STRATEGIC EXERCISE OF OPTIONS ON NON-TRADED ASSETS AND STOCHASTIC VOLATILITY IN AN INCOMPLETE MARKET: INDIFFERENCE PRICING AND ENTROPY METHODS

by

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ABSTRACT

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The first study explores optimal investment policies for strategic option exercise when the underlying project is not traded. A duopoly model captures strategic interactions, while a partial spanning asset models market incompleteness. The option value to invest is obtained through indifference pricing, i.e., certainty equivalent value. I find that incompleteness narrows the gap between leader and follower entry dates. The follower enters much sooner, and the leader delays slightly compared to classic real options models. Modeling investment income stream as an Arithmetic Brownian motion is a better fit than Geometric Brownian motion, while reducing the necessary numerical approximations for obtaining the results in the incomplete market situation. As a

byproduct of modeling two different stochastic income streams, I investigate the impact of market share and uncertainty on the relative investment trigger as well as the option value to invest. Results are sensitive to these factors; thus, it is important to model stochastic processes to accurately reflect the real world circumstances.

The second study explores the valuation consequences of incompleteness resulting from stochastic volatility in a real options setting. The optimal policy is obtained through q-optimal measures as well as indifference pricing. I examine the efficacy of different approaches to finding and justifying a particular martingale measure. Stochastic volatility induced market incompleteness affects the investment/abandonment decision in several important ways. In addition, I demonstrate that indifference prices for the option value to invest and the abandonment option solve quasilinear variational inequalities with obstacle terms. With the exponential utility function, the utility-based indifference price admits a new pricing measure, which is the minimal relative entropy martingale measure minimizing the relative entropy between the historical measure and the Q martingale measure. I also show that the indifference price is non-increasing with respect to risk aversion. As the risk aversion parameter converges to zero, the indifference price converges to the unique bounded viscosity solution of the linear variational inequality with obstacle term.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

One of the primary functions of corporate finance is to properly identify firms' optimal investment policies. Beginning with Myers' (1977) path breaking realization that growth opportunities can be viewed as "real options", numerous developments have appeared in the literature (see, for example, Brennan and Schwartz (1983, 1984, 1985), McDonald and Siegel (1986), Paddock et al (1988)). Dixit and Pindyck's (1994) publication of the first text devoted to real options analysis signaled the growing maturity of the field. The real options theory of corporate investment has developed to the point that it is now in the mainstream of corporate finance.

As important as real options has become to corporate finance, it still rests upon simplifying assumptions that may not hold in the real world. For example, the classical real options model involves decision makers playing against nature rather than against competitors, i.e., "an irreversible investment decision under monopoly". The options game model[1], integrating option pricing theory with game theory, has been developed to address this limitation of the standard real options model. Strategic exercise of real options is now an important topic of current research. Recent work suggests that the fear of pre-emption leads to a significant erosion of the option value to delay investment, with optimal policies approaching that of static net present value analysis.

Apart from the ongoing development of refinements to real options theory and application, deeper unresolved issues still exist that may impact the efficacy of the approach. Perhaps foremost among these issues rests with the question, "What if managers are unable to create the project's replicating portfolio?" In this case project risks cannot be spanned by a portfolio of existing assets. Market incompleteness may limit the utility of contingent claim analysis. The presence of an imperfect hedge still exposes the investor to idiosyncratic risk, thus weakening seriously the risk neutrality assumption lying at the heart of option pricing theory[2]. This incomplete market problem arising from pricing claims written on non-traded assets is to date an unresolved issue. There is considerable disagreement over the practical importance of the non-traded asset issue, with top researchers appearing in both camps. Two general methods are employed to price claims written on non-traded assets. One method, known as mean-variance hedging, originated from work by Follmer and Sonderman (1986). The other method is based on utility maximization. Important theoretical

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[1] See, for example, Dixit and Pindyck (1994, Ch9), Grenadier (1996, 1999, 2002), Smets (1995), Lambrech and Perraudin (1998), Huisman (2001), Smit and Trigeorgis (2004)…etc.

[2] McDonald and Siegel (1986) caveat their model by stating, "Risk aversion by investors is here introduced by supposing that options to invest are owned by well-diversified investors, who need only be compensated for the systematic component of the risk of projects and options to invest….Assuming that investors are well diversified describes publicly owned corporations in the United States and simplifies the computation of the option value." Implicit in their statement and model is the notion that capital markets are "complete", i.e., that any pattern of risky cash flows may be spanned by existing securities. The statement that "Capital markets are sufficiently complete and well diversified investors need only be compensated for the systematic risk…" leads to Henderson's (2005) assertion about McDonald and Siegel's model assumption of risk neutrality to idiosyncratic risk.

groundwork pertaining to the underlying issues includes Follmer and Schweizer (1990), Henderson (2002), Henderson and Hobson (2002), Karatzs and Shreve (2000), Musiela and Zariphopoulou (2003) and Zariphopoulou (2004).

Stochastic volatility is an important issue in contingent claims analysis, and it is especially important to handle volatility correctly in real options, given the long maturities involved. Moreover, stochastic volatility may induce incompleteness because stochastic volatility cannot be traded. Selecting the correct pricing measure in these situations is equivalent to specifying the market price of volatility risk. Research such as Biagini et al. (2000), Heath, Platen and Schweizer (2001), Henderson (2005), Hobson (2004), Laurrent and Pham (1999), and Pham et al. (1998) contain approaches to selecting a single equivalent martingale measure with which to price options. Alternatively, the indifference pricing technique first proposed by Hodges and Neuberger (1998) has been applied to stochastic volatility models in Sircar and Zariphopoulou (2005). Frittelli (2000) analyzes the connection of the pricing rules of agents with exponential utility to the arbitrage-free valuation under minimum entropy martingale measure.

Pinches (1998) states, “one avenue for significant future research is that of valuation of options in incomplete markets”. Extending real options theory to include incompleteness has recently caught more and more attention worldwide (see, for example, Henderson (2005), Hugonnier and Morellec (2004, 2005), Kadam et al (2004), Miao and Wang (2005)) and will continue to be a burgeoning arena to be explored since no clear unifying theory has yet been developed.

1.2 Purpose and Scope of the Study

Assets underlying real options are typically not traded. This characteristic makes integrating incompleteness and risk aversion into the classical real options model vital to the efficacy of the approach. Recent research has shown that optimal investment policies obtained through the classical real options framework are substantially altered if the market is incomplete. Further, if investors are averse to idiosyncratic risks[3], the effects are even greater. This dissertation attempts to extend the existing research by exploring the impact of market incompleteness and managerial risk aversion on the investment timing decision and option value to invest in strategic exercise setting (i.e., duopoly). As an outgrowth of this work, it will be important to explore the effect of stochastic volatility on such models.

Most real options have a maturity of several years, requiring the employment of stochastic volatility. Stochastic volatility in turn induces incompleteness independent of the non-tradability of the underlying asset. This makes it important to examine the order of the option value to invest under q-optimal pricing measures, along with their connection to indifference pricing.

The first issue addressed is the impact of market incompleteness and managerial risk aversion on optimal investment policies if strategic interactions from other market players are integrated into the real options model. Recent work (Henderson (2005), Hugonnier and Morellec (2004, 2005), Kadam et al (2004), Miao and Wang (2005)) investigates optimal investment policies in incomplete markets. The consistent

[3] See footnote 2.

conclusion is that with the lump-sum investment payoffs, incompleteness and managerial risk aversion lower the option value to invest as well as the investment threshold compared with those posited in classical complete market models. Just as consistently, when flow (income stream) investment payoffs are assumed, the abovementioned observation is reversed.[4] Motivated by the extant literature, this dissertation attempts to extend existing real options literature to include strategic interactions. I aim to examine how market incompleteness, risk aversion and strategic competition interact. An interesting question is whether lump-sum investment payoffs and flow payoffs still yield reversed results when strategic interaction is considered simultaneously.

The second issue examines the option value to invest under the class of q-optimal measures with stochastic volatility. I also extend the study to include indifference pricing to determine whether a connection exists between these two different pricing techniques in real options setting.

1.3 Benefits of the Study

Managers may make investment decisions under conditions that resemble an incomplete market due to nontradeability, stochastic volatility, or other factors. Henderson (2005) posits, "There is little evidence that perfect spanning asset exists." Therefore, optimal investment policies obtained from an augmented model integrating

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[4] It is interesting to observe different optimal investment rules due to lump-sum investment payoffs versus flow investment payoffs because they are the same in the standard real options model with complete market setting (see Dixit and Pindyck (1994) Ch5 and Ch6.)

incompleteness and aversion to idiosyncratic risk should result in better capital investment decisions.

In addition, the incompleteness introduced by the stochastic volatility which cannot be traded is an important issue for financial and real options alike. The selection of different arbitrage-free pricing measures, i.e., market price of volatility risk, will yield different investment timing decisions. The links to the alternative pricing technique, indifference pricing, should also deserve study.

This research provides two major benefits. First, it extends the valuation of real options in incomplete markets, which has been done so far by isolating strategic interactions from the model, to include a game-theoretic setting. From the results of this dissertation, both academics and practitioners will know better how the option to invest and the investment timing decision may be distorted if the degree of spanning assets obtainable is wrongly assumed. Second, it provides option value to invest under q-optimal measures and under indifferent pricing technique when the standard real options setting with constant volatility does not hold, rather the volatility is stochastic and cannot be traded.

CHAPTER 2

LITERATURE REVIEW

Literature pertaining to this dissertation is organized along three lines. I begin with research on real options in complete markets. I then summarize research concerning options on non-traded assets and option pricing in incomplete markets. Finally, I review relevant literature on stochastic volatility insofar as it relates to market incompleteness.

2.1 Real Options in Complete Markets

Myers (1977) was apparently the first to write that many corporate assets, particularly growth opportunities, may be viewed as call options. He coined the term, ”real options” to describe these assets. Brennan and Schwartz (1985) applied option pricing techniques to the valuation and optimal operation of a copper mine. The owner of an *operating* copper mine retains a put option to suspend operations should copper prices fall below a threshold value. Similarly, the owner of a *suspended* operation retains a call option to reopen the mine should copper prices rise above a higher threshold. Fixed suspension and resumption costs drive a wedge between the respective thresholds. This in turn leads to a path-dependent optimal policy and a “hysteresis” effect. The threshold prices represent free boundary conditions from the solution to a partial differential equation, so the solutions are sufficiently complex to require numerical techniques.

A considerable volume of real options research appeared during the mid to late 1980s. McDonald and Siegel (1986) showed that if a capital investment project is partially or totally irreversible and if there is flexibility in timing, the value of the option to delay investment may exceed the value of the project in place. The familiar static net present value criterion for capital investment should be replaced in many situations by the criterion that net present value should exceed a project's real option value before assets are put in place. Paddock, Siegel, and Smith (1988) applied option pricing techniques to offshore oil leases, comparing their results to estimates provided by the U.S. Geological Survey. They demonstrate the efficacy of the option approach to lease valuation, and they offer evidence to suggest that their approach is superior to the discounted cash flow approach used by the USGS.

The earlier real options models did not consider optimal exercise policies with the possibility of strategic interactions with competitors. Capital investment problems quite often are more than a simple game against nature. Competitors may have a significant impact on the optimal time to place assets. Game theory was combined with real options to address managerial problems more realistically. Researchers discovered by augmenting the real options framework with strategic considerations, some predictions of the standard real option models are mitigated. The optimal investment rule, as described in the classical real options literature, is to invest when the asset value exceeds the investment cost by a potentially large option premium. However, if firms fear pre-emption, then the option to delay investment is reduced, and project value approaches traditional net present value. Trigeorgis (1991) studied the impact of competition on the optimal timing of project initiation using option methodology, while Ankum and Smit (1993) considered managerial strategies as a sequence of tactical investment projects. Grenadier (1996) used a game-theoretical approach to option exercise in the real estate market to explain cascades and “overbuilding” in real estate markets; he extended the analysis in 1999 and 2002 to consider equilibrium strategies with option exercise games. Smets (1995) provided a treatment of duopoly in a multinational setting. Lambrech and Perraudin (1998) studied strategic behavior under incomplete information. Grenadier (2000) edited a good selection of option games papers. The first textbook covering real option games in continuous time setting appeared in 2001 by Huisman, while Smit and Trigeorgis edited an option games textbook in 2004 mainly focusing on discrete-time models with many practical examples.

Agency problems and information asymmetries have been crucial issues in corporate finance, especially in modern decentralized firm. The standard real options paradigm assumes the options’ owner makes the exercise decision, but this is usually not true in modern decentralized corporations, where options’ owners delegate the investment decision to managers. Such delegation process possibly induces the agency and information asymmetries problems. In view of this, Grenadier and Wang (2005) augment standard real options model with the presence of agency conflicts and information asymmetries, in which an underlying option to invest can be decomposed into a manager's option and an owner's option. Their model predicts that the manager will have a more valuable option to wait than the owner will, and the optimal contracts depend explicitly on two factors, hidden information and hidden action[5]. Grenadier et al. (2004) study the similar issues augmented by recursive optimal contracting and the consideration of manager's risk aversion. They find that the net effect of risk aversion is to delay investment.

2.2 Real Options, Non-Traded Assets, and Incomplete Markets

Lurking in the background of earlier real options research was a concern over the importance of actually being able to construct the delta hedge for a real option. Brennan and Schwartz (1985) employed a self-financing replicating portfolio approach to value a copper mine (see previous section). They cite the advantages of this approach over conventional discounted cash flow techniques, with one important caveat. They state, "When a replicating self-financing portfolio can be constructed, our approach offers several advantages over the market equilibrium approach; not only does it obviate the need for a discount rate derived from an inadequately supported model of market equilibrium but, most important in the current context, it eliminates the need for estimates of the expected rate of change of the underlying cash flow and therefore of the output price." Particularly significant is the phrase, "When a replicating self-financing portfolio can be constructed…." Brennan and Schwartz have chosen an application (i.e., copper mine) that allows them to actually construct a replicating portfolio. Assuming that the convenience yield for copper may be written as a function of output

[5] They are measured by cost/benefit ratio. See Grenadier and Wang (2005).

price alone and that the interest rate is non-stochastic, the presence of a futures market for copper allows Brennan and Schwartz to construct a replicating portfolio.

McDonald and Siegel (1986) caveat their model similarly, stating, "Risk aversion by investors is here introduced by supposing that options to invest are owned by well-diversified investors, who need only be compensated for the systematic component of the risk of projects and options to invest….Assuming that investors are well diversified describes publicly owned corporations in the United States and simplifies the computation of the option value." Implicit in their work the notion that either capital markets are complete, or that investors are risk neutral with respect to idiosyncratic risk.

Paddock et al (1987) are also concerned with the replication / market completeness issue. They state, "Most importantly, we show the necessity of combining option pricing techniques with a model of equilibrium in the market for the underlying asset (petroleum reserves)." Implicit in the above statement is the notion that risk-neutral delta hedging may not be feasible. As with previous real options studies, Paddock et al benefited from the fact that markets existed for both developed and undeveloped petroleum reserves.

Dixit and Pindyck (1994) develop the classic real options models in a parallel mode. That is, they describe a particular model by utilizing contingent claims style delta hedging. They subsequently develop the same model using an optimal stopping approach. It appears that concern over market incompleteness was the reason for the parallel approach. In their Chapter 4, Section 3, they write in objective terms about the pros and cons of optimal stopping versus contingent claims, but then they editorialize. They state that the contingent claims approach offers a better treatment of the discount rate. They then go on to state that the contingent claims approach requires the existence of a sufficiently rich set of markets in risky assets.

Valuing claims on non-traded assets represents a challenge to option pricing theory that has recently attracted much academic attention. A natural way to approach the problem is to choose another traded similar asset or index for use in the delta hedge. If it is not possible to hedge all risk, the calculation of the option price as the expected discounted payoff under a risk-neutral measure may not apply. Several approaches have been suggested to solve this problem. Mean variance hedging (see Follmer and Sonderman (1982)) is one suggestion, and there are two ways to implement the approach. The analyst either minimizes sequential future risk exposure by relaxing the self-financing strategy (also known as mean-self-financing strategy) or he/she minimizes the tracking error at the terminal date and assumes there is a self-financing strategy (see Duffie and Richardson (1991)).

Another well-known approach is based on utility maximization, which can be viewed as a descendant of the seminal Merton (1969) contribution. Hodges and Neuberger (1988) were the first to adapt the static certainty equivalence concept to the expected utility maximization through dynamic hedging/trading, which is then known as indifference pricing and there have been numerous papers study non-traded assets and incomplete market utilizing this technique, see for example Duffie and Zariphopolou (1993), Duffie et al. (1997), Henderson (2001, 2004, 2005), Henderson and Hobson (2002), Zariphopolou (2001, 2003, 2004), Zariphopolou and Sircar (2005) …etc. and also the monograph by Karatzas and Shereve (2000).

Since there are infinitely many admissible pricing measures in the presence of market frictions due to non-tradability, there is a substantial body of literature on how to select the “best” equivalent martingale measure with which to price options (see for example, Biagini et al. (2000), Heath, Platen and Schweizer (2001), Henderson (2005), Hobson (2004), Laurrent and Pham (1999), and Pham et al. (1998)). Other potential approaches include super-replication (see for example Hubalek and Schachermayer (1997)), and convex risk measures (see for example Follmer and Schield (2004)).

Real options represent a particularly important application of the valuation of claims in incomplete markets. Incompleteness is introduced by the fact that the option is usually written on a non-traded asset, and it may be that no ”twin security” can be found. Hubalek and Schachermayer (2001) studied the non-tradability issues indicating that using the assumption of no arbitrage alone would lead to no information about the price of the claim. Observing the results presented by Hubalek and Schachermayer (2001), Henderson and Hobson (2001) consider a utility-based approach to obtain the reservation price as well as the optimal hedging strategy. More recent contributions are Henderson (2005), Hugonnier and Morellec (2004, 2005), Kadam et al (2004), Miao and Wang (2005).

Henderson (2005) considers a lump-sum payoff case for investment and introduces the concept of time consistency utility function for valuation. Hugonnier and Morellec (2004, 2005), Kadam et al (2004), Miao and Wang (2005) analyze both lump-sum and flow payoffs. The consensus is that with the lump-sum investment payoffs, incompleteness and managerial risk aversion lower the option value to invest as well as the investment threshold. When flow payoffs are considered, the consequence is reversed.

2.3 Stochastic Volatility Models, Pricing Measures and Indifference Pricing

Hull and White (1987) introduced stochastic volatility to option pricing. Their work was followed by Stein and Stein (1991) and Heston (1993). Since then, there has been a growing body of research on option pricing techniques with stochastic volatility. Incompleteness induced by stochastic volatility allows infinitely many admissible option prices, consistent with the absence of arbitrage. Each admissible price corresponds to a different martingale measure. Biagini et al. (2000), Heath, Platen and Schweizer (2001), Henderson (2005), Hobson (2004), Laurrent and Pham (1999), and Pham et al. (1998) contain approaches for selecting an equivalent martingale measure with which to price options, where Henderson (2005) compares and analyzes the order of option prices under q-optimal measures. The indifference pricing technique first proposed by Hodges and Neuberger (1998) has been applied to stochastic volatility models in Sircar and Zariphopoulou (2005). Frittelli (2000) analyzes the connection of the pricing rules of agents with exponential utility with the arbitrage-free valuation under minimum entropy martingale measure.

CHAPTER 3

STRATEGIC EXERCISE OF OPTIONS ON NON-TRADED ASSETS IN AN INCOMPLETE MARKET

3.1 Background

This research explores relations between strategic exercise of real options and market completeness. I integrate the non-traded asset / incomplete market model of Henderson (2005) with a Stackelberg leader/follower model similar to Dixit and Pindyck (1994) and Grenadier (1996). From a classic game-theoretic real options setting (see Dixit and Pindyck (1994) and Grenadier (1997)), I also model the investment project value as a flow payoff from a stochastic demand shock. This enables us to study how a market demand shock and demand elasticity interact with the market incompleteness and the manager's risk aversion. With the stochastic flow payoff induced from the stochastic demand following geometric Brownian motion, the stochastic investment stream payoff will be bounded from below by zero given no unit variable costs. In view of the possibility from negative cash flows, I model the stochastic investment income following arithmetic Brownian motion. Arithmetic Brownian Motion allows some analytical closed form solutions for some ordinary differential equations, thus reducing required ordinary differential equation approximations for obtaining results. I find that incompleteness narrows the gap between leader and follower entry dates. Relative to results in Dixit and Pindyck

(1994), the follower enters much sooner, and the leader delays slightly. As a byproduct of modeling two different stochastic income streams, I analyze the impact of market share and uncertainty on the relative investment trigger as well as the option value. Results are sensitive to these factors; thus, it is important to model stochastic processes to reflect real world circumstances.

3.2 Motivation

Real options models depend on simplifying assumptions that may not hold in practice. For example, the classical real options model involves decision makers versus "nature" rather than against competitors, i.e., irreversible investment under monopoly. The options game model[6], integrating option pricing theory with game theory addresses this limitation. Recent work suggests that fear of pre-emption leads to a significant erosion of the option value to delay investment, with optimal policies approaching that of static net present value analysis. This extension to non-traded assets is especially important because assets underlying real options are typically not traded. This characteristic makes integrating incompleteness and risk aversion into real options models important to the efficacy of the approach. Recent research has shown that optimal investment policies obtained through the classical real options framework are substantially altered if the market is incomplete.

This chapter merges two lines of research. In one line, Dixit and Pindyck (1994, Chapter 9), Grenadier (1996, 1999, 2002), and others have examined optimal

[6] See, for example, Dixit and Pindyck (1994, Ch9), Grenadier (1996, 1999, 2002), Smets (1995), Lambrech and Perraudin (1998), Huisman (2001), Smit and Trigeorgis (2004)…etc.

policies for exercise of options in a setting where fear of pre-emption induces one firm to exercise the option to invest sooner than would be the case in a monopoly setting. These models typically assume a Stackelburg leader-follower set up. Grenadier (1996), for example, uses this type of model to explain cascades and overbuilding in real estate markets.

The other line deals with the valuation of contingent claims when the underlying asset is not traded. Under these circumstances, a unique martingale measure does not exist. Various methods have been developed to study the option pricing under such situations, including, on one hand, approaches to selecting an equivalent martingale measure (see, for example, Follmer and Sonderman (1986), Duffie and Richardson (1991), Schweizer (1991), Delbaen and Schachermayer (1996), and Fratelli (2000), among others), and on the other hand, utility maximization and indifference pricing (see, for example, Henderson (2002, 2005), Henderson and Hobson (2002), Musiela and Zariphopoulou (2003), and Zariphopoulou (2004), among others) . Recent research has shown that incompleteness and managerial risk aversion lower the option value to invest as well as the investment threshold compared to in classical complete market models when lump-sum investment payoffs are considered. In contrast, when flow (income stream) investment payoffs are assumed, this observation is reversed.[7] (see, for example, Henderson (2005), Hugonnier and Morellec (2004, 2005), Kadam et al (2004), Miao and Wang (2005)).

[7] It's interesting to observe different optimal investment rules due to lump-sum investment payoffs versus flow investment payoffs because they are the same in the standard real options model with complete market setting (see Dixit and Pindyck (1994) Ch5 and Ch6.)

In this chapter, I consider leader-follower Stackelberg option exercise games for investment in an asset when the underlying asset's risk may only be partially hedged. I employ the model documented in Dixit and Pindyck (1994, Chapter 9) to capture strategic exercise. I combine this model with the partial hedging model of Henderson (2005) to capture market incompleteness. Following Henderson (2005) and Zariphopoulou (2004), I employ an exponential utility function to solve for an "indifference price" for project value and for the option to invest. With constant Sharpe ratio, the minimal martingale measure and minimum entropy martingale measure converge, as does the value of the investment opportunity under both measures. Following much of the game-theoretic real options literature, I assume a stochastic demand for the project's output. This framework allows for a reasonable set of possibilities, including strategic behaviors due to changes in market demand. This particular formulation models investment payoff as as a stochastic cash flow stream. In this case, I again employ the model documented in Dixit and Pindyck (1994, Chapter 9) to capture strategic exercise considerations, and the utility maximization method coupled with indifference pricing. In view of the possibility of negative cash flows, I model the stochastic investment income as an arithmetic Brownian motion. This reduces the required approximations for solution. As a byproduct of modeling two different stochastic income streams, I analyze the impact of market share and uncertainty on the relative investment trigger as well as option value.

3.3 Optimal Policies for Duopolistic Competition by Project Values (Lump-Sum Investment Payoffs)

I consider leader-follower Stackelberg option exercise games for investment in a project when the underlying asset may only be partially hedged. I employ the model documented in Dixit and Pindyck (1994, Chapter 9) to capture strategic exercise. I combine this model with the partial hedging model of Henderson (2005) to capture market incompleteness. Following Henderson (2005) and Zariphopoulou (2004), I employ an exponential utility function to solve for an "indifference price" for project value and for the option value to invest.

3.3.1. Model Set-Up and Assumptions

Two competing firms contemplate entry into a new market where operating profitability is stochastic and the decision to enter the market is completely irreversible. I identify the firms as the Leader (Firm L) and Follower (Firm F), respectively[8]. Entry yields a stochastic payoff resulting in a stochastic project value with which no perfectly correlated portfolios can be found. The absence of perfect spanning assets forces the manager to face unhedgeable idiosyncratic risk and an incomplete market.

Firm L enters the market by investing to receive a monopoly rent until Firm F enters the market. Upon entry by Firm F, I assume that Firm L obtains a fraction $a \in [½,1]$ of project value, leaving $(1\text{-}a)\ V_t$ for Firm F.

[8] I take Firm L and Firm L's management to be synonymous with the implicit assumption that there is no agency problem. I follow the same convention for Firm F.

I fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a fixed σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, where Brownian motion is defined and the expectation $\mathbb{E}\{\bullet\}$ is computed. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the augmented filtration of Brownian motion. The increasing σ-algebras generated by the pair of Brownian motions $(Z_s)_{s \leq t}$ and $(Z^{\perp}_s)_{s \leq t}$, where $Z^{\perp}$ is orthogonal to Z, satisfy the usual conditions of right-continuity and completeness (i.e., include all the sets of probability 0 in $\mathcal{F}$[9]). Let $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ be filtration generated by Z alone.

Non-traded Assets (The Project)[10]

I assume that project value, Vt , evolves exogenously according to a geometric Brownian motion11:

$$dV_t = \alpha V_t dt + \eta V_t dW_t = \eta V_t (\xi dt + dW_t) + r V_t dt ; \quad V_t = v$$

where α is the instantaneous conditional expected percentage change in V per

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[9] The completion by the null sets is important in particular for the following reason. If two random variables X and Y are equal almost surely (X=Y P-a.s. means P{X=Y}=1) and if X is $\mathcal{F}_t$ measurable (meaning that any event $\{X_t \leq x\}$ belongs to $\mathcal{F}_t$) then Y is also $\mathcal{F}_t$-measurable.

[10] Consistent with extant research on real options in incomplete markets, I assume the existence of three assets, a non-traded asset, a traded risky asset and a traded riskless asset.

[11] I have the process able to follow a more general form as $dV_t = \alpha(V_t, t)dt + \eta(V_t, t) dW_t$; $V_t = v$ where $\alpha(V_t, t)$ and $\eta(V_t, t)$ are measurable functions satisfying

(1) $|\alpha(v,t)| + |\eta(v,t)| \leq C(1+|v|)$; $v \in R, 0 \leq t \leq T < \infty$ for some constant C ($0 < C < \infty$)
(2) $|\alpha(v,t) - \alpha(x,t)| + |\eta(v,t) - \eta(x,t)| \leq D|v - x|$; $x, v \in \mathrm{R}, 0 \leq t \leq T < \infty$ for some constant D ($0 < D < \infty$)
, so that the existence and uniqueness solution of V_t process is guaranteed.

unit time, $\xi = \frac{\alpha - r}{\eta}$ is the project's Sharpe ratio, η, is the instantaneous volatility, and W is a standard Brownian motion having correlation $\rho \in (-1,1)$ with Z. Thus, W = ρZ + $\sqrt{1-\rho^2}Z^{\perp}$, or equivalently, dW = ρ dZ + $\sqrt{1-\rho^2}dZ^{\perp}$.

Traded Risky Security

There exists a partial spanning asset which follows the lognormal process[12]:

$dP_t = \mu P_t dt + \sigma P_t dZ_t = \sigma P_t (\lambda dt + dZ_t) + r P_t dt$; $P_t = p$

where μ is the instantaneous conditional expected percentage change in P per unit time, $\lambda = \frac{\mu - r}{\sigma}$ is its Sharpe ratio is its Sharpe ratio, σ is the instantaneous volatility, and Z is a standard Brownian motion.

TradedRiskless Security

I also assume that a riskless bond B is available for trading. The riskless bond with price process B growing in deterministic fashion at risk-free rate satisfies the following dynamics:

$dB_t = rB_t dt$; $B_t = b$

Trading Wealth Process and Utility Function

Realizing that markets are incomplete, the firm's manager may hedge partially using the traded asset P_t and the riskless bond, generating trading wealth X_t that follows:

$dX_t = \theta (dP_t / P_t) + r (X_t - \theta) dt$

[12] Again, I can have the process follows more general forms satisfying all required technical conditions as described in footnote 2 for the guarantee of uniqueness and existence solution to P_t process.

where θ is the cash amount invested in the partial spanning asset P_t, and remaining wealth is invested at riskless rate r.

The utility function employed is a concave mapping U: $\mathbb{R}\rightarrow[-\infty,\infty)$, strictly increasing, strictly concave, and continuously differentiable on its domain satisfying:

(i) The half-line dom(U) = $\{x \in R; U(x) > -\infty\}$ is a non-empty subset of $[0,\infty)$.

(ii) $U'(x)$ is continuous, positive and strictly decreasing on the interior of dom(U), and satisfy the Inada condition:

$$U'(\infty) = \lim_{x \to \infty} U'(x) = 0$$

The standard CARA, CRRA and HARA utility functions satisfy the above properties. Throughout the study, I employ the exponential utility function $U(x) = -\frac{1}{\gamma} e^{-\gamma x}$. I specify $\gamma > 0$, i.e., the manager exhibits constant absolute risk-aversion.

3.3.2 Firm F's Value Function, Investment Timing Decision, and Certainty Equivalence Value

Following Henderson (2005), I assume that investment cost, K, grows at a riskless rate, r. Investment at time τ: $\tau \geq t$, yields the payoff $[(1\text{-}a)\ V_\tau - K\ e^{r(\tau - t)}]^+$. The manager generates trading wealth, X_t, by dynamically adjusting the dollar amount θ_s for s > t in the partial spanning asset, and by consequence, the riskless bond.

Assume Firm L has already entered the market. Firm F will enter optimally without fear of pre-emption. Therefore, the risk-averse manager's problem becomes one of maximizing expected utility of wealth over an infinite horizon. Wealth refers to both the quantity X_t generated by trading, and the payoff $[(1\text{-}a)\ V_\tau - K\ e^{r(\tau - t)}]^+$ received at the time of investment. That is, the manager chooses optimally the time to invest τ, and hedge θ in the partial spanning asset P_t.

3.3.2 Proposition 1:

The value function for Firm F's investment problem is given by the optimal stopping problem:

$$F(x,v) = \sup_{t \le \tau} \sup_{\theta_u, t \le u \le \tau} E_t[U_\tau(X_\tau + ((1-a)V_\tau - Ke^{r(\tau-t)})^+ \mid X_t = x, V_t = v].$$

Employing a time consistent exponential utility function (see Henderson (2005)),

$$U_\tau(x) = -\frac{1}{\gamma} e^{-\gamma e^{-r(\tau-t)} x} e^{\frac{1}{2}\lambda^2(\tau-t)}.$$

, the value function can be rewritten as:

$$F(x,v) = \sup_{t \le \tau} \sup_{\theta_u, t \le u \le \tau} E_t[-\frac{1}{\gamma} e^{-\gamma e^{-r(\tau-t)}(X_\tau + ((1-a)V_\tau - Ke^{r(\tau-t)})^+} e^{\frac{1}{2}\lambda^2(\tau-t)} \mid X_t = x, V_t = v]$$

3.3.2 Proposition 2:

Firm F's value function may be written as:

$$F(x,v) = \begin{cases} -\frac{1}{\gamma} e^{-\gamma x}[1 - (1 - e^{-\gamma((1-a)\tilde{v}^F - K)(1-\rho^2)})(\frac{v}{\tilde{v}^F})^\beta]^{\frac{1}{1-\rho^2}}; v \in [0, \tilde{v}^F) \\ -\frac{1}{\gamma} e^{-\gamma x} e^{-\gamma((1-a)v-k)}; v \in [\tilde{v}^F, \infty)] \end{cases}$$

where $\beta_1 = 1 - \frac{2(\xi - \lambda\rho)}{\eta}$ and $\tilde{v}^F$ is the solution to the following:

$$(1-a)\tilde{v}^F - K = \frac{1}{\gamma(1-\rho^2)} \ln[1 + \frac{\gamma\tilde{v}^F(1-a)(1-\rho^2)}{\beta}].$$

$\tilde{v}^F$ is the solution to the free boundary problem; that is, Firm F's investment problem is to invest as soon as V reaches the threshold $\tilde{v}^F$.

Proof:

Through standard arguments, the corresponding Bellman equation for 3.3.2 Proposition 1 is given by:

$$\frac{1}{2}\lambda^2 F + \xi\eta v F_v + \frac{1}{2}\eta^2 v^2 F_{vv} - \frac{1}{2}\frac{(\lambda F_x + \rho\eta v F_{xv})^2}{F_{xx}} = 0$$

The value function derives by solving the above PDE with the following boundary, value matching and smooth pasting conditions:

$$F(x,0) = -\frac{1}{\gamma}e^{-\gamma x}$$

$$F(x,\tilde{v}^F) = -\frac{1}{\gamma}e^{-\gamma(x+((1-a)\tilde{v}^F - K)^+))}$$

$$F_v(x,\tilde{v}^F) = I_{\{(1-a)\tilde{v}^F \geq K\}} e^{-\gamma(x+((1-a)\tilde{v}^F - K)^+))}.$$

In the stopping region, $F(x,\tilde{v}^F) = -\frac{1}{\gamma}e^{-\gamma(x+((1-a)\tilde{v}^F - K)^+))}$; and the optimal investment τ^* is given by $\tau^* = \inf\{u \geq t : V_u \geq \tilde{v}^F e^{r(u-t)}\}$.

3.3.2 Proposition 3:

Firm F's certainty equivalence valuation of the option value to invest is given by $F_e(v) = -\frac{1}{\gamma(1-\rho^2)}\ln[1-(1-e^{-\gamma((1-a)\tilde{v}^F - K)(1-\rho^2)})(\frac{v}{\tilde{v}^F})^{\beta}]$

where $\beta = 1 - \frac{2(\xi - \lambda\rho)}{\eta}$ and $\tilde{v}^F$ is the solution to the following:

$$(1-a)\tilde{v}^F - K = \frac{1}{\gamma(1-\rho^2)}\ln[1+\frac{\gamma\tilde{v}^F(1-a)(1-\rho^2)}{\beta}]$$

Remarks:

The certainty equivalent option value to invest can be found by equating the value obtained by investing in P and the risk-free asset (receiving the amount $F_e(v)$), to the value obtained by retaining the option, or , $F(x+F_e(v),0)=F(x,v)$.

Using the value function obtained from 3.3.2 Proposition 2 yields the following parity relation:

$$-\frac{1}{\gamma}e^{-\gamma(x+F_e(v))} = -\frac{1}{\gamma}e^{-\gamma x}[1-(1-e^{-\gamma((1-a)\tilde{v}^F-K)(1-\rho^2)})(\frac{v}{\tilde{v}^F})^{\beta}]^{\frac{1}{1-\rho^2}}$$

Solve for $F_e(v)$ by taking natural logarithm of both sides and simplifying:

$$F_e(v) = -\frac{1}{\gamma(1-\rho^2)}\ln[1-(1-e^{-\gamma((1-a)\tilde{v}^F-K)(1-\rho^2)})(\frac{v}{\tilde{v}^F})^{\beta}]\text{~Q.E.D.}$$

3.3.2 Proposition 4:

Firm F's certainty equivalence option value to invest can be expressed in terms of pricing measure Q^0:

$$F_e(v) = \sup_{t\le\tau<\infty} -\frac{1}{\gamma(1-\rho^2)}\ln E^{Q^0}[e^{-\gamma(1-\rho^2)e^{-r(\tau-t)}((1-a)V_\tau-Ke^{r(\tau-t)})^+} \mid V_t = v]$$

where E^{Q^0} denotes expectation with respect to pricing measure Q^0, defined as follows. For each t < ∞, the Radon-Nikodym density of Q^0 with respect to the historical measure P is defined as:

$$\frac{dQ^0}{dP}\Big|_{F_t} = \exp(-\lambda Z_t - \frac{1}{2}\lambda^2 t)$$

Under Q^0, $\frac{dP_t}{P_t} = rdt + \sigma dZ_t^0, where Z_t^0 = Z_t + \lambda t$ is a Q^0-Brownian motion and the independent Brownian motion $Z_t^{\perp}$ is unchanged under Q^0. Under Q^0, the project value V_t follows:

$$\frac{dV_t}{V_t} = (\alpha - \eta\rho\lambda)dt + \eta(\rho dZ_t^0 + \sqrt{1-\rho^2}\, dZ_t^{\perp})$$

The new pricing measure Q^0 is the minimal martingale measure of Follmer and Schweizer (1990), which in this particular case also collapses to the minimal entropy martingale measure.

3.3.3 Firm L's Value Function, Investment Timing Decision, and Certainty Equivalence Value

I begin prior to Firm L's entry, and I assume Firm F will act optimally according to the optimal stopping rule described above. Once Firm L has invested $K\, e^{r\,(T' - t)}$ at time T' it has no further action to take. It enjoys monopolistic rents π until Firm F enters, i.e., $t < \tau^*$, where τ^* is Firm F's entry point. Upon Firm F's entry, Firm L retains the portion $a \in [½, 1]$ of project value leaving (1 – a) of project value to Firm F. If Firm L's management undertakes investment at time T' ($t \leq T' < \tau^*$), the payoff is ($V_{T'} - K\, e^{r\,(T' - t)}$), where $K\, e^{r\,(T' - t)}$ is the investment cost and $V_{T'}$ includes the expected monopoly rent. That is, the expected project value $V_{T'}$ can be decomposed into two parts:(1) expected project value even if the follower jumps in denoted as $a \times V_{T'}$; and (2) expected capitalized monopoly rent prior to Firm F's entry denoted as $V_{T'}\,(\pi)$. The manager's problem is to maximize expected utility of wealth with the investment strategy θ by hedging partially using the traded asset P and the riskless bond.

