)\) respectively.

A.7 HG-mode Overlap Integral for General Scaled Beam Waist and Higher Order Pump

Knowledge of coupling integral in the case of a higher order Gaussian pump \(HG_{q,m}\) with HG modes \(q, m\) along X, Y axis, \(\Lambda_{l,m,n}^{p,q,r}(z)\) at \(z\), allows us to study the effect of increasing pump order on the PSA eigenmodes & spatial bandwidth. We assume the signal spot sizes along X, Y directions are scaled by \(f_{sx}\) and \(f_{sy}\) over the corresponding pump spot sizes.

\[
\Lambda_{l,m,n}^{p,q,r}(z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi_{q,m}(\vec{r}, z, k_p, a_{px}, a_{py}) \Phi_{r,n}^*(\vec{r}, z, k_s, f_{sx}a_{px}, f_{sy}a_{py}) \Phi_{p,l}^*(\vec{r}, z, k_s, f_{sx}a_{px}, f_{sy}a_{py}) d\vec{x} d\vec{y} \quad (A.52)
\]
which can be factorized as

$$\Lambda_{p,q,r}^{l,m,n}(z) = D_{p,q,r}(z)D_{l,m,n}(z).$$  \hspace{1cm} (A.53)

Both independent factors in the integral are similar and can be represented by, (for \((l, m, n)\) replace \(x\) with \(y\))

$$D_{p,q,r}(z) = \frac{1}{\pi^{3/4}} \exp \left(-i \left[(q+1)\psi_{px}(z)/2 - (p+ r + 1)\psi_{sx}(z)\right]\right) \sqrt{2^{p+q+r}p!q!r!a_{sx}^2(z)a_{px}^2(z)}$$

$$\int_{-\infty}^{+\infty} \left[H_q \left(\frac{x}{a_{px}(z)}\right) H_p \left(\frac{x}{f_{sx}a_{px}(z)}\right) H_r \left(\frac{x}{f_{sx}a_{px}(z)}\right) \exp \left(-\frac{x^2}{2a_{px}^2(z)} - \frac{x^2}{a_{sx}^2(z)}\right)\right] dx. \hspace{1cm} (A.54)$$

Again we use the discrete convolution of Hermite polynomials of order \(p,q,l\) to represent their product, through the \((p+r+1) + (q+1) - 1 = p+q+r+1\) coefficients, starting from \(c_0\) to \(c_{p+q+r}\). Using this fact we can evaluate the following integral which forms the RHS of Eq. A.54,

$$\int_{-\infty}^{+\infty} H_q \left(\frac{x}{a_{px}(z)}\right) H_p \left(\frac{x}{f_{sx}a_{px}(z)}\right) H_r \left(\frac{x}{f_{sx}a_{px}(z)}\right) \exp \left(-x^2\xi_x(z)\right) dx,$$

where \(\xi_x(z) = \delta_x(z) - \gamma_x(z)\), \hspace{1cm} (A.55)

$$\delta_x(z) = \frac{a_{sx}^2(z)}{2a_{px}^2(z)} + 1, \quad \gamma_x(z) = \frac{k_p a_{sx}^2(z)}{2} \left(\frac{1}{R_{px}(z)} - \frac{1}{R_{sx}(z)}\right).$$

We note that \(\xi_y(z)\) is also defined identically with \(y\)-dependent spot size and curvature while calculating \(D_{l,m,n}(z)\).

First we convert the Hermite polynomials to be of the same argument, which makes the product of the three Hermite polynomials as a sum of products, which is
equivalent to a polynomial of degree \( p + q + r \) whose \( p + q + r + 1 \) coefficients we can pre-compute.

\[
H_q \left( \frac{x}{a_{px}(z)} \right) \ H_p \left( \frac{x}{f_{sx}a_{px}(z)} \right) \ H_r \left( \frac{x}{f_{sx}a_{px}(z)} \right) = \sum_{k=0}^{q} \left( \begin{array}{c} q \\ k \end{array} \right) H_k \left( \frac{x}{f_{sx}a_{px}(z)} \right) \\
\times H_p \left( \frac{x}{f_{sx}a_{px}(z)} \right) H_r \left( \frac{x}{f_{sx}a_{px}(z)} \right) \left( \frac{2(f_s - 1)}{f_s} \frac{x}{a_{px}(z)} \right)^{q-k} \\
= \sum_{i=0}^{p+q+r} c_i(z) x^i \quad (A.56)
\]

Using this expansion we can replace the product of the Hermite polynomials by the discrete convolution into Eq. A.55, and we can express the integral of sums, as a sum of integrals for which a closed form was known.

