DECOUPLING OF HAMILTONIAN SYSTEM
WITH APPLICATIONS TO
LINEAR QUADRATIC
PROBLEM

by

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ABSTRACT

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This thesis provides a method of decoupling the Hamiltonian system with application to linear quadratic problem in control system. Resulting decoupled Hamiltonian system helps in formulating easy closed form solution to various optimal control problems. These closed form solution are improvement over the problem of solving first order nonlinear differential equations in conventional methods. Decoupling also eliminates the constraint, "$Q = 0$" for certain type of optimal control.

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CHAPTER 1

INTRODUCTION

This thesis deals with the problem of optimization. Optimization is a process by which one seeks to minimize (or maximize) a function by systematically choosing the values of variable from within an allowed set. We apply the theory of optimization to the field of control systems with the objective of minimization to a linear quadratic function under constrain of plant dynamics. The problem of solving linear quadratic problem is equivalent to that of optimization under equality constrains. Problem in general can be stated as minimization of a scalar performance index $L(x,u)$, a function of control vector $u \in \boldsymbol{R}^m$ and a system state vector $x \in \boldsymbol{R}^n$, while satisfying the plant constrain equation $f(x,u)=0$, where f is a set of n scalar equations, $f \in \boldsymbol{R}^n$. There exist many methods for optimization, but two most commonly used are the “Calculus of variation” and “Dynamic programming”.

Dynamic programming was developed by R.E. Bellman in the later 1950s [5], [6], [7]. Dynamic programming is based on Bellman’s principle of optimality:

“An optimal policy has the property that no matter what the previous decisions have been, the remaining decisions must constitute an optimal policy with regard to the state resulting from those previous decisions.”

In dynamic programming problem of finding optimal control is solved by working backward from the final stage. The principle of optimality serves to limit the number of potentially optimal control strategies that must be investigated, thus making backward-in-time choice making an optimal decision.

Calculus of variation deals with the functional, such as those making a function attain a maximum or minimum value. Functional could be formed as integral of function and its derivatives. In this method changes in function, which is to be minimized, is written as sum of independent changes in all of its variables. Values of variable are systematically determined by subjecting variation in function due to that variable to necessary and sufficient conditions. This method is used in this thesis to find optimal control sequence. In next section we shall review this method and derive the control for various types of linear quadratic problem [1], [3], [9], [10].

1.1 Optimization with Equality Constraints

It is desired to minimizes the scalar performance index $L(x, u)$, a function of control vector $u \in \boldsymbol{R}^m$ and a system state vector $x \in \boldsymbol{R}^n$, while satisfying the plant constrain equation $f(x, u) = 0$, where f is a set of n scalar equations, $f \in \boldsymbol{R}^{\boldsymbol{n}}$.

One of the method used to solve such problems is to adjoin constrain to the performance index via use of a Lagrange multiplier to define the Hamiltonian function

$$H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u) \qquad 1.1$$

Increment in H depend on increment in x, u and λ according to

$$dH = H_x^T dx + H_u^T du + H_\lambda^T d\lambda \qquad 1.2$$

where $$H_x = \frac{\partial H}{\partial x}$$

$$H_u = \frac{\partial H}{\partial u}$$

$$H_\lambda = \frac{\partial H}{\partial \lambda}$$

Necessary condition for a minimum point of $L(x,u)$ that also satisfies the constrain $f(x,u)=0$ are

$$\frac{\partial H}{\partial \lambda} = f(x,u) = 0 \qquad 1.2$$

$$\frac{\partial H}{\partial x} = L_x + f_x^T(x,u)\lambda = 0 \qquad 1.3$$

$$\frac{\partial H}{\partial u} = L_u + f_u^T(x,u)\lambda = 0 \qquad 1.4$$

So now we have three set of equation, due to Lagrange multiplier λ. Intuitively we can see that dx and du are not independent increments because of constrain $f(x,u)=0$, but by using an undetermined multiplier λ we can now make dx and du behave as if they were independent increments. Thus by use of Lagrange multiplier we converted the problem of minimizing $L(x,u)$ with constrain $f(x,u)=0$ to that of minimizing $H(x,u,\lambda)$ with no constrain.

We can use these results to determine the optimal control for linear quadratic problems.

1.2 Discrete-Time Linear Quadratic Problem

1.2.1 *Regulation Problem*

Let the plant to be controlled be described by the liner equation

$$x_{k+1} = Ax_k + Bu_k \qquad 1.2\text{-}1$$

with $x_k \in \boldsymbol{R}^n$ and $u \in \boldsymbol{R}^m$. The associated performance index is the quadratic function defined over the interval $[i, N]$

$$J_i = \tfrac{1}{2} x_N^T S_N x_N + \tfrac{1}{2} \sum_{k=i}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right) \qquad 1.2\text{-}2$$

where the matrices Q and S_N are symmetric positive semi-definite and R is symmetric positive definite. It can be shown that the discrete Hamiltonian system for the above system is as follows:

$$\begin{bmatrix} x_{k+1} \\ \lambda_k \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ Q & A^T \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix} \qquad 1.2\text{-}3$$

with $u_k = -R^{-1} B^T \lambda_{k+1}$

1.2.1.1 For free final state, optimal control is given as [1]

$$u_k^* = -K_k x_k \qquad 1.2\text{-}4$$

$$K_k = \left(B^T S_{k+1} B + R \right)^{-1} B^T S_{k+1} A \qquad 1.2\text{-}5$$

$$S_k = A^T \left[S_{k+1} - S_{k+1} B \left(B^T S_{k+1} B + R \right)^{-1} B^T S_{k+1} \right] A + Q \qquad 1.2\text{-}6$$

$$k = 0,\ 1,\ 2,\ 3, \ldots\ldots, N-1$$

1.2.1.2 For fixed final state, the problem becomes difficult to solve analytically due to the coupling $Q \neq 0$ in the Hamiltonian system [1]. If we have $Q = 0$ then Hamiltonian system (1.2-3) is decoupled and we can write optimal control as an open loop control sequence which depends upon desired final state value r_N :

$$u_k = R^{-1}B^T(A^T)^{N-k-1}G_{0,N}^{-1}(r_N - A^N x_0) \qquad 1.2\text{-}6$$

where x_0 is the initial state of the plant and $G_{0,N} = \sum_{i=0}^{N-1} A^{N-i-1}BR^{-1}B^T(A^T)^{N-i-1}$ is the weighted reachability gramian of the system.

1.2.2 *Tracking Problem*

Consider a plant described by the linear equation $x_{k+1} = Ax_k + Bu_k$ with $x_k \in R^n$ and $u_k \in R^m$. It is desired to get a control law that forces a certain linear combination of the states $y_k = Cx_k$ of the plant to track a desired reference trajectory r_k over a specified time interval $[i, N]$, then we have to minimize the cost function

$$J_i = \tfrac{1}{2}(Cx_N - r_N)^T P_N (Cx_N - r_N) + \tfrac{1}{2}\sum_{k=i}^{N-1}\left[(Cx_k - r_k)^T Q (Cx_k - r_k) + u_k^T R u_k\right] \qquad 1.2\text{-}7$$

where P and Q are symmetric positive semi-definite matrices and R is symmetric positive definite matrix. Actual value of x_N is not constrained but we want it to be as close as r_N.

For the above problem it can be shown that the following non-homogenous Hamiltonian system exists

$$\begin{bmatrix} x_{k+1} \\ \lambda_k \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ C^TQC & A^T \end{bmatrix}\begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix} + \begin{bmatrix} 0 \\ -C^TQ \end{bmatrix} r_k \qquad 1.2\text{-}8$$

with $u_k = -R^{-1}B^T\lambda_{k+1}$

We can express u_k as a combination of a linear state variable feedback plus a term depending on reference trajectory r_k as: [1]

$$u_k = \left(B^T S_{k+1} B + R\right)^{-1}\left(-S_{k+1} A x_k + v_{k+1}\right) \qquad 1.2\text{-}9$$

$$S_k = A^T S_{k+1}\left(I + BR^{-1}B^T S_{k+1}\right)A + C^T QC \qquad 1.2\text{-}10$$

$$v_k = \left[A^T - A^T S_{k+1} B\left(B^T S_{k+1} B + R\right)^{-1} B^T\right]v_{k+1} + C^T Q r_k \qquad 1.2\text{-}11$$

$$k = 0,\ 1,\ 2,\ 3, \ldots\ldots, N-1$$

with boundary conditions

$$S_N = C^T PC$$

$$v_N = C^T \mathrm{Pr}_N$$

1.3 Continuous-Time Linear Quadratic Problem

1.3.1 *Regulation Problem*

Consider a plant described by $\dot{x} = Ax + Bu$ with $x \in R^n$ and $u \in R^m$, A and B are constant matrices. The associated performance index is the quadratic function defined over the interval $[t_0, T]$,

$$J(t_0) = \tfrac{1}{2} x^T(T) S(T) x(T) + \tfrac{1}{2}\int_{t_0}^{T}\left(x^T Q x + u^T R u\right)dt \qquad 1.3\text{-}1$$

where matrices $S(T)$ and Q are symmetric positive semi-definite and R is symmetric positive definite. Then it can be shown that the following homogeneous Hamiltonian system holds:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \qquad 1.3\text{-}2$$

with $u(t) = -R^{-1}B^T\lambda(t)$

1.3.1.1 Free final state

For free final state case optimal control is given as: [1]

$$u(t) = -K(t)x(t) \qquad 1.3\text{-}3$$

$$K(t) = R^{-1}B^T S(t) \qquad 1.3\text{-}4$$

$$-\dot{S} = A^T S + SA - SBR^{-1}B^T S + Q \qquad 1.3\text{-}5$$

$$0 \le t \le T$$

1.3.1.2 Fixed final state

Again the problem of fixed final state is difficult to solve analytically due to the coupling $Q \neq 0$ in the Hamiltonian system [1]. So if $Q = 0$ then we can derive optimal control as an open loop control sequence depending upon the desired final state $r(T)$ as:

$$u(t) = R^{-1}B^T e^{A^T(T-t)} G^{-1}(t_0, T)[r(t) - e^{A^T(T-t_0)} x(t_0)] \qquad 1.3\text{-}6$$

where $G(t_0, T) = \int_{t_0}^{T} e^{A^T(T-\tau)} BR^{-1}B^T e^{A^T(T-\tau)} x(t_0) d\tau$ is the weighted continuous reachability gramian for the system.

1.3.2 *Tracking Problem*

Consider a plant described by the linear equation $\dot{x} = Ax + Bu$ with $x \in R^n$ and $u \in R^m$, matrices A and B are constant. It is desired to get a control law that forces a certain linear combination of the states $y = Cx$ of the plant to track a desired reference trajectory $r(t)$ over a specified time interval $[t_0, T]$, then we have to minimize the cost function

$$J(t_0) = \tfrac{1}{2}\left(Cx(T) - r(T)\right)^T P\left(Cx(T) - r(T)\right) + \tfrac{1}{2}\int_{t_0}^{T} \left[\left(Cx(-r\right)^T Q\left(Cx - r\right) + u^T R u\right] \qquad 1.3\text{-}7$$

where P and Q are symmetric positive semi-definite matrices and R is symmetric positive definite matrix. Actual value of $x(T)$ is not constrained but we want it to be as close as $r(T)$.

For the above problem it can be shown that the following non-homogenous Hamiltonian system exists

$$\begin{bmatrix} \dot{x} \\ -\dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ C^T QC & A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ -C^T Q \end{bmatrix} r \qquad 1.3\text{-}8$$

with $u = -R^{-1}B^T \lambda$

We can express u as a combination of a linear state variable feedback plus a term depending on reference trajectory r as: [1]

$$u = -Kx + R^{-1}B^T v \qquad 1.3\text{-}9$$

$$K(t) = R^{-1}B^T S(t) \qquad 1.3\text{-}10$$

$$-\dot{S} = A^T S\left(I + BR^{-1}B^T S\right)A + C^T QC \qquad 1.3\text{-}11$$

$$-\dot{v} = (A - BK)^T v + C^T QC \qquad 1.3\text{-}11$$

$$0 \le t \le T$$

with boundary conditions

$$S(T) = C^T P C$$

$$v(T) = C^T \Pr(T)$$

CHAPTER 2

METHOD FOR DECOUPLING OF THE HAMILTONIAN SYSTEM AND IT'S APPLICATION TO LINEAR QUADRATIC PROBLEM

Here we shall discuss method to decouple the Hamiltonian system formed during solution of the linear quadratic problem. The performance function here is a quadratic and constrains are given by plant equation. With out loss of generality method to decouple involves insertion of an instrumental term in performance index which dose not alters it. Value of this term is then chosen such that the Hamiltonian system is decoupled. Choice of term's value is not a guess as a proper equation is derived in process of decoupling. Remaining portion of this chapter deals with the method of deriving that equation, forming decoupled Hamiltonian system and use of these results for the various linear quadratic problems discussed in chapter 1.

2.1 Discrete-Time Linear Quadratic Problem

2.1.1 *Regulation Problem*

Consider a plant described by the linear equation

$$x_{k+1} = Ax_k + Bu_k \qquad 2.1\text{-}1$$

with $x_k \in R^n$ and $u_k \in R^m$. The associated performance index is the quadratic function defined over the interval $[i, N]$

$$J_i = \tfrac{1}{2} x_N^T S_N x_N + \tfrac{1}{2}\sum_{k=i}^{N-1}\left(x_k^T Q x_k + u_k^T R u_k\right) \qquad 2.1\text{-}2$$

where the matrices Q and S_N are symmetric positive semi-definite and R is symmetric positive definite. The objective is to find closed form solution of control sequence u_k to minimize J_i.

Consider the original cost function

$$J_i = \tfrac{1}{2} x_N^T S_N x_N + \tfrac{1}{2} \sum_{k=i}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right)$$

We can subtract and add a term $\tfrac{1}{2} x_N^T \hat{S} x_N$ to obtain:

$$J_i = \tfrac{1}{2} x_N^T \left(S_N - \hat{S} \right) x_N + \tfrac{1}{2} x_N^T \hat{S} x_N + \tfrac{1}{2} \sum_{k=i}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right) \qquad 2.1\text{-}3$$

where $\hat{S}$ is a symmetric positive semi-definite matrix.

Now observe:

$$\tfrac{1}{2} x_N^T \hat{S} x_N = \tfrac{1}{2} x_i^T \hat{S} x_i + \tfrac{1}{2} \sum_{m=i}^{N-1} \left(x_{m+1}^T \hat{S} x_{m+1} - x_m^T \hat{S} x_m \right) \qquad 2.1\text{-}4$$

Substituting (2.1-4) into (2.1-3) yields

$$\begin{aligned} J_i &= \tfrac{1}{2} x_N^T \left(S_N - \hat{S} \right) x_N + \tfrac{1}{2} x_i^T \hat{S} x_i + \tfrac{1}{2} \sum_{m=i}^{N-1} (x_{m+1}^T \hat{S} x_{m+1} - x_m^T \hat{S} x_m) + \tfrac{1}{2} \sum_{k=i}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right) \\ &= \tfrac{1}{2} x_N^T \left(S_N - \hat{S} \right) x_N + \tfrac{1}{2} x_i^T \hat{S} x_i + \tfrac{1}{2} \sum_{k=i}^{N-1} \left(x_k^T \left(Q - \hat{S} \right) x_k + x_{k+1}^T \hat{S} x_{k+1} + u_k^T R u_k \right) \qquad 2.1\text{-}5 \end{aligned}$$

Using $x_{k+1} = A x_k + B u_k$, (2.1-5) can be rewritten after simplification as:

$$J_i = \tfrac{1}{2} x_N^T \left(\tilde{S}_N \right) x_N + \tfrac{1}{2} x_i^T \hat{S} x_i + \tfrac{1}{2} \sum_{k=i}^{N-1} \left[x_k^T \hat{Q} x_k + 2 x_k^T \hat{G} u_k + 2 u_k^T \hat{H} x_k + u_k^T \hat{R} u_k \right] \quad 2.1\text{-}6$$

where

$\tilde{S}_N = S_N - \hat{S}$,

$$\hat{Q} = Q - \hat{S} + A^T \hat{S} A,$$

$$\hat{G} = \tfrac{1}{2} A^T \hat{S} B, \qquad 2.1\text{-}7$$

$$\hat{H} = \tfrac{1}{2} B^T \hat{S} A,$$

$$\hat{R} = R + B^T \hat{S} B.$$

To solve this *linear quadratic regulation problem*, we begin with the Hamiltonian function for modified cost function (2.1-6)

$$H^k = \tfrac{1}{2}\left(x_k^T \hat{Q} x_k + 2 x_k^T \hat{G} u_k + 2 u_k^T \hat{H} x_k + u_k^T \hat{R} u_k\right) + \lambda_{k+1}^T \left(A x_k + B u_k\right) \qquad 2.1\text{-}8$$

Then state and costate equations are

$$x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = A x_k + B u_k \qquad 2.1\text{-}9$$

$$\lambda_k = \frac{\partial H^k}{\partial x_k} = \hat{Q} x_k + \left(\hat{G} + \hat{H}^T\right) u_k + A^T \lambda_{k+1} \qquad 2.1\text{-}10$$

and the stationary condition

$$0 = \frac{\partial H^k}{\partial u_k} = \hat{R} u_k + \left(\hat{G}^T + \hat{H}\right) x_k + B^T \lambda_{k+1} \qquad 2.1\text{-}11$$

Rearranging (2.1-11), we get

$$u_k = -\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right) x_k - \hat{R}^{-1} B^T \lambda_{k+1} \qquad 2.1\text{-}12$$

Using (2.1-12) to eliminate u_k in (2.1-9) and (2.1-10) we get the following Hamiltonian system

$$\begin{bmatrix} x_{k+1} \\ \lambda_k \end{bmatrix} = \begin{bmatrix} \tilde{A} & -B\hat{R}^{-1}B^T \\ \tilde{Q} & \tilde{A}^T \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix} \qquad 2.1\text{-}13$$

where

$$\tilde{A} = A - B\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right),$$

$$\tilde{Q} = \hat{Q} - \left(\hat{G} + \hat{H}^T\right)\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right). \qquad 2.1\text{-}14$$

In order to decouple the Hamiltonian system (2.1-13), we set $\tilde{Q} = 0$, and obtain

$$\hat{Q} - \left(\hat{G} + \hat{H}^T\right)\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right) = 0 \qquad 2.1\text{-}15$$

Using (2.1-7), (2.1-15) can be rewritten as:

$$Q + A^T\hat{S}A - \hat{S} - A^T\hat{S}B\left(R + B^T\hat{S}B\right)^{-1}B^T\hat{S}A = 0$$

$$\hat{S} = A^T\hat{S}A - \left(A^T\hat{S}B\right)\left(R + B^T\hat{S}B\right)^{-1}\left(B^T\hat{S}A\right) + Q \qquad 2.1\text{-}16$$

which is an algebraic Riccati equation.

There exists a symmetric positive semi-definite solution $\hat{S}^*$ for (2.1-16) as long as (A, B) is a stabilizable pair, $R > 0$ and $Q \geq 0$. Proof of this statement can be obtained from Corollary 13.1.2 in [2].

Corollary 13.1.2: *Assume* $R > 0$, (A, B) *is a d-stabalizable pair, and*

$$\begin{bmatrix} Q & C^* \\ C & R \end{bmatrix} \geq 0$$

Then the maximal hermitian solution X_+ *of (13.1.1) exists, is unique, and is positive semidefinite. Moreover, all eigenvalues of the matrix (13.1.14) lie in the closed unit disk.*

$$X = A^*XA + Q - \left(C + B^*XA\right)^*\left(R + B^*XB\right)^{-1}\left(C + B^*XA\right) \qquad (13.1.1)$$

$$\hat{A} = A - B\left(R + B^*X_+B\right)^{-1}\left(C + B^*X_+A\right) \qquad (13.1.14)$$

Then, for this $\hat{S}^*$, the Hamiltonian system (2.1-13) is decoupled and is given by:

$$\begin{bmatrix} x_{k+1} \\ \lambda_k \end{bmatrix} = \begin{bmatrix} \tilde{A} & -B\hat{R}^{-1}B^T \\ 0 & \tilde{A}^T \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix} \tag{2.1-17}$$

2.1.1.1 Fixed-Final-State optimal control

The state and costate equations are given by (2.1-17), once $\hat{S}^*$ is evaluated. To solve these we need the termination conditions. Let the objective here be to make x_N match exactly the desired final reference state r_N. The boundary conditions thus are:

$$x_N = r_N$$

$$x_0 = \text{Given}.$$

Since both the boundaries of state are fixed, the split boundary conditions are satisfied.

