

**A DYNAMIC MULTIPLE STAGE, MULTIPLE OBJECTIVE
OPTIMIZATION MODEL WITH AN APPLICATION TO A
WASTEWATER TREATMENT SYSTEM**

by
PRASHANT K. TARUN

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To my parents, Vimla and Dinesh, and my wife Srividhya.

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ABSTRACT

A DYNAMIC MULTIPLE STAGE, MULTIPLE OBJECTIVE OPTIMIZATION MODEL WITH AN APPLICATION TO A WASTEWATER TREATMENT SYSTEM

Prashant K. Tarun, Ph.D.

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Co-Supervising Professors: Dr. Victoria C. P. Chen, Dr. H. W. Corley

Decision-making for complex dynamic systems involves multiple objectives. Various methods balance the tradeoffs of multiple objectives, the most popular being weighted-sum and constraint-based methods. Under convexity assumptions an optimal solution to the constraint-based problem can also be obtained by solving the weighted-sum problems, and all Pareto optimal solutions can be obtained by systematically varying the weights or constraint limits. The challenge is to generate meaningful weights or constraint limits that yield practical solutions. In this dissertation, we utilize the Analytic Hierarchy Process (AHP) and develop a methodology to generate weight vectors successively for a dynamic multiple stage, multiple objective (MSMO) problem. Our methodology has three phases: (1) the input phase obtains judgments on pairs of objectives for the first stage and on dependencies from one stage to the next, (2) the matrix generation phase uses the input phase information to compute pairwise comparison matrices for subsequent stages, and (3) the weighting phase applies AHP concepts, with the necessary weight vectors obtained from expert opinions. We develop two new geometric-mean based methods

for computing pairwise comparison matrices in the matrix generation phase. The weight ratios in the pairwise comparison matrices conform to the subjective ratio scale of AHP, and the geometric mean maintains this scale at each stage. Finally, for these two methods, we discuss the consistency of computed pairwise comparison matrices, note the convergence behavior, and apply our three-phase methodology to a problem of evaluating technological processes/units at each stage of an MSMO Wastewater Treatment System (WTS). The WTS is a 20-dimensional, continuous-state, 17-stage, 6-objective, stochastic problem.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

The application of multistage multiobjective optimization is rapidly growing in fields such as operations research, computer science, and environmental engineering, as well as such varied disciplines as chemical engineering, economics, and management. A large portion of control problems exhibit multiple stage, multiple objective (MSMO) characteristics. Despite its prevalence, there are few methods with the capability to solve a general large-scale multistage multiobjective optimization problem. Advancement in computing power in the last two decades has made it possible to solve some computationally intensive problems in various application areas.

The research objective of this dissertation is to scalarize a multistage and multiobjective optimization model so that it can be solved using existing approaches. This dissertation uses weighted-sum of objective functions as an *a priori* method to solve the multiple stage, multiple objective optimization problem. What gives the weighted-sum approach an edge over other contemporary scalarization approaches such as the ε -constraint method, etc., are the following.

- It allows decision makers in a particular MSMO problem domain to participate in the solution process due to its simplicity.
- Decision makers do not need to understand the optimization theory and methodology for an effective participation.

- The weighted-sum can be implemented easily and effectively in general, which is particularly important for a complicated large-scale problem where objectives are conflicting and measured in different units.
- The weighted-sum approach can easily be used with other frequently used approaches in multiobjective decision-making such as Analytic Hierarchy Process (AHP), etc.

A detailed discussion on the contemporary scalarization approaches is given in the literature review. The weighted-sum of objective functions approach converts the multiple objective optimization to single-objective optimization at each stage, which yields a multiple stage single-objective model that can be solved using dynamic-programming based approaches such as in Tsai [113]. The weighted-sum of the objective functions has long been one of the most preferred *a posteriori* methods for solving single stage, multiple objective optimization problems [73, 40]. A major challenge for using the weighted-sum as an *a priori* method in the multiple stage setting is the determination of meaningful weight vectors at each stage.

Figure 1.1 shows the basic multiple stage, multiple objective optimization problem. The weighted-sum of objective functions approach results in a multiple stage, single objective optimization problem as shown in Figure 1.2. We seek to find a method that both determines a weight vector at one stage and modifies this vector appropriately from one stage to the next. An exhaustive set of Pareto optimal solutions can be obtained by varying the weight vectors at each stage of the multiple stage, multiple objective optimization problem.

1.2 Methodology

Traditionally, pairwise comparison matrices are used to compare a pair of objectives in a multiple objective decision-making domain. Pairwise comparison matrices are con-

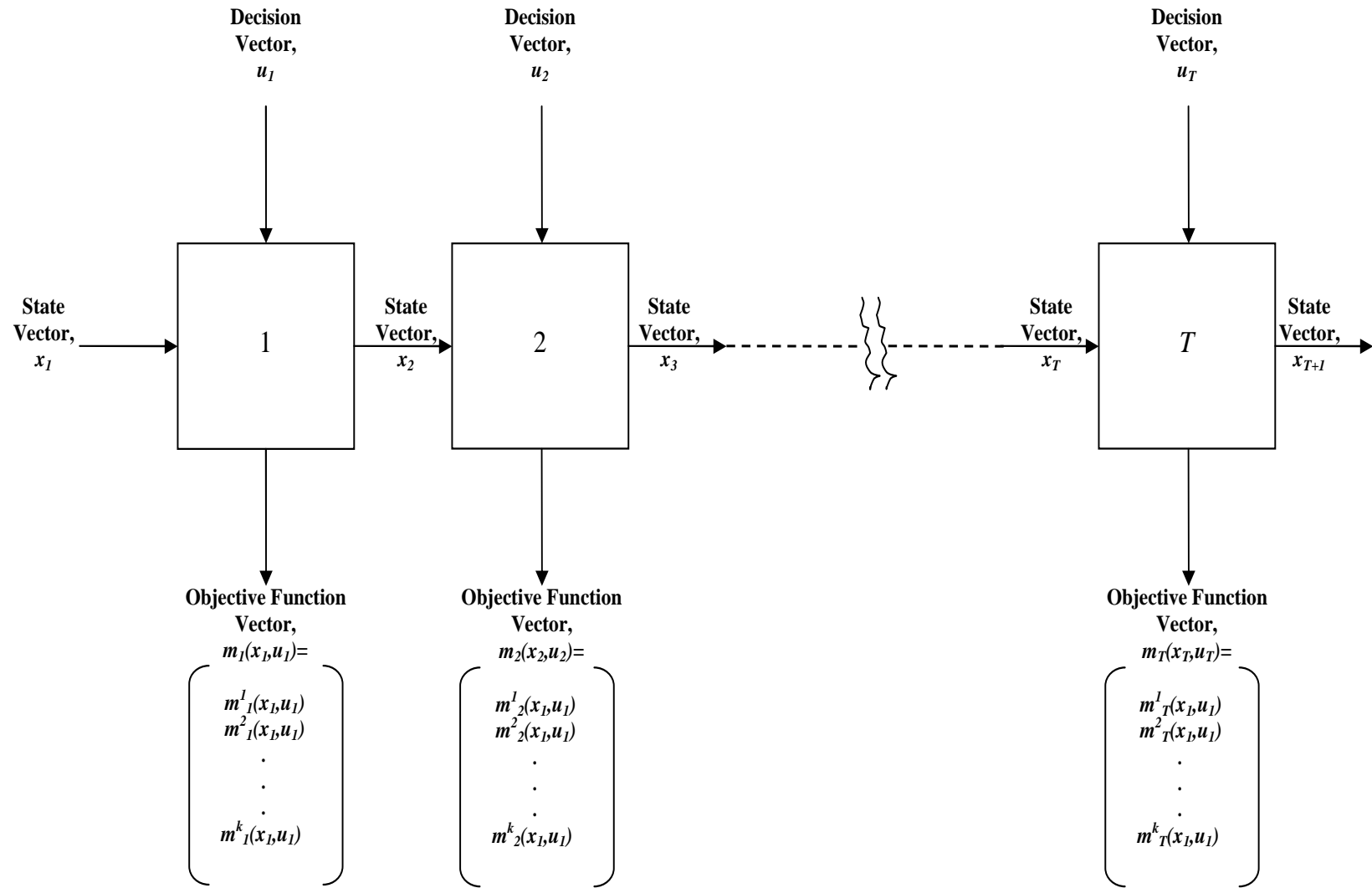


Figure 1.1. Basic formulation of multiple stage multiple objective optimization problem.

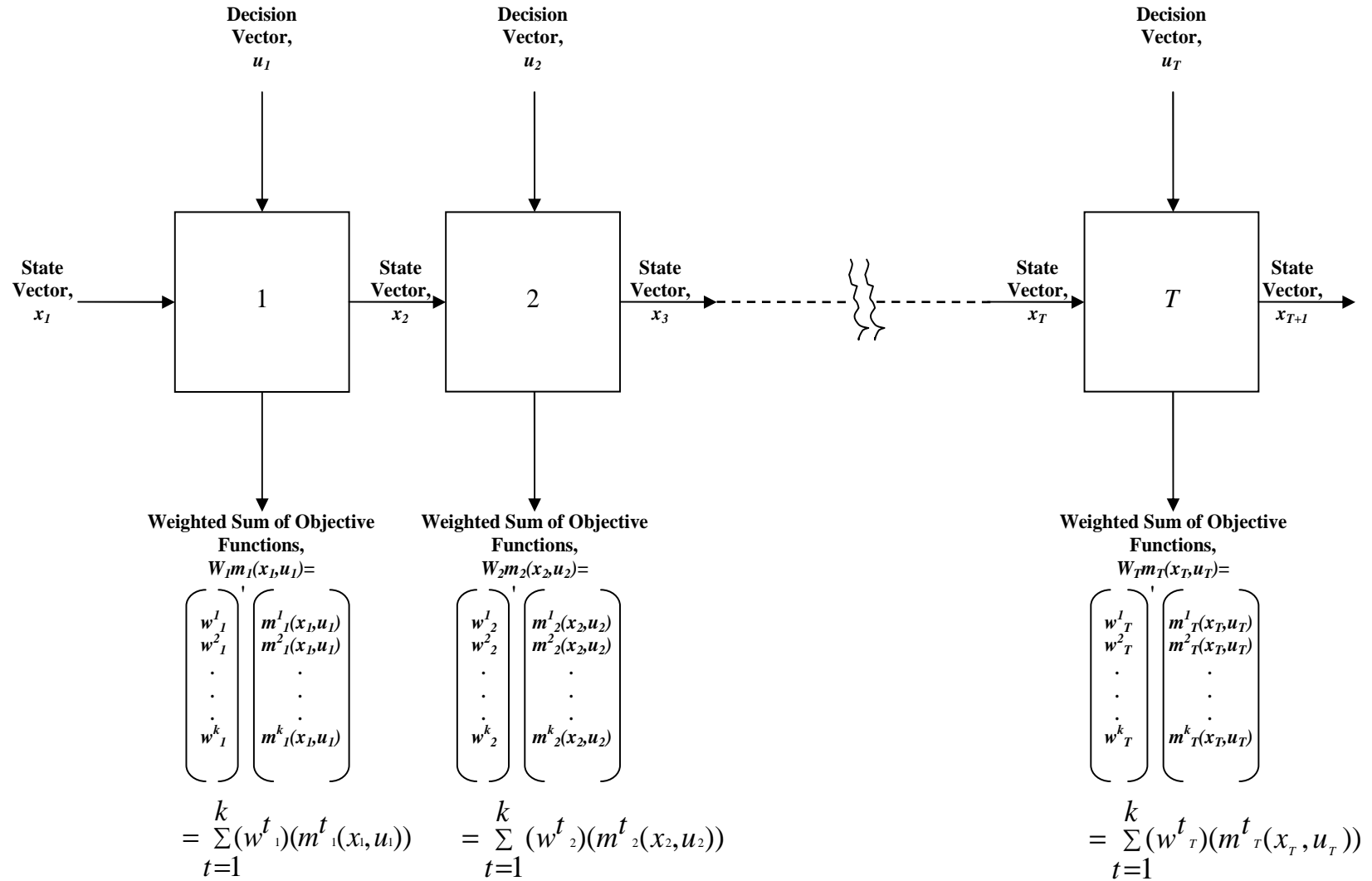


Figure 1.2. Scalarization using weighted-sum of objective functions approach.

structured based on the inputs from the experts/decision makers. The problem with this approach is that we need experts to answer a large number of questions for the complete pairwise comparison matrices. This becomes a larger issue when we deal with multiple stage, multiple objective (MSMO) decision-making. Therefore, for an MSMO decision-making problem we need an approach that reduces the amount of information/input required from the decision makers and yet maintains the consistency in their judgments from one stage to the next. In this dissertation, we introduce a new methodology for computing pairwise comparison matrices at each stage of an MSMO decision-making framework, which can be used to compute stagewise weight vectors and to form the weighted-sum of objective functions at each stage.

The advantage of the new approach in terms of the amount of input required from the decision makers can be seen as follows. In the first stage, decision makers are required to compare $\frac{k \times (k-1)}{2}$ pairs of objectives for obtaining the complete $k \times k$ pairwise comparison matrix in the first stage, where k is the number of objectives at each stage. The assumption is that the number of objectives remains the same for all stages. The benefit of the approach can be seen in subsequent stages, where decision makers need to compare only k pairs of objectives instead of $\frac{k \times (k-1)}{2}$ at each stage. Mathematically, k is less than $\frac{k \times (k-1)}{2}$, for k greater than 3. Hence, a tremendous reduction in the amount of information can be achieved for an MSMO problem with a large number of stages and objectives.

After a multiobjective subproblem has been converted at each stage into a single-objective subproblem, dynamic programming-based approaches, such as that utilized in Tsai [113], can be used to solve the multistage optimization problem. We have developed a methodology that consists of three phases. (1) The *input phase* obtains judgments on pairs of objectives for the first stage and on dependencies from one stage to the next. (2) The *matrix generation phase* uses the input phase information to construct

pairwise comparison matrices for the subsequent stages. (3) The *weighting phase* applies Analytic Hierarchy Process (AHP) concepts to obtain the weight vectors representing expert opinions as in [93]. These three phases are depicted in Figure 1.3 for a typical multistage and multiobjective model.

In this dissertation, we primarily focus on the input phase (1) and matrix generation phase (2). For the input phase (1), we have developed a questionnaire-based approach to elicit inputs from decision makers that are crucial in forming the following two classes of input matrices: *pairwise comparison matrix at stage 1*, and *interstage diagonal transformation matrix*, also referred to as *the matrix of dependencies from one stage to the next*. The questionnaire-based approach is innovative in its implementation to the multiple stage setting. In addition to obtaining the input matrix at stage 1, this also entails asking questions to determine the interstage dependencies between same objective types in consecutive stages. The weighting phase (3) follows the standard AHP approach [49].

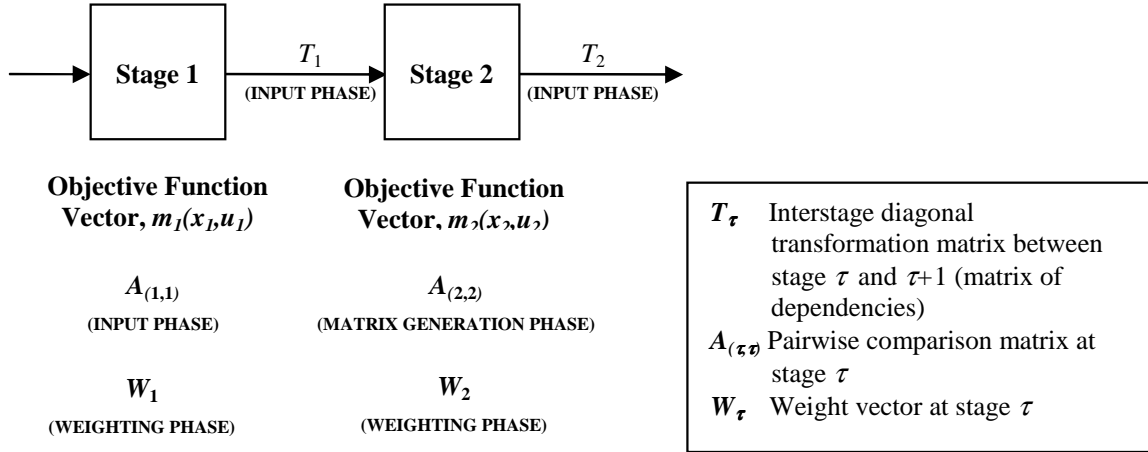


Figure 1.3. A typical multistage multiobjective model highlighting three phases in our methodology.

Two geometric-mean based methods have been developed for the matrix generation phase. These methods compute pairwise comparison matrices in a multistage setting and use AHP methodology to determine weight vectors at each stage from computed stagewise pairwise comparison matrices. Computation of pairwise comparison matrices is a new concept as they have traditionally been formed based on the direct inputs from decision makers. We alleviate the traditional time-consuming process through our new methods, which can reduce the amount of information required from the decision makers thereby reducing the interaction with them in solving a general large-scale MSMO decision-making problem. As a result, we gain a tremendous advantage in terms of time it will take to solve a practical MSMO decision-making problem. Both of these methods use the concept of geometric mean to maintain the component values of computed pairwise comparison matrices in the AHP ratio scale range.

The novelty of our two matrix generation methods can also be seen in the way they help extend AHP to multiple stage, multiple objective decision-making problems. In particular, our approach allows:

- interpretability as a result of the pairwise comparison matrices complying to the AHP ratio scale,
- an improvement in the consistency of the pairwise comparison matrices (shown in Chapter 4),
- the extension of the AHP ratio scale to a continuum,
- the extension of AHP to the MSMO decision-making framework,
- input from the actual decision makers on the relative importance of the different objectives at the initial stage and between two consecutive stages without decision makers needing to understand the optimization concepts in detail.

1.3 The Application

We demonstrate our three-phase methodology on the problem of selecting a set of technologies for a 17-level (or stage) Wastewater Treatment System (WTS) to satisfy the following six objectives: *minimize* [economic cost (in USD, capital cost and operating cost), size (in m^2 , expressed as land area), odor emissions (in mg/min)], *maximize* [robustness (no units), global desirability (on 1-6 scale), and nutrient recovery (on 1-5 scale)]. The multiple stage version is also solved using standard single-stage optimization, which leads to a loss in the significance of stages and stagewise decision-making, the most important feature in a multiple stage decision-making problem. The high-dimensional continuous-state stochastic dynamic programming approach is used for the solution [113] due to the following advantages it holds over contemporary approaches. This method is equipped with efficient design of experiments-based discretization of the state-space and statistical modeling for approximating future value functions, and most importantly it retains the original properties in the multiple stage decision-making problem. The approach has been enhanced to enable the following:

- Augmentation to handle multiple objectives that includes decision makers' tradeoff preferences with respect to objectives,
- Transformation/normalization of non-commensurable (in different units) objectives,
- Addition of new technology processes/units at some levels, which led to new transition equations at these levels and a modification in the cleanliness constraints at these and subsequent levels.

The final solution can be validated from the fact that it satisfies the interstage dependencies between various technologies. Also, the solution results are more or less consistent with those in [113]. Interestingly, the newly-added technologies have been shown to be the

decisions selected in the relevant levels. The big design with 12167 points is considered to provide the “true” solutions to WTS.

1.4 Overview of Dissertation

This dissertation is organized as follows. Chapter 2 presents a literature review and background. Chapter 3 explains the three-phase methodology. Chapter 4 presents the application to an MSMO WTS. Chapter 5 gives the concluding remarks and future research directions. Appendices A and B present questionnaire modeling-related data and computed pairwise comparison matrices in the matrix generation phase, respectively.

CHAPTER 2

BACKGROUND AND LITERATURE REVIEW

2.1 Introduction

The primary objective of this dissertation is to develop an efficient method of generating weight vectors to form a weighted-sum of objective functions at a stage for solving a multiple stage multiple objective (MSMO) problem. The weighted-sum of objective functions method has been used traditionally as an *a posteriori* method to scalarize the vector optimization problem. However, we intend to use the weighted-sum of objective functions method as an *a priori* method for solving MSMO problems. Two critical issues that arise while using the weighted-sum of objective function as an *a priori* method for MSMO problems are:

- Determination of weight vector at one stage,
- Transformation of the weight vector from one stage to the next.

The above issue of determining the weight vector at a stage entails an approach that involves decision makers effectively, which implies that the approach should be simple enough to be understood and followed. In other words, the decision makers should not be expected to understand the optimization concepts for an effective participation. Easy participation of decision makers/technical experts is one of the major advantages the weighted-sum of objective functions approach has over ε -constraint and other contemporary scalarization approaches. Furthermore, the above issue of interstage transformation has its own complication in designing an approach that takes decision makers inputs at one stage and transforms them in such a way that the “true” preferences/judgments of decision makers are reflected in the next stage.

In subsequent sections we will present relevant past research on multiobjective optimization methods, analytic hierarchy process (AHP), weight generation methods, and multistage multiobjective methods respectively.

2.2 Multiobjective Optimization Methods

In this dissertation we solve multiobjective optimization problems at every stage of a general MSMO problem. This section will review some of the prominent solution methods in terms of their popularity, efficiency and applicability from among a vast array of methods to solve multiobjective problems. Despite diverse choices we choose to use the weighted-sum of objective functions approach of solving the multiobjective problem at each stage of MSMO due to the following reasons. It is easy to understand and apply. It uses conventional single objective optimization theory. Local Pareto optimal solutions are easily obtained (unless the objective functions and the feasible region are convex or quasi-convex and convex respectively), and weighted-sum approach can produce an exhaustive set of Pareto optimal solutions by using different weight vectors. The weighted-sum approach scalarizes the multiobjective optimization problem (also referred to as vector optimization problem). Scalarization implies the conversion of a multiobjective (vector) optimization problem into a single (scalar) objective optimization problem.

We first classify multiobjective optimization methods as described in Collette [32] and Miettinen [73]. Such a classification of multiobjective methods is an extremely contentious subject, which can be seen in Cohon [31], Rosenthal [83], Carmichael [20], Hwang [51], and Vedhuizen [118]. Different schools of thought on classification of multiobjective optimization methods can be summarized as follows:

- *generating* (these methods generate an adequately large set of Pareto optimal solutions for the decision maker to choose from), and *preference-based* (these methods

take into account the preferences of the decision maker, and enable the decision maker to select the method that closely maps his/her preferences) [31],

- classes based on *partial generation of the Pareto optimal set, explicit value function maximization* and *interactive implicit value function maximization* [83],
- classes based on a *composite single objective function*, a *single objective function with constraints*, or *many single objective functions* [20]
- classes based on decision maker's participation (a widely used methodology suggested by Hwang and Masud) [51], and
- *interactive methods* (these allow progressive articulation of preferences by decision maker) versus *non-interactive* (do not allow progressive articulation of preferences by decision maker) [73].

Our choice is the most widely used classification methodology based on the participation of the decision maker as in Hwang [51, 18], Buchanan [15], and Lieberman [68, 67]. We prefer this methodology because it:

- covers the majority of current methods,
- has a futuristic value in the sense of its adaptability to the methods under development, and
- utilizes the experience of the decision maker.

This classification includes no participation at all (no-preference methods), participation before (*a priori* methods), participation after (*a posteriori* methods), progressive participation (interactive methods), fuzzy methods, and metaheuristic methods. However, there may be methods that do not belong directly to any of the above classes, or they belong to more than one class simultaneously. For example, it is practical to use *a priori* method first for random computation of preferences and then use *a posteriori* method for generation of a large set of Pareto optimal solutions for the decision maker. We next review various methods in each of these classes.

2.2.1 Methods for no participation at all

No preference methods do not take into consideration the preferences of the decision maker. The methods are: *global criterion*, *exponential sum*, *objective sum*, *min-max*, *Nash arbitration scheme*, *objective product*, *multiobjective proximal bundle* (MPB), etc.

The *global criterion method*, also referred to as compromise programming, minimizes the distance between some reference point and the feasible objective region. In this method, selecting the reference point and the metric for measuring distance is the most significant issue. A typical reference point is the ideal objective vector (final solution). Typical metrics being used are: L_p (values widely used for p are 1, 2, or infinity) and L_∞ is also called Tchebycheff metric [73].

Yu and Zeleny provide a detailed coverage of the global criterion method [131, 134]. This method can be used when nothing is expected other than obtaining a solution. The advantage is an assortment of metrics that guarantee a Pareto optimal solution. The disadvantage [73] is that the solution obtained is seldom used.

Marler explains *exponential sum*, *objective sum*, *min-max*, *Nash arbitration scheme*, and *objective product* methods [70]. The idea behind MPB is to seek for a search direction that optimizes an unconstrained improvement function. MPB can deal with nonlinear and perhaps nondifferentiable functions under the assumption that all the objective and the constraint functions exhibit Lipschitzian behavior locally [73].

2.2.2 *A priori* methods

A priori methods require a decision maker's preferences before the solution process. Unfortunately the decision maker seldom has a good grasp of realities and possibilities of the problem. Yet we must map the decision maker's preferences accurately and maintain the desired level of objectivity while determining a solution.

If the decision maker is supremely confident in his understanding of all aspects of problems and can express his/her understanding in a reliable mathematical form, then the *value function* method is one of the most viable options for obtaining an optimal solution. The value function method takes decision maker's mathematical expression of his/her preferences, formulates a single objective optimization problem with this value function as the objective, and solves it by using a single objective optimization method [57]. Though very simple in formulation, the value function method depends solely on an often unreliable value function expressed by the decision maker. The difficulty in obtaining a value function has been dealt with by deNeuf [35] and Rosinger [84]. The weighting techniques (including weighted-sum and weighted-product) may be seen as a special case of the value function method where the utilities are linear, and additive or multiplicative (for instance, in weighted-product).

The *weighted global criterion* is an extension of the global criterion method discussed in the previous section [134]. A weighted min-max criterion is another approach that utilizes achievement scalarizing functions instead of metrics in the weighted global criterion method as in Wierzbicki [123, 124, 125, 126, 127, 129, 128].

The first step in *lexicographic ordering* requires decision makers to arrange objective functions according to their absolute importance. The ordering is predicated on the assumption that more important means infinitely more important. Subsequent steps involve optimizing the most important objective function subject to the original constraints. Stop if the solution is unique, or else the second most important objective function is optimized subject to the original constraints and a new constraint for maintaining the optimal value of the most important objective function. Again stop if the solution is unique, or else repeat the above step as in Fishburn [41].

The advantages of lexicographic ordering are its simplicity, its orientation toward the decision maker in terms of seeking an ordering of objective functions, and its robust-

ness. Some of the disadvantages include the difficulties in ordering objective functions according to their absolute importance and the fact that tradeoffs between any two objective functions are not allowed.

Goal programming was conceptualized in 1955 [1], and named as such in 1961 by Charnes [22]. This method allows the decision maker to specify goals in terms of assigning aspiration levels to objective functions, and then minimizing any deviations from these goals. Different goal programming formulations are: *weighted* (or *Archimedian*) [23], *lexicographic* (or *preemptive*), *weighted-lexicographic*, *min-max* [42].

Goal programming still ranks high in terms of applicability due to the following reasons: (1) proven methodology, (2) sound rationale of goal-setting, and (3) broad repertoire that allows a variety of formulations and methodologies. Some of the limitations include the difficulty in *a priori* goal-setting without having a clear idea about the feasible region of the Pareto optimal set. Also, tradeoffs are not possible, and there is the underlying restriction of a piecewise linear value function.

Goal programming applications have been in numerous areas such as public works planning [55], portfolio selection [112], wildlife management [86], etc. An extensive collection of references on goal programming can be found in [107].

Bounded objective function methods are frequently used to solve practical multiobjective problems. One of the popular methods of this type is the ε -constraint approach, which minimizes a single objective function while turning all other objective functions into bounded constraints. Marler discusses other *a priori* methods in his dissertation [70].

2.2.3 *A posteriori* methods

A posteriori methods are based on generation of a relatively large set of Pareto optimal solutions that are presented to the decision maker for final selection. They

present several difficulties. First, it is computationally inefficient to generate a large set of Pareto optimal solutions. Second, it is not easy for a decision maker to select from such a large set of solutions. Third, it is hard to find an effective way of presenting the set of Pareto optimal solutions to the decision maker [116, 115, 2, 3].

Many of the previous methods that can be used as *a posteriori* methods are *weighting*, *ε -constraint*, *hybrid* that combines weighting and ε -constraint, *weighted metrics* (a variant of global criterion method), and *weighted min-max*. The approaches that are exclusive to this class are the generalized hyperplane method [106], the envelope approach [65], the non-inferior set estimation method (NISE) method [30, 21, 6], global shooting procedure [12], the normal boundary intersection (NBI) method [34], the normal constraint (NC) method [72], the adaptive search method, and the projection method.

2.2.4 Interactive methods

The interactive method is an iterative way to arrive at the most satisfactory results. The analyst interacts with the decision maker throughout the solution process. This has an obvious advantage over other classes because of the decision maker's high level of confidence in the final solution. The three-step process involves: (1) finding the initial feasible solution, (2) interacting with the decision maker, and (3) obtaining a new solution (or a set of new solutions). If the new solution is acceptable to the decision maker, stop. Otherwise, repeat steps two and three.

Interactive methods eliminate many of the disadvantages of the three classes of methods discussed above. Its advantages include generation of only a part of the Pareto optimal solution set, the flexibility of correcting preferences/judgments as the process moves along, and a better opportunity to understand the problem and improve decision-making.

Consistency of the decisions by the decision maker is one of the basic assumptions overriding the majority of interactive methods. However, inconsistencies are unavoidable because of the subjectivity of the decision makers. Consistency is becoming the focal point in the development of new interactive methods. Various methods deal with inconsistencies in various ways as in [108].

Interactive methods are relatively advanced due to a progressive involvement of decision makers in the solution development process. Numerous such methods are used, and they differ in terms of:

1. how the decision maker receives information,
2. how the decision maker provides information,
3. how the multiple objectives are aggregated to form single objective optimization problem,
4. how the decision maker's inconsistencies are handled,
5. how the decision maker's behavioral issues are accounted for [61, 62],
6. how user-friendly and descriptive they are while dealing with decision maker's inputs, and
7. how and, to what extent, the decision maker's creativity is encouraged.

The decision maker should be able to understand the information easily and utilize it in a meaningful way to come up with a proper response efficiently. This would entail a total commitment from the analyst and the decision maker to this methodology. The amount of information received by the decision maker affects the amount of the information he/she uses. More information does not always lead to better decisions [60].

Convergence of the interactive method is discussed next. According to the conventional definition of convergence, the method is said to converge to Pareto optimal points if the final solution is Pareto optimal [73]. Another definition states that the method is said to converge to a satisficing solution if the final solution is satisficing [73]. In other

words, there is some degree of ambiguity about the meaning and proof of convergence for interactive methods. As a matter of fact there has been research, as described in Stewart [111], Gardiner [44], and Zionts [137, 136], claiming that mathematical convergence is neither necessary nor sufficient to evaluate the validity of an interactive method. The solution process stops when the decision maker is convinced that no significantly better solutions exist.