We then can write the factor \( D_{p,q,r}(z) \) in the overlap integral form as a closed form analytical expression

\[
D_{p,q,r}(z) = \frac{1}{\pi^{3/4}} \frac{\exp \left( -i \left[ (q + 1)\psi_{px}(z)/2 - (p + r + 1)\psi_{sx}(z) \right] \right)}{\sqrt{2^n 2^{n} p! q! r! a_{0px} \sqrt{1 + (z/Z_{Rpx})^2}}} \\
\times \left[ \sum_{i=0,2,4,...}^{i \leq p+q+r} c_i(z) \times (\xi_{x}(z))^{-(1+i)/2} \Gamma((i + 1)/2) \right] , \quad (A.57)
\]

where \( c_i(z) \) are the discrete convolution coefficients representing the RHS of Eq.A.56. A similar equation can be obtained for \( D_{l,m,n} \) by appropriate substitution.
The complete overlap integral is given by,

\[
\Lambda_{p,q,r}^{l,m,n}(z) = \frac{\exp(-i[(g+1)\psi_{pz}(z)/2 - (p+r+1)\psi_{sz}(z)])}{\sqrt{\pi}a_{0x}a_{0y}4\Gamma((i+1)/2)}
\times \left[ \sum_{i=0,2,4,...} c_{i}(z) \times (\xi_{x}(z))^{-(1+i)/2}\right]
\times \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{1 + (z/Z_{Rpx})^2}} \frac{1}{\sqrt{1 + (z/Z_{Rpy})^2}} \frac{1}{\sqrt{2^{l+m+n}l!m!n!a_{sz}^2(z)}}
\times \left[ \sum_{k=0,2,4,...} d_{k}(z) \times (\xi_{y}(z))^{-(1+k)/2}\right],
\]

where \(c_{i}(z)\) and \(d_{k}(z)\) are the discrete convolution coefficients representing products of Hermite polynomials in the X, Y directions.
APPENDIX B

NUMERICAL MODELING OF PSA WITH UNDEPLETED PUMP USING SPLIT STEP FOURIER METHOD
B.1 Introduction

Signal pulse evolution through an Optical Parametric Amplifier (OPA) is known only for the plane wave pump approximation, and not for the case of a Gaussian beam profile of the pump, in the non-degenerate and the degenerate OPA. In this case, the signal pulse evolution can only be numerically calculated using the pulse evolution equation.

In this report, the calculation of signal pulse evolution in degenerate OPA under constant pump approximation with an arbitrary pump wave front (phase profile) using the split-step Fourier method is presented. The split step Fourier method takes after the similar modeling of the nonlinear Schrödinger equation (NLSE). NLSE is used to calculate the spatial evolution of a time-domain soliton equation in the optical fiber.

B.2 Solving NLSE using Split-Step Fourier Method

The NLSE for the optical pulse evolution across the length of the fiber is

\[
\frac{\partial \vec{A}}{\partial z} = \frac{i}{2} \beta_2 \frac{\partial^2 \vec{A}}{\partial t^2} + i\gamma |\vec{A}|^2 \vec{A} \tag{B.1}
\]

where \( \vec{A} = A(z, t) \) is the amplitude of the optical pulse at any point on the optical fiber and at any time instant normalized to the group velocity of the pulse in the propagation media, since the launch of the pulse. The above equation (B.1) is written in an operator notation,

\[
\frac{\partial \vec{A}}{\partial z} = \left[ \hat{D} + \hat{N} \right] \vec{A} \tag{B.2}
\]

where the linear

\[
\hat{D}\vec{A} = \frac{i}{2} \beta_2 \frac{\partial^2 \vec{A}}{\partial t^2} \hat{N}\vec{A} = i\gamma |\vec{A}|^2 \vec{A} \tag{B.3}
\]
and the nonlinear operators are separated.

Treating the system as a ODE in the spatial variable we can write the solution for the NLSE in

\[ \vec{A}(z + dz, t) = \vec{A}(z, t) \exp(\hat{D} z) \exp(\hat{N} z). \] (B.5)

Next step calculates separately the nonlinear and the linear operators using,

\[ \vec{U}(z + dz/2, t) = \vec{A}(z, t) \exp(\hat{D} z) \] and \[ \vec{A}(z + dz, t) = \vec{U}(z + dz/2, t) \exp(\hat{N} z). \] (B.7)

By calculating the eq (B.6) in the fourier domain using a fourier transform applied

\[ \hat{D} \vec{A}(z, \omega) = i \frac{\beta_2}{2} \omega^2 \vec{A} \] (B.8)

to eq (B.3).