As opposed to conventional theory [1], the Hamiltonian system (2.1-17) is decoupled and the problem has an easy solution. Reproducing state and costate equation,

$$x_{k+1} = \tilde{A}x_k - B\hat{R}^{-1}B^T\lambda_{k+1} \qquad \text{2.1-18}$$

$$\lambda_k = \tilde{A}^T\lambda_{k+1} \qquad \text{2.1-19}$$

Rewriting (2.1-19) in term of unknown final costate as

$$\lambda_k = \left(\tilde{A}^T\right)^{N-k}\lambda_N \qquad \text{2.1-20}$$

Use this to eliminate λ_{k+1} in (2.1-18) to get

$$x_{k+1} = \tilde{A}x_k - B\hat{R}^{-1}B^T\left(\tilde{A}^T\right)^{N-k-1}\lambda_N \qquad \text{2.1-21}$$

Considering this as a first order difference equation with second term as input,

$$x_k = \tilde{A}^k x_0 - \sum_{i=0}^{k-1}\tilde{A}^{N-i-1}B\hat{R}^{-1}B^T\left(\tilde{A}^T\right)^{N-i-1}\lambda_N \qquad \text{2.1-22}$$

Evaluate (2.1-22) at $k = N$, to find λ_N

$$x_N = \tilde{A}^N x_0 - \sum_{i=0}^{N-1} \tilde{A}^{N-i-1} B\hat{R}^{-1} B^T \left(\tilde{A}^T\right)^{N-i-1} \lambda_N$$

Therefore, using the fact $x_N = r_N$, the final costate is

$$\lambda_N = -\tilde{G}_{0,N}^{-1}\left(r_N - \tilde{A}^N x_0\right) \quad 2.1\text{-}23$$

where

$$\tilde{G}_{0,N} = \sum_{i=0}^{N-1} \left(\tilde{A}\right)^{N-i-1} B\hat{R}^{-1} B^T \left(\tilde{A}^T\right)^{N-i-1} \quad 2.1\text{-}24$$

Using (2.1-24) in (2.1-20) the costate is

$$\lambda_k = -\left(\tilde{A}\right)^{N-k} \tilde{G}_{0,N}^{-1}\left(r_N - \tilde{A}^N x_0\right) \quad 2.1\text{-}25$$

and so by (2.1-12) the optimal control sequence is

$$u_k = -\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right)x_k + \hat{R}^{-1} B^T \left(\tilde{A}\right)^{N-k-1} \tilde{G}_{0,N}^{-1}\left(r_N - \tilde{A}^N x_0\right) \quad 2.1\text{-}26$$

This is the required minimum-control-energy solution to the fixed-final-state linear quadratic regulator problem.

It is very easy to prove that closed loop control (2.1-26) drives x from x_0 state to $x_N = r_N$ state. Apply the control (2.1-26) to the state equation:

$$x_{k+1} = Ax_k - B\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right)x_k + B\hat{R}^{-1} B^T \left(\tilde{A}\right)^{N-k-1} \tilde{G}_{0,N}^{-1}\left(r_N - \tilde{A}^N x_0\right)$$

$$x_{k+1} = \tilde{A}x_k + B\hat{R}^{-1} B^T \left(\tilde{A}\right)^{N-k-1} \tilde{G}_{0,N}^{-1}\left(r_N - \tilde{A}^N x_0\right) \quad 2.1\text{-}27$$

The solution to (2.1-27) is

$$x_k = \tilde{A}^k x_0 + \sum_{i=0}^{k-1} \left(\tilde{A}\right)^{k-i-1} B\hat{R}^{-1} B^T \left(\tilde{A}^T\right)^{N-i-1} \tilde{G}_{0,N}^{-1}\left(r_N - \tilde{A}^N x_0\right) \quad 2.1\text{-}28$$

Evaluating (2.1-28) at $k = N$ yields

$$x_N = \widetilde{A}^N x_0 + \sum_{i=0}^{N-1} \left(\widetilde{A}\right)^{N-i-1} B\hat{R}^{-1} B^T \left(\widetilde{A}^T\right)^{N-i-1} \widetilde{G}_{0,N}^{-1} \left(r_N - \widetilde{A}^N x_0\right) \qquad 2.1\text{-}29$$

but $\widetilde{G}_{0,N}^{-1}\left(r_N - \widetilde{A}^N x_0\right)$ dose not depend on i, and remaining portion of the sum is just $\widetilde{G}_{0,N}$. This implies that (2.1-29) can be simplified as

$$x_N = \widetilde{A}^N x_0 + \widetilde{G}_{0,N} \widetilde{G}_{0,N}^{-1} \left(r_N - \widetilde{A}^N x_0\right) = r_N \qquad 2.1\text{-}30$$

This is what we desired!

Equation (2.1-24) gives weighted reachability gramian of modified system. In terms of the system reachability matrix $\widetilde{U}_k = \left[B \;\; \widetilde{A}B \quad \dots \quad \widetilde{A}^{k-1}B\right]$, $\widetilde{G}_{0,N}$ can be written as

$$\widetilde{G}_{0,N} = \widetilde{U}_N \begin{bmatrix} \hat{R}^{-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \hat{R}^{-1} \end{bmatrix} \widetilde{U}_N^T \qquad 2.1\text{-}31$$

Since we have assumed $|R| \neq 0$, it is can be easily to show that $|\hat{R}| \neq 0$. So to have $\left|\widetilde{G}_{0,N}\right| \neq 0$ we require $\widetilde{U}_N$ having full rank n, where n is the state dimension. Thus invertiblity of $\widetilde{G}_{0,N}$ is ensured under the condition that the system is reachable, i.e. $\widetilde{U}_n = \left[B \;\; \widetilde{A}B \quad \dots \quad \widetilde{A}^{n-1}B\right]$ has full rank n, and $N \geq n$. Now we know that state feed back (or lack of it) dose not affect controllability of the system. Therefore, we can drive any given x_0 to any desired $x_N = r_N$ for some N if the system is controllable!

TABLE 2-1 Discrete-Time Linear Quadratic Regulator (Final State Fixed)

System model:

$$x_{k+1} = Ax_k + Bu_k, \qquad N > k > i$$

Performance index:

$$J_i = \tfrac{1}{2} x_N^T S_N x_N + \tfrac{1}{2} \sum_{k=i}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right)$$

Assumptions:

$S_N \geq 0$, $Q_k \geq 0$, $R > 0$, and all three are symmetric

Optimal feedback control:

$$u_k = -\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right)x_k + \hat{R}^{-1} B^T \left(\tilde{A}\right)^{N-k-1} \tilde{G}_{0,N}^{-1}\left(r_N - \tilde{A}^N x_0\right)$$

$$\tilde{G}_{0,N} = \sum_{i=0}^{N-1} \left(\tilde{A}\right)^{N-i-1} B\hat{R}^{-1} B^T \left(\tilde{A}^T\right)^{N-i-1}$$

$$\tilde{A} = A - B\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right)$$

$$\tilde{Q} = \hat{Q} - \left(\hat{G} + \hat{H}^T\right)\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right),\ \hat{Q} = Q - \hat{S} + A^T \hat{S} A$$

$$\hat{G} = \tfrac{1}{2} A^T \hat{S} B,\ \hat{H} = \tfrac{1}{2} B^T \hat{S} A,\ \hat{R} = R + B^T \hat{S} B$$

$$\hat{S} = A^T \hat{S} A - \left(A^T \hat{S} B\right)\left(R + B^T \hat{S} B\right)^{-1}\left(B^T \hat{S} A\right) + Q$$

Example 2.1-1: Scalar System

Consider a scalar system described by $x_{k+1} = Ax_k + Bu_k$, where $\mathrm{A} = 1.05, \mathrm{B} = 0.01$, with the initial condition $x_0 = 10$. It is desired to find the optimal control that minimizes the cost function

$$J_0 = \tfrac{1}{2} x_N^T S_N x_N + \tfrac{1}{2} \sum_{k=0}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right) \text{ where } \mathrm{Q} = 1, \mathrm{R} = 1, \mathrm{S_N} = 5, \mathrm{N} = 100.$$

Let $r_N = 12$. Conventional method [1] does not allow an analytical solution because of $Q \neq 0$. Using the new method, control and state sequences are found and are shown in Fig. 2-1 and Fig. 2-2, respectively. Optimal cost in this case is $J_0^* = 51977.98$ and intermediate costs are shown in Fig. 2-3.

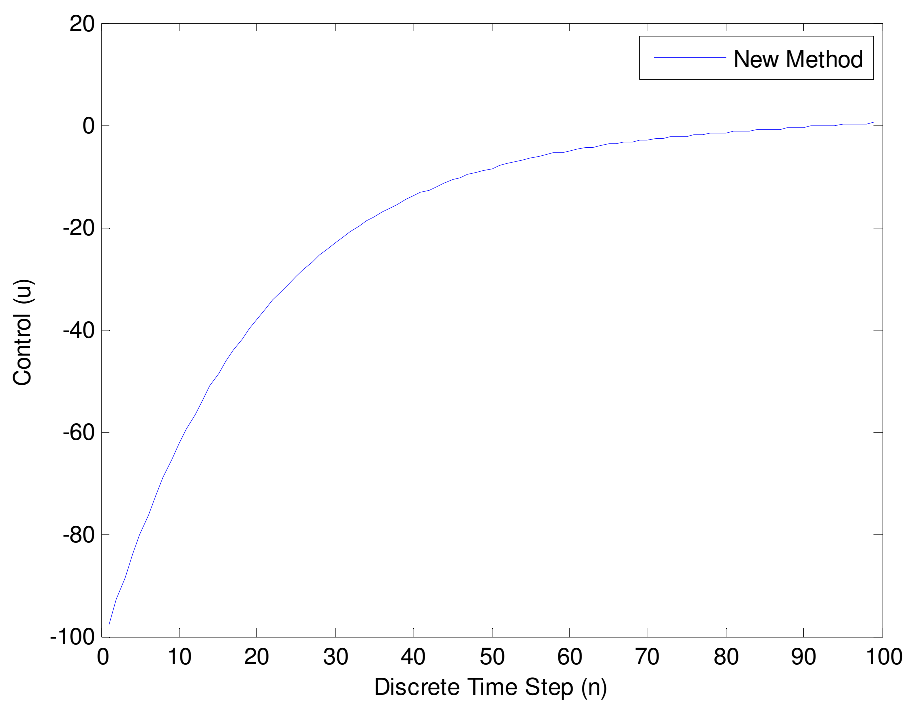


Figure 2-1: Fixed Final State LQR Control Sequence (Scalar case)

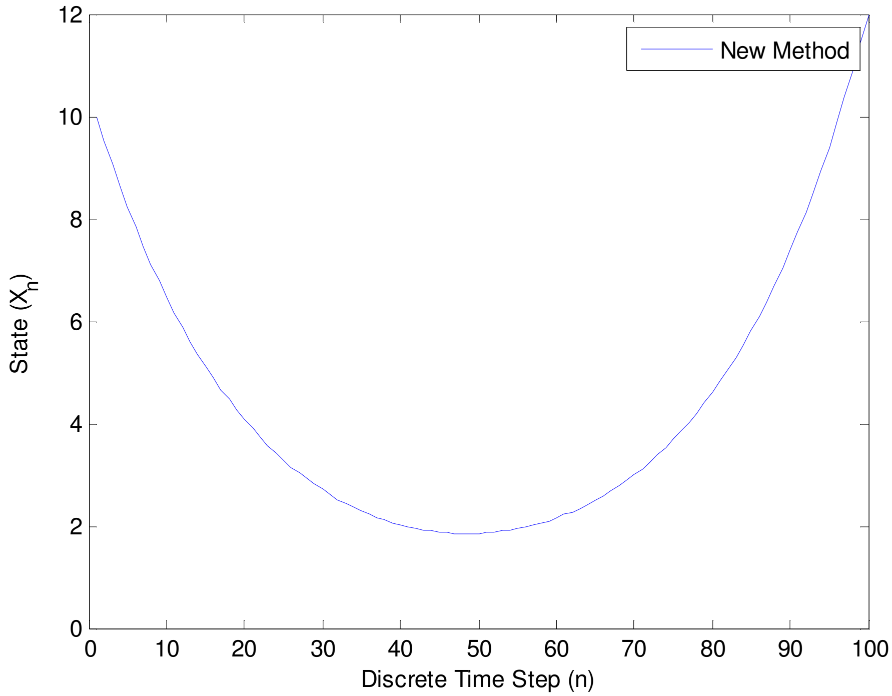


Figure 2-2: Fixed Final State LQR State Sequence (Scalar case)

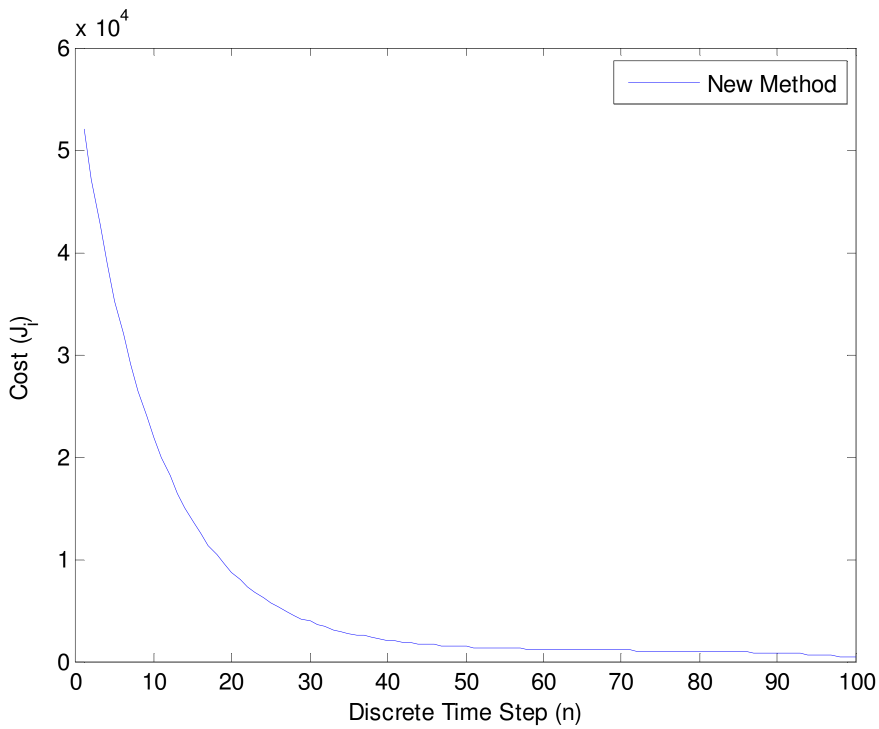


Figure 2-3 Fixed Final State LQR Cost of Controlling (Scalar case)

Examples 2.1-2: Second Order System.

Consider a second order system described by $x_{k+1} = Ax_k + Bu_k$, where $A = \begin{bmatrix} 1 & 0.01 \\ 0.01 & 1.05 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$, with the initial condition $x_0 = \begin{bmatrix} -5 \\ -10 \end{bmatrix}$. It is desired to find the optimal control that minimizes the cost function $J_0 = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=0}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right)$ where $S_N = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $R = 1$.

Let $r_N = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. Again, an analytical solution cannot be derived from the conventional approach owing to $Q \neq 0$. Control and state sequences are shown in Fig. 2-4 and Fig. 2-5, respectively. The optimal cost in this case is $J_0^* = 51977.98$ and intermediate costs are shown in Fig. 2-6.

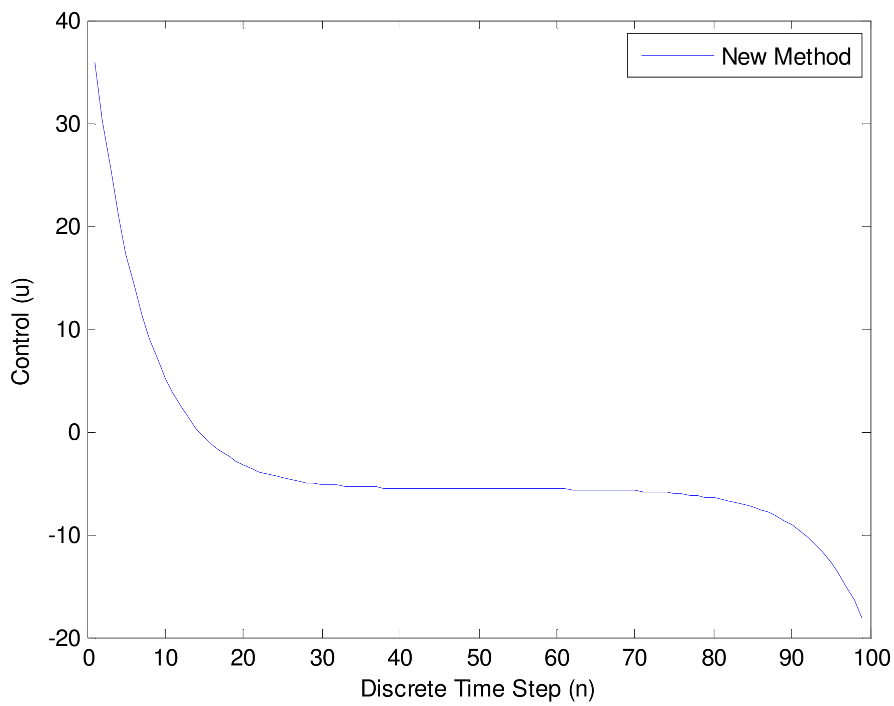


Figure 2-4: Fixed Final State LQR Control Sequence (2nd Order case)

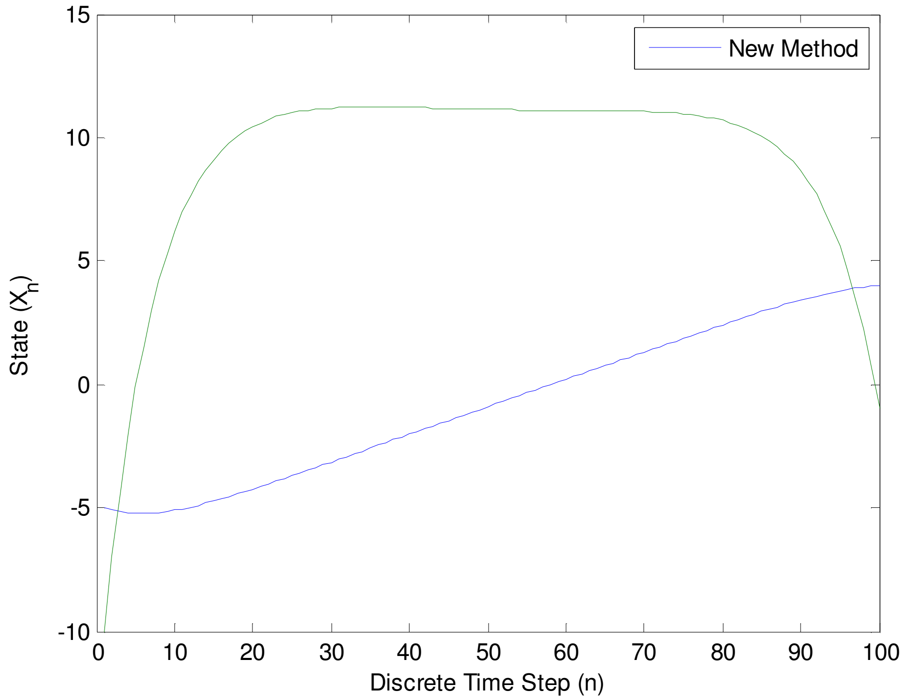


Figure 2-5: Fixed Final State LQR State Sequence (2nd Order case)

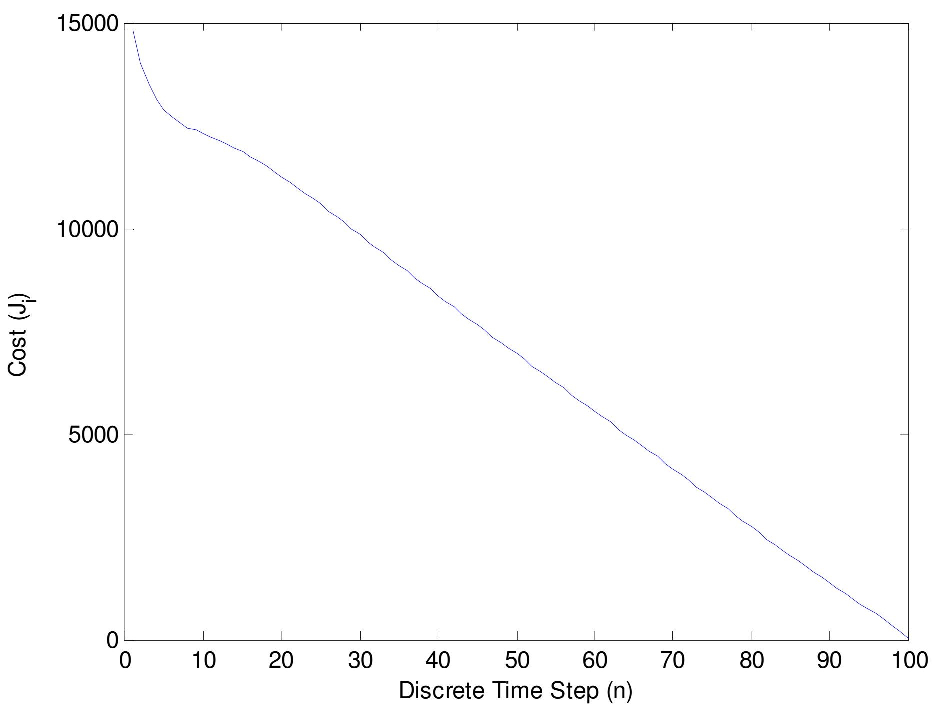


Figure 2-6: Fixed Final State LQR Cost of Controlling (2nd Order System)

2.1.1.2 Free-Final-State optimal control

It is now desired to find an optimal control sequence that drives the system from an initial state x_i, along a trajectory such that the cost function (2.1.1) is minimized, but we are not making any restriction on the final state x_N. This means that x_N can be varied thus $dx_N \neq 0$. Using the split boundary condition we require that

$$\lambda_N = \frac{\partial \phi}{\partial x_N} \qquad 2.1\text{-}32$$

The final state weighting function is $\phi = \frac{1}{2} x_N^T \tilde{S}_N x_N$, so we get

$$\lambda_N = \tilde{S}_N x_N \qquad 2.1\text{-}33$$

This is the new termination condition that we require to solve free final state problem. To solve this two-point boundary-value problem we shall use sweep method [3]. Thus we shall assume linear relation 2.1-33 holds for all times $k \leq N$:

$$\lambda_k = \tilde{S}_k x_k \qquad 2.1\text{-}34$$

for some intermediate sequence of $n \times n$ matrices $\tilde{S}_k$. So now we have to find a formula to populate $\tilde{S}_k$ for all times $k < N$. A valid expression for $\tilde{S}_k$ shall mean that our assumption is valid.

