ON PROGENITIVELY KOSZUL COMMUTATIVE RINGS

by

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ABSTRACT

ON PROGENITIVELY KOSZUL COMMUTATIVE RINGS

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This paper introduces and identifies classes of progenitively Koszul rings. A progenitively Koszul ring is a commutative ring R that admits a Koszul complex $A = K(R)$ that is formal, and such that the homology algebra $H(A)$ generated by A is a Koszul algebra. Local complete intersections, which yield exterior algebras (an example of a Koszul algebra) for their homology algebras, serve as a prototype. It is shown that the local complete intersections occupy only a small portion of the class of progenitively Koszul rings.

The material in Chapter 1 will cover basic definitions and facts regarding free resolutions, differential graded algebras, homology, lattices, Koszul algebras, and PBW basis constructions. The results of Chapter 2 consist of applications of tensors products of differential graded algebras, tensors of Koszul complexes, and tensors of formal algebras to establish that the tensor product of progenitively Koszul rings is itself a progenitively Koszul ring. Chapter 3 will classify several homology algebras

based upon earlier work of Luchezar Avramov, Andrew Kustin, and Matthew Miller. Many of these rings are shown to be progenitively Koszul, and are not restricted to local complete intersections. Applying the results of Chapter 2 shows that any arbitrary tensor of such rings will also yield a progenitively Koszul ring.

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CHAPTER 1

INTRODUCTION, TERMS AND DEFINITIONS

1.1 Introduction

Koszul algebras have been extensively studied over the past 40 years. The initial paper on Koszul algebras by Stewart Priddy [27] investigated resolutions of the Steenrod algebra. In his paper the cohomology of a Lie algebra L is computed by applying an appropriate differential to $U(L) \otimes E(L)$, the tensor of the universal enveloping algebra of L with the exterior algebra of L. The resolutions which result are called Koszul resolutions. The algebras for which these resolutions are defined he called Koszul algebras. For every Koszul algebra A there is a dual Koszul algebra $A^!$. Priddy's paper also established connections between Koszul algebras and PBW algebras. In 1978, I. Bernstein, I. Gelfand, and S. Gelfand [7] demonstrated the symmetric algebra as (Koszul) dual to the exterior algebra. Their paper characterized coherent sheaves on projective spaces in terms of modules over the exterior algebra [25]. The category of modules over a symmetric algebra was shown to be dual to the category of modules over the exterior algebra (Koszul pair). Later extensions of the BGG correspondence have been done by D. Eisenbud [15], A. Beilinson, V. Ginzburg, W. Soergel [6] and others with application to derived categories of graded modules. One area of investigation has been the structure of the free resolutions of finitely generated graded modules over an exterior algebra. In [6] a certain functor establishes an equivalence from the category of graded modules over the symmetric algebra to the category of linear compexes of free modules over the exterior algebra.

Research done by Ralph Fröberg and Jörg Backelin [4], [5], [16] characterized Koszul algebras further by Hilbert and Poincare series, linear minimal resolutions of the field, and lattices. In their papers several equivalent properties of Koszul algebras were demonstrated. Specifically, a quadratic algebra A is a Koszul algebra when $H_A(z)P_A(-z) = 1$, or when $\mathrm{Tor}_i^A(k,k)_j = 0, \;\; i \neq j$, that is, the minimal resolution of the field k over A is linear, or when the product ideal $A_+^{\alpha} I^{\beta} A_+^{\gamma}, \;\; \alpha+\beta+\gamma \geq 0$ generates a distributive lattice. Several other equivalent characteristics of Koszul algebras are known. Koszul algebras are relevant in other areas of mathematics, notably combinatorics [17], [10], [29], [1]; topology [23]; algebraic geometry [24], [19]; and quantum theory [22], [30], [28]. The concept of a Koszul algebra has been extended to graded algebras with relations in more than one degree [11].

The concept of a progenitively Koszul ring is motivated by considering that while a ring may not be Koszul, its homology algebra may yet be a Koszul algebra, and this may still allow one to obtain strong correspondence theories for such rings. Specifically a progenitively Koszul ring is a commutative ring R that admits a Koszul complex $A = K(R)$ that is formal, and such that the homology algebra $H(A)$ generated by A is a Koszul algebra. Local complete intersections, which yield exterior algebras (an example of a Koszul algebra) for their homology algebras, serve as a prototype. It is shown that the local complete intersections occupy only a small portion of the class of progenitively Koszul rings. Current research that involves progenitively Koszul rings in the classification of Ext modules over Ext algebras is developed in "Reverse Homological Algebra over some Local Rings" by L. Avramov and D. Jorgensen [3]. Their paper establishes a duality of the derived category of differential graded modules over a Koszul algebra with the derived category of differential graded modules over the Koszul dual. From there they prove if R is a PK local ring, then one may classify in a strong sense the $\mathrm{Ext}_R(k,k)$-modules $\mathrm{Ext}_R(M,k)$, for finitely generated R-modules M.

Chapter 1 will briefly outline the major concepts that are necessary in identifying classes of progenitively Koszul rings. The main reference for Koszul algebras is the paper by R. Fröberg [16], with supplemental material from standard texts of G. Boffi and D. Buchsbaum [9], D. Eisenbud [14], S. MacLane [21], A. Polischuk and L. Positselski [25], and G. Birkhoff [8]. The seven sections: Graded Algebras; Free Resolutions; Koszul Algebras; Differential Graded Algebras; Homology Algebras and Formal Algebras; Progenitively Koszul Rings; PBW Algebras and Lattices will cover the basic terms and development of the concepts used within the paper.

Chapter 2 will develop the main theorem used to construct progenitively Koszul rings from other progenitively Koszul rings. The theorem will be given in 3 versions: the first two will be formal and the third will be a constructive proof that allows transparency in the relevant characteristics of the rings involved.

Chapter 3 will focus on rings of small projective dimension classifying which are progenitively Koszul and which are not. The results are based upon previous work done by L. Avramov, A. Kustin, and M. Miller. With the exception of two of the classes of the rings considered, all are progenitively Koszul rings and most are not complete intersections. By the results of chapter 2, any arbitrary tensor of such rings will yield a progenitively Koszul ring and these again are not complete intersections if their constituents are not. The classifications in chapter 3 will also use both lattices and PBW basis constructions as developed by Jörg Backelin [4] and A. Polischuk, and L. Positselski [25].

All rings are assumed to have a unit element. All gradings are assumed to be non-negative.

1.2 Graded Algebras

A *graded ring* is a ring A with a direct sum decomposition

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots = \bigoplus_{i \geq 0} A_i$$

such that each component A_i is an additive abelian group whose elements are called *homogeneous of degree* i and where multiplication respects the grading, i.e., for $a \in A_i$ and $b \in A_j$ the product ab is in A_{i+j}. The component A_i is called the *homogeneous component of degree* i. For an element $a \in A_i$ the degree of a is denoted by $|a|$. The component A_0 is itself a ring and $1_A \in A_0$.

Let A be a graded ring. An A-module P is a *graded module*: $P = \bigoplus_{i \geq 0} P_i$ when each P_i is an additive abelian group called the i^{th} *homogeneous component of* P and ring multiplication on P respects the following: for $r \in A_i$ and $m \in P_j$, rm is in P_{i+j}.

Let k be a commutative ring. A k-algebra A is a ring with a ring homomorphism $f : k \longrightarrow A$ (in which $1_k \mapsto 1_A$) and such that $f(k)$ is contained in the center of A. Every A-module M becomes a k-module via the multiplication: $r \in k$, $m \in M$, $r \cdot m = f(r)m$. If A is a commutative graded ring, it is a graded A_0-algebra via the natural inclusion: $A_0 \hookrightarrow A$.

1.2.1 The Tensor Algebra

Let R be a commutative ring and N be any R-module. The *tensor algebra* on N is defined as the free algebra

$$T(N) = R \oplus N \oplus (N \otimes N) \oplus (N \otimes N \otimes N) \oplus \cdots .$$

$T(N)$ is a graded R-algebra with $T_i(N) = \bigotimes^i N$. The definition of multiplication on simple tensors in $T(N)$ is as follows:

Let a and b be simple tensors in $T(N)$ of degree i and j (resp.):

$$a = a_1 \otimes \cdots \otimes a_i \in T(N)_i \ ; \qquad b = b_1 \otimes \cdots \otimes b_j \in T(N)_j \ .$$

The product $a \otimes b$ is given by the following:

$$a \otimes b = a_1 \otimes \cdots \otimes a_i \otimes b_1 \otimes \cdots \otimes b_j \in T(N)_{i+j} \ .$$

Then multiplication in general is defined in $T(N)$ by extending by linearity.