B.3 Degenerate Optical Parametric Amplifier

Calculation of Optical Parametric Amplifier signal pulse evolution in constant pump approximation using Split Step Fourier Method.
The calculation of signal pulse evolution in a degenerate Optical Parametric Amplifier in constant pump approximation using Split Step Fourier Method by the Beam Propagation Method is presented in this report.

B.4 Paraxial Helmholtz Equation with parametric gain

\[ \frac{\partial U_s}{\partial z} = \frac{i\lambda_s}{4\pi n_s} \nabla_p^2 U_s + i \frac{4\pi d_{\text{eff}}}{n_s \lambda_s \sqrt{2\varepsilon_0 n_p c}} U_p U_s^* \exp(i\Delta k z) \]  

(B.9)

where we can define

\[ a = \frac{4\pi d_{\text{eff}}}{n_s \lambda_s \sqrt{2\varepsilon_0 n_p c}}, \]  

(B.10)

and normalization is such that \(|U_p|^2 = I_p(W/m^2)\).

B.5 Split-Step Fourier Beam Propagation Method

Let \( V_s = \mathbf{F}T\{U_s\} \) be the Fourier transform relation for the field.

B.5.1 Free-Space propagation in frequency domain

\[ \frac{\partial V_s}{\partial z} = -\frac{i\lambda_s q^2}{4\pi n_s} V_s \]  

(B.11)

\[ V_s(q, z) = V_s(q, z') \exp\left(-\frac{i\lambda_s q^2}{4\pi n_s} (z - z')\right) \]  

(B.12)

The first step solves the Helmholtz equation in the Fourier domain.
B.5.2 Parametric gain in space-domain

\[ \frac{\partial U_s}{\partial z} = i a U_p(\vec{\rho}, z) \exp (i \Delta k z) U^*_s \]  

(B.13)

where we can define,

\[ \kappa(\vec{\rho}, z) = a |U_p(\vec{\rho}, z)| \]  

(B.14)

\[ \psi(\vec{\rho}, z) = \arg[U_p(\vec{\rho}, z)] + \Delta k z \]  

(B.15)

For \( \kappa, \psi \) independent of \( z \) the analytical solution is

\[ U_s(\vec{\rho}, z = L) = U_s(\vec{\rho}, 0)cosh(\kappa L) + iU^*_s(\vec{\rho}, 0) \exp (i\psi)sinh(\kappa L). \]  

(B.16)

Checking this solution one can obtain

\[ \frac{\partial U_s}{\partial z} = i \kappa \exp (i\psi)U^*_s \]

B.5.3 FFT-BPM algorithm

1. We assume longitudinal discretization

\[ dz = \frac{L}{N}, z_n = ndz, \text{where } n = 0, 1, \ldots N - 1 \]  

(B.17)

with \( z_0 = 0, z_L = L \).

2. \( U_s(z_o = 0) \) is the input, and \( V_s(z_o) \) is the corresponding Fourier transform.

Half-step back in frequency domain:

\[ V_s^{NL}(z_o) = V_s(z_o) \exp \left( \frac{i \lambda_s q^2 dz}{4 \pi n_s 2} \right) \]  

(B.18)
3. Full step forward in frequency domain:

\[ V_s^L(z_n + \frac{dz}{2}) = V_s^{NL}(z_n) \exp\left( -\frac{i\lambda_s q^2}{4\pi n_s} dz \right) \]  \hspace{1cm} (B.19)

4. Full step forward in space domain:

\[ U_s^{NL}(z_{n+1}) = U_s^L \left( z_n + \frac{dz}{2} \right) \cosh \left[ \kappa \left( z_n + \frac{dz}{2} \right) dz \right] + \\
\times iU_s^{L*} \left( z_n + \frac{dz}{2} \right) \exp \left( i\psi(z_n + \frac{dz}{2}) \right) \\
\times \sinh \left[ \kappa(z_n + \frac{dz}{2}) dz \right] \]  \hspace{1cm} (B.20)

5. Repeat previous two steps for \( n = 0, 1, 2, \ldots (N - 1) \).

6. Half-step forward in frequency domain:

\[ V_s(z_N) = V_s^{NL}(z_N) \exp \left( -\frac{i\lambda_s q^2}{4\pi n_s} \frac{dz}{2} \right) \]  \hspace{1cm} (B.21)

7. \( V_s(z_N) \) is the output \( (z_N = L) \).
APPENDIX C

COMPUTATIONAL ISSUES IN SOLVING PSA COUPLED ODE SYSTEM
C.1 Numerical Solver Codes

The results presented in this thesis were calculated by numerical solvers (one for each variant of problem), implemented in Matlab [61] and GNU Octave [62], independently or together with Mathematica [63].