3.3.3 Proposition 1:

Firm L's value function can be written as:

$$L(x,v)=\sup_{\theta_u, t\le\tau\le\infty} E_t[U_\tau(X_\tau + aV_\tau + V_\tau(\pi) - Ke^{r(\tau-t)}) \mid X_t = x, V_t = v]$$

Employing a time consistent exponential utility function (see Henderson (2005)),

$$U_\tau(x) = -\frac{1}{\gamma} e^{-\gamma e^{-r(\tau-t)}x} e^{\frac{1}{2}\lambda^2(\tau-t)}.$$

, the value function can be re-written as:

$$L(x,v)=\sup_{\theta_u, t\le\tau\le\infty} E_t[-\frac{1}{\gamma} e^{-\gamma e^{-r(\tau-t)}(X_\tau+(aV_\tau+V_\tau(\pi)-Ke^{r(\tau-t)})} e^{\frac{1}{2}\lambda^2(\tau-t)} \mid X_t = x, V_t = v]$$

3.3.3 Proposition 2:

Firm L's value function is given by:[13]

$$L(x,v)=\begin{cases} -\frac{1}{\gamma} e^{-\gamma x}[1-(1+v-e^{-\gamma(a\tilde{v}^F-K)(1-\rho^2)})(\frac{v}{\tilde{v}^F})^\beta + v]^{\frac{1}{1-\rho^2}}; v\in[0,\tilde{v}^F] \\ -\frac{1}{\gamma} e^{-\gamma x} e^{-\gamma(av-K)}; v\in[\tilde{v}^F,\infty] \end{cases}$$

where $\beta = 1-\frac{2(\xi-\lambda\rho)}{\eta}$, and $\tilde{v}^F$ is the investment trigger value corresponding to Firm F's optimal investment policy.

Proof

Assume Firm L has already entered the market and given V< $\tilde{v}^F$, the corresponding equation for 3.3.3 Proposition 1 is

$$\frac{1}{2}\lambda^2 L + \xi\eta v L_v + \frac{1}{2}\eta^2 v^2 L_{vv} - \frac{1}{2}\frac{(\lambda L_X + \rho\eta v L_{xv})^2}{L_{xx}} + V(\pi) = 0$$

The value function derives from the solution to the above PDE with the following boundary and value matching conditions:

$$L(x,0) = -\frac{1}{\gamma}e^{-\gamma x}$$

$$L(x,\tilde{v}^F) = -\frac{1}{\gamma}e^{-\gamma(x+a\tilde{v}^F)}$$

3.3.3 Proposition 3:

Firm L's certainty equivalence valuation prior to Firm F's optimal entry is given by:

$$L_e(v) = -\frac{1}{\gamma(1-\rho^2)}\ln[1-(1+v-e^{-\gamma(a\tilde{v}^F-K)^+(1-\rho^2)})(\frac{v}{\tilde{v}^F})^{\beta} + v]$$

where $\beta = 1 - \frac{2(\xi-\lambda\rho)}{\eta}$ and $\tilde{v}^F$ is the solution to the following:

$$(1-a)\tilde{v}^F - K = \frac{1}{\gamma(1-\rho^2)}\ln[1+\frac{\gamma\tilde{w}^F(1-a)(1-\rho^2)}{\beta}]$$

Remarks:

The certainty equivalent option value to invest can be found by equating the value obtained by investing in P and the risk-free asset (receiving the amount $L_e(v)$), to the value obtained by retaining the option, or $L(x+L_e(v),0) = L(x,v)$.

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[13] At the final step, K appears in the exponential component because if firm L exercises the investment option, the project value will decrease by $Ke^{r(T-t)}$ at that point.

Using the value function obtained from 3.3.3 Proposition 2 yields the following parity relation:

$$-\frac{1}{\gamma}e^{-\gamma(x+L_e(v))}=-\frac{1}{\gamma}e^{-\gamma x}[1-(1+v-e^{-\gamma(a\tilde{v}^F-K)^+(1-\rho^2)})(\frac{v}{\tilde{v}^F})^{\beta}+v]^{\frac{1}{1-\rho^2}}$$

Solve for $L_e(v)$ by taking natural logarithm of both sides and simplifying:

$$L_e(v)=-\frac{1}{\gamma(1-\rho^2)}\ln[1-(1+v-e^{-\gamma(a\tilde{v}^F-K)^+(1-\rho^2)})(\frac{v}{\tilde{v}^F})^{\beta}+v]\text{~Q.E.D.}$$

3.3.3 Proposition 4:

Firm L's investment trigger, $\tilde{v}^L$, is the solution to the following equation:

$$-\frac{1}{\gamma(1-\rho^2)}\ln[1-(1-(e^{-\gamma[((1-a)\tilde{v}^F-K)^+](1-\rho^2)})(\frac{\tilde{v}^L}{\tilde{v}^F})^{\beta_1}]$$
$$=-\frac{1}{\gamma(1-\rho^2)}\ln[1-(1+\tilde{v}^L-e^{-\gamma(a\tilde{v}^F-K)(1-\rho^2)})(\frac{\tilde{v}^L}{\tilde{v}^F})^{\beta_1}+\tilde{v}^L]$$

which can be reduced to

$$(e^{-\gamma[((1-a)\tilde{v}^F-K)^+](1-\rho^2)})(\frac{\tilde{v}^L}{\tilde{v}^F})^{\beta_1}$$
$$=(-\tilde{v}^L+e^{-\gamma(a\tilde{v}^F-K)(1-\rho^2)})(\frac{\tilde{v}^L}{\tilde{v}^F})^{\beta_1}+\tilde{v}^L]$$

where $\beta_1=1-\frac{2(\xi-\lambda\rho)}{\eta}$, and $\tilde{v}^F$ is the solution to the following:

$$(1-a)\tilde{v}^F-K=\frac{1}{\gamma(1-\rho^2)}\ln[1+\frac{\gamma\tilde{v}^F(1-a)(1-\rho^2)}{\beta}]$$

Remarks:

The above proposition makes use of the fact that Firm L's trigger value, $\widetilde{v}^{L}$, yields the same value for both firms with $\widetilde{v} < \widetilde{v}^{F}$. That is, $L(x,\widetilde{v}^{L})$ should be equal to $F(x,\widetilde{v}^{L})$.

3.3.4 Model Results Assuming Perfect Spanning (Complete Market)

I now demonstrate that the model reduces to standard results when the market is complete. The market is complete if either perfect spanning holds, i.e., uncertainty over project value, V, may be replicated by asset P (perfect correlation with V), or equivalently, V itself is traded. Such a complete market version of the model creates a "benchmark" for comparison to the incomplete market case.

3.3.4.1 Firm F's Value Function and Investment Timing Decision (Complete Market)

Assuming Firm L has already entered the market, Firm F will enter the market optimally without fear of pre-emption. Under perfect spanning, Firm F's value function and investment trigger can thus be obtained through the following proposition.

3.3.4.1 Proposition 1:

Under perfect spanning, Firm F will exercise the option as soon as the project value approaches the investment threshold, $\widetilde{v}^{(FC)}$, from below. Firm F's option value may be expressed as:

$$F^{C}(v) = \begin{cases} [(1-a)\widetilde{v}^{(FC)} - K](\frac{v}{\widetilde{v}^{(FC)}})^{\beta}; v \in [0, \widetilde{v}^{(FC)}] \\ (1-a)v - K; v \in [\widetilde{v}^{(FC)}, \infty] \end{cases}$$

where:

$$\beta = 1 - \frac{2(\xi - \lambda\rho)}{\eta} \text{ and } \widetilde{v}^{(FC)} = \frac{\beta}{(1-a)(\beta-1)} K .$$

<u>*Proof:*</u>

Under perfect spanning, Firm F's option value to invest at the optimal stopping time τ ($\tau \geq t$) follows from maximizing the expectation of the discounted value of project payoff under the unique equivalent martingale measure, Q:

$$F^C(v) = \sup_{t \leq \tau < \infty} E_t^Q[e^{-r(\tau-t)}((1-a)V_\tau - Ke^{r(\tau-t)})^+ \mid V_t = v].$$

The corresponding Bellman equation for the above equation is given by:

$$F_v^C v\eta(\xi - \lambda) + \frac{1}{2} F_{vv}^C v^2 \eta^2 = 0$$

I solve the above ODE with following boundary, value matching and smooth pasting conditions:

$$F^C(0) = 0$$

$$F^C(\widetilde{v}^{(FC)}) = (1-a)\widetilde{v}^{(FC)} - K$$

$$F_v^C(\widetilde{v}^{(FC)}) = (1-a)$$

The optimal investment $\tau^*$is given by $\tau^* = \inf\{u \geq t : V_u \geq \widetilde{v}^{(FC)} e^{r(u-t)}\}$.

I propose a solution of the form $F^C(v) = Av^\beta$ where A is a constant to be determined. Because $F^C(v) = Av^\beta$, it immediately follows that $F_v^C = A\beta v^{(\beta-1)}$ and $F_{vv}^C = A\beta(\beta-1)v^\beta$. Substituting back into the Bellman equation $F_v^C v\eta(\xi - \lambda) + \frac{1}{2} F_{vv}^C v^2 \eta^2 = 0$; the Bellman equation becomes

$$\frac{1}{2}\beta(\beta-1)\eta^2+\beta\eta(\xi-\lambda)=0$$

Solving for β yields $\beta = 0$ or:

$$\beta=1-\frac{2(\xi-\lambda)}{\eta}.$$

The solution $\beta = 0$ may be rejected given the boundary conditions, and the solution is of the form $F^C(v) = Av^\beta$ with $\beta=1-\frac{2(\xi-\lambda)}{\eta}$. The constant A and investment trigger value $\tilde{v}^{(CF)}$ may be determined by invoking value matching and smooth pasting conditions:

$$F^C(v)=[(1-a)\tilde{v}^{(FC)}-K](\frac{v}{\tilde{v}^{(FC)}})^\beta \qquad ,\tilde{v}^{(FC)}=\frac{\beta}{(1-a)(\beta-1)}K \text{ where}$$

$$\beta=1-\frac{2(\xi-\lambda)}{\eta}.$$

Remarks:

The proposed solution may be verified by comparison with Dixit and Pindyck (1994), Chapter 9, if I assume that the capitalized project value is equivalent to the discounted project cash flow.

3.3.4.2 Firm L's Value Function and Investment Timing Decision (Complete Market)

By design Firm L undertakes investment prior to Firm F's entry. Firm L makes its investment decision conditional on Firm F acting optimally according to the optimal stopping rule described above.

3.3.4.2 Proposition 1:

Firm L's value function is:

$$L^C(v) = \begin{cases} v + [a\tilde{v}^{(FC)} - \tilde{v}^{(FC)}](\frac{v}{\tilde{v}^{(CF)}})^{\beta} - K; v \in [0, \tilde{v}^{(FC)}] \\ av - K; v \in [\tilde{v}^{(FC)}, \infty] \end{cases}$$

where $\beta = 1 - \frac{2(\xi - \lambda)}{\eta}$

and $\tilde{v}^{(FC)} = \frac{\beta}{(1-a)(\beta - 1)} K$,i.e., the investment trigger for Firm F.

Proof (1st Version)

Upon investment at time τ (τ≥t), Firm L expects to receive project value V_τ, which includes the expected monopoly rent. That is, the expected project value can be decomposed into two parts: (1) expected project value even if the follower enters, denoted as a×V_τ and (2) expected capitalized monopoly rent prior to Firm F's entry, denoted as $V_\tau(\pi)$. L's value function is the expected discounted project value conditional on optimal behavior by Firm F under the unique equivalent martingale measure, Q:

$$L^C(v) = E_t^Q[e^{-r(\tau-t)}(aV_\tau + V_\tau(\pi) - Ke^{r(\tau-t)}) \mid V_t = v] \text{ where } \tau \in [\text{t},\propto\}.$$

Assume Firm L has already entered the market and given V<$\tilde{v}^{(FC)}$, the Bellman equation is:

$$L_v^C v\eta(\xi - \lambda) + \frac{1}{2} L_{vv}^C v^2 \eta^2 + V(\pi) = 0$$

With the fact that $V(\pi)$ is a fraction of V, and with a slight abuse of notation:

$$L_v^C v\eta(\xi-\lambda)+\frac{1}{2}L_{vv}^C v^2\eta^2+hv=0 \qquad \text{where } h\in R^+$$

I solve the above ODE with the following boundary and value matching conditions. The value matching condition requires Firm L's value $L^C(V)$ to match the value of simultaneous investment at the boundary V = $\widetilde{v}^{(FC)}$:

$$L^C(\widetilde{v}^{(FC)})=a\widetilde{v}^{(FC)}$$

I propose a solution of the form $L^C(V)=AV^\beta$ for the homogeneous part, plus a particular solution for the non-homogenous part, where A is a constant to be determined. The Bellman equation for the homogenous part is (similar to the previous section):

$$\frac{1}{2}\beta(\beta-1)\eta^2+\beta\eta(\xi-\lambda)=0$$

with solutions $\beta=0$ or $\beta=1-\frac{2(\xi-\lambda)}{\eta}$.

The solution β = 0 may be rejected based on the boundary conditions. One candidate of particular solutions is $(-\frac{hv}{\eta(\xi-\lambda)})$ > 0, which may be interpreted as capitalized project value, V. Therefore, the solution should be the form of $L^C(V)=AV^\beta$ for the homogenous part with $\beta=1-\frac{2(\xi-\lambda)}{\eta}$, plus particular solution, V. From the value matching condition, A may be determined, yielding Firm L's value function:

$$L^C(v)=v+[a\widetilde{v}^{(FC)}-\widetilde{v}^{(FC)}](\frac{v}{\widetilde{v}^{(FC)}})^\beta$$

with $\beta = 1 - \frac{2(\xi - \lambda)}{\eta}$ and $\widetilde{v}^{(FC)} = \frac{\beta}{(1-a)(\beta - 1)} K$

Remarks:

It is easily verified that the value function obtained is the same as the corresponding result from Dixit and Pindyck (1994), Chapter 9 by interpreting the discounted cash flow as the capitalized project value.

<u>*Proof (2nd Version)*</u>

Firm L holds the project and sells an European call on (1-a)V expiring at the stochastic time $\tau^* = \inf \{u \geq t : V_u \geq \widetilde{v}^{(FC)} e^{r(u-t)}\}$ corresponding to Firm F's optimal entry, with zero exercise price. The sale of the European call is to account for the loss of monopolistic rent following Firm F's optimal entry. Therefore, Firm L's portfolio value at time t should be

$$L^C(v) = v - F[(1-a)v]$$

where $F[(1-a)v]$ the value function for the call option

Using Firm F's value function with zero exercise price obtained above, the corresponding Firm L's corresponding value function before and after Firm F's optimal entry is given by:

(i) Before Firm F's optimal entry, that is, V<$\widetilde{v}^{(FC)}$

$$\begin{aligned} L^C(v) &= v - F[(1-a)v] \\ &= v - [(1-a)\widetilde{v}^{(FC)}](\frac{v}{\widetilde{v}^{(FC)}})^{\beta} \\ &= v + [(a\widetilde{v}^{(FC)} - \widetilde{v}^{(FC)}](\frac{v}{\widetilde{v}^{(FC)}})^{\beta} \end{aligned}$$

(ii) After Firm F's optimal entry, that is, V≥$\widetilde{v}^{(FC)}$

$$\begin{aligned} L^C(v) &= v - F[(1-a)v] \\ &= v - [(1-a)v] \\ &= av \end{aligned}$$

In sum, Firm L's value function is:

$$L^C(v) = \begin{cases} v + [a\widetilde{v}^{(FC)} - \widetilde{v}^{(FC)}](\frac{v}{\widetilde{v}^{(CF)}})^{\beta} - K; v \in [0, \widetilde{v}^{(FC)}] \\ av - K \in [\widetilde{v}^{(FC)}, \infty] \end{cases}$$

Q.E.D

3.3.4.2 Proposition 2:

Firm L's investment trigger value $\widetilde{v}^{(LC)}$ is the solution to the following equation:

$$\widetilde{v}^{(LC)} + [a\widetilde{v}^{(FC)} - K](\frac{\widetilde{v}^{(LC)}}{\widetilde{v}^{(FC)}})^{\beta} - K = [(1-a)\widetilde{v}^{(FC)} - K](\frac{\widetilde{v}^{(LC)}}{\widetilde{v}^{(FC)}})^{\beta}$$

where $\beta = 1 - \frac{2(\xi - \lambda)}{\eta}$ and $\widetilde{v}^{(FC)} = \frac{\beta}{(1-a)(\beta - 1)} K$

Remarks:

The above proposition exploits the fact that Firm L's trigger value, $\widetilde{v}^{(LC)}$, yields the same value for both firms with $\widetilde{v} < \widetilde{v}^{(FC)}$. That is, $L^C(\widetilde{v}^{(LC)})$ should be equal to $F^C(\widetilde{v}^{(LC)})$.

3.4 Optimal Policies for Duopolistic Competition by Project Income Stream (Flow Investment Payoffs I – Demand Shock with Geometric Brownian Motion)

Following much of the game-theoretic real options literature, I assume a stochastic demand for the project's output. This framework allows for a reasonable set of possibilities, including strategic behaviors due to changes in market demand. This particular formulation models investment payoff as a stochastic cash flow stream. In this case, I again employ a Dixit and Pindyck (1994, Chapter 9) model to capture strategic exercise considerations, and the utility maximization method coupled with indifference pricing.

Following Dixit and Pindyck (1994) and Grenadier (1997), I model the investment payoff as a stochastic flow arising from a stochastic demand shock. Two competing firms are contemplating entry into a new market. As before, I identify the firms as the Leader (Firm L) and Follower (Firm F), respectively. There are no variable costs of production, and industry demand is assumed sufficiently elastic to ensure capacity production. Project cash flows depend on a stochastic unit output price caused by a demand shock process. The unit output price, $P(t)$, fluctuates stochastically over time so as to clear the market[14]:

$$P(t) = Y(t)D[Q(i)]$$

where $Y(t)$ is a multiplicative demand shock process, $D[\bullet]$ is the inverse demand function, and $Q(i)$ is the industry supply process. The down-sloping inverse demand function ensures the existence of a first mover advantage to investment. $Q(i)$ may be either 0, 1, or 2 depending on the number of active firms. Since the demand shock is not traded, the manager faces unhedgeable idiosyncratic risk and an incomplete market.

[14] This is an indirect way to model the operating cash flow process made via a stochastic demand shock. Alternatively, I can directly model the operating cash flow process with certain type of diffusion processes, say for example arithmetic Brownian motion process to incorporate the "negative" operating cash flow situations or more conventionally the geometric Brownian motion process.

3.4.1 Model Set-Up and Assumptions

I fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a fixed σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, where Brownian motion is defined and the expectation $\mathbb{E}\{\bullet\}$ is computed. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the augmented filtration of Brownian motion. The increasing σ-algebras generated by the pair of Brownian motions $(Z_s)_{s \leq t}$ and $(Z^{\perp}_s)_{s \leq t}$, where $Z^{\perp}$ is orthogonal to Z, satisfy the usual conditions of right-continuity and completeness (i.e., include all the sets of probability 0 in $\mathcal{F}$[15]). Let $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ be filtration generated by Z alone.

Non-traded Assets (The Demand Shock)

Project cash flows depend on a stochastic unit output price caused by a demand shock process. I let the multiplicative demand shock process, Y(t), follow the geometric Brownian motion[16]:

$$dY_t = \alpha Y_t dt + \sigma Y_t \, dW_t; \qquad Y_t = y$$

where α is the instantaneous conditional expected percentage change in Y per unit time, σ is the instantaneous volatility, and W is a standard Brownian motion having

[15] The completion by the null sets is important in particular for the following reason. If two random variables X and Y are equal almost surely (X=Y P-a.s. means P{X=Y}=1) and if X is $\mathcal{F}_t$ measurable (meaning that any event $\{X_t \leq x\}$ belongs to $\mathcal{F}_t$) then Y is also $\mathcal{F}_t$-measurable.

[16] The process follows more general form as $dY_t = \alpha(Y_t, t)dt + \eta(Y_t, t)\, dW_t$; $Y_t = y$ where $\alpha(Y_t, t)$ and $\eta(Y_t, t)$ are measurable functions satisfying

(1) $|\alpha(y,t)| + |\eta(y,t)| \leq C(1+|y|)$; $y \in R, 0 \leq t \leq T < \infty$ for some constant C ($0 < C < \infty$)

(2) $|\alpha(y,t) - \alpha(x,t)| + |\eta(y,t) - \eta(x,t)| \leq D|y-x|; x,y \in R, 0 \leq t \leq T < \infty$ for some constant D ($0 < D < \infty$), so that the existence and uniqueness solution of Y_t process is guaranteed.

correlation $\rho \in (-1,1)$ with Z. For this, I can take W = ρZ + $\sqrt{1-\rho^2}Z^{\perp}$, or equivalently, dW = ρ dZ + $\sqrt{1-\rho^2}dZ^{\perp}$.

Traded Risky Security

There exists a partial spanning asset which follows the lognormal process[17]:

$dS_t = \mu S_t\ dt + \chi S_t dZ_t = \chi S_t (\eta dt + dZt) + r S_t dt$; $S_t = s$

where μ is the instantaneous conditional expected percentage change in S per unit time, $\eta = \frac{\mu - r}{\chi}$ is its Sharpe ratio, η is the instantaneous volatility, and Z is a standard Brownian motion.

Traded Riskless Security

I also assume that a riskless bond B is available for trading. The riskless bond with price process B satisfies the following dynamics:

$dB_t = rB_t dt$; $B_t = b$

Utility Function

Realizing that markets are incomplete, the firm's manager attempts to maximize her expected utility. That is,

Maximize $E[\int_0^{\infty} e^{-\beta t} U(X_t) dt]$

[17] Again, I can have the process follows more general forms satisfying all required technical conditions as described in footnote 2 for the guarantee of uniqueness and existence solution to S_t process.

The utility function is a concave function U: $\mathbb{R} \to [-\infty,\infty)$, which is assumed to be strictly increasing, strictly concave, and continuously differentiable on its domain satisfying:

(i) The half-line dom(U) = $\{x \in R; U(x) > -\infty\}$ is a non-empty subset of $[0,\infty)$.

(ii) $U'(x)$ is continuous, positive, and strictly decreasing on the interior of dom(U), and satisfy the Inada condition:

$$U'(\infty) = \lim_{x \to \infty} U'(x) = 0$$

The standard CARA, CRRA and HARA utility functions satisfy the above properties. Throughout the study, I assume the manager has the exponential utility function $U(x) = -\frac{1}{\gamma} e^{-\gamma x}$. I specify $\gamma > 0$, i.e., the manager exhibits constant absolute risk-aversion.

3.4.2 Firm F's Value Function, Investment Timing Decision, and Certainty Equivalence Value

Assume Firm L has already exercised its option, and thus Firm F may construct its optimal investment policy without fear of pre-emption. Firm F's management may undertake investment at time τ ($\tau \geq$ t), receiving perpetual profit flow $Y_t \times D(2)$. The manager's problem is to maximize expected utility of consumption with respect to stopping time τ and investment strategy θ by hedging partially using the traded asset S and the riskless bond. The dynamics of the wealth process X_t are:

$$dX_t = \theta_t (dS_t / S_t) + r (X_t - \theta_t) dt - C_t dt + [Y_t I_{\{t \geq \tau\}} D(2) - K\delta(t-\tau)] dt$$

$$\delta(t-\tau) = \begin{cases} 1 \text{ if } \tau = t \\ 0 \text{ otherwise} \end{cases}$$

where θ_t is the cash amount invested in the partial spanning asset S, and remaining wealth is invested at riskless rate, r, and C is the consumption rate.

3.4.2 Proposition 1:

The value function for Firm F's investment problem is given by the optimal stopping problem:

$$F(x,y) = \sup_{\theta,C,\tau} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds \mid X_0 = x, Y_0 = y]$$

$$= \sup_{\tau} \sup_{\{C_s,\theta_s,0\le s\le\tau\}} E[\int_0^\tau -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds + e^{-\beta\tau} F_1(X_\tau, Y_\tau) \mid X_0 = x, Y_0 = y]$$

where $F_1(x,y)$ is Firm F's value function after exercising the investment decision.

<u>*Proof*:</u>

(See Appendix)

Firm F's Value Function

I use backward induction to solve Firm F's problem. I first assume that Firm F has already begun receiving the cash flow $Y_t \times D(2)$ and has already paid investment cost K.

The follower maximizes expected utility, given by:

$$F_1(x,y) = \sup_{\theta,C} E[\int_t^\infty -\frac{1}{\gamma} e^{-\beta(s-t)} e^{-\gamma C_s} ds] = \sup_{\theta,C} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds]$$

Proof:

$$F_1(x,y) = \sup_{C,\theta} E[\int_t^{\infty} -\frac{1}{\gamma} e^{-\beta(s-t)} e^{-\gamma C_s} ds];$$

$$= \sup_{\theta,C} E[\int_0^{\infty} -\frac{1}{\gamma} e^{-\beta u} e^{-\gamma C_{t+u}} du]; u = s-t$$

$$= \sup_{\theta,C} E[\int_0^{\infty} -\frac{1}{\gamma} e^{-\beta u} e^{-\gamma C_u} du]; \text{Relabel}\,\{C_{t+u}\}\text{ as}\{C_u\}$$

Fact:

If the process $\{X_t\}_{t\geq 0}$ is time homogeneous, $\{X^{t,x}_{t+u}\}_{u\geq 0}$ and $\{X^{0,x}_u\}_{u\geq 0}$ have the same P^0-distributions. (P^0 is the probability law of Brownian motion, B_t, starting at t = 0).

Define a time homogeneous Ito diffusion process of the form

$dX_t = b(X_t)dt + \sigma(X_t)dB_t$, $s \geq t$; $X_t = x$

satisfying the Lipschitz condition $|b(x)-b(y)|+|\sigma(x)-\sigma(y)| \leq D|x-y|$; D is some constant and $x,y \in R$; therefore, the unique solution $X_s = X^{t,x}_s, s \geq t$ does exist.

Note that

$$X^{t,x}_{t+h} = x + \int_t^{t+h} b(X^{t,x}_u)du + \int_t^{t+h} \sigma(X^{t,x}_u)dB_u$$

$$= x + \int_0^{h} b(X^{t,x}_{t+v})dv + \int_0^{h} \sigma(X^{t,x}_{t+v})d\widetilde{B}_v \quad ; v = u-t \text{ and } \widetilde{B}_v = B_{t+v} - B_t$$

$$X^{0,x}_h = x + \int_0^{h} b(X^{0,x}_v)dv + \int_0^{h} \sigma(X^{0,x}_v)dB_v$$

Since $\{\widetilde{B}_v\}_{v\geq 0}$ and $\{B_v\}_{v\geq 0}$ have the same P^0-distributions, it follows by the weak uniqueness of the solution of the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \ , X_0 = x$$

that $\{X_{t+h}^{t,x}\}_{h\geq 0}$ and $\{X_h^{0,x}\}_{h\geq 0}$ have the same P^0-distributions, equivalently, $\{X_t\}_{t\geq 0}$ is time homogeneous.

Therefore, the manager's problem is:

$$F_1(x,y) = \sup_{\theta,C} E[\int_0^{\infty} -\frac{1}{\gamma}e^{-\beta s}e^{-\gamma C_s}ds]$$

Subject to

$Y_0 = y$, $X_0 = x$

$dX_t = \theta_t (dS_t / S_t) + r (X_t - \theta_t) dt + Y_t D(2) dt - C_t dt$

The Bellman Equation (i.e., Hamilton, Jacobi, Bellman equation) associated with the value function $F_1(x,y)$ is:

$$\beta F_1(x,y) = \sup_{\theta,C}[-\frac{1}{\gamma}e^{-\gamma C_t} + F_{1x}(\theta_t\mu + r(x-\theta_t) + yD(2) - C_t)$$
$$+ F_{1y}\alpha y + \frac{1}{2}F_{1xx}\theta_t^2\chi^2 + \frac{1}{2}F_{1yy}\sigma^2 y^2 + F_{1xy}\rho\sigma\chi y\theta_t]$$

Taking first-order conditions for optimal consumption C and investment strategy θ, I obtain:

$$C_t^* = -\frac{1}{\gamma}\ln F_{1x}$$

$$F_{1x}\mu - F_{1x}r + F_{1xx}\theta_t\chi^2 + F_{1xy}\rho\chi y\sigma = 0$$

$$\theta_t^* = -\frac{F_{1x}(\mu - r)}{F_{1xx}\chi^2} - \frac{F_{1xy}\rho\sigma y}{F_{1xx}\chi}$$

Incorporating first order conditions in the Bellman equation gives

$$\begin{aligned}\beta F_1(x,y) = [&-\frac{1}{\gamma}e^{-\gamma C} + F_{1x}((-\frac{F_{1x}(\mu - r)}{F_{1xx}\chi^2} - \frac{F_{1xy}\rho\sigma y}{F_{1xx}\chi})\mu + \\ &r(x - (-\frac{F_{1x}(\mu - r)}{F_{1xx}\chi^2} - \frac{F_{1xy}\rho\sigma y}{F_{1xx}\chi})) + yD(2)) + \frac{F_{1x}}{\gamma}\ln F_{1x} \\ &+ F_{1y}\alpha y + \frac{1}{2}F_{1xx}(-\frac{F_{1x}(\mu - r)}{F_{1xx}\chi^2} - \frac{F_{1xy}\rho\sigma y}{F_{1xx}\chi})^2\chi^2 \\ &+ \frac{1}{2}F_{1yy}\sigma^2 y^2 + F_{1xy}\rho\sigma y(-\frac{F_{1x}(\mu - r)}{F_{1xx}\chi^2} - \frac{F_{1xy}\rho\sigma y}{F_{1xx}\chi})\chi]\end{aligned}$$

By solving the above equation coupled with transversality condition

$\lim_{T\to\infty} E[e^{-\beta T}e^{-r\gamma W_T - \gamma Y_T D(2)}] = 0$,

I conjecture that the value function takes the form $F_1(x,y) = -\frac{1}{r\gamma}\exp[-r\gamma(x + f(y) + \frac{\eta^2}{2r^2\gamma} + \frac{\beta - r}{r^2\gamma})]]$[18] where $f(y)$ can be interpreted as the implied project value through certainty equivalence value and $f(y)$ solves the following equation

$$(\alpha - \rho\eta\sigma)yf'(y) + \frac{\sigma^2}{2}y^2 f''(y) - \frac{r\gamma\sigma^2}{2}(1-\rho^2)y^2 f'(y)^2 - rf(y) + yD(2) = 0$$

with transversality condition

$\lim_{T\to\infty} E[e^{-\beta T}e^{-r\gamma W_T - \gamma Y_T D(2)}] = 0$

[18] The form of the ansatz solution is based on the form of Merton (1969) solution with the variable separable property of the exponential function.

I now solve the complete problem as follows. Firm F's utility maximization problem is:

$$F\ (x,y) = \sup_{\theta_t,\tau} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds \mid X_0 = x, Y_0 = y]$$

subject to

$dX_t = \theta_t\ (dS_t / S_t) + r\ (X_t - \theta_t)\ dt + [Y_t I_{\{t \geq \tau\}} D(2) - K\delta(t-\tau) - C_t]\ dt$

$$\delta(t-\tau) = \begin{cases} 1 \text{ if } \tau = t \\ 0 \text{ otherwise} \end{cases}$$

By making use of $F_1(x,y)$, I can write the complete problem as:

$$F(x,y) = \sup_{\theta,C,\tau} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds \mid X_0 = x, Y_0 = y]$$

$$= \sup_{\tau} \sup_{\{\theta_s, C, 0 \leq s \leq \tau\}} E[\int_0^\tau -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds + e^{-\beta\tau} F_1(X_\tau, Y_\tau) \mid X_0 = x, Y_0 = y]$$

The Bellman equation associated with the value function $F(x,y)$ is:

$$\beta F(x,y) = \sup_{\theta,C}[-\frac{1}{\gamma} e^{-\gamma C} + F_x(\theta_t \mu + r(x-\theta_t) - C_t) + F_y \alpha y + \frac{1}{2} F_{xx} \theta_t^2 \chi^2 + \frac{1}{2} F_{yy} \sigma^2 y^2 + F_{xy} \rho\sigma y \theta_t \chi]$$

Taking first-order conditions for solving optimal consumption C and investment strategy θ, I obtain:

$$C_t^* = -\frac{1}{\gamma} \ln F_x$$

$$\theta_t^* = -\frac{F_x(\mu - r)}{F_{xx}\chi^2} - \frac{F_{xy}\rho\sigma y}{F_{xx}\chi}$$

I conjecture that the value function takes the form $F(x,y) = -\frac{1}{r\gamma}\exp[-r\gamma(x+g(y)+\frac{\eta^2}{2r^2\gamma}+\frac{\beta-r}{r^2\gamma})]]$ then g(y) may be viewed as the certainty equivalent value of the option value to invest. Using the conjectured solution for C_t^* and θ_t^* and substituting into H-J-B equation, I obtain

$$rg(y) = (\alpha - \rho\eta\sigma)yg'(y) + \frac{\sigma^2}{2}y^2 g''(y) - \frac{r\gamma\sigma^2}{2}(1-\rho^2)y^2 g'(y)^2$$

subject to

$\lim_{y\to-\infty} g(y) = 0$ (absorbing barrier)

$g'(y^F) = f'(y^F)$ (smooth pasting)

$g(y^F) = f(y^F) - K$ (value matching)

The option value to invest, $g(y)$, is solved by invoking certainty equivalence and y^F is the investment trigger for Firm F's optimal entry.

3.4.3 Firm L's Value Function, Investment Timing Decision, and Certainty Equivalence Value

Firm L's management expects to receive profit flow $Y_t \times D(1)$ after undertaking investment and prior to Firm F's entry; after Firm F's entry, Firm L receives perpetual profit flow $Y_t \times D(2)$. The manager's problem is to maximize her expected utility of consumption with the investment strategy θ by hedging partially using the traded asset S and the riskless bond. The dynamics of wealth process X_t are:

$$dX_t = \theta_t (dS_t / S_t) + r (X_t - \theta_t)\, dt + [Y_t I_{\{t<\tau^F\}} D(1) + Y_t I_{\{t\geq\tau^F\}} D(2)$$
$$-K\delta(t-T) - C_t]\, dt$$

$$\delta(t-T)=\begin{cases} 1 \text{ if t = T} \\ 0 \text{ otherwise} \end{cases}$$

where T is the time when Firm L undertakes investment, C_t is the consumption process, and θ_t is the cash amount invested in the partial spanning asset S, and remaining wealth is invested at riskless rate, r.

I first assume that the manager has already invested K, has already started to receive the cash flow $Y_t \times D(1)$, and expects to receive $Y_t \times D(2)$ forever following Firm F's entry at time τ^F.

3.4.3 Proposition 1:

Firm L manager's problem is to maximize her expected utility of consumption with the investment strategy θ by hedging partially using the traded asset S and the riskless bond conditional on Firm F's optimal entry, at time τ^F, where $\tau^F = \inf\{t : Y(t) \geq y^F\}$ and y^F is the investment trigger for Firm F. That is,

$$L(x,y) = \sup_{\theta,C} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds]$$
$$= \sup_{\{C_s,\theta_s,0\leq s\leq\tau\}} E[\int_0^{\tau^F} -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds + e^{-\beta\tau^F} F_1(x,y)]$$

Subject to

$Y_0 = y$, $X_0 = x$

$dX_t = \theta_t (dS_t / S) + r (X_t - \theta_t)\, dt + [Y_t I_{\{t<\tau^F\}} D(1) + Y_t I_{\{t\geq\tau^F\}} D(2) - C]\, dt$

Remarks

The second equality derives from Lemma for the proof of 3.4.2 Proposition 1 (see Appendix) and the fact that after the follower enters, the two firms share the market and hence the value function if homogeneous expectation/utility function is assumed.

Firm L's Value Function

Assume Firm L has already entered the market and given $Y(t) \le y^F$, I then solve the equation by using H-J-B, the standard arguments yield:

$$\beta L(x,y) = \sup_{\theta,C}[-\frac{1}{\gamma}e^{-\beta t}e^{-\gamma C_s} + L_{1x}(\theta_t\mu + r(x-\theta_t) + yD(1) - C_t) + L_{1y}\alpha y + \frac{1}{2}L_{1xx}\theta_t^2\chi^2$$
$$+\frac{1}{2}L_{1yy}\sigma^2 y^2 + L_{1xy}\rho\sigma y\theta_t\chi]$$

Solve for $L(x,y)$

Taking first-order conditions for solving optimal consumption C and investment strategy θ, I obtain:

$$C_t^* = -\frac{1}{\gamma}(\ln L_{1x})$$

$$\theta_t^* = -\frac{L_{1x}(\mu - r)}{L_{1xx}\chi^2} - \frac{L_{1xy}\rho\sigma y}{L_{1xx}\chi}$$

I conjecture that the value function takes the form $L(x,y) = -\frac{1}{r\gamma}\exp[-r\gamma(x + h(y) + \frac{\eta^2}{2r^2\gamma} + \frac{\beta - r}{r^2\gamma})]$ where $h(y)$ has the interpretation as the certainty equivalence value, and solves the following non-linear second order ODE

$$(\alpha-\rho\eta\sigma)y\frac{\partial}{\partial y}h(y)+\frac{\sigma^2}{2}y^2\frac{\partial^2}{\partial y\partial y}h(y)-\frac{r\gamma\sigma^2}{2}(1-\rho^2)y^2(\frac{\partial}{\partial y}h(y))^2-rh(y)$$
$$+yD(1)=0$$

with $h(y^F)=f(y^F)$

where $f(y)$ solves the followig equation (see 3.4.2 Firm F's value function for details)

$$(\alpha-\rho\eta\sigma)yf'(y)+\frac{\sigma^2}{2}y^2f''(y)-\frac{r\gamma\sigma^2}{2}(1-\rho^2)y^2f'(y)^2-rf(y)+yD(2)=0$$

with transversality condition

$$\lim_{T\to\infty}E[e^{-\beta T}e^{-r\gamma W_T-\gamma Y_T D(2)}]=0$$

and y^F is the optimal investment trigger for Firm F's entry.

The investment trigger for Firm L occurs at the point where value function for Firm L is equal to the value function for Firm F with $y<y^F$. That is, $h(y^L)-K=g(y^L)$, where $g(y)$ is the Firm F's option value to invest and y^L is Firm L's investment trigger value.