Miettinen [73] discusses numerous interactive methods. These include an interactive surrogate worth tradeoff (ISWT), the Geoffrion-Dyer-Feinberg (GDF), sequential proxy optimization technique (SPOT), and Tchebycheff's method. He also delves into the step method (STEM), reference point method, GUESS or naive method, and satisficing tradeoff method (STOM), along with light beam search, reference direction approach, reference direction method, and non-differentiable interactive multiobjective bundle based optimization system (NIMBUS). Collette and Siarry [32] discuss Fandel, Jahn, and the simplex methods.

2.2.5 Fuzzy-based methods

The notion of fuzzy sets and logic can be used to model real-world processes. The theory of fuzzy sets, created by Zadeh [133], redefines fuzzy sets to allow one to deal with uncertainty, inaccuracy and progressive transition. The idea is to model each objective function and constraint with a membership function developed by the decision maker using reasoning, experience, interpretation, and perception. Some of the popular fuzzy approaches applied to multiobjective optimization are the Sakawa method [105, 104] and the Reardon method [81, 82].

2.2.6 Methods for metaheuristics

We will now discuss some metaheuristic methods that are inherently different from the above methods. Primary metaheuristic methods include simulated annealing, TABU search and genetic algorithms. Simulated annealing is based on the physical process of metal annealing in terms of how it functions [87]. Glover's TABU search [45] is designed to circumvent local optima. Genetic algorithms are mathematically analogous to the Darwin's theory of natural selection [36]. They converge to a set of Pareto optimal solutions by successively working with a population of points. The vector evaluated genetic algorithm (VEGA) is a non-aggregation method to deal with a multiobjective optimization problem [28, 36]. The drawback with this method is its inability to find Pareto optimal solutions in the non-convex region. The multiple objective genetic algorithm (MOGA) method employs the domination relation to compute the efficiency of an individual [36, 43]. The limitation is its inability to produce diverse solutions approximating the Pareto frontier. Two improvements to the MOGA method are the non-dominated sorting genetic algorithm (NSGA) method [109] and the niched Pareto genetic algorithm (NPGA) method [53].

In summary, no one method is superior to other methods. Selection of a particular method to solve multiobjective optimization problems depends on:

- features of the problem at hand,
- appropriateness to the problem,
- people and the place of implementation,
- knowledge and type of decision maker,
- ease of use for real-world problems,
- theoretical properties of the method,
- computational efficiency, validity and sensitivity of results to the choice of method,
- transparency,

- ease of understanding,
- possibilities of interaction,
- interpretability of the results,
- ease of choosing the most preferred solution from an adequately big set of solution set, and
- ease of showing and interpreting the effect of decision makers' inputs on the solution.

Interestingly, the selection of multiobjective optimization methods is a multiobjective optimization problem itself.

2.3 Analytic Hierarchy Process

The Analytic Hierarchy Process (AHP), developed by Thomas Saaty in the 1970s, has proven remarkably efficient in organizing and resolving complex decision-making problems. The AHP concept is based on human psychology. Humans tend to break down a complex problem in hierarchies or levels, and then make a rather simplistic pairwise comparison between criteria/alternatives at each level. Decomposition of the complexity is one of the first things humans do when they are faced with complex decision-making situations. Having identified various decision elements and the interrelationships among them, the human mind starts synthesizing them to form a judgment.

AHP is hierarchical in structure. For a decision-making problem, the significant factors are identified first. Then, these factors are arranged in a hierarchical structure from an overall goal to objectives, sub-objectives and alternatives in different levels from top to bottom [93]. Hierarchies are not something devised by self-serving bureaucratic organizations, but they are basic to how a human mind breaks a complex situation into clusters and sub-clusters [122]. The hierarchical structure is useful in two ways. First, it gives a detailed account of the intricate relationships in a complex decision-making

problem. Second, it aids decision makers in assessing homogeneity of factors in each level for an accurate comparison. The hierarchy process is illustrated in Saaty [93, 94].

Like any social variable, judgment or preference is subjective in nature. Since judgments/preferences are critical to the use of AHP in a practical problem domain, AHP works on the following assumptions:

- the pairwise comparisons can be obtained by direct questioning of people (meaning a group or individual in the sole authority of the decision making) familiar with the decision making problem;
- inconsistencies are unavoidable in real-world situations;
- priority setting is a must;
- all the alternatives (like technology choices at each level of WTS) are specified in advance, and not all variables need to be under the control of the people involved;
- the expressed judgments are deterministic in nature (although there have been a lot of interest in the study of probabilistic case as in Saaty [101, 102], Ozdemir [77], and Basak [8]); and
- in case of a dispute in a group setting, the separate judgments of the disputing parties may be compared with the judgments of the non-disputing parties.

AHP involves setting up the hierarchical structure of the decision-making problem, pairwise comparisons of all the elements of a lower level with each element of the next higher level, assigning a ratio to each pairwise comparison according to Saaty's fundamental ratio scale [94] in a matrix form, and finding the eigenvector associated with the pairwise comparison matrix giving the largest eigenvalue [93].

Mathematically, AHP works toward computation of principal eigenvector, which when normalized becomes the vector of priorities (or weight vector) for the given pairwise comparison matrix. This in turn is based on an important property of square matrices from algebra stating that associated with a square matrix are its eigenvectors and corre-

sponding eigenvalues [130]. Forming a pairwise comparison matrix is central to AHP's efficacy and is discussed in Saaty [93]. A good discussion on the necessity of the principal eigenvector for the representation of priorities can be found in Saaty [96], and Saaty [98] shows through counter-examples the reason why the principal eigenvector is the only way to obtain correct vector of priorities under inconsistency of judgments. The principal eigenvector is insensitive to small perturbations as explained in Saaty [103].

Saaty [93] described the following four ways to get estimates of the principal eigenvector from a given pairwise comparison matrix.

1. The crudest estimate is obtained by summing the elements in each row and normalizing these by dividing them by the total of all the sums.
2. A relatively better estimate is obtained by summing the elements in each column, forming the reciprocals of these sums, and normalizing by dividing each reciprocal by the sum of the reciprocals.
3. A good estimate is obtained by dividing the elements of each column by the sum of that column (normalizing the column), adding the elements in each row of the normalized column matrix, and dividing this sum by the number of elements in the row (averaging over the normalized columns).
4. An equally good estimate can be obtained by taking the geometric mean of n elements in each row and normalizing the resulting numbers.

We use the method of normalizing the column and averaging over the normalized columns to get an estimate of the weight vector.

The computed eigenvector provides the vector of priorities, and the eigenvalue provides a measure of the consistency (or lack thereof) of judgment. Consistency is not only the requirement that transitivity of preferences gets satisfied but also the actual strength with which the expressed preference transits through the objectives in consideration. The consistency of an $n \times n$ positive reciprocal matrix is equivalent to the requirement that

its maximum eigenvalue should be equal to n . It is possible to estimate the departure from consistency by comparing the *ratio of the difference between maximum eigenvalue and number of elements to (number of elements - 1)* with its value from randomly chosen judgments and corresponding reciprocals in the reverse positions in a matrix of the same order as in [93, 90, 95, 102, 92].

Saaty described in [93] the following method to obtain a crude estimate of consistency of a pairwise comparison matrix:

- multiplying the matrix of pairwise comparisons on the right by the estimated solution vector to obtain a new vector;
- dividing the first component of the new vector by the first component of the estimated solution vector, the second component of the new vector by the second component of the estimated solution vector and so on to obtain another vector; and
- computing average of the components of this vector to get an approximation of maximum (or principal) eigenvalue, which provides an estimate of consistency.

The closer the maximum eigenvalue is to the number of objectives/criteria in the pairwise comparison matrix, the more consistent is the result. New approaches in Pelez [79] and Alonso [5] have recently been developed for studying the consistency.

The next issue is how large the order of pairwise comparison matrix should be to ensure consistency. This should be seven plus or minus two as explained in Saaty [88]. The human mind is more sensitive to improving large inconsistencies, and the inconsistencies are shown to be sufficiently large if the order of pairwise comparison matrix falls in the seven plus or minus two for humans to respond effectively. We note that the wastewater treatment system (WTS) of this dissertation has six objectives/criteria for comparison, which fall in this range.

AHP does an excellent job of eliciting a group or individual judgment and expressing it on the ratio scale: $1/9, 1/8, 1/7, 1/6, 1/5, 1/4, 1/3, 1/2, 1, 2, 3, 4, 5, 6, 7, 8, 9$, where these values suggest the ratio of relative importance of one criterion to another. The relative importance of a particular criterion means the level of its importance to the overall goal of the decision-making process with respect to another criterion. For example, if technology T1 has two different criteria, cost and size, and cost is strongly more important than size. The corresponding ratio value that fits this is 5, implying that $\text{cost} = 5 \times \text{size}$. The interpretation of different values on the ratio scale is given in Saaty [93, 103, 90].

It is important to note here that AHP also uses real numbers other than the 17 rational numbers specified by the ratio scale to form ratios when it is desired to force consistency on the entire matrix from a few judgments. In order to form an $n \times n$ perfectly consistent pairwise comparison matrix, a minimum of $n-1$ judgments are needed [93]. We use this information as an intuitive justification for using continuous ratio values in the computed pairwise comparison matrices.

The ratio scale of AHP raises numerous questions, which include why the ratio scale is necessary, what the rationale is behind the fundamental 1-9 scale, why 9 should be the upper value of the scale, etc. Technically, a scale is a triple consisting of a set of criteria, a binary operation on the criteria and a transformation of the criteria to the real numbers. Ratio scale is used in AHP because it can be shown mathematically that the pairwise comparisons defined by the binary operation map into the ratio scale of real numbers corresponding to the criteria being compared [93]. The fundamental 1-9 scale can be explained through the way we compare two objects in terms of their importance to us. We put the comparisons in either of the following five categories: equally important, weakly more important, strongly more important, demonstrably or very strongly more important, and absolutely more important. This task requires five real numbers, and

thus 1, 3, 5, 7, 9 are chosen to represent the above comparisons. The numbers 2, 4, 6, 8 then represent compromises between differing judgments. A good overview can be found in Saaty [90]. Poyhonen et al. [69] show that there are alternative numerical scales that produce more accurate estimates of judgments than the fundamental 1-9 scale and help reduce the inconsistency of the pairwise comparison matrices.

In order to represent reality in numerical judgments or preferences, AHP makes the following assumptions: (1) physical reality is consistent and can be counted on to produce similar results under controlled conditions; (2) consistency is necessary for capturing reality but not sufficient; (3) improving objectivity and playing down subjectivity will give a better estimate of reality; (4) mathematics can be used to develop a theoretical background for numerical scales of judgments; (5) a scale can be devised that has the power to differentiate various judgments and provide some kind of correspondence between qualitative judgments and the numbers on the scale; (6) measurement theory from physics and economics can be applied; and (7) inconsistency in judgment can be computed [103].

AHP holds some advantages over contemporary methods. It can deal with a large problem easily by breaking it down into subproblems, which in turn helps the decision makers focus and make sound decisions. It can detect the violation in consistency and evaluate it. It can be used by people with little experience, and can translate qualitative and subjective inputs into quantitative values for decision making. It can also point out potential conflicts and tradeoffs. It also exhibits invariance of the solution with respect to the index order of the criteria as explained in Saaty [93], Kablan [56], Nigim [54], and Debeljak [17].

AHP also has its limitations: (1) the inability to take into consideration the variability of confidence in making pairwise comparisons [135]; (2) the inability to explain the use of reciprocal values in the pairwise comparison matrix with regard to human per-

ception and judgment; (3) the unwieldy number of pairwise comparisons for a relatively large problem; (4) there could be better alternatives to the ratio scale, as evidenced by [69]; and (5) invariance is a handicap because of insensitivity to the relative uncertainty of the weight ratios assessed by the decision maker [38].

AHP has been applied in diverse areas since its inception about three and half decades ago. It was developed to solve a contingency planning problem [89]. It had its breakthrough application in designing alternative futures for Sudan [91]. Furthermore, it gained prominence through a flurry of applications [99, 97, 100]. The diversity of its application can be gauged by its use in economic, environmental, social, political and technological areas as described in Saaty [103] and Hobbs et al. [49].

AHP mimics reality since it allows for the inconsistencies in the judgments. AHP has the tools to adapt to complex situations. From the decision maker's standpoint, the ratio scale is simple to use and has all the requisite features to represent complex judgments extensively. As discussed above, AHP readily embraces the unavoidable inconsistencies of the judgments in real-world decision making, provides a way to measure the deviation in consistency, and validates this measurement by putting a given pairwise comparison matrix in a more tangible accept/reject category.

In summary, AHP has the following significant features: the ratio scale, the reciprocal paired comparisons, and the sensitivity of the principal eigenvector. It also allows the extension of scale from 1-9 to 1-infinity as well as group decision making [103].

2.4 Weight Generation Method

Determining weight vectors is one of the most significant elements of this dissertation. We are using AHP method in a multistage framework to compute weight vectors at each stage. Some of the important points in favor of AHP are:

- high level of confidence in the decision makers' judgments for wastewater treatment application to form a valid pairwise comparison matrix,
- a straightforward way to check the consistency of the judgments,
- mathematical basis for a unique weight vector calculation,
- semantic scales used in AHP have an implicit way of describing relative importance of criteria, and
- ease of application.

Nevertheless, the following questions arise. What should be the inherent properties of weight? What are the other methods to generate weight vectors (this also includes other eigenvector-based methods)? Where do they stand versus AHP? Are there methods exclusively designed for multistage setup? If yes, what are they? How do these multistage methods compare with our multistage version? Are there methods to derive weight vectors from pairwise comparison matrices? If yes, what are they? Finally, how do they compare against AHP?

We address each of these issues. We will present a survey of weight vector determination methods assigning pros and cons to each of them. Weight vector determination methods can be classified as: subjective and objective. Subjective methods work on the inputs from decision makers, and objective methods work without any participation from the decision makers. Some methods combine them to achieve some sort of tradeoff between objectivity and subjectivity [27]. Real-world decision making problems are dealt with by a decision making authority (an individual or a group solely in position to make a decision). If the decision making is democratic (takes inputs from all the members in the organization) then a high degree of objectivity can be expected as opposed to a highly subjective autocratic decision making. There is no single best way to make a decision that reflects the essence of democracy.

Since the goal is to develop an approach that can deal with the practical decision-making problems effectively, weights must have certain properties irrespective of the methods used. We will consider a wastewater treatment application to explain these properties. The desirable properties are as below.

- Weight values should be ratio scaled (for example, if the attribute ‘cost’ is three times as important as ‘odor emissions’, then the weight associated with ‘cost’ should be three times as large as the weight associated with ‘odor emissions’).
- Weights should indicate the relative importance of unit changes in their attribute value functions (for example, if a decision maker is indifferent between a change in cost value function of 1 and a change in odor emissions value function of $1/5$, then the weight associated with cost should be 0.2 times weight associated with odor emissions).

In other words, the ratio of the weights of two criteria should be inversely proportional to the rate at which the decision maker makes a tradeoff between them. A criterion’s weight is strongly linked to its value function implying that the value function (or at least the range of value function) definition is necessary for obtaining a meaningful weight.

Hobbs and Rowe [50] present some applications in the power sector that have made the mistake of assigning weights without defining the attribute/criterion value function. Therefore, the validity of weights goes hand in hand with the tradeoffs decision makers are willing to make. The decision makers should understand all the above properties of weights, and they should be able and willing to apply these properties while being asked to answer tradeoff questions. We made the choice to use AHP for weight vector calculations because of our close interaction with the WTS decision maker and a subsequent boost in our confidence in his judgment.

The simplest method to set weights is to use equal weights for the criteria. It is important to understand here that two criteria with equal weights in no way implies they

are equally important. The key point here is that the relative importance of two criteria depends on weights as well as their ranges as explained in [49, 11].

We next discuss subjective methods. The weight vectors obtained through this method reflects decision makers' experience and judgments. It is evident that AHP falls in this category. We will present an assortment of other subjective methods currently in use and under development. Subjective methods begin by collecting judgments from a designated decision maker, and then using these to calculate weights. There are several methods for obtaining decision makers' preferences such as ranking, rating, various versions of pairwise comparisons, successive comparison [19, 39], and Delphi [120]. Further, the methods based on statistical and optimization modeling are multiple regression [49], linear programming [52, 110], and least-squares [93, 63]. In addition, point allocation [49], categorization [49], ratio questioning [93, 49], swing weights [120, 11], method of indifference tradeoff weights [57], and gamble method [49] are the frequently used methods for eliciting decision makers' judgments.

The ranking method ranks each criterion by asking the decision maker to assign a numerical rank. The numerical value 1 indicates most valuable, 2 means next most valuable, and so on.

The rating method rates each criterion by asking the decision maker to draw a line connecting the criterion to a continuous scale marked in units from 0 to 10 (or any equivalent form). This method allows decision makers to select points between numbers and assign more than one criterion to a single point on the scale.

Partial pairwise comparisons version I, proposed by Buel [16], asks the decision maker to indicate in a partial matrix form the number of the more valuable of the pair of criteria forming the row and column. Partial pairwise comparisons version II asks decision maker to circle the more valuable member of each pair of criteria. In this method, each criterion gets paired once with every other criterion. Complete paired

comparisons method is similar in approach as version II, however, the number of paired comparisons gets doubled due to duplication of each pair (for instance A and B appear once as A-B and the second time as B-A at a different place in the pairwise comparison matrix).

The Delphi technique is used to bring a consensus in group decision making through surveys, questionnaires, emails etc. This method has changed so much since its inception in 1950s that more often people have a hard time understanding it. Originally, the Delphi technique was developed to accomplish the following:

- a formal procedure of group decision making in which the group members never met and remained anonymous,
- a contribution by each member of group to produce an estimate of the task of interest supported by written arguments,
- anonymous exchange of these estimates and arguments in sequential rounds leading to new estimates, and
- some form of averaging to determine the group output after the third round typically.

A comparison of the Delphi technique with other group decision making methods is presented in [120].

Multiple regression method uses regression techniques to estimate weights. Linear programming methods use linear optimization techniques to determine weights. Point allocation method asks a decision maker to allocate 100 points among attributes in proportion of their importance. Categorization method assigns attributes to different hierarchies of importance, each having a different weight. Ratio questioning asks decision makers for ratios of the importance of two criteria at a time. AHP is a popular version of ratio questioning.

In the swing weights method, the decision maker considers a hypothetical decision situation where criteria are all at their worst value. Next, a series of swings are made starting with swinging the most preferred attribute from its worst value to its best value first, then swinging the second most preferred attribute, and so on. The swing weights method has the advantage of allowing the decision makers to consider the ranges for each attribute while ranking them [49, 11, 120].

In the method of indifference tradeoff weights, the weights should be consistent with tradeoffs decision makers are willing to make among attributes. This is achieved by asking decision makers to make tradeoffs and then deriving the implied weights.

In the successive comparisons method proposed by Churchman et al. [19], the decision maker takes the following steps. The first step is the ranking of criteria in the order of importance. The second step assigns the value 1 to the most important criterion and other values between 0 and 1 to the other criteria in order of importance. The third step decides whether the criterion with value 1 is more important than all other criteria combined: if it is then increasing the value of most important criterion greater than the sum of all the values associated with all subsequent criteria; else adjusting the value of this criterion to a value less than the sum of the values associated with all subsequent criteria. The fourth step decides whether the second most important criterion is more important than all lower-valued criteria and then proceeding as above, finally continuing until $(n-1)$ criteria have been evaluated.

According to Hobbs et al. [49], of the methods listed above, the most preferred one is indifference tradeoff weights if decision makers are capable and willing to answer tradeoff questions. However, it is not always easy to tradeoff criteria. For example, in the wastewater treatment application both economic costs and odor emissions are to be minimized. It is hard to figure out how much one is willing to increase odor emissions to decrease economic costs, and vice versa. These are the instances when indifference

tradeoff weights assessed may be unstable. In order to minimize instability in the assessed weights a consistency check should be carried out by asking more than the minimum number of questions needed to determine weights. It is highly likely that a good number of decision makers may show discomfort in answering tradeoff questions. In this case, weights may best be assessed by using other methods.

One effective way to ensure the validity of the weights is to combine various methods to reflect the ranges of criteria. Examples of the hybrid methods are: swing-AHP technique [49], Delphi-AHP [58], etc.

According to Hobbs et al. [49], the recommended precautions are as follows. If subjective methods are used: the decision makers (involved in assessing weights) should be well aware of the necessary properties of weights outlined above, decision makers should consider the ranges of criteria when picking weights, tradeoff questions should be asked to validate weights, and use two or more methods for determining weights if time permits.

Since obtaining judgments is not always easy, and for such situations we need objective methods to generate weights. In this method, weight vectors are calculated without seeking judgments of the decision maker. Though the literature is sparse on objective methods, some popular examples include the extreme weight approach [78], the random weight approach [121], and the entropy method [52].

We now look specifically at other eigenvector methods for deriving weight vectors. Some of the prominent methods include: alternative eigenvector method by Cogger and Yu [29], graded eigenvector method (GEM) by Takeda et al. [38], etc. How do these compare with AHP? Cogger and Yu's method is computationally simpler than AHP. The comparison between GEM and AHP is still under development [38].

Krovak [63] presents three methods of deriving weights based on pairwise comparison matrices and gives a simulation study for comparing these methods with AHP.

Batishchev et al. suggested a method to determine weights based on the qualitative information provided by the decision maker [33].

Though the multistage weight determination is not well featured in the literature, Chen and Fu [27] provide an iterative method of obtaining weights in a multistage decision making framework where the objective weights are initially elicited from the information implicit in alternatives and can be interactively adjusted to reflect the dynamic nature of decision situations.

For multiobjective multistage decision making the weight assessment is not straightforward due to dynamic nature of the problem. Every stage has its own characteristics and setting a priority from stage to stage is cumbersome. Despite being criticized for its lack of interpretability and a sound rationale a convenient way to deal with this problem is to have a method that specifies weights objectively (no participation from decision makers) in the beginning, which gets adjusted according to the changes in the situation at different stages [27].

2.5 Multistage Multiobjective Methods

Over past couple of decades a tremendous amount of attention has been directed toward research in the area of multistage multiobjective optimization problems, albeit the progress has been excruciatingly slow. It is evident from the fact that even today not one method can claim to have the capability to solve a general large-scale MSMO problem effectively. The reason for the slow progress can be attributed to a lack of truly efficient and effective method for practical purposes to deal with multistage and multiobjective parts separately.

Dynamic programming is typically used for multistage problems, which however comes with limitations such as curse of dimensionality, inability to satisfy conditions for decomposition: separability and monotonicity, computational intractability for large

problems, etc. An overview of multiobjective methods can be seen in the preceding section. More often than not, multiobjective methods have common drawbacks: problem specific, solves small problems, computationally inefficient for large problems, and lack of real-world application [40].

MSMO problems are prevalent in disparate areas such as operations research and industrial engineering, computer science and engineering, civil engineering, mechanical engineering, electrical engineering, chemical engineering, economics and finance, management, etc. A wide spectrum of contributions stems from its application to a wider area of real-world problems [40]. For example, a large portion of control problems exhibits these characteristics.

We begin with a review of past work on multistage single criterion techniques, and then move on to a discussion on significant past contributions in the multistage multi-objective area. Scalarization techniques such as weighted-sum and ε -constraint convert the multistage, multiobjective formulation into a multistage, single criterion formulation, which can be easily solved using existing approaches for multistage, single criterion problems.

Dynamic programming-based approaches, discussed in [9, 74, 76, 37, 75, 113], are used for solving multistage, single criterion problems. Bellman conceptualized the dynamic programming method, and presents an overview of theory and methodology in [9]. Mitten explains the synthesis of multistage processes using two composition operations: recursive methods to construct the optimal processes, and a state-space dimensionality reduction method [74]. The paper also presents some sufficient conditions for optimal composition and some state-space approximation techniques. Nemhauser does an excellent job of explaining the single criterion, finite stage decision problem [76]. Denardo talks about various aspects of sequential decision processes [37]. Mitten discusses a method for solving finite stage problems based on the preference relations instead of a

real valued utility function measuring the objective in conventional approaches. Mitten's approach generalizes traditional dynamic programming by replacing: transition function with a simple partition of the set of states, and the real valued return function with a set of preference relations [75]. The drawbacks to Mitten's approach are the following: restrictive due to the assumption of complete and transitive preference relation (contrary to the preference relation associated with Pareto optimality, which is not complete and transitive, and the fact that transitivity fails in more general preference relations), and limited practical applicability due to a restriction on the number of states and decisions that may be considered.

We will use a method based on the new dynamic programming-based approach, developed by Tsai [113], to solve our scalarized model. Tsai [113] uses high-dimensional continuous-state Stochastic Dynamic Programming (SDP) for optimizing a system over finite time periods. Solution to the high-dimensional continuous-state SDP problem involves: (1) discretization of the continuous state-space by constructing an orthogonal-array based experimental design, and (2) approximation of the SDP future value function by fitting a statistical model obtained by multivariate adaptive regression splines (MARS). The accurate solution may be obtained in the event the solutions using different number of discretization points converge.

Usually multistage, multiobjective problems are converted into a large single stage, multiobjective problem, which can be solved using the single stage, multiobjective methods discussed above. Lost in the conversion, however, is the innate significance of the stages to the overall problem and perhaps the computational tractability of the problem.

The most significant early contributions in the area of multistage multiobjective optimization are attributed to Brown and Strauch, Henig, and Yu and Seiford. Brown and Strauch extended Bellman's dynamic programming principle to the multicriteria decision making framework [14]. Brown and Strauch's technique has limited applicability due to

the assumption that states use same associative operation to combine the returns from successive stages. It is important to emphasize here that majority of practical problems require operators to possess non-associativity and/or variability from stage to stage.

Henig's dissertation remains a cornerstone in the field of infinite stage multicriteria problems and proposes some of the value and policy improvement techniques [47]. Results of Henig's dissertation can be summarized as: establishing a generalized role of stationary policies among the set of all policies, examining the set of nondominated returns with respect to an acute cone, establishing conditions: for sets in R^n to contain a non-empty nondominated subset, for the non-empty nondominated subset to be characterized and approximated by the set of nondominated exposed points, and for the set of nondominated returns to be characterized and approximated by the set of nondominated stationary policies.

We next discuss some of the pioneering past research in the field of finite stage multiple criteria. Yu et al. [132] discuss finite stage, multiple criteria problems with an unprecedented clarity, which can be outlined as follows: a formulation of serial decision problem with multiple criteria; preliminary results; separability, monotonicity and nondominance boundedness conditions to decompose the problem for stage-wise computation; a 2-objective 3-stage example to illustrate the backward computation; and a list of problems for future research. The approach has the following limitations. It is enumerative and thus computationally inefficient for a real-world problem. It provides an approximate solution if the nondominance boundedness condition is not satisfied. It is restrictive in terms of conditions for decomposing (separability, monotonicity, and nondominance boundedness conditions may not be satisfied simultaneously). Moreover, it suffers from the curse of dimensionality and limited applicability. On the other hand, some of the advantages of this approach are: its simple formulation and its ability to

approximate a set of all nondominated solutions under the conditions of separability and monotonicity.

Another important contribution in finite stage multiple criteria comes from Gomide, who uses the ε -constraint approach, augmented Lagrange multiplier functions, and the concepts of hierarchical optimization to solve multistage, multiobjective optimization problems [46]. Gomide calls his methodology multiobjective, multistage impact analysis method (MMIAM), which essentially is a methodology for multiple stage and multiple objective decision making. For multistage and multiobjective optimization problems satisfying the separability and monotonicity assumptions, multiobjective dynamic programming has been used extensively over last three decades.

Klotzler extends Bellman's dynamic programming to a vector-valued objective function by modifying the separability condition, monotonicity condition and recurrence relations [59]. The modified recurrence relations, referred to as recurrence set relations, are required to satisfy the von Neumann-Morgenstern property. This paper offers a good theoretical account on multiobjective discrete dynamic programming.

Villarreal and Karwan extend the fundamental dynamic programming recursive equations to the multiple criteria framework [119]. Computation results for the binary multiple criteria knapsack problem are reported. The disadvantages are limited applicability and lack of efficiency.

Henig investigates a dynamic programming model with vector-valued returns. He proposes a general theory on dynamic programming with multiple objectives at each stage [48].

Abo-Sinna and Hussein present an algorithm for generating efficient solutions of multiobjective dynamic programming problem [4]. They use a constraint-based method to derive a generalized functional equation of dynamic programming under the separa-

bility and monotonicity conditions. The drawbacks still are: limited applicability and lack of efficiency.

Other significant work on multiobjective dynamic programming is summarized as follows. Li and Haimes review the theoretical concepts behind multi-objective dynamic programming and examine the evolution of its theory and methodology since its inception [66]. Li finds the set of Pareto optimal solutions through the use of a generating approach based on stochastic multi-objective dynamic programming [64]. A general separable class of stochastic programming multi-objective optimization with perfect state information is considered here. Sastry et al. propose a solution methodology to multiple goal control problems with fuzzy goals [117]. These authors consider a decision vector as opposed to a single decision variable at each stage.

CHAPTER 3

METHODOLOGY

3.1 Introduction

In this dissertation, we extend AHP concepts to a multistage setting and use the weighted-sum approach as an *a priori* method at each stage of an MSMO problem. This requires a methodology that both determines a weight vector at one stage and modifies this vector appropriately from one stage to the next. We use AHP to determine stagewise weight vectors. Our motivation behind using AHP can be summarized as follows: provides a way to obtain *a priori* weight vectors at each stage for the weighted-sum approach to be used as an *a priori* method; provides a firm mathematical basis for unique weight vector calculations; decision makers/technical experts easily adapt to AHP due to the following: they do not need to know optimization theory, and they only need to have some idea about the relative importance of objectives/criteria; provides a straightforward way to check the consistency of judgments; provides an efficient way of generating different sets of weight vectors for multiple iterations to navigate through an exhaustive Pareto optimal solution set for an MSMO problem; interpretability as semantic scales used in AHP have an implicit way of describing relative importance of criteria; practical due to its ability to adapt easily to a pairwise comparison matrix with rational number entries; and ease of application. Next, we need some way of transforming the pairwise comparison matrix at one stage to the next to obtain the weight vectors at various stages using AHP while satisfying the properties of pairwise comparison matrix outlined in section 3.2.1.

In this chapter, we introduce a new methodology for computing pairwise comparison matrices at each stage of a multistage decision-making framework, which can be used to compute stagewise weight vectors and to form weighted-sum of objective functions at each stage. After a multiobjective subproblem has been converted at each stage into a single-objective subproblem, dynamic programming-based approaches, such as that utilized in [113], can be used to solve the multistage optimization problem. We have developed a methodology that begins with the *input phase* for obtaining judgments on pairs of objectives for the first stage and on dependencies from one stage to the next, uses the input phase information in the *matrix generation phase* to construct pairwise comparison matrices for the subsequent stages, and applies Analytic Hierarchy Process (AHP) on the information from the first two phases in the *weighting phase* to obtain the stagewise weight vectors representing the expert opinions. In summary, our methodology consists of three phases: (1) The *input phase*, (2) The *matrix generation phase*, and (3) The *weighting phase* [93].