Four main types of solvers to find PSA eigenmodes were developed:

1. **MV9**: PSA is pumped by fundamental HG\(_{0,0}\) elliptical pump, with signal-to-pump spot size ratio of \(\sqrt{2}\).

2. **MV10**: Same as **MV9**, but with a general scale factor between signal-to-pump spot size; the overlap integrals are calculated using pre-computed data from Mathematica to avoid numerical overflow issues.

3. **MV11**: PSA is pumped by higher-order HG\(_{q,m}\) pump beam, with signal-to-pump spot size ratio of \(\sqrt{2}\); the overlap integrals are calculated using pre-computed data from Mathematica to avoid numerical overflow issues.

4. **lgmodesolve**: Same as **MV9**, but analyzed in LG basis.

C.2 Computational Challenge

Calculating the Green’s functions of the DOPA using the coupled-mode formalism becomes intractable when implemented naively, as suggested by equations in Chapter 3. We address issues in developing a solver, that provide large factor improvements in running time, and memory usage compared to the naive scheme. Accurate calculation of the overlap integral coefficient \(B_{mm'}\) for large Hermite-Gaussian modes \(m, m'\) requires careful attention. Such procedures contribute to a fast, and accurate modeling of PSA for practical pump powers, spot sizes requiring large set of HG modes.
C.2.1 Optimizing Green’s Function Calculation

Green’s function of the DOPA provide quantitative information of the modes excited at the output for each mode at input for both the quadratures. Green’s function is theoretically represented as a 4D matrix, with 2D matrix output modes for each of the $M_x M_y$ modes excited at input. However, for calculating the eigenmodes and squeezing spectrum we need to re-index the complex-valued 4D tensor $G$ with dimensions $M_x \times M_y \times M_x \times M_y$, into a 2D matrix with dimensions $(M_x M_y) \times (M_x M_y)$; we also re-index the 2D input matrix of dimension $M_x \times M_y$ into a 1D vector of dimension $M_x M_y$.

With the coupling tensor re-indexed into a 2D matrix $E$ as well, we have

$$\frac{dD(z)}{dz} = k_1 E(z) D^*(z), \quad (C.1)$$

where $D$ is a 1D vector. In the linear ODE system C.1, transfer matrix evolution is independent of initial conditions, and we can propagate all the initial conditions required for Green’s functions at once. By transforming $D$ from 1D vector of length $M_x M_y$, into a 2D matrix of dimension $M_x \times M_y$ with $D(0) = I$, propagating this matrix $D$ over crystal length in the ODE, we will have obtained the total Green’s function. To illustrate the improvement in performance, we could run a case with 32x32 HG-modes in under half-hour, which formerly took a full day on 1GHz a desktop computer. Of course, we do very well in less than few minutes with 8-CPU computer with 96GB RAM.

C.2.2 Block Structure Toeplitz matrix

Block structured Toeplitz matrices are found in the Green’s function and evolution transfer matrices during calculation, as an artifact of the selection rule $\text{mod}(m + \ldots)$.
Figure C.1. Block structured Toeplitz matrix form of Green’s function obtained for 16x16 HG mode case. Hatched regions in blue are filled with non-zero elements, while white blocks are 0s. This matrix is only 25% filled.

Figure C.2. Eigenvalue spectrum for 16x16 HG mode case with pump power 2kW and spot size 25x25µm².
\( m', 2 \) = 0, of coupling integral \( B_{mm'} \), and excitation by a delta input at each mode which preserves this matrix structure at each of the longitudinal steps in our ODE solver. The image in Fig. C.1, shows the sparse nature and block structure of this matrix, which allows us to decompose it into components which are relatively dense, and obtain storage (memory) and computation speed improvements upto a constant factor.