The decoupled state and costate equation with u_k eliminated are reproduced here:

$$x_{k+1} = \tilde{A}x_k - B\hat{R}^{-1}B^T\lambda_{k+1} \qquad 2.1\text{-}35$$

$$\lambda_k = \tilde{A}^T\lambda_{k+1} \qquad 2.1\text{-}36$$

The control is given as:

$$u_k = -\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right)x_k - \hat{R}^{-1}B^T\lambda_{k+1} \qquad 2.1\text{-}37$$

Use (2.1-37) in (2.1-35) to get

$$x_{k+1} = \tilde{A}x_k - B\hat{R}^{-1}B^T\tilde{S}_{k+1}x_{k+1} \qquad 2.1\text{-}38$$

Solving for x_{k+1} yields

$$x_{k+1} = \left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}\tilde{A}x_k \qquad 2.1\text{-}39$$

Now substitute (2.1-34) into costate equation (2.1-36) to get

$$\tilde{S}_k x_k = \tilde{A}^T\tilde{S}_k x_{k+1} \qquad 2.1\text{-}40$$

Use (2.1-39) in (2.1-40) to yield

$$\tilde{S}_k x_k = \tilde{A}^T\tilde{S}_k\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}\tilde{A}x_k \qquad 2.1\text{-}41$$

For (2.1-41) to hold for all state sequences give by any x_i, evidently we have

$$\tilde{S}_k = \tilde{A}^T\tilde{S}_k\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}\tilde{A} \qquad 2.1\text{-}42$$

This is a backward recursion for postulated $\tilde{S}_k$. Though (2.1-42) completely specifies $\tilde{S}_k$, we can modify (2.1-42) to get a closed form solution. If $\left|\tilde{S}_N\right| \neq 0$ and $\left|\tilde{A}\right| \neq 0$ then we can rewrite (2.1-42) as

$$\tilde{S}_k^{-1} = \tilde{A}^{-1}\tilde{S}_{k+1}^{-1}\tilde{A}^{-T} + \tilde{A}^{-1}\left(B\hat{R}^{-1}B^T\right)\tilde{A}^{-T} \qquad 2.1\text{-}43$$

This is a backward-developing Lyapunov equation for $\tilde{S}_k^{-1}$. The solution for (2.1-43) is given by,

$$\tilde{S}_k^{-1} = \tilde{A}^{-(N-k)}\tilde{S}_N^{-1}\left(\tilde{A}^T\right)^{-(N-k)} + \sum_{i=1}^{N-k}\tilde{A}^{-i}B\hat{R}^{-1}B^T\left(\tilde{A}^T\right)^{-i} \qquad 2.1\text{-}44$$

Our work is not quite finished yet. We would like to obtain an expression for u_k in term as a feedback of system state x_k. Use (2.1-34) in (2.1-37) to get

$$u_k = -\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right)x_k - \hat{R}^{-1}B^T\tilde{S}_{k+1}x_{k+1} \qquad 2.1\text{-}45$$

Use the plant equation $x_{k+1} = Ax_k + Bu_k$ in (2.1-45) to yield

$$u_k = -\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right)x_k - \hat{R}^{-1}B^T\tilde{S}_{k+1}\left(Ax_k + Bu_k\right) \qquad 2.1\text{-}46$$

simplifying to get expression for u_k,

$$u_k = \left(\hat{R} + B^T\tilde{S}_{k+1}B\right)^{-1}\left[B^T\tilde{S}_{k+1}A - \left(\hat{G}^T + \hat{H}\right)\right]x_k \qquad 2.1\text{-}47$$

This is the required expression!

TABLE 2-2 Discrete-Time Linear Quadratic Regulator (Final State Free)

System model:

$$x_{k+1} = Ax_k + Bu_k \,, \qquad k > i$$

Performance index:

$$J_i = \tfrac{1}{2} x_N^T S_N x_N + \tfrac{1}{2} \sum_{k=i}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right)$$

Assumptions:

$S_N \geq 0$, $Q_k \geq 0$, $R > 0$, and all three are symmetric

Optimal feedback control:

$$u_k = -K_k x_k \,, \qquad k < N$$

$$K_k = -\left(\hat{R} + B^T \tilde{S}_{k+1} B \right)^{-1} \left[B^T \tilde{S}_{k+1} A - \left(\hat{G}^T + \hat{H} \right) \right]$$

$$\tilde{S}_i^{-1} = \tilde{A}^{-(N-i)} \tilde{S}_N^{-1} \left(\tilde{A}^T \right)^{-(N-i)} + \sum_{\mu=1}^{N-i} \tilde{A}^{-\mu} B \hat{R}^{-1} B^T \left(\tilde{A}^T \right)^{-\mu} \,, \qquad i < N \,, \qquad \tilde{S}_N = S_N - \hat{S}$$

$$\tilde{A} = A - B\hat{R}^{-1} \left(\hat{G}^T + \hat{H} \right)$$

$$\tilde{Q} = \hat{Q} - \left(\hat{G} + \hat{H}^T \right) \hat{R}^{-1} \left(\hat{G}^T + \hat{H} \right)$$

$$\hat{Q} = Q - \hat{S} + A^T \hat{S} A$$

$$\hat{G} = \tfrac{1}{2} A^T \hat{S} B$$

$$\hat{H} = \tfrac{1}{2} B^T \hat{S} A$$

$$\hat{R} = R + B^T \hat{S} B$$

$$\hat{S} = A^T \hat{S} A - \left(A^T \hat{S} B \right) \left(R + B^T \hat{S} B \right)^{-1} \left(B^T \hat{S} A \right) + Q$$

Example 2.1-3: *Example 1*: Scalar System

Consider a scalar system $x_{k+1} = Ax_k + Bu_k$ where $\mathrm{A} = 1.05, \mathrm{B} = 0.01$ with the initial condition $x_0 = 10$. It is desired to find the optimal control that minimizes the cost function

$$J_0 = \tfrac{1}{2} x_N^T S_N x_N + \tfrac{1}{2} \sum_{k=0}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right) \text{ where } \mathrm{Q} = 1, \mathrm{R} = 1, \mathrm{S_N} = 5, \mathrm{N} = 100 .$$

Free final state control and state sequence are shown in Figure. 2.1-7 and Figure. 2.1-8, respectively. Corresponding results for the conventional method are also included and are seen to be the same. Optimal cost is found to be $J_0^* = 51608.16$ in both cases.

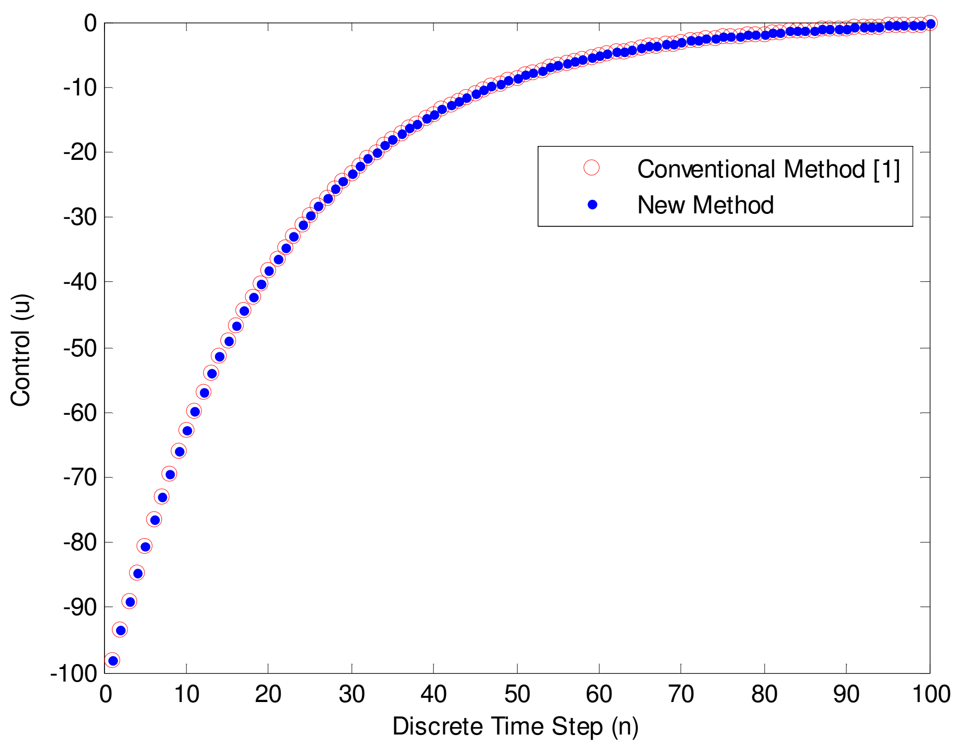


Figure 2-7: Free Final State LQR Control Sequence (Scalar case)

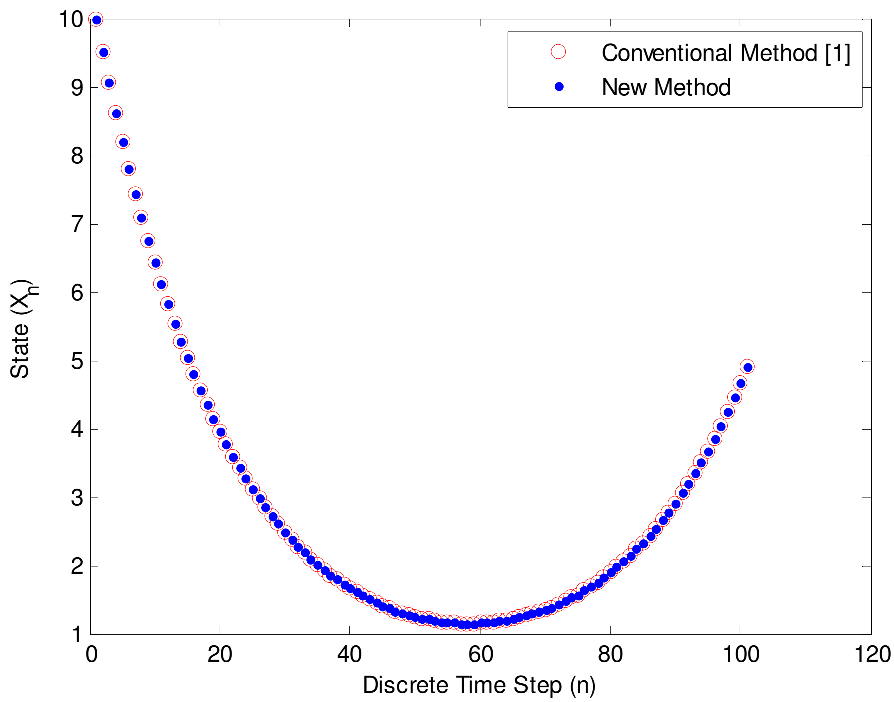


Figure 2-8: Free Final State LQR State Sequence (Scalar Case)

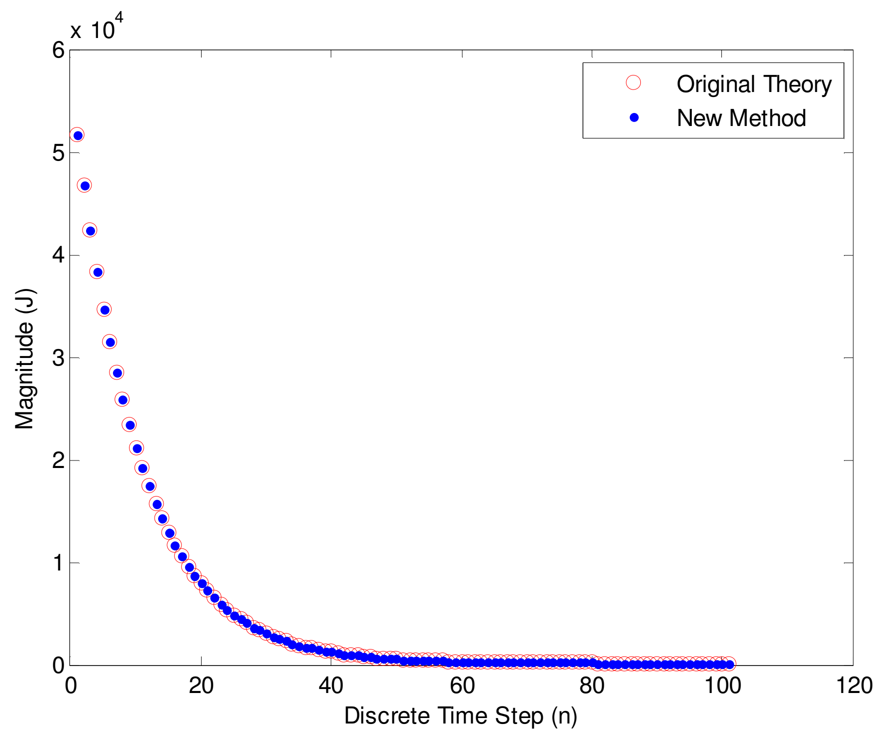


Figure 2-9: Free Final State LQR Cost of Controlling (Scalar case)

Example 2.1-4: Second Order System.

Consider a second order system described by $x_{k+1} = Ax_k + Bu_k$, where $A = \begin{bmatrix} 1 & 0.01 \\ 0.01 & 1.05 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$, with the initial condition $x_0 = \begin{bmatrix} -5 \\ -10 \end{bmatrix}$. It is desired to find the optimal control that minimizes the cost function $J_0 = \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=0}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right)$ where $S_N = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $R = 1$.

Free final state control and state sequences are shown in Figure. 2.1-10 and Figure. 2.1-11, respectively. Optimal cost is found to be $J_0^* = 3472.56$ in both cases.

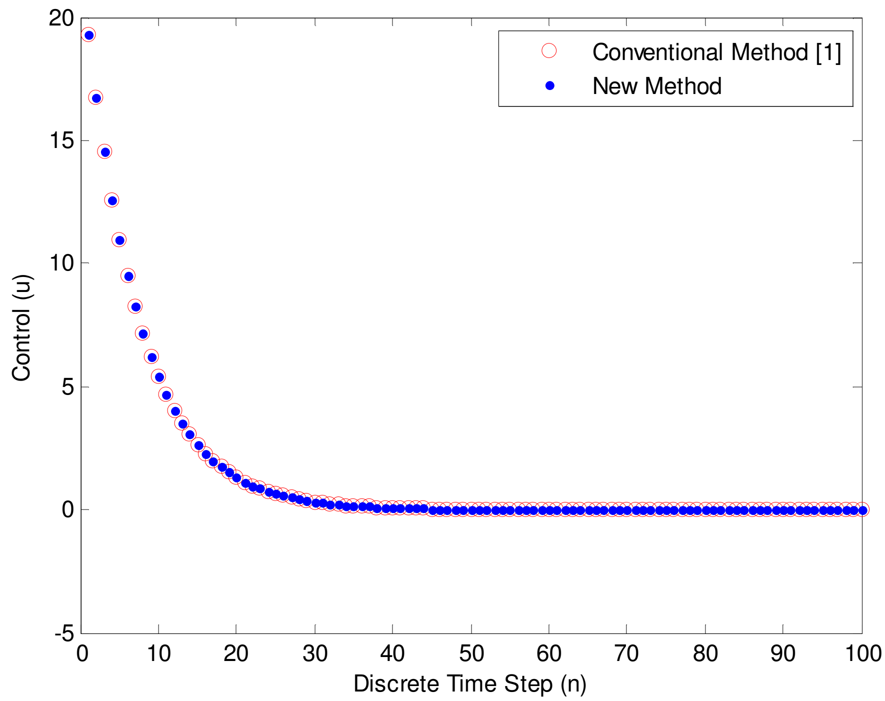


Figure 2-10: Free Final State LQR Control Sequence (2nd Order case)

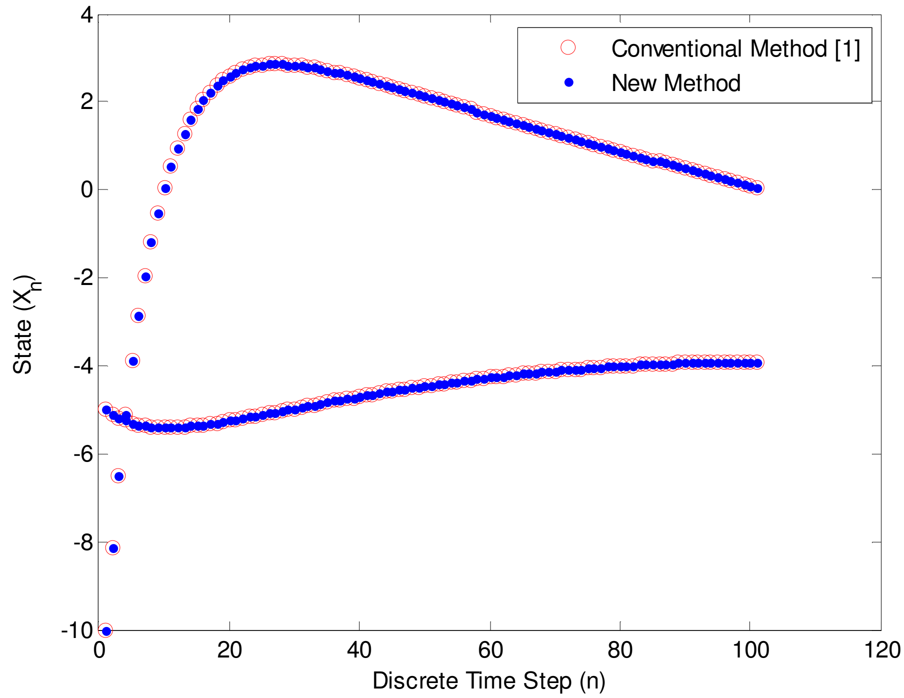


Figure 2-11: Free Final State LQR State Sequence (2nd Order case)

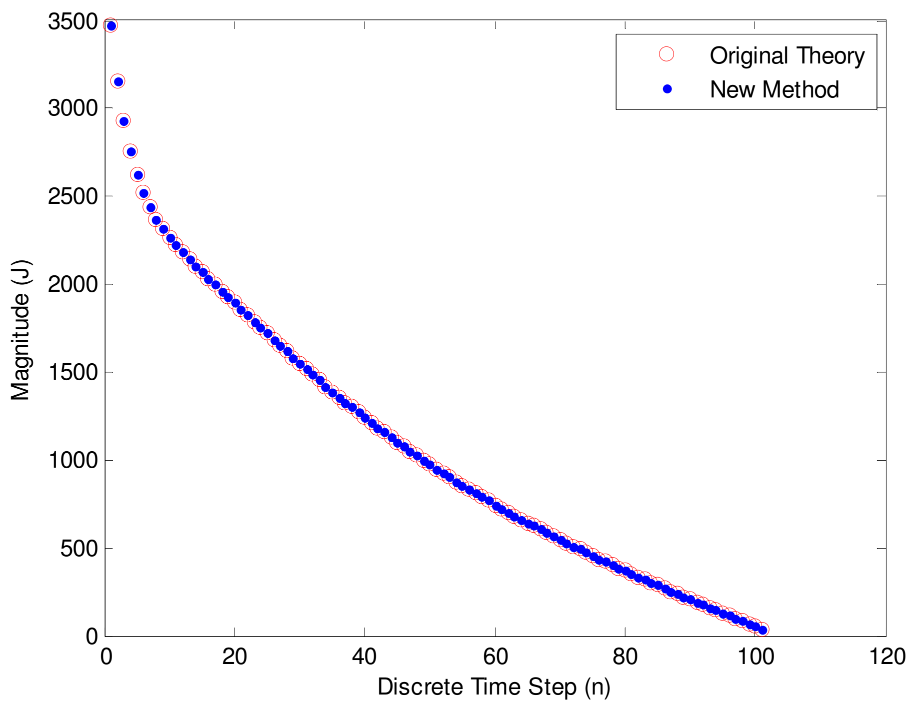


Figure 2-12: Free Final State LQR Cost of Controlling (2nd Order case)

2.1.2 *Tracking Problem*

Consider a plant described by the linear equation $x_{k+1} = Ax_k + Bu_k$ with $x_k \in R^n$ and $u_k \in R^m$. It is desired to get a control law that forces a certain linear combination of the states $y_k = Cx_k$ of the plant to track a desired reference trajectory r_k over a specified time interval $[i, N]$, then problem can be converted to one where we have to minimize the cost function

$$J_i = \tfrac{1}{2}(Cx_N - r_N)^T P_N (Cx_N - r_N) + \tfrac{1}{2}\sum_{k=i}^{N-1}\left[(Cx_k - r_k)^T Q (Cx_k - r_k) + u_k^T R u_k\right]$$

where P and Q are symmetric positive semi-definite matrices and R is symmetric positive definite matrix. Actual value of x_N is not constrained but we want it to be as close as r_N.