The input phase plays a crucial role in controlling the efficacy of the matrix generation phase, which affects the usefulness of weighting phase. Therefore, obtaining high quality data in the input phase is paramount to the effectiveness of our three-phase methodology in solving a general MSMO problem. Since the input phase seeks opinions/judgments of the decision makers, the problem is: *how can we ensure high quality human opinions/judgments?* Our goal is to involve decision makers in finding a practical Pareto optimal solution. One way to improve the odds of getting precise and consistent judgments is by asking questions in a logical sequence so as to instill some degree of intuition for making relatively accurate judgments.

The input phase (1), with the exception of matrix of dependencies from one stage to the next, and the weighting phase (3) follow the standard AHP approach [49]. For the input phase (1), we present a questionnaire-based approach to elicit relatively pre-

cise and consistent judgments on the trade-offs decision makers are willing to make on pairs of objectives for a general MSMO problem. For the matrix generation phase (2), however, we present here two new methods that generate pairwise comparison matrices in a multistage setting and use AHP methodology to determine weight vectors from each pairwise comparison matrix. In both the methods, the geometric mean is used to maintain the component values of resulting pairwise comparison matrices in the AHP ratio scale range. Our two new matrix generation methods extend AHP to multiple objective, multiple stage decision problems. In particular, our approach allows (a) interpretability as a result of the pairwise comparison matrices following the AHP ratio scale, (b) an improvement in the consistency of the pairwise comparison matrices, (c) the extension of the AHP ratio scale to a continuum, and (d) input from the actual decision makers on the relative importance of the different objectives at a stage and between two consecutive stages without decision makers needing to understand the optimization concepts in detail.

This section is organized as follows. Subsection 3.2 explains the *input phase*. Subsection 3.3 describes the *matrix generation phase* and states the necessary definitions. Subsection 3.4 elaborates upon the *weighting phase*. A numerical example is shown in Subsection 3.5. The conditional convergence of the new methods is discussed in Subsection 3.6. The divergence of the SGM method is shown in Subsection 3.7. Finally, some remarks on the convergence behavior of the new methods are given in Subsection 3.8.

3.2 The input phase

Let the matrices $A_{(\tau,\tau)}$ and T_τ be defined as follows.

- $A_{(\tau,\tau)}$ is the $k \times k$ pairwise comparison matrix at stage τ , where k is the number of objective functions at stage τ , and $a_{ij}^{(\tau,\tau)}$ is the value at the intersection of row i and column j of $A_{(\tau,\tau)}$.

- T_τ is the $k \times k$ diagonal transformation matrix between stage τ and $\tau + 1$, and t_{ij}^τ is the value at the intersection of row i and column j of T_τ .

In this phase, two classes of judgments are required from the decision maker (or expert or technical consultant). They are:

1. the judgment on the pairwise comparisons in the first stage to form a complete pairwise comparison matrix at stage one (denoted by matrix $A_{(1,1)}$)
2. the judgments on dependencies of the same classes of objective function from one stage to the next (denoted by matrices T_τ between stage τ and $\tau + 1$)

3.2.1 Judgment on the pairwise comparisons at the first stage

$A_{(1,1)}$ satisfies all the properties of the pairwise comparison matrix specified by the AHP. According to AHP, the following are the properties of a pairwise comparison matrix:

- the value in row i and column j of $A_{(\tau,\tau)}$ (denoted by $a_{ij}^{(\tau,\tau)}$) indicates how much more important objective i is than objective j at stage τ ;
- the *importance* is measured on a ratio scale $[\frac{1}{9}, 9]$ with each number being interpreted according to the AHP philosophy given in [93];
- the value in row i and column j of $A_{\tau,\tau}$ should be positive, i.e., $a_{ij}^{(\tau,\tau)} > 0, \forall i, j$;
- $a_{ii}^{(\tau,\tau)} = 1, \forall i$;
- for consistency it is necessary that $a_{ji}^{(\tau,\tau)} = \frac{1}{a_{ij}^{(\tau,\tau)}}, \forall i, j$;
- transitivity may not hold if the decision maker is inconsistent, i.e., if $\exists i, j, k$ such that $[a_{ij}^{(\tau,\tau)}][a_{jk}^{(\tau,\tau)}] \neq a_{ik}^{(\tau,\tau)}$.

We assume that there are the same k objective functions in every stage. The necessary consistency property implies a need for $\frac{k(k-1)}{2}$ pairwise judgments in order to form a complete pairwise comparison matrix.

3.2.2 Judgments on dependencies from one stage to the next

The $k \times k$ diagonal matrix T_τ implies a need to obtain k pairwise judgments. We assume that dependencies exist between the same objective functions in consecutive stages. This implies k pairwise judgments. Alternately, the matrix of dependencies can also be termed as an interstage diagonal transformation matrix, named from the role it plays in transforming the pairwise comparison matrix in one stage into the pairwise comparison matrix in next stage following the methodologies for matrix generation in the second phase. The properties of the matrix of dependencies (or interstage diagonal transformation matrix), T_τ , are:

- the value in row i and column i of T_τ (denoted by t_{ii}^τ) indicates how much more important objective i in the stage τ is than objective i in the stage $\tau + 1$;
- the importance is measured on a ratio scale $[\frac{1}{9}, 9]$ with each number being interpreted according to AHP philosophy given in [93];
- the value of non-diagonal elements of T_τ should be zero, i.e., $t_{ij}^\tau = 0$, for $i \neq j$;
- diagonal elements of T_τ should be positive, and belong to the AHP ratio scale $[\frac{1}{9}, 9]$, i.e., $t_{ii}^\tau > 0$ and $t_{ii}^\tau \in [\frac{1}{9}, 9]$.

3.2.3 Questionnaire modeling

The questionnaire modeling is used in the input phase to come up with weight ratios that represent the rate at which the decision maker is willing to tradeoff one objective for another considering the improvement from the worst value to the best value. One way to obtain meaningful weight ratios is through ratio questioning based on the tradeoffs reflecting “true” preferences of the decision maker [49]. In the past, there have been numerous instances of erroneous weight assessments due to misrepresentation of preferences [10]. Weight assessment is more complicated for an MSMO problem from the standpoint of decision makers’ participation. We have simplified it by comparing the

same objective types in consecutive stages. However, we consider all pairwise comparisons for the first stage. The questionnaire modeling involves asking two types of questions for each class of judgments, which are:

1. *Value questions* that ask for the worst and best values of objectives in various stages.
2. *Importance questions* that ask for the tradeoff and relative importance between pairs of objectives.

We adopt a sequential questioning process to obtain weight ratios. Each weight ratio requires: 4 value questions to be answered first, and 2 importance questions to be answered second. Moreover, there is a logical order for answering 2 importance questions. Knowledge of lower and upper objective bounds helps decision makers answer tradeoff questions more consistently and precisely. We define the following.

- k is the number of objective functions, which is same for all stages in the MSMO problem,
- T denotes the total number of stages in the MSMO problem,
- $A_{(\tau,\tau)}$ is the $k \times k$ pairwise comparison matrix at stage τ ,
- T_τ is the $k \times k$ diagonal transformation matrix between stage τ and $\tau + 1$,
- w_i is the weight for objective i at stage 1, $i=1,2,\dots,k$, and
- w_τ^i is the weight for objective i at stage τ , $i=1,2,\dots,k$ and $\tau=1,2,\dots,(T-1)$.

The sequential questioning is explained next for both classes of judgments in the input phase. For the *judgments on the pairwise comparisons in the first stage*, the sequential questioning for a pair of objectives i and j involves asking:

1. What is the worst value of objective i in the first stage?
2. What is the best value of objective i in the first stage?
3. What is the worst value of objective j in the first stage?
4. What is the best value of objective j in the first stage?

These value questions lead us to the following importance questions:

5. Which one of the pairs of objectives i and j is more important in terms of improvement from the worst value to the best value?
6. Given the more important objective, how many times is this objective more important than the other?

The answer to question (6) is based on the AHP ratio scale shown in 3.1, which gives the weight ratio $\frac{w_i}{w_j}$. The total number of weight ratios to be determined for a complete

Table 3.1. AHP Ratio Scale

Ratio Values	1	3	5	7	9	2, 4, 6, 8
Interpretation	Equal	Slightly more	Moderately more	Strongly more	Absolutely more	Intermediate Importance Relations

pairwise comparison matrix at stage 1, $A_{(1,1)}$, is $\frac{k(k-1)}{2}$. It does not matter whether these weight ratios come from the upper triangle or lower triangle of the pairwise comparison matrix $A_{(1,1)}$ due to the reciprocal property of the pairwise comparison matrix, which implies that the elements in the upper triangle are the reciprocal of the corresponding elements in the lower triangle. Hence, the total number of questions required to determine $A_{(1,1)}$ is $\frac{6k(k-1)}{2}$.

Similarly for the *judgments on the dependencies from one stage to the next*, the sequential questioning for a pair of objectives, i at stage τ and i at stage $\tau + 1$, involves asking:

1. What is the worst value of objective i at stage $\tau + 1$?
2. What is the best value of objective i at stage $\tau + 1$?
3. What is the worst value of objective i at stage τ ?
4. What is the best value of objective i at stage τ ?

These value questions lead us to the following importance questions:

5. Which one of the pairs of objectives i at stage τ and i at stage $\tau + 1$ is more important in terms of improvement from the worst value to the best value?

6. Given the more important objective, how many times is this objective more important than the other?

The answer to question (6) gives the weight ratio $\frac{w_{\tau+1}^i}{w_{\tau}^i}$. The number of weight ratios to be determined for a diagonal transformation matrix between stage τ and $\tau + 1$, T_{τ} , is k , which implies that the total number of weight ratios required to determine T_{τ} , $\tau=1,2,\dots,(T-1)$, for a T -stage k -objective problem is $k(T-1)$. Therefore, the total number of questions required to determine T_{τ} , $\tau=1,2,\dots,(T-1)$, for a T -stage k -objective problem is $6k(T-1)$. Finally, the input phase for a general T -stage k -objective problem requires,

$$\frac{6k(k-1)}{2} + 6k(T-1),$$

questions to be answered by decision makers. However, the number of unique questions to be answered in the input phase for a general T -stage k -objective problem is: $2kT$ value questions *plus* $k(k-1) + 2k(T-1)$ importance questions.

3.3 The matrix generation phase

This step is crucial for achieving the primary objective of weight vector generation. In this phase, pairwise comparison matrices are computed for all stages.

Initially, we tried various ideas to accomplish this task. One among these was finding a $k \times k$ matrix T_{τ} that could transform pairwise comparison matrix at stage τ , $A_{(\tau,\tau)}$ into pairwise comparison matrix at stage $\tau+1$, $A_{(\tau+1,\tau+1)}$ such that

$$A_{(\tau+1,\tau+1)} = T_{\tau} A_{(\tau,\tau)}, \tau = 1, 2, \dots, (T-1)$$

This was ruled out because of an underlying restriction of invertibility of $A_{(\tau,\tau)}$, which obviously is not invertible if it is a perfectly consistent matrix. The next approach involved using a transformation,

$$A_{(\tau+1,\tau+1)} = (T_{\tau})^{-1} A_{(\tau,\tau)} T_{\tau}, \tau = 1, 2, \dots, (T-1)$$

However, the results of this transformation violated the AHP ratio scale.

Let us denote the expression $[(T_\tau)^{-1}A_{(\tau,\tau)}T_\tau]$ by $\hat{A}_{\tau+1}$ with elements $\hat{a}_{ij}^{\tau+1}$ at the intersection of row i and column j , where $i, j = 1, 2, \dots, k$. Subsequently, we tried the square root of the previous transformation with the purpose of satisfying the AHP ratio scale, which for an arbitrary row i and an arbitrary column j led to

$$a_{ij}^{(\tau+1,\tau+1)} = \sqrt{\hat{a}_{ij}^{\tau+1}},$$

where

$$\hat{a}_{ij}^{\tau+1} = [(\frac{1}{t_{ii}^\tau})(a_{ij}^{(\tau,\tau)})(t_{jj}^\tau)].$$

Following the properties of pairwise comparison matrix the elements with $i < j$ are the reciprocals of the corresponding elements in the lower triangle. In this transformation, square root was applied to each entry of the product matrix $\hat{A}_{\tau+1}$, which is illustrated below.

Numerical Example of Square Root Transformation:

Suppose there are three objectives at each stage.

Let the given pairwise comparison matrix at stage 1 be

$$A_{(1,1)} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{1}{2} \\ 3 & 2 & 1 \end{pmatrix}.$$

Let the interstage diagonal transformation matrix between stage 1 and 2 be

$$T_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Given these input matrices we would like to determine the pairwise comparison matrix at stage 2, $A_{(2,2)}$. In doing that the first step would be to determine the product matrix \hat{A}_2 .

$$\hat{A}_2 = [(T_1)^{-1}A_{(1,1)}T_1] = \begin{pmatrix} (\frac{1}{2})(1)(2) & \frac{1}{(\frac{1}{3})(2)(2)} & \frac{1}{(\frac{1}{2})(3)(3)} \\ (\frac{1}{3})(2)(2) & (\frac{1}{3})(1)(3) & \frac{1}{(\frac{1}{2})(2)(3)} \\ (\frac{1}{2})(3)(3) & (\frac{1}{2})(2)(3) & (\frac{1}{2})(1)(2) \end{pmatrix}.$$

Then the pairwise comparison matrix at stage 2,

$$\begin{aligned} A_{(2,2)} &= \sqrt{\hat{a}_{ij}^2} = \begin{pmatrix} \sqrt{(\frac{1}{2})(1)(2)} & \frac{1}{\sqrt{(\frac{1}{3})(2)(2)}} & \frac{1}{\sqrt{(\frac{1}{2})(3)(3)}} \\ \sqrt{(\frac{1}{3})(2)(2)} & \sqrt{(\frac{1}{3})(1)(3)} & \frac{1}{\sqrt{(\frac{1}{2})(2)(3)}} \\ \sqrt{(\frac{1}{2})(3)(3)} & \sqrt{(\frac{1}{2})(2)(3)} & \sqrt{(\frac{1}{2})(1)(2)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0.866025 & 0.57735 \\ 1.154701 & 1 & 0.57735 \\ 1.732051 & 1.732051 & 1 \end{pmatrix}. \end{aligned}$$

However, the square root transformation also violated the AHP ratio scale. This drawback would be illustrated through an example next.

Numerical Example Illustrating the Violation of AHP Ratio Scale:

Suppose there are three objectives at each stage. Let the given pairwise comparison matrix at stage 1 be

$$A_{(1,1)} = \begin{pmatrix} 1 & \frac{1}{5} & \frac{1}{3} \\ 5 & 1 & 5 \\ 3 & \frac{1}{5} & 1 \end{pmatrix}.$$

Let the interstage diagonal transformation matrix between stage 1 and 2 be

$$T_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Then the product matrix

$$\hat{A}_2 = [(T_1)^{-1}A_{(1,1)}T_1] = \begin{pmatrix} (\frac{1}{4})(1)(4) & \frac{1}{(5)(5)(4)} & \frac{1}{((\frac{1}{4})(3)(4))} \\ (5)(5)(4) & (5)(1)(\frac{1}{5}) & \frac{1}{(\frac{1}{4})(\frac{1}{5})(\frac{1}{5})} \\ (\frac{1}{4})(3)(4) & (\frac{1}{4})(\frac{1}{5})(\frac{1}{5}) & (\frac{1}{4})(1)(4) \end{pmatrix}.$$

Then the pairwise comparison matrix at stage 2 is

$$\begin{aligned} A_{(2,2)} &= \sqrt{\hat{a}_{ij}^2} = \begin{pmatrix} \sqrt{(\frac{1}{4})(1)(4)} & \frac{1}{\sqrt{(5)(5)(4)}} & \frac{1}{\sqrt{((\frac{1}{4})(3)(4))}} \\ \sqrt{(5)(5)(4)} & \sqrt{(5)(1)(\frac{1}{5})} & \frac{1}{\sqrt{(\frac{1}{4})(\frac{1}{5})(\frac{1}{5})}} \\ \sqrt{(\frac{1}{4})(3)(4)} & \sqrt{(\frac{1}{4})(\frac{1}{5})(\frac{1}{5})} & \sqrt{(\frac{1}{4})(1)(4)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0.10 & 0.57735 \\ 10 & 1 & 10 \\ 1.732051 & 0.10 & 1 \end{pmatrix}. \end{aligned}$$

It can be seen above that entries $a_{12}^{(2,2)} = 0.10$, $a_{21}^{(2,2)} = 10$, $a_{23}^{(2,2)} = 10$, and $a_{32}^{(2,2)} = 0.10$ in $A_{2,2}$ do not comply with the AHP ratio scale. In conclusion, the square root transformation leads to the following types of violation of AHP ratio scale:

Violation I: at least one of the entries in the product matrix $(T_\tau)^{-1}A_{(\tau,\tau)}T_\tau$ less than

$\frac{1}{81}$ (In other words, the reciprocal values greater than 81 leading to violation II),

Violation II: at least one of the entries in the product matrix $(T_\tau)^{-1}A_{(\tau,\tau)}T_\tau$ greater

than 81 (In other words, the reciprocal values less than $\frac{1}{81}$ leading to violation I).

Violations I and II occur simultaneously due to the reciprocal property of pairwise comparison matrix, which implies that $a_{ji}^{(\tau,\tau)} = \frac{1}{a_{ij}^{(\tau,\tau)}}$. In other words, the occurrence of violation I at the intersection of row i and column j will lead to the occurrence of violation II at the intersection of row j and column i , and vice versa. In the example above, the violation I occurs at $a_{32}^{(2,2)}$ and $a_{12}^{(2,2)}$, while violation II occurs at $a_{21}^{(2,2)}$ and $a_{23}^{(2,2)}$.

Next we tried the cube root transformation, which for an arbitrary row i and an arbitrary column j led to

$$a_{ij}^{(\tau+1, \tau+1)} = \sqrt[3]{\widehat{a}_{ij}^{\tau+1}},$$

where

$$\widehat{a}_{ij}^{\tau+1} = \left[\left(\frac{1}{t_{ii}^{\tau}} \right) (a_{ij}^{(\tau, \tau)}) (t_{jj}^{\tau}) \right].$$

$\widehat{a}_{ij}^{\tau+1}$ is an element of the matrix $\widehat{A}_{\tau+1}$ at the intersection of row i and column j . Following the properties of pairwise comparison matrix the elements with $i < j$ are the reciprocals of the corresponding elements in the lower triangle. In this transformation cube root was applied to each entry of the product matrix $\widehat{A}_{\tau+1}$, which could be easily understood by substituting cube root for square root in the above square root transformation example. The upside with the cube root transformation was its ability to satisfy AHP ratio scale.

Numerical Example Illustrating the Cube Root Transformation

Suppose there are three objectives at each stage.

Let the given pairwise comparison matrix at stage 1 be

$$A_{(1,1)} = \begin{pmatrix} 1 & 1 & \frac{1}{3} \\ 1 & 1 & \frac{1}{2} \\ 3 & 2 & 1 \end{pmatrix}.$$

Let the interstage diagonal transformation matrix between stage 1 and 2 be

$$T_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the product matrix

$$\begin{aligned}\hat{A}_2 &= [(T_1)^{-1}A_{(1,1)}T_1] = \begin{pmatrix} (\frac{1}{2})(1)(2) & \frac{1}{(1)(1)(2)} & \frac{1}{(1)(3)(2)} \\ (1)(1)(2) & (1)(1)(1) & \frac{1}{(1)(2)(1)} \\ (1)(3)(2) & (1)(2)(1) & (1)(1)(1) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0.5 & 0.166667 \\ 2 & 1 & 0.5 \\ 6 & 2 & 1 \end{pmatrix}.\end{aligned}$$

Then the pairwise comparison matrix at stage 2

$$\begin{aligned}A_{(2,2)} &= \sqrt[3]{\hat{a}_{ij}^2} = \begin{pmatrix} \sqrt[3]{(\frac{1}{2})(1)(2)} & \frac{1}{\sqrt[3]{(1)(1)(2)}} & \frac{1}{\sqrt[3]{(1)(3)(2)}} \\ \sqrt[3]{(1)(1)(2)} & \sqrt[3]{(1)(1)(1)} & \frac{1}{\sqrt[3]{(1)(2)(1)}} \\ \sqrt[3]{(1)(3)(2)} & \sqrt[3]{(1)(2)(1)} & \sqrt[3]{(1)(1)(1)} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0.793701 & 0.550321 \\ 1.259921 & 1 & 0.793701 \\ 1.817121 & 1.259921 & 1 \end{pmatrix}.\end{aligned}$$

Intuitively, it would make more sense to scale as follows:

$$\begin{pmatrix} \sqrt[2]{(\frac{1}{2})(1)(2)} & \frac{1}{\sqrt[1]{(1)(1)(2)}} & \frac{1}{\sqrt[2]{(1)(3)(2)}} \\ \sqrt[1]{(1)(1)(2)} & \sqrt[1]{(1)(1)(1)} & \frac{1}{\sqrt[1]{(1)(2)(1)}} \\ \sqrt[2]{(1)(3)(2)} & \sqrt[1]{(1)(2)(1)} & \sqrt[1]{(1)(1)(1)} \end{pmatrix} = \begin{pmatrix} 1 & 0.5 & 0.408248 \\ 2 & 1 & 0.5 \\ 2.44949 & 2 & 1 \end{pmatrix}.$$

Having compared the entries in this matrix with the corresponding entries in the transformed matrix $A_{(2,2)}$ we found out the following: the relative difference at the intersection of row 2 and column 1 is $\frac{2-1.259921}{2} = 0.37004$, which is larger than the relative difference at the intersection of row 3 and column 1 of $\frac{2.44949-1.817121}{2.44949} = 0.258163$. This implies that more ones distort the results of cube root transformation without contributing to the relative importance of the objectives.

Next we set out to formulate an approach to circumvent limitations with above approaches. Our primary motivation for the new methods was to attain pairwise comparison matrices that comply with the AHP ratio scale. These methods are:

1. Geometric mean (GM)
2. Successive geometric mean (SGM).

Aside from satisfying the AHP ratio scale, these two methods do not distort the scaling. Before describing them, we will first define the new functions g_ν and G that form the basis for GM and SGM methods.

3.3.1 Function Definitions

Let ν_p be a function such that $\nu_p : \Re^p \rightarrow \aleph$ where \Re is the set of all real numbers on the AHP ratio scale $[\frac{1}{9}, 9]$, p is the number of input matrices, \aleph is the set of all natural numbers less than or equal to p , and

$$\nu_p(\alpha_1, \alpha_2, \dots, \alpha_p) = \text{number of non-one } \alpha_i\text{'s, if } \exists \alpha_i \neq 1 \text{ for some } i = 1, 2, \dots, p. \quad (3.1)$$

Then let g_p be a function such that $g_p : \Re^p \rightarrow \Re$, and

$$\begin{aligned} g_p(\alpha_1, \alpha_2, \dots, \alpha_p) &= (\alpha_1 \alpha_2 \cdots \alpha_p)^{\frac{1}{\nu_p(\alpha_1, \alpha_2, \dots, \alpha_p)}}, \text{ if } \exists \alpha_i \neq 1 \text{ for some } i = 1, 2, \dots, p, \\ &= 1, \text{ otherwise.} \end{aligned} \quad (3.2)$$

Let there be a set of p $k \times k$ matrices where p is an odd number greater than or equal to 3. Of the p matrices let there be $p-1$ diagonal matrices $D_q = (d_{ij}^q)$, $q = 1, 2, \dots, (p-1)$ with positive diagonal entries $d_{ii}^q > 0$ for $i = 1, 2, \dots, k$, and $\Theta = (\theta_{ij})$ being a real matrix with positive entries $\theta_{ij} > 0$ for $i, j = 1, 2, \dots, k$. Then let us define a function G such that

$$G(D_1, D_2, \dots, D_{(\frac{p-1}{2})}, \Theta, D_{(\frac{p-1}{2}+1)}, D_{(\frac{p-1}{2}+2)}, \dots, D_{(p-1)}) = \Theta', \quad (3.3)$$

where $\Theta' = (\theta'_{ij})$ is a $k \times k$ matrix, and

$$\theta'_{ij} = g_p(d_{ii}^1, d_{ii}^2, \dots, d_{ii}^{(\frac{p-1}{2})}, \theta_{ij}, d_{jj}^{(\frac{p-1}{2}+1)}, d_{jj}^{(\frac{p-1}{2}+2)}, \dots, d_{jj}^{(p-1)}). \quad (3.4)$$

We next describe the computation of pairwise comparison matrices using these methods.

3.3.2 GM

The GM method calculates the geometric mean of non-ones: the first iteration computes a matrix containing the geometric mean of non-ones of the multiplication of matrices $(T_1)^{-1}$, $A_{(1,1)}$, and T_1 ; the second iteration computes a matrix containing the geometric mean of non-ones of the multiplication of $(T_2)^{-1}$, three matrices from the first iteration, and T_2 ; and an arbitrary $(\tau-1)$ st iteration computes the geometric mean of non-ones of the multiplication of $(T_{\tau-1})^{-1}$, $2\tau-3$ matrices from $(\tau-2)$ nd iteration, and $T_{\tau-1}$. The meaning of the geometric mean of non-ones becomes clear from the second iteration onward. It involves computation of a matrix containing the geometric mean of non-ones of the multiplication of $2i+1$ matrices, where i is the iteration. Given the matrices from the input phase and using the definition of $G(\cdot)$ from the previous section with $p = 2i+1$, the GM computation is as follows:

1st iteration: Pairwise comparison matrix at stage 2

$$A_{(2,2)} = G[(T_1)^{-1}, A_{(1,1)}, T_1],$$

2nd iteration: Pairwise comparison matrix at stage 3

$$A_{(3,3)} = G[(T_2)^{-1}, (T_1)^{-1}, A_{(1,1)}, T_1, T_2],$$

\vdots

$(\tau-1)$ st iteration: Pairwise comparison matrix at stage τ

$$A_{(\tau,\tau)} = G[(T_{\tau-1})^{-1}, (T_{\tau-2})^{-1}, \dots, (T_1)^{-1}, A_{(1,1)}, T_1, T_2, \dots, T_{\tau-1}].$$

For $i > j$ at an arbitrary stage τ , values in the pairwise comparison matrix $A_{(\tau,\tau)}$ can be expressed as:

$$a_{ij}^{(\tau,\tau)} = \left[\left(\left(\frac{t_{jj}^{\tau-1}}{t_{ii}^{\tau-1}} \right) \left(\frac{t_{jj}^{\tau-2}}{t_{ii}^{\tau-2}} \right) \cdots \left(\frac{t_{jj}^1}{t_{ii}^1} \right) \right)^{\frac{1}{N_{ij}^\tau}} \right] \left[(a_{ij}^{(1,1)})^{\frac{1}{N_{ij}^\tau}} \right], \quad (3.5)$$

for $\tau = 2, 3, \dots, T$, where N_{ij}^τ is the number of non-ones involved in the GM computation of $a_{ij}^{(\tau,\tau)}$.

3.3.3 SGM

The SGM method calculates the geometric mean of non-ones successively: the first iteration computes a matrix containing the geometric mean of non-ones of the multiplication of matrices $(T_1)^{-1}$, $A_{(1,1)}$, and T_1 ; the second iteration computes a matrix containing the geometric mean of non-ones of the multiplication of $(T_2)^{-1}$, the resulting matrix from the first iteration, and T_2 ; etc. The meaning of the successive geometric mean of non-ones becomes clear from the second iteration onward. It involves computation of a matrix containing the geometric mean of non-ones of the multiplication of three matrices, one of which is a matrix having geometric mean of non-ones from the previous iteration. Given the matrices from the input phase and using the definition of $G(\cdot)$ from Section 3.3.1 with $p=3$ for all iterations, the SGM computation becomes:

1st iteration: Pairwise comparison matrix at stage 2

$$A_{(2,2)} = G[(T_1)^{-1}, A_{(1,1)}, T_1],$$

2nd iteration: Pairwise comparison matrix at stage 3

$$A_{(3,3)} = G[(T_2)^{-1}, A_{(2,2)}, T_2],$$

\vdots

$(\tau-1)$ st iteration: Pairwise comparison matrix at stage τ

$$A_{(\tau,\tau)} = G[(T_{\tau-1})^{-1}, A_{(\tau-1,\tau-1)}, T_{\tau-1}].$$

For $i > j$ at an arbitrary stage τ , values in the pairwise comparison matrix $A_{(\tau,\tau)}$ can be expressed as:

$$a_{ij}^{(\tau,\tau)} = \left[\left(\frac{t_{jj}^{\tau-1}}{t_{ii}^{\tau-1}} \right)^{\frac{1}{N_{ij}^\tau}} \left(\frac{t_{jj}^{\tau-2}}{t_{ii}^{\tau-2}} \right)^{\frac{1}{N_{ij}^\tau N_{ij}^{\tau-1}}} \cdots \left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{N_{ij}^\tau N_{ij}^{\tau-1} \cdots N_{ij}^2}} \right] \left[(a_{ij}^{(1,1)})^{\frac{1}{N_{ij}^\tau N_{ij}^{\tau-1} \cdots N_{ij}^2}} \right], \quad (3.6)$$

for $\tau = 2, 3, \dots, T$, where N_{ij}^τ is the number of non-ones involved in the SGM computation of $a_{ij}^{(\tau,\tau)}$.

3.4 The weighting phase

This phase is identical for the above two methods of matrix generation. Saaty's eigenvector method is used to approximate the principal eigenvector associated with each pairwise comparison matrix. These principal eigenvectors are referred to as the weight vectors. The weight vector calculation procedure requires one to:

- Normalize the pairwise comparison matrix $A_{(\tau,\tau)}$ at a stage τ by dividing each entry in column j by the sum of entries in column j , which is denoted by $A_{(\tau,\tau)}^{norm}$.
- Approximate the principal eigenvector (termed as weight vector W_τ at stage τ) by finding the average of each row of the normalized matrix.

There are other methods for computing the principal eigenvector. The power method is one of the most popular approaches to determine the principal eigenvector. For a single-stage multiobjective problem, both Saaty's eigenvector method and power method lead to same mean random consistency index [80]. In the next section we present a numerical example.