3.4.4 Model Results Assuming Perfect Spanning (Complete Market)

I consider a complete market version of the model to create a "benchmark" for comparison to the incomplete market cases. The market is complete if either perfect spanning holds, i.e., uncertainty over the income stream, Y, may be replicated by asset S (perfect correlation with Y), or equivalently, Y itself is traded. Such a complete market version of the model creates a "benchmark" for comparison to the incomplete market case.

3.4.4.1 Firm F's Value Function and Investment Timing Decision

Assuming Firm L has already entered the market, Firm F will enter the market optimally without fear of pre-emption. Under the complete market, Firm F's value function and investment trigger can thus be obtained through the following propositions.

3.4.4.1 Proposition 1: (Y itself is traded.)

Firm F manager's value function may be written as:

$$F^C(y) = \begin{cases} [\dfrac{y^F D(2)}{r-\alpha} - K](\dfrac{y}{y^F})^\beta ; y \in [0, y^F] \\ \dfrac{yD(2)}{r-\alpha} - K ; y \in [y^F, \infty] \end{cases}$$

where $\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{\beta}{(\beta-1)} \times \frac{r-\alpha}{D(2)} \times K$

y^F is the solution to the free boundary. Thus Firm F's optimal policy is to invest the first time Y reaches the threshold y^F.

<u>*Proof:*</u>

Following standard arguments, the corresponding Bellman equation for Firm F is

$$\alpha y F_y^C + \frac{1}{2}\sigma^2 y^2 F_{yy}^C - rF^C = 0$$

I solve the above ODE with following boundary, value matching, and smooth pasting conditions:

$$F^C(0) = 0$$

$$F^C(y^F) = \frac{y^F D(2)}{r-\alpha} - K$$

$$F_y^C(y^F) = \frac{D(2)}{r-\alpha}$$

The value matching condition is obtained by the fact that after investment Firm F's expected project value is the discounted expected present value of the duopoly cashflow, YD(2), in perpetuity. That is, assuming that the firm stops at time t, the value of the project equals:

$$\begin{aligned}
V^C(y) &= E[\int_t^\infty e^{-r(s-t)} Y(s) D(2) ds] \\
&= E[\int_t^\infty e^{-r(s-t)} Y(t) e^{(\alpha - \frac{1}{2}\sigma^2)(s-t)+\sigma(W(s-t))} D(2) ds] \\
&= E[\int_0^\infty e^{-ru} Y(t) e^{(\alpha - \frac{1}{2}\sigma^2)u+\sigma(W(u))} D(2) du]; u = s-t \\
&= Y(t) D(2) E[\int_0^\infty e^{-ru} e^{(\alpha - \frac{1}{2}\sigma^2)u+\sigma(W(u))} du] \\
&= \frac{Y(t)D(2)}{r-\alpha}
\end{aligned}$$

The optimal stopping time τ^F is given by τ^F = inf { t : $Y_t \geq y^F$ }. I propose a solution of the form $F^C(y) = Ay^\beta$ where A is a constant to be determined.

Because $F^C(y) = Ay^\beta$, it immediately follows that $F_y^C = A\beta y^{(\beta-1)}$ and $F_{yy}^C = A\beta(\beta-1)y^\beta$. Substituting into the Bellman equation, $\alpha y F_y^C + \frac{1}{2}\sigma^2 y^2 F_{yy}^C - rF^C = 0$, the Bellman equation becomes

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r = 0$$

Solving for β in the above characteristic equation yields

$$\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} > 1 \text{ or}$$

$$\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} < 0$$

The solution $\beta < 0$ may be rejected given the boundary conditions, and the solution is of the form $F^C(y) = Ay^{\beta}$ with $\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$. The constant A and investment trigger value y^F may be determined by invoking value matching and smooth pasting conditions, yielding Firm F's value function and trigger value:

$$F^C(y) = [\frac{y^F D(2)}{r-\alpha} - K](\frac{y}{y^F})^{\beta},$$

$$where\, \beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} \text{ and } y^F = \frac{\beta}{(\beta-1)} \times \frac{r-\alpha}{D(2)} \times K$$

3.4.4.1 Proposition 2: (The perfect spanning asset, S, exists.)

Firm F manager's value function may be written as:

$$F^C(y) = \begin{cases} [\dfrac{y^F D(2)}{r-(\alpha-\sigma\eta)} - K](\dfrac{y}{y^F})^{\beta}; y \in [0, y^F] \\ \dfrac{yD(2)}{r-(\alpha-\sigma\eta)} - K; y \in [y^F, \infty] \end{cases}$$

where $\beta = \frac{1}{2} - \frac{(\alpha-\eta\sigma)}{\sigma^2} + \sqrt{(\frac{(\alpha-\eta\sigma)}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$

and $y^F = \frac{\beta}{(\beta-1)} \times \frac{r-(\alpha-\eta\sigma)}{D(2)} \times K$

y^F is the solution to the free boundary. Thus Firm F's optimal policy is to invest the first time Y reaches the threshold y^F.

Proof:

Following standard arguments, the corresponding Bellman equation for Firm F is

$$(\alpha - \sigma\eta)yF_y^C + \frac{1}{2}\sigma^2 y^2 F_{yy}^C - rF^C = 0$$

I solve the above ODE with following boundary, value matching, and smooth pasting conditions:

$$F^C(0) = 0$$

$$F^C(y^F) = \frac{y^F D(2)}{r - (\alpha - \eta\sigma)} - K$$

$$F_y^C(y^F) = \frac{D(2)}{r - (\alpha - \eta\sigma)}$$

The value matching condition is obtained by the fact that after investment Firm F's expected project value is the discounted expected present value of the duopoly cashflow, YD(2), in perpetuity. That is, assuming that the firm stops at time t, the value of the project equals:

$$
\begin{aligned}
V^C(y) &= E^Q[\int_t^\infty e^{-r(s-t)}Y(s)D(2)ds] \\
&= E^Q[\int_t^\infty e^{-r(s-t)}Y(t)e^{(\alpha-\sigma\eta-\frac{1}{2}\sigma^2)(s-t)+\sigma(Z^Q(s-t))}D(2)ds] \\
&= E^Q[\int_0^\infty e^{-ru}Y(t)e^{(\alpha-\sigma\eta-\frac{1}{2}\sigma^2)u+\sigma(Z^Q(u))}D(2)du]; u = s-t \\
&= Y(t)D(2)E^Q[\int_0^\infty e^{-ru}e^{(\alpha-\sigma\eta-\frac{1}{2}\sigma^2)u+\sigma(Z^Q(u))}du] \\
&= \frac{Y(t)D(2)}{r-(\alpha-\sigma\eta)}
\end{aligned}
$$

where E^Q denotes expectation with respect to unique martingale measure Q, defined as follows. For each t < ∞, the Radon-Nikodym density of Q with respect to the historical measure P is defined as:

$$\frac{dQ}{dP}|_{F_t} = \exp(-\eta Z_t - \frac{1}{2}\eta^2 t)$$

Under Q, $\frac{dS_t}{S_t} = rdt + \chi dZ_t^Q, where Z_t^Q = Z_t + \eta t$ is a Q-Brownian motion and the project stochastic demand follows:

$$dY_t = (\alpha - \eta\sigma)Y_t dt + \sigma Y_t dZ_t^Q$$

The optimal stopping time τ^F is given by τ^F = inf { t : $Y_t \geq y^F$ }. I propose a solution of the form $F^C(y) = Ay^\beta$ where A is a constant to be determined.

Because $F^C(y) = Ay^\beta$, it immediately follows that $F_y^C = A\beta y^{(\beta-1)}$ and $F_{yy}^C = A\beta(\beta-1)y^\beta$. Substituting into the Bellman equation, $(\alpha-\sigma\eta)yF_y^C + \frac{1}{2}\sigma^2 y^2 F_{yy}^C - rF^C = 0$, the Bellman equation becomes

$$\frac{1}{2}\sigma^2\beta(\beta-1)+(\alpha-\sigma\eta)\beta-r=0$$

Solving for β in the above characteristic equation yields

$$\beta=\frac{1}{2}-\frac{(\alpha-\eta\sigma)}{\sigma^2}+\sqrt{(\frac{(\alpha-\eta\sigma)}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}>1 \text{ or}$$

$$\beta=\frac{1}{2}-\frac{(\alpha-\eta\sigma)}{\sigma^2}-\sqrt{(\frac{(\alpha-\eta\sigma)}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}<0$$

The solution β < 0 may be rejected given the boundary conditions, and the solution is of the form $F^C(y)=Ay^\beta$ with $\beta=\frac{1}{2}-\frac{(\alpha-\eta\sigma)}{\sigma^2}+\sqrt{(\frac{(\alpha-\eta\sigma)}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}$. The constant A and investment trigger value y^F may be determined by invoking value matching and smooth pasting conditions, yielding Firm F's value function and trigger value:

$$F^C(y)=[\frac{y^F D(2)}{r-(\alpha-\eta\sigma)}-K](\frac{y}{y^F})^\beta$$

$$,where\ \beta=\frac{1}{2}-\frac{(\alpha-\eta\sigma)}{\sigma^2}+\sqrt{(\frac{(\alpha-\eta\sigma)}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}} \text{ and } y^F=\frac{\beta}{(\beta-1)}\times\frac{r-(\alpha-\eta\sigma)}{D(2)}\times K$$

3.4.4.2 Firm L's Value Function and Investment Timing Decision

By design Firm L undertakes investment prior to Firm F's entry. Firm L makes its investment decision conditional on Firm F acting optimally according to the optimal stopping rule described above.

3.4.4.2 Proposition 1: (Y itself is traded.)

Firm L's value function is:

$$L^C(y) = \begin{cases} \dfrac{yD(1)}{r-\alpha} + [\dfrac{y^F D(2)}{r-\alpha} - \dfrac{y^F}{r-\alpha}](\dfrac{y}{y^F})^\beta - K; y \in [0, y^F] \\ \dfrac{yD(2)}{r-\alpha} - K; y \in [y^F, \infty] \end{cases}$$

where $\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{\beta}{(\beta-1)} \times \frac{r-\alpha}{D(2)} \times K$

<u>*Proof:*</u>

Following standard arguments, with the assumption that Firm L has already entered the market and given $y < y^F$, the corresponding Bellman equation for Firm L is

$$\alpha y L_y^C + \frac{1}{2}\sigma^2 y^2 L_{yy}^C - rL^C + yD(1) = 0$$

I solve the above ODE with following boundary and value matching conditions:

$$L^C(y^F) = \frac{y^F D(2)}{r-\alpha}$$

I propose a solution of the form $L^C(y) = Ay^\beta$ for the homogeneous part, and a particular solution for the non-homogenous part, where A is a constant to be determined. The Bellman equation for the homogenous part is (derivation is the same as the previous section):

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r = 0$$

Solving for β in the above characteristic equation yields

$$\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} > 1 \text{ or}$$

$$\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} < 0$$

The solution $\beta < 0$ may be rejected given the boundary condition. One candidate particular solution is $\frac{yD(1)}{r-\alpha}$. Therefore, the solution should be the form of $L^C(y) = Ay^\beta$ for the homogenous part with $\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$, plus particular solution, $\frac{yD(1)}{r-\alpha}$. From the value matching condition, A may be determined, yielding Firm L's value function:

$$L^C(y) = \frac{yD(1)}{r-\alpha} + [\frac{y^F D(2)}{r-\alpha} - \frac{y^F}{r-\alpha}](\frac{y}{y^F})^\beta - K$$

where $\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{\beta}{(\beta-1)} \times \frac{r-\alpha}{D(2)} \times K$.

3.4.4.2 Proposition 2: (Y itself is traded.)

Firm L's investment trigger value y^L is the solution to the following equation:

$$\frac{y^L D(1)}{r-\alpha} + [\frac{y^F D(2)}{r-\alpha} - \frac{y^F}{r-\alpha}](\frac{y^L}{y^F})^\beta - K = [\frac{y^F D(2)}{r-\alpha} - K](\frac{y^L}{y^F})^\beta$$

where $\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{\beta}{(\beta-1)} \times \frac{r-\alpha}{D(2)} \times K$

Remarks:

The above proposition makes use of the fact that Firm L's trigger value, y^L, yields the same value for both firms with $y < y^F$. That is, $L^C(y^L)$ should be equal to $F^C(y^L)$.

3.4.4.2 Proposition 3: (The perfect spanning asset, S, exists.)

Firm L's value function is:

$$L^C(y) = \begin{cases} \dfrac{yD(1)}{r-(\alpha-\eta\sigma)} + [\dfrac{y^F D(2)}{r-(\alpha-\eta\sigma)} - \dfrac{y^F}{r-(\alpha-\eta\sigma)}](\dfrac{y}{y^F})^{\beta} - K; y \in [0, y^F] \\ \dfrac{yD(2)}{r-(\alpha-\eta\sigma)} - K; y \in [y^F, \infty] \end{cases}$$

where $\beta = \dfrac{1}{2} - \dfrac{(\alpha-\eta\sigma)}{\sigma^2} + \sqrt{(\dfrac{(\alpha-\eta\sigma)}{\sigma^2} - \dfrac{1}{2})^2 + \dfrac{2r}{\sigma^2}}$

and $y^F = \dfrac{\beta}{(\beta-1)} \times \dfrac{r-(\alpha-\eta\sigma)}{D(2)} \times K$

<u>*Proof:*</u>

Following standard arguments, with the assumption that Firm L has already entered the market and given $y < y^F$, the corresponding Bellman equation for Firm L is

$$(\alpha-\eta\sigma)yL_y^C + \frac{1}{2}\sigma^2 y^2 L_{yy}^C - rL^C + yD(1) = 0$$

I solve the above ODE with following boundary and value matching conditions:

$$L^C(y^F) = \frac{y^F D(2)}{r-(\alpha-\eta\sigma)}$$

I propose a solution of the form $L^C(y) = Ay^\beta$ for the homogeneous part, and a particular solution for the non-homogenous part, where A is a constant to be determined. The Bellman equation for the homogenous part is (derivation is the same as the previous section):

$$\frac{1}{2}\sigma^2\beta(\beta-1)+(\alpha-\sigma\eta)\beta-r=0$$

Solving for β in the above characteristic equation yields

$$\beta=\frac{1}{2}-\frac{(\alpha-\eta\sigma)}{\sigma^2}+\sqrt{(\frac{(\alpha-\eta\sigma)}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}>1 \text{ or}$$

$$\beta=\frac{1}{2}-\frac{(\alpha-\eta\sigma)}{\sigma^2}-\sqrt{(\frac{(\alpha-\eta\sigma)}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}<0$$

The solution $\beta < 0$ may be rejected given the boundary condition. One candidate particular solution is $\frac{yD(1)}{r-(\alpha-\sigma\eta)}$. Therefore, the solution should be the form of $L^C(y) = Ay^\beta$ for the homogenous part with $\beta=\frac{1}{2}-\frac{(\alpha-\eta\sigma)}{\sigma^2}+\sqrt{(\frac{(\alpha-\eta\sigma)}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}$, plus particular solution, $\frac{yD(1)}{r-(\alpha-\eta\sigma)}$. From the value matching condition, A may be determined, yielding Firm L's value function:

$$L^C(y)=\frac{yD(1)}{r-(\alpha-\eta\sigma)}+[\frac{y^F D(2)}{r-(\alpha-\eta\sigma)}-\frac{y^F}{r-(\alpha-\eta\sigma)}](\frac{y}{y^F})^\beta-K$$

where $\beta=\frac{1}{2}-\frac{(\alpha-\eta\sigma)}{\sigma^2}+\sqrt{(\frac{(\alpha-\eta\sigma)}{\sigma^2}-\frac{1}{2})^2+\frac{2r}{\sigma^2}}$

and $y^F = \dfrac{\beta}{(\beta-1)} \times \dfrac{r-(\alpha-\eta\sigma)}{D(2)} \times K$.

3.4.4.2 Proposition 4: (The perfect spanning asset, S, exists.)

Firm L's investment trigger value y^L is the solution to the following equation:

$$\frac{y^L D(1)}{r-(\alpha-\eta\sigma)} + [\frac{y^F D(2)}{r-(\alpha-\eta\sigma)} - \frac{y^F}{r-(\alpha-\eta\sigma)}](\frac{y^L}{y^F})^\beta - K =$$

$$[\frac{y^F D(2)}{r-(\alpha-\eta\sigma)} - K](\frac{y^L}{y^F})^\beta$$

where $\beta = \dfrac{1}{2} - \dfrac{(\alpha-\eta\sigma)}{\sigma^2} + \sqrt{(\dfrac{(\alpha-\eta\sigma)}{\sigma^2} - \dfrac{1}{2})^2 + \dfrac{2r}{\sigma^2}}$

and $y^F = \dfrac{\beta}{(\beta-1)} \times \dfrac{r-(\alpha-\eta\sigma)}{D(2)} \times K$

Remarks:

The above proposition makes use of the fact that Firm L's trigger value, y^L, yields the same value for both firms with $y < y^F$. That is, $L^C(y^L)$ should be equal to $F^C(y^L)$.

3.5 Optimal Policies for Duopolistic Competition by Project Income Stream (Flow Investment Payoffs II – Income Stream with Arithmetic Brownian Motion)

The previous section showed that modeling a stochastic investment payoff through a multiplicative demand shock following a geometric Brownian motion process yields many ordinary differential equation approximations. The chosen approximation must be carefully examined to ensure stability and convergence. Motivated by Miao and Wang (2005) and Henderson (2005), I model the stochastic investment payoff as an arithmetic Brownian motion process. Negative values from the arithmetic Brownian motion may be interpreted as a loss generated from operations.

As before, two competing firms are contemplating entry into a new market, and I identify the firms as the Leader (Firm L) and Follower (Firm F), respectively. Firm L enters the market by investing to receive a monopoly rent until Firm F enters the market. Upon entry by Firm F, I assume that Firm L obtains a fraction $a \in [½,1]$ of project income, leaving $(1-a)=a2$ of project income for Firm F.

3.5.1 Model Set-Up and Assumptions

I fix a filtered probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a fixed σ-algebra $\mathcal{G}\subseteq\mathcal{F}$, where

Brownian motion is defined and the expectation $\mathbb{E}\{\bullet\}$ is computed. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the augmented filtration of Brownian motion. The increasing σ-algebras generated by the pair of Brownian motions $(Z_s)_{s\leq t}$ and $(Z^{\perp}_s)_{s\leq t}$, where $Z^{\perp}$ is orthogonal to Z, satisfy the usual conditions of right-continuity and completeness (i.e., include all the sets of probability 0 in $\mathcal{F}$[19]). Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be filtration generated by Z alone.

Non-traded Assets (The Investment Income)

Project cash flows Y(t) follows the arithmetic Brownian motion[20]:

[19] The completion by the null sets is important in particular for the following reason. If two random variables X and Y are equal almost surely (X=Y P-a.s. means P{X=Y}=1) and if X is $\mathcal{F}_t$ measurable (meaning that any event $\{X_t \leq x\}$ belongs to $\mathcal{F}_t$) then Y is also $\mathcal{F}_t$-measurable.

[20] Assume it follows certain regularity conditions to guarantee the uniqueness and existence solution to the Y_t process.

$dY_t = \alpha dt + \sigma dW_t$; $Y_t = y$

where α is the drift rate per unit time, σ is the instantaneous volatility, and W is a standard Brownian motion having correlation $\rho \in (-1,1)$ with Z. For this, I can take W = ρZ + $\sqrt{1-\rho^2} Z^{\perp}$, or equivalently, dW = ρ dZ + $\sqrt{1-\rho^2}\, dZ^{\perp}$.

Traded Risky Security

There exists a partial spanning asset which follows the lognormal process21:

$dS_t = \mu S_t\, dt + \chi S_t\, dZt = \chi S_t (\eta dt + dZt) + r S_t dt$; $S_t = s$

where μ is the instantaneous conditional expected percentage change in S per unit time, $\eta = \frac{\mu - r}{\chi}$ is its Sharpe ratio, χ is the instantaneous volatility, and Z is a standard Brownian motion.

Traded Riskless Security

I also assume that a riskless bond B is available for trading. The riskless bond with price process B satisfies the following dynamics:

$dB_t = rB_t dt$; $B_t = b$

Utility Function

Realizing that markets are incomplete, the firm's manager attempts to maximize her expected utility. That is,

Maximize $E[\int_0^{\infty} e^{-\beta t} U(X_t) dt]$

[21] Again, I can have the process follows more general forms satisfying all required technical conditions as described in footnote 2 for the guarantee of uniqueness and existence solution to S_t process.

The utility function is a concave function U: $\mathbb{R} \rightarrow [-\infty,\infty)$, which is assumed to be strictly increasing, strictly concave, and continuously differentiable on its domain satisfying:

(i) The half-line dom(U) = $\{x \in R; U(x) > -\infty\}$ is a non-empty subset of $[0,\infty)$.

(ii) $U'(x)$ is continuous, positive, and strictly decreasing on the interior of dom(U), and satisfy the Inada condition:

$$U'(\infty) = \lim_{x \to \infty} U'(x) = 0$$

The standard CARA, CRRA and HARA utility functions satisfy the above properties. Throughout the study, I assume the manager has the exponential utility function $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$. I specify $\gamma > 0$, i.e., the manager exhibits constant absolute risk-aversion.

3.5.2 Firm F's Value Function, Investment Timing Decision, and Certainty Equivalence Value

Assume Firm L has already exercised its option, and thus Firm F may construct its optimal investment policy without fear of pre-emption. Firm F's management may undertake investment at time τ ($\tau \geq t$), receiving perpetual profit flow $Y_t \times (1\text{-}a)$. The manager's problem is to maximize expected utility of consumption with respect to stopping time τ and investment strategy θ by hedging partially using the traded asset S and the riskless bond. The dynamics of the wealth process X_t are:

$$dX_t = \theta_t\,(dS_t / S_t) + r\,(X_t - \theta_t)\,dt - C_t dt + [(1-a)Y_t I_{\{t \geq \tau\}} - K\delta(t-\tau)]\,dt$$

$$\delta(t-\tau) = \begin{cases} 1 \text{ if } \tau = t \\ 0 \text{ otherwise} \end{cases}$$

where θ_t is the cash amount invested in the partial spanning asset S, and remaining wealth is invested at riskless rate, r, and C is the consumption rate.

3.5.2 Proposition 1:

The value function for Firm F's investment problem is given by the optimal stopping problem:

$$F(x,y) = \sup_{\theta,C,\tau} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds \mid X_0 = x, Y_0 = y]$$
$$= \sup_{\tau} \sup_{\{C_s,\theta_s,0\le s\le\tau\}} E[\int_0^\tau -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds + e^{-\beta\tau} F_1(X_\tau, Y_\tau) \mid X_0 = x, Y_0 = y]$$

where $F_1(x,y)$ is Firm F's value function after exercising the investment decision.

<u>*Proof*:</u>

(See Appendix)

Firm F's Value Function

I use backward induction to solve Firm F's problem. I first assume that Firm F has already begun receiving the cash flow $Y_t \times (1\text{-}a)$ and has already paid investment cost K.

The follower maximizes expected utility, given by:

$$F_1(x,y) = \sup_{\theta,C} E[\int_t^\infty -\frac{1}{\gamma} e^{-\beta(s-t)} e^{-\gamma C_s} ds] = \sup_{\theta,C} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds]$$

Proof:

$$F_1(x,y) = \sup_{C,\theta} E_{t,x}[\int_t^\infty -\frac{1}{\gamma} e^{-\beta(s-t)} e^{-\gamma C_s} ds];$$

$$= \sup_{\theta,C} E_x[\int_0^\infty -\frac{1}{\gamma} e^{-\beta u} e^{-\gamma C_{t+u}} du]; u = s-t$$

$$= \sup_{\theta,C} E_x[\int_0^\infty -\frac{1}{\gamma} e^{-\beta u} e^{-\gamma C_u} du]; \text{Relabel } \{C_{t+u}\} \text{ as } \{C_u\}$$

Therefore, the manager's problem is:

$$F_1(x,y) = \sup_{\theta} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds]$$

Subject to

$Y_0 = y$, $X_0 = x$

$dX_t = \theta_t (dS_t / S_t) + r (X_t - \theta_t) dt + (1-a) Y_t dt - C_t dt$

The Bellman Equation (i.e., Hamilton, Jacobi, Bellman equation) associated with the value function $F_1(x,y)$ is:

$$\beta F_1(x,y) = \sup_{\theta,C}[-\frac{1}{\gamma} e^{-\gamma C_t} + F_{1x}(\theta_t \mu + r(x-\theta_t) + (1-a)y - C_t)$$
$$+ F_{1y}\alpha + \frac{1}{2} F_{1xx}\theta_t^2 \chi^2 + \frac{1}{2} F_{1yy}\sigma^2 + F_{1xy}\rho\chi\theta_t\sigma]$$

Taking first-order conditions for optimal consumption C and investment strategy θ, I obtain:

$$C_t^* = -\frac{1}{\gamma} \ln F_{1x}$$

$$F_{1x}\mu - F_{1x}r + F_{1xx}\theta_t \chi^2 + F_{1xy}\rho\chi\sigma = 0$$

$$\theta_t^* = -\frac{F_{1x}(\mu - r)}{F_{1xx}\chi^2} - \frac{F_{1xy}\rho\sigma}{F_{1xx}\chi}$$

Incorporating first order conditions in the Bellman equation gives

$$\begin{aligned}\beta F_1(x,y) = [&-\frac{1}{\gamma}e^{-\gamma C} + F_{1x}((-\frac{F_{1x}(\mu-r)}{F_{1xx}\chi^2} - \frac{F_{1xy}\rho\sigma}{F_{1xx}\chi})\mu + \\ & r(x - (-\frac{F_{1x}(\mu-r)}{F_{1xx}\chi^2} - \frac{F_{1xy}\rho\sigma}{F_{1xx}\chi})) + (1-a)y) + \frac{F_{1x}}{\gamma}\ln F_{1x} \\ & + F_{1y}\alpha + \frac{1}{2}F_{1xx}(-\frac{F_{1x}(\mu-r)}{F_{1xx}\chi^2} - \frac{F_{1xy}\rho\sigma}{F_{1xx}\chi})^2\chi^2 + \frac{1}{2}F_{1yy}\sigma^2 \\ & + F_{1xy}\rho\chi(-\frac{F_{1x}(\mu-r)}{F_{1xx}\chi^2} - \frac{F_{1xy}\rho\sigma}{F_{1xx}\chi})\sigma]\end{aligned}$$

By solving the above equation coupled with transversality condition

$\lim_{T\to\infty} E[e^{-\beta T} e^{-r\gamma W_T - \gamma Y_T D(2)}] = 0$,

I conjecture that the value function takes the form $F_1(x,y) = -\frac{1}{r\gamma}\exp[-r\gamma(x + f(y) + \frac{\eta^2}{2r^2\gamma} + \frac{(\beta - r)}{r^2\gamma})]$[22] where $f(y)$ can be interpreted as the implied project value through certainty equivalence value and $f(y)$ solves the followig equation

$$(\alpha - \rho\eta\sigma)f'(y) + \frac{\sigma^2}{2}y^2 f''(y) - \frac{r\gamma\sigma^2}{2}(1-\rho^2)f'(y)^2 - rf(y) + (1-a)y = 0$$

with transversality condition

$\lim_{T\to\infty} E[e^{-\beta T} e^{-r\gamma W_T - \gamma Y_T D(2)}] = 0$

[22] The form of ansatz solution is based on the form of Merton (1969) solution with the variable separable property of the exponential function.

I can find $f(y) = \frac{(1-a)}{r} y + \frac{(\alpha - \rho\eta\sigma)(1-a)}{r^2} - \frac{\gamma\sigma^2(1-\rho^2)(1-a)^2}{2r^2}$

I now solve the complete problem as follows. Firm F's utility maximization problem is:

$$F(x,y) = \sup_{\theta_t,\tau} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds \mid X_0 = x, Y_0 = y]$$

subject to

$dX_t = \theta_t (dS_t / S) + r (X_t - \theta_t) dt + [Y_t I_{\{t \geq \tau\}}(1-a) - K\delta(t-\tau) - C_t] dt$

$$\delta(t-\tau) = \begin{cases} 1 \text{ if } \tau = t \\ 0 \text{ otherwise} \end{cases}$$

By making use of $F_1(x,y)$, I can write the complete problem as:

$$F(x,y) = \sup_{\theta,C,\tau} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds \mid X_0 = x, Y_0 = y]$$

$$= \sup_{\tau} \sup_{\{\theta_s, C, 0 \leq s \leq \tau\}} E[\int_0^\tau -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds + e^{-\beta\tau} F_1(X_\tau, Y_\tau) \mid X_0 = x, Y_0 = y]$$

The Bellman equation associated with the value function $F(x,y)$ is:

$$\beta F(x,y) = \sup_{\theta,C} [-\frac{1}{\gamma} e^{-\gamma C} + F_x(\theta_t \mu + r(x-\theta_t) - C_t) + F_y \alpha + \frac{1}{2} F_{xx} \theta_t^2 \chi^2 + \frac{1}{2} F_{yy} \sigma^2 + F_{xy} \rho \chi \theta_t \sigma]$$

Taking first-order conditions for solving optimal consumption C and investment strategy θ, I obtain:

$$C_t^* = -\frac{1}{\gamma} \ln F_x$$

$$\theta_t^* = -\frac{F_x(\mu - r)}{F_{xx}\chi^2} - \frac{F_{xy}\rho\sigma}{F_{xx}\chi}$$

I conjecture that the value function takes the form $F(x,y) = -\frac{1}{r\gamma}\exp[-r\gamma(x + g(y) + \frac{\eta^2}{2r^2\gamma} + \frac{(\beta - r)}{r^2\gamma})]$ then g(y) may be viewed as the certainty equivalent value of the option value to invest. Using the ansatz solution for C_t^* and θ_t^* and substituting into H-J-B equation, I obtain

$$rg(y) = (\alpha - \rho\eta\sigma)g'(y) + \frac{\sigma^2}{2}g''(y) - \frac{r\gamma\sigma^2}{2}(1-\rho^2)g'(y)^2$$

subject to

$\lim_{y \to -\infty} g(y) = 0$ (absorbing barrier)

$g'(y^F) = \frac{(1-a)}{r}$ (smooth pasting)

$$g(y^F) = f(y^F) - K = \frac{(1-a)}{r}y^F + \frac{(\alpha - \rho\eta\sigma)(1-a)}{r^2} - \frac{\gamma\sigma^2(1-\rho^2)(1-a)^2}{2r^2} - K$$

(value matching)

The option value to invest, $g(y)$, is solved by invoking certainty equivalence and y^F is the investment trigger for Firm F's optimal entry.

3.5.3 Firm L's Value Function, Investment Timing Decision, and Certainty Equivalence Value

Firm L's management expects to receive profit flow Y_t after undertaking investment and prior to Firm F's entry; after Firm F's entry, Firm L receives perpetual profit flow $Y_t \times a$. The manager's problem is to maximize her expected utility of consumption with the investment strategy θ by hedging partially using the traded asset S and the riskless bond. The dynamics of wealth process X_t are:

$dX_t = \theta_t\ (dS_t / S_t) + r\ (X_t - \theta_t)\ dt + [Y_t I_{\{t<\tau^F\}} + Y_t I_{\{t\geq\tau^F\}} a$

$-K\delta(t-T) - C_t]\ dt$

$$\delta(t-T) = \begin{cases} 1 & \text{if } t = T \\ 0 & \text{otherwise} \end{cases}$$

where T is the time when Firm L undertakes investment, C_t is the consumption process, and θ_t is the cash amount invested in the partial spanning asset S, and remaining wealth is invested at riskless rate, r.

I first assume that the manager has already invested K, has already started to receive the cash flow Y_t, and expects to receive $Y_t \times a$ forever following Firm F's entry at time τ^F.

3.5.3 Proposition 1:

Firm L manager's problem is to maximize her expected utility of consumption with the investment strategy θ by hedging partially using the traded asset S and the riskless bond conditional on Firm F's optimal entry, at time τ^F, where $\tau^F = \inf\{t : Y(t) \geq y^F\}$ and y^F is the investment trigger for Firm F. That is,

$$L(x,y) = \sup_{\theta, C} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds]$$

$$= \sup_{\{C_s, \theta_s, 0\leq s\leq\tau\}} E[\int_0^{\tau^F} -\frac{1}{\gamma} e^{-\beta s} e^{-\gamma C_s} ds + e^{-\beta\tau^F} F_1(x,y)]$$

Subject to

$Y_0 = y$, $X_0 = x$

$dX_t = \theta_t\ (dS_t / S) + r\ (X_t - \theta_t)\ dt + [Y_t I_{\{t<\tau^F\}} a1 + Y_t I_{\{t\geq\tau^F\}} a - C]\ dt$

Remarks

The second equality derives from Lemma 2, Section 3.5.2 and the fact that after follower enters, the two firms share the market and hence the value function if homogeneous expectation/utility function is assumed.

Firm L's Value Function

Assume Firm L has already entered the market and given $Y(t) \leq y^F$, I then solve the equation by using H-J-B, the standard arguments yield:

$$\beta L(x,y) = \sup_{\theta,C}[-\frac{1}{\gamma}e^{-\beta t}e^{-\gamma C_t} + L_{1x}(\theta_t \mu + r(x-\theta_t) + y - C_t) + L_{1y}\alpha + \frac{1}{2}L_{1xx}\theta_t^2\chi^2$$
$$+\frac{1}{2}L_{1yy}\sigma^2 + L_{1xy}\rho y\sigma\theta_t\chi]$$

Solve for $L(x,y)$

Taking first-order conditions for solving optimal consumption C and investment strategy θ, I obtain:

$$C_t^* = -\frac{1}{\gamma}(\ln L_{1x})$$

$$\theta_t^* = -\frac{L_{1x}(\mu-r)}{L_{1xx}\chi^2} - \frac{L_{1xy}\rho\sigma}{L_{1xx}\chi}$$

I conjecture that the value function takes the form $L(x,y) = -\frac{1}{r\gamma}\exp[-r\gamma(x+h(y)+\frac{\eta^2}{2r^2\gamma}+\frac{(\beta-r)}{r^2\gamma})]$ where $h(y)$ has the interpretation as the certainty equivalence value, and solves the following non-linear second order ODE

$$(\alpha-\rho\eta\sigma)\frac{\partial}{\partial y}h(y)+\frac{\sigma^2}{2}\frac{\partial^2}{\partial y\partial y}h(y)-\frac{r\gamma\sigma^2}{2}(1-\rho^2)(\frac{\partial}{\partial y}h(y))^2-rh(y)$$
$$+y=0$$

with $h(y^F)=f(y^F)=\frac{a}{r}y^F+\frac{(\alpha-\rho\eta\sigma)a}{r^2}-\frac{\gamma\sigma^2(1-\rho^2)a^2}{2r^2}$. This is the value matching condition indicating Firm L's project value after Firm F's entry. It differs from Firm F's project value obtained in the previous section because of the difference in market share.

where $f(y)=\frac{a}{r}y+\frac{(\mu-\rho\eta\sigma)a}{r^2}-\frac{\gamma\sigma^2(1-\rho^2)a^2}{2r^2}$.(see 3.5.2 Firm F's value function for details[23])

and y^F is the optimal investment trigger for Firm F's entry.

The investment trigger for Firm L occurs at the point where value function for Firm L is equal to the value function for Firm F with $y<y^F$. That is, $h(y^L)-K=g(y^L)$, where $g(y)$ is Firm F's option value to invest and y^L is Firm L's investment trigger value.

3.5.4 Model Results Assuming Perfect Spanning (Complete Market)

I consider a complete market version of the model to create a "benchmark" for comparison to the incomplete market cases. The market is complete if either perfect spanning holds, i.e., uncertainty over the income stream, Y, may be replicated by asset S (perfect correlation with Y), or equivalently, Y itself is traded. Such a complete

[23] It differs from Firm F's project value obtained in the previous section due to the difference in the market share.

market version of the model creates a "benchmark" for comparison to the incomplete market case.

3.5.4.1 Firm F's Value Function and Investment Timing Decision

Assuming Firm L has already entered the market, Firm F will enter the market optimally without fear of pre-emption. Under the complete market, Firm F's value function and investment trigger can thus be obtained through the following proposition.

3.5.4.1 Proposition 1: (Y itself is traded.)

Firm F manager's value function may be written as:

$$F^C(y) = \begin{cases} \dfrac{1-a}{\beta r} e^{\beta(y-y^F)}; y \in [0, y^F] \\ \dfrac{(1-a)y}{r} + \dfrac{(1-a)\alpha}{r^2} - K; y \in [y^F, \infty] \end{cases}$$

where $\beta = -\frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{1}{\beta} - \frac{\alpha}{r} + \frac{rK}{1-a}$

y^F is the solution to the free boundary. Thus Firm F's optimal policy is to invest the first time Y reaches the threshold y^F.

<u>*Proof*:</u>

Following standard arguments, the corresponding Bellman equation for Firm F is

$$\alpha F_y^C + \frac{1}{2}\sigma^2 F_{yy}^C - rF^C = 0$$

I solve the above ODE with following boundary, value matching, and smooth pasting conditions:

$$\lim_{y->-\infty} F^C(y) = 0$$

$$F^C(y^F) = \frac{(1-a)y^F}{r} + \frac{(1-a)\alpha}{r^2} - K$$

$$F_y^C(y^F) = \frac{1-a}{r}$$

The value matching condition obtains from the fact that after investment Firm F's expected project value is the discounted expected present value of the duopoly cashflow, (1-a)Y, in perpetuity. That is, assuming that the firm stops at time t, the value of the project equals:

$$\begin{aligned}
V^C(y) &= E[\int_t^\infty e^{-r(s-t)} Y(s)(1-a)ds] \\
&= E[\int_t^\infty e^{-r(s-t)}(1-a)(Y(t) + (\alpha - \frac{1}{2}\sigma^2)(s-t) + \sigma W(s-t))ds] \\
&= E[\int_0^\infty (1-a)e^{-ru}(Y(t) + (\alpha - \frac{1}{2}\sigma^2)u + \sigma W(u))du]; u = s-t \\
&= (1-a)[\int_0^\infty e^{-ru} Y(t)du + \int_0^\infty e^{-ru}\alpha u du] \\
&= (1\text{-}a)[-\frac{Y(t)}{r} e^{-ru} |_0^\infty + (-\alpha \frac{e^{-ru}(1+rt)}{r^2} |_0^\infty] \\
&= (1-a)(\frac{Y(t)}{r} + \frac{\alpha}{r^2})
\end{aligned}$$

The optimal stopping time τ^F is given by τ^F = inf { t : $Y_t \geq y^F$ }. I propose a solution of the form $F^C(y) = Ae^{y\beta}$ where A is a constant to be determined.