Consider the original cost function

$$J_i = \tfrac{1}{2}(Cx_N - r_N)^T P_N (Cx_N - r_N) + \tfrac{1}{2}\sum_{k=i}^{N-1}\left[(Cx_k - r_k)^T Q (Cx_k - r_k) + u_k^T R u_k\right] \qquad 2.1\text{-}48$$

We can subtract and add a term $\tfrac{1}{2}(Cx_N - r_N)^T \hat{S}(Cx_N - r_N)$ to obtain:

$$\begin{aligned} J_i &= \tfrac{1}{2}(Cx_N - r_N)^T (P_N - \hat{S})(Cx_N - r_N) + \tfrac{1}{2}(Cx_N - r_N)^T \hat{S}(Cx_N - r_N) \\ &\quad + \tfrac{1}{2}\sum_{k=i}^{N-1}\left[(Cx_k - r_k)^T Q (Cx_k - r_k) + u_k^T R u_k\right] \end{aligned} \qquad 2.1\text{-}49$$

where $\hat{S}$ is a symmetric positive semi-definite.

Now observe

$$\tfrac{1}{2}\left(Cx_N - r_N\right)^T \hat{S}\left(Cx_N - r_N\right) = \tfrac{1}{2}\left(Cx_i - r_i\right)^T \hat{S}\left(Cx_i - r_i\right) + \tfrac{1}{2}\sum_{m=i}^{N-1}\begin{bmatrix}\left(Cx_{m+1} - r_{m+1}\right)^T \hat{S}\left(Cx_{m+1} - r_{m+1}\right) \\ - \tfrac{1}{2}\left(Cx_m - r_m\right)^T \hat{S}\left(Cx_m - r_m\right)\end{bmatrix} \qquad 2.1\text{-}50$$

Substituting (2.1-50) into (2.1-49) yields

$$\begin{aligned} J_i &= \tfrac{1}{2}\left(Cx_N - r_N\right)^T\left(P_N - \hat{S}\right)\left(Cx_N - r_N\right) + \tfrac{1}{2}\left(Cx_i - r_i\right)^T \hat{S}\left(Cx_i - r_i\right) \\ &\quad + \tfrac{1}{2}\sum_{m=i}^{N-1}\left(Cx_{m+1} - r_{m+1}\right)^T \hat{S}\left(Cx_{m+1} - r_{m+1}\right) - \tfrac{1}{2}\left(Cx_m - r_m\right)^T \hat{S}\left(Cx_m - r_m\right) \\ &\quad + \tfrac{1}{2}\sum_{k=i}^{N-1}\left[\left(Cx_k - r_k\right)^T Q\left(Cx_k - r_k\right) + u_k^T R u_k\right] \\ &= \tfrac{1}{2}\left(Cx_N - r_N\right)^T\left(P_N - \hat{S}\right)\left(Cx_N - r_N\right) + \tfrac{1}{2}\left(Cx_i - r_i\right)^T \hat{S}\left(Cx_i - r_i\right) \\ &\quad + \tfrac{1}{2}\sum_{k=i}^{N-1}\left[\left(Cx_k - r_k\right)^T\left(Q - \hat{S}\right)\left(Cx_k - r_k\right) + \left(Cx_{k+1} - r_{k+1}\right)^T \hat{S}\left(Cx_{k+1} - r_{k+1}\right) + u_k^T R u_k\right] \end{aligned} \qquad 2.1\text{-}51$$

Using $x_{k+1} = Ax_k + Bu_k$, (2.1-51) can be rewritten after simplification as:

$$\begin{aligned} J_i &= \tfrac{1}{2}\left(Cx_N - r_N\right)^T \hat{P}_N\left(Cx_N - r_N\right) + \tfrac{1}{2}\left(Cx_0 - r_0\right)^T \hat{S}\left(Cx_N - r_N\right) \\ &\quad + \tfrac{1}{2}\sum_{k=i}^{N-1}\begin{bmatrix} x_k^T \hat{Q} x_k + 2x_k^T \hat{G} u_k + 2u_k^T \hat{H} x_k + u_k^T \hat{R} u_k + 2x_k^T \hat{E} r_k + 2r_k^T \hat{F} x_k \\ + 2x_k^T \hat{L} r_{k+1} + 2r_{k+1}^T \hat{M} x_k + 2u_k^T \hat{O} r_{k+1} + 2r_{k+1}^T \hat{T} u_k + r_k^T \hat{W} r_k + r_{k+1}^T \hat{S} r_{k+1}\end{bmatrix} \end{aligned} \qquad 2.1\text{-}52$$

where

$$\hat{P}_N = P_N - \hat{S}$$

$$\hat{Q} = C^T\left(Q - \hat{S}\right)C + A^T C^T \hat{S} C A$$

$$\hat{G} = \tfrac{1}{2} A^T C^T \hat{S} C B$$

$$\hat{H} = \tfrac{1}{2} B^T C^T \hat{S} C A$$

$$\hat{R} = R + B^T C^T \hat{S} C B$$

$$\hat{E} = -\tfrac{1}{2} C^T\left(Q - \hat{S}\right)$$

$$\hat{F} = -\tfrac{1}{2}\left(Q - \hat{S}\right)C \qquad 2.1\text{-}53$$

$$\hat{L} = -\tfrac{1}{2} A^T C^T \hat{S}$$

$$\hat{M} = -\tfrac{1}{2} \hat{S} C A$$

$$\hat{O} = -\tfrac{1}{2} B^T C^T \hat{S}$$

$$\hat{T} = -\tfrac{1}{2} \hat{S} C B$$

$$\hat{W} = Q - \hat{S}$$

Hamiltonian function for modified cost function (2.1-52)

$$H^k = \tfrac{1}{2}\begin{bmatrix} x_k^T \hat{Q} x_k + 2x_k^T \hat{G} u_k + 2u_k^T \hat{H} x_k + u_k^T \hat{R} u_k + 2x_k^T \hat{E} r_k + 2r_k^T \hat{F} x_k \\ + 2x_k^T \hat{L} r_{k+1} + 2r_{k+1}^T \hat{M} x_k + 2u_k^T \hat{O} r_{k+1} + 2r_{k+1}^T \hat{T} u_k + r_k^T \hat{W} r_k + r_{k+1}^T \hat{S} r_{k+1} \end{bmatrix} \qquad 2.1\text{-}54$$
$$+ \lambda_{k+1}^T \left(A x_k + B u_k\right)$$

Then state and costate equations are

$$x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = A x_k + B u_k \qquad 2.1\text{-}55$$

$$\lambda_k = \frac{\partial H^k}{\partial x_k} = \hat{Q} x_k + \left(\hat{G} + \hat{H}^T\right) u_k + \left(\hat{E} + \hat{F}^T\right) r_k + \left(\hat{L} + \hat{M}^T\right) r_{k+1} + A^T \lambda_{k+1} \qquad 2.1\text{-}56$$

and the stationary condition

$$0 = \frac{\partial H^k}{\partial u_k} = \hat{R} u_k + \left(\hat{G}^T + \hat{H}\right) x_k + \left(\hat{O} + \hat{T}^T\right) + B^T \lambda_{k+1} \qquad 2.1\text{-}57$$

Rearranging (2.1-57), we get

$$u_k = -\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right) x_k - \hat{R}^{-1}\left(\hat{O} + \hat{T}^T\right) r_{k+1} - \hat{R}^{-1} B^T \lambda_{k+1} \qquad 2.1\text{-}58$$

Using (2.1-58) to eliminate u_k in (2.1-55) and (2.1-56) we get the following Hamiltonian system

$$\begin{bmatrix} x_{k+1} \\ \lambda_k \end{bmatrix} = \begin{bmatrix} \tilde{A} & -B\hat{R}^{-1}B^T \\ \tilde{Q} & \tilde{A}^T \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix} + \begin{bmatrix} 0 & -B\hat{R}^{-1}\left(\hat{O} + \hat{T}^T\right) \\ \left(\hat{E}^T + \hat{F}\right) & \tilde{Z} \end{bmatrix} \begin{bmatrix} r_k \\ r_{k+1} \end{bmatrix} \qquad 2.1\text{-}59$$

where

$$\tilde{A} = A - B\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right),$$

$$\tilde{Q} = \hat{Q} - \left(\hat{G} + \hat{H}^T\right)\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right). \qquad 2.1\text{-}60$$

$$\tilde{Z} = \left(\hat{L} + \hat{M}^T\right) - \left(\hat{G} + \hat{H}^T\right)\hat{R}^{-1}\left(\hat{O} + \hat{T}^T\right)$$

In order to decouple the Hamiltonian system (2.1-59) we set $\tilde{Q} = 0$

$$\hat{Q} - \left(\hat{G} + \hat{H}^T\right)\hat{R}^{-1}\left(\hat{H} + \hat{G}^T\right) = 0 \qquad 2.1\text{-}61$$

using (2.1-53), (2.1-61) can be rearranged as,

$$C^T\hat{S}C = A^TC^T\hat{S}CA - A^TC^T\hat{S}B\left(R + B^TC^T\hat{S}CB\right)^{-1}B^TC^T\hat{S}CA + C^TQC \qquad 2.1\text{-}62$$

Equation (2.1-62) is an algebraic riccati equation. There exits a symmetric positive semi-definite solution $\hat{S}^*$ for (2.1-62) as long as (A, B) is a stabalizable pair, $R > 0$ and $Q \geq 0$. Proof of this can be obtained from Collary 13.1.2 in [2].

Corollary 13.1.2: *Assume* $R > 0$, (A, B) *is a d-stabalizable pair, and*

$$\begin{bmatrix} Q & C^* \\ C & R \end{bmatrix} \geq 0$$

Then the maximal hermitian solution X_+ *of (13.1.1) exists, is unique, and is positive semidefinite. Moreover, all eigenvalues of the matrix (13.1.14) lie in the closed unit disk.*

$$X = A^*XA + Q - \left(C + B^*XA\right)^*\left(R + B^*XB\right)^{-1}\left(C + B^*XA\right) \qquad (13.1.1)$$

$$\hat{A} = A - B\left(R + B^* X_+ B\right)^{-1}\left(C + B^* X_+ A\right) \qquad \textit{(13.1.14)}$$

Then for this $\hat{S}^*$, the Hamiltonian system (2.1-59) is decoupled and is given by:

$$\begin{bmatrix} x_{k+1} \\ \lambda_k \end{bmatrix} = \begin{bmatrix} \tilde{A} & -B\hat{R}^{-1}B^T \\ 0 & \tilde{A}^T \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix} + \begin{bmatrix} 0 & -B\hat{R}^{-1}\left(\hat{O} + \hat{T}^T\right) \\ \left(\hat{E}^T + \hat{F}\right) & \tilde{Z} \end{bmatrix} \begin{bmatrix} r_k \\ r_{k+1} \end{bmatrix} \qquad 2.1\text{-}63$$

To solve this problem we need boundary conditions. First one is the initial condition x_0, which is given. The final state condition x_N, which we want to be as closed as r_N, is not fixed. Thus $dx_N \neq 0$ and therefore from split boundary conditions we require

$$\lambda_N = \frac{\partial \phi}{\partial x_N} \qquad 2.1\text{-}64$$

The final state weighting function is $\phi = \frac{1}{2}\left(Cx_N - r_N\right)^T \hat{P}_N \left(Cx_N - r_N\right)$, so we get

$$\lambda_N = C^T \hat{P}_N \left(Cx_N - r_N\right) \qquad 2.1\text{-}65$$

We assume that linear relation (2.1-65) holds for all $k \leq N$ [3], thus we can write:

$$\lambda_k = \tilde{S}_k x_k - v_k \qquad 2.1\text{-}66$$

where

$$\tilde{S}_N = C^T \hat{P}_N C$$

$$v_N = C^T \hat{P}_N r_N$$

Use (2.1-66) in state equation portion of decoupled system (2.1-63) to get

$$x_{k+1} = \tilde{A}x_k - B\hat{R}^{-1}B^T \tilde{S}_{k+1} x_{k+1} + B\hat{R}^{-1}B^T v_{k+1} - B\hat{R}^{-1}\left(\hat{O} + \hat{T}^T\right) r_{k+1} \qquad 2.1\text{-}67$$

Solve for x_{k+1} to yield

$$x_{k+1} = \left(I + B\hat{R}^{-1}B^T \tilde{S}_{k+1}\right)^{-1} \left[\tilde{A}x_k + B\hat{R}^{-1}B^T v_{k+1} - B\hat{R}^{-1}\left(\hat{O} + \hat{T}^T\right) r_{k+1}\right] \qquad 2.1\text{-}68$$

Now use (2.1-66) again in costate portion of decoupled system (2.1-63) to get

$$\tilde{S}_k x_k - v_k = \tilde{A}^T \tilde{S}_{k+1} x_{k+!} - \tilde{A}^T v_{k+1} + \left(\hat{E} + \hat{F}^T\right) r_k + \tilde{Z} r_{k+1} \qquad 2.1\text{-}69$$

Use expression for x_{k+1} from (2.1-70) in (2.1-71) to simplify as

$$\begin{aligned} &\left[\tilde{S}_k - \tilde{A}^T \tilde{S}_{k+1}\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}\tilde{A}\right]x_k - v_k \\ &+ \tilde{A}^T v_{k+1} - \tilde{A}^T\tilde{S}_{k+1}\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}B\hat{R}^{-1}B^T v_{k+1} \\ &+ \left[\tilde{A}^T\tilde{S}_{k+1}\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}B\hat{R}^{-1}\left(\hat{O} + \hat{T}^T\right) - \tilde{Z}\right] r_{k+1} - \left(\hat{E} + \hat{F}^T\right) r_k = 0 \end{aligned} \qquad 2.1\text{-}72$$

Equation (2.1-72) must hold for all state sequences x_k given any x_0, thus we have following set of equations

$$\tilde{S}_k - \tilde{A}^T \tilde{S}_{k+1}\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}\tilde{A} = 0$$

$$\begin{aligned} &- v_k + \tilde{A}^T v_{k+1} - \tilde{A}^T\tilde{S}_{k+1}\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}B\hat{R}^{-1}B^T v_{k+1} \\ &+ \left[\tilde{A}^T\tilde{S}_{k+1}\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}B\hat{R}^{-1}\left(\hat{O} + \hat{T}^T\right) - \tilde{Z}\right] r_{k+1} - \left(\hat{E} + \hat{F}^T\right) r_k = 0 \end{aligned}$$

or

$$\tilde{S}_k = \tilde{A}^T \tilde{S}_{k+1}\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}\tilde{A} \qquad 2.1\text{-}73$$

$$\begin{aligned} v_k = &\tilde{A}^T v_{k+1} - \tilde{A}^T\tilde{S}_{k+1}\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}B\hat{R}^{-1}B^T v_{k+1} \\ &+ \left[\tilde{A}^T\tilde{S}_{k+1}\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1}B\hat{R}^{-1}\left(\hat{O} + \hat{T}^T\right) - \tilde{Z}\right] r_{k+1} - \left(\hat{E} + \hat{F}^T\right) r_k \end{aligned} \qquad 2.1\text{-}74$$

Equation (2.1-73) and (2.1-74) are the required consistent equations that validate our assumptions. Boundary condition for these set of equation are given by

$$\tilde{S}_N = C^T \hat{P}_N C$$

$$v_N = C^T \hat{P}_N r_N$$

We can further modify (2.1-73) to get a closed form solution for it. If $\left|\tilde{S}_N\right| \neq 0$ and $\left|\tilde{A}\right| \neq 0$ then we can rewrite (2.1-73) as

$$\tilde{S}_k^{-1} = \tilde{A}^{-1}\tilde{S}_{k+1}^{-1}\tilde{A}^{-T} + \tilde{A}^{-1}\left(B\hat{R}^{-1}B^T\right)\tilde{A}^{-T} \quad 2.1\text{-}75$$

This is a backward-developing Lyapunov equation for $\tilde{S}_k^{-1}$. The solution for (2.1-75) is given by,

$$\tilde{S}_i^{-1} = \tilde{A}^{-(N-i)}\tilde{S}_N^{-1}\left(\tilde{A}^T\right)^{-(N-i)} + \sum_{\mu=1}^{N-i}\tilde{A}^{-\mu}B\hat{R}^{-1}B^T\left(\tilde{A}^T\right)^{-\mu} \quad 2.1\text{-}76$$

$i = 0,\ 1,\ 2,\ 3, \ldots\ldots, N-1$

We are yet to find a expression for u_k in term of system state. To do this we first put (2.1.-66) in (2.1-58) to get

$$u_k = -\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right)x_k - \hat{R}^{-1}\left(\hat{O} + \hat{T}^T\right)r_{k+1} - \hat{R}^{-1}B^T\tilde{S}_{k+1}x_{k+1} + \hat{R}^{-1}B^T v_{k+1} \quad 2.1\text{-}77$$

Use plant equation $x_{k+1} = Ax_k + Bu_k$ to solve for u_k in (2.1-77) to yield,

$$u_k = -\left(\hat{R} + B^T\tilde{S}_{k+1}B\right)^{-1}\left[\left(B^T\tilde{S}_{k+1}A + \hat{G}^T + \hat{H}\right)x_k + \left(\hat{T}^T + \hat{O}\right)r_{k+1} - B^T v_{k+1}\right] \quad 2.1\text{-}78$$

TABLE 2-3 Discrete-Time Linear Quadratic Tracker

System model:

$$x_{k+1} = Ax_k + Bu_k, \qquad k > i$$

$$y_k = Cx_k$$

Performance index:

$$J_i = \tfrac{1}{2}(Cx_N - r_N)^T P_N (Cx_N - r_N) + \tfrac{1}{2}\sum_{k=i}^{N-1}\left[(Cx_k - r_k)^T Q(Cx_k - r_k) + u_k^T R u_k\right]$$

Assumptions:

$P_N \ge 0$, $Q_k \ge 0$, $R > 0$, and all three are symmetric

Optimal feedback control:

$$u_k = -\left(\hat{R} + B^T \tilde{S}_{k+1} B\right)^{-1}\left[\left(B^T \tilde{S}_{k+1} A + \hat{G}^T + \hat{H}\right)x_k + \left(\hat{T}^T + \hat{O}\right)r_{k+1} - B^T v_{k+1}\right], \qquad k < N$$

$$\tilde{S}_i^{-1} = \tilde{A}^{-(N-i)}\tilde{S}_N^{-1}\left(\tilde{A}^T\right)^{-(N-i)} + \sum_{\mu=1}^{N-i}\tilde{A}^{-\mu} B\hat{R}^{-1}B^T\left(\tilde{A}^T\right)^{-\mu}, \quad i < N, \quad \tilde{S}_N = C^T\left(P_N - \hat{S}\right)C$$

$$v_k = \tilde{A}^T v_{k+1} - \tilde{A}^T \tilde{S}_{k+1}\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1} B\hat{R}^{-1}B^T v_{k+1} + \left[\tilde{A}^T\tilde{S}_{k+1}\left(I + B\hat{R}^{-1}B^T\tilde{S}_{k+1}\right)^{-1} B\hat{R}^{-1}\left(\hat{O} + \hat{T}^T\right) - \tilde{Z}\right]r_{k+1} - \left(\hat{E} + \hat{F}^T\right)r_k,$$

$$v_N = C^T\left(P_N - \hat{S}\right)r_N$$

$$\tilde{A} = A - B\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right)$$

$$\tilde{Q} = \hat{Q} - \left(\hat{G} + \hat{H}^T\right)\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right)$$

$$\hat{Q} = Q - \hat{S} + A^T\hat{S}A$$

$$\hat{G} = \tfrac{1}{2}A^T\hat{S}B$$

$$\hat{H} = \tfrac{1}{2}B^T\hat{S}A$$

$$\hat{R} = R + B^T\hat{S}B$$

$$\hat{O} = -\tfrac{1}{2}B^T C^T\hat{S}$$

$$\hat{T} = -\tfrac{1}{2}\hat{S}CB$$

$$\hat{E} = -\tfrac{1}{2}C^T\left(Q - \hat{S}\right)$$

$$\hat{F} = -\tfrac{1}{2}\left(Q - \hat{S}\right)C$$

$$\hat{S} = A^T\hat{S}A - \left(A^T\hat{S}B\right)\left(R + B^T\hat{S}B\right)^{-1}\left(B^T\hat{S}A\right) + Q$$

Example 2.1-5: Scalar Case

Consider a scalar system described by $x_{k+1} = Ax_k + Bu_k$, where $\mathrm{A} = 1.05, \mathrm{B} = 0.01$, with the initial condition $x_0 = 10$. It is desired to find the optimal control that minimizes the cost function $J_i = \frac{1}{2}(Cx_N - r_N)^T P_N (Cx_N - r_N) + \frac{1}{2}\sum_{k=i}^{N-1}\left[(Cx_k - r_k)^T Q(Cx_k - r_k) + u_k^T Ru_k\right]$

where $Q = 100000, \quad R = 1, \quad P_N = 5, \quad N = 100 \quad C = 1$.

It is desired to make system track a specific trajectory given by $r_k = 5\sin(k/\pi)$.