3.5 Numerical example on GM and SGM methods

We illustrate GM and SGM methods with a numerical example that has 3 stages and 3 objective functions at each stage. Suppose the input phase results in the following matrices,

$$A_{(1,1)} = \begin{pmatrix} 1 & 0.5 & 0.333333 \\ 2 & 1 & 0.5 \\ 3 & 2 & 1 \end{pmatrix}, T_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, T_2 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}.$$

For both GM and SGM methods stage 1 and 2 yield the same weight vectors and pairwise comparison matrices with the calculations are shown below:

The normalized matrix and weight vector at stage 1 thus become,

$$A_{(1,1)}^{norm} = \begin{pmatrix} 0.1667 & 0.1428 & 0.1818 \\ 0.3333 & 0.2857 & 0.2727 \\ 0.5 & 0.5714 & 0.5454 \end{pmatrix}, W_1 = \begin{pmatrix} 0.1638 \\ 0.2973 \\ 0.5389 \end{pmatrix}.$$

Similarly the computed pairwise comparison matrix, the normalized matrix, and weight vector at stage 2 are

$$A_{(2,2)} = \begin{pmatrix} 1 & 0.9085 & 0.8736 \\ 1.1006 & 1 & 0.8736 \\ 1.1447 & 1.1447 & 1 \end{pmatrix}, A_{(2,2)}^{norm} = \begin{pmatrix} 0.3081 & 0.2975 & 0.3179 \\ 0.3391 & 0.3275 & 0.3179 \\ 0.3527 & 0.3749 & 0.3640 \end{pmatrix},$$

$$W_2 = \begin{pmatrix} 0.3079 \\ 0.3282 \\ 0.3639 \end{pmatrix}.$$

However, stage 3 results differ for GM and SGM due to their algorithmic structures. Stage 3 calculations are shown in Table 3.2, and the detailed layout outlining various phases of GM and SGM methods can be seen in the Figure 1.3 of the introduction.

Table 3.2. Stage 3 Calculations for GM and SGM Methods.

	$A_{(3,3)}$	$A_{(3,3)}^{norm}$	W_3
GM	$\begin{pmatrix} 1 & 0.9791 & 0.9863 \\ 1.0213 & 1 & 0.951 \\ 1.0139 & 1.0515 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.3295 & 0.3231 & 0.3358 \\ 0.3365 & 0.3299 & 0.3237 \\ 0.3340 & 0.3469 & 0.3404 \end{pmatrix}$	$\begin{pmatrix} 0.3294 \\ 0.3301 \\ 0.3405 \end{pmatrix}$
SGM	$\begin{pmatrix} 1 & 1.0292 & 1.0694 \\ 0.9716 & 1 & 1.0063 \\ 0.9351 & 0.9937 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.344 & 0.3405 & 0.3477 \\ 0.3343 & 0.3308 & 0.3272 \\ 0.3217 & 0.3287 & 0.3251 \end{pmatrix}$	$\begin{pmatrix} 0.3441 \\ 0.3307 \\ 0.3252 \end{pmatrix}$

3.6 Conditional convergence of the GM and SGM methods

This section is focused on developing a theory for the convergence behavior of new methods described in the matrix generation phase. For practical purposes we desire to have a method that results in distinct pairwise comparison matrices from stage to stage (implying distinct stagewise weight vectors) in order to explore the Pareto optimal solution set exhaustively. Hence, a method that quickly converges as it progresses through the stages is less desirable. As discussed above in the input phase, the input matrices integral to our approaches are: input pairwise comparison matrix for stage 1 $A_{(1,1)}$, and the interstage diagonal transformation matrices T_τ s between stage τ and stage $(\tau + 1)$. The matrix generation phase shows the functioning of GM and SGM methods. Next we will analyze them to gain an insight on their convergence behavior. We first define the notion of divergence vs. convergence of the stage-wise sequence of pairwise comparison matrices obtained using GM and SGM methods.

Definition 3.1: Convergence and Divergence of Pairwise Comparison Matrix

Let $A_{(\tau,\tau)}$ be a $k \times k$ pairwise comparison matrix having the properties outlined in Section 3.2.1. Let $A_{ij}^\tau = ((a_{ij})_\tau)$ be a sequence of real numbers where $(a_{ij})_\tau$ is the element at the intersection of an arbitrary row i and an arbitrary column j of $A_{(\tau,\tau)}$. Then $A_{(\tau,\tau)}$ is divergent if and only if there exists (i, j) such that sequence $A_{ij}^\tau = ((a_{ij})_\tau)$ of real numbers

is divergent. Equivalently, $A_{(\tau,\tau)}$ is convergent if and only if sequence $A_{ij}^\tau = ((a_{ij})_\tau)$ of real numbers is convergent for all (i, j) .

We will present a case when the dependencies of the same classes of objective function from one stage to the next are equal i.e. T_τ 's are equal. The reason for considering this case lies in the general human tendency to assign same level of relative importance to the same class of objectives from one stage to the next.

We assume the following: the interstage diagonal transformation matrices are equal, which implies $T_1 = T_2 = \dots = T_{(\tau-1)} = T$; all the AHP assumptions are still valid; and T_τ satisfies the properties given in the input phase. For both the methods we will show the calculation involving the lower triangle values of the pairwise comparison matrix, which can be used to compute the corresponding upper triangle values (being reciprocals from AHP). For illustration purposes we use the following $k \times k$ real-valued matrices:

$$T = \begin{pmatrix} t_{11} & 0 & . & . & . & 0 \\ 0 & t_{22} & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & t_{kk} \end{pmatrix} \quad A_{(1,1)} = \begin{pmatrix} 1 & \frac{1}{a_{21}^{(1,1)}} & . & . & . & \frac{1}{a_{k1}^{(1,1)}} \\ a_{21}^{(1,1)} & 1 & . & . & . & \frac{1}{a_{k2}^{(1,1)}} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{k1}^{(1,1)} & a_{k2}^{(1,1)} & . & . & . & 1 \end{pmatrix} \quad (3.7)$$

3.6.1 Convergence behavior of the GM method

Using this method the pairwise comparison matrix at stage 2 is:

$$A_{(2,2)} = G[T^{-1}, A_{(1,1)}, T] = \begin{pmatrix} 1 & R_{21}^2 & \cdot & \cdot & \cdot & R_{k1}^2 \\ ((\frac{1}{t_{22}})(a_{21}^{(1,1)})(t_{11}))^{\frac{1}{N_{21}^2}} & 1 & \cdot & \cdot & \cdot & R_{k2}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ ((\frac{1}{t_{kk}})(a_{k1}^{(1,1)})(t_{11}))^{\frac{1}{N_{k1}^2}} & ((\frac{1}{t_{kk}})(a_{k2}^{(1,1)})(t_{22}))^{\frac{1}{N_{k2}^2}} & \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad (3.8)$$

where N_{ij}^2 , for $i > j$ means the number of non-ones in $A_{(2,2)}$ at the intersection of row i and column j , and R_{ij}^2 for $i < j$ means $\frac{1}{a_{ij}^{(2,2)}}$, where $a_{ij}^{(2,2)} = ((\frac{1}{t_{ii}})(a_{ij}^{(1,1)})(t_{jj}))^{\frac{1}{N_{ij}^2}}$ for $i, j = 1, 2, \dots, k$. This notation is extended to other stages.

For the intersection of row 2 and column 1, the values associated with pairwise comparison matrix from stage 3 through t are as follows:

$$\begin{aligned} a_{21}^{(3,3)} &= [(\frac{1}{t_{22}})(\frac{1}{t_{22}})(a_{21}^{(1,1)})(t_{11})(t_{11}))^{(\frac{1}{N_{21}^3})}] = [(\frac{t_{11}}{t_{22}})^{(\frac{2}{N_{21}^3})}] [(a_{21}^{(1,1)})^{(\frac{1}{N_{21}^3})}], \\ a_{21}^{(4,4)} &= [(\frac{1}{t_{22}})(\frac{1}{t_{22}})(\frac{1}{t_{22}})(a_{21}^{(1,1)})(t_{11})(t_{11})(t_{11}))^{(\frac{1}{N_{21}^4})}] = [(\frac{t_{11}}{t_{22}})^{(\frac{3}{N_{21}^4})}] [(a_{21}^{(1,1)})^{(\frac{1}{N_{21}^4})}], \\ &\vdots \\ a_{21}^{(t,t)} &= [(\frac{t_{11}}{t_{22}})^{(\frac{t-1}{N_{21}^t})}] [(a_{21}^{(1,1)})^{(\frac{1}{N_{21}^t})}]. \end{aligned}$$

For $i > j$ at stage t and intersection of row i and column j , we can generalize the pairwise comparison matrix as:

$$a_{ij}^{(t,t)} = (\frac{t_{jj}}{t_{ii}})^{(\frac{t-1}{N_{ij}^t})} * (a_{ij}^{(1,1)})^{(\frac{1}{N_{ij}^t})}, \quad (3.9)$$

for $t = 2, 3, \dots$

The next task is to find out an exhaustive list of cases that will be studied for its infinite convergence behavior. The results of the analysis are the following cases that converge infinitely to the same value using this approach:

1. All ones i.e. $N_{ij}^t = 1$, for all t .
2. Arithmetic series with a common difference of one:
 - Examples: $N_{ij}^t = t - 1$ OR $N_{ij}^t = t$, for $t = 2, 3, \dots$
3. Arithmetic series with a common difference of two:
 - Examples: $N_{ij}^t = 2t - 2 = 2(t - 1)$ OR $N_{ij}^t = 2t - 1$, for $t = 2, 3, \dots$

CASE 1: All ones i.e. $N_{ij}^t = N_{ij}^{t-1} = \dots = N_{ij}^2 = 1 = N$ (say).

Substituting into equation (3.9) we get

$$a_{ij}^{(t,t)} = \left[\left(\frac{t_{jj}}{t_{ii}} \right)^{(t-1)} \right] [(a_{ij}^{(1,1)})^1] = \left[\left(\frac{t_{jj}}{t_{ii}} \right)^{(t-1)} \right] [a_{ij}^{(1,1)}].$$

However, for $N=1$ to be possible exactly one of the following two conditions must be satisfied:

1. $\frac{1}{t_{ii}} = 1$, $a_{ij}^{(1,1)} = 1$, and $t_{jj} = 1$. In other words, if the case has all ones (or no non-ones).
2. $\frac{1}{t_{ii}} = 1$, $a_{ij}^{(1,1)} \neq 1$, and $t_{jj} = 1$.

From above conditions we infer that $t_{ii} = 1$ and $t_{jj} = 1$, which implies $\frac{t_{jj}}{t_{ii}} = 1$. Consequently we get $a_{ij}^{(t,t)} = a_{ij}^{(1,1)}$, which confirms an immediate convergence.

We conclude that CASE 1 converges immediately to $a_{ij}^{(1,1)}$.

CASE 2: Arithmetic series with a common difference of one.

For convergence analysis we select the case $N_{ij}^t = t - 1$, for $t = 2, 3, \dots$. Substituting into equation (3.9) we get

$$a_{ij}^{(t,t)} = \left[\left(\frac{t_{jj}}{t_{ii}} \right)^{\left(\frac{t-1}{t-1} \right)} \right] [(a_{ij}^{(1,1)})^{\left(\frac{1}{t-1} \right)}] = \left[\left(\frac{t_{jj}}{t_{ii}} \right) \right] [(a_{ij}^{(1,1)})^{\left(\frac{1}{t-1} \right)}].$$

It can also be shown that as t goes to ∞ , $\frac{1}{t-1}$ converges to 0 (By the definition of the limit of a sequence and the Archimedean property).

This implies that as t goes to ∞ , $a_{ij}^{(t,t)}$ converges to $[\frac{t_{jj}}{t_{ii}}][(a_{ij}^{(1,1)})^0] = \frac{t_{jj}}{t_{ii}}$.

We conclude that CASE 2 converges to $\frac{t_{jj}}{t_{ii}}$.

CASE 3: Arithmetic series with a common difference of two.

For convergence analysis we select the case $N_{ij}^t = 2t - 2 = 2(t - 1)$, for $t = 2, 3, \dots$

Substituting into equation (3.9) we get

$$a_{ij}^{(t,t)} = \left[\left(\frac{t_{jj}}{t_{ii}}\right)^{\left(\frac{t-1}{2t-2}\right)}\right][(a_{ij}^{(1,1)})^{\left(\frac{1}{2t-2}\right)}] = \left[\left(\frac{t_{jj}}{t_{ii}}\right)^{\left(\frac{1}{2}\right)}\right][(a_{ij}^{(1,1)})^{\left(\frac{1}{2t-2}\right)}].$$

It can also be shown that $\frac{1}{2t-2}$ converges to 0 as t goes to ∞ (By the definition of the limit of a sequence and the Archimedean property).

This implies that as t goes to ∞ , $a_{ij}^{(t,t)}$ converges to $[(\frac{t_{jj}}{t_{ii}})^{\left(\frac{1}{2}\right)}][(a_{ij}^{(1,1)})^0] = (\frac{t_{jj}}{t_{ii}})^{\left(\frac{1}{2}\right)}$.

We conclude that CASE 3 converges to $(\frac{t_{jj}}{t_{ii}})^{\left(\frac{1}{2}\right)}$.

3.6.2 Convergence behavior of the SGM method

We will use matrices in equation (3.7) for illustrating SGM method. Using this approach the pairwise comparison matrix at stage 2 is same as equation (3.8) above. For the intersection of row 2 and column 1, the values associated with pairwise comparison matrix from stage 3 through t are as follows:

$$\begin{aligned} a_{21}^{(3,3)} &= \left[\left(\frac{1}{t_{22}}\right)\left[\left(\frac{1}{t_{22}}\right)(a_{21}^{(1,1)})(t_{11})\right]^{\left(\frac{1}{N_{21}^3}\right)}(t_{11})\right]^{\left(\frac{1}{N_{21}^3}\right)} = \left[\left(\frac{t_{11}}{t_{22}}\right)^{\left(\frac{1}{N_{21}^3} + \frac{1}{N_{21}^3 N_{21}^2}\right)}\right][(a_{21}^{(1,1)})^{\left(\frac{1}{N_{21}^3 N_{21}^2}\right)}], \\ a_{21}^{(4,4)} &= \left[\left(\frac{1}{t_{22}}\right)\left[\left(\frac{1}{t_{22}}\right)\left[\left(\frac{1}{t_{22}}\right)(a_{21}^{(1,1)})(t_{11})\right]^{\left(\frac{1}{N_{21}^2}\right)}(t_{11})\right]^{\left(\frac{1}{N_{21}^3}\right)}(t_{11})\right]^{\left(\frac{1}{N_{21}^4}\right)} \\ &= \left[\left(\frac{t_{11}}{t_{22}}\right)^{\left(\frac{1}{N_{21}^4} + \frac{1}{N_{21}^4 N_{21}^3} + \frac{1}{N_{21}^4 N_{21}^3 N_{21}^2}\right)}\right][(a_{21}^{(1,1)})^{\left(\frac{1}{N_{21}^4 N_{21}^3 N_{21}^2}\right)}], \\ &\vdots \\ a_{21}^{(t,t)} &= \left[\left(\frac{t_{11}}{t_{22}}\right)^{\left(\frac{1}{N_{21}^t} + \frac{1}{N_{21}^t N_{21}^{t-1}} + \dots + \frac{1}{N_{21}^t N_{21}^{t-1} \dots N_{21}^2}\right)}\right][(a_{21}^{(1,1)})^{\left(\frac{1}{N_{21}^t N_{21}^{t-1} \dots N_{21}^2}\right)}]. \end{aligned}$$

For $i > j$ at stage t and intersection of row i and column j , we can generalize the entries in pairwise comparison matrix $A_{(t,t)}$ as:

$$a_{ij}^{(t,t)} = \left[\left(\frac{t_{jj}}{t_{ii}} \right)^{\left(\frac{1}{N_{ij}^t} + \frac{1}{N_{ij}^t N_{ij}^{t-1}} + \dots + \frac{1}{N_{ij}^t N_{ij}^{t-1} \dots N_{ij}^2} \right)} \right] \left[a_{ij}^{(1,1)} \right]^{\left(\frac{1}{N_{ij}^t N_{ij}^{t-1} \dots N_{ij}^2} \right)}, \quad (3.10)$$

for $t = 2, 3, \dots$

The next task is to find out an exhaustive list of cases that will be studied for its infinite convergence behavior. In order to accomplish this we develop scenarios for the values of N_{ij}^t , for $t = 2, 3, \dots, T$. We start with all possible values of N_{ij}^t at stage $t=2$. Without loss in generality, N_{ij}^t can be 1, 2, or 3. We start with $N_{ij}^2=1, 2$, or 3, and for each of these values we develop various scenarios. The next step is the analysis of these scenarios to ascertain possible corresponding values for N_{ij}^3, N_{ij}^4 , and so on. The results of the analysis are the following cases that converge infinitely to the same value using this approach:

1. All ones i.e. $N_{ij}^t = 1$, for all t .
2. No threes *AND* at least one two:
 - (a) mix of 1 and 2, *OR*
 - Some of the examples are: $N_{ij}^t = \{1 \text{ for } t=2, \text{ and } 2 \text{ otherwise}\}; N_{ij}^t = \{1 \text{ for } t=3, \text{ and } 2 \text{ otherwise}\}$, etc.
 - (b) all twos i.e. $N_{ij}^t = 2$, for all t .
3. No ones *AND* at least one three:
 - (a) mix of 2 and 3, *OR*
 - Some of the examples are: $N_{ij}^t = \{2 \text{ for } t=2, \text{ and } 3 \text{ otherwise}\}; N_{ij}^t = \{2 \text{ for } t=3, \text{ and } 3 \text{ otherwise}\}$, etc.
 - (b) all threes i.e. $N_{ij}^t = 3$, for all t .

CASE 1: All ones i.e. $N_{ij}^t = N_{ij}^{t-1} = \dots = N_{ij}^2 = 1 = N$ (say).

Substituting into equation (3.10) we get

$$\begin{aligned} a_{ij}^{(t,t)} &= \left[\left(\frac{t_{jj}}{t_{ii}} \right)^{\left(\frac{1}{N} + \frac{1}{N^2} + \dots + \frac{1}{N^{(t-1)}} \right)} \right] [(a_{ij}^{(1,1)})^{\frac{1}{N^{(t-1)}}}] \\ &= \left[\left(\frac{t_{jj}}{t_{ii}} \right)^{(1+1+\dots+1)} \right] [(a_{ij}^{(1,1)})^{(1)}] = \left[\left(\frac{t_{jj}}{t_{ii}} \right)^{(t-1)} \right] [a_{ij}^{(1,1)}]. \end{aligned}$$

Since we get the same result as case 1 of GM method we follow the same procedure for proof of convergence. Therefore as before, *we conclude that CASE 1 converges immediately to $a_{ij}^{(1,1)}$.*

CASE 2: No threes AND at least one two.

For convergence analysis we select the case $N_{ij}^t = 2$, for all t (all twos), due to its simplicity. Substituting into equation (3.10) we get

$$a_{ij}^{(t,t)} = \left[\left(\frac{t_{jj}}{t_{ii}} \right)^{\left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{(t-1)}} \right)} \right] [(a_{ij}^{(1,1)})^{\frac{1}{2^{(t-1)}}}].$$

Using the formula for summing a geometric series we get

$$\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{(t-1)}} = \frac{\left(\frac{1}{2} \right) \left(1 - \frac{1}{2^{(t-1)}} \right)}{\left(1 - \frac{1}{2} \right)}.$$

It can be shown that as t goes to ∞ , $\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{(t-1)}}$ converges to 1 (By the convergence of geometric series with the absolute value of common ratio less than 1).

Also, it can be shown that as t goes to ∞ , $\frac{1}{2^{(t-1)}}$ converges to 0 (Using the definition of the limit of a sequence (b^n) converges to 0 if $0 < b < 1$).

This implies that as t goes to ∞ , $a_{ij}^{(t,t)}$ converges to $\frac{t_{jj}}{t_{ii}}$.

We conclude that CASE 2 converges to $\frac{t_{jj}}{t_{ii}}$.

CASE 3: No ones AND at least one three.

For convergence analysis we select the case $N_{ij}^t = 3$, for all t (all threes) due to its simplicity. Substituting into equation (3.10) we get

$$a_{ij}^{(t,t)} = \left[\left(\frac{t_{jj}}{t_{ii}} \right)^{\left(\frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{(t-1)}} \right)} \right] [(a_{ij}^{(1,1)})^{\frac{1}{3^{(t-1)}}}].$$

Using the sum of geometric series we get

$$\frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{(t-1)}} = \frac{(\frac{1}{3})(1 - \frac{1}{3^{(t-1)}})}{(1 - \frac{1}{3})}.$$

It can be shown that as t goes to ∞ , $\frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{(t-1)}}$ converges to $\frac{1}{2}$ (By the convergence of geometric series with the absolute value of common ratio less than 1).

Also, it can be shown that as t goes to ∞ , $\frac{1}{3^{(t-1)}}$ converges to 0 (Using the definition of the limit of a sequence (b^n) converges to 0 if $0 < b < 1$).

This implies that as t goes to ∞ , $a_{ij}^{(t,t)}$ converges to $(\frac{t_{jj}}{t_{ii}})^{(\frac{1}{2})}$.

We conclude that CASE 3 converges to $(\frac{t_{jj}}{t_{ii}})^{(\frac{1}{2})}$.

Thus the convergence behavior of GM and SGM methods is established. Following interesting observations can be gathered from the convergence behavior of these approaches:

- For both the approaches, the case **all ones** converges to $a_{ij}^{(1,1)}$.
- Cases **arithmetic series with a common difference of one** for GM approach and **no threes AND at least one two** for SGM approach converge to $\frac{t_{jj}}{t_{ii}}$.
- Cases **arithmetic series with a common difference of two** for GM approach and **no ones AND at least one three** for SGM approach converge to $(\frac{t_{jj}}{t_{ii}})^{(\frac{1}{2})}$.

There are instances when SGM method does not converge at all. We will discuss the divergence of SGM method in the next section.

3.7 Divergence of the SGM method

In this section we will present a case that leads to the divergence of SGM method. We will use $A_{(1,1)}$ matrix in equation (3.7). Let's suppose T_τ 's oscillate between T^* and T^{**} , where

$$T_\tau = \begin{pmatrix} t_{11}^1 & 0 & . & . & . & 0 \\ 0 & t_{22}^1 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & t_{kk}^1 \end{pmatrix} = T^*, \tau = 1, 3, 5, \dots \quad (3.11)$$

$$= \begin{pmatrix} t_{11}^2 & 0 & . & . & . & 0 \\ 0 & t_{22}^2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & t_{kk}^2 \end{pmatrix} = T^{**}, \tau = 2, 4, 6, \dots \quad (3.12)$$

For $i > j$, using SGM method the elements of pairwise comparison matrix from stage 2 through t at the intersection of an arbitrary row i and an arbitrary column j are calculated as follows:

$$\begin{aligned} a_{ij}^{(2,2)} &= \left[\left(\frac{1}{t_{ii}^1} \right) (a_{ij}^{(1,1)}) (t_{jj}^1) \right]^{\frac{1}{N_{ij}^2}} = \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{N_{ij}^2}} \right] [(a_{ij}^{(1,1)})^{\frac{1}{N_{ij}^2}}], \\ a_{ij}^{(3,3)} &= \left[\left(\frac{1}{t_{ii}^2} \right) (a_{ij}^{(2,2)}) (t_{jj}^2) \right]^{\frac{1}{N_{ij}^3}} = \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{1}{N_{ij}^3}} \right] \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{N_{ij}^3 N_{ij}^2}} \right] [(a_{ij}^{(1,1)})^{\frac{1}{N_{ij}^3 N_{ij}^2}}], \end{aligned}$$

$$\begin{aligned}
a_{ij}^{(4,4)} &= \left[\left(\frac{1}{t_{ii}^1} \right) (a_{ij}^{(3,3)}) (t_{jj}^1) \right]^{\frac{1}{N_{ij}^4}} = \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{1}{N_{ij}^4 N_{ij}^3}} \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{N_{ij}^4} + \frac{1}{N_{ij}^4 N_{ij}^3 N_{ij}^2}} \right] \left[(a_{ij}^{(1,1)})^{\frac{1}{N_{ij}^4 N_{ij}^3 N_{ij}^2}} \right] \right], \\
a_{ij}^{(5,5)} &= \left[\left(\frac{1}{t_{ii}^2} \right) (a_{ij}^{(4,4)}) (t_{jj}^2) \right]^{\frac{1}{N_{ij}^5}} \\
&= \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{1}{N_{ij}^5} + \frac{1}{N_{ij}^5 N_{ij}^4 N_{ij}^3}} \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{N_{ij}^5 N_{ij}^4} + \frac{1}{N_{ij}^5 N_{ij}^4 \dots N_{ij}^2}} \right] \left[(a_{ij}^{(1,1)})^{\frac{1}{N_{ij}^5 N_{ij}^4 \dots N_{ij}^2}} \right] \right], \\
a_{ij}^{(6,6)} &= \left[\left(\frac{1}{t_{ii}^1} \right) (a_{ij}^{(5,5)}) (t_{jj}^1) \right]^{\frac{1}{N_{ij}^6}} \\
&= \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{1}{N_{ij}^6 N_{ij}^5} + \frac{1}{N_{ij}^6 N_{ij}^5 \dots N_{ij}^3}} \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{N_{ij}^6} + \frac{1}{N_{ij}^6 N_{ij}^5 N_{ij}^4} + \frac{1}{N_{ij}^6 N_{ij}^5 \dots N_{ij}^2}} \right] \left[(a_{ij}^{(1,1)})^{\frac{1}{N_{ij}^6 N_{ij}^5 \dots N_{ij}^2}} \right] \right], \\
a_{ij}^{(7,7)} &= \left[\left(\frac{1}{t_{ii}^2} \right) (a_{ij}^{(6,6)}) (t_{jj}^2) \right]^{\frac{1}{N_{ij}^7}} \\
&= \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{1}{N_{ij}^7} + \frac{1}{N_{ij}^7 N_{ij}^6 N_{ij}^5} + \frac{1}{N_{ij}^7 N_{ij}^6 \dots N_{ij}^3}} \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{N_{ij}^7 N_{ij}^6} + \frac{1}{N_{ij}^7 N_{ij}^6 \dots N_{ij}^4} + \frac{1}{N_{ij}^7 N_{ij}^6 \dots N_{ij}^2}} \right] \left[(a_{ij}^{(1,1)})^{\frac{1}{N_{ij}^7 N_{ij}^6 \dots N_{ij}^2}} \right] \right], \\
&\vdots
\end{aligned}$$

For the even-numbered stage 2τ

$$\begin{aligned}
a_{ij}^{(2\tau, 2\tau)} &= \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{1}{N_{ij}^{2\tau} N_{ij}^{2\tau-1}} + \frac{1}{N_{ij}^{2\tau} N_{ij}^{2\tau-1} \dots N_{ij}^{2\tau-3}} + \dots + \frac{1}{N_{ij}^{2\tau} N_{ij}^{2\tau-1} \dots N_{ij}^3}} \right] \\
&\quad \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{N_{ij}^{2\tau}} + \frac{1}{N_{ij}^{2\tau} N_{ij}^{2\tau-1} N_{ij}^{2\tau-2}} + \dots + \frac{1}{N_{ij}^{2\tau} N_{ij}^{2\tau-1} \dots N_{ij}^2}} \right] \\
&\quad \left[(a_{ij}^{(1,1)})^{\frac{1}{N_{ij}^{2\tau} N_{ij}^{2\tau-1} \dots N_{ij}^2}} \right].
\end{aligned} \tag{3.13}$$

For the odd-numbered stage $2\tau + 1$

$$\begin{aligned}
a_{ij}^{(2\tau+1, 2\tau+1)} &= \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{1}{N_{ij}^{2\tau+1}} + \frac{1}{N_{ij}^{2\tau+1} N_{ij}^{2\tau} N_{ij}^{2\tau-1}} + \dots + \frac{1}{N_{ij}^{2\tau+1} N_{ij}^{2\tau} \dots N_{ij}^3}} \right] \\
&\quad \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{N_{ij}^{2\tau+1} N_{ij}^{2\tau}} + \frac{1}{N_{ij}^{2\tau+1} N_{ij}^{2\tau} \dots N_{ij}^{2\tau-2}} + \dots + \frac{1}{N_{ij}^{2\tau+1} N_{ij}^{2\tau} \dots N_{ij}^2}} \right] \\
&\quad \left[(a_{ij}^{(1,1)})^{\frac{1}{N_{ij}^{2\tau+1} N_{ij}^{2\tau} \dots N_{ij}^2}} \right].
\end{aligned} \tag{3.14}$$

Subsequences are frequently used to establish the divergence of a sequence [7]. Next we define subsequence of a sequence, and then we give the divergence criteria that has been used in this section to establish the divergence of SGM method.

Definition 3.2: Subsequence of a Sequence

Let $A = (a_n)$ be a sequence of real numbers and let $n_1 < n_2 < \dots < n_i < \dots$ be a strictly increasing sequence of natural numbers. Then the sequence $A' = (a_{n_i})$ given by

$$(a_{n_1}, a_{n_2}, \dots, a_{n_i}, \dots)$$

is called a subsequence of A .

Theorem 3.1: Divergence Criteria

If a sequence $A = (a_n)$ of real numbers has two convergent subsequences $A' = (a_{n_i})$ and $A'' = (a_{n_j})$ whose limits are not equal, then A is divergent.

For the proof of the theorem 3.1 refer [7]. Now we will discuss the conditions under which SGM method satisfies the above divergence criteria. We will give one example to demonstrate the divergence of SGM method.

Divergence condition I:

$$1. t_{ii}^1 \neq 1, t_{jj}^1 \neq 1, t_{ii}^2 \neq 1, t_{jj}^2 = 1, a_{ij}^{(\tau, \tau)} \neq 1, \forall \tau,$$

OR

$$t_{ii}^1 \neq 1, t_{jj}^1 \neq 1, t_{ii}^2 = 1, t_{jj}^2 \neq 1, a_{ij}^{(\tau, \tau)} \neq 1, \forall \tau.$$

$$2. N_{ij}^\tau = \{3, \text{ for } \tau = 2, 4, 6, \dots; 2, \text{ for } \tau = 3, 5, 7, \dots\}.$$

$$3. \left(\frac{t_{jj}^1}{t_{ii}^1}\right) \neq \left(\frac{t_{jj}^2}{t_{ii}^2}\right)^2.$$

Numerical Example for Divergence Condition I:

$$\text{Let } t_{ii}^1 = \frac{1}{9}, t_{jj}^1 = 9, t_{ii}^2 = 3, t_{jj}^2 = 1, a_{ij}^{(1,1)} = 0.5.$$

These values satisfy condition (3) as $\left(\frac{t_{jj}^1}{t_{ii}^1}\right) = \frac{9}{\frac{1}{9}} = 81$, which is not equal to $\left(\frac{t_{jj}^2}{t_{ii}^2}\right)^2 = \frac{1}{3}$.

We will use theorem (7.2) to prove the divergence of SGM under divergence condition I.