Because $F^C(y) = Ae^{y\beta}$, it immediately follows that $F_y^C = A\beta e^{y\beta}$ and $F_{yy}^C = A\beta^2 e^{y\beta}$. Substituting into the Bellman equation, $\alpha F_y^C + \frac{1}{2}\sigma^2 F_{yy}^C - rF^C = 0$, it becomes

$$\frac{1}{2}\sigma^2\beta^2 + \alpha\beta - r = 0$$

Solving for β in the above characteristic equation yields

$$\beta = -\frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}} > 0 \text{ or}$$

$$\beta = -\frac{\alpha}{\sigma^2} - \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}} < 0$$

The solution $\beta < 0$ may be rejected given the boundary conditions, and the solution is of the form $F^C(y) = Ae^{y\beta}$ with $\beta = -\frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$. The constant A and investment trigger value y^F may be determined by invoking value matching and smooth pasting conditions, yielding Firm F's value function and trigger value:

$$F^C(y) = \frac{1-a}{\beta r} e^{\beta(y-y^F)} \text{ ,} where\, \beta = -\frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}} \text{ and } y^F = \frac{1}{\beta} - \frac{\alpha}{r} + \frac{rK}{1-a}$$

3.5.4.1 Proposition 2: (The perfect spanning asset, S, exists.)

Firm F manager's value function may be written as:

$$F^C(y) = \begin{cases} \dfrac{1-a}{\beta r} e^{\beta(y-y^F)}; y \in [0, y^F] \\ \dfrac{(1-a)y}{r} + \dfrac{(1-a)(\alpha - \eta\sigma)}{r^2} - K; y \in [y^F, \infty] \end{cases}$$

where $\beta = -\dfrac{(\alpha - \sigma\eta)}{\sigma^2} + \sqrt{(\dfrac{\alpha - \sigma\eta}{\sigma^2})^2 + \dfrac{2r}{\sigma^2}}$ and $y^F = \dfrac{1}{\beta} - \dfrac{(\alpha - \eta\sigma)}{r} + \dfrac{rK}{1-a}$

y^F is the solution to the free boundary. Thus Firm F's optimal policy is to invest the first time Y reaches the threshold y^F.

Proof:

Following standard arguments, the corresponding Bellman equation for Firm F is

$$(\alpha - \sigma\eta)F_y^C + \frac{1}{2}\sigma^2 F_{yy}^C - rF^C = 0$$

I solve the above ODE with following boundary, value matching, and smooth pasting conditions:

$$\lim_{y->-\infty} F^C(y) = 0$$

$$F^C(y^F) = \frac{(1-a)y^F}{r} + \frac{(1-a)(\alpha - \eta\sigma)}{r^2} - K$$

$$F_y^C(y^F) = \frac{1-a}{r}$$

The value matching condition is obtained by the fact that after investment Firm F's expected project value is the discounted expected present value of the duopoly cashflow, (1-a)Y, in perpetuity. That is, assuming that the firm stops at time t, the value of the project equals:

$$
\begin{aligned}
V^C(y) &= E^Q[\int_t^\infty e^{-r(s-t)} Y(s)(1-a)ds] \\
&= E^Q[\int_t^\infty e^{-r(s-t)}(1-a)(Y(t)+(\alpha-\sigma\eta-\frac{1}{2}\sigma^2)(s-t)+\sigma Z^Q(s-t))ds] \\
&= E^Q[\int_0^\infty (1-a)e^{-ru}(Y(t)+(\alpha-\sigma\eta-\frac{1}{2}\sigma^2)u+\sigma Z^Q(u))du]; u=s-t \\
&= (1-a)[\int_0^\infty e^{-ru}Y(t)du+\int_0^\infty e^{-ru}(\alpha-\sigma\eta)udu] \\
&= (1\text{-}a)[-\frac{Y(t)}{r}e^{-ru}\,|_0^\infty + (-(\alpha-\sigma\eta)\frac{e^{-ru}(1+rt)}{r^2})\,|_0^\infty] \\
&= (1-a)(\frac{Y(t)}{r}+\frac{\alpha-\sigma\eta}{r^2})
\end{aligned}
$$

where E^Q denotes expectation with respect to unique martingale measure Q, defined as follows. For each t < ∞, the Radon-Nikodym density of Q with respect to the historical measure P is defined as:

$$\frac{dQ}{dP}|_{F_t} = \exp(-\eta Z_t - \frac{1}{2}\eta^2 t)$$

Under Q, $\frac{dS_t}{S_t} = rdt + \chi dZ_t^Q, where Z_t^Q = Z_t + \eta t$ is a Q-Brownian motion and the project stochastic income follows:

$$dY_t = (\alpha - \eta\sigma)dt + \sigma dZ_t^Q$$

The optimal stopping time τ^F is given by τ^F = inf { t : $Y_t \geq y^F$ }. I propose a solution of the form $F^C(y) = Ae^{y\beta}$ where A is a constant to be determined.

Because $F^C(y) = Ae^{y\beta}$, it immediately follows that $F_y^C = A\beta e^{y\beta}$ and $F_{yy}^C = A\beta^2 e^{y\beta}$. Substituting into the Bellman equation, $(\alpha - \sigma\eta)F_y^C + \frac{1}{2}\sigma^2 F_{yy}^C - rF^C = 0$, the Bellman equation becomes

$$\frac{1}{2}\sigma^2\beta^2 + (\alpha - \eta\sigma)\beta - r = 0$$

Solving for β in the above characteristic equation yields

$$\beta = -\frac{(\alpha - \sigma\eta)}{\sigma^2} + \sqrt{(\frac{\alpha - \sigma\eta}{\sigma^2})^2 + \frac{2r}{\sigma^2}} > 0 \text{ or}$$

$$\beta = -\frac{(\alpha - \sigma\eta)}{\sigma^2} - \sqrt{(\frac{\alpha - \sigma\eta}{\sigma^2})^2 + \frac{2r}{\sigma^2}} < 0$$

The solution $\beta < 0$ may be rejected given the boundary conditions, and the solution is of the form $F^C(y) = Ae^{y\beta}$ with $\beta = -\frac{(\alpha - \sigma\eta)}{\sigma^2} + \sqrt{(\frac{\alpha - \sigma\eta}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$. The constant A and investment trigger value y^F may be determined by invoking value matching and smooth pasting conditions, yielding Firm F's value function and investment trigger:

$$F^C(y) = \frac{1-a}{\beta r} e^{\beta(y - y^F)}$$

$$where\, \beta = -\frac{(\alpha - \sigma\eta)}{\sigma^2} + \sqrt{(\frac{\alpha - \sigma\eta}{\sigma^2})^2 + \frac{2r}{\sigma^2}} \text{ and } y^F = \frac{1}{\beta} - \frac{(\alpha - \sigma\eta)}{r} + \frac{rK}{1-a}$$

3.5.4.2 Firm L's Value Function and Investment Timing Decision

By design Firm L undertakes investment prior to Firm F's entry. Firm L makes its investment decision conditional on Firm F acting optimally according to the optimal stopping rule described above.

3.5.4.2 Proposition 1: (Y itself is traded.)

Firm L's value function is:

$$L^C(y) = \begin{cases} \left(\frac{(a-1)y^F}{r} + \frac{(a-1)\alpha}{r^2}\right)e^{\beta(y-y^F)} + \frac{y}{r} + \frac{\alpha}{r^2} - K; y \in [0, y^F] \\ \frac{ay}{r} + \frac{a\alpha}{r^2} - K; y \in [y^F, \infty] \end{cases}$$

where $\beta = -\frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{1}{\beta} - \frac{\alpha}{r} + \frac{rK}{1-a}$.

Proof:

Following standard arguments, the corresponding Bellman equation for Firm L is

$$\alpha L_y^C + \frac{1}{2}\sigma^2 L_{yy}^C - rL^C + y = 0$$

I solve the above ODE subject to the following boundary condition:

$$L^C(y^F) = \frac{ay^F}{r} + \frac{a\alpha}{r^2}$$

I propose a solution of the form $L^1(y) = Ae^{y\beta}$ for the homogeneous part, and a particular solution for the non-homogenous part, where A is a constant to be determined. The Bellman equation for the homogenous part is (derivation is the same as the previous section):

$$\frac{1}{2}\sigma^2\beta^2 + (\alpha - \eta\sigma)\beta - r = 0$$

Solving for β in the above characteristic equation yields

$$\beta = -\frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}} > 0 \text{ or}$$

$$\beta = -\frac{\alpha}{\sigma^2} - \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}} < 0$$

The solution $\beta < 0$ may be rejected given the boundary condition. One candidate particular solution is $\frac{y}{r} + \frac{\alpha}{r^2}$. Therefore, the solution should be the form of $L^C(y) = Ae^{y\beta}$ for the homogenous part with $\beta = -\frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$, plus particular solution, $\frac{y}{r} + \frac{\alpha}{r^2}$. From the value matching condition, A may be determined, yielding Firm L's value function:

$L^C(y) = (\frac{(a-1)y^F}{r} + \frac{(a-1)\alpha}{r^2})e^{\beta(y-y^F)} + \frac{y}{r} + \frac{\alpha}{r^2} - K$ with the consideration of investment costs, *where* $\beta = -\frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{1}{\beta} - \frac{\alpha}{r} + \frac{rK}{1-a}$.

3.5.4.2 Proposition 2: (Y itself is traded.)

Firm L's investment trigger value y^L is the solution to the following equation:

$$(\frac{(a-1)y^F}{r} + \frac{(a-1)\alpha}{r^2})e^{\beta(y^L-y^F)} + \frac{y^L}{r} + \frac{\alpha}{r^2} - K = \frac{1-a}{\beta r}e^{\beta(y^L-y^F)}$$

where $\beta = -\frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{1}{\beta} - \frac{\alpha}{r} + \frac{rK}{1-a}$.

Remarks:

The above proposition makes use of the fact that Firm L's trigger value, y^L, yields the same value for both firms. That is, $L^C(y^L)$ should be equal to $F^C(y^L)$.

3.5.4.2 Proposition 3: (The perfect spanning asset, S, exists.)

Firm L's value function is:

$$L^C(y) = \begin{cases} (\frac{(a-1)y^F}{r} + \frac{(a-1)(\alpha - \sigma\eta)}{r^2})e^{\beta(y-y^F)} + \frac{y}{r} + \frac{(\alpha - \sigma\eta)}{r^2} - K; y \in [0, y^F] \\ \frac{ay}{r} + \frac{a(\alpha - \sigma\eta)}{r^2} - K; y \in [y^F, \infty] \end{cases}$$

where $\beta = -\frac{(\alpha - \sigma\eta)}{\sigma^2} + \sqrt{(\frac{\alpha - \sigma\eta}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{1}{\beta} - \frac{(\alpha - \eta\sigma)}{r} + \frac{rK}{1-a}$.

<u>*Proof:*</u>

Following standard arguments, the corresponding Bellman equation for Firm L is

$$(\alpha - \sigma\eta)L_y^C + \frac{1}{2}\sigma^2 L_{yy}^C - rL^C + y = 0$$

I solve the above ODE subject to the following boundary condition:

$$L^C(y^F) = \frac{ay^F}{r} + \frac{a(\alpha - \eta\sigma)}{r^2}$$

I propose a solution of the form $L^1(y) = Ae^{y\beta}$ for the homogeneous part, and a particular solution for the non-homogenous part, where A is a constant to be determined. The Bellman equation for the homogenous part is (derivation is the same as the previous section):

$$\frac{1}{2}\sigma^2\beta^2+(\alpha-\eta\sigma)\beta-r=0$$

Solving for β in the above characteristic equation yields

$$\beta=-\frac{(\alpha-\sigma\eta)}{\sigma^2}+\sqrt{(\frac{\alpha-\sigma\eta}{\sigma^2})^2+\frac{2r}{\sigma^2}}>0 \text{ or}$$

$$\beta=-\frac{(\alpha-\sigma\eta)}{\sigma^2}-\sqrt{(\frac{\alpha-\sigma\eta}{\sigma^2})^2+\frac{2r}{\sigma^2}}<0$$

The solution $\beta<0$ may be rejected given the boundary condition. One candidate particular solution is $\frac{y}{r}+\frac{(\alpha-\sigma\eta)}{r^2}$. Therefore, the solution should be the form of $L^C(y)=Ae^{y\beta}$ for the homogenous part with $\beta=-\frac{(\alpha-\sigma\eta)}{\sigma^2}+\sqrt{(\frac{\alpha-\sigma\eta}{\sigma^2})^2+\frac{2r}{\sigma^2}}$, plus particular solution, $\frac{y}{r}+\frac{(\alpha-\eta\sigma)}{r^2}$. From the value matching condition, A may be determined, yielding Firm L's value function:

$L^C(y)=(\frac{(a-1)y^F}{r}+\frac{(a-1)(\alpha-\sigma\eta)}{r^2})e^{\beta(y-y^F)}+\frac{y}{r}+\frac{(\alpha-\sigma\eta)}{r^2}-K$ with the consideration of investment costs, *where* $\beta=-\frac{(\alpha-\sigma\eta)}{\sigma^2}+\sqrt{(\frac{\alpha-\sigma\eta}{\sigma^2})^2+\frac{2r}{\sigma^2}}$ and $y^F=\frac{1}{\beta}-\frac{(\alpha-\eta\sigma)}{r}+\frac{rK}{1-a}$.

3.5.4.2 Proposition 4: (The perfect spanning asset, S, exists.)

Firm L's investment trigger value y^L is the solution to the following equation:

$$(\frac{(a-1)y^F}{r}+\frac{(a-1)(\alpha-\eta\sigma)}{r^2})e^{\beta(y^L-y^F)}+\frac{y^L}{r}+\frac{(\alpha-\eta\sigma)}{r^2}-K=\frac{1-a}{\beta r}e^{\beta(y^L-y^F)}$$

where $\beta = -\frac{(\alpha - \sigma\eta)}{\sigma^2} + \sqrt{(\frac{\alpha - \sigma\eta}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{1}{\beta} - \frac{(\alpha - \eta\sigma)}{r} + \frac{rK}{1-a}$.

Remarks:

The above proposition makes use of the fact that Firm L's trigger value, y^L, yields the same value for both firms with $y < y^F$. That is, $L^C(y^L)$ should be equal to $F^C(y^L)$.

3.6 Quantitative Results Analysis

I consider a capital investment project with the following base case parameter values:

Investment cost (K) = 1,

Project volatility (η) =0.4,

$\xi(=\frac{\alpha - r}{\eta}) - \rho \times \lambda(=\frac{\mu - r}{\sigma})$ is set at –0.3, fixing $\beta = 2.5$,

Leader market share, a, ranges from 0.5 to 0.8 upon entry by Firm F.

All trigger values reported below are "discounted" values.

3.6.1 The Impact of Market Completeness on the Investment Timing Decision and Option Value to Invest

Holding constant ρ and γ, I find that the lower Firm L's market share following Firm F's entry, the lower the trigger investment value for the follower. That is, Firm F has greater incentive to enter the market the greater the anticipated market share. When the firms are able to hedge completely the project risk, Firm L's investment trigger value is negatively correlated with its market share. Moreover, option value to invest becomes smaller as L's market share increases. That is, Firm L enters immediately to secure the pre-emptive advantage.

I find that the higher the degree of completeness (measured by ρ), the greater is the follower's option value to invest, a result consistent with Henderson (2005). The leader's option value for investment, as well as project value, is higher than is the case when perfect hedging is possible. Considering simultaneously the market incompleteness and the leader's fear of pre-emption, it appears that Firm L displays behavior closer to the classic real option models relative to the case in which perfect hedging is possible. I conjecture that Firm L's management has greater concern for the risk involved in the imperfect hedge than for the risk of pre-emption.

I next focus on Firm L's market share, a. If Firms L and F expect to share the market equally, they will enter the market nearly simultaneously. This result conflicts with classical model results in a complete market setting. However, if Firm L anticipates a market share greater than 50% upon F's entry, Firm L will enter the market slightly earlier than F but not as fast as would be the case in a complete market. These results reflect in part our specification of the leader's value function. I anticipate verifying and refining this specification in future versions of the paper. Results are summarized in Table 1.

3.6.2 The Impact of Risk Aversion on Investment Timing Decision and Option Value to Invest

The greater the risk aversion coefficient, the lower is the investment option value for both firms. This result suggests that the more risk-averse managers may be more concerned about the unhedgeable risks, placing relatively less value in the option to delay investment. Results are summarized in Table 2.

3.7 Does Stochastic Process Matter? Geometric Brownian Motion vs. Arithmetic Brownian Motion (GBM vs. ABM)

I investigate how the stochastic process specification impacts the decision rule. It may not be appropriate to compare the static trigger values and option values for two different stochastic processes. Schwartz (1997) indicates the importance of mean-reverting process vs. non-reverting process in the capital budgeting investment problem. Trajanowska and Kort (2005) compare the equilibrium in a strategic real-option game under arithmetic Brownian motion with Grenadier (2002) under geometric Brownian motion. They exponentiate arithmetic Brownian motion results to make "reasonable" static comparisons between the two.

3.7.1 Firm F's and Firm L's Value Functions under Geometric Brownian Motion versus Arithmetic Brownian Motion

As presented in section 3.4, modeling stochastic investment payoff through a multiplicative demand shock following a GBM process assuming zero variable cost yields a GBM investment income stream. I model directly the stochastic income stream following (first) GBM and (second) ABM respectively. That is, the stochastic income stream either follows process (A) or the process (B):

(A) GBM: $dY_t = \alpha Y_t dt + \sigma Y_t dW_t$; $Y_t = y$

(B) ABM: $dY_t = \alpha dt + \sigma dW_t$; $Y_t = y$

As before, two competing firms are contemplating entry into a new market, and I identify the firms as the Leader (Firm L) and Follower (Firm F), respectively. Firm L enters the market by investing to receive a monopoly rent until Firm F enters the market. Upon entry by Firm F, I assume that Firm L obtains a fraction a ∈ [½,1] of project income, leaving (1-a)=a2 of project income for Firm F.

The corresponding Firm F's and Firm L's value functions under two different stochastic processes are summarized as:

(C) GBM:

$$F^C(y) = \begin{cases} [\frac{y^F(1-a)}{r-\alpha} - K](\frac{y}{y^F})^\beta; y \in [0, y^F] \\ \frac{y(1-a)}{r-\alpha} - K; y \in [y^F, \infty] \end{cases}$$

$$L^C(y) = \begin{cases} \frac{y}{r-\alpha} + \frac{(a-1)y^F}{r-\alpha}(\frac{y}{y^F})^\beta - K; y \in [0, y^F] \\ \frac{ya}{r-\alpha} - K; y \in [y^F, \infty] \end{cases}$$

where $\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{\beta}{(\beta-1)} \times \frac{r-\alpha}{D(2)} \times K$

(D) ABM:

$$F^C(y) = \begin{cases} \frac{1-a}{\beta r} e^{\beta(y-y^F)}; y \in [0, y^F] \\ \frac{(1-a)y}{r} + \frac{(1-a)\alpha}{r^2} - K; y \in [y^F, \infty] \end{cases}$$

$$L^C(y) = \begin{cases} (\frac{(a-1)y^F}{r} + \frac{(a-1)\alpha}{r^2})e^{\beta(y-y^F)} + \frac{y}{r} + \frac{\alpha}{r^2} - K; y \in [0, y^F] \\ \frac{ay}{r} + \frac{a\alpha}{r^2} - K; y \in [y^F, \infty] \end{cases}$$

where $\beta = -\frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$ and $y^F = \frac{1}{\beta} - \frac{\alpha}{r} + \frac{rK}{1-a}$

3.7.2 Uncertainty Impact

I focus on the uncertainty variable, i.e., income stream volatility. Holding all else constant, I find that increasing uncertainty delays Firm L's investment trigger in both stochastic processes. However, the sensitivity is quite different. Firm L's investment trigger value is more sensitive to the uncertainty in the GBM case, while Firm L's option value is more sensitive to the uncertainty in the ABM case. However, for Firm F's investment trigger and option value to invest, they are both more sensitive to the uncertainty in the GBM case and less sensitive in the ABM case. (See Table 6, 7, 8.)

3.7.3 Market Share Impact

Focus on varying Firm L's market share after Firm F's entry, holding all else constant, I find that the lower the Firm F's market share after its entry, the higher the investment trigger for Firm F in both stochastic processes. However, with the GBM investment income stream, Firm F's option value to invest does not vary with the market share as its investment trigger appears. On the contrary, with the ABM investment income stream, Firm F's option value to invest varies with the market share as its investment trigger appears. The phenomenon can be justified through the following equation:

With GBM, Firm F's option value to invest is

$$F^C(y) = [\frac{y^F(1-a)}{r-\alpha} - K](\frac{y}{y^F})^\beta$$

$$, where\, \beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} \text{ and } y^F = \frac{\beta}{(\beta - 1)} \times \frac{r - \alpha}{1 - a} \times K$$

To observe how the market share impacts the option value to invest, take derivative of $F^C(y)$ with respect to the market share parameter, (1-a), evaluated at the trigger point, yielding:

$$F^C(y) = [\frac{y^F(1-a)}{r-\alpha} - K](\frac{y}{y^F})^\beta = \frac{K}{\beta - 1} \times (\frac{y}{y^F})^\beta$$

Thus, $\frac{\partial F^C(y)}{\partial(1-a)}|_{y=y^F} = 0$

That is, Firm F's option value to invest is independent of the market share though the investment trigger increases as its market share decreases.

With ABM, Firm F's option value to invest is

$$F^C(y) = \frac{1-a}{\beta r} e^{\beta(y-y^F)},\ \beta = -\frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2})^2 + \frac{2r}{\sigma^2}} \text{ and } y^F = \frac{1}{\beta} - \frac{\alpha}{r} + \frac{rK}{1-a}$$

To observe how the market share impacts the option value to invest, take derivative of $F^C(y)$ with respect to the market share parameter, (1-a), evaluated at the trigger point, yielding:

$$\frac{\partial F^C(y)}{\partial(1-a)}|_{y=y^F} = \frac{1}{\beta r} > 0$$

That is, Firm F's option value to invest decreases as its market share decreases upon entry, whereas its investment trigger increases as its market share decreases. Moreover, it shows that Firm F's option value to invest and its market share decrease at approximately the same rate.

Firm F's trigger exhibits the same sensitivity to market share in both stochastic processes. However, Firm L's investment trigger is more sensitive to the market share in the GBM case, while Firm L's option value to invest is more sensitive to market share in the ABM case. Results are summarized in Table 1, 2, and 3.

3.7.4 Conclusion

Sensitivity analyses indicate that the choice of process specification, ABM or GBM, has a material impact on project value and option value to invest.

CHAPTER 4

REAL OPTIONS UNDER STOCHASTIC VOLATILITY

4.1 Background

This chapter explores the valuation consequences of incompleteness resulting from stochastic volatility in a real options setting. I examine the efficacy of different approaches to finding and justifying a particular martingale measure. This research provides insight into how choice of equivalent martingale measures impacts the real option values, relative to complete market models.

Stochastic volatility induced market incompleteness affects the investment/abandonment decision in important ways. The optimal investment/ abandonment decision rule changes, as do the corresponding option values, as follows:

(1) With non-zero correlation between the project randomness and volatility randomness, the option values to invest/abandonment option value under q-optimal measures will decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ increases (respectively, decreases) in σ. Thus, the difference between NPV and option value to invest (also project value with option to abandon[24]) under q-optimal measures will decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ increases (respectively, decreases) in σ.

[24] Project value with option to abandon = Static net present value + Abandonment option premium

(2) With zero correlation between the project randomness and volatility randomness, $\lambda^{(q)}(t,\sigma)$ will be non-decreasing (respectively, non-increasing) in q if $\xi(t,\sigma)^2$ is non-decreasing (respectively, non-increasing) in σ. As a result, option values to invest/abandonment option value under q-optimal measures will decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ increases (respectively, decreases) in σ. The difference between NPV and the option values to invest (also project value with option to abandon) under q-optimal measures will increase (respectively, decrease) in q if $\xi(t,\sigma)^2$ is non- increasing (respectively, non-decreasing) in σ.

(3) With a non-zero correlation between the project randomness and volatility randomness, the optimal investment trigger under q-optimal measures decreases (respectively, increases) in q if $\xi(t,\sigma)^2$ increases (respectively, decreases) in σ. The optimal abandonment trigger is reversed.

(4) With zero correlation between the project randomness and volatility randomness, the optimal investment trigger under q-optimal measures decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ increases (respectively, decreases) in σ. The optimal abandonment trigger is reversed.

(5) There are no conclusive relations between the option value to invest/ abandonment option value under q-optimal measures and the correlation between the project randomness and volatility randomness.

I demonstrate the indifference prices for the option value to invest and the abandonment option solve quasilinear variational inequalities with obstacle terms. With the choice of the exponential utility function, the utility-based indifference price admits a new pricing measure, which is the minimal relative entropy martingale measure minimizing the relative entropy between the historical measure and the equivalent Q martingale measure. I also show that the indifference price for the option value to invest and the abandonment option (also, project value with abandonment option) is non-increasing with respect to the risk aversion parameter. As the risk aversion parameter converges to zero, the indifference price converges to the unique bounded viscosity solution of the linear variational inequality with obstacle term.

4.2 Motivation

Stochastic volatility is important in contingent claims analysis because it represents an unhedgeable risk. It is especially important with respect to real options, given long maturities and concern over tradability of the underlying asset. There is no longer a unique martingale measure, and the choice of pricing measure is no longer preference free. It depends on the utility of investors, or on a criterion depending on some measure of pricing error. Relatively little attention has been paid to comparisons between proposed measures. Heath et al. (2001) examined numerically obtained option price orderings from different martingale measures. Henderson (2005) and Henderson et al. (2004) obtained an ordering of option prices under several q-optimal measures, minimizing the qth moment of the Radon-Nikodym derivative of the pricing measure with respect to the original real-world measure. Henderson et al demonstrated that the ordering proposed in Heath et al. (2001) was incorrect.

Indifference pricing technique represents an alternative to the q-optimal approach. An arbitrage free price is selected according to the investment optimality criteria of a risk averse investor. First proposed by Hodges and Neuberger (1998), indifference pricing has been applied to stochastic volatility models. (see Sircar and Zariphopoulou (2005)).

Classic real options models typically assume constant volatility (e.g., Dixit and Pindyck (1994)). Long investment horizons make the assumption problematic in real options. Motivated by this concern, I extend the classical real options model to incorporate stochastic volatility. I focus on the option value to invest and abandonment option value under the class of q-optimal measures. Henderson (2005) and Henderson et al. (2004) show that when volatility is stochastic, option prices with convex payoffs decrease in q. That is, option prices under the minimal martingale measure ($q = 0$) are at least as large as option prices under the minimal entropy martingale measures ($q =1$), which in turn are at least as large as option prices under the variance optimal martingale measure ($q = 2$). The comparisons were derived for European options, and I contend that they also apply to American options.

This chapter examines three issues pertaining to the option value to invest and also abandonment option under both q-optimal measures and indifference pricing. First, I examine how the choice of q-optimal measure affects the optimal investment/abandonment policy. Second, I explore how the correlation between the volatility and project value alters option value under q-optimal measures. Third, I compare option values under q-optimal measures to the investment decision derived from traditional net present value. Henderson (2005) and Henderson et al. (2004) propose the existence of links between pricing under the q-optimal measure and utility indifference pricing under a power-law utility for $q \neq 0, 1$. For $q < 1$, the price under the q-optimal measure corresponds to the marginal utility indifference price for an agent with power-law utility with constant relative risk aversion. For $q = 0$, it corresponds to logarithmic utility (i.e., unity risk aversion coefficient in the power utility function).

Motivated by Sircar and Zariphopoulou (2005), I employ indifference pricing for an agent with absolute risk aversion to determine: (1) whether a connection exists between these two pricing techniques in a real options setting, and (2) the nature of the interaction among risk aversion, correlation and q-optimal measures. It has been noted that the zero risk aversion limit of the indifference price corresponds to the minimal entropy martingale measure price.

4.3 The General Standard Stochastic Volatility Model and the Class of q-Optimal Measures

I fix a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},P)$ with $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$, the increasing σ-algebras generated by the pair of Brownian motions $(B_s)_{s\leq t}$ and $(B^{\perp}{}_s)_{s\leq t}$,where $B^{\perp}$ is orthogonal to B, satisfying the usual conditions of right-continuity and completeness. Let V be the project value, with volatility σ. Assuming a non-stochastic risk-free interest rate, there is no loss of generality using discounted quantities. Thus, V represents the discounted project value process. Under the real world measure, P, V and σ follow a stochastic process with coefficients satisfying sufficient regularity conditions to ensure the existence of a unique solution with the Strong Markov Property, as follows:

$$\frac{dV_t}{V_t} = \sigma_t(\xi(\sigma_t, t)dt + dB_t) \text{ -- } (1)$$

$$d\sigma_t = \alpha(\sigma_t, t)dt + \beta(\sigma_t, t)dW_t = \alpha(\sigma_t, t)dt + \beta(\sigma_t, t)(\rho dB_t + \overline{\rho} dB_t^{\perp}) \text{ ---------- } (2)$$

where B and W are independent Brownian motions having correlation $\rho \in (-1,1)$. For this, I can take W = ρB + $\sqrt{1-\rho^2}B^{\perp} = \rho B + \overline{\rho}B^{\perp}$, or equivalently, dW = ρdB + $\overline{\rho}dB^{\perp}$. $\xi(\sigma_t, t)$ may be interpreted as the Sharpe ratio or equity risk premium.

I assume the existence of perfect spanning assets, so I do not consider incompleteness caused by the non-tradability of the project. However, σ is not traded, so the market is incomplete and there is no unique martingale measure. Following the analysis in Frey (1997), I characterize the family of equivalent local martingale measures. Let Θ denote the set of such measures and Θ ≠ ∅.

Under the proposed project value process, a probability measure Q ∈ Θ equivalent to P on F_T is a local martingale measure for V on F_T if and only if there is a progressively measurable process λ $=(\lambda_t)_{0\le t\le T}$ with $\int_0^T \lambda_s^2 ds < \infty$ P a.s. such that the local martingale $(Z_t)_{0\le t\le T}$ with

$$Z_t = \exp(-\int_0^t \xi(u,\sigma_u)dB_u - \frac{1}{2}\int_0^t \xi(u,\sigma_u)^2 du - \int_0^t \lambda_u dB_u^{\perp} - \frac{1}{2}\lambda_u^2 du)$$

satisfies $E[Z_T]=1$ and Z_T = $\dfrac{dQ}{dP}$ on F_T. If Z_t is of the form of the above equation, V is a Q-local martingale, and if Z is a true P-martingale, Q is a probability measure. K_t = $\int_0^t \xi(u,\sigma_u)^2 du$ is the mean-variance tradeoff process typical in the finance literature.

By Girsanov's theorem, Brownian motions B and $B^{\perp}$ under Q are given as:

$B_t^Q = B_t + \int_0^t \xi(u,\sigma_u)du$ and $B_t^{\perp,Q} = B_t^{\perp} + \int_0^t \lambda_u du$

Under Q ∈ Θ, V and σ follow the processes:

$$\frac{dV_t}{V_t} = \sigma_t dB_t^Q \text{ ---(3)}$$

$$d\sigma_t = [\alpha(\sigma_t,t) - \rho\xi(\sigma_t,t)\beta(\sigma_t,t) - \overline{\rho}\lambda_t\beta(\sigma_t,t)]dt + \rho\beta(\sigma_t,t)dB_t^Q + \overline{\rho}\beta(\sigma_t,t)dB_t^{\perp,Q} \text{ -(4)}$$

Under Q, the change of drift λ_t on the Brownian motion $B^{\perp}$ is called the market price of $B^{\perp}$ risk, the associated change of drift on W is $\rho\xi(\sigma_t,t) + \overline{\rho}\lambda_t$ and the change of drift on σ is $(\rho\xi(\sigma_t,t) + \overline{\rho}\lambda_t)\beta(\sigma_t,t)$. I call the stochastic process $\rho\xi(\sigma_t,t) + \overline{\rho}\lambda_t$ the volatility risk premium, or the market price of volatility risk. The first term represents the effect of the market price of B risk, and the second term represents the effect of the market price of $B^{\perp}$ risk. From Girsanov's theorem, there is a one-to-one correspondence between $(\lambda_t)_{0\le t\le T}$ (such that $\int_0^T \lambda_s^2 ds < \infty$) and the generic martingale measure. Thus the option pricing problem reduces to the selection of the volatility risk premium, or equivalently, the associated martingale pricing measure.

Remark:

The case $\lambda_t = 0$ corresponds to the minimal martingale measure of Follmer and Schweizer (1991), known as the local risk minimization measure. In this case, the Q is defined via $\frac{dQ}{dP} | F_T =$ $\exp(-\int_0^t \xi(u,\sigma_u)dB_u - \frac{1}{2}\int_0^t \xi(u,\sigma_u)^2 du)$, which makes the traded assets into martingales, leaving the drifts of Brownian motions which are orthogonal to the traded assets unchanged. In other words, it means the unhedgeable risk is not priced.

The q-optimal measure is the equivalent martingale measure which is closest to the original real world measure P based on a distance metric of the qth moment of the relative density. In order to calculate the q-optimal measure, it is necessary to know the real world dynamics and the real world probability measure P. Following the techniques in Hobson (2004) and the analysis in Henderson (2005) and Henderson et al. (2004), I define the following such that for $q \in R$ the q-optimal measure is the measure $Q^{(q)}$ minimizing relative entropy $H_q(P,Q)$.

For $q \in R \setminus \{0,1\}$

$$H_q(P,Q) = \{ \begin{matrix} E[\frac{q}{q-1}(Z_T)^q] \text{ if } Q << P \\ \infty \text{ otherwise} \end{matrix}$$

For $q \in \{0,1\}$

$$H_q(P,Q) = \{ \begin{matrix} E[(-1)^{1+q} Z_T^q \ln(Z_T)] \text{ if } Q << P \\ \infty \text{ otherwise} \end{matrix}$$

Remarks:

Through this definition, I infer[25]:

For q = 0, by definition, I have $H_0(P,Q) = E[-1Z_T^0 \ln(Z_T)]$ $= E[-1\ln(Z_T)]$ $= E[-\ln(\frac{dQ}{dP})]$ $= H(P,Q)$, which is the reverse relative entropy of $H(Q,P)$.

The idea of considering $H(P,Q)$ instead of $H(Q,P)$ is first presented by Platen et al. (1996). They define arbitrage information as the information obtained from the difference between the objective (real world) probability measure and the minimal equivalent martingale (risk neutral) pricing measure for the contingent claim processes.

Following Platen et al., I define Radon-Nikodym derivative as $(\frac{dQ}{dP})$, and Kullback-Leibler information process as $h = \{h_t : t_0 \le t < \infty\}$ where $h_t = (\frac{dQ}{dP})^{-1} \ln(\frac{dQ}{dP})^{-1}$; then for time $t \in [t_0, \infty)$, the total information functional at time t_0 of P with respect to Q can be defined as

$$I_{t_0,t}(P,Q) = \begin{cases} E_Q[h_t \mid F_{t_0}] & \text{when } h_t \text{ is } Q\text{-integrable} \\ \infty & \text{otherwise} \end{cases}$$

Consequently, $I_{t_0,t}(P,Q) = E_Q[h_t \mid F_{t_0}] = E[-\log(\frac{dQ}{dP}) \mid F_{t_0}] \quad \forall t \ge t_0$ represents the information up to time t at time t_0 of the objective probability measure P with respect to the martingale measure Q. $I_{t_0,t}(P,Q)$ is called the arbitrage information up to

25 For detailed proofs see Hobson 2004.

time t at time t_0. It can be seen that the arbitrage information is equivalent to negative relative entropy, and thus I can interpret it as a measure of free energy in terms of relative entropy. The relative entropy can be related to the characterization of the minimal equivalent martingale measure defined in Follmer and Schweizer (1991).

Schweizer (1999) shows that the minimal martingale measure, defined through the Radon-Nikodym derivative $\exp(-\int_0^t \xi(u,\sigma_u)dB_u - \frac{1}{2}\int_0^t \xi(u,\sigma_u)^2 du)$ in my diffusion model with respect to objective probability measure, minimizes the reverse relative entropy of $H(P,Q)$. In my setting the minimal martingale measure $Q^M = Q^0$ corresponding to $\lambda_t = 0$.

I have $H_1(P,Q) = E[1Z_T^1 \ln(Z_T)] = E[\frac{dQ}{dP}\ln(\frac{dQ}{dP})] = H(Q,P)$ when q = 1, which is the relative entropy. By invoking results from Delbaen and Schachermayer (1996), Grandits and Rheinlander (2002), and Frittelli (2000), Hobson (2004) shows the existence and optimality of the candidate measure $Q^{(q)}$ defined through the representative form of q-optimal measure. If q=1, it is the minimal relative entropy martingale measure.

For q = 2, I have-the variance-optimal measure. The variance optimal measure is defined as the density $\frac{dQ^{vom}}{dP}$ having minimal $L^2(P)$ -norm. In fact,

$Var(\frac{dQ^{vom}}{dP}) = E[\frac{dQ^{vom}}{dP} - E[\frac{dQ^{vom}}{dP}]]^2 \;=\; E[\frac{dQ^{vom}}{dP}]^2 - 1$, that is $E[\frac{dQ^{vom}}{dP}]^2 = Var(\frac{dQ^{vom}}{dP}) + 1$; equivalently, $\left\| \frac{dQ^{vom}}{dP} \right\|_{L^2(P)} = \sqrt{Var(\frac{dQ^{vom}}{dP}) + 1}$.

To calculate the option price under the q-optimal measure, it is necessary to be able to characterize the measure. Following Hobson's (2004) q-optimal representation equations, the identification of the q-optimal measure $Q^{(q)}$ is given via the market price of unhedgeable randomness $B^{\perp}$ risk,

$$\lambda^{(q)}(t,\sigma_t) = \beta(\sigma_t,t)\bar{\rho}g_{\sigma}(t,\sigma_t)$$

where

$$g(t,\sigma) = \begin{cases} 0 \quad \text{if q} = 0 \\ -\frac{1}{R}\log \hat{E}[\exp(-\frac{q}{2}R\int_t^T \xi(u,\sigma_u)^2 du) \mid \sigma_t = \sigma] \quad \text{otherwise with R} \neq 0 \text{ --} \\ \hat{E}[\exp(\frac{q}{2}\int_t^T \xi(u,\sigma_u)^2 du) \mid \sigma_t = \sigma] \quad \text{R} = 0 \end{cases}$$

---------(5)

where $R = 1 - q\rho^2$

Under measure $\hat{P}$, the dynamics of σ are modified to become

$$d\sigma_t = [\alpha(\sigma_t,t) - q\rho\xi(\sigma_t,t)\beta(\sigma_t,t)]dt + \beta(\sigma_t,t)d\hat{W}_t$$

Note that $\hat{P}$ corresponds to the real world probability measure if q=0 or ρ=0.