For choice of \( M_x \) and \( M_y \) to be even numbers, we find transfer matrix in quadrature representation can be decomposed into 4 distinct block matrices which are 50\% filled, and evolve independently of each other. Data parallelism is using multiple CPU’s to process different data sets on same code, simultaneously. This is an instance of explicit data parallelism, as the 4 sets of data have to be integrated by the same kind of program. Thus our 4 block matrices are integrated simultaneously to provide a 4\times improvement in the integration loop computation time compared to standard case, while the storage requirements go down by a similar factor of 2. This essentially extends the reach of our problem size, using limited resources which otherwise would not be solvable in a reasonable amount of time or available memory.

C.2.3 Numerical Accuracy

Coupling integral \( B_{mm'} \) calculation is complicated by the overflow of an intermediate value (the double factorial calculation) for higher mode orders, typically for \( m' + m \geq 160 \), even if the final value is within the range of double precision. The double precision, 8-byte representation used in the calculations can represent real numbers upto \( \approx 1.7977 \times 10^{308} \), after which a numerical overflow occurs. In this particular case we have overcome the problem by resorting to calculations in the logarithmic domain; using the numerical routine, \( \text{gammaln} \) (calculates \( \ln(\Gamma(x)) \)) function which can be used to evaluate logarithm of factorials and hence the double factorial.
Figure C.3. Coupling integral calculations with and without overflow; (top) with numerical overflow, restricting accurate calculation of $B_{m,m}$ to values of $m \leq 80$, (bot) accurately calculated in logarithmic domain.
To illustrate the differences we have plotted the coupling integral for sum of mode indices ($m = m'$), using routines with and without overflow errors in Fig.C.3.

C.2.4 Mode Field Visualization

Spatial mode field calculations in (x,y)-domain also suffer from numerical overflow, since higher order Hermite polynomials have large coefficients. The only way around this problem seems to be using higher precision arithmetic or calculating mode patterns in symbolic packages like Mathematica.

Right now our approach is to use Mathematica to evaluate the troublesome higher order HG-modes in a specific spatial grid, and use this data in the inverse HG-transformation with the eigenvector mode coefficients for generating the spatial eigenmode.

C.3 Rules of Thumb for Selecting Parameters

Even though the HG-modes form a complete and orthogonal normalized basis set of modes, for calculation purposes we have to use only a finite number. This brings us to a question: how many modes is good enough to solve the problem accurately? In this section we show rules of thumb developed to select the number of modes, and discretization step in the z-direction for ODE solver, the Runge-Kutta order 4 (RK4) integrator.

C.3.1 Cutoff Mode Number

We develop the estimate for maximum number of modes (cutoff mode number) required in modeling, by relating phase accumulated by the largest spatial mode over the entire crystal should be $\frac{\pi}{2}$, as otherwise the mode would change from gain to
deamplification. Using this fact on the phase part of the coupling integral $B_{m m'}$, we derive the cutoff mode $m_{\text{max}}$ as,

$$2m_{\text{max}} \frac{L/2}{k_p a_0^2} = \frac{\pi}{2},$$

$$m_{\text{max}} = \frac{a_0^2 \pi k_p}{2L} = \frac{z_R \pi}{L/2}. \quad \text{(C.2)}$$

This estimate provides an order of magnitude of cutoff mode number, and indicates how it scales quadratically with pump spot size; i.e. when we choose for a pump spot size $a_0 = 100\mu m$ a cutoff mode number $m_{\text{max}} = 32$, the corresponding cutoff mode for a pump spot size $a_0 = 400\mu m$ becomes at least $m_{\text{max}} = 512$. Our empirical method is to choose a cutoff mode based on the order of magnitude estimate above, and quadruple the mode number along each dimension for a doubling in the corresponding pump spot size. Naturally the memory requirements outpace what is physically available on the computer, and sets the upper limit on the achievable pump spot size.

C.3.2 Integration Step Size

We choose step size $dz$ over which maximum order HG mode undergoes a phase change less than a few degrees, in coupling integral $B_{m m'}$. The discretization of step
size relates to the crystal length \( dz = L/N_L \), by number of points \( N_L \). Using the phase part of the coupling integral, we arrive at

\[
2m \times \tan^{-1}\left( \frac{L}{N Z_R} \right) > \frac{1}{30}
\]

\[
\frac{L}{N Z_R} < 60m_{\text{max}}
\]

\[
= 60 \frac{Z_R \pi}{L} \frac{1}{2}
\]

\( N_L = 30\pi \approx 100 \)

We use a RK-4 ODE-IVP solver with \( N_L = 300 \) points over crystal length, and note that the axial step is independent of the cutoff mode number.