1.2.2 The Exterior Algebra

Let S be the two sided ideal in $T(N)$ generated by the elements: $x \otimes y + y \otimes x$, $x \otimes x$ for $x, y \in N$. The *exterior algebra* $\bigwedge N$ is defined as the free algebra $T(N)$ modulo S, i.e. $\bigwedge N = T(N)/S$. Since $T(N)$ is graded and S is a homogeneous two-sided ideal the exterior algebra is also a graded R-algebra. The coset $a \otimes b + S$ is denoted by $a \wedge b$. The homogeneous component of degree i in $\bigwedge N$ is denoted by $\bigwedge^i N$ and is generated by products of exactly i elements of N. Let $a \in \bigwedge^i N$ and $b \in \bigwedge^j N$ be simple wedges, that is, $a = a_1 \wedge \cdots \wedge a_i$ and $b = b_1 \wedge \cdots \wedge b_j$. Multiplication is induced from that in $T(N)$, thus the product $a \wedge b$ is given by the following:

$$a \wedge b = a_1 \wedge \cdots \wedge a_i \wedge b_1 \wedge \cdots \wedge b_j \in \bigwedge^{i+j} N, \ .$$

and the product in $\bigwedge N$ in general is obtained by extending this rule by linearity. For an R-module homomorphism $f : N \to M$ there is an induced map:

$$\bigwedge f : \bigwedge^p N \to \bigwedge^p M$$
$$a_1 \wedge \cdots \wedge a_p \to f(a_1) \wedge \cdots \wedge f(a_p)\ .$$

EXAMPLE: Let N be a free R-module of rank n with basis $\{e_1, \cdots, e_n\}$. Then $\bigwedge^i N$ is a free R-module of rank $\binom{n}{i}$ with basis $\{e_{j_1} \wedge \cdots \wedge e_{j_i} :\ j_1 < \cdots < j_i\}$.

1.2.3 The Hilbert Series

Let $A = \bigoplus_{i \geq 0} A_i$ be a graded Noetherian ring with A_0 a field k. In this case each A_i is a finite dimensional vector space over k for all $i \geq 0$.

The *Hilbert series* of A is defined as the generating function for the k-vector space dimensions of the A_i

$$\mathrm{H}(z) = \sum_{i \geq 0} \dim_k A_i \cdot z^i$$

where z is a formal variable.

1.3 Free Resolutions

It is assumed that the reader is familiar with the definition of a projective module and that free modules are projective. In what follows M will denote a module over a ring R. Consider a sequence of homomorphisms of R-modules:

$$\mathbf{C} : \cdots \to C_{n+1} \xrightarrow{\phi_{n+1}} C_n \xrightarrow{\phi_n} C_{n-1} \to \cdots\ .$$

The sequence $\mathbf{C}$ is a *complex* if the image of ϕ_{n+1} is contained in the kernel of ϕ_n for all n. The maps ϕ_i are referred to as the *differentials*. The *homology* $\mathrm{H}(\mathbf{C})$ of $\mathbf{C}$ is the direct sum of the modules: $\mathrm{H}_i(\mathbf{C}) = \mathrm{Ker}(\phi_i)/\,\mathrm{Im}(\phi_{i+1})$; $\mathrm{H}_i(\mathbf{C})$ is the i^{th} *homology of* $\mathbf{C}$.

The sequence $\mathbf{C}$ is called *exact* when the image of ϕ_{i+1} equals the kernel of ϕ_i for all i, in other words, $\mathrm{H}_i(C) = 0$ for all i. A *projective resolution* of the R-module M is an exact sequence

$$\mathbf{P} : \cdots \to P_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} P_1 \xrightarrow{\phi_1} P_0 \xrightarrow{\phi_0} M \to 0$$

such that all P_i are projective. If M above is replaced by 0, then the resulting sequence is called a *deleted* projective resolution of M. In the case that the projective modules are free R-modules the above is called a *free resolution* of M. It is a standard fact that if R is commutative and Noetherian, and M is finitely generated, then there exists a free resolution of M of the form:

$$\mathbf{F} : \cdots \to R^{b_n} \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} R^{b_1} \xrightarrow{\phi_1} R^{b_0} \xrightarrow{\phi_0} M \to 0$$

where the b_i in the expression are all non-negative integers, called the *Betti numbers* of $\mathbf{F}$.

By fixing bases of the R^{b_i} the maps ϕ_i can be represented by matrices with entries in R. In the case when R is graded, and all nonzero entries of the matrices are homogeneous, $\mathbf{F}$ is called a *graded resolution of M.* Such a resolution exists if and only if M is a graded R-module. When R is a graded ring, and all non-zero entries of each map representing ϕ_i have positive degree, then $\mathbf{F}$ is a *graded minimal free resolution.* When R is a local ring, then $\mathbf{F}$ is a *minimal free resolution*, uniquely determined by M, when all entries of the matrices representing the maps lie in the maximal ideal of

A. In this case the Betti numbers of $\mathbf{F}$ are uniquely determined by M and are called the *Betti numbers of M*.

1.3.1 The Poincaré Series

Let M and N both be R-modules. The derived functor $\operatorname{Tor}_*^R(M,N)$ is defined by tensoring (over R) each module in a deleted projective resolution $\mathbf{P}$ of M with N and subsequently evaluating homology of the induced complex:

$$\operatorname{Tor}_i^R(M,N) = \mathrm{H}_i(\mathbf{P} \otimes N) = \operatorname{Ker}(\phi_i \otimes N)/\operatorname{Im}(\phi_{i+1} \otimes N)\ .$$

Since free modules are also projective modules, the (above) definition applies to free resolutions. Assume R is a local ring with residue field k. Consider the minimal free R-resolution of M represented above by $\mathbf{F}$. Let N be the field k. From the definition of a minimal free resolution, $\phi_i \otimes k = 0$ for all $i \geq 1$, hence $\operatorname{Tor}_i^R(M,k) \cong R^{b_i} \otimes k \cong k^{b_i}$. Thus, each free module in the resolution becomes a finite dimensional vector space after tensoring with k. The *Poincaré series of A*, $\mathrm{P}_R^M(z)$, is the generating function for the k-vector space dimensions (i.e. Betti numbers) of $\operatorname{Tor}_*^R(M,k)$

$$\mathrm{P}_R^M(z) = \sum_{i\geq 0} \dim_k \operatorname{Tor}_i^R(M,k) \cdot z^i = \sum_{i \geq 0} b_i \cdot z^i\ .$$

1.3.2 Graded Resolutions

Suppose that A is a graded ring. Define $A(-j)$ as the graded ring with $(A(-j))_n = A_{n-j}$. In this case each module F_i in the graded free resolution of a graded A-module

M is a direct sum of b_i shifted copies of A. We let $b_{i,j}$ denote the number of copies of $A(-j)$ in F_i. Thus $F_i = \bigoplus_{j\geq 0} A(-j)^{b_{i,j}}$. A homogeneous element $a \in F_i$ is said to have *homological degree* i, and supposing that $a \in (F_i)_n$, a is said to have *internal degree* n. The resolution is now written as

$$\mathbf{F}: \cdots \to \bigoplus_{j\geq 0} A(-j)^{b_{n,j}} \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} \bigoplus_{j\geq 0} A(-j)^{b_{1,j}} \xrightarrow{\phi_1} \bigoplus_{j\geq 0} A(-j)^{b_{0,j}} \xrightarrow{\phi_0} M \to 0\,.$$

In this graded case there is a refinement to the (above) Poincare series:

$$\mathrm{P}_A^M(x,y) = \sum_{i,j} \dim_k(\mathrm{Tor}_i^A(M,k))_j \cdot x^i y^j = \sum_{i,j\geq 0} b_{i,j} \cdot x^i y^j$$

where the formal variable x records the homological degree i and y records the internal degree j.

1.3.3 The Koszul Complex

The following development of the Koszul complex closely follows the presentation in Boffi and Buchsbaum [9, pp.42-44]. Let R be a ring, L an R-module, and $f: L \to R$ an R-linear map. Let $l_1, \ldots, l_n$ be a sequence of elements of L. Recall the definition of $\bigwedge L$ from section 1.1. The assignment

$$(l_1, \ldots, l_n) \to \sum_{i=1}^{n} (-1)^{i+1} f(l_i) \cdot l_1 \wedge \cdots \wedge \widehat{l_i} \wedge \cdots \wedge l_n$$

defines an alternating n-multilinear map $L^n \to \bigwedge^{n-1} L$ where $\widehat{l_i}$ refers to the omission of the element l_i from the exterior product. By the universal mapping property [13, p.342] applied to the n^{th} power of an exterior algebra, the following R-multilinear map exists:

$$\partial_{(f,n)} : \bigwedge^n L \to \bigwedge^{n-1} L$$

$$\partial_{(f,n)}(l_1 \wedge \cdots \wedge l_n) \to \sum_{i=1}^{n}(-1)^{i+1} f(l_i) \cdot l_1 \wedge \cdots \wedge \widehat{l_i} \wedge \cdots \wedge l_n \,.$$

One can show $\partial_{(f,n-1)} \circ \partial_{(f,n)} = 0$. For $x \in \bigwedge^p L$ and $y \in \bigwedge^q L$,

$$\partial_f(x \wedge y) = \partial_f(x) \wedge y + (-1)^{|x|} \cdot x \wedge \partial_f(y)$$

which is referred to as the *Leibniz rule*.