If R = 0, from the Feynman-Kac formula g solves the following representation equation

$$\frac{q}{2}\xi(t,\sigma)^2 - q\rho\beta(t,\sigma)\xi(t,\sigma)g_\sigma + \alpha(t,\sigma)g_\sigma + \frac{1}{2}\beta(t,\sigma)^2 g_{\sigma\sigma} + \dot{g} = 0 \text{ ---(6)}$$

with $g(T,\sigma) = 0$.

If R ≠ 0, from the Feynman-Kac formula g solves the following representation equation:

$$\begin{aligned} &\frac{q}{2}\xi(t,\sigma)^2 - q\rho\beta(t,\sigma)\xi(t,\sigma)g_\sigma - \frac{R}{2}\beta(t,\sigma)^2(g_\sigma)^2 + \alpha(t,\sigma)g_\sigma \\ &+\frac{1}{2}\beta(t,\sigma)^2 g_{\sigma\sigma} + \dot{g} = 0 \end{aligned} \quad \text{--(7)}$$

with $g(T,\sigma) = 0$.

Under $Q^{(q)}$, the dynamics of σ and V in equation (3) and (4) become

$$\frac{dV_t}{V_t} = \sigma_t dB_t^{Q^{(q)}}$$

$$\begin{aligned} d\sigma_t = &[\alpha(\sigma_t,t) - \rho\xi(\sigma_t,t)\beta(\sigma_t,t) - \bar{\rho}^2\beta^2(\sigma_t,t)g_\sigma(t,\sigma_t)]dt + \rho\beta(\sigma_t,t)dB_t^{Q^{(q)}} \\ &+ \bar{\rho}\beta(\sigma_t,t)dB_t^{\perp,Q^{(q)}} \end{aligned}$$

where $B_t^{Q^{(q)}} = B_t + \int_0^t \xi(u,\sigma_u)du$ and $B_t^{\perp,Q^{(q)}} = B_t^{\perp} + \int_0^t \bar{\rho}\beta(u,\sigma_u)g_\sigma(u,\sigma_u)du$.

4.4 Option Value to Invest – q-Optimal Measures

This section investigates the option value to invest under q-optimal measures. I first formulate the investment problem and propose the corresponding model. I then show the ordering results for option value to invest under q-optimal measures.

4.4.1 Investment Problem Formulation and the Model

I assume that the manager faces an investment timing problem in which the investment cost, K, grows at the risk free rate. I specify the manager's investment problem, and I find the option value to invest by solving the following maximization problem (fixing some Q ∈ Θ):

$$p^1(v,\sigma) = \sup_{t\le\tau<\infty} E_t^{Q^{(q)}}[e^{-r(\tau-t)}(\widetilde{V}_\tau - Ke^{-r(\tau-t)})^+ \mid V_t = v, \sigma_t = \sigma]$$ [26]

$$= \sup_{t\le\tau<\infty} E_t^{Q^{(q)}}[(V_\tau - K)^+ \mid V_t = v, \sigma_t = \sigma]$$

where $\widetilde{V}$ denotes the forward process of discount process V_t, that is the original process.

The option value to invest will vary with different choices for Q, i.e., different choices for the market price of volatility risk. I focus on the market price of volatility

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[26] (1) It will never be optimal to early exercise American call option if the underlying asset does not pay dividends.

(2) To capture the early exercise property of the American call option, it is assumed that the asset pays "continuous dividend yield", (in the investment opportunity problem formulation presented, i.e., the real options problem, it is termed "service flow".) The general discounted process defined in equation (1) $\frac{dV_t}{V_t} = \sigma_t(\xi(\sigma_t,t)dt + dB_t)$ will have $\sigma_t(\xi(\sigma_t,t)) = \mu(\sigma_t,t) - \kappa(\sigma_t,t)$, where $\kappa(\sigma_t,t)$ is the continuous dividend yield. For tractability, we keep $\kappa(\sigma_t,t)$ constant. For the Girsanov's transformation for equivalent martingale measure, if we specify

(i) $B_t^Q = B_t + \int_0^t \xi(u,\sigma_u)du$, then V_t and σ_t process are still the same specified in equation (3) and (4) with different $\xi(\sigma_t,t)$ specification. $\xi(\sigma_t,t) = \frac{\mu(\sigma_t,t)-\kappa}{\sigma_t}$ rather than the one with no dividend yield $\xi(\sigma_t,t) = \frac{\mu(\sigma_t,t)}{\sigma_t}$.

(ii) $B_t^Q = B_t + \int_0^t \frac{\mu(u,\sigma_u)}{\sigma_t}du = B_t + \int_0^t \varsigma_u du$, then V_t and σ_t in equation (3) and (4) become $\frac{dV_t}{V_t} = -\kappa dt + \sigma_t dB_t^Q$ with

$$d\sigma_t = [\alpha(\sigma_t,t) - \rho\varsigma(\sigma_t,t)\beta(\sigma_t,t) - \bar{\rho}\lambda_t\beta(\sigma_t,t)]dt + \rho\beta(\sigma_t,t)dB_t^Q + \bar{\rho}\beta(\sigma_t,t)dB_t^{\perp,Q}$$

risk, and in particular, the market price of unhedgeable randomness (i.e., $B^{\perp}$) risk. The q-optimal measure market price of volatility risk is related to q through the λ^{q} equation and fundamental representation equation presented in the previous section.

Consistent with Hobson (2004), I assume a finite time horizon. Later, I will consider the perpetual American option in the classical real option setting. The manager's investment problem is revised as follows

$$\begin{aligned} p^{1}(v,\sigma,t) &= \sup_{t\le\tau<T} E_t^{Q(q)}[e^{-r(\tau-t)}(\widetilde{V}_\tau - Ke^{-r(\tau-t)})^{+} \mid V_t = v, \sigma_t = \sigma] \\ &= \sup_{t\le\tau<T} E_t^{Q(q)}[(V_\tau - K)^{+} \mid V_t = v, \sigma_t = \sigma] \end{aligned}$$

where $\widetilde{V}$ denotes the forward process of discount process V_t, that is the original process.

Invoking the general model dynamics for the class of q-optimal measures, the problem then reduces to satisfying the following variational inequality[27]:

The parameterization is different (also presented in the corresponding variational inequality), but it will not affect the analysis result.

[27] As pointed out in the footnote 26, the variational inequality will be a little different due to the parameterization.

$$
\begin{cases}
p_t^1 + \frac{1}{2}\sigma_t^2 V_t^2 p_v^1 + [\alpha(\sigma_t, t) - \rho\xi(\sigma_t, t)\beta(\sigma_t, t) - \bar{\rho}^2\beta^2(\sigma_t, t)g_\sigma(t, \sigma_t)]p_\sigma^1 \\
+ \rho\sigma_t\beta(\sigma_t, t)p_{v\sigma}^1 + \frac{1}{2}\beta^2(\sigma_t, t)p_{\sigma\sigma}^1 \le 0, t \in [0, T[,\ v > 0,\ p^1(v, \sigma, t) \ge (v-K)^+ \\
\{p_t^1 + \frac{1}{2}\sigma_t^2 V_t^2 p_v^1 + [\alpha(\sigma_t, t) - \rho\xi(\sigma_t, t)\beta(\sigma_t, t) - \bar{\rho}^2\beta^2(\sigma_t, t)g_\sigma(t, \sigma_t)]p_\sigma^1 \\
+ \rho\sigma_t\beta(\sigma_t, t)p_{v\sigma}^1 + \frac{1}{2}\beta^2(\sigma_t, t)p_{\sigma\sigma}^1\} \times [p^1(v, \sigma, T) - (v-K)^+] = 0, t \in [0, T[,\ v > 0 \\
[p^1(v, \sigma, T) - (v-K)^+] = 0
\end{cases}
$$

where $g(t,\sigma)$ is defined in equation (5), f or R=0 $g(t,\sigma)$ solves the representation equation (6) , for R≠0, $g(t,\sigma)$ solves the representation equation (7).

It reduces to solving the following equations

$p^1(t, v, \sigma) = v - K$ for $v \ge V_{fb}(t,\sigma)$

For $v \le V_{fb}(t,\sigma)$

$$
p_t^1 + \frac{1}{2}\sigma_t^2 V_t^2 p_v^1 + [\alpha(\sigma_t, t) - \rho\xi(\sigma_t, t)\beta(\sigma_t, t) - \bar{\rho}^2\beta^2(\sigma_t, t)g_\sigma(t, \sigma_t)]p_\sigma^1
$$
$$
+ \rho\sigma_t\beta(\sigma_t, t)p_{v\sigma}^1 + \frac{1}{2}\beta^2(\sigma_t, t)p_{\sigma\sigma}^1 = 0
$$

with

$p^1(T, v, \sigma) = (v-K)^+$

$V_{fb}(T,\sigma) = K$

In addition, the following boundary and smooth pasting conditions:

$$p^1(t, v^*(=V_{fb}(t,\sigma),\sigma) = (v^* - K)^+$$

$$\frac{\partial p^1(t, v^*(=V_{fb}(t,\sigma)),\sigma)}{\partial v^*} = 1$$

$$\frac{\partial p^1(t, v^*(=V_{fb}(t,\sigma),\sigma)}{\partial \sigma} = 0$$

assure that $p^1(V_u, \sigma_u, u)$, $\frac{\partial p^1(t, v^*(=V_{fb}(t,\sigma),\sigma)}{\partial v^*}$, and $\frac{\partial p^1(t, v^*(=V_{fb}(t,\sigma),\sigma)}{\partial \sigma}$ are continuous across the boundary $V_{fb}(t,\sigma)$.

The partial differential equation cannot be solved analytically for a closed form solution. I must resort to numerical techniques. By considering different q (i.e., different market prices of volatility risk), the investment trigger and option value to invest will vary.

4.4.2 Ordering Results for Option Value to Invest under q-Optimal Measures

The convexity of option value is used for generating the ordering results. The American call option value with stochastic volatility is strictly convex in the continuation region.[28]

4.4.2 Proposition 1:

The convex option prices are decreasing in the market price of $B^{\perp}$ risk parameter λ_t, or equivalently the market price of volatility risk. The reason is that, with a selected pricing measure, the dynamic presented in equation (4) shows that an

[28] This statement can be proven by applying Touzi (1999) Lemma 2.3 for the American put option with stochastic volatility.

increase in either λ_t or the market price of volatility risk corresponds to a decrease in the drift of the volatility.

Proof:

I prove the proposition in two steps. I first show that the model with higher volatility drift term yields the higher option price. I then induce the relationship between option prices and market price of volatility risk.

(1) Let the asset price process V and volatility σ satisfy

$$\frac{dV_t}{V_t} = \sigma_t dB_t$$

$$d\sigma_t = \alpha(\sigma_t, t)dt + \beta(\sigma_t, t)dW_t = \alpha(\sigma_t, t)dt + \beta(\sigma_t, t)(\rho dB_t + \overline{\rho} dB_t^{\perp})$$

Suppose that the drift on the volatility either takes the form of $\alpha(\sigma_t, t) = \alpha^+(\sigma_t, t)$ or $\alpha(\sigma_t, t) = \alpha^-(\sigma_t, t)$ where $\alpha^+(\sigma_t, t) > \alpha^-(\sigma_t, t)$, and let E^+ (respectively E^-) denote the model with drift $\alpha^+(\sigma_t, t)$ (respectively $\alpha^-(\sigma_t, t)$). For a payoff function h define

$$J^+(t, v, \sigma) = \sup_{t \le \tau \le T} E^+[h(V_\tau)] \mid V_t = v, \sigma_t = \sigma],$$

and $J^-(t, v, \sigma) = \sup_{t \le \tau \le T} E^-[h(V_\tau)] \mid V_t = v, \sigma_t = \sigma]$

$J^+(t, v, \sigma)$ and $J^-(t, v, \sigma)$ are the price of American options. It's known that American option, V^{AO}, may be expressed as the counterpart of a European option, denoted V^{EO}, plus the early exercise premium, denoted V^{erp}. That is, $V^{AO} = V^{EO} + V^{erp}$.

In addition, the early exercise premium, V^{erp}, is itself the value process of a European option (Karatzas and Shreve (1998) Ch2 Theorem 5.8, Remark 5.10).

Therefore, I re-express $J^{+}(t,v,\sigma)$ and $J^{-}(t,v,\sigma)$ as:

$$
\begin{aligned}
J^{+}(t,v,\sigma) &= \sup_{t\le\tau\le T} E^{+}[h(V_{\tau})] \mid V_t = v, \sigma_t = \sigma] = \mathrm{V}^{\mathrm{EO}} + V^{erp} \\
&= E^{+}[h(V_T)] \mid V_t = v, \sigma_t = \sigma] + V(t)E^{+}[\int_t^T \frac{V(u)d\Lambda(u)}{V(u)} | V_t = v, \sigma_t = \sigma] \\
&= E^{+}[h(V_T)] \mid V_t = v, \sigma_t = \sigma] + E^{+}_{erp}[h(V_T)] \mid V_t = v, \sigma_t = \sigma] \\
&= L^{+}(t,v,\sigma) + L^{+}_{erp}(t,v,\sigma)
\end{aligned}
$$

where $d\Lambda(t) = 1_{\{V(u)>V_{fb(}(u)\}} d(t)$

where the third equality makes use of the definition of European options and the early exercise premium from Karatzas and Shreve (1998) Ch 2 Theorem 5.8, Remark 5.10. The fourth equality replaces the second term of the third equality with the fact that the early exercise premium is the value process of the European options (e.g. Karatzas and Shreve (1998) Ch2 Proposition 2.3).

Fact:

Touzi (1999, p. 415) shows the suitability of applying Karatzas and Shreve's theorem and proposition in complete market case (constant volatility) to the model with stochastic volatility by assuming some options are traded on the market additionally to the underlying risky asset and the riskless one.

Similarly,

$$J^{-}(t,v,\sigma) = \sup_{t\le\tau\le T} E^{-}[h(V_{\tau})] \mid V_t = v, \sigma_t = \sigma] = \mathrm{V}^{\mathrm{EO}} + V^{erp}$$

$$= E^{-}[h(V_T)] \mid V_t = v, \sigma_t = \sigma] + V(t) E^{-}[\int_t^T \frac{V(u) d\Lambda(u)}{V(u)} | V_t = v, \sigma_t = \sigma]$$

$$= E^{-}[h(V_T)] \mid V_t = v, \sigma_t = \sigma] + E^{-}_{erp}[h(V_T)] \mid V_t = v, \sigma_t = \sigma]$$

$$= L^{-}(t,v,\sigma) + L^{-}_{erp}(t,v,\sigma)$$

$$J^{+}(t,v,\sigma) - J^{-}(t,v,\sigma) = L^{+}(t,v,\sigma) - L^{-}(t,v,\sigma) + L^{+}_{erp}(t,v,\sigma) - L^{-}_{erp}(t,v,\sigma)$$

Following Henderson et. al. (2004), by making use of the fact from the European option that $L^{+}(t,v,\sigma)$ ($L^{-}(t,v,\sigma)$ respectively) satisfies $A^{+}L^{+}(t,v,\sigma) = 0$ ($A^{-}L^{-}(t,v,\sigma) = 0$ respectively) subject to $L^{+}(T,v,\sigma) = h(v)$ ($L^{-}(T,v,\sigma) = h(v)$ respectively), where $A^{+}(A^{-})$ is the infinitesmall generator, and by defining a new function $\hat{L}(t,v,\sigma) = L^{+}(t,v,\sigma) - L^{-}(t,v,\sigma)$ subject to $\hat{L}(T,v,\sigma) = 0$

with

$$A^{-}\hat{L}(t,v,\sigma) = A^{-}(L^{+}(t,v,\sigma) - L^{-}(t,v,\sigma))$$

$$= A^{+}L^{+}(t,v,\sigma) - A^{-}L^{-}(t,v,\sigma) - (A^{+} - A^{-})L^{+}(t,v,\sigma)$$

$$= -(\alpha^{+}(\sigma,t) - \alpha^{-}(\sigma,t))L^{+}_{\sigma}.$$

From the application of Feynman-Kac formula,

$$\hat{L}(t,v,\sigma) = E^{-}[\int_t^T (\alpha^{+}(\sigma,u) - \alpha^{-}(\sigma,u))L^{+}_{\sigma}(u,V_u,\sigma_u)du \mid V_t = v, \sigma_t = \sigma].$$

Thus $\hat{L}(t,v,\sigma) = L^{+}(t,v,\sigma) - L^{-}(t,v,\sigma) \ge 0$ because $L^{+}_{\sigma}(t,v,\sigma) \ge 0$ (e.g. Romano and Touzi (1997)).

I can apply the same proof to $L^{+}_{erp}(t,v,\sigma)-L^{-}_{erp}(t,v,\sigma)$ since $L^{+}_{erp}(t,v,\sigma)$ ($L^{-}_{erp}(t,v,\sigma)$ respectively) is a value process of a European option.

Therefore, $J^{+}(t,v,\sigma)-J^{-}(t,v,\sigma)\geq 0$. In other words, the model with higher volatility drift term yields the higher option price.

(2) I now proceed to the induction of the relationship between option prices and market price of volatility risk.

Define two candidate pricing measures, Q^{+} and Q^{-}, with associated market price of $B^{\perp}$ risk parameter λ^{+} and λ^{-} respectively, where $\lambda^{+}>\lambda^{-}$.

By definition, with the stochastic volatility model specified in equation (1) and (2), the volatility drift terms corresponding to Q^{+} and Q^{-} are $[\alpha(\sigma_t,t)-\rho\xi(\sigma_t,t)\beta(\sigma_t,t)-\overline{\rho}\lambda^{+}{}_t\beta(\sigma_t,t)]$ and $[\alpha(\sigma_t,t)-\rho\xi(\sigma_t,t)\beta(\sigma_t,t)-\overline{\rho}\lambda^{-}{}_t\beta(\sigma_t,t)]$ respectively. It follows that $[\alpha(\sigma_t,t)-\rho\xi(\sigma_t,t)\beta(\sigma_t,t)-\overline{\rho}\lambda^{+}{}_t\beta(\sigma_t,t)]<$ $[\alpha(\sigma_t,t)-\rho\xi(\sigma_t,t)\beta(\sigma_t,t)-\overline{\rho}\lambda^{-}{}_t\beta(\sigma_t,t)]$ since $\lambda^{+}>\lambda^{-}$.

With the result from the step 1 that the model with higher volatility drift term yields the higher option price, it shows that the pricing measure with higher market price of $B^{\perp}$ risk, λ ,yields lower option prices.

Alternatively, since the whole process $\rho\xi(\sigma_t,t)+\overline{\rho}\lambda_t$ is termed volatility risk premium or market price of volatility risk, it can be said that the pricing measure with higher market price of volatility risk yields lower option prices. □

From the above proposition, I know that the convex option prices are decreasing in the market price of $B^{\perp}$ risk parameter λ_t, or equivalently the market price of volatility risk. Since the identification of the q-optimal measure $Q^{(q)}$ is given via the market price of $B^{\perp}$ risk with $\lambda^{(q)}(t,\sigma_t) = \beta(\sigma_t,t)\overline{\rho}g_{\sigma}(t,\sigma_t)$, the comparison of option prices under q-optimal measures can be performed through $\lambda^{(q)}(t,\sigma)$. As pointed out in the Henderson et al (2004) Theorem 2, the sign of $\lambda^{(q)}(t,\sigma_t)(=\beta(\sigma_t,t)\overline{\rho}g_{\sigma}(t,\sigma_t))$ is related to $\dfrac{\partial q\xi(t,\sigma)^2}{\partial\sigma}$ through the link to the first derivative of the function $g(t,\sigma)$ with respect to σ[29]. $\xi(t,\sigma)$ is the project Sharpe ratio and $g(t,\sigma)$ is the solution to the

---

[29] (A) Henderson et al (2004) shows that with volatility dynamics specified in equation (2) and $\lambda^{(q)}(t,\sigma_t) = \beta(\sigma_t,t)\overline{\rho}g_{\sigma}(t,\sigma_t)$,

(1) If $R \neq 0$ with the transformation $f = e^{-Rg}$ subject to $f(T,\sigma)=1$ and Feynman-Kac formula, it is able to arrive $g_{\sigma} = \dfrac{q}{f}\hat{E}[\int_t^T \xi\xi_{\sigma} f \exp(\int_t^s (\alpha_{\sigma} - q\rho\beta\xi_{\sigma} - q\rho\xi\beta_{\sigma} - \frac{q}{2}R\alpha^2)du)ds]$. Since $f > 0$, it is shown that (i) $q\xi\xi_{\sigma} > 0 \to g_{\sigma} > 0$ (ii) $q\xi\xi_{\sigma} < 0 \to g_{\sigma} < 0$ (iii) $q\xi\xi_{\sigma} = 0 \to g_{\sigma} = 0$.

(2) If $R = 0$ subject to $g(T,\sigma)=1$ and Feynman-Kac formula, it is able to arrive $g_{\sigma} = q\hat{E}[\int_t^T \xi\xi_{\sigma}\exp(\int_t^s(\alpha_{\sigma} - q\rho\beta\xi_{\sigma} - q\rho\xi\beta_{\sigma})du)ds]$. It is shown that (i) $q\xi\xi_{\sigma} > 0 \to g_{\sigma} > 0$ (ii) $q\xi\xi_{\sigma} < 0 \to g_{\sigma} < 0$ (iii) $q\xi\xi_{\sigma} = 0 \to g_{\sigma} = 0$.

(3) Since $\lambda^{(q)}(t,\sigma_t) = \beta(\sigma_t,t)\overline{\rho}g_{\sigma}(t,\sigma_t)$ with $\beta > 0$ and $\overline{\rho} > 0$, the sign of $\lambda^{(q)}(t,\sigma_t)$ hinges on $q\xi\xi_{\sigma}$.

representative equation. $\lambda^{(q)}(t,\sigma) \geq 0$ (respectively ≤ 0) iff $q\xi(t,\sigma)^2$ is non-decreasing in σ (respectively non-increasing in σ). The strict equality holds iff $q\xi(t,\sigma)^2$ is strictly non-decreasing in σ (respectively strictly non-increasing in σ).

4.4.2 Proposition 1 shows that the convex option payoffs in American options yield the same relationship displayed in European options between option prices and the market price of $B^{\perp}$ risk parameter λ_t, or equivalently the market price of volatility risk. It shows that the convex option prices are decreasing in the market price of $B^{\perp}$ risk parameter λ_t, or equivalently the market price of volatility risk. Incorporating 4.4.2 proposition 1 and Theorem 2 in Henderson et al (2004), I am able to arrive following propositions.

4.4.2 Proposition 2:

With non-zero correlation between the project randomness and volatility randomness, I conjecture that the option values to invest (i.e., prices of American call options) under q-optimal measures will be decreasing (respectively. increasing) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ as conjectured in Henderson et al (2004) and postulated in Henderson (2005) for European option case.

(B) It is straightforward that if $q\xi(t,\sigma)^2$ is non-decreasing in σ, i.e., $\frac{\partial q\xi(t,\sigma)^2}{\partial\sigma} \geq 0$, then it has $2q\xi(t,\sigma)\xi_\sigma \geq 0$, equivalently $q\xi(t,\sigma)\xi_\sigma \geq 0$. Therefore, $\lambda^{(q)}(t,\sigma_t) \geq 0$ if $q\xi(t,\sigma)^2$ is non-decreasing in σ.

(C) The reverse inequality and strict inequality is proven by the same argument.

Proof:

(1) Henderson et al (2004) Corollary 3 proposes that (i) if $\xi(t,\sigma)^2$ is increasing in σ, then for $q > 0$, European option prices under the q-optimal measure are less than those under minimal martingale measure, and for $q < 0$, European option prices under the q-optimal measure are greater than those under minimal martingale measure.[30] (ii) if $\xi(t,\sigma)^2$ is decreasing in σ, then for $q > 0$, European option prices under the q-optimal measure are greater than those under minimal martingale measure, and for $q < 0$, European option prices under the q-optimal measure are less than those under minimal martingale measure.[31]

(D) To sum up, the sign of $\lambda^{(q)}(t,\sigma_t)(=\beta(\sigma_t,t)\bar{\rho}g_\sigma(t,\sigma_t))$ is related to $\frac{\partial q\xi(t,\sigma)^2}{\partial\sigma}$.

[30] (1) $\xi(t,\sigma)^2$ is increasing in $\sigma \rightarrow \frac{\partial\xi(t,\sigma)^2}{\partial\sigma} > 0 \rightarrow \xi(t,\sigma)\xi_\sigma > 0$. The sign of $\lambda^{(q)}(t,\sigma_t)$ hinges on $q\xi\xi_\sigma$. For $q > 0$ and $\xi(t,\sigma)\xi_\sigma > 0$, it implies that $\lambda^{(q)}(t,\sigma_t) > 0$.

(2) In the q-optimal measure setting, minimal martingale measure means $\lambda^{(q)}(t,\sigma_t) = 0$.

(3) For the volatility dynamic specified in equation (2), it shows under the q-optimal measure, the volatility drift under q-optimal measure $[\alpha(\sigma_t,t) - \rho\xi(\sigma_t,t)\beta(\sigma_t,t) - \bar{\rho}\lambda_t\beta(\sigma_t,t)]$ is lower than that under minimal martingale measure, $[\alpha(\sigma_t,t) - \rho\xi(\sigma_t,t)\beta(\sigma_t,t)]$.

(4) Therefore, the option prices under the q-optimal measure are less than those under minimal martingale measure if $\xi(t,\sigma)^2$ is increasing in σ and $q > 0$, meaning $\lambda^{(q)}(t,\sigma_t) > 0$.

(5) The same argument applies to the case where $q < 0$. In this case, $\lambda^{(q)}(t,\sigma_t) < 0$, therefore, all the arguments are reversed.

[31] (1) $\xi(t,\sigma)^2$ is decreasing in $\sigma \rightarrow \frac{\partial\xi(t,\sigma)^2}{\partial\sigma} < 0 \rightarrow \xi(t,\sigma)\xi_\sigma < 0$. The sign of $\lambda^{(q)}(t,\sigma_t)$ hinges on $q\xi\xi_\sigma$. For $q > 0$ and $\xi(t,\sigma)\xi_\sigma < 0$, it implies that $\lambda^{(q)}(t,\sigma_t) < 0$.

(2) As in Henderson et al (2004), linking the relationship observed in the zero correlation case in Henderson (2005), they conjecture that European option prices under q-optimal measures will be decreasing (respectively, increasing) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ.

(3)

(a) I apply the proof for 4.4.2 Proposition 1 showing that the convex option prices are decreasing in the market price of $B^{\perp}$ risk parameter λ_t (or equivalently the market price of volatility risk given American options setting), I have

$$p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_1)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_1)}(t,\sigma));\lambda^{(q_1)}(t,\sigma)) \leq$$
$$p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_2)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_2)}(t,\sigma));\lambda^{(q_2)}(t,\sigma))$$

if $\lambda^{(q_1)}(t,\sigma) \geq \lambda^{(q_2)}(t,\sigma)$, where p(;•) is the option price under market price of volatility risk •.

(b) As in Henderson et al (2004), I employ the strong relations observed in the zero correlation case in Henderson (2005) that $\lambda^{(q)}(t,\sigma)$ will be non-decreasing

---

(2) In the q-optimal measure setting, minimal martingale measure means $\lambda^{(q)}(t,\sigma_t) = 0$.

(3) For the volatility dynamic specified in equation (2), it shows under the q-optimal measure, the volatility drift under q-optimal measure $[\alpha(\sigma_t,t) - \rho\xi(\sigma_t,t)\beta(\sigma_t,t) - \bar{\rho}\lambda_t\beta(\sigma_t,t)]$ is greater than that under minimal martingale measure, $[\alpha(\sigma_t,t) - \rho\xi(\sigma_t,t)\beta(\sigma_t,t)]$.

(4) Therefore, the option prices under the q-optimal measure are less than those under minimal martingale measure if $\xi(t,\sigma)^2$ is decreasing in σ and $q > 0$, meaning $\lambda^{(q)}(t,\sigma_t) < 0$.

(5) The same argument applies to the case where $q < 0$. In this case, $\lambda^{(q)}(t,\sigma_t) > 0$, therefore, all the arguments are reversed.

(respectively, non-increasing) in q if $\xi(t,\sigma)^2$ is non-decreasing (respectively, non-increasing) in σ.

(c) From (b) I find that $\lambda^{(q_1)}(t,\sigma) \geq \lambda^{(q_2)}(t,\sigma)$ (respectively, $\lambda^{(q_1)}(t,\sigma) \leq \lambda^{(q_2)}(t,\sigma)$) with $q_1 \geq q_2$ (respectively $q_1 \leq q_2$) if $\xi(t,\sigma)^2$ is non-decreasing (respectively non-increasing) in σ. Incorporate this relation with (a), I have

$$p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_1)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_1)}(t,\sigma));\lambda^{(q_1)}(t,\sigma)) \leq$$
$$p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_2)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_2)}(t,\sigma));\lambda^{(q_2)}(t,\sigma))$$

(respectively, $\begin{matrix} p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_1)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_1)}(t,\sigma));\lambda^{(q_1)}(t,\sigma)) \geq \\ p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_2)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_2)}(t,\sigma));\lambda^{(q_2)}(t,\sigma)) \end{matrix}$)

if $\lambda^{(q_1)}(t,\sigma) \geq \lambda^{(q_2)}(t,\sigma)$ (respectively, $\lambda^{(q_1)}(t,\sigma) \leq \lambda^{(q_2)}(t,\sigma)$).

(d) Therefore, I conjecture that the option values to invest (i.e., prices of American call options) under q-optimal measures will be decreasing (respectively. increasing) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ. □

4.4.2 Corollary 1:

With non-zero correlation between the project randomness and volatility randomness, the difference between NPV and option value to invest under q-optimal measures will be will be decreasing (respectively. increasing) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ.

4.4.2 Corrollary 2:

With non-zero correlation between the project randomness and volatility randomness, there are no conclusive relations between the option value to invest under q-optimal measures and the correlation between the project randomness and volatility randomness.

Proof:

(1) Under the general stochastic volatility setting, the stochastic volatility process under q-optimal measures is:

$$
\begin{aligned}
d\sigma_t = {} & [\alpha(\sigma_t, t) - \rho\xi(\sigma_t, t)\beta(\sigma_t, t) - \bar{\rho}^2\beta(\sigma_t, t)g_\sigma(t, \sigma_t)]dt \\
& + \rho\beta(\sigma_t, t)dB_t^{Q^{(q)}} + \bar{\rho}\beta(\sigma_t, t)dB_t^{\perp, Q^{(q)}}
\end{aligned}
$$

The change in correlation changes $(\rho\xi(\sigma_t, t) + \bar{\rho}\lambda_t^q)\beta(\sigma_t, t)$, where $\lambda^{(q)}(t, \sigma_t) = \beta(\sigma_t, t)\bar{\rho}g_\sigma(t, \sigma_t)$, and $g(t, \sigma_t)$ is defined in equation (5).

(2) $\forall \rho \in (0,1), \rho \uparrow \rightarrow (1-\rho^2) = \bar{\rho}^2 \downarrow$. It indicates that increases in ρ increases the first term, $\rho\xi(\sigma_t, t)\beta(\sigma_t, t)$ while decreasing the second term, $\bar{\rho}\lambda_t^q\beta(\sigma_t, t)$.

(3) $\forall \rho \in (0,1)$, given q, as ρ increases and the change in first term, $\rho\xi(\sigma_t, t)\beta(\sigma_t, t)$, dominates the change in the second term, $\bar{\rho}\lambda_t^q\beta(\sigma_t, t)$, by applying the proof of 4.4.2 Proposition 1 result (1), it shows that the option price due to increases in the correlation between the project randomness and volatility randomness decreases, and vice versa.

(4) $\forall \rho \in (-1,0)$, above results are reversed.

(5) Thus, there are no conclusive relations between the project randomness and volatility randomness, there are no conclusive relations between the option value to invest under q-optimal measures and the correlation between the project randomness and volatility randomness. □

4.4.2 Proposition 3:

With zero correlation between the project randomness and volatility randomness, I propose that $\lambda^{(q)}(t,\sigma)$ will be non-decreasing (respectively, non-increasing) in q if $\xi(t,\sigma)^2$ is non-decreasing (respectively, non-increasing) in σ as postulated in Henderson (2005). As a result, option values to invest under q-optimal measures will decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ.

Proof:

(1) Henderson (2005) Theorem 4 shows that $\lambda^{(q)}(t,\sigma)$ will be non-decreasing (respectively, non-increasing) in q if $\xi(t,\sigma)^2$ is non-decreasing (respectively, non-increasing) in σ.[32]

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(1) For $\rho = 0$, $\lambda^{(q)}(t,\sigma_t) = \beta(\sigma_t,t)g_\sigma(t,\sigma_t)$, where $g(t,\sigma)$ solves the following representation equation:
$\frac{q}{2}\xi(t,\sigma)^2 - \frac{1}{2}\beta(t,\sigma)^2(g_\sigma)^2 + \alpha(t,\sigma)g_\sigma + \frac{1}{2}\beta(t,\sigma)^2 g_{\sigma\sigma} + \dot{g} = 0$, or equivalently by Feynman-Kac formula $g(t,\sigma) = -\log \hat{E}[\exp(-\frac{q}{2}\int_t^T \xi(u,\sigma_u)^2 du) \mid \sigma_t = \sigma]$. The main idea is to analyze the q dependence of g_σ.

(2) $\dfrac{\partial g(t,\sigma)}{\partial q} = \dfrac{\hat{E}[\frac{1}{2}\int_t^T \xi(u,\sigma_u)^2 du \ \exp(-\frac{q}{2}\int_t^T \xi(u,\sigma_u)^2 du]}{\hat{E}[\exp(-\frac{q}{2}\int_t^T \xi(u,\sigma_u)^2 du)]}$

$\rightarrow 2\dfrac{\partial g(t,\sigma)}{\partial q} = \dfrac{\hat{E}[\int_t^T \xi(u,\sigma_u)^2 du \ \exp(-\frac{q}{2}\int_t^T \xi(u,\sigma_u)^2 du]}{\hat{E}[\exp(-\frac{q}{2}\int_t^T \xi(u,\sigma_u)^2 du)]}$ → Show the dependence of $\dfrac{\partial g}{\partial q}$ on σ.

(3) Following Henderson (2005),

(a) Fix t=0 and re-write $\int_0^T \xi(u,\sigma_u)^2 du$ as mean-variance tradeoff process K_T, I obtain

$$2\frac{\partial g(0,\sigma)}{\partial q} = \frac{\hat{E}[K_T \exp(-\frac{q}{2}K_T]}{\hat{E}[\exp(-\frac{q}{2}K_T]}.$$

(b) From Hobson (2004), there exists a process ς and a finite constant such that

$$\frac{q}{2}K_T = \int_0^T \varsigma(u,\sigma_u)^2 dW_u + \frac{1}{2}\int_0^T \varsigma(u,\sigma_u)^2 du + c.$$

(c) Define a new measure $\widetilde{P}$ by $\widetilde{Z}_T = \dfrac{d\widetilde{P}}{d\hat{P}} = \dfrac{\exp(-\frac{q}{2}K_T)}{\exp(-c)} = \dfrac{\exp(-\frac{q}{2}K_T)}{\hat{E}[\exp(-\frac{q}{2}K_T]}$. Then

$\widetilde{Z}_t = \hat{E}[\widetilde{Z}_T \mid F_t] = \exp(-\int_0^t \varsigma(u,\sigma_u)^2 dW_u - \frac{1}{2}\int_0^t \varsigma(u,\sigma_u)^2 du)$ by making use of the equation in (b).

(d) Therefore $2\dfrac{\partial g(0,\sigma)}{\partial q} = \dfrac{\hat{E}[K_T \exp(-\frac{q}{2}K_T]}{\hat{E}[\exp(-\frac{q}{2}K_T]} = \hat{E}K_T \dfrac{d\widetilde{P}}{d\hat{P}} = \widetilde{E}K_T = \widetilde{E}\int_0^T \xi(u,\sigma_u)^2 du.$

(2) The market price of $B^{\perp}$ risk parameter λ_t, or equivalently the market price of volatility risk may be negative if q<0. [33]However, this will not affect the relations shown in step (1).

(3)

(a) By step (1): $\lambda^{(q)}(t,\sigma)$ will be non-decreasing in q if $\xi(t,\sigma)^2$ is non-decreasing in σ. That is, $\lambda^{(q_1)}(t,\sigma) \geq \lambda^{(q_2)}(t,\sigma)$ with $q_1 \geq q_2$ if $\xi(t,\sigma)^2$ is non-decreasing in σ.

(b) From 4.4.2 Proposition 1, I show that the convex option prices are decreasing in the market price of $B^{\perp}$ risk parameter λ_t, (or equivalently the market

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(e) Invoking Henderson (2005) Lemma 3 with $m(f) = \int_0^T \xi(u,f)^2 du$, it states that if $m(f) = \int_0^T \xi(u,f)^2 du$ is non-decreasing (respectively, non-increasing) in f, $\widetilde{E}m(f_1) = \widetilde{E}\int_0^T \xi(u,f_1)^2 du \geq \widetilde{E}m(f_2) = \widetilde{E}\int_0^T \xi(u,f_2)^2 du$ (respectively, $\widetilde{E}m(f_1) = \widetilde{E}\int_0^T \xi(u,f_1)^2 du \leq \widetilde{E}m(f_2) = \widetilde{E}\int_0^T \xi(u,f_2)^2 du$) with $f_2 \geq f_1$. That is the dependence of $\frac{\partial g}{\partial q}$ on σ is non-decreasing (respectively, non-increasing) in σ if $\xi(t,\sigma)^2$ is non-decreasing (respectively, non-increasing) in σ. The proof is completed.

[33] Henderson (2005) Remrk 5 points out the index option data for a negative market price of volatility risk $\lambda^q(\sigma,t)$ (Bakshi and Kapadia (2003)). (1) When $\xi^2(\sigma,t)$ (project Sharpe ratio) non-decreases in σ, the negative market price of volatility risk (i.e., $\lambda^q(\sigma,t)$ <0) is consistent with pricing under q-optimal measures with q<0. (2) When $\xi^2(\sigma,t)$ (project Sharpe ratio) non-increases in σ, the negative market price of volatility risk (i.e., $\lambda^q(\sigma,t)$ <0 is consistent with q>0.

price of volatility risk given an American options setting). That is,

$$p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_1)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_1)}(t,\sigma));\lambda^{(q_1)}(t,\sigma)) \leq$$
$$p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_2)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_2)}(t,\sigma));\lambda^{(q_2)}(t,\sigma))$$

if $\lambda^{(q_1)}(t,\sigma) \geq \lambda^{(q_2)}(t,\sigma)$, where p(;•) means the option price under market price of volatility risk •.