We need to show that the subsequence of real numbers formed at stage $2\tau, \tau = 1, 2, \dots$

converges to a value different from the limit of the subsequence of real numbers formed at stage $2\tau + 1, \tau = 1, 2, \dots$. From equation (3.13), we have at even-numbered stages

$$\begin{aligned} a_{ij}^{(2\tau, 2\tau)} &= \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{1}{(3)(2)} + \frac{1}{(3)^2(2)^2} + \dots + \frac{1}{(3)^{(\tau-1)}(2)^{(\tau-1)}}} \right] \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{3} + \frac{1}{(3)^2(2)} + \dots + \frac{1}{(3)^{(\tau)}(2)^{(\tau-1)}}} \right] \\ &\quad \left[(a_{ij}^{(1,1)})^{(3)^{(\tau)}(2)^{(\tau-1)}} \right] \\ &= \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{(\frac{1}{(3)(2)})(1 - (\frac{1}{(3)(2)})^{(\tau-1)})}{(1 - (\frac{1}{(3)(2)})^{\tau})}} \right] \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{(\frac{1}{3})(1 - (\frac{1}{(3)(2)})^{\tau})}{(1 - (\frac{1}{(3)(2)})^{\tau})}} \right] \left[(a_{ij}^{(1,1)})^{(3)^{(\tau)}(2)^{(\tau-1)}} \right]. \end{aligned}$$

As $\tau \longrightarrow \infty$,

$$a_{ij}^{(2\tau, 2\tau)} \longrightarrow \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{(\frac{1}{(3)(2)})(1-0)}{(1 - (\frac{1}{(3)(2)})^{\tau})}} \right] \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{(\frac{1}{3})(1-0)}{(1 - (\frac{1}{(3)(2)})^{\tau})}} \right] \left[(a_{ij}^{(1,1)})^0 \right] = \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{1}{5}} \right] \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{2}{5}} \right].$$

Substituting the above values we get

$$a_{ij}^{(2\tau, 2\tau)} \longrightarrow \left[\left(\frac{1}{3} \right)^{\frac{1}{5}} \right] \left[\left(\frac{9}{1} \right)^{\frac{2}{5}} \right] = 4.655536722.$$

From equation (3.14), we have at odd-numbered stages

$$\begin{aligned} a_{ij}^{(2\tau+1, 2\tau+1)} &= \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{1}{2} + \frac{1}{(2)^2(3)} + \dots + \frac{1}{(2)^{(\tau)}(3)^{(\tau-1)}}} \right] \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{(2)(3)} + \frac{1}{(2)^2(3)^2} + \dots + \frac{1}{(2)^{(\tau)}(3)^{(\tau)}}} \right] \\ &\quad \left[(a_{ij}^{(1,1)})^{(2)^{(\tau)}(3)^{(\tau)}} \right] \\ &= \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{(\frac{1}{2})(1 - (\frac{1}{(2)(3)})^{\tau})}{(1 - (\frac{1}{(2)(3)})^{\tau})}} \right] \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{(\frac{1}{(2)(3)})(1 - (\frac{1}{(2)(3)})^{\tau})}{(1 - (\frac{1}{(2)(3)})^{\tau})}} \right] \left[(a_{ij}^{(1,1)})^{(2)^{(\tau)}(3)^{(\tau)}} \right]. \end{aligned}$$

As $\tau \longrightarrow \infty$,

$$a_{ij}^{(2\tau+1, 2\tau+1)} \longrightarrow \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{(\frac{1}{2})(1-0)}{(1 - (\frac{1}{(2)(3)})^{\tau})}} \right] \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{(\frac{1}{(2)(3)})(1-0)}{(1 - (\frac{1}{(2)(3)})^{\tau})}} \right] \left[(a_{ij}^{(1,1)})^0 \right] = \left[\left(\frac{t_{jj}^2}{t_{ii}^2} \right)^{\frac{3}{5}} \right] \left[\left(\frac{t_{jj}^1}{t_{ii}^1} \right)^{\frac{1}{5}} \right].$$

Substituting the above values we get

$$a_{ij}^{(2\tau, 2\tau)} \longrightarrow \left[\left(\frac{1}{3} \right)^{\frac{3}{5}} \right] \left[\left(\frac{9}{1} \right)^{\frac{1}{5}} \right] = 1.24573094.$$

Since subsequence of even stages converges to 4.655536722, while subsequence of odd stages converges to 1.24573094, hence the sequence consisting of these two subsequences is divergent according to the above theorem.

We next present some examples from among several divergence conditions for SGM method:

- Divergence condition II:**
1. $t_{ii}^1 \neq 1, t_{jj}^1 \neq 1, t_{ii}^2 \neq 1, t_{jj}^2 \neq 1, a_{ij}^{(\tau, \tau)} \neq 1, \forall \tau$
 2. $N_{ij}^\tau = 3, \forall \tau$
 3. $(\frac{t_{jj}^1}{t_{ii}^1}) \neq (\frac{t_{jj}^2}{t_{ii}^2})$.

- Divergence condition III:**
1. $a_{ij}^{(1,1)} = 1, t_{ii}^1 \neq 1, t_{jj}^1 \neq 1, t_{ii}^2 \neq 1, t_{jj}^2 \neq 1, a_{ij}^{(\tau, \tau)} \neq 1$ for $\tau = 2, 3, \dots$
 2. $N_{ij}^\tau = \{2, \text{ for } \tau = 2; 3, \text{ otherwise}\}$
 3. $(\frac{t_{jj}^1}{t_{ii}^1}) \neq (\frac{t_{jj}^2}{t_{ii}^2})$.

3.8 Remarks on the Convergence Behavior

Previous two sections established the conditional convergence of both GM and SGM methods while also proving that the SGM method diverges conditionally. These information are valuable in the following ways. Since SGM method displays divergence at some points, it holds an edge over GM method in generating distinct weight vectors at each stage of a large-scale MSMO optimization model. The distinct weight vectors at each stage allow an exhaustive exploration of the Pareto optimal solution set.

CHAPTER 4

APPLICATION TO A MULTIPLE STAGE, MULTIPLE OBJECTIVE WASTEWATER TREATMENT SYSTEM (WTS)

4.1 Introduction

Rapid population growth and continued industrial development have created enormous challenges in conserving water, one of the most precious natural resources, and keeping it clean for the survival of living beings on this planet. Despite these challenges, a wastewater treatment system (WTS) needs to be designed to meet the economic, environmental, space, and performance requirements. There is a need for wastewater treatment technologies that meet the demands of today. The wastewater treatment system considered here is based on the system presented by Chen and Beck [24], which involves cleaning the liquid and solid pollutants in several levels of processing of domestic wastewater. There are eleven levels of liquid processing, and six levels of solid processing shown in Figure 4.1 and Figure 4.2 respectively. This WTS differs from the one used in Tsai's dissertation [113] in the following:

- Yellow Water Separation, and Yellow and Black Water Separation are new treatment technologies added in WTS level 1.
- UASB System *plus* Activated Sludge Process (C, P, N) is a new treatment technology added in WTS level 5.
- There are multiple objectives at each level of WTS. These objectives are: *minimize* (economic cost, size, odor emissions), and *maximize* (nutrient recovery, robustness, global desirability).

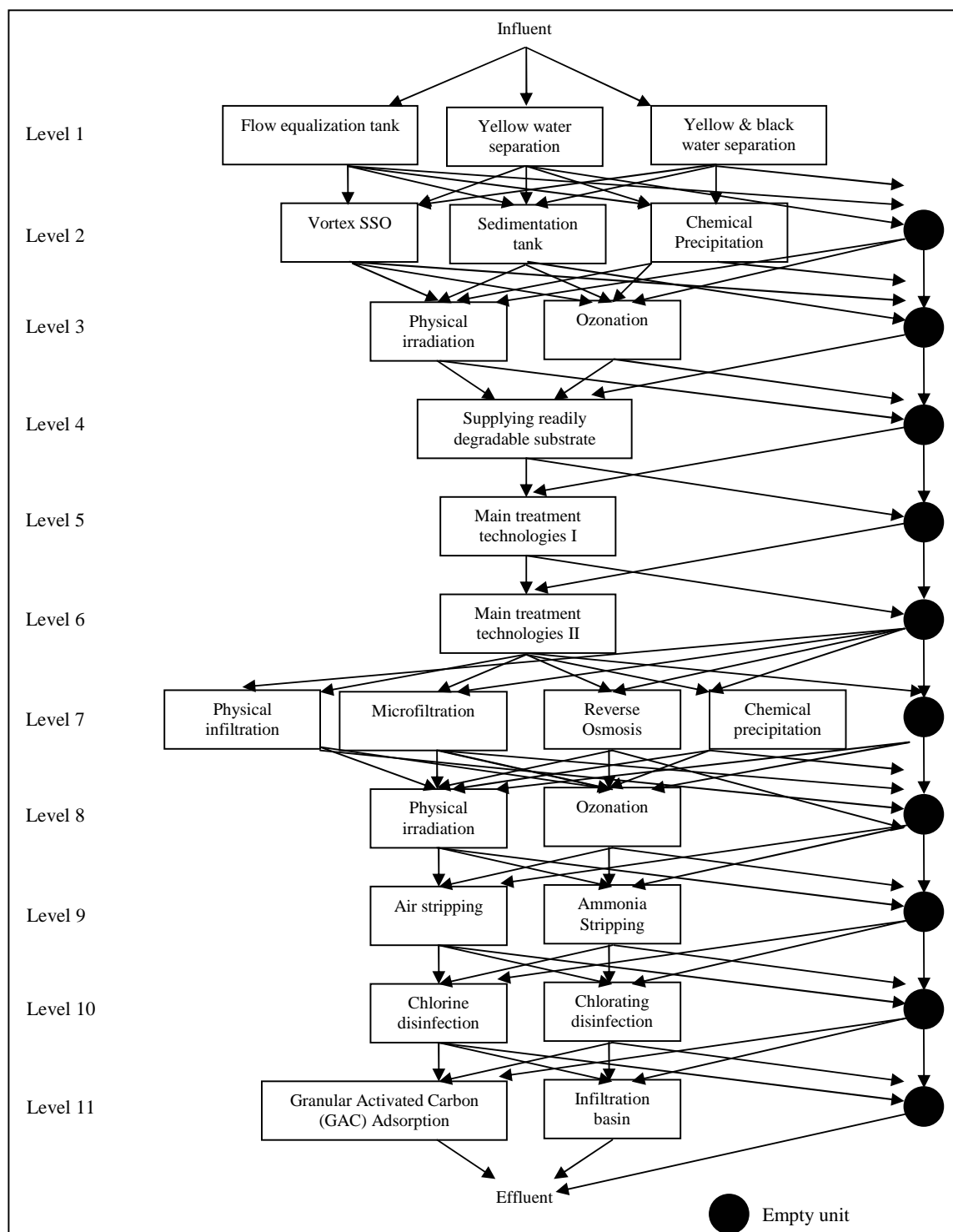


Figure 4.1. Levels and unit processes for the liquid line of the wastewater treatment system..

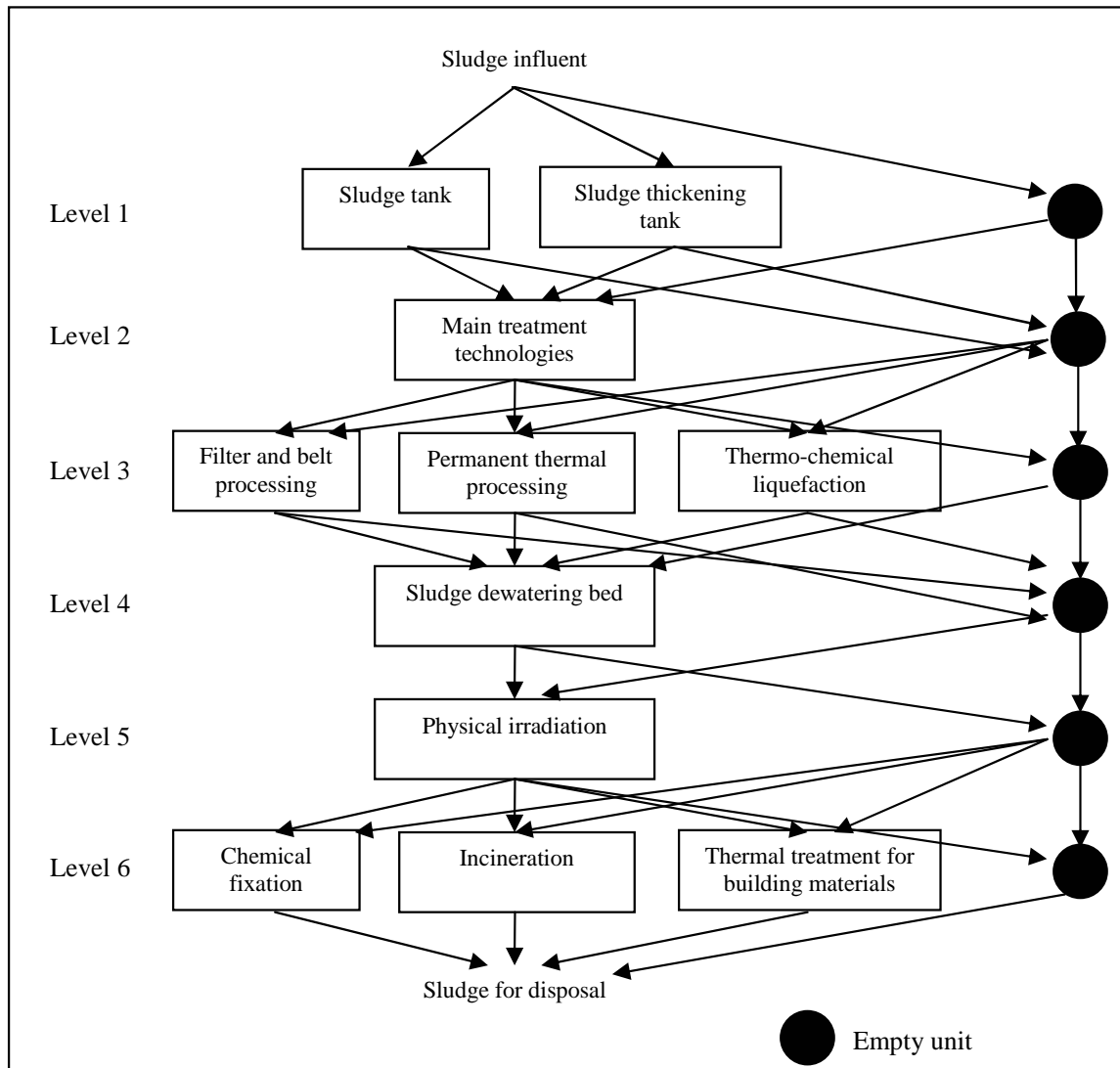


Figure 4.2. Levels and unit processes for the solid line of the wastewater treatment system..

In order to evaluate the treatment technology options for this multiple objective version of WTS, the decision-making elements - state variables, decision variables, transition functions, objectives, and constraints, are defined in the next section.

4.2 Decision-Making Elements in WTS

The multiple stage, single objective decision-making framework presented in Tsai [113] is extended to a multiple stage, multiple objective (MSMO) decision-making domain. The stages correspond to different levels of processing in WTS. Various elements of decision-making are specified next.

4.2.1 State and decision variables

The *state variables* represent the state of the wastewater treatment system going across various levels of treatment. The state variables are continuous. The following ten state variables were considered for the liquid processing in the wastewater treatment system:

1. chemical oxygen demand (**Liq-COD**).
2. suspended solids (**Liq-SS**).
3. organic-nitrogen (**Liq-orgN**).
4. ammonia-nitrogen (**Liq-ammN**).
5. nitrate-nitrogen (**Liq-nitN**).
6. total phosphorus (**Liq-totP**).
7. heavy metals (**Liq-HM**).
8. synthetic organic chemicals (**Liq-SOCs**).
9. pathogens (**Liq-pathogens**).
10. viruses (**Liq-viruses**).

Similarly, the ten sludge state variables considered for both liquid and solid processing are:

11. sludge volume (**Sl-Vol**).
12. sludge water content (**Sl-WC**).
13. sludge organic-carbon (**Sl-orgC**).

14. sludge inorganic-carbon (**Sl-inorgC**).
15. sludge organic-nitrogen (**Sl-orgN**).
16. sludge ammonia-nitrogen (**Sl-ammN**).
17. sludge total phospho-rus (**Sl-totP**).
18. sludge heavy metals (**Sl-HM**).
19. sludge synthetic organic chemicals (**Sl-SOCs**).
20. sludge pathogens (**Sl-pathogens**).

In the liquid line, all 20 of these state variables must be monitored. In particular, sludge is generated in levels 1, 2, 5, 6, and 7 of the liquid line.

The *decision variables* are the control variables, which are the technology units being evaluated at each level of the multiple stage, multiple objective WTS. In order to connect the liquid line and the solid line, the levels are numbered 1 through 17. The levels 1 through 11 form the solid line, and 12 through 17 form the solid line. The decision variables in this 17-level WTS are shown in Table 4.1. At each stage, a decision has to be made regarding the selection of a technology unit. At any level, technology units can be added or removed depending on current or future needs. Except for the first level of the liquid line, an “empty unit” can be seen in all other WTS levels. The selection of “empty unit” at a particular level implies that no treatment is performed at that level. Further, there are interstage dependencies for the usage of technology units at certain levels. Specifically, the list of interstage dependencies between the technology units is as follows:

1. In level 5, “Reed Bed System” requires using a technology unit in level 2.
2. In level 13, “Sludge C-G Drying” requires using of a technology unit in level 12.
3. In level 13, “Anaerobic Digestion” requires using a technology unit in level 12.
4. In level 14, “Filter and Belt” requires using the “Sludge Thickening Tank” in level 12 AND one of the following in level 13: Sludge Vertech + Ammonia Stripping,

CWOP-UASB + Ammonia Stripping, Sludge Hydrolysis + UASB, Anaerobic Digestion, OR Aerobic-Anaerobic Digestion.

Also, uncertainty of the following two types are modeled:

1. Uncertainty in the values of the influent and state variables at subsequent levels of WTS, and
2. Uncertainty in the performance of technology, denoted by u_t in level t , including the *objectives* associated with a technology.

Type (1) is represented by range limits, initiated by those in Table 4.2. In order to solve the SDP, we will describe later an approach that utilizes a *statistical experimental design* to efficiently represent the possible values and model over the continuous ranges of the state variables. Type (2) is represented by the stochastic vector ϵ_{t,u_t} . The dimension of this vector depends on the performance parameters for a particular technology, specified in the database by Jining Chen [25]. Since nothing is known about the appropriate probability distributions to represent the stochasticity, only the ranges of the performance parameters are specified, and sampling based on a uniform distribution is utilized. Narrower ranges are assigned to well-known technologies and wider ranges are assigned to newer, emerging technologies.

4.2.2 Transition functions

The transition function for a particular level determines how the state variables change at the exit point of this level. For the multivariate transition function at level t , given \mathbf{x}_t , the state entering level t , if u_t is the treatment technology selected in level t , and ϵ_{t,u_t} is the uncertainty associated with the transition, then the new state exiting level t , $\mathbf{x}_{t+1} = f_t(\mathbf{x}_t, u_t, \epsilon_{t,u_t})$.

Table 4.1. Decision Variables in WTS

LIQUID LINE OF WTS							
Level	1	2	3	4	5		
Technology Units	<ul style="list-style-type: none"> Flow Equalization Tank Yellow Water Separation Yellow & Black Water Separation 	<ul style="list-style-type: none"> Empty Unit Vortex SSO Sedimentation Tank Chemical Precipitation 	<ul style="list-style-type: none"> Empty Unit Physical Irradiation Ozonation 	<ul style="list-style-type: none"> Empty Unit 	<ul style="list-style-type: none"> Empty Unit Activated Sludge (C) Activated Sludge (C, N) Activated Sludge (C, P) Activated Sludge (C, P, N) High Biomass Act. Sludge (C, N) Activated Sludge (N) Multi-reactor/Deep 	<ul style="list-style-type: none"> A-B System Trickling Filter Rotating Biological Contractors UASB System UASB System + Activated Sludge (C, P, N) Reed Bed System Lagoons and Ponds 	
Level	6	7	8	9	10	11	
Technology Units	<ul style="list-style-type: none"> Empty Unit Secondary Settler Microfiltration Reverse Osmosis Chemical Precipitation 	<ul style="list-style-type: none"> Empty Unit Physical Filtration Microfiltration Reverse Osmosis Chemical Precipitation 	<ul style="list-style-type: none"> Empty Unit Physical Irradiation Ozonation 	<ul style="list-style-type: none"> Empty Unit Air Stripping Ammonia Stripping 	<ul style="list-style-type: none"> Empty Unit Chlorine Disinfection Chlorating Disinfection 	<ul style="list-style-type: none"> Empty Unit GAC Adsorption Infiltration Basin 	
SOLID LINE OF WTS							
Level	12	13	14	15	16	17	
Technology Units	<ul style="list-style-type: none"> Empty Unit Sludge Storage Tank Sludge Thickening Tank 	<ul style="list-style-type: none"> Empty Unit Sludge Dewatering Bed Sludge C-G Drying Sludge V + A Stripping Sludge CWOP-UASB + A Stripping Sludge Hydrolysis + UASB Anaerobic Digestion Aerobic Digestion Aerobic-Anaerobic Digestion 	<ul style="list-style-type: none"> Empty Unit Filter and Belt Permanent Thermal Process Thermo-Chemical Liquefaction 	<ul style="list-style-type: none"> Empty Unit Sludge Dewatering Bed (II) 	<ul style="list-style-type: none"> Empty Unit Physical Irradiation 	<ul style="list-style-type: none"> Empty Unit Chemical Fixation Incineration Thermal Building Materials 	

4.2.3 Objectives and constraints

As mentioned above, the six objective functions that are considered for the multiple stage, multiple objective wastewater treatment system are:

1. Minimize economic cost (in *US Dollars*), capital and operating cost of the treatment technology units
2. Minimize size (in m^2), the land area occupied by the treatment technology units
3. Minimize odor emissions (in mg/min), obtained by multiplying the concentration of the discharged gas (in the unit of mg/l or mg/m^3) and the flow of the gas (in the unit of l/min or m^3/min)
4. Maximize nutrient recovery (on 1-5 scale), characterizing the rating of the treatment technology units in cleaning liquid or sludge pollutants
5. Maximize robustness (no units), characterizing the insensitivity to the variation of the inputs
6. Maximize global desirability (on 1-6 scale), characterizing the impact of wastewater treatment outputs on the global environment

The basic constraints are cleanliness targets that are specified at levels 11 and 17 for the liquid and solid lines respectively. Also, constraints are added on the cleanliness of the liquid/sludge entering each level to handle a situation when liquid/sludge in the WTS are too polluted to be processed by any treatment technology units available in a particular level (as a result of empty units being selected too often in earlier levels). These range limits on the state variables entering each level are shown in Tables 4.2, 4.3, and 4.4. They define the state space for each level of liquid and solid lines. The weighted-sums of objective functions are calculated for each selected treatment technology unit, and are subject to uncertainty modeled similar to the performance parameters for each technological unit. Attainment of the constraints is achieved via a penalty function added to the weighted-sum of the objective functions. The same penalty function

was utilized to achieve cleanliness targets and to maintain state space limits. The mathematics behind the formulation of the target penalty function can be seen in Chen et al. [85] and Tsai [113]. The purpose of utilizing a penalty function is to assess penalty for violating liquid/sludge cleanliness targets. Therefore, the exact form of the penalty function is not that important, and it was chosen to facilitate modeling by multivariate adaptive regression splines (MARS). The cleanliness penalty is assessed in level 11 for the liquid line and in level 17 for the solid line. Tables 4.5 and 4.6 present the state variable ranges of the liquid/sludge exiting the wastewater treatment system, target values, cost smoothing values (denoted by Δ), and penalty coefficients. Target values can be easily adjusted to satisfy any desired requirements.

Table 4.2. Lower and Upper Limits (in mg/l) on the Ten Liquid State Variables of the Wastewater Treatment System for the Liquid Line (Levels 1-11)

Entering ↓	Liq-COD	Liq-SS	Liq-orgN	Liq-ammN	Liq-nitN
Level 1	200 210	220 231	15 15.75	25 26.25	0 0.01
Level 2	52 210	59.4 231	4.5 15.75	2 26.25	0 0.01
Level 3	33.8 210	5.94 115.5	1.35 14.9625	1.8 28.875	0 0.01
Level 5	27.04 199.5	5.346 115.5	1.08 14.214375	1.08 23.1	1.314 42.1675
Level 6	0.5408 69.825	0.10692 7000	0.162 100	0 21.945	0.1314 122.26675
Level 7	0.05408 69.825	1.07(10 ⁻³) 350	0.0162 100	0 21.945	0.01314 122.26675
Level 8	5.408(10 ⁻³) 62.8425	1.07(10 ⁻⁵) 52.5	0.00162 70	0 21.945	1.314(10 ⁻³) 122.26675
Level 9	4.3264(10 ⁻³) 59.700375	9.6228(10 ⁻⁶) 52.5	1.296(10 ⁻³) 66.5	0 17.556	1.314(10 ⁻³) 170.3263
Level 10	8.6528(10 ⁻⁴) 47.7603	9.6228(10 ⁻⁶) 52.5	1.296(10 ⁻³) 66.5	0 14.9226	1.314(10 ⁻³) 170.3263
Level 11	8.6528(10 ⁻⁴) 47.7603	9.6228(10 ⁻⁶) 52.5	1.296(10 ⁻³) 66.5	0 14.9226	1.314(10 ⁻³) 170.3263
Entering ↓	Liq-totP	Liq-HM	Liq-SOCs	Liq-pathogens	Liq-viruses
Level 1	8 8.4	0.01 0.0105	15 15.75	5.00(10 ⁷) 5.25(10 ⁷)	100 105
Level 2	2 8.4	0.0035 0.0105	3.75 15.75	1.75(10 ⁷) 5.25(10 ⁷)	40 105
Level 3	0.2 7.98	0.000175 0.00945	0.375 14.175	1.75(10 ⁶) 3.15(10 ⁷)	4 63
Level 5	0.2 7.98	0.000175 0.00945	0.05625 7.0875	0 3.15(10 ⁶)	0.08 6.3
Level 6	0.01 10	1.75(10 ⁻⁶) 0.00567	5.625(10 ⁻⁶) 4.2525	0 1.89(10 ⁶)	8(10 ⁻⁶) 3.78
Level 7	0 10	3.5(10 ⁻⁸) 0.00567	2.8125(10 ⁻⁷) 4.2525	0 1.89(10 ⁶)	4(10 ⁻⁸) 3.78
Level 8	0 8	7(10 ⁻¹⁰) 0.001701	1.4063(10 ⁻⁸) 1.063125	0 283500	2(10 ⁻¹⁰) 0.567
Level 9	0 8	7(10 ⁻¹⁰) 0.001701	2.1094(10 ⁻⁹) 0.5315625	0 28350	4(10 ⁻¹²) 0.0567
Level 10	0 8	7(10 ⁻¹⁰) 0.001701	3.164(10 ⁻¹⁰) 0.26578125	0 4252.5	4(10 ⁻¹⁴) 0.008505
Level 11	0 8	7(10 ⁻¹⁰) 0.001701	3.164(10 ⁻¹⁰) 0.3189375	0 212.625	8(10 ⁻¹⁵) 0.0025515

Table 4.3. Lower and Upper Limits (in mg/l) on the Ten Sludge State Variables of the Wastewater Treatment System for the Liquid Line (Levels 1-11)

Entering ↓	Sl-Vol	Sl-WC	Sl-orgC	Sl-inorgC	Sl-orgN
Level 2	0.0575 0.453	0.0566 0.451	197.36 5328.72	56.4 1998.27	3.595 169.853
Level 3	1.0575 100.405	0.86 99.9	3248.395 9.23(10 ⁶)	3167.25 9.41(10 ⁶)	86.93 6.66(10 ⁵)
Level 6	2.057 203.69	1.842 203.086	8048.395 1.03(10 ⁷)	4367.25 9.826(10 ⁶)	206.93 7.07(10 ⁵)
Level 7	2.057 2728.69	1.842 2725.56	8048.47 2.4(10 ⁷)	4367.266 1.89(10 ⁷)	206.932 1.48(10 ⁶)
Level 8	3.057 3078.69	2.802 3075.22	8348.47 2.455(10 ⁷)	4667.266 1.953(10 ⁷)	221.932 1.533(10 ⁶)
Entering ↓	Sl-ammN	Sl-totP	Sl-HM	Sl-SOCs	Sl-pathogens
Level 2	0.024 1.698	507.94 37705.5	0 0.83	152.381 14883.75	0 3.721(10 ⁹)
Level 3	0.3545 3999.77	552.1 5.14(10 ⁵)	0.126 629.252	341.62 9.08(10 ⁵)	3.81(10 ⁸) 1.79(10 ¹²)
Level 6	1.1545 4402.6	553.014 8.194(10 ⁵)	0.1285 1121.44	342.3 1.3(10 ⁶)	3.81(10 ⁸) 1.87(10 ¹²)
Level 7	1.1545 59813.71	553.014 4.72(10 ¹¹)	0.1285 5.583(10 ⁸)	342.3 2.932(10 ¹¹)	3.81(10 ⁸) 7.445(10 ¹⁶)
Level 8	1.1545 67494.46	553.014 4.72(10 ¹¹)	0.1285 5.583(10 ⁸)	342.3 2.93(10 ¹¹)	3.81(10 ⁸) 7.445(10 ¹⁶)

Table 4.4. Lower and Upper Limits (in mg/l) on the Ten Liquid State Variables of the Wastewater Treatment System for the Solid Line (Levels 12-17)

Entering ▼	SI-Vol	SI-WC	SI-orgC	SI-inorgC	SI-orgN
Level 12	3.0575 3078.691	0.001 0.9999	2.712 8.03(10 ⁶)	1.516 6.387(10 ⁶)	7.21(10 ⁻²) 5.0125(10 ⁵)
Level 13	1 3078.691	0.001 0.996	2.441 8.03(10 ⁶)	1.516 6.387(10 ⁶)	6.5(10 ⁻²) 5.0125(10 ⁵)
Level 14	1 3078.691	9.1(10 ⁻⁵) 0.999	0.122 8.03(10 ⁶)	0.455 6.387(10 ⁶)	3.24(10 ⁻³) 5.0125(10 ⁵)
Level 15	1 3078.691	0.05 0.8	0.0122 8.03(10 ⁶)	0.273 6.387(10 ⁶)	4.866(10 ⁻⁴) 5.0125(10 ⁵)
Level 16	1 3078.691	0.005 0.6	0.0122 8.03(10 ⁶)	0.273 6.387(10 ⁶)	4.866(10 ⁻⁴) 5.0125(10 ⁵)
Level 17	1 3078.691	0.005 0.6	0.0122 8.03(10 ⁶)	0.273 6.387(10 ⁶)	4.866(10 ⁻⁴) 5.0125(10 ⁵)
Entering ▼	SI-ammN	SI-totP	SI-HM	SI-SOCs	SI-pathogens
Level 12	3.75(10 ⁻⁴) 2.21(10 ⁴)	0.1796 1.544(10 ¹¹)	4.173(10 ⁻⁵) 1.826(10 ⁸)	0.1112 9.6(10 ¹⁰)	1.24(10 ⁵) 2.435(10 ¹⁶)
Level 13	3.75(10 ⁻⁴) 2.21(10 ⁴)	0.1796 1.544(10 ¹¹)	4.173(10 ⁻⁵) 1.826(10 ⁸)	0.1056 9.6(10 ¹⁰)	6.187(10 ⁴) 2.435(10 ¹⁶)
Level 14	3.75(10 ⁻⁵) 2.21(10 ⁴)	0.028 1.544(10 ¹¹)	6.506(10 ⁻⁶) 1.826(10 ⁸)	0.0317 9.6(10 ¹⁰)	61.87 2.435(10 ¹⁶)
Level 15	3.75(10 ⁻⁶) 2.21(10 ⁴)	0.0168 1.544(10 ¹¹)	6.506(10 ⁻⁶) 1.826(10 ⁸)	0 9.6(10 ¹⁰)	18.561 2.435(10 ¹⁶)
Level 16	3.56(10 ⁻⁶) 2.21(10 ⁴)	0.0168 1.544(10 ¹¹)	6.506(10 ⁻⁶) 1.826(10 ⁸)	0 9.6(10 ¹⁰)	0.18561 2.435(10 ¹⁵)
Level 17	3.56(10 ⁻⁶) 2.21(10 ⁴)	0.0168 1.544(10 ¹¹)	6.506(10 ⁻⁶) 1.826(10 ⁸)	0 9.6(10 ¹⁰)	1.856(10 ⁻⁴) 2.435(10 ¹³)