In the case that L is a free R-module of rank s with basis $\{e_1, \ldots, e_n\}$, a map $f : L \to R$ defined as: $e_i \mapsto x_i$ with $x_1, \ldots, x_s$ a fixed sequence of elements of R, then $\bigwedge L$, together with the maps ∂_f above, is called the *Koszul complex on* $x_1, \ldots, x_s$ and is denoted by $K(x_1, \ldots, x_s)$. Note: The degree n component $K_n(x_1, \ldots, x_s)$ of $K(x_1, \ldots, x_s)$ is a free R-module of rank $\binom{s}{n}$ with basis $\{e_{i_1} \wedge \cdots \wedge e_{i_n} : i_1 < i_2 < \ldots < i_n\}$ and has the following degree -1 differential:

$$\partial = \partial_{(f,n)} :$$

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_n}) = \sum_{r=1}^{n} (-1)^{r+1}(x_{i_r}) \cdot e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_r} \wedge \cdots \wedge e_{i_n} \,.$$

In the case that a maximal ideal m of a regular local ring R is generated by a sequence of elements $x_1, \ldots, x_s$, it is well-known [14, 19.12] that the Koszul complex $K(x_1, \ldots, x_s)$ is a minimal free R-resolution of the residue field $k = R/m$.

1.4 Koszul Algebras

Let V denote a vector space over a field k generated by a basis $\{x_1, ..., x_n\}$. Recall the definition of $T(V)$ from section 1.1. Let I denote a two sided homogeneous ideal in $T(V)$. The quotient algebra $A = T(V)/I$ denotes a graded algebra with grading induced by $T(V)$. If I is generated by homogeneous elements of degree 2, then A is a *quadratic algebra*. A *Koszul algebra* is defined to be a quadratic algebra such that $\mathrm{Tor}_i^A(k,k)_j = 0$ for $i \neq j$. This is equivalent to k having a linear free resolution, as above, in which the nonzero entries of all matrices representing the differentials have degree 1. The following are equivalent [16, p.4]:

1. $\mathrm{H}_A(z)\,\mathrm{P}_A(-z) = 1$;
2. $\mathrm{H}_A(xy)\,\mathrm{P}_A(-x,y) = 1$;
3. A is Koszul.

1.5 Differential Graded Algebras

A *differential graded algebra* over a commutative ring k ("DG k-algebra") [21] is a graded k-algebra A equipped with a graded k-module homomorphism $\partial : A \to A$ of degree -1 with $\partial^2 = 0$ such that the Leibniz formula :

$$\partial(a_1a_2) = (\partial a_1)a_2 + (-1)^{|a_1|}a_1(\partial a_2)$$

is satisfied.

Let R be a commutative ring and $x_1,\ldots,x_s$ a sequence of elements of R. The Koszul complex on $x_1,\ldots,x_s$ is an example of a DG R-algebra. Multiplication is defined by concatenation of the wedge symbol "$\wedge$".

EXAMPLE:

Consider the polynomial ring in five variables: $R = k[x_1,x_2,x_3,x_4,x_5]$.

Let $\{e_1,\ldots,e_5\}$ be the standard basis vectors of R^5 and define $f : R^5 \to R$ by $f(e_i) = x_i$. Let $a = e_1 \wedge e_3$ and let $b = e_2 \wedge e_4 \wedge e_5$.

The Leibniz rule will be verified in an example:

$\partial(ab) = \partial((e_1 \wedge e_3)\cdot(e_2 \wedge e_4 \wedge e_5))$

$$= \partial(e_1 \wedge e_3)(e_2 \wedge e_4 \wedge e_5) + (-1)^{|e_1 \wedge e_3|}\ (e_1 \wedge e_3)\partial(e_2 \wedge e_4 \wedge e_5)$$
$$= ((x_1)e_3 - (x_3)e_1)(e_2 \wedge e_4 \wedge e_5) + (-1)^2(e_1 \wedge e_3)((x_2)e_4 \wedge e_5 - (x_4)e_2 \wedge e_5 + (x_5)e_2 \wedge e_4)$$
$$= (x_1)e_3 \wedge e_2 \wedge e_4 \wedge e_5 - (x_3)e_1 \wedge e_2 \wedge e_4 \wedge e_5 + (x_2)e_1 \wedge e_3 \wedge e_4 \wedge e_5$$
$$-(x_4)e_1 \wedge e_3 \wedge e_2 \wedge e_5 + (x_5)e_1 \wedge e_3 \wedge e_2 \wedge e_4$$
$$= \partial(e_1 \wedge e_3 \wedge e_2 \wedge e_4 \wedge e_5)\ . \qquad \text{Q.E.D.}$$

1.6 The Homology Algebra

Associated to every DG A-algebra is its *homology algebra* $H(A)$. Multiplication in $H(A)$ is induced by the multiplication in A and is well defined. Let $x_1, \ldots, x_s$ be a sequence of elements in a commutative ring R. For the purposes of this thesis the homology algebra $H\big(K(x_1, \ldots, x_s)\big)$ of the Koszul complex $K(x_1, \ldots, x_s)$ will be referred to as *the homology algebra of* $x_1, \ldots, x_s$. It is a standard fact about the homology algebra that $x_i \cdot H\big(K(x_1, \ldots, x_s)\big) = 0$ for all i, and thus $H_i(K(x_1, \ldots, x_s))$ is a module over $R/(x_1, \ldots, x_s)$ for all i.

1.6.1 Formal Algebras

Let F_1 and F_2 denote two complexes of R-modules. A chain map $\Phi : F_1 \to F_2$ is said to be a *quasi-isomorphism* if Φ induces an isomorphism on homology, i.e. $\mathrm{H}(\Phi) : \mathrm{H}(F_1) \xrightarrow{\cong} \mathrm{H}(F_2)$ is an isomorphism in each degree. If two complexes have a quasi-isomorphism between them they are said to be *quasi-isomorphic* [14, p.662]. Two complexes linked by a sequence of quasi-isomorphisms are said to be *weakly* *equivalent.* Quasi-isomorphisms will be denoted by $\simeq$.

EXAMPLE: Every deleted free resolution of an R-module establishes a quasi-isomorphism from the resolution to the module.

$$\begin{array}{lccccccccc}
\mathbf{F}: & \cdots & \xrightarrow{\phi_3} & F_2 & \xrightarrow{\phi_2} & F_1 & \xrightarrow{\phi_1} & F_0 & \to & 0 \\
\downarrow\simeq & & & \downarrow & & \downarrow & & \downarrow\pi & & \\
\mathbf{F}': & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{M} & \to & 0
\end{array}$$

$\mathrm{H}_0(\mathbf{F}) = \mathrm{Ker}(F_0 \to 0)/\operatorname{Im}(\phi_1) = F_0/\operatorname{Im}(\phi_1) \cong M = \mathrm{H}_0(\mathbf{F}')$

EXAMPLE: Let R be a local ring with residue field k. For an ideal $I \subset Q$, let $R = Q/I$. Let F_1 and F_2 be minimal free resolutions of k and R (resp.) over Q. Then the following complexes are quasi-isomorphic:

$$k \otimes_Q F_2 \xleftarrow{\simeq} F_1 \otimes_Q F_2 \xrightarrow{\simeq} F_1 \otimes_Q R$$

$$\mathrm{H}(k \otimes_Q F_2) \cong \mathrm{H}(F_1 \otimes_Q F_2) \cong \mathrm{H}(F_1 \otimes_Q R)$$

In this example the isomorphisms are the result of the balanced functor: $\mathrm{Tor}^Q(k, R)$.

A *formal algebra* is a DG k-algebra A which is *weakly equivalent* to its homology algebra $H(A)$. Specifically, A is a formal algebra when there exists a sequence of quasi-isomorphisms linking A to $H(A)$:

$$A \overset{\simeq}{\longleftrightarrow} A_1 \overset{\simeq}{\longleftrightarrow} A_2 \overset{\simeq}{\longleftrightarrow} \cdots \overset{\simeq}{\longleftrightarrow} A_m \overset{\simeq}{\longleftrightarrow} H(A)\,.$$

Note that the symbol $\overset{\simeq}{\longleftrightarrow}$ indicates the quasi-isomorphism may occur in either direction.