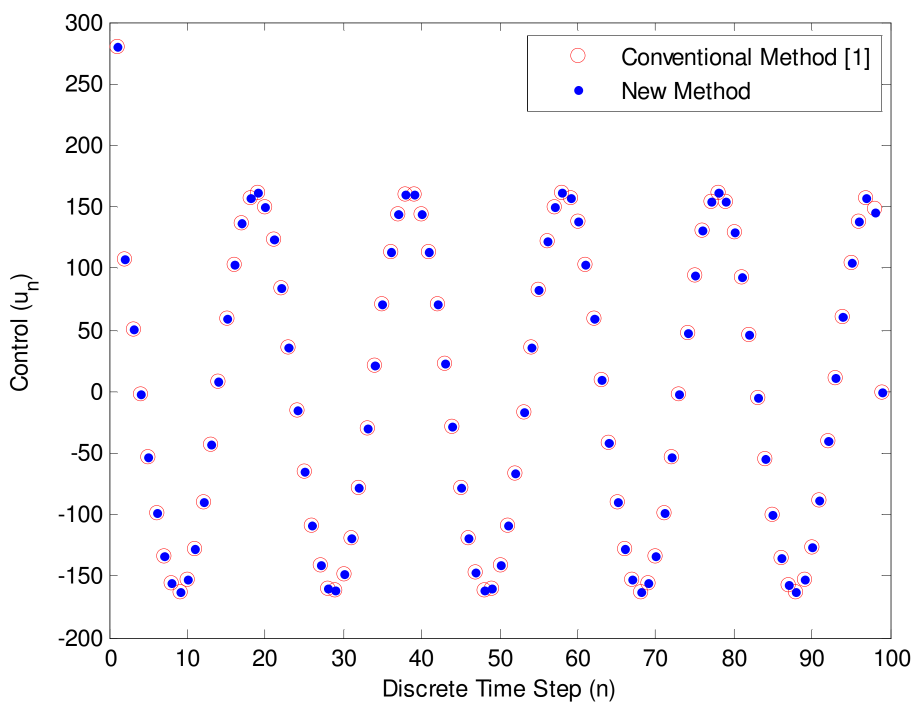


Figure 2-13: Discrete LQ Tracker Control Sequence

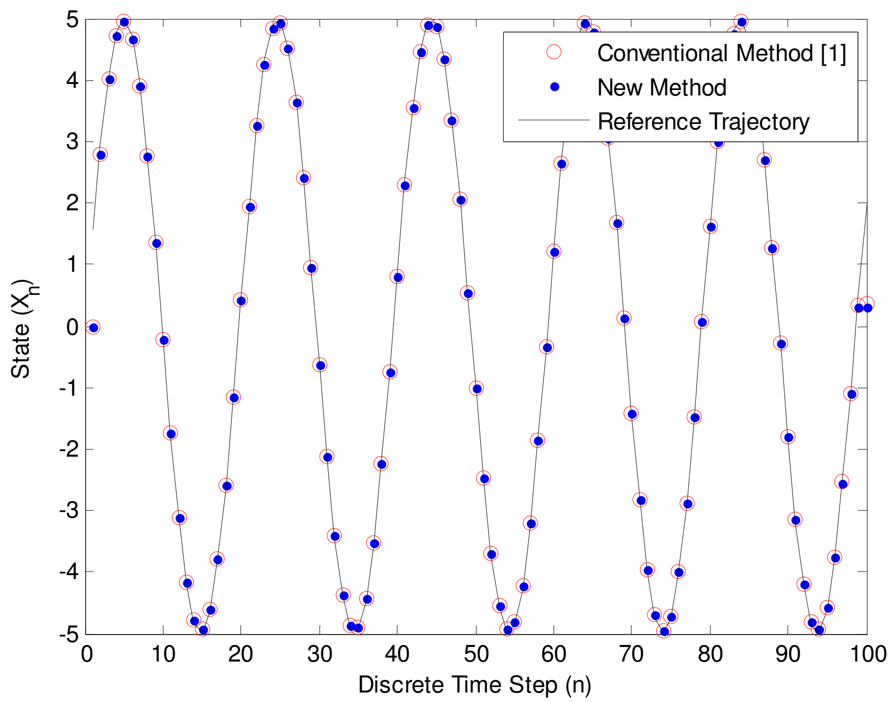


Figure 2-14: Discrete LQ Tracker State Sequence

2.2 Continuous-Time Linear Quadratic Problem

2.2.1 *Regulation Problem*

In this section we would consider continuous-time linear plant

$$\dot{x} = Ax + Bu\,, \qquad 2.2\text{-}1$$

where $x \in R^n$ and $u \in R^m$ with associated quadratic performance index defined over the time interval $[t_0, T]$

$$J(t_0) = \tfrac{1}{2}x^T(T)S(T)x(T) + \tfrac{1}{2}\int_{t_0}^{T}\left(x^T Qx + u^T Ru\right)dt \qquad 2.2\text{-}2$$

where matrices $S(T)$ and Q are positive semi-definite and R is symmetric positive definite. We have to determine the control sequence u_k such that it minimizes $J(t_0)$ on

$[t_0, T]$. As in discrete regulator case we have two cases: fixed final state and free final state. But before that we shall apply a transformation to quadratic performance index similar to that discrete case to enable a closed form solution of problem.

Consider the original cost function (quadratic performance index)

$$J(t_0) = \tfrac{1}{2} x^T(T) S(T) x(T) + \tfrac{1}{2}\int_{t_0}^{T}\left(x^T Q x + u^T R u\right) dt$$

We can subtract and add a term $\tfrac{1}{2} x^T(T)\hat{S}x(T)$ to obtain

$$J(t_0) = \tfrac{1}{2} x^T(T)\left(S(T) - \hat{S}\right)x(T) + \tfrac{1}{2} x^T(T)\hat{S}x(T) + \tfrac{1}{2}\int_{t_0}^{T}\left(x^T Q x + u^T R u\right) dt \qquad 2.2\text{-}3$$

Now observe

$$\tfrac{1}{2} x^T(T)\hat{S}x(T) = \tfrac{1}{2} x^T(t_0)\hat{S}x(t_0) + \tfrac{1}{2}\int_{t_0}^{T}\left(\dot{x}^T(t)\hat{S}x(t) + x^T(t)\hat{S}\dot{x}(t)\right) dt \qquad 2.2\text{-}4$$

Substituting (2.2-4) into (2.2-3) yields

$$\begin{aligned} J(t_0) = {} & \tfrac{1}{2} x^T(T)\left(S(T) - \hat{S}\right)x(T) + \tfrac{1}{2} x^T(t_0)\hat{S}x(t_0) \\ & + \tfrac{1}{2}\int_{t_0}^{T}\left(\dot{x}^T(t)\hat{S}x(t) + x^T(t)\hat{S}\dot{x}(t)\right) dt + \tfrac{1}{2}\int_{t_0}^{T}\left(x^T Q x + u^T R u\right) dt \end{aligned} \qquad 2.2\text{-}5$$

Using plant equation $\dot{x} = Ax + Bu$, (2.2-5) can be written as after simplification as

$$\begin{aligned} J(t_0) = {} & \tfrac{1}{2} x^T(T)\tilde{S}(T)x(T) + \tfrac{1}{2} x^T(t_0)\hat{S}x(t_0) \\ & + \tfrac{1}{2}\int_{t_0}^{T}\left(x^T \hat{Q} x + 2x^T \hat{G} u + 2u^T \hat{H} x + u^T R u\right) dt \end{aligned} \qquad 2.2\text{-}6$$

where

$$\tilde{S}(T) = S(T) - \hat{S}$$

$$\hat{Q} = Q + \hat{S}A + A^T\hat{S}$$

$$\hat{G} = \tfrac{1}{2}\hat{S}B \qquad 2.2\text{-}7$$

$$\hat{H} = \tfrac{1}{2}B^T\hat{S}$$

Hamiltonian function for this modified cost function (2.2-6)

$$H(t) = \tfrac{1}{2}\left(x^T\hat{Q}x + 2x^T\hat{G}u + 2u^T\hat{H}x + u^TRu\right) + \lambda^T(Ax + Bu) \qquad 2.2\text{-}8$$

Then the state and costate equation are

$$\dot{x} = \frac{\partial H}{\partial \lambda} = Ax + Bu \qquad 2.2\text{-}9$$

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \hat{Q}x + \left(\hat{G} + \hat{H}\right)u + A^T\lambda \qquad 2.2\text{-}10$$

and the stationary condition is

$$0 = \frac{\partial H}{\partial u} = Ru + \left(\hat{G} + \hat{H}\right)x + B^T\lambda \qquad 2.2\text{-}11$$

Rearrange (2.2-11) to yield control $u(t)$ in term of costate $\lambda(t)$

$$u(t) = -R^{-1}\left(\hat{G} + \hat{H}\right)x - R^{-1}B^T\lambda \qquad 2.2\text{-}12$$

Using (2.2-12) to eliminate $u(t)$ from (2.2-9) and (2.2-10) we get the following Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \tilde{A} & -BR^{-1}B^T \\ -\tilde{Q} & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \qquad 2.2\text{-}13$$

where

$$\widetilde{A} = A - BR^{-1}\left(\hat{G}^T + \hat{H}\right),$$

$$\widetilde{Q} = \hat{Q} - \left(\hat{G} + \hat{H}^T\right)R^{-1}\left(\hat{G}^T + \hat{H}\right)$$

In order to decouple the Hamiltonian system (2.2-13) we set $\widetilde{Q} = 0$

$$\widetilde{Q} = \hat{Q} - \left(\hat{G} + \hat{H}^T\right)\hat{R}^{-1}\left(\hat{G}^T + \hat{H}\right) = 0 \qquad 2.2\text{-}14$$

Using (2.2-7), (2.2-14) can rewritten after simplification as

$$\hat{S}BR^{-1}B^T\hat{S} - \hat{S}A - A^T\hat{S} - Q = 0 \qquad 2.2\text{-}15$$

Equation (2.2-15) is an continuous algebraic riccati equation. There exits a symmetric positive semi-definite solution $\hat{S}^*$ for (2.2-15) as long as (A, B) is a stabalizable pair, $R > 0$ and $Q \geq 0$. Proof of this can be obtained from Theorem 9.1.2 in [2].

> ***Theorem 9.1.2*** : *If* $D \geq 0$, $C \geq 0$ *and the pair* (A, D) *is stabalizable then there exist hermitian solutions of* $\mathcal{R}(X) = 0$. *Moreover, the maximal hermitian solution* X_+ *also satisfies* $X_+ \geq 0$. *If, in addition,* (C, A) *is detectable then* $A - DX_+$ *is stable.*

$$\mathcal{R}(X) = XDX - XA - A^*X - C = 0 \qquad (9.1.1)$$

Then for this $\hat{S}^*$, Hamiltonian system (2.2-13) is decoupled and given by

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \widetilde{A} & -BR^{-1}B^T \\ 0 & \widetilde{A}^T \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix} \qquad 2.2\text{-}16$$

2.2.1.1 Fixed-Final-State optimal control

Objective here is to make $x(T)$ exactly match reference final state $r(T)$. The decoupled Hamiltonian system is given by (2.2-16), to solve it we first need to state boundary condition and check for validity of split boundary conditions. The boundary conditions are

$$x(T) = r(T)$$

$$x(t_0) = \text{Given}.$$

Since both conditions are fixed, split boundary equations are also satisfied.

State and costate equation from decoupled Hamiltonian system (2.2-16) are reproduced

$$\dot{x} = \tilde{A}x - BR^{-1}B^T\lambda \qquad 2.2\text{-}17$$

$$\dot{\lambda} = \tilde{A}^T\lambda \qquad 2.2\text{-}18$$

Solution to (2.2-18) is

$$\lambda(t) = e^{\tilde{A}^T(T-t)}\lambda(T) \qquad 2.2\text{-}19$$

where we still don't know $\lambda(T)$. Using this expression (2.2-19) in state equation (2.2-17) yields

$$\dot{x} = \tilde{A}x - BR^{-1}B^T e^{\tilde{A}^T(T-t)}\lambda(T) \qquad 2.2\text{-}20$$

We can obtain a solution to (2.2-20) considering second term as input as

$$x(t) = e^{\tilde{A}(t-t_0)}x(t_0) - \int_{t_0}^{t} e^{\tilde{A}(t-\tau)}BR^{-1}B^T e^{\tilde{A}^T(T-\tau)}\lambda(T)d\tau \qquad 2.2\text{-}21$$

Evaluate (2.2-21) at $t = T$, to get expression for $\lambda(T)$

$$x(T)=e^{\tilde{A}(T-t_0)}x(t_0)-\tilde{G}(t_0,T)\lambda(T) \qquad 2.2\text{-}22$$

where the weighted continuous reachability gramian of modified system is:

$$\tilde{G}(t_0,T)=\int_{t_0}^{T}e^{\tilde{A}(T-\tau)}BR^{-1}B^{T}e^{\tilde{A}^{T}(T-\tau)}\lambda(T)d\tau \qquad 2.2\text{-}23$$

Using final condition $x(T)=r(T)$ in (2.2-22) to get

$$\lambda(T)=-\tilde{G}^{-1}(t_0,T)[r(T)-e^{\tilde{A}(T-t_0)}x(t_0)] \qquad 2.2\text{-}24$$

Using (2.2-24) in (2.2-19) to get costate expression as

$$\lambda(t)=-e^{\tilde{A}^{T}(T-t)}\tilde{G}^{-1}(t_0,T)[r(T)-e^{\tilde{A}(T-t_0)}x(t_0)] \qquad 2.2\text{-}25$$

Finally using (2.2-25) in (2.2-12) to get optimal control as

$$u(t)=-R^{-1}\left(\hat{G}+\hat{H}\right)x+R^{-1}B^{T}e^{\tilde{A}^{T}(T-t)}\tilde{G}^{-1}(t_0,T)[r(T)-e^{\tilde{A}(T-t_0)}x(t_0)] \qquad 2.2\text{-}26$$

TABLE 2-4 Continuous-Time Linear Quadratic Regulator (Final State Fixed)

System model:

$$\dot{x}=Ax+Bu\,, \qquad t\geq t_0$$

Performance index:

$$J(t_0)=\tfrac{1}{2}x^{T}(T)S(T)x(T)+\tfrac{1}{2}\int_{t_0}^{T}\left(x^{T}Qx+u^{T}Ru\right)dt$$

Assumptions:

$S(T)\geq 0$, $Q\geq 0$, $R>0$, and all three are symmetric

Optimal feedback control:

$$u(t)=-R^{-1}\left(\hat{G}+\hat{H}\right)x+R^{-1}B^{T}e^{\tilde{A}^{T}(T-t)}\tilde{G}^{-1}(t_0,T)[r(T)-e^{\tilde{A}(T-t_0)}x(t_0)]$$

$$\tilde{G}(t_0,T)=\int_{t_0}^{T}e^{\tilde{A}(T-\tau)}BR^{-1}B^{T}e^{\tilde{A}^{T}(T-\tau)}\lambda(T)d\tau$$

$$\tilde{A}=A-BR^{-1}\left(\hat{G}^{T}+\hat{H}\right)$$

$$\tilde{Q}=\hat{Q}-\left(\hat{G}+\hat{H}^{T}\right)R^{-1}\left(\hat{G}^{T}+\hat{H}\right)$$

$$\hat{Q}=Q+\hat{S}A+A^{T}\hat{S}$$

$$\hat{G}=\tfrac{1}{2}\hat{S}B\,,\ \hat{H}=\tfrac{1}{2}B^{T}\hat{S}$$

$$\hat{S}BR^{-1}B^{T}\hat{S}-\hat{S}A-A^{T}\hat{S}-Q=0$$

2.2.1.2 Free-Final-State optimal control

The decoupled state and costate equations are reproduced here

$$\dot{x} = \tilde{A}x - BR^{-1}B^T\lambda \qquad 2.2\text{-}27$$

$$\dot{\lambda} = \tilde{A}^T\lambda \qquad 2.2\text{-}28$$

and the control input is

$$u = -R^{-1}\left(\hat{G} + \hat{H}\right)x - R^{-1}B^T\lambda \qquad 2.2\text{-}29$$

We are required to find the optimal control that minimizes the quadratic performance index (2.2-2), while having no constrain on final system state. In other words final state $x(T)$ is free. Thus $dx(T) \neq 0$, so from split boundary condition we require

$$\lambda(T) = \left.\frac{\partial\phi}{\partial x}\right|_T \qquad 2.2\text{-}30$$

The final state weighting function is $\phi = \frac{1}{2}x^T(T)\tilde{S}(T()x(T)$, so we get

$$\lambda(T) = \tilde{S}(T)x(T) \qquad 2.2\text{-}31$$

This is the required termination condition. Now we again use sweep method [3] and that the linear relation (2.2-31) holds for all $t \in [t_0, T]$, for some yet unknown matrix function $\tilde{S}(t)$.

$$\lambda(t) = \tilde{S}(t)x(t) \qquad 2.2\text{-}32$$

Now we have to find intermediate function $\tilde{S}(t)$, a valid expression shall mean our assumption was correct. Differentiate (2.2-32) with respect to time to yield

$$\dot{\lambda} = \dot{\tilde{S}}x + \tilde{S}\dot{x} \qquad 2.2\text{-}33$$

Use the state equation (2.2-27) and costate equation (2.2-28) in (2.2-33) to get

$$-\tilde{A}^T\lambda = \dot{\tilde{S}}x + \tilde{S}\left(\tilde{A}x - BR^{-1}B^T\lambda\right) \qquad 2.2\text{-}34$$

again use (2.2-32) in (2.2-34) to get

$$-\tilde{A}^T\tilde{S}x = \dot{\tilde{S}}x + \tilde{S}\left(\tilde{A}x - BR^{-1}B^T\tilde{S}x\right)$$

or

$$-\dot{\tilde{S}} = \tilde{A}^T\tilde{S} + \tilde{S}\tilde{A} - \tilde{S}BR^{-1}B^T\tilde{S} \qquad 2.2\text{-}35$$

This specifies $\tilde{S}$ entirely. But we can modify (2.2-35) to get a closed form solution for $\tilde{S}$. If $\left|\tilde{S}(T)\right| \neq 0$ we can rewrite (2.2-35) as

$$-\tilde{S}^{-1}\dot{\tilde{S}}\tilde{S}^{-1} = \tilde{S}^{-1}\tilde{A}^T + \tilde{A}\tilde{S}^{-1} - BR^{-1}B^T$$

or

$$\dot{\tilde{S}}^{-1} = \tilde{S}^{-1}\tilde{A}^T + \tilde{A}\tilde{S}^{-1} - BR^{-1}B^T \qquad 2.2\text{-}36$$

This is the Lyapunov equation in $\tilde{S}^{-1}$ and has a closed form solution given as

$$\tilde{S}^{-1}(t) = e^{\tilde{A}(t-T)}\tilde{S}^{-1}(T)e^{\tilde{A}^T(t-T)} + \int_T^t e^{\tilde{A}(t-\tau)}BR^{-1}B^Te^{\tilde{A}^T(t-\tau)}d\tau \qquad 2.2\text{-}37$$

We now are going to find optimal control sequence. Use (2.2-32) in (2.2-29) to get expression for $u(t)$ in term of state feedback as

$$u(t) = -[R^{-1}\left(\hat{G} + \hat{H}\right) + R^{-1}B^T\tilde{S}]x(t) \qquad 2.2\text{-}38$$

TABLE 2-5 Continuous-Time Linear Quadratic Regulator (Final State Free)

System model:

$$\dot{x} = Ax + Bu\,, \qquad t \ge t_0$$

Performance index:

$$J(t_0) = \tfrac{1}{2} x^T(T)S(T)x(T) + \tfrac{1}{2}\int_{t_0}^{T}\left(x^T Q x + u^T R u\right)dt$$

Assumptions:

$S(T) \ge 0$, $Q \ge 0$, $R > 0$, and all three are symmetric

Optimal feedback control:

$$u(t) = -[R^{-1}\left(\hat{G} + \hat{H}\right) + R^{-1}B^T\tilde{S}]x$$

$$\tilde{S}^{-1}(t) = e^{\tilde{A}(t-T)}\tilde{S}^{-1}(T)e^{\tilde{A}^T(t-T)} + \int_{T}^{t} e^{\tilde{A}(t-\tau)}BR^{-1}B^T e^{\tilde{A}^T(t-\tau)}d\tau\,, \quad \tilde{S}(T) = S(T) - \hat{S}$$

$$\tilde{A} = A - BR^{-1}\left(\hat{G}^T + \hat{H}\right)$$

$$\hat{G} = \tfrac{1}{2}\hat{S}B$$

$$\hat{H} = \tfrac{1}{2}B^T\hat{S}$$

$$\hat{S}BR^{-1}B^T\hat{S} - \hat{S}A - A^T\hat{S} - Q = 0$$

Example 2.2-1: Scalar system

Consider a scalar system described by $\dot{x} = Ax + Bu$ where $A = 1.05$ and $B = 0.01$, with initial condition $x(0) = 10$. It is desired to find optimal control that minimizes the cost function $J(t_0) = \tfrac{1}{2}x^T(T)S(T)x(T) + \tfrac{1}{2}\int_{t_0}^{T}\left(x^T Qx + u^T Ru\right)dt$ where $Q = 1$, $R = 1$, and $S(T) = 1; t_0 = 0, T = 10\,\text{sec}$.

Free final state.

Free final state control and state sequence are shown in Figure 2.2-1 and Figure 2.2-2 respectively. Corresponding results for the conventional method [1] are also included and are seen to be the same.