Table 4.5. Minimum and Maximum Values (in mg/l) of the Ten Liquid State Variables of the Wastewater Exiting the Liquid Line

Entering ▼	Liq-COD	Liq-SS	Liq-orgN	Liq-ammN	Liq-nitN
Minimum	8.6528(10^{-4})	9.6228(10^{-6})	1.296(10^{-3})	0	1.314(10^{-3})
Maximum	47.7603	52.5	66.5	14.9226	170.3263
Target	5	5.5	7	1.5	16
Δ	21.4	24	30	6.7	77.2
Penalty	47	43	34	149	13
Entering ▼	Liq-totP	Liq-HM	Liq-SOCs	Liq-pathogens	Liq-viruses
Minimum	0	7(10^{-10})	3.164(10^{-10})	0	8(10^{-15})
Maximum	8	0.001701	0.32	212.625	0.0025515
Target	0.8	0.00015	0.04	20	0.0003
Δ	3.6	0.00078	0.1395	96.313	0.001126
Penalty	278	1.29(10^6)	7.2(10^3)	10.4	8.88(10^5)

4.3 Multiple Objective Stochastic Dynamic Programming Formulation of WTS and Weighted-Sum Scalarization

The MSMO version of WTS is formulated as a stochastic dynamic programming (SDP), which has traditionally been used for optimization of multiperiod problems in areas of application such as engineering, finance, economics, etc. [13] The WTS is a continuous-state, 20-dimensional, 17-stage (time periods or levels), 6-objective MSMO optimization problem. Since state variables are continuous, there is a need to discretize the state space for finding approximate solutions. We use the orthogonal array (OA) based Latin hypercube designs for state space discretization, and the MARS [114] algorithm is used for future value function approximation. The details on the orthogonal array based Latin hypercubes, OA-LHD and OA-LHS, can be seen in the Tsai's dissertation [113]. A discussion on the MARS algorithm used for WTS can also be seen in the Tsai's dissertation [113]. Next, the multiple objective stochastic dynamic programming

Table 4.6. Minimum and Maximum Values (in mg/l) of the Ten Solid State Variables for the Sludge Exiting the Solid Line

Entering ↓	Sl-Vol	Sl-WC	Sl-orgC	Sl-inorgC	Sl-orgN
Minimum	1	0	0.0122	0.082	4.87(10 ⁻⁴)
Maximum	3078.691	0.3	8.03(10 ⁶)	8.942(10 ⁶)	5.0125(10 ⁵)
Target	306	0.03	6.08(10 ⁵)	6.83(10 ⁵)	3.81(10 ⁴)
Δ	1386	0.135	3.71(10 ⁶)	4.13(10 ⁶)	2.316(10 ⁵)
Penalty	0.72	7407.41	2.7(10 ⁻⁴)	2.4(10 ⁻⁴)	4.32(10 ⁻³)
Entering ↓	Sl-ammN	Sl-totP	Sl-HM	Sl-SOCs	Sl-pathogens
Minimum	1.425(10 ⁻⁶)	0.0168	6.5(10 ⁻⁶)	0	1.856(10 ⁻⁶)
Maximum	2.21(10 ⁴)	1.5442(10 ¹¹)	1.826(10 ⁸)	9.588(10 ¹⁰)	2.435(10 ¹³)
Target	1685	1.472(10 ⁵)	453	2.74(10 ⁵)	1.38(10 ⁸)
Δ	1.02(10 ⁴)	7.721(10 ¹⁰)	9.13(10 ⁷)	4.794(10 ¹⁰)	1.22(10 ¹³)
Penalty	0.0981	1.3(10 ⁻⁸)	1.1(10 ⁻⁵)	2.09(10 ⁻⁸)	8.21(10 ⁻¹¹)

formulation of WTS is elaborated. The multiple objective SDP can be formulated as (shown schematically in Figure 1.1-Chapter 1):

$$\begin{aligned}
 V - \min_{u_1, \dots, u_T} \quad & E \{ M \{ \mathbf{m}_1(\mathbf{x}_1, u_1, \epsilon_{1,u_1}), \mathbf{m}_2(\mathbf{x}_2, u_2, \epsilon_{2,u_2}), \dots, \mathbf{m}_T(\mathbf{x}_T, u_T, \epsilon_{T,u_T}) \} \} \\
 \text{s.t.} \quad & \mathbf{x}_{\tau+1} = f_{\tau}(\mathbf{x}_{\tau}, u_{\tau}, \epsilon_{\tau,u_{\tau}}), \text{ for } \tau = 1, \dots, T-1 \text{ and} \\
 & \mathbf{x}_{\tau} \in S_{\tau}, \quad u_{\tau} \in \Gamma_{\tau}, \text{ for } \tau = 1, \dots, T
 \end{aligned} \tag{4.1}$$

where T is time horizon, \mathbf{x}_{τ} is the state vector (attributes of the liquid/sludge), u_{τ} is the index of the selected technology unit, $\epsilon_{\tau,u_{\tau}}$ is the stochastic component on the performance parameters of unit u_{τ} , $\mathbf{x}_{\tau+1}$ is determined by the transition function $f_{\tau}(\cdot)$, S_{τ} contains the range limits on the state variables, Γ_{τ} contains the indices of the available technology units, the function $M(\cdot)$ denotes the multiple objective return function, and $V-min$ is used to differentiate the vector minimization problem from the minimization problem. Also, the objective vector at stage τ is

$$\mathbf{m}_{\tau}(\mathbf{x}_{\tau}, u_{\tau}, \epsilon_{\tau,u_{\tau}}) = \begin{pmatrix} m_{\tau}^1(\mathbf{x}_{\tau}, u_{\tau}, \epsilon_{\tau,u_{\tau}}) \\ m_{\tau}^2(\mathbf{x}_{\tau}, u_{\tau}, \epsilon_{\tau,u_{\tau}}) \\ \vdots \\ m_{\tau}^k(\mathbf{x}_{\tau}, u_{\tau}, \epsilon_{\tau,u_{\tau}}) \end{pmatrix},$$

where k is the number of objective functions at each stage.

The equation 4.1 is the original multiple stage, multiple objective problem formulation. The next task is to generate Pareto optimal solutions. Some of the methods for generating Pareto optimal points include, weighted-sum [21], ε -constraint [21], hybrid that combines both weighted-sum and ε -constraint [73], norm or weighted metrics, and minimax. The weighted-sum method is used here for its ease of application and involvement of decision makers in the solution process. The weighted-sum transforms multiple objective functions into a single objective function, and optimizes the weighted-sum of the objectives. This conversion process is usually referred to as scalarization. The weighted-sum scalarized form of the problem (shown schematically in Figure 1.2-Chapter 1) becomes

$$\begin{aligned} \min_{u_1, \dots, u_T} \quad & E \left\{ \sum_{\tau=1}^T \mathbf{W}_\tau \mathbf{m}_\tau(\mathbf{x}_\tau, u_\tau, \epsilon_{\tau, u_\tau}) \right\} \\ \text{s.t.} \quad & \mathbf{x}_{\tau+1} = f_\tau(\mathbf{x}_\tau, u_\tau, \epsilon_{\tau, u_\tau}), \text{ for } \tau = 1, \dots, T-1 \text{ and} \\ & \mathbf{x}_\tau \in S_\tau, \quad u_\tau \in \Gamma_\tau, \text{ for } \tau = 1, \dots, T \end{aligned} \tag{4.2}$$

where \mathbf{W}_τ is the *weight vector* at stage τ , $(w_\tau^1, w_\tau^2, \dots, w_\tau^k)$, and $\mathbf{m}_\tau(\mathbf{x}_\tau, u_\tau, \epsilon_{\tau, u_\tau})$ is the *objective vector* at stage τ , $(m_\tau^1(\mathbf{x}_\tau, u_\tau, \epsilon_{\tau, u_\tau}), m_\tau^2(\mathbf{x}_\tau, u_\tau, \epsilon_{\tau, u_\tau}), \dots, m_\tau^k(\mathbf{x}_\tau, u_\tau, \epsilon_{\tau, u_\tau}))'$. We next demonstrate the application of our three-phase methodology for computing weight vectors at each stage of the wastewater treatment system.

4.4 Normalization/transformation of objective functions

It is practical to transform or normalize the objective functions so that they all have comparable orders of magnitude. We use the *upper-lower-bound approach* recommended in Marler and Arora [71]. For a given stage τ , this approach uses the transformation,

$$mt^i = \frac{m^i(\mathbf{u}) - m_0^i}{m_{\max}^i - m_0^i}, \quad (4.3)$$

$$m_{\max}^i = \max_{1 \leq j \leq k} m^i(\mathbf{u}_j^*), \quad (4.4)$$

for $i = 1, \dots, k$ where k is the number of objective functions and \mathbf{u}_j^* is the point that minimizes the j th objective function,

$$m_0^i = \min_{\mathbf{u}} \{m^i(\mathbf{u}) | \mathbf{u} \in \Gamma\}, \quad (4.5)$$

where Γ is the feasible decision space.

Therefore, the normalized/transformed form of the stochastic dynamic programming problem formulation in equation 4.2 is,

$$\begin{aligned} \min_{u_1, \dots, u_T} \quad & E \left\{ \sum_{\tau=1}^T \mathbf{W}_\tau \mathbf{M} \mathbf{T}_\tau \mathbf{m}_\tau(\mathbf{x}_\tau, u_\tau, \epsilon_{\tau, u_\tau}) \right\} \\ \text{s.t.} \quad & \mathbf{x}_{\tau+1} = f_\tau(\mathbf{x}_\tau, u_\tau, \epsilon_{\tau, u_\tau}), \text{ for } \tau = 1, \dots, T-1 \text{ and} \\ & \mathbf{x}_\tau \in S_\tau, \quad u_\tau \in \Gamma_\tau, \text{ for } \tau = 1, \dots, T \end{aligned} \quad (4.6)$$

where $\mathbf{M} \mathbf{T}_\tau$ is the *objective transformation vector* at stage τ , $(mt_\tau^1, mt_\tau^2, \dots, mt_\tau^k)$.

4.5 Implementation of Three-Phase Methodology

We are applying our three-phase methodology [26] on a conceptual multiple stage, multiple objective wastewater treatment system. This extends the multiple stage, single objective decision-making framework presented in Tsai [113] to a multiple stage, multiple objective (MSMO) decision-making domain. Table 4.1 shows choices of treatment technology units at all levels of WTS. The stages correspond to different levels of processing in WTS. The MSMO version of WTS has two new technologies at level 1 and

one new technology at level 5 of the liquid processing line. As mentioned above, the goal is to select the treatment technology units at each level of WTS for *minimizing* economic cost, size, and odor emissions, while *maximizing* nutrient recovery, robustness, and global desirability. We will present the results of the implementation of our three-phase methodology on the MSMO version of the WTS in the following sections.

4.5.1 Input phase implementation

For the input phase, we use the questionnaire-modeling approach discussed in chapter 3. Since a single technology unit at level 4 of WTS implies that no optimization is needed at this level, we do not include it in the questionnaire. In other words, our questionnaire concerns levels 1 through 3, and 5 through 17. Dr. Feng Jiang, who specializes in wastewater treatment system and works at Georgia Department of Natural Resources (GADNR), provided us with the answers to our questionnaire. Dr. Jiang was a student of Dr. M. Bruce Beck at the University of Georgia, and is well-versed with the original WTS co-developed by Dr. Beck and Dr. Jining Chen. Moreover, both Drs. Beck and Chen have worked with Dr. Victoria Chen on the SDP formulation of WTS. Dr. Jiang responded quickly and enthusiastically to a total of 390 questions. The information provided by the questionnaire is used to compute practical weight vectors relating the objectives at each WTS level. We denote the WTS objectives at level τ as follows:

- EC_τ denotes the Economic Cost (in *USD*),
- S_τ denotes the Land Area (in m^2),
- O_τ denotes the Odor Emissions (in *mg/min*),
- NR_τ denotes the Nutrient Recovery (on 1-5 scale),
- R_τ denotes the Robustness (no units), and
- GD_τ denotes the Global Desirability (on 1-6 scale)

We asked Dr. Jiang questions on: the worst/best values for the objectives at each level, the relative importance for different objectives within level 1, and the relative importance for same types of objectives across different levels, of the WTS. The WTS questionnaire is organized in the following tables:

- Table 4.7 presents the worst and the best values for all six objectives at levels 1 through 3. These values are based on the WTS code developed by Jining Chen [25, 24]. The worst and best values for all six objectives at all 17 levels can be seen in the Appendix A.
- Table 4.8 presents importance questions for comparing the different objectives within Level 1 of WTS.
- Table 4.9 presents interlevel importance questions for comparing the same objective types across different levels of WTS. The complete tables for all six objectives can be seen in the Appendix A.

The worst and best values in Table 4.7 help decision makers answer tradeoff questions (or importance questions) in Tables 4.8 and 4.9 with a relatively high level of precision and consistency. We use Tables 4.8 and 4.9 to get the complete pairwise comparison matrix at level 1 and interstage diagonal matrices, respectively.

Table 4.7. The Worst and the Best Objective Values in WTS Levels 1-3

The Worst Value	The Best Value	The Worst Value	The Best Value	The Worst Value	The Best Value
Economic Cost (in USD)		Nutrient Recovery (on 1-5 ordinal scale)		Robustness (No units)	
$EC_1=43200550$	$EC_1=9643.3121$	$NR_1=1$	$NR_1=4$	$R_1=3821656.051$	$R_1=15286624.2$
$EC_2=14500$	$EC_2=2500$	$NR_2=1$	$NR_2=5$	$R_2=2000$	$R_2=9000$
$EC_3=9300$	$EC_3=6600$	$NR_3=1$	$NR_3=1$	$R_3=675$	$R_3=1500$
Size (in m^3)		Odor Emissions (in mg/min)		Global Desirability (on 1-6 ordinal scale)	
$S_1=96000$	$S_1=1274$	$O_1=384000$	$O_1=3821.6561$	$GD_1=2.5$	$GD_1=6$
$S_2=1000$	$S_2=100$	$O_2=3500$	$O_2=250$	$GD_2=2.5$	$GD_2=4$
$S_3=600$	$S_3=75$	$O_3=2880$	$O_3=330$	$GD_3=4$	$GD_3=4.8$

Table 4.8. The Importance Questions for Level 1 of Wastewater Treatment System

Which one of the following pairs of objectives is more important in terms of improvement from the worst value to the best value?			Given the more important objective, how many times is this objective more important than the other? (Note: "Equal" = 1)									
			1	2	3	4	5	6	7	8	9	
<input checked="" type="checkbox"/> EC_1	<input type="checkbox"/> S_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input checked="" type="checkbox"/> EC_1	<input type="checkbox"/> O_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	
<input checked="" type="checkbox"/> EC_1	<input type="checkbox"/> NR_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input checked="" type="checkbox"/> EC_1	<input type="checkbox"/> R_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input checked="" type="checkbox"/> EC_1	<input type="checkbox"/> GD_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input checked="" type="checkbox"/> S_1	<input type="checkbox"/> O_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input checked="" type="checkbox"/> S_1	<input type="checkbox"/> NR_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input checked="" type="checkbox"/> S_1	<input type="checkbox"/> R_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input checked="" type="checkbox"/> S_1	<input type="checkbox"/> GD_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input type="checkbox"/> O_1	<input checked="" type="checkbox"/> NR_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input type="checkbox"/> O_1	<input checked="" type="checkbox"/> R_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input type="checkbox"/> O_1	<input checked="" type="checkbox"/> GD_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input checked="" type="checkbox"/> NR_1	<input type="checkbox"/> R_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input checked="" type="checkbox"/> NR_1	<input type="checkbox"/> GD_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	
<input checked="" type="checkbox"/> R_1	<input type="checkbox"/> GD_1	<input type="checkbox"/> Equal	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	

The questionnaire-based approach results in the pairwise comparison matrix at stage 1, $A_{(1,1)}$, and interstage diagonal transformation matrices (or matrices of dependencies from one stage to the next), T_τ , for $\tau = 1, 2, \dots, (T - 1)$, where $T=17$. These matrices are utilized as inputs to the matrix generation phase. The resulting pairwise comparison matrix at stage 1,

$$A_{(1,1)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1 & 3 & \frac{1}{4} & \frac{1}{2} & 4 & 4 \\ \mathbf{S} & \frac{1}{3} & 1 & \frac{1}{6} & \frac{1}{4} & 2 & \frac{1}{2} \\ \mathbf{O} & 4 & 6 & 1 & 2 & 8 & 7 \\ \mathbf{GD} & 2 & 4 & \frac{1}{2} & 1 & 6 & 6 \\ \mathbf{EC} & \frac{1}{4} & \frac{1}{2} & \frac{1}{8} & \frac{1}{6} & 1 & \frac{1}{3} \\ \mathbf{NR} & \frac{1}{4} & 2 & \frac{1}{7} & \frac{1}{6} & 3 & 1 \end{pmatrix}.$$

The sixteen interstage diagonal transformation matrices are,

$$T_1 = \begin{pmatrix} \frac{1}{7} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}, T_2 = \begin{pmatrix} \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} \end{pmatrix}, T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, T_5 = \begin{pmatrix} \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}, T_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_7 = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} \end{pmatrix}, T_8 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, T_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} \end{pmatrix},$$

$$T_{10} = \begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}, T_{11} = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, T_{12} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{13} = \begin{pmatrix} \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}, T_{14} = \begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{15} = \begin{pmatrix} \frac{1}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} \end{pmatrix}, T_{16} = \begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}.$$

The above matrices are used as inputs to our two new matrix generation methods, GM and SGM, for computing pairwise comparison matrices at each stage of the MSMO version of the WTS in the matrix generation phase.

4.5.2 Matrix generation phase implementation

For both GM and SGM methods, the computed pairwise comparison matrix at stage 2 is,

$$A_{(2,2)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.912931 & 0.704730 & 1.052727 & 1.518294 & 4.820285 \\ \mathbf{S} & 0.522758 & 1.000000 & 0.464159 & 0.629961 & 0.908560 & 1.817121 \\ \mathbf{O} & 1.418983 & 2.154435 & 1.000000 & 1.493802 & 1.709976 & 5.192494 \\ \mathbf{GD} & 0.949914 & 1.587401 & 0.669433 & 1.000000 & 1.310371 & 4.160168 \\ \mathbf{EC} & 0.658634 & 1.100642 & 0.584804 & 0.763143 & 1.000000 & 2.201285 \\ \mathbf{NR} & 0.207457 & 0.550321 & 0.192586 & 0.240375 & 0.454280 & 1.000000 \end{pmatrix},$$

The computed pairwise comparison matrices for WTS levels 3 through 17 using the methods, geometric mean (GM) of non-ones and successive geometric mean (SGM) of non-ones, for matrix generation can be seen in Appendix B.

4.5.3 Weighting Phase Implementation

Saaty's eigenvector approach is used to compute weight vectors for both GM and SGM method. These are principal eigenvectors for the pairwise comparison matrices obtained in the matrix generation phase. Tables 4.10 and 4.11 are the weight vectors for the GM and SGM methods, respectively.

4.5.4 Computational results on the consistency of computed pairwise matrices

In this section, results for consistency indices are presented. Lower values of consistency indices indicate better consistencies of corresponding pairwise comparison matrices. These results show that the weight vectors computed are extremely meaningful, and re-

Table 4.10. Weight Vectors for GM Method

	GM Method
\mathbf{W}_1	(0.154237, 0.058427, 0.421142, 0.256685, 0.036603, 0.072906)
\mathbf{W}_2	(0.218300, 0.117765, 0.273487, 0.195936, 0.139935, 0.054577)
\mathbf{W}_3	(0.244031, 0.106403, 0.214016, 0.119192, 0.203495, 0.112864)
\mathbf{W}_4	(0.244031, 0.106403, 0.214016, 0.119192, 0.203495, 0.112864)
\mathbf{W}_5	(0.244031, 0.106403, 0.214016, 0.119192, 0.203495, 0.112864)
\mathbf{W}_6	(0.231888, 0.137621, 0.225310, 0.101896, 0.219167, 0.084118)
\mathbf{W}_7	(0.231888, 0.137621, 0.225310, 0.101896, 0.219167, 0.084118)
\mathbf{W}_8	(0.221956, 0.142302, 0.204020, 0.089873, 0.225492, 0.116357)
\mathbf{W}_9	(0.211932, 0.147220, 0.196833, 0.104796, 0.214795, 0.124425)
\mathbf{W}_{10}	(0.193660, 0.147616, 0.188594, 0.128934, 0.196043, 0.145154)
\mathbf{W}_{11}	(0.191603, 0.154427, 0.187325, 0.131833, 0.189292, 0.145520)
\mathbf{W}_{12}	(0.186320, 0.161504, 0.186168, 0.132277, 0.174369, 0.159363)
\mathbf{W}_{13}	(0.183047, 0.161063, 0.179718, 0.138594, 0.167228, 0.170350)
\mathbf{W}_{14}	(0.186209, 0.166115, 0.183183, 0.124514, 0.171779, 0.168199)
\mathbf{W}_{15}	(0.176766, 0.160269, 0.174110, 0.146163, 0.163228, 0.179465)
\mathbf{W}_{16}	(0.180087, 0.164718, 0.177622, 0.127089, 0.167485, 0.182998)
\mathbf{W}_{17}	(0.178291, 0.167024, 0.177371, 0.135758, 0.164072, 0.177484)

Table 4.11. Weight Vectors for SGM Method

	SGM Method
\mathbf{W}_1	(0.154237, 0.058427, 0.421142, 0.256685, 0.036603, 0.072906)
\mathbf{W}_2	(0.218300, 0.117765, 0.273487, 0.195936, 0.139935, 0.054577)
\mathbf{W}_3	(0.249305, 0.087228, 0.150558, 0.076308, 0.253163, 0.183438)
\mathbf{W}_4	(0.249305, 0.087228, 0.150558, 0.076308, 0.253163, 0.183438)
\mathbf{W}_5	(0.249305, 0.087228, 0.150558, 0.076308, 0.253163, 0.183438)
\mathbf{W}_6	(0.223666, 0.164886, 0.212225, 0.063104, 0.266724, 0.069395)
\mathbf{W}_7	(0.223666, 0.164886, 0.212225, 0.063104, 0.266724, 0.069395)
\mathbf{W}_8	(0.192707, 0.152082, 0.165406, 0.060352, 0.242291, 0.187161)
\mathbf{W}_9	(0.174404, 0.160278, 0.165265, 0.138625, 0.189626, 0.171803)
\mathbf{W}_{10}	(0.097431, 0.144250, 0.145907, 0.243649, 0.102808, 0.265956)
\mathbf{W}_{11}	(0.149496, 0.181108, 0.171078, 0.177404, 0.138297, 0.18261)
\mathbf{W}_{12}	(0.134220, 0.185263, 0.156601, 0.131842, 0.097592, 0.294482)
\mathbf{W}_{13}	(0.132221, 0.149261, 0.126301, 0.152176, 0.097900, 0.342142)
\mathbf{W}_{14}	(0.183986, 0.191534, 0.181199, 0.076517, 0.166317, 0.200446)
\mathbf{W}_{15}	(0.107444, 0.114419, 0.106874, 0.274970, 0.099282, 0.297011)
\mathbf{W}_{16}	(0.173007, 0.176721, 0.172702, 0.066514, 0.168428, 0.242628)
\mathbf{W}_{17}	(0.163754, 0.188783, 0.173912, 0.171678, 0.141776, 0.160098)

spresent a high level of consistency in the decision makers' judgments derived in the input phase.

Table 4.12. Consistency Indices (CI) for GM and SGM Methods

	GM Method	SGM Method
CI₁	0.054527	0.054527
CI₂	0.005730	0.005730
CI₃	0.003273	0.012704
CI₄	0.003273	0.012704
CI₅	0.003273	0.012704
CI₆	0.002739	0.010679
CI₇	0.002739	0.010679
CI₈	0.001420	0.001159
CI₉	0.001117	0.000310
CI₁₀	0.000546	0.002617
CI₁₁	0.000380	0.000291
CI₁₂	0.000247	0.001748
CI₁₃	0.000196	0.001932
CI₁₄	0.000148	0.000214
CI₁₅	0.000140	0.004674
CI₁₆	0.000109	0.000517
CI₁₇	0.000091	0.000057

For GM method, consistency indices decrease as we go from one stage to the next. This can be seen in Figure 4.3, the plot for the consistency index over various WTS levels. It can be observed that the consistency indices obtained for the SGM method are higher than the ones obtained using the GM method, which implies that the GM method performs better in terms of consistency of judgments elicited from decision makers.

4.6 Solution to the Scalarized WTS

The MSMO version of WTS was scalarized using weighted-sum of objective functions at each stage, which resulted in a stochastic dynamic programming (SDP) formu-

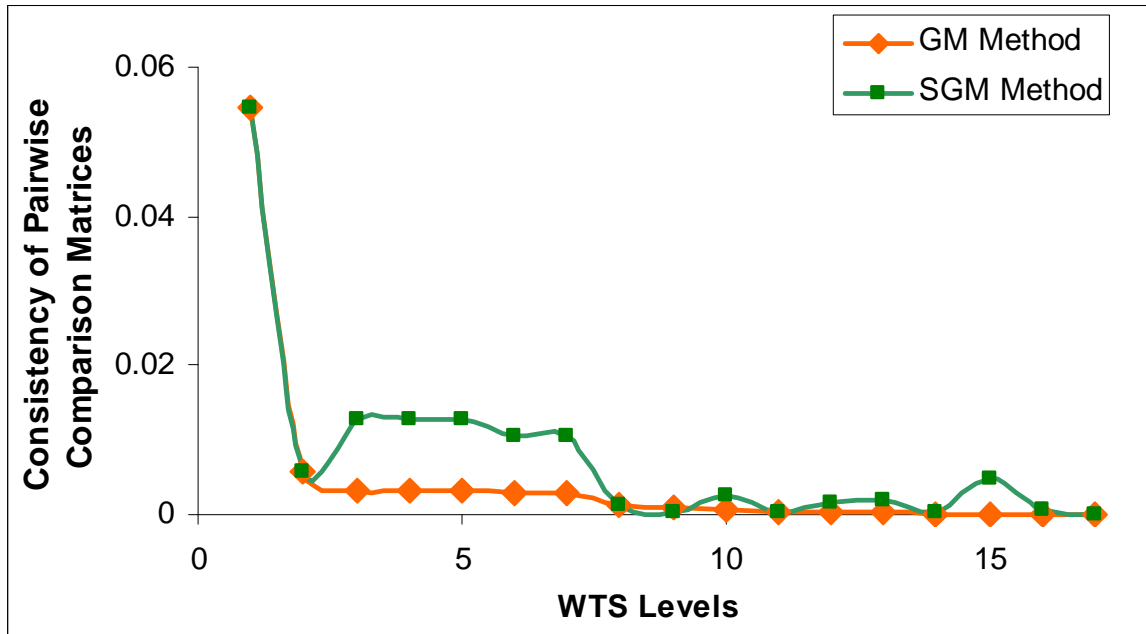


Figure 4.3. Consistency indices of pairwise comparison matrices over WTS levels..

lation of wastewater treatment system. In this section, we present the SDP solution results, which are obtained using an approach that modified the methods described in Tsai [113] and Chen et al. [85] to deal with multiple objectives. In particular, the following modifications were made that helped scalarize the MSMO form of WTS (i.e. to convert it into Multiple Stage, Single Objective (MSSO) problem):

- A routine was added to capture decision makers' preferences, which get translated into weights using the three-phase methodology.
- A routine was added to normalize/transform the objective functions with different units to have similar units and orders of magnitude.
- State transition equations were added for new technology units- Yellow Water Separation and Yellow and Black Water Separation in level 1, and UASB System *plus* Activated Sludge Process (C, P, N) in level 5.
- Constraints (both liquid and sludge state variable limits in each level) were modified to reflect the addition of new technology units.

- Penalty coefficients and costsmooth parameters were modified due to addition of new technologies.

For demonstration purposes, we have considered a small design with 2209 design points and a big design with 12167 design points. These experimental designs are elucidated in Tsai's dissertation [113].

We next explain how these design points will be used to evaluate technology processes/units in WTS. We solve for the future value functions backward. Starting in the last level, for each discretization/design point we solve the optimization problem that minimizes the weighted-sum of objective functions in the last level. Then, we approximate the solution to this minimization problem through a future value function, for all the discretization points, obtained by fitting a statistical model to the data obtained in the previous step. This results in the selection of optimal technologies for each of the design points in the last level. Moving to the next level (last but one), for each discretization point we solve the optimization problem that minimizes the (*the weighted-sum of objective functions in the current level + the future value function obtained from the last level*). Then, we approximate the solution to optimization problem through a future value function, for all the discretization points in the current level, obtained by fitting a statistical model to the data obtained in the previous step. This results in the selection of optimal technologies for each of the design points in the current level. The process is repeated until the future value function for the level 1 is obtained.

4.6.1 Solution for the small design

Tables 4.13 and 4.14 present the resulting counts for a solution with the GM and the SGM methods respectively, using an OA-LHD with $N=2209$ generated from a 47-level strength two orthogonal array. For the MARS modeling to estimate the future value functions, the maximum number of basis functions and number of eligible knots

considered are 200 and 35, respectively. The dependencies between technology units are evident in Tables 4.13 and 4.14. For instance, the technology unit *Sludge Thickening Tank* gets picked most of the time in level 12 and *Aerobic-Anaerobic Digestion* quite a significant number of times in level 13, thereby enabling the selection of *Filter and Belt* in level 14.

Yellow Water Separation, the new technology in level 1, is a clear winner for both GM and SGM. Vortex SSO and Chemical Precipitation look promising in level 2. Ozonation and Physical Irradiation are picked up in level 3. In level 5, A-B System is only selected using the GM method while Activated Sludge (C, P, N) is selected only with the SGM method. The use of a technology is necessary for levels 6 and 7. Both Physical Irradiation and Ozonation appear to be promising in level 8. Air Stripping gets selected more often in level 9. None of the technology units in level 10 appears to be a good candidate. GAC Adsorption is a clear winner in level 11. In the solid line, Sludge Thickening Tank in level 12 appears to be highly promising. Sludge Dewatering Bed and Aerobic-Anaerobic Digestion are effective in level 13. In level 14, Filter and Belt is favored in counts. However, Permanent Thermal Process and Thermo-Chemical Liquefaction look promising as well. Sludge Dewatering Bed and Physical Irradiation work well in levels 15 and 16 respectively. In level 17, Chemical Fixation appears to win by count with GM while settling for a second place with SGM.