EXAMPLE: Let Q, F_1, and F_2 be as in the previous example. Let $(F_2)_i$ denote the i^{th} free Q- module in the complex F_2.

$$H(k \otimes_Q F_2) \xleftarrow{\cong} k \otimes_Q F_2 \xleftarrow{\simeq} F_1 \otimes_Q F_2 \xrightarrow{\simeq} F_1 \otimes_Q R$$

(Note: The left arrow $\xleftarrow{\cong}$ is an isomorphism.)

In this example the isomorphism linking $H(k \otimes_Q F_2)$ to $k \otimes_Q F_2$ is the result of tensoring the minimal free resolution F_2 with k, which forces all free Q-modules to become vector spaces and all differentials to be the 0 map. Each free module $k \otimes_Q (F_2)_i$ is the kernel of $Id_k \otimes \phi_i$ at degree i, and 0 is the image of $Id_k \otimes \phi_{i+1}$. Thus,

$$\mathrm{H}_i(k \otimes_Q F_2) = (k \otimes_Q (F_2)_i/0) \cong k \otimes_Q (F_2)_i.$$

1.7 Progenitively Koszul Commutative Rings (PK Rings)

Let R be a ring and $x_1, \ldots, x_s$ be a sequence of elements of R generating a maximal ideal m. Let $k = R/m$. Let $A = K(x_1, \ldots, x_s)$ be the Koszul complex on the sequence $x_1, \ldots, x_s$. If A is formal such that its homology algebra $H(A)$ is a Koszul algebra, then the ring R is said to be *progenitively Koszul with respect to* $x_1, \ldots, x_s$. Henceforth, these rings will be denoted as *PK rings.*

1.8 PBW Algebras and Lattices

The classification of PK rings in chapter 3 will depend upon the construction of Poincare-Birkhoff-Witt bases as developed in ([25, pp.81-82]). The presentation has been condensed to fit the specifics of this paper.

Let V be a vector space with a basis $z_1, \ldots, z_m$. Let $T(V)$ be the tensor algebra on V and $J \subset T(V)$ be an ideal generated by homogeneous elements of degree 2. Then $A = T(V)/J$ is a quadratic algebra. Let $I = \{1, 2, \ldots, m\}$. Denote by α the multiindex $(i_1, .., i_n) \in I^n = I \times I \times \cdots \times I$ (n-times) and denote by z^α the monomial $z_{i_1} z_{i_2} \cdots z_{i_n} \in T(V)$. For $\alpha = \emptyset$ set $z^\emptyset = 1$. Consider the lexicographic order on the set of multiindices of length n: $(i_1, .., i_n) < (j_1, ...j_n)$ if and only if there exists a k such that $i_1 = j_1, ..., i_{k-1} = j_{k-1}$ and $i_k < j_k$.

LEMMA 1.1 [25, p.81]: *Let V, $T(V)$, I and J be as stated (above). Consider the basis of $V \otimes V$ generated by the degree 2 tensors: $z_{i_1} \otimes z_{i_2}$ (henceforth written as degree 2 monomials: $z_{i_1} z_{i_2}$). Let S_J be the subset of I^2 consisting of all $(i_1, i_2) \in I^2$ such that $z_{i_1} z_{i_2}$ can not be represented as a linear combination of $z_{j_1} z_{j_2}$ with $(j_1, j_2) < (i_1, i_2)$ modulo J. Then the images of the elements $z_{i_1} z_{i_2}$ with $(i_1, i_2) \in S_J$ form a basis of $(V \otimes V)/J_2$ where J_2 is the degree 2 component of J. The subset S_J is also uniquely characterized by the property that there exists a basis of J_2 of the form:*

$$z_{j_1} z_{j_2} - \sum_{\substack{(i_1,i_2)<(j_1,j_2) \\ (i_1,i_2)\in S_J}} c^{i_1,i_2}_{j_1,j_2} z_{i_1} z_{i_2}\,, \quad (j_1, j_2) \in I^2 - S_J.$$

Let $S^{(0)} = \emptyset$ and $S^{(1)} = \{1, 2, \ldots, m\}$. For $n \geq 2$, let $S^{(n)}$ denote the set of multiindices:

$$S^{(n)} = \{(i_1, \cdots, i_n) | (i_1, i_2) \in S_J, (i_2, i_3) \in S_J, \cdots (i_{n-1}, i_n) \in S_J\}.$$

Elements $z_1, \ldots, z_m \in V$ are called *PBW generators of A* if the monomials z^α with $\alpha \in \bigcup_{n\geq 0} S^{(n)}$ form a basis of A (PBW-basis of A). *A PBW-algebra is a quadratic algebra admitting a PBW basis.* The monomial orderings determine the choice for generators of the bases ([25, p.82]).

For the classifications in Chapter 3, the following Theorems will also be required:

Theorem [25, p.82]: *If the cubic monomials $z_{i_1}z_{i_2}z_{i_3}$ with $(i_1, i_2, i_3) \in S^{(3)}$ are linearly independent in A_3 then the same is true in any degree n. Therefore, in this case the elements $z_1, \ldots, z_m$ are PBW generators of A."*

Theorem [27, p.51]: *A quadratic PBW algebra is a Koszul algebra.*

Note: Not every Koszul algebra will have a PBW basis ([25], p.84).

J.Backelin developed an alternative definition for Koszul algebras using the concept of a distributive lattice from earlier work by V.E. Govorov. In his work ([4], pp.21-30) an algebra A is Koszul when it is quadratic and the product ideal: $A_+^{\alpha} I^{\beta} A_+^{\gamma}$ generates a distributive lattice for non-negative values of α, β, γ. The ideal $A_+^0 I^0 A_+^0$ is defined in the lattice to be the tensor algebra $T(V)$ on the indeterminates in the ring $R = T(V)/I$, and $\alpha+\beta+\gamma > 0$ in the lattice ([4], pp.11,15). The following lattice definition is due to G. Birkhoff ([8], pp.6-8): Let P denote a partially ordered set. A lattice is a partially ordered set such that any two of its elements have a greatest lower bound ("meet") denoted by $x \cap y$ and a least upper bound ("join") denoted by $x \cup y$. The operations $\cap$ and $\cup$ satisfy the following properties:

(i). $x \cap x = x$; $x \cup x = x$.

(ii). $x \cap y = y \cap x$; $x \cup y = y \cup x$.

(iii). $x \cap (y \cap z) = (x \cap y) \cap z$; $x \cup (y \cup z) = (x \cup y) \cup z$.

(iv). $x \cap (x \cup y) = x \cup (x \cap y) = x$.

A lattice is "modular" when it satisfies the following property:

If $x \leq z$, then $x \cup (y \cap z) = (x \cup y) \cap z$.

A lattice is "distributive" when it satisfies the following property:

$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$; $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$.

For purposes of this paper when considering lattices, "$\cap$" will denote "intersection" (of ideals) and "$\cup$" will denote "sum" (of ideals).

CHAPTER 2

TENSOR PRODUCTS OF PK RINGS

2.1 Introduction and the Main Theorem

The purpose of this chapter is to derive a method of generating PK rings from other PK rings. The main theorem of this section is the following:

THEOREM (2.1): *Let R_1 and R_2 both be commutative, Noetherian k-algebras where k is a field. Let $x_1, \ldots, x_s$ and $y_1, \ldots, y_t$ be sequences of elements generating maximal ideals m_1 and m_2 of R_1 and R_2 (resp.). Assume that R_1/m_1 and R_2/m_2 are both isomorphic to k. If R_1 and R_2 are PK rings with respect to $x_1, \ldots, x_s$ and $y_1, \ldots, y_t$ (resp.), then $R_1 \otimes_k R_2$ is a PK ring with respect to $\{x_1 \otimes 1_{R_2}, \ldots, x_s \otimes 1_{R_2}, 1_{R_1} \otimes y_1, \ldots, 1_{R_1} \otimes y_t\}$.*

The results of this chapter rely upon the special case of the Kunneth Theorem [18, p.184]:

If C and D are complexes of vector spaces over a field k, then for all n there is an isomorphism:

$$\mathrm{H}_n(C \otimes D) \to \bigoplus_{p+q=n} \mathrm{H}_p(C) \otimes_k \mathrm{H}_q(D).$$

2.2 The Tensor of DG k-Algebras is a DG k-Algebra

In this section k denotes any commutative ring. Let C and D be complexes of k-modules. We define $C \otimes_k D$, the tensor product of the two complexes C and D, as follows:

$$(C \otimes_k D)_n = \bigoplus_{p+q=n} C_p \otimes_k D_q.$$

Let $c \in C_p$, $d \in D_q$. The differential $\partial = \partial^{C \otimes D}$ for $C \otimes_k D$ is defined as follows:

$$\partial(c \otimes d) = \partial^C(c) \otimes d + (-1)^p c \otimes \partial^D(d),$$

where the symbol ∂^* denotes the differential associated with the complex $(*)$. It is trivial to show that, with the definition of the differential (above), $C \otimes_k D$ is a complex and ∂ is k-linear.