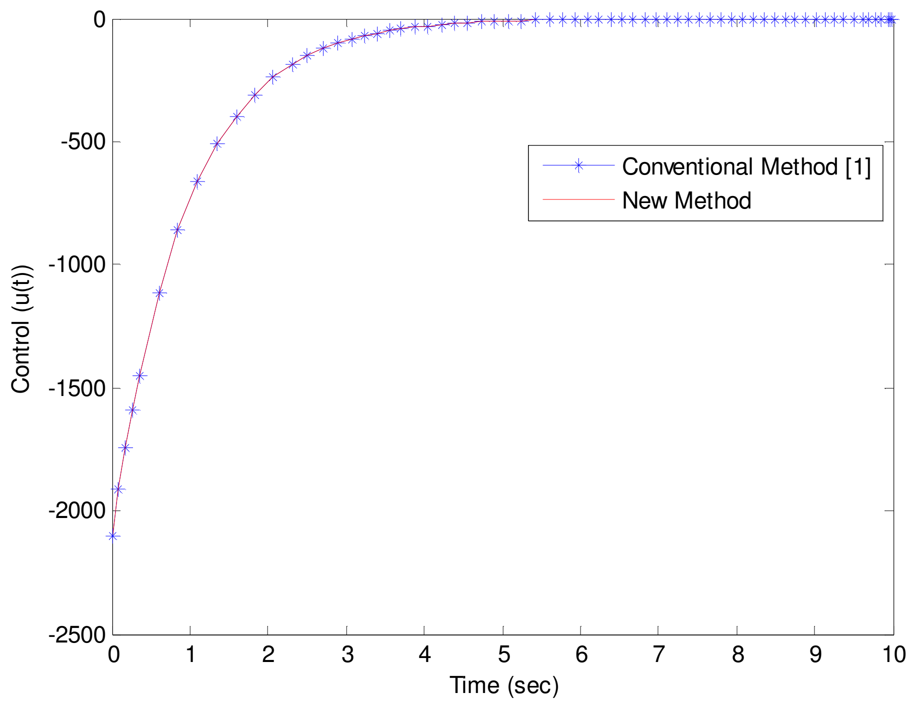


Figure 2-15: Free Final State CLQR Control Sequence (Scalar case)

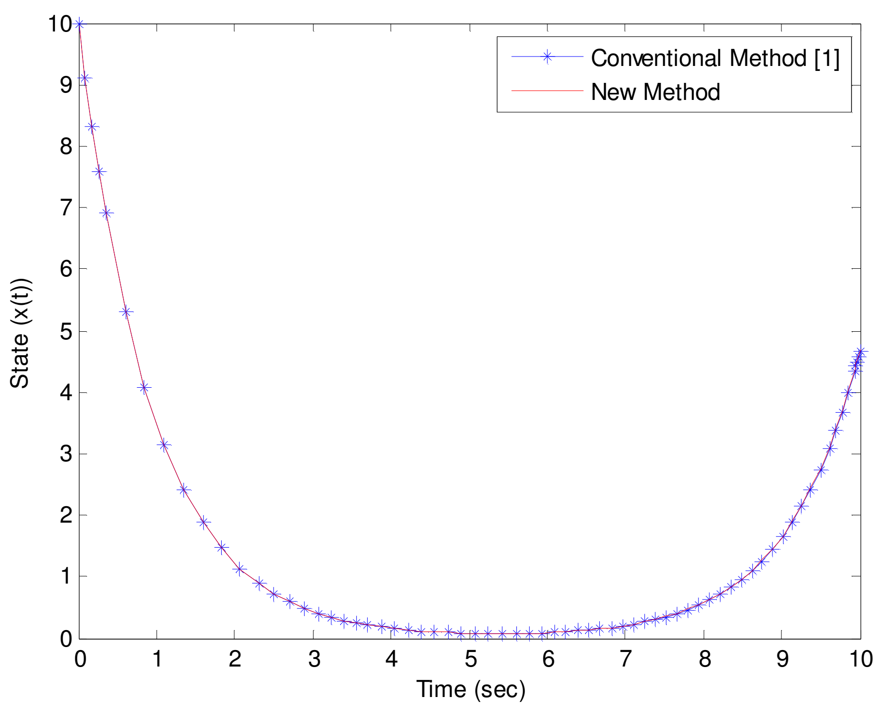


Figure 2-16: Free Final State CLQR State Sequence (Scalar case)

Fixed final state

Let $r(T) = 12$. Conventional method does not allow an analytical solution because $Q \neq 0$. Using the new method, control and state sequence are found and are shown in Figure 2.2-3 and Figure 2.2-4, respectively.

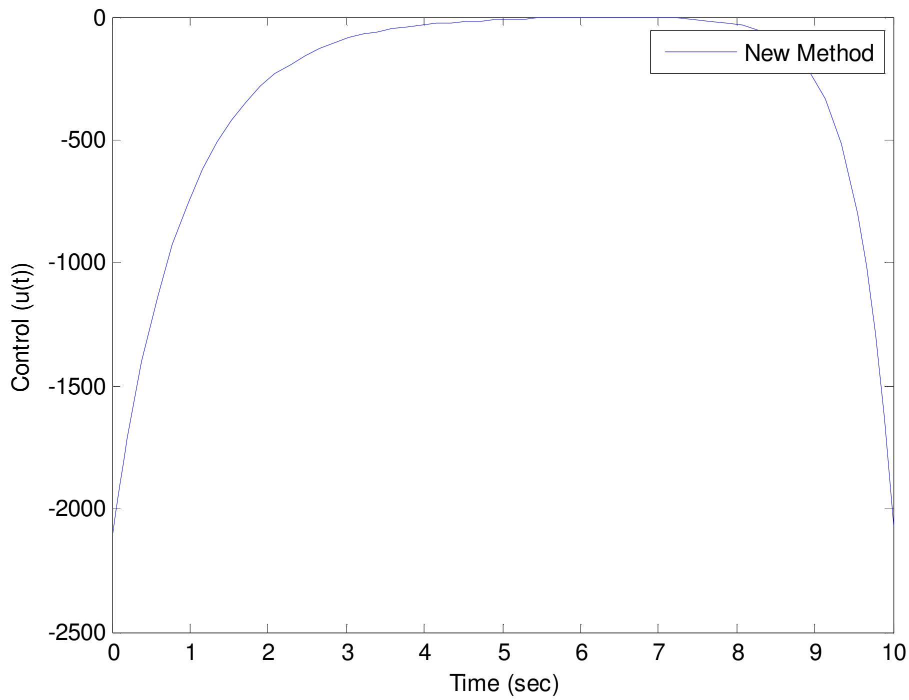


Figure 2-17: Fixed Final State CLQR Control Sequence (2^{nd} Order case)

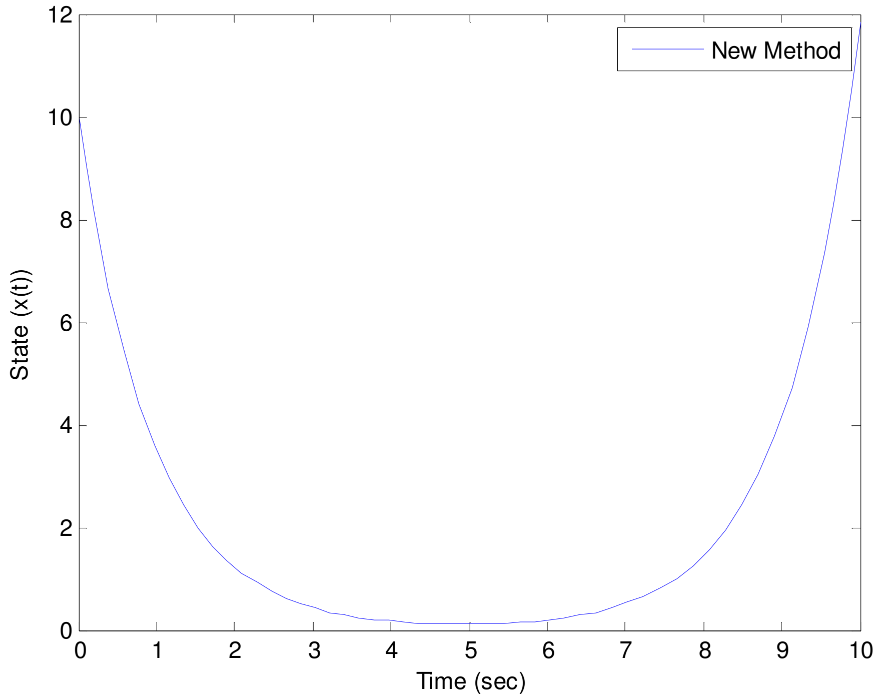


Figure 2-18: Fixed Final State CLQR State Sequence (2nd Order case)

Example 2.2.-2: Second order system

Consider a scalar system described by $\dot{x} = Ax + Bu$ where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, with initial condition $x(0) = \begin{bmatrix} 4 & -10 \end{bmatrix}^T$. It is desired to find optimal control that minimizes the cost function $J(t_0) = \frac{1}{2} x^T(T)S(T)x(T) + \frac{1}{2}\int_{t_0}^{T} \left(x^T Qx + u^T Ru\right) dt$ where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 1$, and $S(T) = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$; $t_0 = 0, T = 10\,\text{sec}$.

Free final state.

Free final state control and state sequence are shown in Figure 2.2-5 and Figure

2.2-6 respectively. Corresponding results for the conventional method [1] are also included and are seen to be the same.

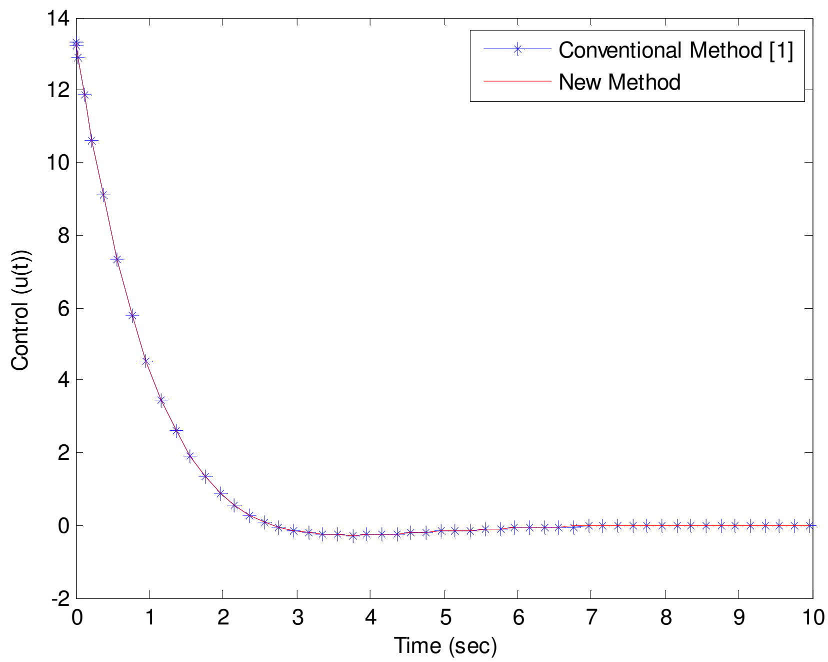


Figure 2-19: Free Final State CLQR Control Sequence (2^{nd} Order Case)

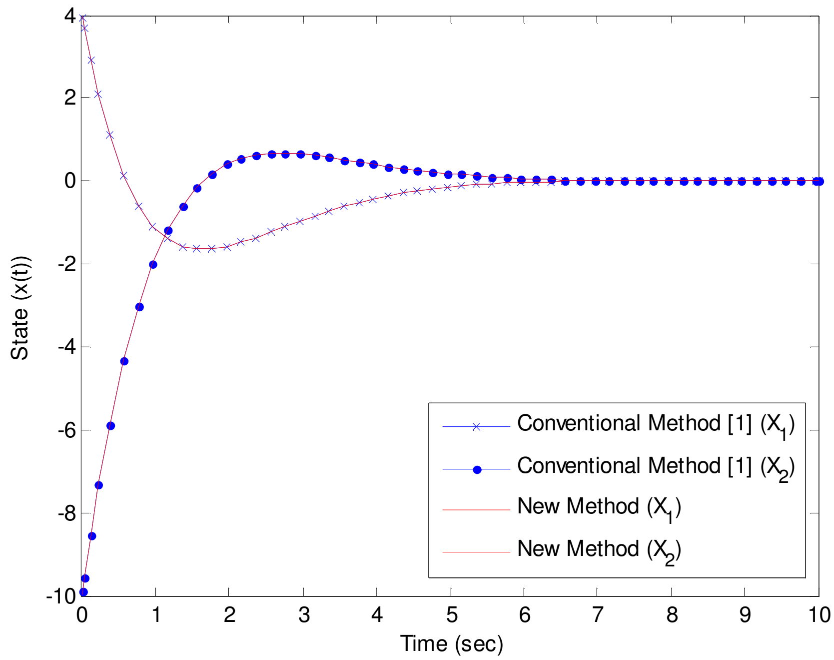


Figure 2-20: Free Final State CLQR State Sequence (2nd Order)

Fixed final state

Let $r(T) = \begin{bmatrix} -4 & -1 \end{bmatrix}^T$ .Conventional method does not allow an analytical solution because $Q \neq 0$. Using the new method, control and state sequence are found and are shown in Figure 2.2-7 and Figure 2.2-8, respectively.

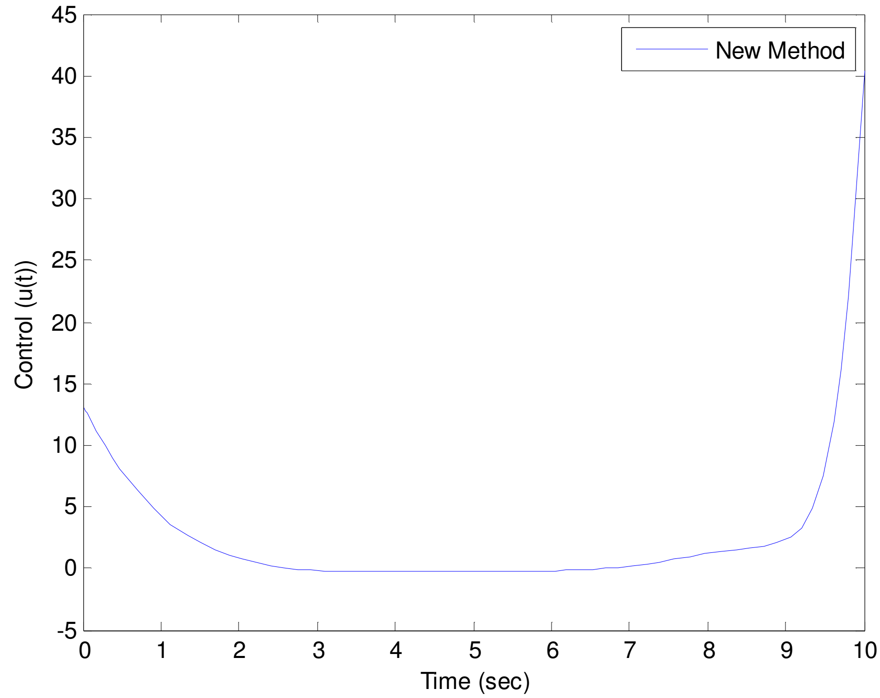


Figure 2-21 Fixed Final State CLQR Control Sequence (2^{nd} Order case)

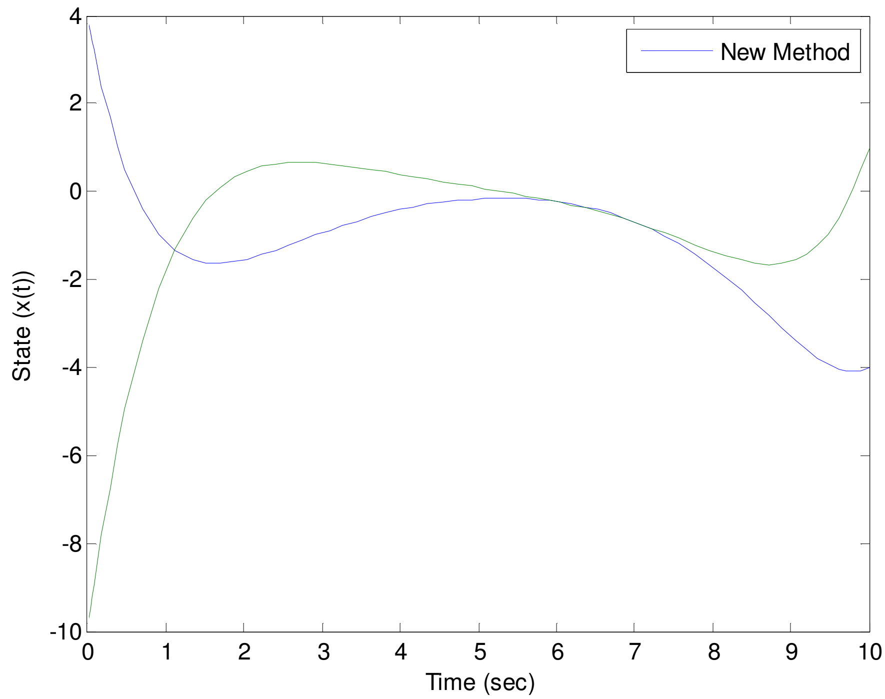


Figure 2-22: Fixed Final State CLQR State Sequence (2^{nd} Order System)

2.2.2 *Tracking Problem*

Consider a plant described by the linear equation $\dot{x} = Ax + Bu$ with $x \in R^n$ and $u \in R^m$. It is desired to get a control law that forces a certain linear combination of the states $y(t) = Cx(t)$ of the plant to track a desired reference trajectory $r(t)$ over a specified time interval $[i, N]$, then problem can be converted to one where we have to minimize the cost function

$$J_i = \tfrac{1}{2}\left(Cx(T) - r(T)\right)^T P\left(Cx(T) - r(T)\right) + \tfrac{1}{2}\int_{t_0}^{T}\left[\left(Cx - r\right)^T Q\left(Cx - r\right) + u^T Ru\right]dt$$

where P and Q are symmetric positive semi-definite matrices and R is symmetric positive definite matrix. Actual value of $x(T)$ is not constrained but we want it to be as close as $r(T)$.

Consider the original cost function

$$J_i = \tfrac{1}{2}\left(Cx(T) - r(T)\right)^T P\left(Cx(T) - r(T)\right) + \tfrac{1}{2}\int_{t_0}^{T}\left[\left(Cx - r\right)^T Q\left(Cx - r\right) + u^T Ru\right]dt \qquad 2.2\text{-}39$$

We can subtract and add a term $\tfrac{1}{2}\left(Cx(T) - r(T)\right)^T \hat{S}\left(Cx(T) - r(T)\right)$ to obtain:

$$\begin{aligned} J_i = {} & \tfrac{1}{2}\left(Cx(T) - r(T)\right)^T \left(P - \hat{S}\right)\left(Cx(T) - r(T)\right) + \tfrac{1}{2}\left(Cx(T) - r(T)\right)^T \hat{S}\left(Cx(T) - r(T)\right) \\ & + \tfrac{1}{2}\int_{t_0}^{T}\left[\left(Cx - r\right)^T Q\left(Cx - r\right) + u^T Ru\right]dt \end{aligned} \qquad 2.2\text{-}40$$

where $\hat{S}$ is a symmetric positive semi-definite.