(c) Combine (a) and (b): If $\xi(t,\sigma)^2$ is non-decreasing in σ, then $\lambda^{(q_1)}(t,\sigma) \geq \lambda^{(q_2)}(t,\sigma)$ with $q_1 \geq q_2$, which in turn leads to

$$p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_1)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_1)}(t,\sigma));\lambda^{(q_1)}(t,\sigma)) \leq$$
$$p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_2)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_2)}(t,\sigma));\lambda^{(q_2)}(t,\sigma)).$$

In other words, the option price, equivalently option value to invest, under q-optimal measures decreases in q if $\xi(t,\sigma)^2$ is non-decreasing in σ.

(d) Applying the same procedures to the case where $\xi(t,\sigma)^2$ is non- increasing in σ, I am able to derive the option values to invest under q-optimal measures will increase in q. □

4.4.2 Corollary 3:

With zero correlation between the project randomness and volatility randomness, the difference between NPV and the option values to invest under q-optimal measures will increase (respectively, decrease) in q if $\xi(t,\sigma)^2$ is non-increasing (respectively, non-decreasing) in σ.

4.4.2 Proposition 4:

(1) With a non-zero correlation between the project randomness and volatility randomness, from 4.4.2 Proposition 2, I conjecture that the optimal investment trigger under q-optimal measures decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ.

(2) With zero correlation between the project randomness and volatility randomness, from 4.4.2 Proposition 3, I propose that the optimal investment trigger under q-optimal measures decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ.

Proof:

(1) From 4.4.2 Proposition 2 and Proposition 3 I know that if $\xi(t,\sigma)^2$ is non-decreasing in σ, then $\lambda^{(q_1)}(t,\sigma) \geq \lambda^{(q_2)}(t,\sigma)$ with $q_1 \geq q_2$, which in turn leads to

$$\begin{aligned}&p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_1)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_1)}(t,\sigma));\lambda^{(q_1)}(t,\sigma)) \leq \\ &p(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_2)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_2)}(t,\sigma));\lambda^{(q_2)}(t,\sigma))\end{aligned}.$$

(2) By the definition of exercise boundary, $,\bar{v}(t,\sigma^{-1}(\lambda^{(q_1)}(t,\sigma)) \leq \bar{v}(t,\sigma^{-1}(\lambda^{(q_2)}(t,\sigma))$. In other words, the optimal investment trigger under q-optimal measures decreases in q if $\xi(t,\sigma)^2$ is non-decreasing in σ.

(3) Applying the same the same procedure to the case where $\xi(t,\sigma)^2$ is non-increasing in σ, I am able to show the optimal investment trigger under q-optimal measures increases in q if $\xi(t,\sigma)^2$ is non-increasing in σ. □

4.4.2 Proposition 5:

From the above considerations, I deduce that the optimal investment policy varies with choice of q-optimal measures (which directly link to different market price of non-hedgeable volatility randomness, $B^{\perp}$, risk).

<u>4.5 Abandonment Option Value – q-Optimal Measures</u>

This section investigates the abandonment option value (also project value with option to abandon) under q-optimal measures. I first formulate the investment problem and propose the corresponding model. I then show the ordering results for the abandonment option value (also project value with option to abandon) under q-optimal measures.

4.5.1 Investment Problem Formulation and the Model

I assume that the manager faces the abandonment timing problem in which the salvage value, K, grows at the risk free rate. I specify the manager's abandonment problem, and I find the abandonment option value by solving the following maximization problem (fixing some Q $\in$ Θ):

$$\begin{aligned} p^1(v,\sigma) &= \sup_{t \le \tau < \infty} E_t^{Q(q)}[e^{-r(\tau-t)}(Ke^{r(\tau-t)} - \widetilde{V}_\tau)^+ \mid V_t = v, \sigma_t = \sigma] \\ &= \sup_{t \le \tau < \infty} E_t^{Q(q)}[(K - V_\tau)^+ \mid V_t = v, \sigma_t = \sigma] \end{aligned}$$

where $\widetilde{V}$ denotes the forward process of discount process V_t, that is the original process.

Again, as in section 4.4, the abandonment option value will vary with different choices for Q, i.e., different choices for the market price of volatility risk. The market price of volatility risk, and in particular, the market price of unhedgeable randomness (i.e., $B^{\perp}$) risk, is my focus. The q-optimal measure market price of volatility risk is related to q through the λ^q equation and fundamental representation equation presented in the previous section.

Consistent with Hobson (2004), I assume a finite time horizon. Later, I will consider the perpetual American option in the classical real option setting. The manager's investment problem is revised as follows

$$
\begin{aligned}
p^{abn}(v,\sigma,t) &= \sup_{t\le\tau<T} E_t^{Q^{(q)}}[e^{-r(\tau-t)}(Ke^{r(\tau-t)}-\widetilde{V}_\tau)^+ \mid V_t = v, \sigma_t = \sigma] \\
&= \sup_{t\le\tau<T} E_t^{Q^{(q)}}[(K-V_\tau)^+ \mid V_t = v, \sigma_t = \sigma]
\end{aligned}
$$

where $\widetilde{V}$ denotes the forward process of discount process V_t, that is the original process.

Invoking the general model dynamics for the class of q-optimal measures, the problem then reduces to satisfying the following variational inequality:

$$\begin{cases} p_t^{abn} + \frac{1}{2}\sigma_t^2 V_t^2 p_v^{abn} + [\alpha(\sigma_t, t) - \rho\xi(\sigma_t, t)\beta(\sigma_t, t) - \overline{\rho}^2\beta^2(\sigma_t, t)g_\sigma(t, \sigma_t)]p_\sigma^{abn} \\ + \rho\sigma_t\beta(\sigma_t, t)p_{v\sigma}^{abn} + \frac{1}{2}\beta^2(\sigma_t, t)p_{\sigma\sigma}^{abn} \leq 0, t \in [0, T[,\ v > 0,\ p^{abn}(v, \sigma, t) \geq (K - v)^+ \\ \{p_t^{abn} + \frac{1}{2}\sigma_t^2 V_t^2 p_v^{abn} + [\alpha(\sigma_t, t) - \rho\xi(\sigma_t, t)\beta(\sigma_t, t) - \overline{\rho}^2\beta^2(\sigma_t, t)g_\sigma(t, \sigma_t)]p_\sigma^{abn} \\ + \rho\sigma_t\beta(\sigma_t, t)p_{v\sigma}^{abn} + \frac{1}{2}\beta^2(\sigma_t, t)p_{\sigma\sigma}^{abn}\} \times [p^{abn}(v, \sigma, T) - (K - v)^+] = 0, t \in [0, T[,\ v > 0 \\ [p^{abn}(v, \sigma, T) - (K - v)^+] = 0 \end{cases}$$

where $g(t,\sigma)$ is defined in equation (5), f or R=0 $g(t,\sigma)$ solves the representation equation (6) , for R≠0, $g(t,\sigma)$ solves the representation equation (7).

The problem reduces to solving the following equations

$p^{abn}(t, v, \sigma) = K - v$ for $v \leq V_{fb}(t, \sigma)$

For $v \geq V_{fb}(t, \sigma)$

$$p_t^{abn} + \frac{1}{2}\sigma_t^2 V_t^2 p_v^{abn} + [\alpha(\sigma_t, t) - \rho\xi(\sigma_t, t)\beta(\sigma_t, t) - \overline{\rho}^2\beta^2(\sigma_t, t)g_\sigma(t, \sigma_t)]p_\sigma^{abn}$$
$$+ \rho\sigma_t\beta(\sigma_t, t)p_{v\sigma}^{abn} + \frac{1}{2}\beta^2(\sigma_t, t)p_{\sigma\sigma}^{abn} = 0$$

with

$p^{abn}(T, v, \sigma) = (K - v)^+$

$V_{fb}(T, \sigma) = K$

In addition, the following boundary and smooth pasting conditions:

$$p^{abn}(t, v^*(=V_{fb}(t,\sigma), \sigma) = (K - v^*)^+$$

$$\frac{\partial p^{abn}(t, v^*(=V_{fb}(t,\sigma)), \sigma)}{\partial v^*} = -1$$

$$\frac{\partial p^1(t, v^*(=V_{fb}(t,\sigma), \sigma)}{\partial \sigma} = 0$$

assure that $p^{abn}(V_u, \sigma_u, u)$, $\frac{\partial p^{abn}(t, v^*(=V_{fb}(t,\sigma), \sigma)}{\partial v^*}$, and $\frac{\partial p^{abn}(t, v^*(=V_{fb}(t,\sigma), \sigma)}{\partial \sigma}$ are continuous across the boundary $V_{fb}(t,\sigma)$. The free boundary becomes a surface.

The partial differential equation cannot be solved analytically for a closed form solution. I must resort to numerical techniques. By considering different q (i.e., different market prices of volatility risk), the optimal abandonment trigger and abandonment option value will vary.

4.5.2 Ordering Results for Abandonment Option Value under q-Optimal Measures

The convexity of option value is used for generating the ordering results. The American put option value with stochastic volatility is strictly convex in the continuation region.[34]

4.5.2 Proposition 1:

The convex option prices are decreasing in the market price of $B^{\perp}$ risk parameter λ_t, or equivalently the market price of volatility risk. The reason is that, with

[34] See Touzi (1999) Lema 2.3 for the American put option with stochastic volatility.

a selected pricing measure, the dynamic presented in equation (4) shows that an increase in either λ_t or the market price of volatility risk corresponds to a decrease in the drift of the volatility.

Proof:

(1) With the convexity of the American put option premium, Touzi (1999) Proposition 3.1 shows that in the continuation region, the American put option premium is increasing with respect to the volatility variable. That is $p_{\sigma}^{abn} > 0$.

(2) With a selected pricing measure, the dynamic presented in equation 4 shows that an increase in either λ_t or the market price of volatility risk corresponds to a decrease in the drift of the volatility.

(3) Combine step (1) and (2); the convex American put option premium decreases in the market price of $B^{\perp}$ risk parameter λ_t, or equivalently the market price of volatility risk. □

4.5.2 Proposition 2:

With a non-zero correlation between the project randomness and volatility randomness, corresponding to 4.4.2 Proposition 2, I conjecture that the abandonment option values (i.e., prices of American put options) under q-optimal measures will be decreasing (respectively. increasing) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ.

Proof:

(1) Make use of the observed relations between $\lambda^{(q)}(t,\sigma)$ and $\xi(t,\sigma)^2$ under different q.

(2) Incorporate 4.5.2 Proposition 1. □

4.5.2 Corollary 1:

With non-zero correlation between the project randomness and volatility randomness, the difference between NPV and project value with abandonment option under q-optimal measures will be will be decreasing (respectively. increasing) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ.

4.5.2 Corollary 2:

With non-zero correlation between the project randomness and volatility randomness, there are no conclusive relations between the abandonment option value (also project value with option to abandon) under q-optimal measures and the correlation between the project randomness and volatility randomness.

Proof:

Same as 4.4.2 Proposition 2. □

4.5.2 Proposition 3:

With zero correlation between the project randomness and volatility randomness, corresponding to 4.4.2 Proposition 3, I propose that abandonment option values under q-optimal measures will decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ.

Proof:

Same as 4.5.2 Proposition 2. □

4.5.2 Corollary 3:

With zero correlation between the project randomness and volatility randomness, the difference between NPV and the project values with option to abandon under q-optimal measures will increase (respectively, decrease) in q if $\xi(t,\sigma)^2$ is non-increasing (respectively, non-decreasing) in σ.

4.5.2 Proposition 4:

(1) With non-zero correlation between the project randomness and volatility randomness, from 4.5.2 Proposition 2, I conjecture that the optimal abandonment trigger under q-optimal measures increase (respectively, decrease) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ.

(2) With zero correlation between the project randomness and volatility randomness, from 4.5.2 Proposition 3, I propose that the optimal abandonment trigger under q-optimal measures increase (respectively, decrease) in q if $\xi(t,\sigma)^2$ is increasing (respectively, decreasing) in σ.

Proof:

(1) From 4.5.2 Proposition 2 and Proposition 3 I know that if $\xi(t,\sigma)^2$ is non-decreasing in σ, then $\lambda^{(q_1)}(t,\sigma) \geq \lambda^{(q_2)}(t,\sigma)$ with $q_1 \geq q_2$, which in turn leads to

$$\begin{aligned} &p^{abn}(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_1)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_1)}(t,\sigma));\lambda^{(q_1)}(t,\sigma)) \leq \\ &p^{abn}(t,\bar{v}(t,\sigma^{-1}(\lambda^{(q_2)}(t,\sigma)),\sigma^{-1}(\lambda^{(q_2)}(t,\sigma));\lambda^{(q_2)}(t,\sigma)) \end{aligned}.$$

(2) By the definition of exercise boundary, $,\bar{v}(t,\sigma^{-1}(\lambda^{(q_1)}(t,\sigma)) \geq \bar{v}(t,\sigma^{-1}(\lambda^{(q_2)}(t,\sigma))$. In other words, the optimal abandonment trigger under q-optimal measures increases in q if $\xi(t,\sigma)^2$ is non-decreasing in σ.

(3) Apply the same the same procedure to the case where $\xi(t,\sigma)^2$ is non-increasing in σ; I am able to arrive the optimal abandonment trigger under q-optimal measures decreases in q if $\xi(t,\sigma)^2$ is non-increasing in σ. □

4.5.2 Proposition 5:

From the above considerations, I deduce that different choice of q-optimal measures, which directly link to different market price of non-hedgeable volatility randomness, $B^{\perp}$, risk, will alter the optimal abandonment policy.

4.6 General Stochastic Volatility Model and Indifference Pricing

Delbaen et al. (2002) show that the indifference price of a contingent claim under exponential utility is related via the dual problem to the solution of a minimum entropy problem. As the coefficient of risk aversion approaches zero, the utility indifference price approaches the expected value under the minimum entropy martingale measure. Sircar and Zariphopoulou (2005) apply indifference pricing in a stochastic volatility setting. I extend this technique to a real options setting, and I investigate the interaction among risk aversion, and certain q-optimal measures.

4.6.1 Option Value to Invest – Investment Problem Formulation and the Model

I fix a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},P)$ with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, the increasing σ-algebras generated by the pair of Brownian motions $(B_s)_{s\leq t}$ and $(B^{\perp}_s)_{s\leq t}$, where $B^{\perp}$ is orthogonal to B, satisfying the usual conditions of right-continuity and completeness. Let V be the project value and σ be the volatility of project value. Throughout the analysis, I assume that the riskless interest rate is equal to zero. Under the real world measure P, V and σ follow a stochastic process with coefficients satisfying sufficient regularity conditions to ensure the existence of a unique solution with the Strong Markov Property, as follows:

$$\frac{dV_t}{V_t} = \sigma_t(\xi(\sigma_t,t)dt + dB_t)^{35};\ \ V_t = v \text{ --(1)}$$

$$d\sigma_t = \alpha(\sigma_t,t)dt + \beta(\sigma_t,t)dW_t = \alpha(\sigma_t,t)dt + \beta(\sigma_t,t)(\rho dB_t + \overline{\rho} dB_t^{\perp})\,;\ \sigma_t = \sigma \text{ --------(2)}$$

---

[35] (1) It will never be optimal to early exercise American call option if the underlying asset does not pay dividends.

(2) To capture the "early exercise" characteristics of the American call option, $\sigma_t(\xi(\sigma_t,t)) = \mu(\sigma_t,t) - \kappa(\sigma_t,t)$, where $\kappa(\sigma_t,t)$ is the continuous dividend yield. For simplicity and tractability, I keep μ and κ constant.

(3) For the Girsanov's transformation for equivalent martingale measure, if we specify

(i) $B_t^Q = B_t + \int_0^t \xi(u,\sigma_u)du$, then V_t will be a martingale with $\xi(\sigma_t,t) = \dfrac{\mu-\kappa}{\sigma_t}$ rather than the one with no dividend yield $\xi(\sigma_t,t) = \dfrac{\mu}{\sigma_t}$.

I further assume a twin security exists, and the manager generates trading wealth through dynamically adjusting the amount invested in the twin security for θ_s for s > t and riskless bond, yielding constant interest rate $r = 0$. I assume that no intermediate consumption nor infusion of extraneous funds is allowed. The trading wealth process under real world probability measure P is:

$$dX_t = \theta_t \mu(\sigma_t, t)dt + \theta_t \sigma_t dB_t \quad ; \; X_t = x \text{ --------------------------(3)}$$

The single control variable θ_s is called admissible if it is $\mathcal{F}_s$ measurable and satisfies the integrability constraint $E\int_t^T \sigma(s,\sigma_s)^2 \theta_s^2 ds < \infty$.

I assume the manager has exponential utility function $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$. I specify γ > 0, i.e., the manager exhibits constant absolute risk-aversion.

The risk-averse manager's investment problem is to maximize her expected utility of wealth with optimally exercising investment decision with investment cost K during the investment horizon, [t,T]. I assume the manager has a one time irreversible investment opportunity. Prior to exercising the investment option, time τ, the wealth

---

(ii) $B_t^Q = B_t + \int_0^t \frac{\mu(u,\sigma_u)}{\sigma_t} du = B_t + \int_0^t \varsigma_u du$, then V_t becomes $\frac{dV_t}{V_t} = -\kappa dt + \sigma_t dB_t^Q$ which itself is not a martingale, rather the process with dividend revisted in the underlying asset becomes the martingale.

(4) For tractability, we adopt (i).

refers to the quantity X_τ the manager generates by following investment policy θ, at the time of investment, τ, the manager's wealth X_τ increases to $X_\tau + [V_\tau - K]^{+}$[36], and after time τ, the manager faces the same investment opportunities and continues trading between the twin security and the bond till the end of investment horizon T.

4.6.2 Option Value to Invest – Indifference Pricing

In order to construct the indifference price, I introduce two stochastic optimization problems. I first define the value function with no investment opportunities presented, i.e., a classical Merton portfolio problem modified to accommodate stochastic volatility:

$$V(X,\sigma,t) = \sup_{\theta} E[U(X_T) \mid X_t = x, \sigma_t = \sigma] \text{ ----------------------------------(4)}$$

where the single control variable θ_s is called admissible if it is $\mathcal{F}_s$ measurable and satisfies the integrability constraint $E\int_t^T \sigma(s,\sigma_s)^2 \theta_s^2 ds < \infty$.

I then introduce the investment opportunity into the optimization problem. When the manager faces the investment opportunity, the Dynamic Programming

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(5) In the American put option case, I leave $\sigma_t(\xi(\sigma_t,t)) = \mu(\sigma_t,t)$, that is, $\xi(\sigma_t,t)$ is the project Sharpe ratio. Again for simplicity and tractability, I keep μ and κ constant.

[36] (1) It is assumed the payoff is bounded. For an unbounded payoff, e.g. traditional calls, regularization technique, that is, regularizing payoff, is resorted (See for example, Fouque et. al . (2003), Ilhan and Sircar (2004)). It does not affect the analysis framework.

(2) Or we may follow Frey and Sin (1999) by assuming $(V_T\text{-}K)^+$ is locally bounded.

Optimality Principle[37] yields that at time τ, the manager's expected utility payoff is given by:

$$J(x,v,\sigma,t) = E[V(X_\tau + (V_\tau - K)^+, \tau) \mid X_t = x, V_t = v, \sigma_t = \sigma] \text{-------------------(5)}$$

Therefore, the manager's value function is then defined for $0 \le t \le T$ as

$$\begin{aligned} F(x,v,\sigma,t) &= \sup_A J(x,v,\sigma,t) \\ &= \sup_A E[V(X_\tau + (V_\tau - K)^+, \tau) \mid X_t = x, V_t = v, \sigma_t = \sigma] \end{aligned} \text{--------------(6)}$$

where

$A = \{(\theta,\tau) : \theta_s \text{ is } \mathcal{F}_s - \text{progressively measurable}, E\int_t^T \theta_s \sigma(\sigma_s, s)^2 < \infty, \text{and } \tau \in I_{[t,T]}\}$, and

$I_{[t,T]}$ is the set of all stopping times of Filtration $\mathcal{F}$.

Combined with the classical Merton portfolio problem, appropriately modified to accommodate stochastic volatility, defined in equation (4), the manager's indifference price of this investment opportunity, $h(x,v,\sigma,t)$, is defined by:

$$V(x,\sigma,t) = F(x - h(x,v,\sigma,t), v, \sigma, t) \text{---(7)}$$

It states that the manager is indifferent between paying nothing and not having the investment opportunity versus paying $h(x,v,\sigma,t)$ to hold the investment opportunity. Equivalently, equation (7) can be expressed as $V(x + h(x,v,\sigma,t), x, \sigma, t) = F(x,v,\sigma,t)$ which states that the manager is indifferent between having the wealth

[37] Bellman and Dreyfus (1962), P15 "An optimal sequence of decisions in a multistage decision process problem has the property that whatever the initial stage, state, and decision are, the remaining decisions must constitute an optimal sequence of decisions for the remaining problem, with the stage and state resulting from the first decision considered as initial conditions.

$x+h(x,v,\sigma,t)$ and having wealth x while simultaneously holding the investment opportunity.

When early exercise is not allowed, that is the investment can only be made at time T, the manager's optimization problem in equation (6) becomes

$$\begin{aligned}\overline{F}(x,v,\sigma,t) &= \sup_{A_0} E[V(X_T+(V_T-K)^+,T)\mid X_t=x,V_t=v,\sigma_t=\sigma] \\ &= \sup_{A_0} E[-\frac{1}{\gamma}e^{-\gamma(X_T+(V_T-K)^+)}\mid X_t=x,V_t=v,\sigma_t=\sigma]\end{aligned} \qquad \text{-----------}(8)$$

where the set of A_0 takes the set of A by restricting $\tau=T$.

Thus, the manager's indifference price of this investment opportunity fixed at time T is defined by

$$V(x,\sigma,t)=\overline{F}(x-H(x,v,\sigma,t),v,\sigma,t) \text{ --}(9)$$

$H(x,v,\sigma,t)$ is the manager's indifference price of investment opportunity fixed at time T.

To facilitate presentation, I introduce the following operators and Hamiltonians:

$$\begin{aligned}A^{v,\sigma}F &= \frac{1}{2}\sigma(\sigma,t)^2v^2F_{vv}+\rho\sigma(\sigma,t)\beta(\sigma,t)vF_{v\sigma}+\frac{1}{2}\beta(\sigma,t)^2F_{\sigma\sigma} \\ &\quad +\sigma(\sigma,t)\xi(\sigma,t)vF_v+\alpha(\sigma,t)F_\sigma\end{aligned}$$

$$A^{\sigma}F=\frac{1}{2}\beta(\sigma,t)^2F_{\sigma\sigma}+\alpha(\sigma,t)F_\sigma$$

where A^{σ} and $A^{v,\sigma}$ are actually the infinitesimal generators of the Markov process σ and (V,σ) respectively.

$$\mathsf{H}^{v,\sigma}(F_{xx},F_{xv},F_{x\sigma},F_x)=\sup_{\theta}\{\frac{1}{2}\theta^2\sigma(\sigma,t)^2F_{xx}+\theta\sigma(\sigma,t)^2vF_{xv}+\theta\rho\sigma(\sigma,t)\beta(\sigma,t)F_{x\sigma}$$
$$+\theta\mu(\sigma,t)F_x\}$$

$$L^{v,\sigma}F=\frac{1}{2}\sigma(\sigma,t)^2v^2F_{vv}+\rho\sigma(\sigma,t)\beta(\sigma,t)vF_{v\sigma}+\frac{1}{2}\beta(\sigma,t)^2F_{\sigma\sigma}-\kappa(\sigma,t)vF_v$$
$$+(\alpha(\sigma,t)-\rho\frac{\mu(\sigma,t)}{\sigma(\sigma,t)}\beta(\sigma,t))F_\sigma$$

$$L^{\sigma}F=A^{\sigma}F-\rho\frac{\mu(\sigma,t)}{\sigma(\sigma,t)}\beta(\sigma,t)F_\sigma$$

$$R(\Gamma_v,\Gamma_\sigma,\Gamma)=\frac{1}{2}\sigma(\sigma,t)^2v^2\frac{\Gamma_v^2}{\Gamma}+\rho\sigma(\sigma,t)\beta(\sigma,t)v\frac{\Gamma_v\Gamma_\sigma}{\Gamma}+\frac{1}{2}\rho^2\beta(\sigma,t)^2\frac{\Gamma_\sigma^2}{\Gamma}$$

4.6.2 Theorem 1:

From Sircar and Zariphopoulou 2005 Theorem 2.7, the indifference price for the manager's investment opportunity fixed at time T, $H(x,v,\sigma,t)$, (more specifically $H(v,\sigma,t)$ because the exponential utility function allows us to separate "wealth" from utility indifference pricing), is the unique $C^{2,2,1}(\mathbb{R}^+\times\mathbb{R}\times[0,T])$ bounded solution of the following pricing equation

$$\begin{cases} H_t+L^{v,\sigma}H+(1-\rho^2)\beta(\sigma,t)\phi_\sigma H_\sigma-\frac{1}{2}\gamma(1-\rho^2)\beta(\sigma,t)^2H_\sigma^2=0 \\ H(v,\sigma,T)=(v-K)^+ \end{cases}$$

where $\phi(\sigma,t,T)$ solves

$$\begin{cases} \phi_t+L^{\sigma}\phi+\frac{1}{2}(1-\rho^2)\beta(\sigma,t)^2\phi_\sigma^2=\frac{\mu(\sigma,t)}{2\sigma(\sigma,t)^2} \\ \phi(\sigma,T,T)=0 \end{cases}$$

It is given by

$$H(v,\sigma,t) = -\frac{1}{\gamma}\ln\frac{J(v,\sigma,t)}{e^{\phi(\sigma,t)}} = -\frac{1}{\gamma}\ln\frac{e^{\gamma\psi(v,\sigma,t)}}{e^{\phi(\sigma,t)}} = -\psi(v,\sigma,t) + \frac{1}{\gamma}\phi(\sigma,t)$$

where $\phi(\sigma,t,T)$ is defined above and $\psi(\sigma,t,T)$ solves

$$\begin{cases} \psi_t + L^{v,\sigma}\psi + \frac{1}{2}\gamma(1-\rho^2)\beta(\sigma,t)^2\psi_\sigma^2 = \frac{\mu(\sigma,t)}{2\gamma\sigma(\sigma,t)^2} \\ \psi(v,\sigma,T) = (v-K)^+ \end{cases}$$

4.6.2 Proposition 1:

As pointed out in equation (4), the classical Merton portfolio problem, appropriately modified to accommodate stochastic volatility, defines the following value function:

$$V(X,\sigma,t) = \sup_\theta E[U(X_T) \mid X_t = x, \sigma_t = \sigma]$$

The value function $V(x,\sigma,t)$ solves the Hamilton-Jacobi-Bellman equation

$$\begin{cases} V_t + \sup_\theta\{\frac{1}{2}\theta^2\sigma(\sigma,t)^2V_{xx} + \theta\rho\sigma(\sigma,t)\beta(\sigma,t)V_{x\sigma} + \theta\mu(\sigma,t)V_x\} + \frac{1}{2}\beta(\sigma,t)^2V_{\sigma\sigma} \\ \quad + \alpha(\sigma,t)V_\sigma) = 0 \\ V(x,\sigma,T) = -\frac{1}{\gamma}e^{-\gamma x} \end{cases}$$

With the optimal θ, the above becomes

$$\begin{cases} V_t - \frac{(\mu(\sigma,t)V_x + \rho\sigma(\sigma,t)\beta(\sigma,t)V_{x\sigma})^2}{2\sigma(\sigma,t)^2V_{xx}} + \frac{1}{2}\beta(\sigma,t)^2V_{\sigma\sigma} + \alpha(\sigma,t)V_\sigma) = 0 ----(10) \\ V(x,\sigma,T) = -\frac{1}{\gamma}e^{-\gamma x} \end{cases}$$

Propose the form of the solution $V(x,\sigma,t) = -\frac{1}{\gamma}e^{-\gamma x}G(\sigma,t)$

After simplification, I have

$$\begin{cases} G_t + L^{\sigma}G - \frac{1}{2}\frac{\mu(\sigma,t)}{\sigma(\sigma,t)^2}G - \frac{1}{2}\rho^2\beta(\sigma,t)^2\frac{G_{\sigma}^2}{G} = 0 \\ G(\sigma,T) = 1 \end{cases}$$

Linearize the above non-linear HJB by Hopf-Cole-type transformation[38] and set

$G(\sigma,t) = g(\sigma,t)^{\delta}$ with $\delta = \frac{1}{1-\rho^2}$, yielding

$$\begin{cases} g_t + L^{\sigma}g - \frac{1}{2}(1-\rho^2)\frac{\mu(\sigma,t)^2}{\sigma(\sigma,t)^2}g = 0 \\ g(\sigma,T) = 1 \end{cases}$$

To sum up, the value function is given by

$$V(x,\sigma,t) = -\frac{1}{\gamma}e^{-\gamma x}G(\sigma,t) = -\frac{1}{\gamma}e^{-\gamma x}g(\sigma,t)^{\frac{1}{1-\rho^2}} \text{---------------------------------}(11)$$

where $g(\sigma,t)$ solves

$$\begin{cases} g_t + L^{\sigma}g - \frac{1}{2}(1-\rho^2)\frac{\mu(\sigma,t)^2}{\sigma(\sigma,t)^2}g = 0 \\ g(\sigma,T) = 1 \end{cases} \text{ for } (\sigma,t) \in R\times[t,T].$$

4.6.2 Proposition 2:

Following Oberman and Zariphopoulou (2003) Theorem 6 and Zariphopoulou and Davis (1995), the manager's value function $F(x,v,\sigma,t)$ defined in equation (5) is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation

[38] The idea was introduced by Zariphopoulou (1983) with the terminology distortion.

$$
\begin{cases}
\min(-F_t - \sup_{\theta}\{\frac{1}{2}\theta^2\sigma(\sigma,t)^2 F_{xx} + \theta\sigma(\sigma,t)^2 vF_{xv} + \theta\rho\sigma(\sigma,t)\beta(\sigma,t)F_{x\sigma} \\
\quad + \theta\mu(\sigma,t)F_x\} - (\frac{1}{2}\sigma(\sigma,t)^2 v^2 F_{vv} + \rho\sigma(\sigma,t)\beta(\sigma,t)vF_{v\sigma} \\
\quad + \frac{1}{2}\beta(\sigma,t)^2 F_{\sigma\sigma} + \mu(\sigma,t)vF_v - \kappa(\sigma,t)vF_v + \alpha(\sigma,t)F_\sigma), \\
\quad F(x,v,\sigma,t) - V(x+(v-K)^+,t) = 0 ------------------ (12) \\
\quad\quad F(x,v,\sigma,T) = V(x+(v-K)^+,T) = -\frac{1}{\gamma}e^{-\gamma(x+(v-K)^+)}
\end{cases}
$$

where $F(x,v,\sigma,t) = V(x+(v-K)^+,t)$ is in the class of functions that are concave and increasing in the spatial argument x and bounded in $(v-K)^+$.[39]

4.6.2 Proposition 3:

The manager's early exercise indifference price (i.e., option value to invest) is the unique bounded viscosity solution of the quasilinear variational inequality with terminal conditions

$$
\begin{cases}
\min\,(-h_t - L^{v,\sigma}h - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma + \frac{1}{2}\gamma(1-\rho^2)\beta(\sigma,t)^2 h_\sigma^2, h(v,\sigma,t) \\
-(v-K)^+) = 0 \\
\quad\quad h(v,\sigma,T) = (v-K)^+
\end{cases}
$$

Proof:

Using the pricing equality (7) and the HJB equation (12), evaluated at the point $(x-h(v,\sigma,t),v,\sigma,t)$, the HJB becomes

[39] Following Frey and Sin (1999), we assume that $(V_T - K)^+$ is locally bounded.

$$\min(-V_t - \sup_\theta\{\frac{1}{2}\theta^2\sigma(\sigma,t)^2V_{xx} + \theta\rho\sigma(\sigma,t)\beta(\sigma,t)F_{x\sigma} + \theta\mu(\sigma,t)F_x\}$$
$$+V_x dh(v,\sigma,t) - \frac{1}{2}\beta(\sigma,t)^2F_{\sigma\sigma} - \alpha(\sigma,t)F_\sigma), \mathrm{V}(x,v,\sigma,t) - V(x-h(v,\sigma,t)$$
$$+(v-K)^+,t)) = 0 ----(13)$$

With the optimal θ, the above becomes

$$\min(-V_t + \frac{(\mu(\sigma,t)V_x + \rho\sigma(\sigma,t)\beta(\sigma,t)V_{x\sigma})^2}{2\sigma(\sigma,t)^2V_{xx}} + V_x dh(v,\sigma,t) - \frac{1}{2}\beta(\sigma,t)^2V_{\sigma\sigma} - \alpha(\sigma,t)V_\sigma$$
$$,\mathrm{V}(x,v,\sigma,t) - V(x-h(v,\sigma,t)+(v-K)^+,t)) = 0$$

That is,

$$\min(-V_t + \frac{(\mu(\sigma,t)V_x + \rho\sigma(\sigma,t)\beta(\sigma,t)V_{x\sigma})^2}{2\sigma(\sigma,t)^2V_{xx}} - \frac{1}{2}\beta(\sigma,t)^2V_{\sigma\sigma} - \alpha(\sigma,t)V_\sigma$$
$$+V_x(-h_t - L^{v,\sigma}h - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma + \frac{1}{2}\gamma(1-\rho^2)\beta(\sigma,t)^2h_\sigma^2),$$
$$\mathrm{V}(x,v,\sigma,t) - V(x-h(v,\sigma,t)+(v-K)^+,t)) = 0$$

For the first part of the above variational inequality, when evaluated at the optimal θ, the term

$$-V_t + \frac{(\mu(\sigma,t)V_x + \rho\sigma(\sigma,t)\beta(\sigma,t)V_{x\sigma})^2}{2\sigma(\sigma,t)^2V_{xx}} - \frac{1}{2}\beta(\sigma,t)^2V_{\sigma\sigma} - \alpha(\sigma,t)V_\sigma = 0 \text{ (see}$$

equation (10))

By equation (11) $V(x,\sigma,t) = -\frac{1}{\gamma}e^{-\gamma x}g(\sigma,t)^{\frac{1}{1-\rho^2}}$, $g(\sigma,t)$ must be positive since $V(x,\sigma,t)$ is negative. As a result, I see that $V_x = e^{-\gamma x}g(\sigma,t)^{\frac{1}{1-\rho^2}} > 0$. Combining the above two arguments, $h(v,\sigma,t)$ satisfies

$$(-h_t - L^{v,\sigma}h - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma + \frac{1}{2}\gamma(1-\rho^2)\beta(\sigma,t)^2 h_\sigma^2) \geq 0 \text{ --------------(14)}$$

for $(x,v,\sigma,t) \in R \times R \times R \times [0,T]$.

On the other hand, the monotonicity of $V(x,\sigma,t)$ with respect to the spatial argument x and the form of the obstacle term in equation (13) yield

$$x - (x + h - (v-K)^+) \geq 0 \rightarrow h(v,\sigma,t) \geq (v-K)^+$$

for $(x,v,\sigma,t) \in R \times R \times R \times [0,T]$.---(15)

Combining the inequalities (14) and (15) yields:

$$\begin{cases} \min\,(-h_t - L^{v,\sigma}h - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma + \frac{1}{2}\gamma(1-\rho^2)\beta(\sigma,t)^2 h_\sigma^2 \\ \qquad , h(v,\sigma,t) - (v-K)^+) = 0 \\ \qquad\qquad\qquad h(v,\sigma,T) = (v-K)^+ \end{cases}$$

Remark: Viscosity Solution and Some Properties[40]

Frequently, the value function might not be smooth, and we must relax the notion of solutions to the H-J-B equation. A rich class of weak solutions to the H-J-B are known as viscosity solutions. They were introduced by Crandall and Lions (1983) for the first order non-linear PDE and by Lions (1983) for the second order case. The viscosity solution provides rigorous characterization of the value function as the unique solution to the H-J-B equation. Strong stability properties provide excellent

[40] References:

(1) Fleming, Wendell H. and H. Mete Soner (1992): Controlled Markov Processes and Viscosity Solutions. Springer-Verlag

(2) Nizar Touzi (2002): Stochastic Control Problems, Viscosity Solutions and Application to Finance

(3) Thaleia Zariphopolou (2001): Stochastic Control Methods in Asset Pricing

convergence results for a large class of numerical schemes for the value function and the optimal policies. The basic ideas are presented as follows:

Consider a nonlinear second order PDE of the form

$F(x,u(x),Du(x),D^2u(x)) = 0$ for $x \in \Omega$

where Ω is an open subset of R^n, $Du(x)$ and $D^2u(x)$ denote the gradient vector and the second derivative matrix of $u(x)$, and the function F is continuous in all its arguments and degenerate elliptic; that is

$F(x,r,p,A) \le F(x,r,p,B)$ whenever $A \ge B$

Definition (1): A function $u:\Omega \to R$ is a classical supersolution (respectively, subsolution) of $F(x,u(x),Du(x),D^2u(x)) = 0$ if $u \in C^2(\Omega)$ and

$F(x,u(x),Du(x),D^2u(x)) \ge (respectively, \le) 0$ for $x \in \Omega$

Definition (2): Let u be a $C^2(\Omega)$ function. Then the following claims are equivalents:

(I) u is a classical supersolution (respectively, subsolution) of $F(x,u(x),Du(x),D^2u(x)) = 0$.

(II) For all pairs $(x_0,\varphi) \in \Omega \times C^2(\Omega)$ such that x_0 is a minimizer (respectively, maximizer) of the difference $(u-\varphi)$ on Ω, we have $F(x_0,u(x_0),D\varphi(x_0),D^2\varphi(x_0)) \ge (respectively, \le) 0$.

The formal definition of viscosity solution is presented as follows:

For a locally bounded function $u:\Omega \to R$, denote $\underline{u}$ and $\overline{u}$ the lower and upper semicontinuous envelopes of u and

$\underline{u} = \liminf_{x' \to x} u(x')$, $\overline{u} = \limsup_{x' \to x} u(x')$

Observe that the definition (II) does not involve the regularity of u. It therefore suggests the following weak notion of solution to $F(x,u(x),Du(x),D^2u(x)) = 0$.