### C.4 Accuracy Checks on Numerical Model

Checking if the Green’s function \( G \) for the problem turns out to be symplectic.

\[
G J^T G^T J = I
\]

where

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Symmetry in eigenvalue spectrum of correlator matrix \( R = GG^T \), ensures for each squeezed mode with eigenvalue \( \lambda \), an eigenvalue \( 1/\lambda \) exists corresponding to anti-squeezed mode. This is another indicator of accuracy.

Other accuracy measures include checking normally ordered Green’s correlator \( R_{\text{norm}} = R - I \) has coupling-integral like dependence on higher order mode index \( m \) and mode index difference \( m - m' \).
The properties of normally ordered correlator matrix $R_{\text{norm}}$ elements, for elliptic gaussian beam pump have a spatial-bandwidth-related roll-off for higher order modes $m = m'$ and a roll-off following a gaussian envelope for $m \neq m'$ positions.

For higher order modes, $m = m'$, the roll-off proportional to $1/\sqrt{m}$. For mode mismatch $m \neq m'$, envelope dependence looks like $1/(m - m)^2$. Comparing the $32 \times 32$ HG mode solution for pump power $P_0 = 20\text{kW}$, with pump spot size $100 \times 25\text{µm}^2$, and for pump power $P_0 = 20\text{kW}$, but spot size $400 \times 100\text{µm}^2$ we see the accuracy checks indicate insufficient cutoff mode for the latter case.

C.5 Largest Tractable Problem Size

The largest pump power and spot size (problem size) are limited by the CPU power, and available memory (RAM) for the simulations. Memory requirements of calculated (with double precision) Green’s matrix $G$ of size $(2M_xM_y) \times (2M_xM_y)$. The size of $G$ quadruples for doubling of mode number along either X or Y dimension. CPU time increases with the cutoff mode number due to longer times spent in the integration loop and in eigenvalue-vector solver. The integration routine scales linearly with problem size $(M_xM_y)$ but the eigenvalue-vector solver takes an order of magnitude times longer for every doubling of number of modes in transverse dimension.

Currently for a $512 \times 32$ HG-mode problem with $400 \times 100\text{µm}^2$ spot size, and $20\text{kW}$ power our best solver takes about 4 days and 20GB RAM for solution. Resources currently available to us are: an 8 CPU Intel Xeon processor with 96GB of RAM. We estimated that we could study upto $2 \times$ larger pump spot sizes (e.g. $800 \times 100\text{µm}^2$) in one dimension and higher resolutions that come with larger pump spot size, at $2048 \times 32$ HG-modes required to solve such a problem, but the diagonalization procedure requires more memory than just to fit the matrix in memory.
Figure C.4. Comparing normally ordered correlators, of $32 \times 32$ HG mode solution for pump power $P_0 = 20\text{kW}$, with pump spot size $100 \times 25\mu\text{m}^2$ (a), (b), and for pump power $P_0 = 20\text{kW}$, but spot size $400 \times 100\mu\text{m}^2$ (c), (d), we see effect of cutoff on convergence of calculations.
We have solved the model for larger pump spot sizes, until the point where largest solvable problem is defined by limited resources.
REFERENCES


BIOGRAPHICAL STATEMENT

Muthiah Annamalai has been with the Nonlinear Optics & Nanophotonics group led by Prof. Vasilyev at The University of Texas, at Arlington, since 2005. He started on the Ph.D program after getting the M.S (by research) in 2007 for his work on plasmonic modeling. Previously he got the B.Tech in Electronics and Communication Engineering in 2005 from National Institute of Technology-Tiruchirapalli, India.

During the Ph.D. program, he spent the Summer-2010 as intern at Raytheon BBN Technologies in Cambridge, MA. He got a IEEE Photonics Society student travel award, to present his research work at the Photonics-2010 conference in Guwahati, India.

He is interested in Nonlinear Optics, Quantum Optics, and Quantum Information Processing. Muthiah likes it outdoors, tennis, hiking, and amusement park rides even with the tummy suspended in free-fall.

In my previous life, I was involved in freelance software development, including work on the numerical package Octave, only finishing with a toy programming language in the native Indian language Tamil with much of the software work written up as unpublished reports on ArXiV.

When not involved in thinking about engineering work, or reading popular science works, he likes to look at the night sky sometimes with a Dobsonian (reflector) or refractor, and wonder about alien life.