Now observe

$$\tfrac{1}{2}\left(Cx(T)-r(T)\right)^T \hat{S}\left(Cx(T)-r(T)\right)=\tfrac{1}{2}\left(Cx(t_0)-r(t_0)\right)^T \hat{S}\left(Cx(t_0)-r(t_0)\right) + \tfrac{1}{2}\int_{t_0}^{T}\left[\left(C\dot{x}-\dot{r}\right)^T \hat{S}\left(Cx-r\right)+\left(Cx-r\right)^T \hat{S}\left(C\dot{x}-\dot{r}\right)\right]dt \qquad 2.2\text{-}41$$

Use (2.2-41) in (2.2-40) to get:

$$\begin{aligned} J_i = {} & \tfrac{1}{2}\left(Cx(T)-r(T)\right)^T \left(P-\hat{S}\right)\left(Cx(T)-r(T)\right)+\tfrac{1}{2}\left(Cx(t_0)-r(t_0)\right)^T \hat{S}\left(Cx(t_0)-r(t_0)\right) \\ & + \tfrac{1}{2}\int_{t_0}^{T}\left[\left(C\dot{x}-\dot{r}\right)^T \hat{S}\left(Cx-r\right)+\left(Cx-r\right)^T \hat{S}\left(C\dot{x}-\dot{r}\right)\right]dt \\ & + \tfrac{1}{2}\int_{t_0}^{T}\left[\left(Cx-r\right)^T Q\left(Cx-r\right)+u^T Ru\right]dt \end{aligned} \qquad 2.2\text{-}42$$

Use $\dot{x}=Ax+Bu$ in (2.2-42) to get after simplification:

$$\begin{aligned} J(t_0) = {} & \tfrac{1}{2}x^T(T)\hat{P}(T)x(T)+\tfrac{1}{2}x^T(t_0)\hat{S}x(t_0) \\ & +\tfrac{1}{2}\int_{t_0}^{T}\begin{pmatrix} x^T\hat{Q}x+2x^T\hat{G}u+2u^T\hat{H}x+u^TRu-2x^T\hat{E}r-2r^T\hat{F}x-2u^T\hat{L}r \\ -2r^T\hat{M}x+r^TQr-\dot{r}^T\hat{S}Cx-x^TC^T\hat{S}\dot{r}+\dot{r}^T\hat{S}r+r^T\hat{S}\dot{r} \end{pmatrix}dt \end{aligned} \qquad 2.2\text{-}43$$

where

$$\hat{P}(T)=P-\hat{S}$$

$$\hat{Q}=C^TQC+C^T\hat{S}AC+C^TA^T\hat{S}C$$

$$\hat{G}=\tfrac{1}{2}B^TC^T\hat{S}C$$

$$\hat{H}=\tfrac{1}{2}C^T\hat{S}CB$$

$$\hat{E}=\tfrac{1}{2}[C^TQ+A^TC^T\hat{S}] \qquad 2.2\text{-}44$$

$$\hat{F}=\tfrac{1}{2}[QC+\hat{S}CA]$$

$$\hat{L}=\tfrac{1}{2}B^TC^T\hat{S}$$

$$\hat{M}=\tfrac{1}{2}\hat{S}CB$$

Hamiltonian function for this modified cost function (2.2-43)

$$H(t) = \tfrac{1}{2}\begin{pmatrix} x^T\hat{Q}x + 2x^T\hat{G}u + 2u^T\hat{H}x + u^T Ru - 2x^T\hat{E}r - 2r^T\hat{F}x - 2u^T\hat{L}r \\ -2r^T\hat{M}x + r^T Qr - \dot{r}^T\hat{S}Cx - x^T C^T\hat{S}\dot{r} + \dot{r}^T\hat{S}r + r^T\hat{S}\dot{r} \end{pmatrix} + \lambda^T\left(Ax + Bu\right)$$

2.2-45

Then the state and costate equation are:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = Ax + Bu \qquad 2.2\text{-}46$$

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \hat{Q}x + \left(\hat{G} + \hat{H}\right)u - (\hat{E} + \hat{F}^T)r + A^T\lambda - 2C^T\hat{S}\dot{r} \qquad 2.2\text{-}47$$

and the stationary condition is

$$0 = \frac{\partial H}{\partial u} = Ru + \left(\hat{G} + \hat{H}\right)x + B^T\lambda - (\hat{L} + \hat{M}^T)r \qquad 2.2\text{-}48$$

Rearrange (2.2-48) to yield control $u(t)$ in term of costate $\lambda(t)$

$$u(t) = -R^{-1}\left(\hat{G} + \hat{H}\right)x - R^{-1}B^T\lambda + R^{-1}(\hat{L} + \hat{M}^T)r \qquad 2.2\text{-}49$$

Using (2.2-49) to eliminate $u(t)$ from (2.2-46) and (2.2-47) we get the following Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \tilde{A} & -BR^{-1}B^T \\ -\tilde{Q} & -\tilde{A}^T \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 & -BR^{-1}(\hat{L} + \hat{M}^T) \\ 2C^T\hat{S} & \tilde{Z} \end{bmatrix}\begin{bmatrix} \dot{r} \\ r \end{bmatrix} \qquad 2.2\text{-}50$$

where

$$\tilde{A} = A - BR^{-1}\left(\hat{G}^T + \hat{H}\right),$$

$$\tilde{Q} = \hat{Q} - \left(\hat{G} + \hat{H}^T\right)R^{-1}\left(\hat{G}^T + \hat{H}\right)$$

$$\tilde{Z} = (\hat{E} + \hat{F}^T) - (\hat{G}^T + \hat{H})R^{-1}(\hat{L} + \hat{M}^T)$$

To decouple the Hamiltonian system (2.2-50) we set $\tilde{Q} = 0$

$$\tilde{Q} = \hat{Q} - \left(\hat{G} + \hat{H}^T\right)R^{-1}\left(\hat{G}^T + \hat{H}\right) = 0$$

Use (2.2-44) to simplify as:

$$C^T QC + C^T \hat{S}AC + C^T A^T \hat{S}C - C^T \hat{S}CBR^{-1}B^T C^T \hat{S}C = 0 \qquad 2.2\text{-}51$$

Equation (2.2-51) is an continuous algebraic riccati equation. There exits a symmetric positive semi-definite solution $\hat{S}^*$ for (2.2-51) as long as (A, B) is a stabalizable pair, $R > 0$ and $Q \geq 0$. Proof of this can be obtained from Theorem 9.1.2 in [2].

> ***Theorem 9.1.2*** : *If* $D \geq 0$, $C \geq 0$ *and the pair* (A, D) *is stabalizable then there exist hermitian solutions of* $\mathcal{R}(X) = 0$. *Moreover, the maximal hermitian solution* X_+ *also satisfies* $X_+ \geq 0$. *If, in addition,* (C, A) *is detectable then* $A - DX_+$ *is stable.*

$$\mathcal{R}(X) = XDX - XA - A^* X - C = 0 \qquad (9.1.1)$$

Then for this $\hat{S}^*$, Hamiltonian system (2.2-50) is decoupled and given by

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \tilde{A} & -BR^{-1}B^T \\ 0 & -\tilde{A}^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 & -BR^{-1}(\hat{L} + \hat{M}^T) \\ 2C^T \hat{S} & \tilde{Z} \end{bmatrix} \begin{bmatrix} \dot{r} \\ r \end{bmatrix} \qquad 2.2\text{-}51$$

We have to still find the optimal control for the problem. For this we require boundary conditions, first one is the given initial system state $x(t_0)$ while second condition is:

$$\lambda(T) = C^T \hat{P}(T)\big(Cx(T) - r(T)\big) \qquad 2.2\text{-}52$$

We again shall make assumption that linear relation (2.2-53) holds for all time $t_0 \le t \le T$, thus we can say

$$\lambda(t) = \tilde{S}(t)x(t) - v(t) \qquad 2.2\text{-}53$$

where $\tilde{S}(T) = C^T \hat{P}(T)C$

$v(T) = C^T \hat{P}(T)r(T)$

Differentiating (2.2-53) with respect to time to get:

$$\dot{\lambda} = \dot{\tilde{S}}x + \tilde{S}\dot{x} - \dot{v} \qquad 2.2\text{-}54$$

Use the Hamiltonian System (2.2-51) and (2.2-53) again in (2.2-54) to get:

$$-\tilde{A}^T\tilde{S}x + \tilde{A}^T v + 2C^T\hat{S}\dot{r} + \tilde{Z}r = \dot{\tilde{S}}x + \tilde{S}\tilde{A}x - \tilde{S}BR^{-1}B^T\tilde{S}x + \tilde{S}BR^{-1}B^T v - \tilde{S}BR^{-1}(\hat{L} + \hat{M}^T)r - \dot{v} \qquad 2.2\text{-}55$$

$$-\dot{\tilde{S}}x - \tilde{A}^T\tilde{S}x - \tilde{S}\tilde{A}x + \tilde{S}BR^{-1}B^T\tilde{S}x + \dot{v} + \tilde{A}^T v - \tilde{S}BR^{-1}B^T v + 2C^T\hat{S}\dot{r} + \tilde{Z}r + \tilde{S}BR^{-1}(\hat{L} + \hat{M}^T)r = 0 \qquad 2.2\text{-}56$$

For (2.2-56) to hold for all $x(t)$ under any given $x(t_0)$ condition we shall have:

$$-\dot{\tilde{S}} - \tilde{A}^T\tilde{S} - \tilde{S}\tilde{A} + \tilde{S}BR^{-1}B^T\tilde{S} = 0 \qquad 2.2\text{-}57$$

$$\dot{v} + \tilde{A}^T v - \tilde{S}BR^{-1}B^T v + 2C^T\hat{S}\dot{r} + \tilde{Z}r + \tilde{S}BR^{-1}(\hat{L} + \hat{M}^T)r = 0 \qquad 2.2\text{-}58$$

or on further simplification:

$$-\dot{\tilde{S}} = +\tilde{A}^T\tilde{S} + \tilde{S}\tilde{A} - \tilde{S}BR^{-1}B^T\tilde{S} \qquad 2.2\text{-}59$$

$$-\dot{v} = \left(\tilde{A}^T - \tilde{S}BR^{-1}B^T\right)v + 2C^T\hat{S}\dot{r} + \left[\tilde{Z} + \tilde{S}BR^{-1}(\hat{L} + \hat{M}^T)\right]r \qquad 2.2\text{-}60$$

Equation (2.2-59) and (2.2-60) are the required equation, boundary condition for them are:

$$\tilde{S}(T) = C^T \hat{P}(T) C$$

$$v(T) = C^T \hat{P}(T) r(T)$$

But we can modify (2.2-59) to get a closed form solution for $\tilde{S}$. If $\left|\tilde{S}(T)\right| \neq 0$ we can rewrite (2.2-59) as

$$-\tilde{S}^{-1}\dot{\tilde{S}}\tilde{S}^{-1} = \tilde{S}^{-1}\tilde{A}^T + \tilde{A}\tilde{S}^{-1} - BR^{-1}B^T$$

or $$\dot{\tilde{S}}^{-1} = \tilde{S}^{-1}\tilde{A}^T + \tilde{A}\tilde{S}^{-1} - BR^{-1}B^T \qquad 2.2\text{-}61$$

This is the Lyapunov equation in $\tilde{S}^{-1}$ and has a closed form solution given as

$$\tilde{S}^{-1}(t) = e^{\tilde{A}(t-T)}\tilde{S}^{-1}(T)e^{\tilde{A}^T(t-T)} + \int_T^t e^{\tilde{A}(t-\tau)}BR^{-1}B^T e^{\tilde{A}^T(t-\tau)}d\tau \qquad 2.2\text{-}62$$

Optimal control for this tracker is obtained by substituting (2.2-53) in (2.2-49)

$$u(t) = -R^{-1}\left[\left(\hat{G} + \hat{H}\right) + B^T\tilde{S}(t)\right]x + R^{-1}v + R^{-1}(\hat{L} + \hat{M}^T)r \qquad 2.2\text{-}63$$

TABLE 2-6 Continuous-Time Linear Quadratic Tracker

System model:

$$\dot{x} = Ax + Bu\,, \qquad t \ge t_0$$

Performance index:

$$J_i = \tfrac{1}{2}\big(Cx(T) - r(T)\big)^T P\big(Cx(T) - r(T)\big) + \tfrac{1}{2}\int_{t_0}^{T}\Big[\big(Cx - r\big)^T Q\big(Cx - r\big) + u^T Ru\Big]dt$$

Assumptions:

$S(T) \ge 0$, $\quad Q \ge 0$, $R > 0$, $\quad$ and all three are symmetric

Optimal feedback control:

$$u(t) = -R^{-1}\Big[\big(\hat{G} + \hat{H}\big) + B^T \tilde{S}(t)\Big]x + R^{-1}v + R^{-1}(\hat{L} + \hat{M}^T)r$$

$$\tilde{S}^{-1}(t) = e^{\tilde{A}(t-T)}\tilde{S}^{-1}(T)e^{\tilde{A}^T(t-T)} + \int_{T}^{t} e^{\tilde{A}(t-\tau)}BR^{-1}B^T e^{\tilde{A}^T(t-\tau)}d\tau\,, \quad \tilde{S}(T) = C^T\hat{P}(T)C$$

$$-\dot{v} = \big(\tilde{A}^T - \tilde{S}BR^{-1}B^T\big)v + 2C^T\hat{S}\dot{r} + \Big[\tilde{Z} + \tilde{S}BR^{-1}(\hat{L} + \hat{M}^T)\Big]r$$

$$\tilde{A} = A - BR^{-1}\big(\hat{G}^T + \hat{H}\big)$$

$$\hat{G} = \tfrac{1}{2}B^T C^T \hat{S}C\,,\; \hat{H} = \tfrac{1}{2}C^T\hat{S}CB$$

$$\hat{E} = \tfrac{1}{2}[C^T Q + A^T C^T \hat{S}]\,,\; \hat{F} = \tfrac{1}{2}[QC + \hat{S}CA]\,,\; \hat{L} = \tfrac{1}{2}B^T C^T \hat{S}\,,\; \hat{M} = \tfrac{1}{2}\hat{S}CB$$

$$C^T QC + C^T \hat{S}AC + C^T A^T \hat{S}C - C^T \hat{S}CBR^{-1}B^T C^T \hat{S}C = 0$$

CHAPTER 3

CONCLUSION AND FUTURE WORK

3.1 Conclusion

In this work through chapter 2 we have explored a way to decouple the Hamiltonian system formed during the solution to linear quadratic problem in control theory. Resulting decoupled system proves to be helpful in solving linear quadratic problem. As opposed to conventional theory [1], [3], [8], [9], [10] we are able to reduce the problem of solving first order nonlinear differential equation and get a closed form solutions to those equations. Control vector is still combination of solution to those equations. For some type of linear quadratic problem we derive a closed loop control as opposed to open loop control given by conventional theory [1], [3], [8], [9], [10]. Work done here is for Time-invariant plant and performance index.

3.2 Future Work

This thesis deals with decoupling of the Hamiltonian system and its application in linear quadratic problem. Work done represents time- invariant plant system and performance index. Here are few suggestions for future work

1) Formulate method for time variant plant system and performance index.
2) Sub-optimal control and infinite horizon control.

APPENDIX A

MATLAB® CODE FOR THE EXAMPLES

Example 2.1-1:

```
close all
clear all
clc
A=1.05; B=0.01; Q=1; R=1; Sn=5; N=101;
x0=10;
S=zeros(1,N);
Ss=zeros(1,N);
K=zeros(1,N-1);
Ks=zeros(1,N-1);
u=zeros(1,N-1);
us=zeros(1,N-1);
x=zeros(1,N);
xs=zeros(1,N);
x(1)=x0;
xs(1)=x0;
S(N)=Sn;
Sc=dare(A,B,Q,R);
Ss(N)=Sn-Sc;
Qc=Q-Sc+(A')*Sc*A;
Gc=(1/2)*((A')*Sc*B);
Hc=(1/2)*((B')*Sc*A);
Rc=R+((B')*Sc*B);
As=A-B*(Rc^-1)*((Gc')+Hc);
Qs=Qc-(Gc+(Hc'))*(Rc^-1)*((Gc')+Hc);
for i=N-1:-1:1
    S(i)=(A^2)*(S(i+1)-(S(i+1)^2)*(B^2)*(((B^2)*S(i+1)+R)^-1))+Q;
    K(i)=(((B^2)*S(i+1)+R)^-1)*(B)*S(i+1)*A;
end
t=zeros(1,1);
for m=1:N-1
    t=t+((As)^-(m))*B*(Rc^-1)*(B')*((As')^-(m));
end
Stemp=((As)^-(N-1))*((Ss(N))^-1)*((As')^-(N-1))+t;
Ss(1)=Stemp^-1;
for i=1:N-1
    u(i)=-K(i)*x(i);
    x(i+1)=A*x(i)+B*u(i);
    t=zeros(1,1);
    Stemp=ones(1,1);
    for m=1:N-i-1
        t=t+((As)^-(m))*B*(Rc^-1)*(B')*((As')^-(m));
    end
    Stemp=((As)^-(N-i-1))*((Ss(N))^-1)*((As')^-(N-i-1))+t;
    Ss(i+1)=Stemp^-1;
    Ks(i)=((Rc+B'*Ss(i+1)*B)^-1)*((B'*Ss(i+1)*A)+(Gc'+Hc));
    us(i)=-Ks(i)*xs(i);
    xs(i+1)=A*xs(i)+B*us(i);
end
J=zeros(1,N);
Js=zeros(1,N);
i=0;
```

```
for i=N:-1:1
    temp2=zeros(1,i-1);
    temp3=zeros(1,N-i);
    temp4=zeros(1,N-i);
    m=0;
    for m=1:i-1
        temp2(m)=(xs(:,m+1)')*Sc*xs(:,m+1)-(xs(:,m)')*Sc*xs(:,m);
    end
    if i==N
        J(i)=(1/2)*(x(i)')*S(i)*x(i);

Js(i)=(1/2)*(xs(i)')*Ss(i)*xs(i)+(1/2)*(xs(1)')*Sc*xs(1)+(1/2)*sum(tem
p2);
        tt=(1/2)*sum(temp2);
    else
        k=0;
        for k=i:N-1
            temp3(k)=(x(k)')*Q*x(k)+(u(k)')*R*u(k);

temp4(k)=((xs(k)')*Qc*xs(k)+(xs(k)')*2*Gc*us(k)+(us(k)')*2*Hc*xs(k)+(u
s(k)')*Rc*us(k));
        end
        J(i)=J(N)+(1/2)*sum(temp3);
        Js(i)=Js(N)+(1/2)*sum(temp2)+(1/2)*sum(temp4)-tt;
    end
end
u_plot=plot(1:N-1,u,'or',1:N-1,us,'.b');
legend(u_plot,'Conventional Method [1]','New Method')
ylabel('Control (u)')
xlabel('Discrete Time Step (n)')
figure
x_plot=plot(1:N,x(1:N),'or',1:N,xs(1:N),'.b');
legend(x_plot,'Conventional Method [1]','New Method')
ylabel('State (X_n)')
xlabel('Discrete Time Step (n)')
figure
J_plot=plot(1:N,J,'or',1:N,Js,'.b');
legend(J_plot,'Original Theory','New Method')
ylabel('Magnitude (J)')
xlabel('Discrete Time Step (n)')
```

<u>Example 2.1-2</u>:

```
close all
clear all
clc
A=[1 0.01;0.01 1];
B=[0;0.1];
Sn=[5 0;0 5];
Q=[2 0;0 2];
R=1;
N=101;
```

```
x0=[-5;-10];
N2=2*N;
S=zeros(2,N2);
Ss=zeros(2,N2);
K=zeros(N-1,2);
Ks=zeros(N-1,2);
u=zeros(1,N-1);
us=zeros(1,N-1);
x=zeros(2,N);
xs=zeros(2,N);
x(:,1)=x0;
xs(:,1)=x0;
Sc=dare(A,B,Q,R);
Qc=Q-Sc+(A')*Sc*A;
Gc=(1/2)*((A')*Sc*B);
Hc=(1/2)*((B')*Sc*A);
Rc=R+(B')*Sc*B;
S(:,N2-1:N2)=Sn;
Ss(:,N2-1:N2)=Sn-Sc;
As=A-B*(Rc^-1)*((Gc')+Hc);
Qs=Qc-(Gc+(Hc'))*(Rc^-1)*((Gc')+Hc);

for i=N-1:-1:1
    S(:,2*(i-1)+1:2*i)=(A')*(S(:,2*i+1:2*(i+1))-
S(:,2*i+1:2*(i+1))*(B)*(((B')*S(:,2*i+1:2*(i+1))*B+R)^-
1)*(B')*S(:,2*i+1:2*(i+1)))*A+Q;
    K(i,:)=(((B')*S(:,2*i+1:2*(i+1))*B+R)^-
1)*(B')*S(:,2*i+1:2*(i+1))*A;
end

t=zeros(2,2);
for m=1:N-1
    t=t+((As)^-(m))*B*(Rc^-1)*(B')*((As')^-(m));
end
Stemp=((As)^-(N-1))*((Ss(:,N2-1:N2))^-1)*((As')^-(N-1))+t;
Ss(:,1:2)=Stemp^-1;

for i=1:N-1
    u(i)=-K(i,:)*x(:,i);
    x(:,i+1)=A*x(:,i)+B*u(i);

    t=zeros(2,2);
    Stemp=ones(2,2);
    for m=1:N-i-1
        t=t+((As)^-(m))*B*(Rc^-1)*(B')*((As')^-(m));
    end
    Stemp=((As)^-(N-i-1))*((Ss(:,N2-1:N2))^-1)*((As')^-(N-i-1))+t;
    Ss(:,2*i+1:2*(i+1))=Stemp^-1;
    Ks(i,:)=((Rc+B'*Ss(:,2*i+1:2*(i+1))*B)^-
1)*((B'*Ss(:,2*i+1:2*(i+1))*A)+(Gc'+Hc));

    us(i)=-Ks(i,:)*xs(:,i);
    xs(:,i+1)=A*xs(:,i)+B*us(i);
end
```

```
J=zeros(1,N);
Js=zeros(1,N);
for i=N:-1:1
    temp2=zeros(1,i-1);
    temp3=zeros(1,N-i);
    temp4=zeros(1,N-i);
    m=0;
    for m=1:i-1
        temp2(m)=(xs(:,m+1)')*Sc*xs(:,m+1)-(xs(:,m)')*Sc*xs(:,m);
    end
    if i==N
        J(i)=(1/2)*(x(:,i)')*S(:,2*(i-1)+1:2*i)*x(:,i);
        Js(i)=(1/2)*(xs(:,i)')*Ss(:,2*(i-
1)+1:2*i)*xs(:,i)+(1/2)*(xs(:,1)')*Sc*xs(:,1)+(1/2)*sum(temp2);
        tt=(1/2)*sum(temp2);
    else
        k=0;
        for k=i:N-1
            temp3(k)=(x(:,k)')*Q*x(:,k)+(u(k)')*R*u(k);

temp4(k)=((xs(:,k)')*Qc*xs(:,k)+(xs(:,k)')*2*Gc*us(k)+(us(k)')*2*Hc*xs
(:,k)+(us(k)')*Rc*us(k));
        end
        J(i)=J(N)+(1/2)*sum(temp3);
        Js(i)=Js(N)+(1/2)*sum(temp2)+(1/2)*sum(temp4)-tt;
    end
end
plot(1:N-1,u,'or',1:N-1,us,'.b')
legend('Conventional Method [1]','New Method')
ylabel('Control (u)')
xlabel('Discrete Time Step (n)')
figure
xplot=plot(1:N,x(:,1:N),'or',1:N,xs(:,1:N),'.b');
legend(xplot(1:2:3),'Conventional Method [1]','New Method')
ylabel('State (X_n)')
xlabel('Discrete Time Step (n)')
figure
plot(1:N,J,'or',1:N,Js,'.b')
legend('Original Theory','New Method')
ylabel('Magnitude (J)')
xlabel('Discrete Time Step (n)')
```