Table 4.13. Selected Technologies using GM Method with 2209 Design Points

Level	Technology Unit	Count	Level	Technology Unit	Count
1	Flow Equalization Tank	0	10	Empty Unit	2205
	<i>Yellow Water Separation</i>	2209		Chlorine Disinfection	4
	<i>Yellow & Black Water Separation</i>	0		Chlorating Disinfection	0
2	Empty Unit	0	11	Empty Unit	0
	<i>Vortex SSO</i>	1471		GAC Adsorption	2209
	Sedimentation Tank	0		Infiltration Basin	0
	Chemical Precipitation	738	12	Empty Unit	0
3	Empty Unit	0		Sludge Storage Tank	6
	Physical Irradiation	8		Sludge Thickening Tank	2203
	Ozonation	2201	13	Empty Unit	0
4	Empty Unit	2209		Sludge Dewatering Bed	1461
5	Empty Unit	1		Sludge C-G Drying	0
	Activated Sludge (C)	0		Sludge V + A Stripping	0
	Activated Sludge (C, N)	10		CWOP-UASB + A	0
	Activated Sludge (C, P)	0		Stripping	0
	Activated Sludge (C, P, N)	0		Sludge Hydrolysis + UASB	0
	<i>High Biomass Act. Sludge (C, N)</i>	0		Anaerobic Digestion	0
	Activated Sludge (N)	11		Aerobic Digestion	0
	<i>Multi-reactor/Deep</i>	0		Aerobic-Anaerobic Digestion	748
	A-B System	1	14	Empty Unit	688
	Trickling Filter	65		Filter and Belt	981
	Rotating Biological Contractors	0		Permanent Thermal Process	332
	UASB System	0		Thermo-Chemical Liquefaction	208
	Reed Bed System	0	15	Empty Unit	712
	Lagoons and Ponds	0		Sludge Dewatering Bed (II)	1497
	<i>UASB+Activated Sludge (C, P, N)</i>	2121	16	Empty Unit	28
				Physical Irradiation	2181
6	Empty Unit	0	17	Empty Unit	980
	Secondary Settler	26		Chemical Fixation	1002
	Microfiltration	50		Incineration	33
	Reverse Osmosis	13		Thermal Building Materials	194
	Chemical Precipitation	2120			
7	Empty Unit	0			
	Physical Filtration	220			
	Microfiltration	178			
	Reverse Osmosis	1677			
8	Chemical Precipitation	134			
	Empty Unit	2			
	<i>Physical Irradiation</i>	1356			
9	Ozonation	851			
	Empty Unit	436			
	Air Stripping	1739			
	Ammonia Stripping	34			

Table 4.14. Selected Technologies using SGM Method with 2209 Design Points

Level	Technology Unit	Count	Level	Technology Unit	Count
1 [0.999]	Flow Equalization Tank <i>Yellow Water Separation</i> <i>Yellow & Black Water Separation</i>	0 12167 0	10 [0.832]	Empty Unit Chlorine Disinfection Chlorating Disinfection	12162 5 0
2 [0.999]	Empty Unit <i>Vortex SSO</i> Sedimentation Tank Chemical Precipitation	0 12167 0 0	11 [0.999]	Empty Unit GAC Adsorption Infiltration Basin	0 12167 0
3 [0.431]	Empty Unit Physical Irradiation Ozonation	0 0 12167	12 [0.999]	Empty Unit Sludge Storage Tank Sludge Thickening Tank	0 18 12149
4	Empty Unit	12167	13 [0.996]	Empty Unit Sludge Dewatering Bed Sludge C-G Drying Sludge V + A Stripping CWOP-UASB + A Stripping Sludge Hydrolysis + UASB Anaerobic Digestion Aerobic Digestion Aerobic-Anaerobic Digestion	0 7809 0 0 0 0 0 0 4358
5 [0.989]	Empty Unit Activated Sludge (C) Activated Sludge (C, N) Activated Sludge (C, P) Activated Sludge (C, P, N) <i>High Biomass Act. Sludge (C, N)</i> Activated Sludge (N) <i>Multi-reactor/Deep</i> A-B System Trickling Filter Rotating Biological Contractors UASB System Reed Bed System Lagoons and Ponds <i>UASB+Activated Sludge (C, P, N)</i>	0 0 0 0 0 0 0 0 0 55 0 0 0 0 12112	14 [0.972]	Empty Unit Filter and Belt Permanent Thermal Process Thermo-Chemical Liquefaction	3922 5396 1730 1119
6 [0.84]	Empty Unit Secondary Settler Microfiltration Reverse Osmosis Chemical Precipitation	2 3504 7617 10 1034	15 [0.99]	Empty Unit Sludge Dewatering Bed (II)	3993 8174
7 [0.971]	Empty Unit Physical Filtration Microfiltration Reverse Osmosis Chemical Precipitation	1 25 4469 6980 692	16 [0.91]	Empty Unit Physical Irradiation	148 12019
8 [0.935]	Empty Unit <i>Physical Irradiation</i> Ozonation	283 522 11362	17 [0.985]	Empty Unit Chemical Fixation Incineration Thermal Building Materials	5648 5127 219 1173
9 [0.77]	Empty Unit Air Stripping Ammonia Stripping	4121 8041 5			

4.6.2 Solution for the big design

In order to validate the results using the small design, a solution was obtained using an OA-LHD with 12167 design points generated from a 23-level strength three orthogonal array. The results, which should be more precise than the small design, can be seen in Tables 4.15 and 4.16. The coefficients of multiple determination (R^2) are shown inside the square brackets. The dependencies between technology units can be easily seen.

In level 1, Yellow Water Separation is a clear winner for both GM and SGM. In level 2, Vortex SSO is a winner by count with GM, and is a clear winner with SGM. Ozonation is a clear winner in level 3. For GM, both Multi-reactor/Deep and UASB+Activated Sludge (C, P, N) appear to be promising in level 5. However, for SGM UASB+Activated Sludge (C, P, N) looks to be a heavy favorite. Microfiltration and Secondary settler appear to have potential in level 6, while Reverse Osmosis and Microfiltration are shown to be effective in level 7. Ozonation is dominant in level 8. Air stripping outshines other units in level 9. Results for levels 10 and 11 are identical to the small design. Results for the solid line look similar to the results using the small design.

Table 4.15. Selected Technologies using GM Method with 12167 Design Points, where the Fractions in Square Brackets denote the R^2 Values

Level	Technology Unit	Count	Level	Technology Unit	Count
1 [0.91]	Flow Equalization Tank <i>Yellow Water Separation</i> <i>Yellow & Black Water Separation</i>	0 12167 0	10 [0.832]	Empty Unit Chlorine Disinfection Chlorating Disinfection	12162 5 0
2 [0.984]	Empty Unit <i>Vortex SSO</i> Sedimentation Tank Chemical Precipitation	96 12007 3 61	11 [0.999]	Empty Unit GAC Adsorption Infiltration Basin	0 12167 0
3 [0.975]	Empty Unit Physical Irradiation Ozonation	0 0 12167	12 [0.999]	Empty Unit Sludge Storage Tank Sludge Thickening Tank	0 44 12123
4	Empty Unit	12167	13 [0.996]	Empty Unit Sludge Dewatering Bed Sludge C-G Drying Sludge V + A Stripping CWOP-UASB + A Stripping Sludge Hydrolysis + UASB Anaerobic Digestion Aerobic Digestion Aerobic-Anaerobic Digestion	0 8094 0 0 0 0 0 0 4073
5 [0.983]	Empty Unit Activated Sludge (C) Activated Sludge (C, N) Activated Sludge (C, P) Activated Sludge (C, P, N) <i>High Biomass Act. Sludge (C, N)</i> Activated Sludge (N) <i>Multi-reactor/Deep A-B System</i> Trickling Filter Rotating Biological Contractors UASB System Reed Bed System Lagoons and Ponds <i>UASB+Activated Sludge (C, P, N)</i>	0 0 0 0 0 0 6016 0 20 0 113 0 0 6018	14 [0.972]	Empty Unit Filter and Belt Permanent Thermal Process Thermo-Chemical Liquefaction	3903 5379 1780 1105
6 [0.94]	Empty Unit Secondary Settler Microfiltration Reverse Osmosis Chemical Precipitation	6 3216 7547 88 1310	15 [0.995]	Empty Unit Sludge Dewatering Bed (II)	3900 8267
7 [0.974]	Empty Unit Physical Filtration Microfiltration Reverse Osmosis Chemical Precipitation	1 45 4678 6647 796	16 [0.921]	Empty Unit Physical Irradiation	152 12015
8 [0.94]	Empty Unit <i>Physical Irradiation</i> Ozonation	286 516 11365	17 [0.988]	Empty Unit Chemical Fixation Incineration Thermal Building Materials	5415 5493 212 1047
9 [0.77]	Empty Unit Air Stripping Ammonia Stripping	4125 8037 5			

Table 4.16. Selected Technologies using SGM Method with 12167 Design Points, where the Fractions in Square Brackets denote the R^2 Values

Level	Technology Unit	Count	Level	Technology Unit	Count
1 [0.999]	Flow Equalization Tank <i>Yellow Water Separation</i> <i>Yellow & Black Water Separation</i>	0 12167 0	10 [0.832]	Empty Unit Chlorine Disinfection Chlorating Disinfection	12162 5 0
2 [0.999]	Empty Unit <i>Vortex SSO</i> Sedimentation Tank Chemical Precipitation	0 12167 0 0	11 [0.999]	Empty Unit GAC Adsorption Infiltration Basin	0 12167 0
3 [0.431]	Empty Unit Physical Irradiation Ozonation	0 0 12167	12 [0.999]	Empty Unit Sludge Storage Tank Sludge Thickening Tank	0 18 12149
4	Empty Unit	12167	13 [0.996]	Empty Unit Sludge Dewatering Bed Sludge C-G Drying Sludge V + A Stripping CWOP-UASB + A Stripping Sludge Hydrolysis + UASB Anaerobic Digestion Aerobic Digestion Aerobic-Anaerobic Digestion	0 7809 0 0 0 0 0 0 4358
5 [0.989]	Empty Unit Activated Sludge (C) Activated Sludge (C, N) Activated Sludge (C, P) Activated Sludge (C, P, N) <i>High Biomass Act. Sludge (C, N)</i> Activated Sludge (N) <i>Multi-reactor/Deep</i> A-B System Trickling Filter Rotating Biological Contractors UASB System Reed Bed System Lagoons and Ponds <i>UASB+Activated Sludge (C, P, N)</i>	0 0 0 0 0 0 0 0 0 55 0 0 0 0 0 12112	14 [0.972]	Empty Unit Filter and Belt Permanent Thermal Process Thermo-Chemical Liquefaction	3922 5396 1730 1119
6 [0.84]	Empty Unit Secondary Settler Microfiltration Reverse Osmosis Chemical Precipitation	2 3504 7617 10 1034	15 [0.99]	Empty Unit Sludge Dewatering Bed (II)	3993 8174
7 [0.971]	Empty Unit Physical Filtration Microfiltration Reverse Osmosis Chemical Precipitation	1 25 4469 6980 692	16 [0.91]	Empty Unit Physical Irradiation	148 12019
8 [0.935]	Empty Unit <i>Physical Irradiation</i> Ozonation	283 522 11362	17 [0.985]	Empty Unit Chemical Fixation Incineration Thermal Building Materials	5648 5127 219 1173
9 [0.77]	Empty Unit Air Stripping Ammonia Stripping	4121 8041 5			

4.6.3 Discussion on the solution

In this section, a detailed account on the comparison of results for the big design with the small design will be given. First, a study of the results using GM method will be presented. Then, the SGM method results will be elaborated.

In the GM method results, Yellow Water Separation wins clearly for both the designs in level 1. It is interesting to see the newly-added Yellow Water Separation unit get selected in level 1. In level 2, the small design selects Vortex SSO and Chemical Precipitation while the big design also selects Sedimentation Tank and Empty Unit. Nonetheless, there is a consistency in the selection of Vortex SSO as a heavy favorite in level 2. In level 3, Ozonation wins the selection on count for both the designs. Levels 5, 6, 7, and 8 have the following interesting observations:

- In level 5, Multi-reactor/Deep emerges as another legitimate option for the big design in addition to UASB + Activated Sludge (C, P, N). It is encouraging to see UASB + Activated Sludge (C, P, N), the newly-added unit in level 5, as the most promising technology.
- In level 5, UASB System gets selected for the big design as opposed to the small design.
- In level 5, The set of technological units selected for the big design is smaller in comparison with the small design.
- Results for levels 6 and 7 confirm a necessity for the use of technological units.
- Level 6 for the big design selects Microfiltration as the winner followed by Secondary Settler, Chemical Precipitation, and Reverse Osmosis. On the other hand, level 6 for the small design declares Chemical Precipitation as a clear winner on count while selecting other technology units occasionally.
- In level 7, both big and small design concur on Reverse Osmosis as the major selection. However, the big design has a better balance in terms of how often

other technology units get selected. Microfiltration and Chemical Precipitation are selected frequently for the big design, while Physical Filtration, Microfiltration, and Chemical Precipitation are the choices for the small design.

- In level 8, the big design selects Ozonation as the winner on count followed by Physical Irradiation, which swaps the results for small design interestingly.

Level 9 selects Air Stripping far more often for both the designs. It is apparent that level 10 does not require the use of a technology unit. In level 11, GAC Adsorption is the clear winner. In the solid line, the results are more or less consistent for both the designs.

In the SGM method results, Yellow Water Separation wins clearly for both small and big designs in level 1. The big design clearly declares Vortex SSO and Ozonation as winners in levels 2 and 3 respectively. The small design, however, also selects Chemical Precipitation in level 2 and Physical Irradiation (rather infrequently) in level 3. For both the designs, level 5 selects UASB + Activated Sludge (C, P, N) to be the winner. Interestingly, the small design selects Activated Sludge (C, N), Activated Sludge (C, P, N), and Activated Sludge (N) in level 5, which do not get picked up at all for the big design. Trends similar to the GM method can be seen in levels 6 through 17.

In summary, both the methods show the new technologies in levels 1 and 5 to be promising. This justifies the decision to include these technologies in the evaluation process. Also, the solution obtained satisfies the dependencies between technology units.

CHAPTER 5

CONCLUSIONS AND FUTURE WORK

5.1 Contributions

The contributions made in this dissertation can be summarized as follows. First, we address the disconnect between the decision makers and the solution development process through a questionnaire-based approach, which is unique in its application to multiple-stage decision making problems. Second, we extend Analytic Hierarchy Process (AHP) to a multiple-stage decision-making framework using our novel three-phase methodology. Third, we develop theories and methodologies for computing pairwise comparison matrices, which have traditionally been constructed using direct inputs from decision makers. This reduces the amount of information required from the decision makers, thereby increasing the efficiency of the solution process. More importantly, the new methods in the matrix generation phase, GM and SGM, result in consistent pairwise comparison matrices. Fourth, we augment the high-dimensional continuous-state stochastic dynamic programming (SDP) approach in Tsai [113] to handle multiple objectives. This was done through two new routines as follows. One, the routine that allows inputs from decision makers in form of weight vectors at each stage. Two, the routine that normalizes/transforms the objective functions with different units and orders of magnitude, which is crucial to a practical application of weighted-sum of objective functions approach. Finally, we solve a 20-dimensional, 17-stage, 6-objective, continuous-state wastewater treatment system (WTS), which is larger than any numerically solved problem in the literature.

5.2 Concluding Remarks

We use our three-phase methodology with the augmented high dimensional, continuous state SDP to solve a multiple stage, multiple objective (MSMO) model. This technique is quite *general* in its application to a variety of large-scale MSMO practical problems. Also, the technique is *pragmatic* in the sense of involving decision makers in the solution development process. We have demonstrated the solution technique on a 20-dimensional, 17-stage, 6-objective, continuous-state wastewater treatment system (WTS). The solution obtained satisfies all the constraints and complications such as dependencies between technologies, etc. The final solution to WTS selects the new technologies, Yellow Water Separation and UASB System *plus* Activated Sludge Process (C, P, N), in levels 1 and 5, respectively. This result justifies the decision to include these new technologies in the evaluation process. However, our three-phase methodology might benefit from the following refinements:

- Exploring the involvement of multiple decision makers in order to achieve a desired level of objectivity in the *Input phase*,
- Developing a theoretical basis for the consistency behavior of computed pairwise comparison matrices at a stage using the GM and SGM methods in the *Matrix Generation Phase*,
- Exploring alternate weight determination methods in the *Weighting Phase* in a quest to get a better weight estimate.

These and other extensions are discussed next.

5.3 Future Work

The results using the consistency index in Chapter 4 indicate that the consistency indices decrease along the stages (or levels), implying that the judgments seem to improve

as we move from one stage to the next for GM method. However, the consistency results are not as conclusive for SGM method. A theoretical method for comparing the results should be developed. The idea is to construct a mathematical proof to show that the consistency indices for the computed pairwise comparison matrices using the GM and the SGM methods, in general, go down while moving from one stage to the next. It is based on the AHP-based fact that a lower value of consistency index indicates a more consistent judgment from the decision maker.

In addition, we will investigate the use of alternate methods for weight determination in the weighting phase of the three-phase methodology. It is possible that these methods could provide weight estimates better than the Saaty's eigenvector method. Some of the promising weight-determination methods that could be explored are: the alternative eigenvector method by Cogger and Yu [29], the graded eigenvector method (GEM) of Takeda et al. [38], the three methods for weight derivation based on pairwise comparison matrices of Krovak [63], and the weight determination based on the decision makers' qualitative information by Batishchev et al. [33].

Group decision making is fast becoming the reality of today's organizational decision making. The input phase in our three-phase methodology depends on the responses from single expert thereby making it subjective. If the questionnaire could account for responses from multiple decision makers, the input phase could be improved in terms of consistency, accuracy, and representation of experts' preferences. In other words, the goal would be to collect inputs from multiple decision makers at a time in order to achieve some level of objectivity in the judgment.

Currently, our three-phase methodology is based on inputs from one decision maker thereby making it more or less deterministic. Though AHP has its value in terms of maintaining and ensuring consistency in decision makers' judgments/preferences, nonetheless it could do much better if the pairwise comparison matrices were stochastic reflecting

uncertainties or subjectivities in human judgments. In addition to asking the decision makers *a priori* about their preferences, we could also investigate the possibility of presenting them with multiple Pareto optimal solutions for multiple sets of stagewise weight vectors. In solving the weighted-sum version of the stochastic dynamic programming, we could also investigate various designs of experiments for the discretization of state space and other statistical modeling approaches for the future value approximation.

APPENDIX A
QUESTIONNAIRE MODELING-RELATED DATA

In this appendix, we present:

- the table for the worst and best values for all six objectives at all 17 levels,
- the table with interlevel importance questions for the objective *Economic Cost*,
- the table with interlevel importance questions for the objective *Size*,
- the table with interlevel importance questions for the objective *Nutrient Recovery*,
- the table with interlevel importance questions for the objective *Odor Emissions*,
- the table with interlevel importance questions for the objective *Robustness*, and
- the table with interlevel importance questions for the objective *Global Desirability*.

Table A.1. The worst and the best objective values for all six objectives at all 17 levels

The worst value of	The best value of	The worst value of	The best value of
Economic Cost (in US Dollars)		Nutrient Recovery (on 1-5 ordinal scale)	
EC ₁ = 43200550	EC ₁ = 9643.3121	NR ₁ = 1	NR ₁ = 4
EC ₂ = 14500	EC ₂ = 2500	NR ₂ = 1	NR ₂ = 5
EC ₃ = 9300	EC ₃ = 6600	NR ₃ = 1	NR ₃ = 1
EC ₅ = 865500	EC ₅ = 4650	NR ₅ = 1	NR ₅ = 4
EC ₆ = 38500	EC ₆ = 5218.75	NR ₆ = 1	NR ₆ = 5
EC ₇ = 38500	EC ₇ = 5040.8163	NR ₇ = 1	NR ₇ = 5
EC ₈ = 9300	EC ₈ = 6600	NR ₈ = 1	NR ₈ = 1
EC ₉ = 10500	EC ₉ = 6400	NR ₉ = 1	NR ₉ = 5
EC ₁₀ = 8400	EC ₁₀ = 5500	NR ₁₀ = 1	NR ₁₀ = 1
EC ₁₁ = 758500	EC ₁₁ = 8250	NR ₁₁ = 2	NR ₁₁ = 4
EC ₁₂ = 109203182	EC ₁₂ = 11.1495	NR ₁₂ = 1	NR ₁₂ = 5
EC ₁₃ = 251458510	EC ₁₃ = 16.7	NR ₁₃ = 2	NR ₁₃ = 4
EC ₁₄ = 257187.84	EC ₁₄ = 4.9	NR ₁₄ = 1	NR ₁₄ = 4
EC ₁₅ = 239125010	EC ₁₅ = 2578.3421	NR ₁₅ = 3	NR ₁₅ = 3
EC ₁₆ = 32148.48	EC ₁₆ = 7	NR ₁₆ = 1	NR ₁₆ = 1
EC ₁₇ = 1821134.8	EC ₁₇ = 4	NR ₁₇ = 1	NR ₁₇ = 5
Size (in m ²)		Odor Emissions (in mg/min)	
S ₁ = 96000	S ₁ = 1274	O ₁ = 384000	O ₁ = 3821.6561
S ₂ = 1000	S ₂ = 100	O ₂ = 3500	O ₂ = 250
S ₃ = 600	S ₃ = 75	O ₃ = 2880	O ₃ = 330
S ₅ = 4000000	S ₅ = 3.3333	O ₅ = 16200000	O ₅ = 5
S ₆ = 1667	S ₆ = 41.67	O ₆ = 7500	O ₆ = 145.83333
S ₇ = 1667	S ₇ = 40.82	O ₇ = 7500	O ₇ = 122.44898
S ₈ = 600	S ₈ = 75	O ₈ = 2880	O ₈ = 330
S ₉ = 2000	S ₉ = 200	O ₉ = 9600	O ₉ = 900
S ₁₀ = 800	S ₁₀ = 200	O ₁₀ = 2400	O ₁₀ = 500
S ₁₁ = 500000	S ₁₁ = 20.83	O ₁₁ = 2500000	O ₁₁ = 93.75
S ₁₂ = 6000000	S ₁₂ = 0.716	O ₁₂ = 18719233	O ₁₂ = 1.7894737
S ₁₃ = 500000000	S ₁₃ = 0.005	O ₁₃ = 1.63E+09	O ₁₃ = 0.02
S ₁₄ = 18371	S ₁₄ = 0.05	O ₁₄ = 81646.933	O ₁₄ = 0.1
S ₁₅ = 400000000	S ₁₅ = 10737	O ₁₅ = 1.55E+09	O ₁₅ = 16105.263
S ₁₆ = 3062	S ₁₆ = 0.25	O ₁₆ = 14696.448	O ₁₆ = 1.1
S ₁₇ = 500000	S ₁₇ = 0	O ₁₇ = 1543127	O ₁₇ = 0
Robustness (no units)		Global Desirability (on 1-6 ordinal scale)	
R ₁ = 3821656.051	R ₁ = 15286624.2	GD ₁ = 2.5	GD ₁ = 6
R ₂ = 2000	R ₂ = 9000	GD ₂ = 2.5	GD ₂ = 4
R ₃ = 675	R ₃ = 1500	GD ₃ = 4	GD ₃ = 4.8
R ₅ = 540	R ₅ = 2160000	GD ₅ = 1	GD ₅ = 4.7
R ₆ = 93.75	R ₆ = 15000	GD ₆ = 2.8	GD ₆ = 4
R ₇ = 35.71428571	R ₇ = 15000	GD ₇ = 2.8	GD ₇ = 4
R ₈ = 675	R ₈ = 1500	GD ₈ = 4	GD ₈ = 4.8
R ₉ = 1200	R ₉ = 4000	GD ₉ = 4.3	GD ₉ = 4.8
R ₁₀ = 1600	R ₁₀ = 3680	GD ₁₀ = 1.3	GD ₁₀ = 2
R ₁₁ = 875	R ₁₁ = 1750000	GD ₁₁ = 4.5	GD ₁₁ = 5
R ₁₂ = 5.726315789	R ₁₂ = 140394249.2	GD ₁₂ = 2	GD ₁₂ = 3
R ₁₃ = 6	R ₁₃ = 628630966.6	GD ₁₃ = 1.5	GD ₁₃ = 4.5
R ₁₄ = 0.7	R ₁₄ = 110223.36	GD ₁₄ = 1.5	GD ₁₄ = 5
R ₁₅ = 8589.473684	R ₁₅ = 597797215.5	GD ₁₅ = 1.5	GD ₁₅ = 3
R ₁₆ = 1.125	R ₁₆ = 7348.224	GD ₁₆ = 4	GD ₁₆ = 4.5
R ₁₇ = 0	R ₁₇ = 2571878.4	GD ₁₇ = 3.5	GD ₁₇ = 5.5

Table A.2. The interlevel importance questions for the objective Economic Cost

	Which one of the following pairs of objectives is more important in terms of improvement from the worst value to the best value?			Given the more important objective, how many times is this objective more important than the other? (Note: “Equally important” = 1)								
EC vs. EC				1	2	3	4	5	6	7	8	9
Level 1-Level 2	<input checked="" type="checkbox"/> EC ₁	<input type="checkbox"/> EC ₂	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>
Level 2-Level 3	<input checked="" type="checkbox"/> EC ₂	<input type="checkbox"/> EC ₃	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 5-Level 6	<input checked="" type="checkbox"/> EC ₅	<input type="checkbox"/> EC ₆	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>
Level 6-Level 7	<input type="checkbox"/> EC ₆	<input type="checkbox"/> EC ₇	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 7-Level 8	<input checked="" type="checkbox"/> EC ₇	<input type="checkbox"/> EC ₈	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 8-Level 9	<input type="checkbox"/> EC ₈	<input checked="" type="checkbox"/> EC ₉	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 9-Level 10	<input type="checkbox"/> EC ₉	<input type="checkbox"/> EC ₁₀	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 10-Level 11	<input type="checkbox"/> EC ₁₀	<input checked="" type="checkbox"/> EC ₁₁	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>
Level 11-Level 12	<input type="checkbox"/> EC ₁₁	<input checked="" type="checkbox"/> EC ₁₂	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 12-Level 13	<input type="checkbox"/> EC ₁₂	<input checked="" type="checkbox"/> EC ₁₃	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 13-Level 14	<input checked="" type="checkbox"/> EC ₁₃	<input type="checkbox"/> EC ₁₄	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>
Level 14-Level 15	<input type="checkbox"/> EC ₁₄	<input checked="" type="checkbox"/> EC ₁₅	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 15-Level 16	<input checked="" type="checkbox"/> EC ₁₅	<input type="checkbox"/> EC ₁₆	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 16-Level 17	<input type="checkbox"/> EC ₁₆	<input checked="" type="checkbox"/> EC ₁₇	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>

Table A.3. The interlevel importance questions for the objective Size

	Which one of the following pairs of objectives is more important in terms of improvement from the worst value to the best value?			Given the more important objective, how many times is this objective more important than the other? (Note: “Equally important” = 1)								
S vs. S				1	2	3	4	5	6	7	8	9
Level 1-Level 2	<input checked="" type="checkbox"/> S ₁	<input type="checkbox"/> S ₂	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 2-Level 3	<input type="checkbox"/> S ₂	<input type="checkbox"/> S ₃	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 5-Level 6	<input checked="" type="checkbox"/> S ₅	<input type="checkbox"/> S ₆	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 6-Level 7	<input type="checkbox"/> S ₆	<input type="checkbox"/> S ₇	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 7-Level 8	<input checked="" type="checkbox"/> S ₇	<input type="checkbox"/> S ₈	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 8-Level 9	<input type="checkbox"/> S ₈	<input checked="" type="checkbox"/> S ₉	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 9-Level 10	<input checked="" type="checkbox"/> S ₉	<input type="checkbox"/> S ₁₀	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 10-Level 11	<input type="checkbox"/> S ₁₀	<input checked="" type="checkbox"/> S ₁₁	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 11-Level 12	<input type="checkbox"/> S ₁₁	<input checked="" type="checkbox"/> S ₁₂	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 12-Level 13	<input type="checkbox"/> S ₁₂	<input checked="" type="checkbox"/> S ₁₃	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 13-Level 14	<input checked="" type="checkbox"/> S ₁₃	<input type="checkbox"/> S ₁₄	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>
Level 14-Level 15	<input type="checkbox"/> S ₁₄	<input checked="" type="checkbox"/> S ₁₅	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 15-Level 16	<input checked="" type="checkbox"/> S ₁₅	<input type="checkbox"/> S ₁₆	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 16-Level 17	<input type="checkbox"/> S ₁₆	<input checked="" type="checkbox"/> S ₁₇	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>

Table A.4. The interlevel importance questions for the objective Nutrient Recovery

	Which one of the following pairs of objectives is more important in terms of improvement from the worst value to the best value?			Given the more important objective, how many times is this objective more important than the other? (Note: “Equally important” = 1)								
NR vs. NR				1	2	3	4	5	6	7	8	9
Level 1-Level 2	<input type="checkbox"/> NR ₁	<input checked="" type="checkbox"/> NR ₂	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 2-Level 3	<input checked="" type="checkbox"/> NR ₂	<input type="checkbox"/> NR ₃	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 5-Level 6	<input type="checkbox"/> NR ₅	<input checked="" type="checkbox"/> NR ₆	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 6-Level 7	<input type="checkbox"/> NR ₆	<input type="checkbox"/> NR ₇	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 7-Level 8	<input checked="" type="checkbox"/> NR ₇	<input type="checkbox"/> NR ₈	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 8-Level 9	<input type="checkbox"/> NR ₈	<input checked="" type="checkbox"/> NR ₉	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 9-Level 10	<input checked="" type="checkbox"/> NR ₉	<input type="checkbox"/> NR ₁₀	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 10-Level 11	<input type="checkbox"/> NR ₁₀	<input checked="" type="checkbox"/> NR ₁₁	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 11-Level 12	<input type="checkbox"/> NR ₁₁	<input type="checkbox"/> NR ₁₂	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 12-Level 13	<input type="checkbox"/> NR ₁₂	<input type="checkbox"/> NR ₁₃	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 13-Level 14	<input checked="" type="checkbox"/> NR ₁₃	<input type="checkbox"/> NR ₁₄	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 14-Level 15	<input type="checkbox"/> NR ₁₄	<input type="checkbox"/> NR ₁₅	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 15-Level 16	<input checked="" type="checkbox"/> NR ₁₅	<input type="checkbox"/> NR ₁₆	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 16-Level 17	<input type="checkbox"/> NR ₁₆	<input checked="" type="checkbox"/> NR ₁₇	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>