Now let C and D be DG k-algebras. Multiplication in $C \otimes_k D$ is defined as follows: For $(c \otimes d) \in C_p \otimes_k D_q$ and $(c' \otimes d') \in C_{p'} \otimes_k D_{q'}$,

$$(c \otimes d)(c' \otimes d') = (-1)^{p'q}(cc') \otimes (dd').$$

With this multiplication, $C \otimes_k D$ is also a DG k-algebra, as demonstrated in the following proposition:

PROPOSITION (2.2): *If C and D are DG k-algebras, then $C \otimes_k D$ is also a DG k-algebra.*

PROOF: It will be shown that $C \otimes_k D$ with this multiplication satisfies the Leibniz rule. In the first computation, the product will first be evaluated, followed by application of the differential; the second computation will first apply the differential, followed by the evaluation of the product.

Let $c \in C_p$, $c' \in C_{p'}$, $d \in D_q$, and $d' \in D_{q'}$.

$$\begin{aligned}
\partial((c \otimes d)(c' \otimes d')) &= (-1)^{p'q} \partial\, (cc' \otimes dd') \\
&= (-1)^{p'q} \Big(\partial(cc') \otimes (dd') + (-1)^{p+p'} (cc') \otimes \partial(dd') \Big) \\
&= (-1)^{p'q} ((\partial(c)c' + (-1)^p c\partial(c')) \otimes dd' \\
&\quad + (-1)^{p+p'} (cc' \otimes (\partial(d)d' + (-1)^q d\partial(d'))) \\
&= (-1)^{p'q} (\partial(c)c' \otimes dd' + (-1)^p c\partial(c') \otimes dd' \\
&\quad + (-1)^{p+p'} cc' \otimes \partial(d)d' + (-1)^{p+p'+q} cc' \otimes d\partial(d')) \\
&= (-1)^{p'q} \partial(c)c' \otimes dd' + (-1)^{p'q+p} c\partial(c') \otimes dd' \\
&\quad + (-1)^{p'q+p+p'} cc' \otimes \partial(d)d' + \ (-1)^{p'q+p+p'+q} cc' \otimes d\partial(d').
\end{aligned}$$

$$\begin{aligned}
\partial((c \otimes d)(c' \otimes d')) &= \partial\, (c \otimes d)\, (c' \otimes d') + (-1)^{|c \otimes d|}\ (c \otimes d)\partial(c' \otimes d') \\
&= (\partial(c) \otimes d + (-1)^p c \otimes \partial(d))(c' \otimes d') \\
&\quad + (-1)^{p+q} (c \otimes d) \Big(\partial(c') \otimes d' + (-1)^{p'} c' \otimes \partial(d') \Big) \\
&= (\partial(c) \otimes d)(c' \otimes d') + (-1)^p (c \otimes \partial(d))(c' \otimes d') \\
&\quad + (-1) p + q (c \otimes d)\, (\partial(c') \otimes d') + \ (-1)^{p+q+p'} (c \otimes d)\, (c' \otimes \partial(d')) \\
&= (-1)^{p'q} \partial(c)c' \otimes dd' + (-1)^{p+p'(q-1)} cc' \otimes \partial(d)d' \\
&\quad + (-1)^{p+q+(p'-1)q} c\partial(c') \otimes dd' + (-1)^{p+q+p'+p'q} cc' \otimes d\partial(d').
\end{aligned}$$

The coefficients of all four terms match, as indicated in the following table. The function notation, d.m., refers to the (composite) operation of first evaluating the

Table 2.1. Coefficients of Terms

$*$	$\partial(c)c'\otimes dd'$	$cc'\otimes\partial(d)d'$	$c\partial(c')\otimes dd'$	$cc'\otimes d\partial(d')$
d.m.	$(-1)^{p'q}$	$(-1)^{p'q+p+p'}$	$(-1)^{p'q+p}$	$(-1)^{p'q+p+p'+q}$
m.d.	$(-1)^{p'q}$	$(-1)^{p'q+p-p'}$	$(-1)^{p'q+p}$	$(-1)^{p'q+p+p'+q}$

product by the definition for multiplication in the complex, followed by differentiation; m.d. refers to the reverse order of the same two operations. The specific terms are listed as column headers. The sign coefficients of the terms are listed beneath for comparison with respect to the two different methods of evaluation. Q.E.D.

2.2.1 Extension to DG R_1- and DG R_2-Algebras

Let R_1 and R_2 both be k-algebras. The previous theorem will be extended to DG R_1- and DG R_2-algebras.

PROPOSITION (2.3): *Let F_1 be a DG R_1-algebra. Let F_2 be a DG R_2-algebra. Then $F_1 \otimes_k F_2$ is a DG $R_1 \otimes_k R_2$-algebra.*

PROOF: Let $r_1 \in R_1$, $r_2 \in R_2$, $f_1 \in F_1$, and $f_2 \in F_2$. By Proposition (2.2) it is known that $F_1 \otimes_k F_2$ is a DG k-algebra. Multiplication by $R_1 \otimes_k R_2$ on $F_1 \otimes_k F_2$ is defined as follows:

$$(r_1 \otimes r_2)(f_1 \otimes f_2) = r_1 f_1 \otimes r_2 f_2$$

and is extended by linearity.

The preservation of addition by the map ∂ is trivial and will be omitted. The $R_1 \otimes_k R_2$-linearity of the differential will now be demonstrated.

$$\begin{aligned}
\partial((r_1 \otimes r_2)(f_1 \otimes f_2)) &= \partial(r_1 f_1 \otimes r_2 f_2) \\
&= \partial(r_1 f_1) \otimes r_2 f_2 + (-1)^{|r_1 f_1|} r_1 f_1 \otimes \partial(r_2 f_2) \\
&= r_1 \partial(f_1) \otimes r_2 f_2 + (-1)^{|f_1|} r_1 f_1 \otimes r_2 \partial(f_2) \\
&= (r_1 \otimes r_2)\left(\partial f_1 \otimes f_2 + (-1)^{|f_1|} f_1 \otimes \partial f_2\right) \\
&= (r_1 \otimes r_2)\partial(f_1 \otimes f_2).
\end{aligned}$$

By Proposition (2.2) and the $R_1 \otimes_k R_2$- linearity of the differential, it is demonstrated that differentiation commutes with multiplication; hence, Proposition (2.2) is extended to DG R_1- and DG R_2-algebras. Q.E.D.

2.2.2 Tensors of Koszul Complexes over the field k

As a special case of the previous theorems the following proposition will be required for later results.

PROPOSITION (2.4): *Let R_1 and R_2 both be commutative k-algebras where k is a field. Let $S_1 = x_1, \ldots, x_s$ and $S_2 = y_1, \ldots, y_t$ be sequences of elements from R_1 and R_2 (resp.) such that S_1 generates a maximal ideal m_1 in R_1 and S_2 generates a maximal ideal m_2 in R_2. Assume that R_1/m_1 and R_2/m_2 are both isomorphic to k.*

Consider the sequence of elements:

$$S = x_1 \otimes 1_{R_2}, x_2 \otimes 1_{R_2}, \cdots, x_s \otimes 1_{R_2}, 1_{R_1} \otimes y_1, 1_{R_1} \otimes y_2, \cdots, 1_{R_1} \otimes y_t.$$

Let $K(S_1)$, $K(S_2)$, and $K(S)$ denote the Koszul complexes generated by S_1, S_2, and S (resp.). Then $K(S_1) \otimes_k K(S_2) \cong K(S)$ as DG k-algebras.

The following Lemma is required for the construction of the isomorphism

$$\Phi : K(S_1) \otimes_k K(S_2) \xrightarrow{\cong} K(S).$$

LEMMA (2.5): *Let S_1 and S_2 be as stated in Proposition (2.4). Let m_1 and m_2 be the maximal ideals generated by the sequences of elements S_1 and S_2 (resp.) and let k be the field (as in Proposition (2.4)). Let $m = m_1 \otimes_k R_2 + R_1 \otimes_k m_2$. Then m is a maximal ideal in $R_1 \otimes_k R_2$.*

PROOF: Consider the canonical projection map $\pi = \pi_1 \otimes \pi_2$:

$$R_1 \otimes_k R_2 \to k \otimes k$$
$$\pi(a \otimes b) = \bar{a} \otimes \bar{b}\,.$$

Since $\operatorname{Ker}(\pi) = m_1 \otimes_k R_2 + R_1 \otimes_k m_2$,

$$R_1 \otimes_k R_2/(m_1 \otimes_k R_2 + R_1 \otimes_k m_2) \cong k \otimes k \cong k\,.$$

Thus $\operatorname{Ker}(\pi) = m_1 \otimes_k R_2 + R_1 \otimes_k m_2$ is a maximal ideal in $R_1 \otimes_k R_2$. Q.E.D.