<u>Example 2.1-3</u>:

```
close all
clear all
clc
A=1.05; B=0.01; Q=1; R=1; Sn=5; N=100;
x0=10;
rn=12;
u=zeros(1,N-1);
x=zeros(1,N);
x(1)=x0;
Sc=dare(A,B,Q,R);
```

```
Qc=Q-Sc+(A')*Sc*A;
Gc=(1/2)*((A')*Sc*B);
Hc=(1/2)*((B')*Sc*A);
Rc=R+((B')*Sc*B);
As=A-B*(Rc^-1)*((Gc')+Hc);
Qs=Qc-(Gc+(Hc'))*(Rc^-1)*((Gc')+Hc);
Ss(N)=Sn-Sc;
Gs=0;
U=B;
for i=1:N-1
    U=cat(2,U,(As^i)*B);
end
Gs=U*((Rc^-1)*eye(N))*(U');
for i=1:N-1
    u(i)=-(Rc^-1)*((Gc'+Hc)*x(i)-(B')*((As')^(N-i-1))*(Gs^-1)*(rn-
(As^(N-1))*x(1)));
    x(i+1)=A*x(i)+B*u(i);
end
J=zeros(1,N);
i=0;
for i=N:-1:1
    temp2=zeros(1,i-1);
    temp4=zeros(1,N-i);
    m=0;
    for m=1:i-1
        temp2(m)=(x(:,m+1)')*Sc*x(:,m+1)-(x(:,m)')*Sc*x(:,m);
    end
    if i==N

Js(i)=(1/2)*(x(i)')*Ss(i)*x(i)+(1/2)*(x(1)')*Sc*x(1)+(1/2)*sum(temp2);
        tt=(1/2)*sum(temp2);
    else
        k=0;
        for k=i:N-1

temp4(k)=((x(k)')*Qc*x(k)+(x(k)')*2*Gc*u(k)+(u(k)')*2*Hc*x(k)+(u(k)')*
Rc*u(k));
        end
        Js(i)=Js(N)+(1/2)*sum(temp2)+(1/2)*sum(temp4)-tt;
    end
end
plot(1:N-1,u)
legend('New Method')
ylabel('Control (u)')
xlabel('Discrete Time Step (n)')
figure
plot(1:N,x)
legend('New Method')
ylabel('State (X_n)')
xlabel('Discrete Time Step (n)')
figure
plot(1:N,Js)
legend('New Method')
ylabel('Cost (J_i)')
```

```
xlabel('Discrete Time Step (n)')
```

Example 2.1-4:

```
close all;clear all;clc
A=[1 0.01;0.01 1.05];
B=[0;0.1];
Sn=[1 0;0 2];
Q=[2 0;0 2];
R=1;
N=100;
x0=[-5;-10];
rn=[4;-1];
N2=2*N;
xs=zeros(2,N);
us=zeros(1,N-1);
xs(:,1)=x0;
Sc=dare(A,B,Q,R);
Qc=Q-Sc+A'*Sc*A;
Gc=(1/2)*((A')*Sc*B);
Hc=(1/2)*((B')*Sc*A);
Rc=R+B'*Sc*B;
As=A-B*(Rc^-1)*(Gc'+Hc);
Qs=Qc-(Gc+Hc')*(Rc^-1)*(Gc'+Hc);
Ss(:,N2-1:N2)=Sn-Sc;
S(:,N2-1:N2)=Sn;
Gs=0;
Us=B;
for i=1:N-1
    Us=cat(2,Us,(As^i)*B);
end
Gs=Us*((Rc^-1)*eye(N))*(Us');
for i=1:N-1
    us(i)=-(Rc^-1)*((Gc'+Hc)*xs(:,i)-(B')*((As')^(N-i-1))*(Gs^-1)*(rn-
(As^(N-1))*xs(:,1)));
    xs(:,i+1)=A*xs(:,i)+B*us(i);
end
Js=zeros(1,N);
for i=N:-1:1
    temp2=zeros(1,i-1);
    temp4=zeros(1,N-i);
    m=0;
    for m=1:i-1
        temp2(m)=(xs(:,m+1)')*Sc*xs(:,m+1)-(xs(:,m)')*Sc*xs(:,m);
    end
    if i==N
        Js(i)=(1/2)*(xs(:,i)')*Ss(:,2*(i-
1)+1:2*i)*xs(:,i)+(1/2)*(xs(:,1)')*Sc*xs(:,1)+(1/2)*sum(temp2);
        tt=(1/2)*sum(temp2);
    else
        k=0;
        for k=i:N-1
```

```
temp4(k)=((xs(:,k)')*Qc*xs(:,k)+(xs(:,k)')*2*Gc*us(k)+(us(k)')*2*Hc*xs
(:,k)+(us(k)')*Rc*us(k));
        end
        Js(i)=Js(N)+(1/2)*sum(temp2)+(1/2)*sum(temp4)-tt;
    end
end
plot(1:N-1,us)
legend('New Method')
ylabel('Control (u)')
xlabel('Discrete Time Step (n)')
figure
xplot=plot(1:N,xs);
legend(xplot(1),'New Method')
ylabel('State (X_n)')
xlabel('Discrete Time Step (n)')
figure
plot(1:N,Js);
legend(xplot(1),'New Method')
ylabel('Cost (J_i)')
xlabel('Discrete Time Step (n)')
```

<u>Example 2.1-5</u>:

```
close all
clear all
clc
A=1.05;
I=eye(size(A));
B=0.01;
C=1;
P=1;
Q=100000;
R=1;
N=100;
v=zeros(1,N);
S=zeros(1,N);
x=zeros(1,N);
u=zeros(1,N-1);
xs=zeros(1,N);
us=zeros(1,N-1);
x(1)=0;
xs(1)=0;
r=5*sin((1:N)/pi);
v(N)=C'*P*r(N);
S(N)=C'*P*C;
Sc=dare(A,B,(C')*Q*C,R);
Sc=((C')^-1)*Sc*(C^-1);
Ss=zeros(1,N);
vs=zeros(1,N);
Ss(N)=(C')*(P-Sc)*C;
vs(N)=(C')*(P-Sc)*r(N);
```

```
Qc=(C')*(Q-Sc)*C+(A')*(C')*Sc*C*A;
Gc=(1/2)*(A')*(C')*Sc*C*B;
Hc=(1/2)*(B')*(C')*Sc*C*A;
Ec=-(1/2)*(C')*(Q-Sc);
Fc=-(1/2)*(Q-Sc)*C;
Oc=-(1/2)*(B')*(C')*Sc;
Tc=-(1/2)*Sc*C*B;
Mc=-(1/2)*Sc*C*A;
Lc=-(1/2)*(A')*(C')*Sc;
Rc=R+(B')*(C')*Sc*C*B;
Wc=Q-Sc;
As=A-B*(Rc^-1)*((Gc')+Hc);
Qs=(C')*(Q-Sc)*C+(A')*(C')*Sc*C*A-(A')*(C')*Sc*C*B*(Rc^-
1)*(B')*(C')*Sc*C*A
Zs=(Lc+(Mc'))-(Gc+(Hc'))*(Rc^-1)*(Oc+(Tc'));
i=0;
Kx=zeros(1,N);
Kv=zeros(1,N);
Ks=zeros(1,N);
for i=1:N-1
    Stemp=0;
    sumtemp=0;
    for m=1:(N-i)
        sumtemp=sumtemp+(As^-m)*(B*(Rc^-1)*(B'))*((As')^-m);
    end
    Stemp=(As^-(N-i))*Ss(N)*((As')^-(N-i))+sumtemp;
    Ss(i)=Stemp^-1;
end
for i=N-1:-1:1
    v(i)=[(A')-(A')*S(i+1)*B*(((B')*S(i+1)*B+R)^-
1)*(B')]*v(i+1)+(C')*Q*r(i);
    S(i)=(A')*S(i+1)*[(I+B*(R^-1)*(B')*S(i+1))^-1]*A+(C')*Q*C;
    Kx(i)=[((B')*S(i+1)*B+R)^-1]*(B')*S(i+1)*A;
    Kv(i)=[((B')*S(i+1)*B+R)^-1]*(B');
%     Ss(i)=(As')*Ss(i+1)*[(I+B*(Rc^-1)*(B')*Ss(i+1))^-1]*As;
    vs(i)=[(As')-(As')*Ss(i+1)*((I+B*(Rc^-1)*(B')*Ss(i+1))^-1)*B*(Rc^-
1)*(B')]*vs(i+1)-(Ec+(Fc'))*r(i)+[(As')*Ss(i+1)*((I+B*(Rc^-
1)*(B')*Ss(i+1))^-1)*B*(Rc^-1)*(Oc+(Tc'))-Zs]*r(i+1);
    Ks(i)=(Rc+(B')*Ss(i+1)*B)^-1;
end
i=0;
for i=1:N-1
    u(i)=-Kx(i)*x(i)+Kv(i)*v(i+1);
    x(i+1)=A*x(i)+B*u(i);
    us(i)=-Ks(i)*[((B')*Ss(i+1)*A+(Gc')+Hc)*xs(i)+((Tc')+Oc)*r(i+1)-
(B')*vs(i+1)];
    xs(i+1)=A*xs(i)+B*us(i);
end
u_plot=plot(1:N-1,u,'or',1:N-1,us,'.b');
legend(u_plot,'Conventional Method [1]','New Method')
ylabel('Control (u_n)')
xlabel('Discrete Time Step (n)')
figure
x_plot=plot(1:N,x(1:N),'or',1:N,xs(1:N),'.b',1:N,r,'-k');
```

```
legend(x_plot,'Conventional Method [1]','New Method','Reference
Trajectory')
ylabel('State (X_n)')
xlabel('Discrete Time Step (n)')
```

Example 2.2-1:

Free Final State

```
close all
clear all
clc
A=1.05; B=0.01; Q=1; R=1; Sn=1;
tf=10;
x0=10;
Sc=care(A,B,Q,R);
Qc=Q+Sc*A+A'*Sc;
Gc=(1/2)*Sc*B;
Hc=(1/2)*B'*Sc;
As=A-B*(R^-1)*(Gc'+Hc);
Qs=Qc-(Gc+Hc')*(R^-1)*(Gc'+Hc);
Ssn=Sn-Sc;
x(1)=x0;
xs(1)=x0;
sd=@(t,s)(-A'*s-s*A+s*B*(R^-1)*(B')*s-Q);
[tb,S]=ode45(sd,[10 0],Sn);
t=flipud(tb)';
S=flipud(S)';
for i=1:length(t)-1
    K(i)=(R^-1)*(B')*S(i+1);
    u(i)=-K(i)*x(i);
    x(i+1)=expm((A-B*K(i))*(t(i+1)-t(i)))*x(i);
    fun=@(y)(exp(As.*(t(i+1)-y)).*B.*(R^-1).*(B').*exp((As').*(t(i+1)-
y)));
    Stemp=exp(As.*(t(i+1)-tf)).*(Ssn^-1).*exp((As').*(t(i+1)-tf))-
quad(fun,tf,t(i+1));
    Ss(i+1)=Stemp^-1;
    Ks(i)=(R^-1)*((Gc')+Hc)+(R^-1)*((B')*Ss(i+1));
    us(i)=-Ks(i)*xs(i);
    xs(i+1)=expm((A-B*Ks(i))*(t(i+1)-t(i)))*xs(i);
end
plot(t(1:end-1),u,'-b',t(1:end-1),us,'-r');
legend('Conventional Method [1]','New Method')
ylabel('Control (u(t))')
xlabel(' Time (sec)')
figure
plot(t,x,'-b',t,xs,'-r')
legend('Conventional Method [1]','New Method')
ylabel('State (x(t))')
xlabel(' Time (sec)')
```

Fixed Final state case

```
close all
clear all
clc
A=1.05; B=0.01; Q=1; R=1; Sn=1;
tf=10;
x0=10;
r=12;
Sc=care(A,B,Q,R);
Qc=Q+Sc*A+A'*Sc;
Gc=(1/2)*Sc*B;
Hc=(1/2)*B'*Sc;
As=A-B*(R^-1)*(Gc'+Hc);
Qs=Qc-(Gc+Hc')*(R^-1)*(Gc'+Hc);
Ssn=Sn-Sc;
xs(1)=x0;
Gsf=@(t)(exp(As.*(tf-t)).*B.*(R^-1).*(B').*exp((As').*(tf-t)));
Gs=quadl(Gsf,0,tf);
xsd=@(t,xs1)(A.*xs1-B.*(R^-1).*(Gc'+Hc)*xs1+B.*(R^-
1).*(B').*exp((As').*(tf-t)).*(Gs^-1).*(r-exp(As.*(tf)).*xs(1)));
[tb xs]=ode45(xsd,[0 10],xs(1));
for i=1:length(tb)
    us(i)=-(R^-1)*(Gc'+Hc)*xs(i)+(R^-1)*(B')*exp((As')*(tf-
tb(i)))*(Gs^-1)*(r-exp(As*(tf-tb(i)))*xs(1));
end
plot(tb,us);
legend('New Method')
ylabel('Control (u(t))')
xlabel(' Time (sec)')
figure
plot(tb,xs)
legend('New Method')
ylabel('State (x(t))')
xlabel(' Time (sec)')
```

Example 2.2-2:

Free Final state case

```
close all
clear all
clc
A=[0 1;0 0];
B=[0;1];
Sn=[5 0;0 5];
Q=[1 0;0 1];
R=1;
x0=[4;-10];
r=[-4;1];
tf=10;
xs(:,1)=x0;
Sc=care(A,B,Q,R);
```

```
Qc=Q+Sc*A+A'*Sc;
Gc=(1/2)*Sc*B;
Hc=(1/2)*B'*Sc;
As=A-B*(R^-1)*(Gc'+Hc);
Qs=Qc-(Gc+Hc')*(R^-1)*(Gc'+Hc);
Ssn=Sn-Sc;
sim('graman',[0 tf]);
Gs=g(:,:,end);
xsd=@(t,xs1)(A*xs1-B*(R^-1)*(Gc'+Hc)*xs1+B*(R^-1)*(B')*expm((As')*(tf-
t))*(Gs^-1)*(r-expm(As*(tf))*xs(:,1)));
[tb xs]=ode45(xsd,[0 10],xs(:,1));
xs=xs';
for i=1:length(tb)
    us(i)=-(R^-1)*(Gc'+Hc)*xs(:,i)+(R^-1)*(B')*expm((As').*(tf-
tb(i)))*(Gs^-1)*(r-expm(As.*(tf-tb(i)))*xs(:,1));
end
plot(tb,us);
legend('New Method')
ylabel('Control (u(t))')
xlabel(' Time (sec)')
figure
plot(tb,xs)
legend('New Method')
ylabel('State (x(t))')
xlabel(' Time (sec)')
```

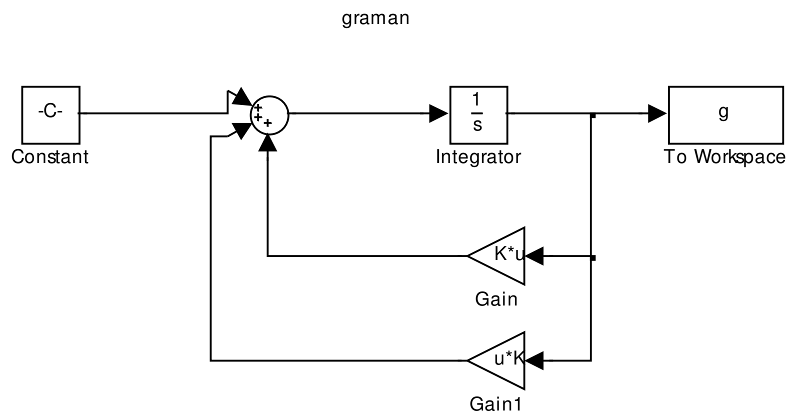
Fixed final state case

```
close all
clear all
clc
A=[0 1;0 0];
B=[0;1];
Sn=[5 0;0 5];
Q=[1 0;0 1];
R=1;
x0=[4;-10];
tf=10;
xs(:,1)=x0;
x(:,1)=x0;
Sc=care(A,B,Q,R);
Qc=Q+Sc*A+A'*Sc;
Gc=(1/2)*Sc*B;
Hc=(1/2)*B'*Sc;
As=A-B*(R^-1)*(Gc'+Hc);
Qs=Qc-(Gc+Hc')*(R^-1)*(Gc'+Hc);
Ssn=Sn-Sc;
t=sim('conth',[0 tf]);
sim('newth',t);
n=length(t);
for i=1:n-1
    K(i,:)=(R^-1)*(B')*S(:,:,n-i);
```

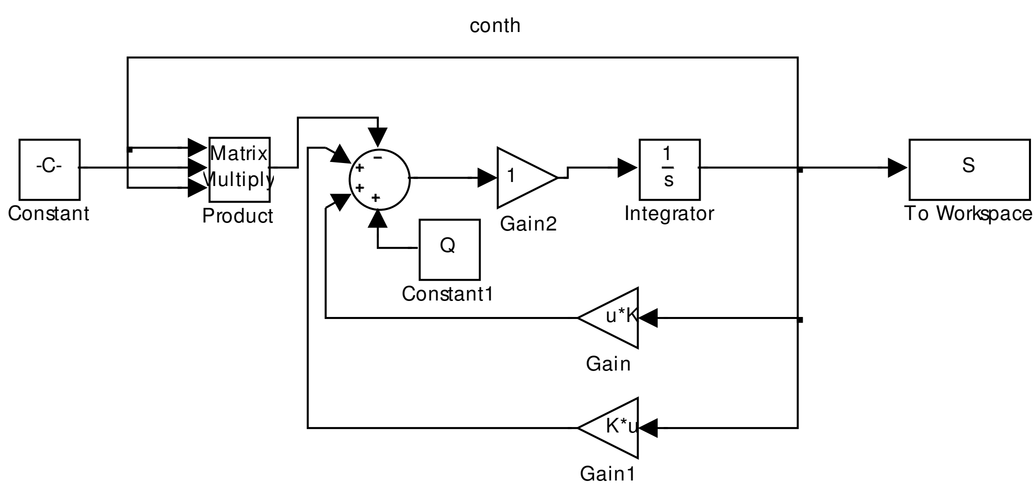
```
    u(i)=-K(i,:)*x(:,i);
    x(:,i+1)=expm((A-B*K(i,:))*(t(i+1)-t(i)))*x(:,i);
    Ss(:,:,n-i+1)=Ss(:,:,n-i)^-1;
    Ks(i,:)=(R^-1)*((Gc')+Hc)+(R^-1)*((B')*Ss(:,:,n-i));
    us(i)=-Ks(i,:)*xs(:,i);
    xs(:,i+1)=expm((A-B*Ks(i,:))*(t(i+1)-t(i)))*xs(:,i);
end
plot(t(1:end-1),u,'-b',t(1:end-1),us,'-r');
legend('Conventional Method [1]','New Method')
ylabel('Control (u(t))')
xlabel(' Time (sec)')
figure
plot(t,x,'-b',t,xs,'-r')
legend('Conventional Method [1]','New Method')
ylabel('State (x(t))')
xlabel(' Time (sec)')
```

APPENDIX B

SIMULINK® MODEL USED IN CODES

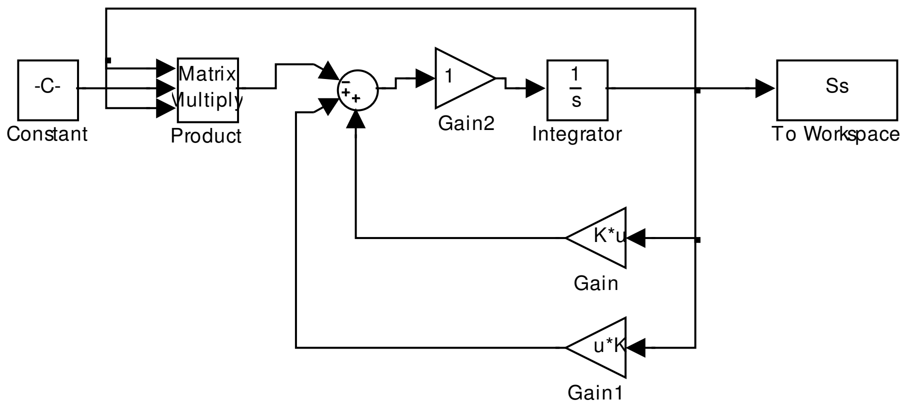


Model “graman”



Model “conth”

newth



Model “newth”

REFERENCES

[1] Frank L. Lewis and Vassilis L. Syrmos, *Optimal Control*, Second Edition , John Wiley & Sons, New York, 1995.

[2] Peter Lancaster and Leiba Rodman, *Algebraic Riccati Equations*, Clarendon Press, Oxford, 1995.

[3] A. E. Bryson and Y. C. Ho*, Applied Optimal Control*, New York, Hemisphere, 1975

[4] Roger W. Brockett (1965), " Poles, Zeros, and Feedback: State Space Interpretation", *IEEE Transaction on Automatic Control*, AC-10, 129-135.

[5] R. E. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, New Jersey, 1957.

[6] R. E. Bellman and S. E. Dreyfus, *Applied Dynamics Programming*, Princeton University Press, Princeton, New Jersey, 1962.

[7] R. E. Bellman and R. E. Kalaba, *Dynamics Programming and Modern Control Therapy*, Academic Press, Orlando, Florida, 1965.

[8] L. S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze and E. F. Mishchenko, *The Mathematical Theory of Optimal Process*, Wiley-Interscience, New York, 1962.

[9] Donald E. Kirk, *Optimal Control Theory – An Introduction*, Pretince-Hall Inc., Englewood Cliffs, New Jersy, 1970.

[10] Magnus R. Hestenes, *Calculus of Varation and Optimal Control Theory*, John Wiley & Sons, Inc. New York, 1966.

BIOGRAPHICAL INFOARMATION

Vishal Gupta graduated from Punjab Technical University, India with a Bachelor of Technology degree in Instrumentation and Control Engineering in August 2004. Later he started his own company dealing in electrical switchgear and industrial lightning fixtures. He has been active member of technical societies, IEEE and ISA since his undergraduate years. He joined University of Texas at Arlington with interest in control system design and analysis. He earned his Master of Science degree in Electrical Engineering in December 2007.