Definition (3): For a locally bounded function $u:\Omega \to R$,

(I) u is a (discontinuous) viscosity supersolution of $F(x,u(x),Du(x),D^2u(x)) = 0$ if $F(x_0,\underline{u}(x_0),D\varphi(x_0),D^2\varphi(x_0)) \geq 0$ For all pairs $(x_0,\varphi) \in \Omega \times C^2(\Omega)$ such that x_0 is a minimizer of the difference $(\underline{u} - \varphi)$ on Ω.

(II) u is a (discontinuous) viscosity subsolution of $F(x,u(x),Du(x),D^2u(x)) = 0$ if $F(x_0,\overline{u}(x_0),D\varphi(x_0),D^2\varphi(x_0)) \leq 0$ For all pairs $(x_0,\varphi) \in \Omega \times C^2(\Omega)$ such that x_0 is a maximizer of the difference $(\overline{u} - \varphi)$ on Ω.

(III) u is a (discontinuous) viscosity solution of $F(x,u(x),Du(x),D^2u(x)) = 0$ if it is both a viscosity supersolution and subsolution of $F(x,u(x),Du(x),D^2u(x)) = 0$.

4.6.3 Option Value to Invest, Indifference Pricing and Pricing Measure

The indifference pricing equation for the manager's investment opportunity fixed at time T,

$$H_t + L^{v,\sigma} H + (1-\rho^2)\beta(\sigma,t)\phi_\sigma H_\sigma - \frac{1}{2}\gamma(1-\rho^2)\beta(\sigma,t)^2 H_\sigma^2 = 0$$

or the corresponding early exercise indifference price, $h(v,\sigma,t)$, the solution of the unique bounded viscosity solution of the quasilinear variational inequality with terminal conditions

$$\begin{cases} \min\,(-h_t - L^{v,\sigma}h - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma + \frac{1}{2}\gamma(1-\rho^2)\beta(\sigma,t)^2 h_\sigma^2 \\ \quad , h(v,\sigma,t) - (v-K)^+) = 0 \\ \qquad\qquad h(v,\sigma,T) = (v-K)^+ \end{cases}$$

admits a new pricing measure Q:

$$\frac{dQ}{dP} = \exp(-\int_0^T \xi(\sigma_t,t)dB_t - \frac{1}{2}\int_0^T \xi(\sigma_t,t)^2 dt + \int_0^T \lambda(\sigma_t,t)dB_t^\perp - \frac{1}{2}\int_0^T \lambda(\sigma_t,t)^2 dt)$$

where $-\lambda(\sigma_t,t)$ is a particular market price of volatility risk. In the above pricing formula, $\lambda(\sigma_t,t) = \bar{\rho}\beta(\sigma_t,t)\phi_\sigma(\sigma_t,t)$. By Girsanov's theorem, Brownian motions B and $B^\perp$ under Q are given as:

$$B_t^Q = B_t + \int_0^t \xi(u,\sigma_u)du \text{ and } B_t^{\perp,Q} = B_t^\perp - \int_0^t \lambda_u du$$

Under Q, V and σ follow the processes:

$$\frac{dV_t}{V_t} = \sigma_t dB_t^Q$$

$$\begin{aligned} d\sigma_t &= [\alpha(\sigma_t,t) - \rho\xi(\sigma_t,t)\beta(\sigma_t,t) + \bar{\rho}\lambda_t\beta(\sigma_t,t)]dt + \rho\beta(\sigma_t,t)dB_t^Q + \bar{\rho}\beta(\sigma_t,t)dB_t^{\perp,Q} \\ &= [\alpha(\sigma_t,t) - \rho\xi(\sigma_t,t)\beta(\sigma_t,t) + \bar{\rho}^2\beta(\sigma_t,t)\phi_\sigma(\sigma_t,t)]dt + \rho\beta(\sigma_t,t)dB_t^Q + \bar{\rho}\beta(\sigma_t,t)dB_t^{\perp,Q} \\ &= [\alpha(\sigma_t,t) - (\rho\xi(\sigma_t,t) - \bar{\rho}^2\phi_\sigma(\sigma_t,t))\beta(\sigma_t,t)]dt + \rho\beta(\sigma_t,t)dB_t^Q + \bar{\rho}\beta(\sigma_t,t)dB_t^{\perp,Q} \end{aligned}$$

Under the new pricing measure Q, the market price of volatility risk premium, or the market price of volatility risk is $(\rho\xi(\sigma_t,t) - \bar{\rho}^2\phi_\sigma(\sigma_t,t))$. This new pricing measure is actually the minimal entropy martingale measure which minimizes the relative entropy of pricing measure Q with respect to the historical pricing measure P In terms of q-optimal measures discussed previously, it yields q=1. Several papers describe the link between maximizing exponential utility function and the minimal entropy martingale measure (see for example Fritelli (2000), Delbaen et. al. (2002) etc.). There are also several studies involving this issue with the stochastic volatility presented (e.g. Sircar and Zariphopoulou (2005) and Ilhan and Sircar (2005)). The following briefly sketches the proof and assertion from Sircar and Zariphopoulou (2005) and Ilhan and Sircar (2005).

Sircar and Zariphopoulou (2005) present the view from "relative entropy penalization". The indifference pricing obtained admits a new pricing measure under which V and σ follow the processes:

$$\frac{dV_t}{V_t} = \sigma_t dB_t^Q$$

$$\begin{aligned}
d\sigma_t &= [\alpha(\sigma_t,t) - \rho\xi(\sigma_t,t)\beta(\sigma_t,t) + \overline{\rho}\lambda_t\beta(\sigma_t,t)]dt + \rho\beta(\sigma_t,t)dB_t^Q + \overline{\rho}\beta(\sigma_t,t)dB_t^{\perp,Q} \\
&= [\alpha(\sigma_t,t) - \rho\xi(\sigma_t,t)\beta(\sigma_t,t) + \overline{\rho}^2\beta(\sigma_t,t)\phi_\sigma(\sigma_t,t)]dt + \rho\beta(\sigma_t,t)dB_t^Q + \overline{\rho}\beta(\sigma_t,t)dB_t^{\perp,Q} \\
&= [\alpha(\sigma_t,t) - (\rho\xi(\sigma_t,t) - \overline{\rho}^2\phi_\sigma(\sigma_t,t))\beta(\sigma_t,t)]dt + \rho\beta(\sigma_t,t)dB_t^Q + \overline{\rho}\beta(\sigma_t,t)dB_t^{\perp,Q}
\end{aligned}$$

That is

$$\frac{dQ}{dP} = \exp(-\int_0^T \xi(\sigma_t,t)dB_t - \frac{1}{2}\int_0^T \xi(\sigma_t,t)^2 dt + \int_0^T \lambda(\sigma_t,t)dB_t^\perp - \frac{1}{2}\int_0^T \lambda(\sigma_t,t)^2 dt)$$

with $B_t^Q = B_t + \int_0^t \xi(u,\sigma_u)du$ and $B_t^{\perp,Q} = B_t^\perp - \int_0^t \lambda_u du$

Now let P^L be any equivalent local martingale measure with some progressively measurable process $L = (L_t)_{0 \le t \le T}$ with $\int_0^T L_s^2 ds < \infty$ Q a.s. such that

$$\frac{dP^L}{dQ} = \exp(-\int_0^T L(\sigma_t, t) dB_t^{\perp,Q} - \frac{1}{2}\int_0^T L(\sigma_t, t)^2 dt) \text{ with } B_t^{\perp,L} = B_t^{\perp,Q} + \int_0^t L_u du$$

The dynamics of σ becomes

$$\begin{aligned} d\sigma_t &= [\alpha(\sigma_t, t) - \rho\xi(\sigma_t, t)\beta(\sigma_t, t) + \bar{\rho}\beta(\sigma_t, t)(\lambda_t - L_t)]dt \\ &\quad + \rho\beta(\sigma_t, t)dB_t^Q + \bar{\rho}\beta(\sigma_t, t)dB_t^{\perp,L} \end{aligned}$$

Through the direct calculation of relative entropy $H(P^L \mid Q)$, it yields

$$H(P^L \mid Q) = E^{P^L}[\int_0^T \frac{1}{2} L_t^2 dt]$$

It shows that when picking up the other new pricing measure rather than Q, it will yield a quadratic penalization on the additional volatility risk premium L_t.

Ilhan and Sircar (2005) approach the problem by finding the minimal entropy martingale measure. Let Q be any equivalent local martingale measure with some progressively measurable process $\lambda = (\lambda_t)_{0 \le t \le T}$ with $\int_0^T \lambda_s^2 ds < \infty$ P a.s. such that

$$\frac{dQ}{dP} = \exp(-\int_0^T \xi(\sigma_t, t) dB_t - \frac{1}{2}\int_0^T \xi(\sigma_t, t)^2 dt + \int_0^T \lambda(\sigma_t, t) dB_t^{\perp} - \frac{1}{2}\int_0^T \lambda(\sigma_t, t)^2 dt)$$

with $B_t^Q = B_t + \int_0^t \xi(u, \sigma_u) du$ and $B_t^{\perp,Q} = B_t^{\perp} - \int_0^t \lambda_u du$

The entropy of measure Q with respect to P is

$$H(Q \mid P) = E^Q[\int_0^T \frac{1}{2}(\xi(\sigma_t, t)^2 + \lambda_t^2) dt]$$

The stochastic control problem related to maximizing the negative of relative entropy gives:

$$\phi(\sigma,t)=\sup_{\lambda\in\mathcal{H}^2(Q)} E^Q[-\int_0^T \frac{1}{2}(\xi(\sigma_t,t)^2+\lambda_t^2)dt]$$

Through H-J-B and direct calculation, it yields that

$\lambda(\sigma_t,t)=\bar{\rho}\beta(\sigma_t,t)\phi_\sigma(\sigma_t,t)$, with $\phi(\sigma,t)$ solving

$$\begin{cases}\phi_t+L^\sigma\phi+\frac{1}{2}(1-\rho^2)\beta(\sigma,t)^2\phi_\sigma^2=\frac{\mu(\sigma,t)}{2\sigma(\sigma,t)^2}\\ \phi(\sigma,T,T)=0\end{cases}$$

Next invoking the Proposition 3.2 of Grandits and Rheinlander (2002)[41], applying Ito's formula to $\phi(\sigma,t)$ and through direct calculation, it yields that λ_t is equal to the parameter solution to the minimal entropy martingale measure and Q is the minimal entropy martingale measure.

[41] Grandits and Rheinlander (2002) Proposition 3.2:
Notation

$M^s(P)$: Space of all signed martingale measures.

$M(P)$: The elements of $M^s(P)$ eith non-negative density.

$M^e(P)$: The subset of $M(P)$ consisting probability measures which are equivalent to P.

Proposition 3.2: Assume there exists $\overline{Q}\in M^e(P)$ with $H(Q,P)<\infty$. Then $\overline{Q}=Q^E$, where Q^E is the minimal martingale measure iff the following holds

(i) $\frac{d\overline{Q}}{dP}=c\exp((\int\eta dX)_T)$ for a constant c and an X-integrable η.

(ii) $E^Q[(\int\eta dX)_T]=0$ for $Q=\overline{Q},Q^E$

4.6.4 Option Value to Invest, Indifference Pricing and Risk Aversion

With the obtained result that the manager's early exercise indifference price (i.e., option value to invest) solves the variational inequality with obstacle terms, this section shows the relationship between the manager's early exercise indifference price (i.e., option value to invest) and the risk aversion parameter.

4.6.4 Proposition 1:

The manager's early exercise indifference price (i.e., option value to invest) is decreasing with respect to the risk aversion parameter. As $\gamma \to 0$, the manager's early exercise indifference price (i.e., option value to invest) satisfies the unique bounded viscosity solution of the variational inequality with termianl conditions[42]

$$\begin{cases} \min\,(-h_t^0 - L^{v,\sigma}h^0 - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma^0, h^0(v,\sigma,t) - (v-K)^+) = 0 \\ h^0(v,\sigma,T) = (v-K)^+ \end{cases}$$

<u>*Proof:*</u>

(1) The comparison principle for viscosity solutions implies that subsolutions of the relevant equation are dominated by its solution.

(2) Denote the manager's early exercise indifference by h^{γ_1} and h^{γ_2}, where γ_1 and γ_2 represent the corresponding risk aversion coefficients satisfying $0 \le \gamma_1 \le \gamma_2$.

(3) The nonlinear term in the variational inequality

[42] The superscript for $h(v,\sigma,t)$ represents for the risk aversion parameter.

$$\min\,(-h_t - L^{v,\sigma}h - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma + \frac{1}{2}\gamma(1-\rho^2)\beta(\sigma,t)^2 h_\sigma^2$$
$$, h(v,\sigma,t) - (v-K)^+) = 0$$

is monotone with respect to γ while the rest of the differential expression is independent of γ. Thus,

$$0 = \min\,(-h_t^{\gamma_1} - L^{v,\sigma}h^{\gamma_1} - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma^{\gamma_1} + \frac{1}{2}\gamma_1(1-\rho^2)\beta(\sigma,t)^2 (h_\sigma^{\gamma_1})^2$$
$$, h^{\gamma_1}(v,\sigma,t) - (v-K)^+)$$
$$\leq \min\,(-h_t^{\gamma_1} - L^{v,\sigma}h^{\gamma_1} - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma^{\gamma_1} + \frac{1}{2}\gamma_2(1-\rho^2)\beta(\sigma,t)^2 (h_\sigma^{\gamma_1})^2,$$
$$h^{\gamma_1}(v,\sigma,t) - (v-K)^+)$$

The terminal condition does not depend on risk aversion. Combining the above differential inequality, h^{γ_1} is a subsolution to the variational inequality satisfied by h^{γ_2}.

(4) Combining (3) with (1), I find that the manager's early exercise indifference price is decreasing with respect to the risk aversion parameter.

(5) Since h^γ are uniformly bounded with respect to γ, $\{h^\gamma\}$ converge along with subsequences. It is thus observed that as $\gamma \to 0$ the pricing equation

$$\min\,(-h_t - L^{v,\sigma}h - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma + \frac{1}{2}\gamma(1-\rho^2)\beta(\sigma,t)^2 h_\sigma^2$$
$$, h(v,\sigma,t) - (v-K)^+) = 0$$

converges, locally uniformly bounded in γ, to the linear variational inequality

$$\min\,(-h_t^0 - L^{v,\sigma}h^0 - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma^0, h^0(v,\sigma,t) - (v-K)^+) = 0\,.$$

Classical optimal stopping results (see Ishii and Lions (1990)) imply that the above variational inequality has a unique viscosity solution in the class of bounded functions. In addition, the stability of viscosity solutions (see Lions(1983)) yields that as

$\gamma \to 0$ $\{h^\gamma\}$, along with subsequences, converge to h^0 locally uniformly bounded in γ. Therefore, the desired result that as $\gamma \to 0$, the manager's early exercise indifference price satisfies the unique bounded viscosity solution of the variational inequality with terminal conditions

$$\begin{cases} \min\,(-h_t^0 - L^{v,\sigma}h^0 - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma^0,\, h^0(v,\sigma,t) - (v-K)^+) = 0 \\ h^0(v,\sigma,T) = (v-K)^+ \end{cases}$$ is obtained. □

4.6.5 Abandonment Option - Investment Problem Formulation and the Model

As a by-product of the previous section, I propose the indifference price for the abandonment option. The project value with the option to abandon can be viewed as the summation of static net present value and the abandonment option premium. I modify one assumption and the manager's investment problem holding all else the same as the previous section.

First, the manager's investment problem is modified as follows. The risk-averse manager's investment problem is to maximize her expected utility of wealth with respect to the optimal exercise abandonment decision. She receives salvage value K if she exercises during the investment horizon, [t,T]. It is a one time irreversible investment decision. Prior to exercising the investment decision, time τ, the wealth refers to the quantity X_τ the manager generates by following investment policy θ, at the time of investment, τ, the manager's wealth X_τ increases to X_τ+[K-V_τ]$^+$, and after time τ, the manager faces the same investment opportunities and continues trading between the twin security and the bond till the end of investment horizon T.

Second, I do not consider the "dividend yield"[43] for the project value process, equivalently, the twin security process. Thus the, wealth process and the twin security process will not differ by the dividend yield in the drift rate.

$$\frac{dV_t}{V_t} = \sigma_t(\xi(\sigma_t,t)dt + dB_t)\,; V_t = v$$

$$d\sigma_t = \alpha(\sigma_t,t)dt + \beta(\sigma_t,t)dW_t = \alpha(\sigma_t,t)dt + \beta(\sigma_t,t)(\rho dB_t + \overline{\rho} dB_t^{\perp})\,;\ \sigma_t = \sigma$$

$$dX_t = \theta_t\mu(\sigma_t,t)dt + \theta_t\sigma_t dB_t\,;\ X_t = x$$

4.6.6 Abandonment Option Value – Indifference Pricing

By making use of the indifference pricing result for the manager's option value to invest, it is able to establish the indifference pricing for the manager's abandonment option value.

4.6.6 Proposition 1:

The manager's early exercise indifference price (i.e., the abandonment option value) is the unique bounded viscosity solution of the quasilinear variational inequality with terminal conditions

$$\begin{cases} \min(-h_t - L^{v,\sigma}h - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma + \frac{1}{2}\gamma(1-\rho^2)\beta(\sigma,t)^2 h_\sigma^2 \\ \qquad , h(v,\sigma,t) - (K-v)^+) = 0 \\ \qquad\qquad h(v,\sigma,T) = (K-v)^+ \end{cases}$$

[43] That is $\kappa(\sigma,t)$ =0.

Proof:

Same as 4.6.2 Proposition 3.

4.6.7 Abandonment Option Value, Indifference Pricing and Pricing Measure

The indifference pricing equality (for fixed time abandonment decision (i.e., traditional European put) or the indifferencing pricing variational inequality does not change (except for the obstacle term presented in the early exercise situation); therefore, as the discussion shown in 4.5.4, the indifference pricing admits a new pricing measure which minimize the relative entropy between the new pricing measure and the historical measure.

4.6.8 Abandonment Option Value, Indifference Pricing and Risk Aversion

With the obtained result that the manager's early exercise indifference price (i.e., the abandonment option value) solves the variational inequality with obstacle terms, this section shows the relationship between the manager's early exercise indifference price (i.e., the abandonment option value) and the risk aversion parameter.

4.6.8 Proposition 1:

The manager's early exercise indifference price (i.e., the abandonment option value)[44] is decreasing with respect to the risk aversion parameter. As $\gamma \to 0$, the manager's early exercise indifference price (i.e., the abandonment option value) satisfies the unique bounded viscosity solution of the variational inequality with terminal conditions

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[44] The same relationship can be inferred for the project value with the abandonment option by the equality Project value with option to abandon = Static net present value + Abandonment option premium

$$\begin{cases} \min\left(-h_t^0 - L^{v,\sigma} h^0 - (1-\rho^2)\beta(\sigma,t)\phi_\sigma h_\sigma^0,\ h^0(v,\sigma,t) - (K-v)^+\right) = 0 \\ h^0(v,\sigma,T) = (K-v)^+ \end{cases}$$

Proof:

Same as 4.6.6 Proposition 1

CHAPTER 5

CONCLUSION

5.1 Conclusion

The real options theory of corporate investment has developed to the point that it is now in the mainstream of corporate finance. The classical real options approach relies on one of the following assumptions: (1) the tradability of real investment opportunity, (2) the existence of perfect spanning traded assets, or (3) the risk neutrality. It is noted that if a capital investment project is partially or totally irreversible and if there is flexibility in timing, the value of the option to delay investment may exceed the value of the project in place. The familiar static net present value criterion for capital investment should be replaced in many situations by the criterion that net present value should exceed a project's real option value before assets are put in place.

If one or more assumptions is violated, the classical options valuation technique may require modification. Hubalek and Schachermayer (2001) studied the non-tradability issues indicating that using the assumption of no arbitrage alone would lead to no information about the price of the claim. There has been much study in pricing claims written on non-traded assets.

This research explores the market incompleteness issue presented in the capital budgeting problem with real options setting. The incompleteness is presented in two different scenarios. The first incomplete market problem arises from pricing claims written on non-traded assets with the existence of partial spanning assets. This problem has been studied by many researchers and has been extended specifically to the capital budgeting problems in real options setting. Recent papers include Henderson (2005), Hugonnier and Morellec (2004, 2005), Kadam et al (2004), Miao and Wang (2005) etc. I extend existing real options literature in an incomplete market setting to include strategic interactions for exploring relations between market incompleteness and strategic exercise of real options in a Stackelberg model. I find that incompleteness narrows the gap between leader and follower entry dates. Relative to results in Dixit and Pindyck (1994), the follower enters much sooner, and the leader delays slightly. I conjecture that Firm L's management has greater concern for the risk involved in the imperfect hedge than for the risk of pre-emption. Thus, the incompleteness coupled with strategic interactions alters the corporate capital investment decision. The more detailed results are summarized as:

(A) *Market share, investment timing decision, and option value to invest:* Holding market completeness correlation coefficient and risk parameter constant, I find that the lower Firm L's market share following Firm F's entry, the lower the trigger investment value for the follower. That is, Firm F has greater incentive to enter the market the greater the anticipated market share. When the firms are able to hedge completely the project risk, Firm L's investment trigger value is negatively correlated with its market share. Moreover, option value to invest becomes smaller as L's market share increases. That is, Firm L enters immediately to secure the pre-emptive advantage.

(B) <u>*Degree of completeness, investment timing and option value to invest:*</u> Focusing on the degree of completeness, I find that the higher the degree of completeness, the greater is the follower's option value to invest, a result consistent with Henderson (2005). The leader's option value for investment is higher than is the case when perfect hedging is possible. Considering simultaneously the market incompleteness and the leader's fear of pre-emption, it appears that Firm L displays behavior closer to the classic real option models relative to the case in which perfect hedging is possible. I next focus on Firm L's market share. If Firms L and F expect to share the market equally, they will enter the market nearly simultaneously. This result conflicts with classical model results in a complete market setting. However, if Firm L anticipates a market share greater than 50% upon F's entry, Firm L will enter the market slightly earlier than F but not as fast as would be the case in a complete market. These results reflect in part our specification of the leader's value function.

(C) <u>*Managerial Risk aversion, investment timing and option value to invest:*</u> Regarding managerial risk preferences, the greater the risk aversion coefficient, the lower is the investment option value for both firms. This result suggests that the more risk-averse managers may be more concerned about the unhedgeable risks, placing relatively less value in the option to delay investment.

In addition, as a byproduct of this research, I analyze the impact of market share and uncertainty on the relative investment trigger as well as option value through modeling two different stochastic income streams. The senstivity analysis indicates that the choice of the process specification has a material impact both on Firm L's and Firm F's project value and option value to invest. This result is consistent with Schwartz's (1997) assertion about the importance of mean-reverting process vs. non-reverting process in the capital budgeting investment problem. Therefore, it is important to model stochastic processes to reflect the real world circumstances.

The second incomplete market problem arises from the stochastic volatility since the volatility itself is not traded. I explore this problem through two approaches. Since it is noted that there are infinitely many admissible pricing measures in the presence of market frictions due to non-tradability, the option pricing problem reduces to selection of a measure with which to price options. Therefore, I first work through q-optimal measures selected to investigate optimal investment/ abandonment decision rule and the the corresponding option values as well as project value. The optimal investment/ abandonment decision rule changes, so do the corresponding option values as well as project value as follows:

(1) With non-zero correlation between the project randomness and volatility randomness, the option values to invest/abandonment option value under q-optimal measures will decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ increases (respectively, decreases) in σ. Thus, the difference between NPV and option value to invest (also

project value with option to abandon[45]) under q-optimal measures will decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ increases (respectively, decreases) in σ.

(2) With zero correlation between the project randomness and volatility randomness, $\lambda^{(q)}(t,\sigma)$ will be non-decreasing (respectively, non-increasing) in q if $\xi(t,\sigma)^2$ is non-decreasing (respectively, non-increasing) in σ. As a result, option values to invest/abandonment option value under q-optimal measures will decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ increases (respectively, decreases) in σ. The difference between NPV and the option values to invest (also project value with option to abandon) under q-optimal measures will increase (respectively, decrease) in q if $\xi(t,\sigma)^2$ is non- increasing (respectively, non-decreasing) in σ.

(3) With a non-zero correlation between the project randomness and volatility randomness, the optimal investment trigger under q-optimal measures decreases (respectively, increases) in q if $\xi(t,\sigma)^2$ increases (respectively, decreases) in σ. The optimal abandonment trigger is reversed.

(4) With zero correlation between the project randomness and volatility randomness, the optimal investment trigger under q-optimal measures decrease (respectively, increase) in q if $\xi(t,\sigma)^2$ increases (respectively, decreases) in σ. The optimal abandonment trigger is reversed.

[45] Project value with option to abandon = Static net present value + Abandonment option premium

(5) There are no conclusive relations between the option value to invest/ abandonment option value under q-optimal measures and the correlation between the project randomness and volatility randomness.

I then use utility maximization approach to price the option value to invest and the abandonment option. I demonstrate the indifference prices for the option value to invest and the abandonment option solve quasilinear variational inequalitites with obstacle terms. Assuming an exponential utility function, the utility-based indifference price admits a new pricing measure, which is the minimal relative entropy martingale measure minimizing the relative entropy between the historical measure and the Q martingale measure. I also show that the indifference price for the option value to invest and the abandonment option (also, project value with abandonment option)[46] is non-increasing with respect to the risk aversion parameter. As the risk aversion parameter converges to zero, the indifference price converges to the unique bounded viscosity solution of the linear variational inequality with obstacle term.

5.2 Limitation and Future Research

The first study presents the model incorporating market incompleteness with strategic behavior through the Stackleberg leader follower model. I will extend the analysis to incorporate the "collusion" and "collaboration" strategy and also extend the model to consider other game-theoretic settings. I will consider how the stochastic interest rate affects the result of current analysis. An efficient and stable approximation numerical scheme will be developed.

The second study presents the model with stochastic volatility. For q-optimal measure valuation, I currently focus on the finite time horizon under current model. There are three items for the future research: (1) Find out the sufficient/necessary conditions for current model with q-optimal measure valuations to be extended to the infinite time horizon. (2) Find out manager's optimal hedging strategies under q-optimal measures with the investment opportunity and or abandonment option presented. (3) Develop the efficient and stable computational scheme for approximations.

For indifference pricing, I will investigate in detail how other q-optimal measures penalize the indifference pricing other than the minimal entropy martingale measure, that is q =0, under the exponential utility function. Also what is the impact with other class of HARA utility functions? Finally, I would like to develop the efficient and stable computational scheme for approximations.

Since assets underlying real options are typically not traded and the fact that long investment horizons make the constant volatility assumption problematic in real options, the study and development of valuation technique under market incompleteness are inevitable for providing better capital investment policy and hedging strategies for the modern corporation.

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[46] The project value with the abandonment option is inferred from the valuation equality: Project value with the abandonment option = Static Net Present Value + Abandonment Option Premium.

APPENDIX A

PROOF OF 3.4.2 PROPOSITION 1

Appendix A **Proof of 3.4.2 Proposition 1**

Lemma[47]

Define: $J(t,x;u) = E_{t,x}[\int_t^T e^{-\beta(s-t)} P(s, X(s), u(s))ds + e^{-\beta(T-t)} \psi(T, X(T))]$

$V(t,x) = \sup_{u(\cdot)\in U(t,x)} J(t,x;u)$

Let $(t,x) \in [0,T) \times R$ be given. Then for every stopping time r valued in [t, T] I have

$$V(t,x) = \sup_{u(\cdot)\in U(t,x)} E_{t,x}[\int_t^{r.} e^{-\beta(s-t)} P(s, X(s), u(s))ds + e^{-\beta(r-t)} V(r, X(r))]$$

Proof :

(1) By the tower law, I have

$$J(t,x;u) = E_{t,x}[\int_t^r e^{-\beta(s-t)} P(s, X(s), u(s))ds + e^{-\beta(r-t)} J(r, X(r), u)]$$

(2) By definition I have $V(r, X(r)) = \sup_{u(\cdot)\in U(t,x)} J(r, X(r); u)$, implying

$V(r, X(r)) \geq J(r, X(r); u)$.

(3) Therefore, I have

$$\begin{aligned} V(t,x) &= \sup_{u(\cdot)\in U(t,x)} E_{t,x}[\int_t^r e^{-\beta(s-t)} P(s, X(s), u(s))ds + e^{-\beta(r-t)} J(r, X(r), u)] \\ &\leq \sup_{u(\cdot)\in U(t,x)} E_{t,x}[\int_t^r e^{-\beta(s-t)} P(s, X(s), u(s))ds + e^{-\beta(r-t)} V(r, X(r))] \end{aligned}$$

[47] References:

(1) Fleming, Wendell H. and H. Mete Soner (1992): Controlled Markov Processes and Viscosity Solutions. Springer-Verlag

(4) Definition for δ-optimal control $X^1(s)$ for $V(r, X(r))$

For any $\delta > 0$, choose an admissible control $u^1(\cdot) \in U(r, X(r))$ such that

$$V(r, X(r)) - \delta \le E_{r,X(r)}[\int_r^T e^{-\beta(s-r)} P(s, X^1(s), u^1(s))ds + e^{-\beta(T-r)} \psi(T, X^1(T))]$$
$$= J(r, X(r), u^1)$$

Here $X^1(s)$ is the state at time s corresponding to the control $u^1(\cdot)$ and initial condition $(r, X(r))$. Such a control $u^1(\cdot)$ is called δ-optimal.

Define an admissible control $u(\cdot) \in U(t, x)$ by

$$\widetilde{u}(s) = \{ \begin{matrix} u(s), \mathrm{s} \le r \\ u^1(s), \mathrm{s} > r \end{matrix}$$

Let $\widetilde{x}(s)$ be the state corresponding to $\widetilde{u}(\cdot)$ with initial condition (t, x).

I have

$$V(t, x) \ge J(t, \widetilde{x}; \widetilde{u})$$
$$= \mathrm{E}_{t,x}[\int_t^T e^{-\beta(s-t)} P(s, \widetilde{X}(s), \widetilde{u}(s))ds + e^{-\beta(T-t)} \psi(T, \widetilde{X}(T))]$$
$$= E_{t,x}[\int_t^r e^{-\beta(s-t)} P(s, x(s), u(s))ds + e^{-\beta(r-t)} J(r, X(r), u)]$$
$$\ge E_{t,x}[\int_t^r e^{-\beta(s-t)} P(s, x(s), u(s))ds + e^{-\beta(r-t)} V(r, x(r))] - e^{-\beta(r-t)} \delta$$

(5) For a small positive δ, choose a δ-optimal admissible control $u(\cdot) \in U(t, x)$. Then I have

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(2) Nizar Touzi (2002): Stochastic Control Problems, Viscosity Solutions and Application to Finance

$$V(t,x) \geq J(t,x;u)$$
$$= E_{t,x}[\int_t^T e^{-\beta(s-t)} P(s,x(s),u(s))ds + e^{-\beta(T-t)}\psi(T,X(T))]$$
$$= E_{t,x}[\int_t^r e^{-\beta(s-t)} P(s,x(s),u(s))ds + e^{-\beta(r-t)} J(r,X(r),u)]$$
$$\geq E_{t,x}[\int_t^r e^{-\beta(s-t)} P(s,x(s),u(s))ds + e^{-\beta(r-t)} V(r,X(r)] - e^{-\beta(r-t)}\delta$$

(6) Since δ is arbitrary, I have proved the following:

$$V(t,x) = \sup_{u(\cdot)\in U(t,x)} J(t,x;u)$$
$$= \sup_{u(\cdot)\in U(t,x)} E_{t,x}[\int_t^T e^{-\beta(s-t)} P(s,x(s),u(s))ds + e^{-\beta(T-t)}\psi(T,X(T))]$$
$$= \sup_{u(\cdot)\in U(t,x)} E_{t,x}[\int_t^r e^{-\beta(s-t)} P(s,x(s),u(s))ds + e^{-\beta(r-t)} J(r,X(r),u)]$$
$$\geq \sup_{u(\cdot)\in U(t,x)} E_{t,x}[\int_t^r e^{-\beta(s-t)} P(s,x(s),u(s))ds + e^{-\beta(r-t)} V(r,X(r)]$$

(7) Since I have already shown the reverse inequality, the equality holds, i.e.,

$$V(t,x) = \sup_{u(\cdot)\in U(t,x)} E_{t,x}[\int_t^{T} e^{-\beta(s-t)}(s,X(s),u(s))ds + e^{-\beta(T-t)}\psi(T,X(T))]$$
$$= \sup_{u(\cdot)\in U(t,x)} E_{t,x}[\int_t^{r} e^{-\beta(s-t)}(s,X(s),u(s))ds + e^{-\beta(r-t)} V(r,X(r))]$$

Q.E.D.

In addition, it can be shown that an optimal control $u^*(\cdot) \in U(t,x)$ maximizes the above equation at every r. Therefore, I can choose r arbitrarily close to t.

(2) By invoking the lemma with the specification of t = 0, T = ∞, no bequest function, and optimal stopping time τ, Firm F's value function:

$$F(x,y) = \sup_{\theta,\tau} E[\int_0^\infty -\frac{1}{\gamma} e^{-\beta s} e^{-C_s} ds \mid X_0 = x, Y_0 = y]$$

may be written as:

$$F(x,y)=\sup_{\tau}\sup_{\{\theta_s,0\le s\le\tau\}}E[\int_0^{\tau}-\frac{1}{\gamma}e^{-\beta s}e^{-\gamma C_s}ds+e^{-\beta\tau}F_1(X_\tau,Y_\tau)\,|\,X_0=x,Y_0=y]$$

where $F_1(x,y)$ is Firm F manager's value function after exercising the investment decision. Q.E.D.

APPENDIX B

TABLES

Table 1 The Impact of Market Incompleteness on the Investment Timing Decision and the Option Value to Invest ~Simulation Results

Correlation =1, Risk Aversion=10				
Market Share	Follower's Trigger	Follower's Option Value	Leader's Trigger	Leader's Option Value
0.5	3.33	0.67	1.17	0.0487
0.6	4.14	0.67	1.08	0.02279
0.7	5.56	0.67	1.03	0.00998
0.8	8.33	0.67	1.01	0.0034

Correlation =0.99, Risk Aversion=10				
Market Share	Follower's Trigger	Follower's Option Value	Leader's Trigger	Leader's Option Value
0.5	3.207	0.60372	3.207	0.60372
0.6	4.009	0.60372	3.944	0.578073
0.7	5.345	0.60372	5.193	0.55958
0.8	8.019	0.60372	7.743	0.550373

Correlation =0.90, Risk Aversion=10				
Market Share	Follower's Trigger	Follower's Option Value	Leader's Trigger	Leader's Option Value
0.5	2.7538	0.37689	2.7538	0.37689
0.6	3.4424	0.37689	3.4173	0.367071
0.7	4.5897	0.37689	4.554	0.3664
0.8	6.88448	0.37689	6.844	0.368975

Table 2 The Impact of Risk Aversion on the Investment Timing Decision and the Option Value to Invest

~ Simulation Results

Correlation =0.90, Risk Aversion=10				
Market Sahre	Follower's Trigger	Follower's Option Value	Leader's Trigger	Leader's Option Value
0.5	2.7538	0.37689	2.7538	0.37689
0.6	3.4424	0.37689	3.4173	0.367071
0.7	4.5897	0.37689	4.554	0.3664
0.8	6.88448	0.37689	6.844	0.368975
Correlation =0.90, Risk Aversion=5				
Market Sahre	Follower's Trigger	Follower's Option Value	Leader's Trigger	Leader's Option Value
0.5	2.9323	0.466159	2.9323	0.466159
0.6	3.6554	0.466159	3.60197	0.441411
0.7	4.8872	0.466159	4.77157	0.433634
0.8	7.3308	0.466159	7.19127	0.439004

Table 3 Geometric Brownian Motion Process v.s. Arithmetic Brownian Motion Process ~ Volatility v.s. Option Value v.s. Investment Trigger (Leader Market Share: 50%)

(A) After F's Entry: L and F co-share the market.