PROOF of (2.4):

Let $\{g_1, g_2, \ldots, g_{s+t}\}$ be a basis for the free module $(R_1 \otimes_k R_2)^{s+t}$. Then a basis for $K_{p+q}(S)$ is given by $\{g_{l_1} \wedge g_{l_2} \wedge \cdots \wedge g_{l_{p+q}} \mid l_1 < l_2 < \cdots < l_{p+q}\}$. The Koszul differential is given by:

$$\partial^{K(S)}(g_{l_1} \wedge g_{l_2} \wedge \cdots \wedge g_{l_{p+q}}) = \sum_{r=1}^{p+q} (-1)^{r+1} z_{l_r} \cdot g_{l_1} \wedge g_{l_2} \wedge \cdots \wedge \widehat{g}_{l_r} \wedge \cdots \wedge g_{l_{p+q}}$$

where $z_r = x_r \otimes_k 1_{R_2}$, $1 \leq r \leq s$, and $z_r = 1_{R_1} \otimes_k y_{(r-s)}$, $s+1 \leq r \leq s+t$. Let $\{e_1, \dots, e_s\}$ be a basis of the free module R_1^s. A basis for $K_p(S_1)$ is given by $\{e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_p} | \, i_1 < i_2 < \cdots < i_p\}$. The Koszul differential for this complex is:

$$\partial^{K(S_1)}(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}) = \sum_{r=1}^{p} (-1)^{r+1} x_{i_r} \cdot e_{i_1} \wedge e_{i2} \wedge \cdots \wedge \widehat{e}_{ir} \wedge \cdots \wedge e_{i_p}$$

Let $\{f_1, \dots, f_t\}$ be a basis of the free module R_2^t. A basis for $K_q(S_2)$ is given by $\{f_{j_1} \wedge f_{j_2} \cdots \wedge f_{j_q} | \, j_1 < j_2 < \cdots < j_q\}$. The Koszul differential for this complex is:

$$\partial^{K(S_2)}(f_{j_1} \wedge f_{j_2} \wedge \cdots \wedge f_{j_q}) = \sum_{r=1}^{q} (-1)^{r+1} y_{j_r} \cdot f_{j_1} \wedge f_{j2} \wedge \cdots \wedge \widehat{f}_{jr} \wedge \cdots \wedge f_{j_q}$$

By the definition for the n^{th} degree of the total complex

$$(K(S_1) \otimes_k K(S_2))_n = \bigoplus_{p+q=n} K_p(S_1) \otimes_k K_q(S_2) \, .$$

A basis for the free module $\big(K(S_1) \otimes_k K(S_2)\big)_n$ is

$$\bigcup_{p+q=n} \{e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_p} \ \otimes_k f_{j_1} \wedge f_{j_2} \cdots \wedge f_{j_q} | \, i_1 < i_2 < \cdots < \ i_p, \ j_1 < j_2 < \cdots < j_q\}$$

Let $\Phi : K(S_1) \otimes_k K(S_2) \to K(S)$ be defined as:

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \otimes f_{j_1} \wedge f_{j_2} \wedge \cdots \wedge f_{j_q} \mapsto g_{l_1} \wedge g_{l_2} \wedge \cdots \wedge g_{l_{p+q}}$$

$$\Phi(x_{i_r}) \mapsto x_{i_r} \otimes 1_{R_2} \qquad \Phi(y_{j_r}) \mapsto 1_{R_1} \otimes y_{j_r}$$

where $l_\alpha = i_\alpha$ for $1 \leq \alpha \leq p$ and $l_\alpha = s + j_{(\alpha - p)}$ for $p+1 \leq \alpha \leq p+q$.

It will now be demonstrated that the differentials commute with Φ.

From the definitions of the Koszul differential and Φ,

$$\begin{aligned}
&\Phi \circ \partial^{K(S_1) \otimes_k K(S_2)}(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \otimes f_{j_1} \wedge f_{j_2} \wedge \cdots \wedge f_{j_q}) \\
&\quad = \Phi\big((\partial^{K(S_1)} e_{i_1} \wedge \cdots \wedge e_{i_p}) \otimes f_{j_1} \wedge \cdots \wedge f_{j_q} + (-1)^p e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes (\partial^{K(S_2)} f_{j_1} \wedge \cdots \wedge f_{j_q})\big) \\
&\quad = \Phi\big((\textstyle\sum (-1)^{r+1} x_{i_r} \cdot e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_r} \wedge \cdots e_{i_p}) \otimes f_{j_1} \wedge \cdots \wedge f_{j_q} \\
&\qquad + (-1)^p e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes (\textstyle\sum (-1)^{r+1} y_{j_r} \cdot f_{j_1} \wedge \cdots \widehat{f}_{j_r} \wedge \cdots \wedge f_{j_q})\big)
\end{aligned}$$

$= \sum_{r=1}^{p}(-1)^{r+1}(x_{i_r} \otimes_k 1_{R_2}) \cdot g_{l_1} \wedge \cdots \widehat{g}_{l_r} \wedge \cdots \wedge g_{l_{p+q}}$

$\quad + \sum_{r=p+1}^{p+q}(-1)^{r+1}(1_{R_1} \otimes_k y_{j_r}) \cdot g_{l_1} \wedge \cdots \widehat{g}_{l_r} \wedge \cdots \wedge g_{l_{p+q}}$

$= \sum_{r=1}^{p+q}(-1)^{r+1} z_{l_r} \cdot g_{l_1} \wedge \cdots \widehat{g}_{l_r} \wedge \cdots \wedge g_{l_{p+q}}.$

$\partial^{K(S)} \circ \Phi(e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes f_{j_1} \wedge \cdots \wedge f_{j_q})$

$= \partial^{K(S)}(g_{l_1} \wedge \cdots \wedge g_{l_p} \wedge g_{l_{p+1}} \cdots \wedge g_{l_{p+q}})$

$= \sum_{r=1}^{p+q}(-1)^{r+1} z_{l_r} \cdot g_{l_1} \wedge \cdots \widehat{g}_{l_r} \wedge \cdots \wedge g_{l_{p+q}}$

The previous computations show that $\Phi \circ \partial^{K(S_1) \otimes_k K(S_2)} = \partial^{K(S)} \circ \Phi$. Thus, Φ is a chain map.

It remains to be shown that $\Phi(ab) = \Phi(a)\Phi(b)$.

Let $a = e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes f_{j_1} \wedge \cdots \wedge f_{j_q}$ and $b = e_{i'_1} \wedge \cdots \wedge e_{i'_{p'}} \otimes f_{j'_1} \wedge \cdots \wedge f_{j'_{q'}}$.

Let $\Phi(a) = g_{l_1} \wedge \cdots \wedge g_{l_{p+q}}$ and $\Phi(b) = g_{l'_1} \wedge \cdots \wedge g_{l'_{p'+q'}}$.

For the elements of $\Phi(b)$ a similar assignment as that used in the previous demonstration will also be required. Specifically:

$l'_{\alpha'} = i'_{\alpha'}$ for $1 \leq \alpha' \leq p' \leq s$ and $l'_{\alpha'} = s + j'_{(\alpha' - p')}$ for $s+1 \leq p'+1 \leq \alpha' \leq p'+q' \leq s+t$.

For purposes of economy in representation of those components of $\Phi(a)$ and $\Phi(b)$ that are mapped by the action of Φ on $K(S_1)$ and $K(S_2)$, let

$$g_{l_{(1 \to p)}} = \Phi(e_{i_1} \wedge \cdots \wedge e_{i_p});$$

$$g_{l_{(p+1 \to p+q)}} = \Phi(f_{j_1} \wedge \cdots \wedge f_{j_q});$$

$$g_{l'_{(1 \to p')}} = \Phi(e_{i'_1} \wedge \cdots \wedge e_{i'_{p'}});$$

$$g_{l'_{(p'+1 \to p'+q')}} = \Phi(f_{j'_1} \wedge \cdots \wedge f_{j'_{q'}}).$$

Thus

$$\Phi(a) = g_{l_{(1\to p)}} \wedge g_{l_{(p+1\to p+q)}} = g_{l_{(1\to p+q)}} \text{ and } \Phi(b) = g_{l'_{(1\to p')}} \wedge g_{l'_{(p'+1\to p'+q')}} = g_{l'_{(1\to p'+q')}}.$$

$$\begin{aligned}
\Phi(ab) &= \Phi\big((-1)^{p'q} \cdot e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_{i'_1} \wedge \cdots \wedge e_{i'_{p'}} \otimes f_{j_1} \wedge \cdots \wedge f_{j_q} \wedge f_{j'_1} \wedge \cdots \wedge f_{j'_{q'}}\big) \\
&= (-1)^{p'q} \cdot g_{l_{(1\to p)}} \wedge g_{l'_{(1\to p')}} \wedge g_{l_{(p+1\to p+q)}} \wedge g_{l'_{(p'+1\to p'+q')}} \\
&= (-1)^{2p'q} \cdot g_{l_{(1\to p)}} \wedge g_{l_{(p+1\to p+q)}} \wedge g_{l'_{(1\to p')}} \wedge g_{l'_{(p'+1\to p'+q')}} \\
&= g_{l_{(1\to p+q)}} \wedge g_{l'_{(1\to p'+q')}} \\
&= \Phi(a)\Phi(b). \text{ Q.E.D.}
\end{aligned}$$