Table A.5. The interlevel importance questions for the objective Odor Emissions

	Which one of the following pairs of objectives is more important in terms of improvement from the worst value to the best value?			Given the more important objective, how many times is this objective more important than the other? (Note: “Equally important” = 1)								
O vs. O				1	2	3	4	5	6	7	8	9
Level 1-Level 2	<input checked="" type="checkbox"/> O ₁	<input type="checkbox"/> O ₂	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 2-Level 3	<input type="checkbox"/> O ₂	<input type="checkbox"/> O ₃	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 5-Level 6	<input checked="" type="checkbox"/> O ₅	<input type="checkbox"/> O ₆	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 6-Level 7	<input type="checkbox"/> O ₆	<input type="checkbox"/> O ₇	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 7-Level 8	<input checked="" type="checkbox"/> O ₇	<input type="checkbox"/> O ₈	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 8-Level 9	<input type="checkbox"/> O ₈	<input checked="" type="checkbox"/> O ₉	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 9-Level 10	<input checked="" type="checkbox"/> O ₉	<input type="checkbox"/> O ₁₀	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 10-Level 11	<input type="checkbox"/> O ₁₀	<input checked="" type="checkbox"/> O ₁₁	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 11-Level 12	<input type="checkbox"/> O ₁₁	<input checked="" type="checkbox"/> O ₁₂	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 12-Level 13	<input type="checkbox"/> O ₁₂	<input checked="" type="checkbox"/> O ₁₃	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 13-Level 14	<input checked="" type="checkbox"/> O ₁₃	<input type="checkbox"/> O ₁₄	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>
Level 14-Level 15	<input type="checkbox"/> O ₁₄	<input checked="" type="checkbox"/> O ₁₅	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>
Level 15-Level 16	<input checked="" type="checkbox"/> O ₁₅	<input type="checkbox"/> O ₁₆	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 16-Level 17	<input type="checkbox"/> O ₁₆	<input checked="" type="checkbox"/> O ₁₇	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>

Table A.6. The interlevel importance questions for the objective Robustness

	Which one of the following pairs of objectives is more important in terms of improvement from the worst value to the best value?			Given the more important objective, how many times is this objective more important than the other? (Note: “Equally important” = 1)								
R vs. R				1	2	3	4	5	6	7	8	9
Level 1-Level 2	<input checked="" type="checkbox"/> R ₁	<input type="checkbox"/> R ₂	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 2-Level 3	<input checked="" type="checkbox"/> R ₂	<input type="checkbox"/> R ₃	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 5-Level 6	<input checked="" type="checkbox"/> R ₅	<input type="checkbox"/> R ₆	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 6-Level 7	<input type="checkbox"/> R ₆	<input type="checkbox"/> R ₇	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 7-Level 8	<input checked="" type="checkbox"/> R ₇	<input type="checkbox"/> R ₈	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 8-Level 9	<input type="checkbox"/> R ₈	<input checked="" type="checkbox"/> R ₉	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 9-Level 10	<input type="checkbox"/> R ₉	<input type="checkbox"/> R ₁₀	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 10-Level 11	<input type="checkbox"/> R ₁₀	<input checked="" type="checkbox"/> R ₁₁	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 11-Level 12	<input type="checkbox"/> R ₁₁	<input checked="" type="checkbox"/> R ₁₂	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 12-Level 13	<input type="checkbox"/> R ₁₂	<input checked="" type="checkbox"/> R ₁₃	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 13-Level 14	<input checked="" type="checkbox"/> R ₁₃	<input type="checkbox"/> R ₁₄	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>
Level 14-Level 15	<input type="checkbox"/> R ₁₄	<input checked="" type="checkbox"/> R ₁₅	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>
Level 15-Level 16	<input checked="" type="checkbox"/> R ₁₅	<input type="checkbox"/> R ₁₆	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 16-Level 17	<input type="checkbox"/> R ₁₆	<input checked="" type="checkbox"/> R ₁₇	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>

Table A.7. The interlevel importance questions for the objective Global Desirability

	Which one of the following pairs of objectives is more important in terms of improvement from the worst value to the best value?			Given the more important objective, how many times is this objective more important than the other? (Note: “Equally important” = 1)								
GD vs. GD				1	2	3	4	5	6	7	8	9
Level 1-Level 2	<input checked="" type="checkbox"/> GD ₁	<input type="checkbox"/> GD ₂	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 2-Level 3	<input type="checkbox"/> GD ₂	<input checked="" type="checkbox"/> GD ₃	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 5-Level 6	<input type="checkbox"/> GD ₅	<input type="checkbox"/> GD ₆	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 6-Level 7	<input type="checkbox"/> GD ₆	<input type="checkbox"/> GD ₇	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 7-Level 8	<input type="checkbox"/> GD ₇	<input checked="" type="checkbox"/> GD ₈	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 8-Level 9	<input type="checkbox"/> GD ₈	<input type="checkbox"/> GD ₉	<input checked="" type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 9-Level 10	<input checked="" type="checkbox"/> GD ₉	<input type="checkbox"/> GD ₁₀	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 10-Level 11	<input type="checkbox"/> GD ₁₀	<input checked="" type="checkbox"/> GD ₁₁	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>
Level 11-Level 12	<input checked="" type="checkbox"/> GD ₁₁	<input type="checkbox"/> GD ₁₂	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 12-Level 13	<input type="checkbox"/> GD ₁₂	<input checked="" type="checkbox"/> GD ₁₃	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 13-Level 14	<input type="checkbox"/> GD ₁₃	<input checked="" type="checkbox"/> GD ₁₄	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 14-Level 15	<input checked="" type="checkbox"/> GD ₁₄	<input type="checkbox"/> GD ₁₅	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 15-Level 16	<input type="checkbox"/> GD ₁₅	<input checked="" type="checkbox"/> GD ₁₆	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Level 16-Level 17	<input type="checkbox"/> GD ₁₆	<input checked="" type="checkbox"/> GD ₁₇	<input type="checkbox"/> Equally important	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>

APPENDIX B

**COMPUTED PAIRWISE COMPARISON MATRICES IN THE MATRIX
GENERATION PHASE**

In this appendix, we present the computed pairwise comparison matrices using Geometric Mean (GM) of non-ones and Successive Geometric Mean (SGM) of non-ones.

B.0.0.1 Computed pairwise comparison matrices using GM method

$$A_{(3,3)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 2.432299 & 1.150163 & 1.877567 & 1.169471 & 2.284489 \\ \mathbf{S} & 0.411134 & 1.000000 & 0.464159 & 1.000000 & 0.553341 & 0.903602 \\ \mathbf{O} & 0.869442 & 2.154435 & 1.000000 & 1.910886 & 0.889140 & 1.985964 \\ \mathbf{GD} & 0.532604 & 1.000000 & 0.523318 & 1.000000 & 0.588040 & 1.148698 \\ \mathbf{EC} & 0.855087 & 1.807204 & 1.124683 & 1.700566 & 1.000000 & 1.568105 \\ \mathbf{NR} & 0.437735 & 1.106682 & 0.503534 & 0.870551 & 0.637712 & 1.000000 \end{pmatrix},$$

$$A_{(4,4)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 2.432299 & 1.150163 & 1.877567 & 1.169471 & 2.284489 \\ \mathbf{S} & 0.411134 & 1.000000 & 0.464159 & 1.000000 & 0.553341 & 0.903602 \\ \mathbf{O} & 0.869442 & 2.154435 & 1.000000 & 1.910886 & 0.889140 & 1.985964 \\ \mathbf{GD} & 0.532604 & 1.000000 & 0.523318 & 1.000000 & 0.588040 & 1.148698 \\ \mathbf{EC} & 0.855087 & 1.807204 & 1.124683 & 1.700566 & 1.000000 & 1.568105 \\ \mathbf{NR} & 0.437735 & 1.106682 & 0.503534 & 0.870551 & 0.637712 & 1.000000 \end{pmatrix},$$

$$A_{(5,5)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 2.432299 & 1.150163 & 1.877567 & 1.169471 & 2.284489 \\ \mathbf{S} & 0.411134 & 1.000000 & 0.464159 & 1.000000 & 0.553341 & 0.903602 \\ \mathbf{O} & 0.869442 & 2.154435 & 1.000000 & 1.910886 & 0.889140 & 1.985964 \\ \mathbf{GD} & 0.532604 & 1.000000 & 0.523318 & 1.000000 & 0.588040 & 1.148698 \\ \mathbf{EC} & 0.855087 & 1.807204 & 1.124683 & 1.700566 & 1.000000 & 1.568105 \\ \mathbf{NR} & 0.437735 & 1.106682 & 0.503534 & 0.870551 & 0.637712 & 1.000000 \end{pmatrix},$$

$$A_{(6,6)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.754477 & 1.037891 & 2.210503 & 1.045693 & 2.767830 \\ \mathbf{S} & 0.569970 & 1.000000 & 0.611802 & 1.430969 & 0.642449 & 1.587401 \\ \mathbf{O} & 0.963492 & 1.634517 & 1.000000 & 2.477464 & 0.904304 & 2.753218 \\ \mathbf{GD} & 0.452386 & 0.698827 & 0.403639 & 1.000000 & 0.454280 & 1.414214 \\ \mathbf{EC} & 0.956304 & 1.556543 & 1.105822 & 2.201285 & 1.000000 & 2.262430 \\ \mathbf{NR} & 0.361294 & 0.629961 & 0.363211 & 0.707107 & 0.442003 & 1.000000 \end{pmatrix},$$

$$A_{(7,7)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.754477 & 1.037891 & 2.210503 & 1.045693 & 2.767830 \\ \mathbf{S} & 0.569970 & 1.000000 & 0.611802 & 1.430969 & 0.642449 & 1.587401 \\ \mathbf{O} & 0.963492 & 1.634517 & 1.000000 & 2.477464 & 0.904304 & 2.753218 \\ \mathbf{GD} & 0.452386 & 0.698827 & 0.403639 & 1.000000 & 0.454280 & 1.414214 \\ \mathbf{EC} & 0.956304 & 1.556543 & 1.105822 & 2.201285 & 1.000000 & 2.262430 \\ \mathbf{NR} & 0.361294 & 0.629961 & 0.363211 & 0.707107 & 0.442003 & 1.000000 \end{pmatrix},$$

$$A_{(8,8)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.603697 & 1.081746 & 2.385880 & 0.978233 & 1.953767 \\ \mathbf{S} & 0.623559 & 1.000000 & 0.704010 & 1.668510 & 0.639936 & 1.171827 \\ \mathbf{O} & 0.924432 & 1.420434 & 1.000000 & 2.469439 & 0.826977 & 1.771042 \\ \mathbf{GD} & 0.419132 & 0.599337 & 0.404950 & 1.000000 & 0.394440 & 0.858946 \\ \mathbf{EC} & 1.022252 & 1.562656 & 1.209223 & 2.535237 & 1.000000 & 1.767735 \\ \mathbf{NR} & 0.511832 & 0.853368 & 0.564639 & 1.164218 & 0.565696 & 1.000000 \end{pmatrix},$$

$$A_{(9,9)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.459143 & 1.064879 & 2.005580 & 0.982155 & 1.729764 \\ \mathbf{S} & 0.685334 & 1.000000 & 0.761115 & 1.435189 & 0.699696 & 1.135248 \\ \mathbf{O} & 0.939074 & 1.313862 & 1.000000 & 2.022529 & 0.859003 & 1.579731 \\ \mathbf{GD} & 0.498609 & 0.696772 & 0.494430 & 1.000000 & 0.472411 & 0.943519 \\ \mathbf{EC} & 1.018169 & 1.429193 & 1.164140 & 2.116800 & 1.000000 & 1.593794 \\ \mathbf{NR} & 0.578114 & 0.880865 & 0.633019 & 1.059862 & 0.627434 & 1.000000 \end{pmatrix},$$

$$A_{(10,10)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.323773 & 0.994150 & 1.501729 & 0.982155 & 1.376050 \\ \mathbf{S} & 0.755417 & 1.000000 & 0.799843 & 1.148698 & 0.769794 & 0.980561 \\ \mathbf{O} & 1.005884 & 1.250245 & 1.000000 & 1.511453 & 0.927602 & 1.291373 \\ \mathbf{GD} & 0.665899 & 0.870551 & 0.661615 & 1.000000 & 0.634326 & 0.953546 \\ \mathbf{EC} & 1.018169 & 1.299048 & 1.078049 & 1.576477 & 1.000000 & 1.276564 \\ \mathbf{NR} & 0.726718 & 1.019824 & 0.774370 & 1.048718 & 0.783353 & 1.000000 \end{pmatrix},$$

$$A_{(11,11)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.250207 & 0.995048 & 1.451557 & 1.006917 & 1.353341 \\ \mathbf{S} & 0.799867 & 1.000000 & 0.839495 & 1.178812 & 0.830916 & 1.025478 \\ \mathbf{O} & 1.004977 & 1.191193 & 1.000000 & 1.459385 & 0.959386 & 1.281637 \\ \mathbf{GD} & 0.688916 & 0.848312 & 0.685220 & 1.000000 & 0.677638 & 0.960549 \\ \mathbf{EC} & 0.993130 & 1.203491 & 1.042333 & 1.475715 & 1.000000 & 1.243218 \\ \mathbf{NR} & 0.738912 & 0.975155 & 0.780252 & 1.041071 & 0.804364 & 1.000000 \end{pmatrix},$$

$$A_{(12,12)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.158731 & 0.976792 & 1.398418 & 1.061875 & 1.209232 \\ \mathbf{S} & 0.863013 & 1.000000 & 0.882853 & 1.229313 & 0.941522 & 0.977528 \\ \mathbf{O} & 1.023759 & 1.132691 & 1.000000 & 1.434047 & 1.038012 & 1.171582 \\ \mathbf{GD} & 0.715094 & 0.813463 & 0.697327 & 1.000000 & 0.747093 & 0.858701 \\ \mathbf{EC} & 0.941730 & 1.062110 & 0.963380 & 1.338522 & 1.000000 & 1.058347 \\ \mathbf{NR} & 0.826971 & 1.022989 & 0.853547 & 1.164550 & 0.944870 & 1.000000 \end{pmatrix},$$

$$A_{(13,13)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.138820 & 0.996211 & 1.307455 & 1.086565 & 1.115662 \\ \mathbf{S} & 0.878102 & 1.000000 & 0.911179 & 1.168016 & 0.977145 & 0.913958 \\ \mathbf{O} & 1.003804 & 1.097479 & 1.000000 & 1.312739 & 1.047121 & 1.063765 \\ \mathbf{GD} & 0.764845 & 0.856153 & 0.761766 & 1.000000 & 0.820491 & 0.828293 \\ \mathbf{EC} & 0.920331 & 1.023389 & 0.954999 & 1.218782 & 1.000000 & 0.953681 \\ \mathbf{NR} & 0.896329 & 1.094143 & 0.940058 & 1.207303 & 1.048568 & 1.000000 \end{pmatrix},$$

$$A_{(14,14)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.123343 & 0.996609 & 1.480423 & 1.077111 & 1.145447 \\ \mathbf{S} & 0.890200 & 1.000000 & 0.920144 & 1.339214 & 0.979526 & 0.959382 \\ \mathbf{O} & 1.003403 & 1.086786 & 1.000000 & 1.485741 & 1.042058 & 1.097960 \\ \mathbf{GD} & 0.675482 & 0.746706 & 0.673065 & 1.000000 & 0.718995 & 0.749353 \\ \mathbf{EC} & 0.928409 & 1.020902 & 0.959639 & 1.390831 & 1.000000 & 0.996358 \\ \mathbf{NR} & 0.873021 & 1.042338 & 0.910780 & 1.334485 & 1.003655 & 1.000000 \end{pmatrix},$$

$$A_{(15,15)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.103927 & 0.996931 & 1.196988 & 1.075533 & 1.019389 \\ \mathbf{S} & 0.905857 & 1.000000 & 0.933383 & 1.101050 & 0.993274 & 0.867881 \\ \mathbf{O} & 1.003078 & 1.071372 & 1.000000 & 1.200857 & 1.043816 & 0.979308 \\ \mathbf{GD} & 0.835430 & 0.908224 & 0.832739 & 1.000000 & 0.888932 & 0.822424 \\ \mathbf{EC} & 0.929771 & 1.006772 & 0.958024 & 1.124946 & 1.000000 & 0.887719 \\ \mathbf{NR} & 0.980979 & 1.152231 & 1.021129 & 1.215917 & 1.126482 & 1.000000 \end{pmatrix},$$

$$A_{(16,16)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.094476 & 0.997198 & 1.400028 & 1.068745 & 1.017527 \\ \mathbf{S} & 0.913679 & 1.000000 & 0.938995 & 1.297635 & 0.993857 & 0.879673 \\ \mathbf{O} & 1.002810 & 1.064968 & 1.000000 & 1.404141 & 1.039931 & 0.981260 \\ \mathbf{GD} & 0.714271 & 0.770633 & 0.712179 & 1.000000 & 0.755737 & 0.693307 \\ \mathbf{EC} & 0.935677 & 1.006181 & 0.961603 & 1.323212 & 1.000000 & 0.897846 \\ \mathbf{NR} & 0.982775 & 1.136786 & 1.019098 & 1.442362 & 1.113777 & 1.000000 \end{pmatrix},$$

$$A_{(17,17)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.069119 & 0.990174 & 1.300405 & 1.080458 & 1.034060 \\ \mathbf{S} & 0.935349 & 1.000000 & 0.952197 & 1.233617 & 1.027130 & 0.921457 \\ \mathbf{O} & 1.009923 & 1.050203 & 1.000000 & 1.313850 & 1.061341 & 1.008318 \\ \mathbf{GD} & 0.768991 & 0.810625 & 0.761122 & 1.000000 & 0.823614 & 0.767500 \\ \mathbf{EC} & 0.925534 & 0.973587 & 0.942204 & 1.214161 & 1.000000 & 0.906298 \\ \mathbf{NR} & 0.967062 & 1.085237 & 0.991751 & 1.302931 & 1.103389 & 1.000000 \end{pmatrix}.$$

B.0.0.2 Computed pairwise comparison matrices using SGM method

$$A_{(3,3)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 3.092678 & 1.877139 & 2.761310 & 0.982680 & 1.388673 \\ \mathbf{S} & 0.323344 & 1.000000 & 0.464159 & 1.587401 & 0.337002 & 0.449335 \\ \mathbf{O} & 0.532726 & 2.154435 & 1.000000 & 2.444424 & 0.462328 & 0.759568 \\ \mathbf{GD} & 0.362147 & 0.629961 & 0.409094 & 1.000000 & 0.344679 & 0.487083 \\ \mathbf{EC} & 1.017626 & 2.967346 & 2.162967 & 2.901251 & 1.000000 & 1.250762 \\ \mathbf{NR} & 0.720112 & 2.225509 & 1.316538 & 2.053039 & 0.799513 & 1.000000 \end{pmatrix},$$

$$A_{(4,4)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 3.092678 & 1.877139 & 2.761310 & 0.982680 & 1.388673 \\ \mathbf{S} & 0.323344 & 1.000000 & 0.464159 & 1.587401 & 0.337002 & 0.449335 \\ \mathbf{O} & 0.532726 & 2.154435 & 1.000000 & 2.444424 & 0.462328 & 0.759568 \\ \mathbf{GD} & 0.362147 & 0.629961 & 0.409094 & 1.000000 & 0.344679 & 0.487083 \\ \mathbf{EC} & 1.017626 & 2.967346 & 2.162967 & 2.901251 & 1.000000 & 1.250762 \\ \mathbf{NR} & 0.720112 & 2.225509 & 1.316538 & 2.053039 & 0.799513 & 1.000000 \end{pmatrix},$$

$$A_{(5,5)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 3.092678 & 1.877139 & 2.761310 & 0.982680 & 1.388673 \\ \mathbf{S} & 0.323344 & 1.000000 & 0.464159 & 1.587401 & 0.337002 & 0.449335 \\ \mathbf{O} & 0.532726 & 2.154435 & 1.000000 & 2.444424 & 0.462328 & 0.759568 \\ \mathbf{GD} & 0.362147 & 0.629961 & 0.409094 & 1.000000 & 0.344679 & 0.487083 \\ \mathbf{EC} & 1.017626 & 2.967346 & 2.162967 & 2.901251 & 1.000000 & 1.250762 \\ \mathbf{NR} & 0.720112 & 2.225509 & 1.316538 & 2.053039 & 0.799513 & 1.000000 \end{pmatrix},$$

$$A_{(6,6)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.371043 & 1.102697 & 3.715717 & 0.850023 & 3.028378 \\ \mathbf{S} & 0.729372 & 1.000000 & 0.735484 & 3.086164 & 0.632263 & 2.209330 \\ \mathbf{O} & 0.906868 & 1.359649 & 1.000000 & 4.136540 & 0.739581 & 2.770607 \\ \mathbf{GD} & 0.269127 & 0.324027 & 0.241748 & 1.000000 & 0.207569 & 1.395826 \\ \mathbf{EC} & 1.176439 & 1.581620 & 1.352116 & 4.817676 & 1.000000 & 3.420646 \\ \mathbf{NR} & 0.330210 & 0.452626 & 0.360932 & 0.716422 & 0.292342 & 1.000000 \end{pmatrix},$$

$$A_{(7,7)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 1.371043 & 1.102697 & 3.715717 & 0.850023 & 3.028378 \\ \mathbf{S} & 0.729372 & 1.000000 & 0.735484 & 3.086164 & 0.632263 & 2.209330 \\ \mathbf{O} & 0.906868 & 1.359649 & 1.000000 & 4.136540 & 0.739581 & 2.770607 \\ \mathbf{GD} & 0.269127 & 0.324027 & 0.241748 & 1.000000 & 0.207569 & 1.395826 \\ \mathbf{EC} & 1.176439 & 1.581620 & 1.352116 & 4.817676 & 1.000000 & 3.420646 \\ \mathbf{NR} & 0.330210 & 0.452626 & 0.360932 & 0.716422 & 0.292342 & 1.000000 \end{pmatrix},$$

$$A_{(8,8)} = \left(\begin{array}{c|cccccc} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \hline \mathbf{R} & 1.000000 & 1.271689 & 1.182631 & 3.221774 & 0.798964 & 1.003143 \\ \mathbf{S} & 0.786356 & 1.000000 & 0.902660 & 2.645595 & 0.632391 & 0.788889 \\ \mathbf{O} & 0.845573 & 1.107836 & 1.000000 & 2.916953 & 0.666319 & 0.850722 \\ \mathbf{GD} & 0.310388 & 0.377987 & 0.342823 & 1.000000 & 0.240081 & 0.372525 \\ \mathbf{EC} & 1.251621 & 1.581299 & 1.500783 & 4.165268 & 1.000000 & 1.238640 \\ \mathbf{NR} & 0.996867 & 1.267605 & 1.175473 & 2.684381 & 0.807337 & 1.000000 \end{array} \right),$$

$$A_{(9,9)} = \left(\begin{array}{c|cccccc} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \hline \mathbf{R} & 1.000000 & 1.083412 & 1.057506 & 1.269207 & 0.927917 & 1.001047 \\ \mathbf{S} & 0.923010 & 1.000000 & 0.966440 & 1.150129 & 0.858345 & 0.924000 \\ \mathbf{O} & 0.945621 & 1.034726 & 1.000000 & 1.207674 & 0.873428 & 0.947536 \\ \mathbf{GD} & 0.787893 & 0.869467 & 0.828038 & 1.000000 & 0.692937 & 0.863163 \\ \mathbf{EC} & 1.077683 & 1.165032 & 1.144913 & 1.443133 & 1.000000 & 1.073944 \\ \mathbf{NR} & 0.998954 & 1.082251 & 1.055369 & 1.158529 & 0.931147 & 1.000000 \end{array} \right),$$

$$A_{(10,10)} = \left(\begin{array}{c|cccccc} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \hline \mathbf{R} & 1.000000 & 0.736007 & 0.727154 & 0.375530 & 0.927917 & 0.333508 \\ \mathbf{S} & 1.358683 & 1.000000 & 0.988686 & 0.634617 & 1.310225 & 0.589956 \\ \mathbf{O} & 1.375224 & 1.011444 & 1.000000 & 0.645029 & 1.321687 & 0.594923 \\ \mathbf{GD} & 2.662901 & 1.575755 & 1.550318 & 1.000000 & 2.497284 & 0.952133 \\ \mathbf{EC} & 1.077683 & 0.763228 & 0.756609 & 0.400435 & 1.000000 & 0.345438 \\ \mathbf{NR} & 2.998431 & 1.695041 & 1.680888 & 1.050273 & 2.894879 & 1.000000 \end{array} \right),$$

$$A_{(11,11)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 0.849637 & 0.899240 & 0.825871 & 1.073534 & 0.793839 \\ \mathbf{S} & 1.176973 & 1.000000 & 1.058636 & 1.045349 & 1.279839 & 1.020228 \\ \mathbf{O} & 1.112050 & 0.944612 & 1.000000 & 0.989062 & 1.207876 & 0.962759 \\ \mathbf{GD} & 1.210843 & 0.956618 & 1.011059 & 1.000000 & 1.304483 & 0.983783 \\ \mathbf{EC} & 0.931503 & 0.781348 & 0.827899 & 0.766587 & 1.000000 & 0.729750 \\ \mathbf{NR} & 1.259701 & 0.980173 & 1.038682 & 1.016485 & 1.370332 & 1.000000 \end{pmatrix},$$

$$A_{(12,12)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 0.751740 & 0.876958 & 1.010665 & 1.341733 & 0.445488 \\ \mathbf{S} & 1.330247 & 1.000000 & 1.166664 & 1.377422 & 1.792487 & 0.714223 \\ \mathbf{O} & 1.140305 & 0.857145 & 1.000000 & 1.181292 & 1.535965 & 0.566498 \\ \mathbf{GD} & 0.989447 & 0.725994 & 0.846530 & 1.000000 & 1.329139 & 0.443572 \\ \mathbf{EC} & 0.745305 & 0.557884 & 0.651057 & 0.752367 & 1.000000 & 0.284751 \\ \mathbf{NR} & 2.244728 & 1.400123 & 1.765232 & 2.254423 & 3.511836 & 1.000000 \end{pmatrix},$$

$$A_{(13,13)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 0.909263 & 1.053511 & 0.876675 & 1.307690 & 0.385352 \\ \mathbf{S} & 1.099792 & 1.000000 & 1.158675 & 0.971983 & 1.440240 & 0.487929 \\ \mathbf{O} & 0.949207 & 0.863055 & 1.000000 & 0.839027 & 1.242883 & 0.376330 \\ \mathbf{GD} & 1.140673 & 1.028825 & 1.191857 & 1.000000 & 1.492233 & 0.470942 \\ \mathbf{EC} & 0.764707 & 0.694329 & 0.804581 & 0.670136 & 1.000000 & 0.238643 \\ \mathbf{NR} & 2.595031 & 2.049480 & 2.657241 & 2.123404 & 4.190368 & 1.000000 \end{pmatrix},$$

$$A_{(14,14)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 0.968790 & 1.017528 & 2.411679 & 1.093541 & 0.916845 \\ \mathbf{S} & 1.032215 & 1.000000 & 1.050317 & 2.496086 & 1.129306 & 0.991887 \\ \mathbf{O} & 0.982774 & 0.952093 & 1.000000 & 2.376650 & 1.075169 & 0.909633 \\ \mathbf{GD} & 0.414649 & 0.400627 & 0.420760 & 1.000000 & 0.453495 & 0.389009 \\ \mathbf{EC} & 0.914461 & 0.885500 & 0.930086 & 2.205097 & 1.000000 & 0.781495 \\ \mathbf{NR} & 1.090697 & 1.008180 & 1.099344 & 2.570638 & 1.279599 & 1.000000 \end{pmatrix},$$

$$A_{(15,15)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 0.946410 & 1.005809 & 0.422399 & 1.071509 & 0.338534 \\ \mathbf{S} & 1.056625 & 1.000000 & 1.062766 & 0.446718 & 1.132362 & 0.376428 \\ \mathbf{O} & 0.994225 & 0.940941 & 1.000000 & 0.420344 & 1.065475 & 0.337200 \\ \mathbf{GD} & 2.367428 & 2.238547 & 2.379003 & 1.000000 & 2.536831 & 1.247411 \\ \mathbf{EC} & 0.933263 & 0.883110 & 0.938549 & 0.394193 & 1.000000 & 0.294674 \\ \mathbf{NR} & 2.953909 & 2.656550 & 2.965595 & 0.801660 & 3.393581 & 1.000000 \end{pmatrix},$$

$$A_{(16,16)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 0.981808 & 1.001933 & 2.668775 & 1.023290 & 0.696949 \\ \mathbf{S} & 1.018529 & 1.000000 & 1.020499 & 2.719039 & 1.042306 & 0.722039 \\ \mathbf{O} & 0.998071 & 0.979913 & 1.000000 & 2.664440 & 1.021365 & 0.696032 \\ \mathbf{GD} & 0.374704 & 0.367777 & 0.375313 & 1.000000 & 0.383436 & 0.302644 \\ \mathbf{EC} & 0.977240 & 0.959411 & 0.979082 & 2.607996 & 1.000000 & 0.665448 \\ \mathbf{NR} & 1.434825 & 1.384967 & 1.436715 & 3.304210 & 1.502748 & 1.000000 \end{pmatrix},$$

$$A_{(17,17)} = \begin{pmatrix} & \mathbf{R} & \mathbf{S} & \mathbf{O} & \mathbf{GD} & \mathbf{EC} & \mathbf{NR} \\ \mathbf{R} & 1.000000 & 0.868251 & 0.941642 & 0.961753 & 1.153533 & 1.014918 \\ \mathbf{S} & 1.151741 & 1.000000 & 1.084528 & 1.107801 & 1.328595 & 1.175568 \\ \mathbf{O} & 1.061975 & 0.922060 & 1.000000 & 1.021461 & 1.225043 & 1.078038 \\ \mathbf{GD} & 1.039768 & 0.902689 & 0.978990 & 1.000000 & 1.199412 & 1.108447 \\ \mathbf{EC} & 0.866902 & 0.752675 & 0.816298 & 0.833742 & 1.000000 & 0.873048 \\ \mathbf{NR} & 0.985302 & 0.850653 & 0.927611 & 0.902163 & 1.145413 & 1.000000 \end{pmatrix}.$$

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BIOGRAPHICAL STATEMENT

Prashant K. Tarun was born in Begusarai, Bihar, India, in 1974. He received his Bachelor of Engineering (B.E.) in Mechanical Engineering from Birla Vishwakarma Mahavidyalaya, Sardar Patel University, Anand, Gujarat, India, in 1996. He received his Master of Science (2002) and Doctor of Philosophy (2008) degrees in Industrial Engineering from the University of Texas at Arlington. From 1996 to 2000 he worked for a premier Indian engineering conglomerate Larsen and Toubro Ltd., India. His current research interests include Multicriteria Optimization, Operations Research, Stochastic Dynamic Programming, Stochastic Processes, Statistical Modeling, Design of Experiments, and Statistical Data Mining.