Geometric Brownian Motion

Volatility	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.25	4.21697	0.563793	11.6157	10.5677
0.5	5.54222	3.2679	17.0298	24.8152
0.75	7.53495	8.01664	24.3534	44.088
% Change in Volatility	% Change in L's Trigger	% Change in L's Option Value	% Change in F's Trigger	% Change in F's Option Value
100.00%	31.43%	479.63%	46.61%	134.82%
200.00%	78.68%	1321.91%	109.66%	317.20%

Arithmetic Brownian Motion				
Volatility	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.25	3.95012	0.000029	8.37107	1.05269
0.5	3.963382	0.0056683	8.76596	2.03991
0.75	4.02502	0.0435887	9.16112	3.02779
% Change in Volatility	% Change in L's Trigger	% Change in L's Option Value	% Change in F's Trigger	% Change in F's Option Value
100.00%	0.34%	19445.86%	4.72%	93.78%
200.00%	1.90%	150205.86%	9.44%	187.62%

Base Parameters: r=0.2, α=0.01,K=20

Table 4 Geometric Brownian Motion Process v.s. Arithmetic Brownian Motion Process~ Volatility v.s. Option Value v.s. Investment Trigger (Leader Market Share: 60%)

(B) After F's Entry: L owns 60% maket share, while F owns 40% market share

Geometric Brownian Motion				
Volatility	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.25	3.98576	0.251167	14.5196	10.5677
0.5	4.64708	1.58888	21.2872	24.8152
0.75	5.51594	3.68076	30.4418	44.088
% Change in Volatility	% Change in L's Trigger	% Change in L's Option Value	% Change in F's Trigger	% Change in F's Option Value
100.00%	16.59%	532.60%	46.61%	134.82%
200.00%	38.39%	1365.46%	109.66%	317.20%

Arithmetic Brownian Motion				
Volatility	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.25	3.95	0.0000002	10.3711	0.842149
0.5	3.9511	0.000385044	10.766	1.63193
0.75	3.96303	0.00635414	11.1611	2.4224
% Change in Volatility	% Change in L's Trigger	% Change in L's Option Value	% Change in F's Trigger	% Change in F's Option Value
100%	0.03%	192422.00%	3.81%	93.78%
200%	0.33%	3176970.00%	7.62%	187.65%

Base Parameters: r=0.2, α=0.01,K=20

Table 5 Geometric Brownian Motion Process v.s. Arithmetic Brownian Motion Process~ Volatility v.s. Option Value v.s. Investment Trigger (Leader Market Share: 70%)

After F's Entry: L owns 70% maket share, while F owns 30% market share

Geometric Brownian Motion				
Volatility	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.25	3.87447	0.100692	19.3595	10.5677
0.5	4.22407	0.795429	28.383	24.8152
0.75	4.6931	1.91573	40.5891	44.088
% Change in Volatility	% Change in L's Trigger	% Change in L's Option Value	% Change in F's Trigger	% Change in F's Option Value
100%	9.02%	689.96%	46.61%	134.82%
200%	21.13%	1802.56%	109.66%	317.20%
Arithmetic Brownian Motion				
Volatility	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.25	3.95	**close to zero**	13.7044	0.631612
0.5	3.95002	**4.8511E-06**	14.0993	1.22395
0.75	3.95078	**0.000300902**	14.4945	1.81668
% Change in Volatility	% Change in L's Trigger	% Change in L's Option Value	% Change in F's Trigger	% Change in F's Option Value
100%	0.00%	**Huge**	2.88%	93.78%
200%	0.02%	**Huge**	5.77%	187.63%

Base Parameters: r=0.2, α=0.01,K=20

Table 6 Geometric Brownian Motion Process v.s. Arithmetic Brownian Motion Process ~ Market Share v.s. Option Value v.s. Investment Trigger (Volatility at 25%)

(A) Volatility= 0.25				
Geometric Brownian Motion				
F's Market Share	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.5	4.21697	0.563793	11.6157	10.5677
0.4	3.98576	0.251167	14.5196	10.5677
0.3	3.87447	0.100692	19.3595	10.5677
% Change in Market Share	% Change in L's Trigger	% Change in L's Option Value	% Change inF's Trigger	% Change in F's Option Value
-20.00%	-5.48%	-55.45%	25.00%	0.00%
-40.00%	-8.12%	-82.14%	66.67%	0.00%

Arithmetic Brownian Motion

F's Market Share	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.5	3.95012	0.000029	8.37107	1.05269
0.4	3.95	0.0000002	10.3711	0.842149
0.3	3.95	close to zero	13.7044	0.631612
% Change in Market Share	% Change in L's Trigger	% Change in L's Option Value	% Change inF's Trigger	% Change in F's Option Value
-20.00%	0.00%	-99.31%	23.89%	-20.00%
-40.00%	0.00%	nearly -100%	63.71%	-40.00%

Base Parameters: r=0.2, α=0.01,K=20

Table 7 Geometric Brownian Motion Process v.s. Arithmetic Brownian Motion Process~ Market Share v.s. Option Value v.s. Investment Trigger (Volatility at 50%)

(B) Volatility=0.5

Geometric Brownian Motion				
F's Market Share	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.5	5.54222	3.2679	17.0298	24.8152
0.4	4.64708	1.58888	21.2872	24.8152
0.3	4.22407	0.795429	28.383	24.8152
% Change in Market Share	% Change in L's Trigger	% Change in L's Option Value	% Change inF's Trigger	% Change in F's Option Value
-20.00%	-16.15%	-51.38%	25.00%	0.00%
-40.00%	-23.78%	-75.66%	66.67%	0.00%

Arithmetic Brownian Motion

F's Market Share	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.5	3.963382	0.0056683	8.76596	2.03991
0.4	3.9511	0.000385044	10.766	1.63193
0.3	3.95002	4.8511E-06	14.0993	1.22395
% Change in Market Share	% Change in L's Trigger	% Change in L's Option Value	% Change inF's Trigger	% Change in F's Option Value
-20.00%	-0.31%	-93.21%	22.82%	-20.00%
-40.00%	-0.34%	-99.91%	60.84%	-40.00%

Base Parameters: r=0.2, α=0.01,K=20

Table 8 Geometric Brownian Motion Process v.s. Arithmetic Brownian Motion Process ~ Market Share v.s. Option Value v.s. Investment Trigger (Volatility at 75%)

(C) Volatility = 0.75

Geometric Brownian Motion				
F's Market Share	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.5	7.53495	8.01664	24.3534	44.088
0.4	5.51594	3.68076	30.4418	44.088
0.3	4.6931	1.91573	40.5891	44.088
% Change in Market Share	% Change in L's Trigger	% Change in L's Option Value	% Change inF's Trigger	% Change in F's Option Value
-20.00%	-26.80%	-54.09%	25.00%	0.00%
-40.00%	-37.72%	-76.10%	66.67%	0.00%

Arithmetic Brownian Motion

F's Market Share	L's Trigger	L's Option Value	F's Trigger	F's Option Value
0.5	4.02502	0.0435887	9.16112	3.02779
0.4	3.96303	0.00635414	11.1611	2.4224
0.3	3.95078	0.000300902	14.4945	1.81668
% Change in Market Share	% Change in L's Trigger	% Change in L's Option Value	% Change inF's Trigger	% Change in F's Option Value
-20.00%	-1.54%	-85.42%	21.83%	-19.99%
-40.00%	-1.84%	-99.31%	58.22%	-40.00%

Base Parameters: r=0.2, α=0.01,K=20

APPENDIX C

ILLUSTRATIONS

Figure 1 Leader's/Follower's Value Functions in the Complete Market

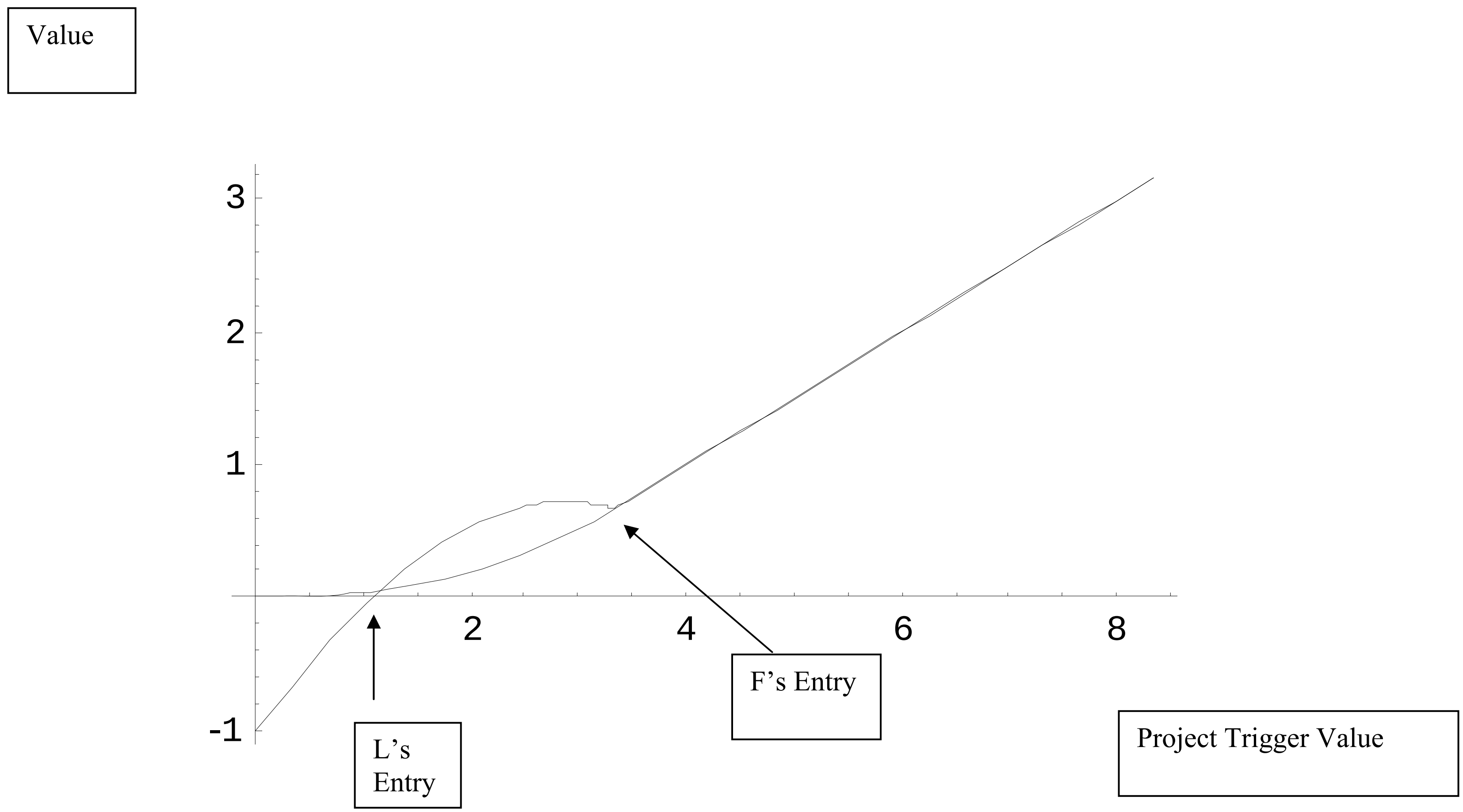


Figure 2 Follower's Value Function in the Incomplete Market (Representative)

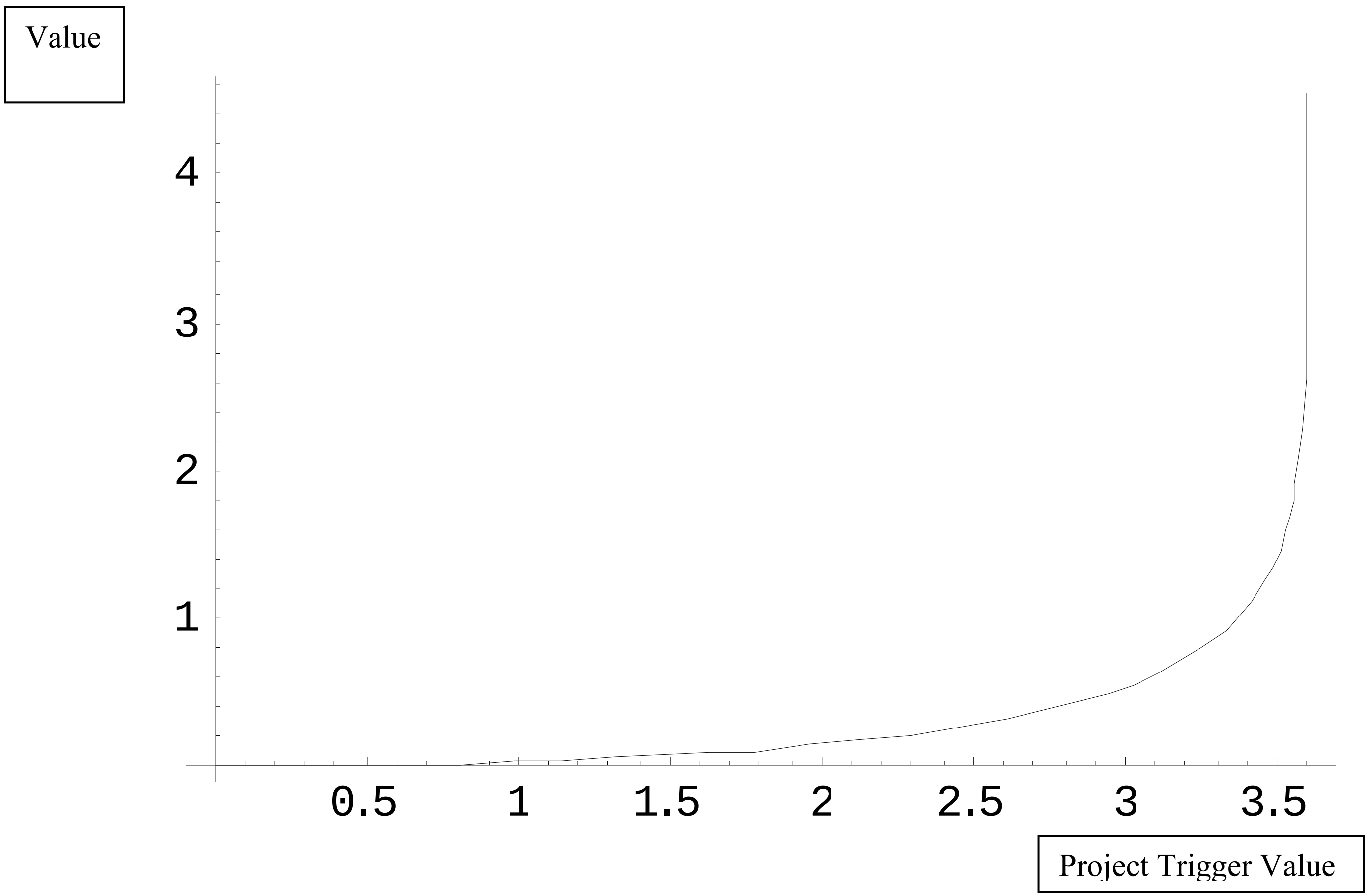


Figure 3 Leader's/Follower's Value Functions in the Incomplete Market
(If Leader keeps larger market share upon Follower's Entry)

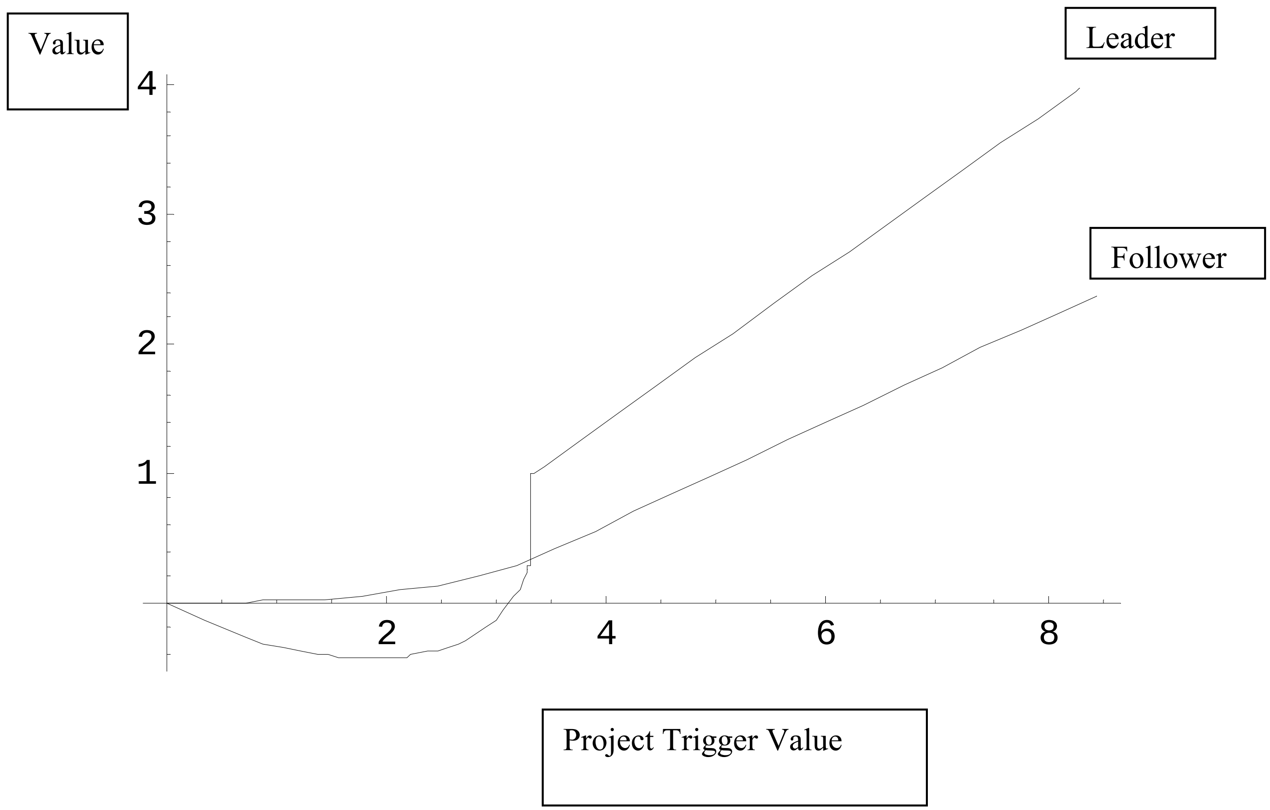
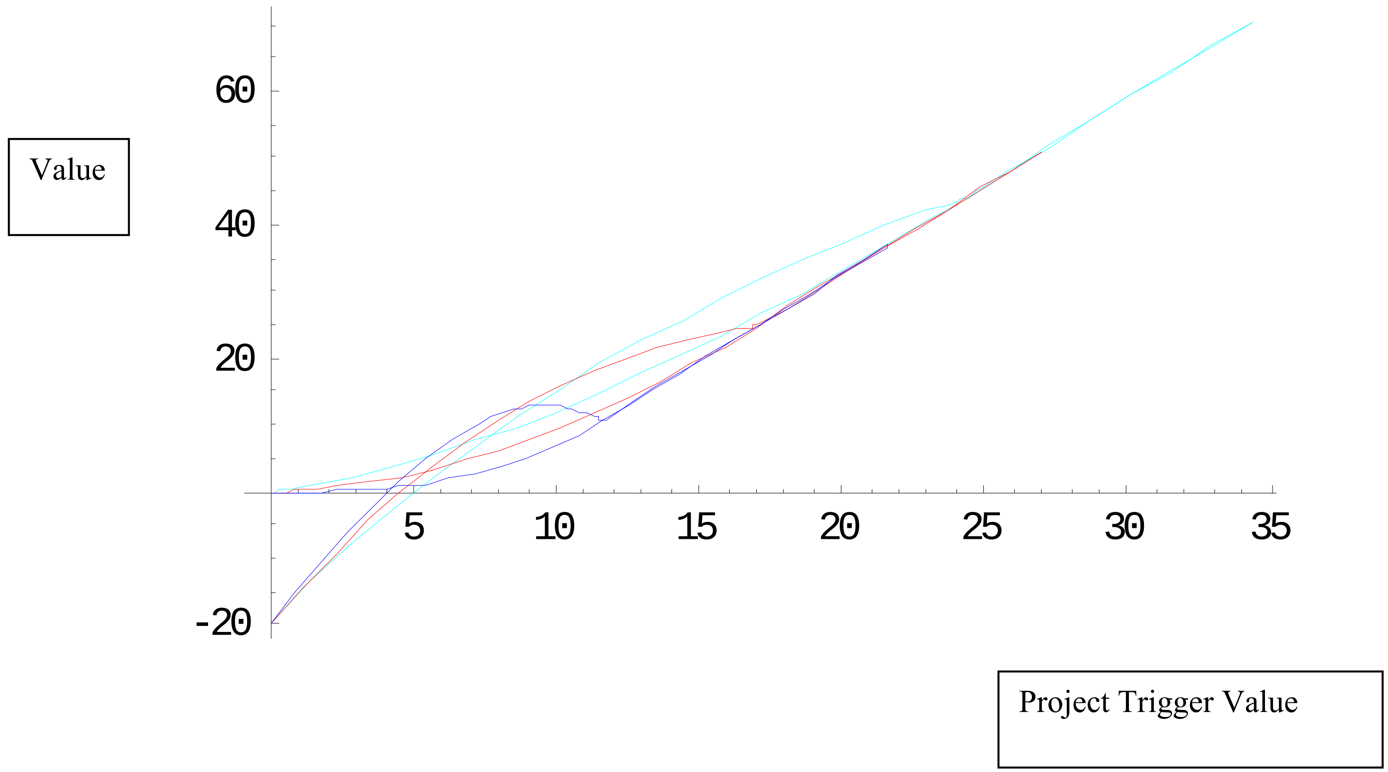
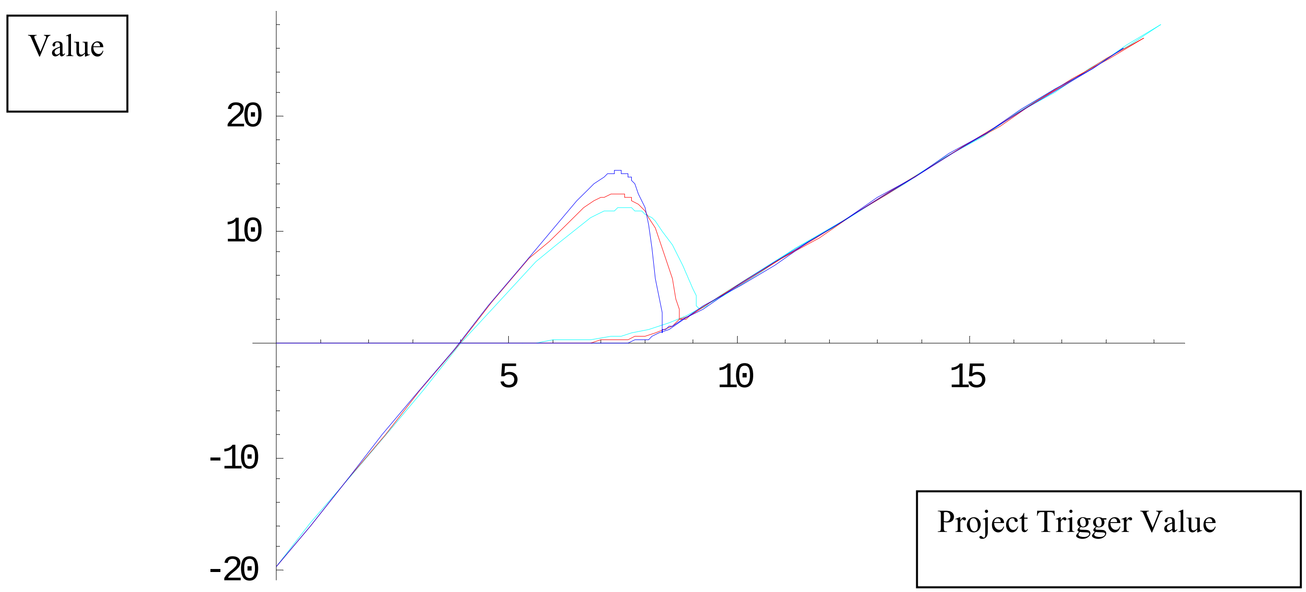


Figure 4 Geometric Brownian Motion Process ~ Volatility v.s. Option Value v.s. Investment Trigger (Co-share Market)



Blue- Volatility:0.25; Red- Volatility:0.5; Light Blue- Volatility:0.75

Figure 5 Arithmetic Brownian Motion Process ~ Volatility v.s. Option Value v.s. Investment Trigger (Co-share Market)



Blue- Volatility:0.25; Red- Volatility:0.5; Light Blue- Volatility:0.75
PS: The figure assumes the lower bound of the investment income stream is bounded below from zero.

REFERENCES

Avellaneda, M., 1998,"Minimum-Entropy Calibration of Asset Pricing Models," *International Journal of Theoretical and Applied Finance*, 1, 447-472.

Berk, J., R. Green, and V Naik, 2003, "Valuation and Return Dynamics of New Ventures," *Review of Financial Studies,* 17, 1-35.

Black, F., and M. Scholes, 1973, "The Pricing of Corporate Liabilities," *Journal of Political Economy*, 81, 637-659.

Biagini F., P. Guasoni and M. Prattelli, 2000, "Mean Variance Hedging for Stochastic Volatility models", *Mathematical* Finance, 10(2), 109-123.

Boyle, P., 1988, "A Lattice Framework for Option Pricing with Two States Variables," *Journal of Financial and Quantitative Analysis*, 23, 1988.

Brealey, R., and S.C. Myers, *Principles of Corporate Finance*, McGraw-Hill, 2000.

Brennan, M. & E. Schwartz, 1982, "Regulation and Corporate Investment Policy", *Journal of Finance*, 37(2), 289-300.

Brennan, M. & E. Schwartz, 1982, "Consistent Regulatory Policy Under Uncertainty", *Bell Journal of Economics*, 13(2), 506-524.

Brennan, M. & E. Schwartz, 1985, "Evaluating Natural Resource Investments," *Journal of Business*, 58(2), 135-157.

Childs, P., and A. Triantis, 1999, "Dynamic R&D Investment Policies, *Management Science*, 45, 1359-1377.

Clarke, N., and K. Parrot, 1999, "Multigrid for American Option Pricing with Stochastic Volatility," *Applied Mathematical Finance*, 6, 177-195.

Copeland, T., and V. Antikarov, *Real Options: A Practitioner's Guide*, TEXERE LLC, 2001.

Davis, M. H. A., and T. Zariphopoulou, 1995, "American Options and Transaction Fees," *Mathematical Finance, IMA Volume in Mathematics and Its Applications,* Springer-Verlag.

Delbaen, F. , Grandits P., Sampri D., Schweizer M. and Stricker C., 2002, "Exponential Hedging and Entropic Penalties", *Mathematical Finance*, 12(2), 99-123.

Delbaen, F., Schachermayer W., Schweizer M,. Stricker C.,, 1997, "Weighted Norm Inequalities and Hedging in Incomplete Markets" *Finance and Stochastics,* 1, 181-227.

Delbaen, F. & W. Schachermayer, 1996, "The Variance Optimal Martingale Measure for Continuous Processes," Bernoulli, 2(1), 81-105.

Delbaen, F. & W. Schachermayer, 1996, "The Variance Optimal Martingale Measure for Continuous Processes," Bernoulli, 2(1), 81-105.

Dimasi, J.A., R.W. Hansen, L Grabowski, and Lasagna L., 1991, "Cost of Innovation in the Pharmaceutical Industry," *Journal of Health Economics*, 10, 107-142.

Dimasi, J.A., R.W. Hansen, L Grabowski, and Lasagna L., 1995, "Research and Development Costs for New Drugs by Therapeutic Category: A Study of the US Pharmaceutical Industry," *Pharmaco Economics*, 7, 152-169.

Dixit, A., and R.S. Pindyck, *Investment Under Uncertainty*, Princeton University Press, 1994.

Duffie, D., *Dynamic Asset Pricing Theory*, Princeton University Press, 1996.

Duffie, D. & H. Richardson, 1991, "Mean-Variance Hedging in Continuous Time", *Annals of Applied Probability*, 1(1), 1-15.

Duffie, D. & T. Zariphopolou, 1993, "Optimal Investment with Undiversifiable Income Risk", *Mathematical Finance*, 3, 135-148.

El Karoui, N., S. Peng, and M.C. Quenez, 1997, "Backward Stochastic Differential Equations in Finance," *Mathematical Finance*, 75, 1-71.

Follmer, H. & M. Schweizer, 1991, "Hedging of Contingent Claims Under Incomplete Information," in M.H.A. Davis and R.J. Elliot (eds.), *Applied Stochastic Analysis*, Gordon and Breach, London.

Follmer, H. & D. Sonderman, 1986, "Hedging of Non-Redundant Contingent Claims," in *Contributions to Mathematical Economics*, W. Hildenbrand and A. Mas Collel (eds.).

Fouque, J. P., G. Papanicolaou, and K. R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge, 2000.

Frey R., 1997, “Derivative Asset Analysis in models with level-dependent and stochastic volatility,” *CWI Quarterly*, 10(1), 1-34.Frittelli, M., 2000, “The Minimal Entropy Martingale Measure and the Valuation Problem in Incomplete Markets,” Mathematical Finance, 10(1), 39-52.

Frittelli, M., 2000, “The Minimal Entropy Martingale Measure and the Valuation Problem in Incomplete Markets,” Mathematical Finance, 10(1), 39-52.

Grenadier, S., 1996, "The Strategic Exercise Of Options: Development Cascades And Overbuilding In Real Estate Markets," *Journal of Finance*, 51(5), 1653-1679.

Grenadier, S., 1999, "Information Revelation Through Option Exercise," *Review of Financial Studies*, 12(1), 95-129.

Grenadier, S., 2002, "Option Exercise Games: An Application To The Equilibrium Investment Strategies Of Firms," *Review of Financial Studies*, 15(3), 691-721.

Grenadier, S., and N. Wang, 2005, "Investment timing, Agency and Information," *Journal of Financial Economics*, 75, 493-533.

Heath, D., E. Platen and M. Schweizer, 2001, “A Comparison of Two Quadratic Approaches to Hedging in Incomplete Markets,” Mathematical Finance, 11(4), 385-413.

Henderson, V and D. Hobson, 2005, “A Note on Irreversible Investment, Hedging and Optimal Consumption Problems,” to appear in International Journal of Theoretical and Applied Finance .

Henderson, V., 2005, “Valuing the Option to Invest in an Incomplete Market,” Princeton University Working Paper.

Henderson, V., 2005, “Analytical Comparisons of Option Prices in Stochastic Volatility models,” *Mathematical Finance*, 15(1), 49-59.

Henderson, V., D Hobson, S Howison and T Kluge, 2005, “A Comparison of Option Prices under Different Pricing Measures in a Stochastic Volatility Model with Correlation,” Review of Derivatives Research, 8, 5-25.

Henderson, V., 2002, “Valuation of Claims on Non-Traded Assets Using Utility Maximization,” *Mathematical Finance*, 12(4), 351-373.

Henderson, V. & D. Hobson, 2002, "Real Options with Constant Relative Risk Aversion," *Journal of Economic Dynamics and Control*, 27(2), 329-355.

Heston, S., 1993,"A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies,* 6, 327-343.

Hobson D., 2004, "Stochastic Volatility Models, Correlation and the q-optimal Measure," *Mathematical Finance*, 14, 537-556.

Hodges, S. and A. Neuberger, 1989,"Optimal Replication Contingent Claims under Transaction Costs," *Review of Futures Markets,* 8, 222-239.

Hofmann, N., E. Platen and M. Schweizer (1992), "Option Pricing under Incompleteness and Stochastic Volatility" *Mathematical Finance* ***2***, 153-187

Hsu, J.C. and E. Schwartz, 2003, "A model of R&D Valuation and the Design of Research Incentives," NBER Working Paper 10041.

Hugonnier, J. and E. Morellec, 2004, "Investment Under Uncertainty and Incomplete Markets", Working Paper

Hull, John C., *Options, Futures, & Other Derivatives*, Prentice-Hall, Inc., 2000.

Hull, J. and A. White, 1987, "The Pricing of Options on Assets with Stochastic Volatility," *Journal of Finance*, 42, 281-300.

Ilhan, A. and K. R. Sircar, 2006, "Optimal Dynamic Hedges for Barrier Options", *Mathematical Finance,* 16(2), 359-385.

Jonsson, M., and K. R. Sircar, 2002, "Partial Hedging in a Stochastic Volatility Environment", *Mathematical Finance*, 12(4), 375-409.

Karatzas I. and S. Shreve, *Brownian Motion and Stochastc Calculus,* Springer, 1991.

Karatzas I. and S. Shreve, *Methods of Mathematical Finance,* Springer, 1998.

Karatzas I. and H. Wang, 2000, "Utility Maximiztion with Discretionary Stopping," *Journal on Control and Optimization*, 39, 306-329.

Kim, I. J., 1990, "The Analytic Valuation of American Options," *Review of Financial Studies*, 3, 547-572.

Kramkov, D., and W. Schachermayer, 2003, “Necessary and sufficient conditions in the problem of optimal investment in incomplete markets,” *Annals of Applied Probability*, 13 (4), 1504-1516.

Laurent J.P and H. Pham, 1999, “Dynamic Programming and Mean-Variance Hedging,” *Finance and Stochastics*, 3, 83-101.

Lerner, J., and R. P. Merges, 1998, “The Control of Technology Alliances: An Empirical Analysis of The Biotechnology Industry,” *Journal of Industrial Economics*, 46, 125-156.

Lweis, A.L., *Option Valuation under Stochastic Volatility*, Finance Press, California, 2000.

Lin,B., and J. Darling, 1999 “An Analysis of the Formulation of Strategic Alliances: A Focus on the Pharmaceutical Industry,” *Industrial Management & Data Systems*, 99, 121-127.

Longstaff, F., and E. Schwartz, 2001, “Valuing American Options by Simulation: A Simple Least-Squares Approach,” *Review of Financial Studies*, 14, 113-147.

Mania, M., M. Santacroce and R. Tevzadze, 2003, “A Semi-martingale BSDE Related to the Minimal Entropy Martingale Measure,”*Finance and Stochastics,*7, 385-402.

Mauer, D., and A., Triantis, 1994, “Interactions of Corporate Financing and Investment Decisions: A Dynamic Framework,” *Journal of Finance*, 49, 1253-1277.

McDonald, Robert L., *Derivatives Markets*, Pearson Education, Inc., 2003.

McDonald, R. & D. Siegel, 1986, “The Value of Waiting to Invest,” *Quarterly Journal of Economics,* 101(4), 708-728.

McKean, H. P., 1965, “A Free Boundary Value Problem for the Heat Equation Arising from a Problem in Mathematical Economics,” *Industrial Management Review*, 6, 32-39.

Merton, R.C., 1969, “Lifetime Portfolio Selection Under Uncertainty: The Continuous Time Case”, *Review of Economics and Statistics,* 51, 247-257.

Merton, R.C., 1971, “Optimum Consumption and Portfolio Rules in a Continuous Time Model”, *Journal of Economic Theory,* 3, 373-413.

Merton, R.C., 1973, “Theory of Rational Option Pricing”, *Bell Journal of Economics and Management,* 4, 141-183.

Miao, J. and N. Wang, 2005, “Investment, Consumption and Hedging under Incomplete Markets”, Working Paper.

Milterson, K. and E. Schwartz, 2002, “R&D Investments with Competitive Interactions,” UCLA Working Paper.

Musiela, M. & M. Rutkowski, *Martingale Methods in Financial Modelling*, Springer-Verlag, 1997.

Musiela, M. & T. Zariphopoulou, 2001, “Pricing and Risk Management of Derivatives Written on Non-Traded Assets,” Working Paper.

Musiela, M. & T. Zariphopoulou, 2004, “An Example of Indifference Prices Under Exponential Preferences,” *Finance and Stochastics*, 8(2), 229-239.

Myers, S.C., and C.D. Howe, 1997, “A Life-Cycle Financial Model of Pharmaceutical R&D,” MIT Working Paper.

Oberman, A. and T. Zariphopoulou, 2003, “Pricing Early Exercise Contracts in Incomplete Markets,” *Computational Management Science*, 75-107.

Ottoo, Richard E., *Valuation of Corporate Growth Opportunities: A Real Options Approach*, Garland Publishing, Inc., 2000.

Ottoo, Richard E., 1998, “Valuation of Internal Growth Opportunities: The Case of a Biotechnology Company,” *The Quarterly Review of Economics and Finance*, 38, Special Issue, 615-633.

Paddock, J., D. Siegel & J. Smith, 1988, “Option Valuation of Claims on Real Assets: The Case of Offshore Petroleum Leases,” *Quarterly Journal of Economics*, 103(3), 479-508.

Pham H., T. Rheinlander and M. Schweizer, 1998, “Mean-Variance Hedging for Continuous Process: New Proofs and Examples,” *Finance and Stochastics*, 2(2), 173-198.

Pindyck, R., 1993, “Investments of Uncertain Cost,” *Journal of Financial Economics*, 34, 53-76.

Rheinländer, T., and M. Schweizer, 1997, "On L^2-Projections on a Space of Stochastic Integrals," *Annals of Probability* ***25***, 1810-1831.

Rheinlander T., 2005, “An Entropy Approach to the Strein and Stein Model with Correlation,” *Finance and Stochastics*, 9, 399-413.

Robinson, D.T., and T.E. Stuart, 2003, “Contractual, Relational, and Property Rights-based Control in Biotech Strategic Alliances,” Working Paper.

Robinson, D.T., and T.E. Stuart, 2002, “Just How Incomplete Are Incomplete Contracts? Evidence from Biotech Strategic Alliances,” Working Paper.

Romano, M., and N. Touzi, 1996, “Contingent Claims and Market Completeness in a Stochastic Volatility Model”, *Mathematical Finance,* 7, 399-412.

Rouge, R., and N. El Karoui, 2000, “Pricing via Utility Maximization and Entropy,” *Mathematical Finance*, 10, 259-276.

Schwartz, E., 2003, “Patents and R&D as Real Options,” NBER Working Paper 10114.

Schwartz, E., 2004, “Patents and R&D as Real Options,” *Economic Notes*, 33, 23-54.

Schwartz, E., and M. Moon, 2000, “Evaluating Research and Development Investments”, in *Innovation, Infrastructure and Strategic Options*, M. Brennan and L. Trigeorgis, eds., Oxford University Press, 85-106.

Schweizer, M., 1991, “Option Hedging for Semimartingales,” *Stochastic Processes and their Applications*, 37(2), 339-363.

Schweizer, M.,1992, "Mean-Variance Hedging for General Claims" *Annals of Applied Probability* ***2***, 171-179.

Schweizer, M., 1994, "Approximating Random Variables by Stochastic Integrals," *Annals of Probability* ***22***, 1536-1575.

Schweizer, M., 1995, “On the Minimal Martingale Measure and the Follmer-Schweizer Decomposition,” *Stochastic Analysis and Applications*, 13, 573-599.

Schweizer, M., 1999, “A Minimality Property of the Minimal Martingale Measure,” *Statistics and Probability Letters* ***42***, 27-31.

Shenkar, O., and J. J. Reuer, eds., *Handbook of Strategic Alliances*, Sage Publications Ltd., 2005.

Sircar, K. R. and T. Zariphopoulou, 2004, “Bounds and Asymptotic Approximations for Utility Prices When Volatility is Random,” *Journal of Control and Optimization*, 43, 1328-1353.

Sircar, K. R. and G. C. Papanicolaou, 1999, "Stochastic Volatility, Smile & Asymptotics," *Applied Mathematical Finance*. 6(2), 107-145.

Smit, Han T.J. and Lenos Trigeorgis, 2001, "R&D Option Strategies," 5th Annual Real Options Conference.

Stein, E. M. and J. C. Stein, 1991, "Stock Price Distributions with Stochastic Volatility: An Analytical Approach," *Review of Financial Studies*, 4, 727-752.

Touzi, N., 1999, "American Options Exercise Boundary When the Volatility Changes Randomly," *Applied Mathematics Optimization*, 39, 411-412.

Touzi , N. (2002): Stochastic Control Problems, Viscosity Solutions and Application to Finance

Trigeorgis,L., *Real Options: Managerial Flexibility and Strategy in Resource Allocation,* MIT Press, 1996.

Trigeorgis, L., 1993, "Real Options and Interactions with Financial Flexibility," *Financial Management*, 22, 202-224.

Willner, Ram, 1995, "Valuing Start-up Venture Growth Options," in *Real Options in Capital Investment: Models, Strategies and Applications*, Trigeorgis, L., ed., Praeger Publishers, 221-239.

Zariphopoulou, T., 2001, "A Solution Approach to Valuation with Unhedgeable Risks," Finance and Stochastics, 5(1), 61-82.

Zvan, R., P.A. Forsyth, and K.R. Vetzal, 1998, "Penalty Methods for American Options with Stochastic Volatility," *Journal of Computational and Applied Mathematics*, 91, 199-218

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