PROPOSITION (2.6): *Let A, B, C, be DG k-algebras where k is a field. Additionally, suppose that A is quasi-isomorphic to B. Then $A \otimes_k C$ is quasi-isomorphic to $B \otimes_k C$.*

PROOF: Suppose, without loss of generality, that there exists a quasi-isomorphism $\Phi : A \xrightarrow{\simeq} B$. Then $\mathrm{H}(\Phi)$ is an isomorphism $\mathrm{H}(A) \xrightarrow{\cong} \mathrm{H}(B)$. Also note that $\Phi \otimes \mathrm{Id}_C$ is a chain map: $A \otimes C \xrightarrow{\simeq} B \otimes C$. By the Kunneth relation there exist isomorphisms: $\Psi_1 : \mathrm{H}(A \otimes_k C) \xrightarrow{\cong} \mathrm{H}(A) \otimes_k \mathrm{H}(C)$ and $\Psi_2 : \mathrm{H}(B \otimes_k C) \xrightarrow{\cong} \mathrm{H}(B) \otimes_k \mathrm{H}(C)$.
Since $\mathrm{H}(\Phi)$ is an isomorphism, $\mathrm{H}(\Phi) \otimes_k \mathrm{Id}_{\mathrm{H}(C)}$ is also an isomorphism:

$$\mathrm{H}(\Phi) \otimes_k \mathrm{Id}_{\mathrm{H}(C)} : \mathrm{H}(A) \otimes_k \mathrm{H}(C) \xrightarrow{\cong} \mathrm{H}(B) \otimes_k \mathrm{H}(C).$$

It is clear that $\mathrm{H}(\Phi \otimes_k \mathrm{Id}_C)$ is the isomorphism:

$$\Psi_2^{-1} \circ \mathrm{H}(\Phi) \otimes_k \mathrm{Id}_{\mathrm{H}(C)} \circ \Psi_1 : \ \mathrm{H}(A \otimes_k C) \xrightarrow{\cong} \mathrm{H}(B \otimes_k C)$$

described by the commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}(A \otimes_k C) & \xrightarrow{\mathrm{H}(\Phi \otimes \mathrm{Id}_c)} & \mathrm{H}(B \otimes_k C) \\
\Psi_1 \downarrow & & \downarrow \Psi_2 \\
\mathrm{H}(A) \otimes_k \mathrm{H}(C) & \xrightarrow{\mathrm{H}(\Phi) \otimes \mathrm{Id}_{\mathrm{H}(C)}} & \mathrm{H}(B) \otimes_k \mathrm{H}(C)
\end{array}$$

This demonstrates that $\Phi \otimes_k \mathrm{Id}_C$ is a quasi-isomorphism between complexes $A \otimes_k C$ and $B \otimes_k C$. Q.E.D.

2.2.3 Proof of the Main Theorem

PROOF of (2.1):

Theorems (2.4) and (2.6) established the following sequence of maps:

$$K(S) \xrightarrow{\cong} K(S_1) \otimes_k K(S_2) \xleftrightarrow{\simeq} \mathrm{H}(K(S_1)) \ \otimes_k K(S_2) \xleftrightarrow{\simeq} \mathrm{H}(K(S_1)) \otimes_k \mathrm{H}(K(S_2))$$
$$\xrightarrow{\cong} \mathrm{H}(K(S_1 \otimes_k S_2)) \xrightarrow{\cong} \ \mathrm{H}(K(S)) \ .$$

The first isomorphism is Proposition (2.4), the second quasi-isomorphism is by substitution of $K(S_1)$ for A and $K(S_2)$ for C in Proposition (2.6), the third quasi-isomorphism is by substituting $\mathrm{H}(S_1)$ for C and $K(S_2)$ for A (with $A \otimes_k B \cong B \otimes_k A$), the fourth isomorphism is by the Kunneth relation, and the fifth isomorphism was determined by Theorem (2.4).

The previous sequence of maps proves only that if $K(S_1)$ and $K(S_2)$ are formal, then $K(S_1 \otimes_k S_2) \cong K(S)$ is formal. Theorem (2.1) is completed by the following theorem of Jörg Backelin and Ralph Fröberg. Since $H(K(S_1))$ and $H(K(S_2))$ are Koszul, then $H(K(S_1)) \otimes H(K(S_2) \cong H(K(S_1 \otimes S_2)) \cong H(K(S))$ is Koszul.Thus, $R_1 \otimes_k R_2$ is PK and Theorem (2.1) is proved. Q.E.D.

THEOREM 2 [16, p.7] [5, p.91]: *$A \otimes_k B$ is a Koszul algebra if and only if A and B are both Koszul algebras* (or, for this special case) *$H(K(S_1)) \otimes_k H(K(S_2))$ is a Koszul algebra if and only if $H(K(S_1))$ and $H(K(S_2))$ are both Koszul algebras.*

As a special case, the following will be proved as a consequence of Theorem (2.1):

COROLLARY (2.7): *Let Q_1 and Q_2 be regular local rings over a common residue field k. Let m_{Q_1} and m_{Q_2} be maximal ideals in Q_1 and Q_2 (resp.). Let I_1 and I_2 be ideals of Q_1 and Q_2 (resp.) such that $I_1 \subset m_{Q_1}^2$ and $I_2 \subset m_{Q_2}^2$. Let $R_1 = Q_1/I_1$ and $R_2 = Q_2/I_2$. Let $m_1 = (x_1, \ldots, x_s)$ and $m_2 = (y_1, \ldots, y_t)$ be the maximal ideals in R_1 and R_2 (resp.). Assume that the minimal free resolutions G_1 of R_1 over Q_1 and G_2 of R_2 over Q_2 have algebra structures, that is, they are DG Q-algebras. Further assume that their respective homology algebras: $H(K(x_1, \ldots, x_s))$ and $H(K(y_1, \ldots, y_t))$ are Koszul. Then $R_1 \otimes_k R_2$ is a PK ring with respect to the sequence of elements $x_1 \otimes_k 1_{R_2}, \ldots, x_s \otimes_k 1_{R_2},\ 1_{R_1} \otimes_k y_1, \ldots, 1_{R_1} \otimes_k y_t$.*

This corollary will require the following lemma:

LEMMA (2.8): *Let Q be a Noetherian commutative ring having a maximal ideal $m_Q = (x_1, \ldots, x_s)$ generated by a Q-regular sequence $x_1, \ldots, x_s$, and $k = Q/m_Q$ (a field). Let I be an ideal of Q such that $m_R = (\bar{x_1}, \ldots, \bar{x_s})$ is a maximal ideal of $R = Q/I$. Then $k \cong R/m_R$. Assume that (G, d) is a DG Q-algebra quasi-isomorphic to R such that $\operatorname{Im}(d_i) \subseteq (x_1, \ldots, x_s)G_{i-1}$ for all i. If the homology algebra $H(K(\bar{x_1}, \ldots, \bar{x_s}))$ is Koszul, then R is a PK ring with respect to $\bar{x_1}, \ldots, \bar{x_s}$.*

PROOF: Recall from section 1.2 that the Koszul complex on $x_1, \ldots, x_s$ is isomorphic to the minimal free resolution of the field k over Q. Let F denote this minimal free resolution of the field k over Q; let G denote the free resolution of R over Q with an algebra structure, that is, a DG Q-algebra quasi-isomorphic to R. The following maps, in conjunction with the definition of a PK ring, justify the lemma.

$$\begin{array}{ccccc} k \otimes_Q G & \xleftarrow{\simeq} & F \otimes_Q G & \xrightarrow{\simeq} & F \otimes_Q R \\ \downarrow\cong & & \downarrow & & \downarrow \\ \mathrm{H}(k \otimes_Q G) & \cong & \mathrm{H}(F \otimes_Q G) & \cong & \mathrm{H}(F \otimes_Q R) \end{array}$$

The top row is the sequence of quasi-isomorphisms. The Koszul complex is $F \otimes_Q R$, and therefore the homology algebra of R is $H(F \otimes_Q R)$. The bottom row is justified by the balanced Tor functor. As indicated in an earlier example (section 1.6) the isomorphism $k \otimes_Q G \xrightarrow{\cong} \mathrm{H}(k \otimes_Q G)$ is forced by tensoring G over Q with the field k, making all maps in $\mathrm{H}(k \otimes_Q G)$ trivial, and thus isomorphic to $k \otimes_Q G$. Q.E.D.

PROOF of Corollary (2.7): Assume that the conditions of the corollary are satisfied. Since the homology algebras, $H(K(x_1, \ldots, x_s)$ and $H(K(y_1, \ldots, y_t)$ of the corollary are both Koszul, therefore, by the lemma, both R_1 and R_2 are PK rings. Then, by Theorem (2.1), $R_1 \otimes_k R_2$ is PK ring. Q.E.D.

It will be useful for future applications to have a direct and constructive proof of the Corollary. The following Propositions will allow for a direct substitution of terms from the statement of the Corollary.

2.3 Isomorphisms of Tensored Complexes

Let R_1 and R_2 be k-algebras. Let F_1, G_1, be DG R_1-algebras; let F_2, G_2 be DG R_2-algebras. Then it is immediate that F_1, G_1, F_2, G_2 are all DG k-algebras. The following are a consequence of Propositions (2.2) and (2.3):

(i). $F_1 \otimes_k F_2$ is a DG $R_1 \otimes_k R_2$-algebra;

(ii). $G_1 \otimes_k G_2$ is a DG $R_1 \otimes_k R_2$-algebra;

(iii). $F_1 \otimes_{R_1} G_1$ is a DG R_1-algebra;

(iv). $F_2 \otimes_{R_2} G_2$ is a DG R_2-algebra;

(v). $(F_1 \otimes_k F_2) \otimes_{R_1 \otimes_k R_2} (G_1 \otimes_k G_2)$ is a DG $R_1 \otimes_k R_2$-algebra;

(vi). $(F_1 \otimes_{R_1} G_1) \otimes_k (F_2 \otimes_{R_2} G_2)$ is a DG $R_1 \otimes_k R_2$-algebra.

For the proof of the Corollary, (v) and (vi) will be shown to be isomorphic complexes of $R_1 \otimes_k R_2$-algebras.

THEOREM (2.9): *Let F_1 and G_1 be R_1-modules; let F_2 and G_2 be R_2-modules. Then with the $R_1 \otimes_k R_2$ coefficient action, there exists an $R_1 \otimes_k R_2$-module isomorphism,*

$$\Phi : (F_1 \otimes_k F_2) \otimes_{R_1 \otimes_k R_2} (G_1 \otimes_k G_2) \longrightarrow (F_1 \otimes_{R_1} G_1) \otimes_k (F_2 \otimes_{R_2} G_2)$$

$$\Phi(f_1 \otimes f_2 \otimes g_1 \otimes g_2) = f_1 \otimes g_1 \otimes f_2 \otimes g_2.$$

PROOF: Both the map Φ and its inverse:

$$\Psi : (F_1 \otimes_{R_1} G_1) \otimes_k (F_2 \otimes_{R_2} G_2) \longrightarrow (F_1 \otimes_k F_2) \otimes_{R_1 \otimes_k R_2} (G_1 \otimes_k G_2)$$

$$\Psi((f_1 \otimes g_1) \otimes (f_2 \otimes g_2)) = (f_1 \otimes f_2) \otimes (g_1 \otimes g_2).$$

are constructed by repeated use of the Universal Mapping Property of tensor products.

Proposition (2.10), which follows, is required for the completion of Theorem (2.9).

PROPOSITION (2.10): *Φ is an $R_1 \otimes_k R_2$-linear map.*

PROOF: Let $r_1 \otimes r_2$ and $r_1' \otimes r_2'$ be elements of $R_1 \otimes_k R_2$.

$$\Phi((r_1 \otimes r_2 + r_1' \otimes r_2')(f_1 \otimes f_2 \otimes g_1 \otimes g_2))$$
$$= \Phi\left((r_1 \otimes r_2)f_1 \otimes f_2 \otimes g_1 \otimes g_2 + (r_1' \otimes r_2')f_1 \otimes f_2 \otimes g_1 \otimes g_2\right)$$
$$= \Phi\left((r_1 f_1 \otimes r_2 f_2 \otimes g_1 \otimes g_2) + (r_1' f_1 \otimes r_2' f_2 \otimes g_1 \otimes g_2)\right)$$
$$= (-1)^{|f_2||g_1|} r_1 f_1 \otimes r_2 f_2 \otimes g_1 \otimes g_2 + (-1)^{|f_2||g_1|} r_1 f_1 \otimes r_2 f_2 \otimes g_1 \otimes g_2$$
$$= (-1)^{|f_2||g_1|}(r_1 \otimes r_2) f_1 \otimes f_2 \otimes g_1 \otimes g_2 + (-1)^{|f_2||g_1|}(r_1' \otimes r_2') f_1 \otimes f_2 \otimes g_1 \otimes g_2$$
$$= (r_1 \otimes r_2 + r_1 \otimes r_2)\left((-1)^{|f_2||g_1|} f_1 \otimes f_2 \otimes g_1 \otimes g_2\right)$$
$$= (r_1 \otimes r_2 + r_1 \otimes r_2)\Phi(f_1 \otimes f_2 \otimes g_1 \otimes g_2). \qquad \text{Q.E.D.}$$

2.3.1 Extending the Theorem from Modules to Complexes

The previous proof will be extended from modules to complexes. This is accomplished with minor changes in statement and architecture of proof. Additionally, it will be demonstrated that the isomorphisms commute with the differential for total complexes.

PROPOSITION (2.11): *Let $\mathbf{F}_1$ and $\mathbf{G}_1$ be complexes of R_1 modules; let $\mathbf{F}_2$ and $\mathbf{G}_2$ be complexes of R_2 modules. Then $(\mathbf{F}_1 \otimes_k \mathbf{F}_2) \otimes_{R_1 \otimes_k R_2} (\mathbf{G}_1 \otimes_k \mathbf{G}_2)$ and $(\mathbf{F}_1 \otimes_{R_1} \mathbf{G}_1) \otimes_k (\mathbf{F}_2 \otimes_{R_2} \mathbf{G}_2)$ are isomorphic as complexes of modules over $R_1 \otimes_k R_2$.*

PROOF: Substitute complexes for the modules in the proof of Proposition (2.9). Define the isomorphisms Φ and Ψ acting on an arbitrary degree n simple tensor in the total complex as before except with coefficient adjustments of $(-1)^{|f_2|\cdot|g_1|}$ for Φ ($(-1)^{|g_1|\cdot|f_2|}$ for Ψ) to account for the transpositions across the tensor symbols.

$$\Phi:\ f_1\otimes f_2\otimes g_1\otimes g_2 \longrightarrow\ (-1)^{|f_2|\cdot|g_1|} f_1\otimes g_1\otimes f_2\otimes g_2$$

$$\Psi:\ f_1\otimes g_1\otimes f_2\otimes g_2 \longrightarrow\ (-1)^{|g_1|\cdot|f_2|} f_1\otimes f_2\otimes g_1\otimes g_2. \qquad \text{Q.E.D.}$$

2.3.2 Φ is a Chain Map

PROPOSITION (2.12): *Let Φ be the map defined on complexes from the previous theorem:*

$$\Phi : (\mathbf{F}_1\otimes_k \mathbf{F}_2)\otimes_{R_1\otimes_k R_2}(\mathbf{G}_1\otimes_k \mathbf{G}_2) \longrightarrow (\mathbf{F}_1\otimes_{R_1}\mathbf{G}_1)\otimes_k(\mathbf{F}_2\otimes_{R_2}\mathbf{G}_2)\,.$$

Let ∂ be the total differential defined on $(\mathbf{F}_1\otimes_k \mathbf{F}_2)\otimes_{R_1\otimes_k R_2}(\mathbf{G}_1\otimes_k \mathbf{G}_2)$. Let ∂' be the total differential defined on $(\mathbf{F}_1\otimes_{R_1}\mathbf{G}_1)\otimes_k(\mathbf{F}_2\otimes_{R_2}\mathbf{G}_2)$. Then $\Phi\circ\partial = \partial'\circ\Phi$ (the differential commutes with the isomorphism).

PROOF: Consider the sequence of maps: $\Phi\circ\partial$ acting on a simple degree n element, $f_1\otimes f_2\otimes g_1\otimes g_2$ of

$$((\mathbf{F}_1\otimes_k \mathbf{F}_2)\otimes_{R_1\otimes_k R_2}(\mathbf{G}_1\otimes_k \mathbf{G}_2))_n = \bigoplus_{p+q+r+s=n}(\mathbf{F}_{1,p}\otimes_k \mathbf{F}_{2,q})\otimes_{R_1\otimes_k R_2}(\mathbf{G}_{1,r}\otimes_k \mathbf{G}_{2,s}):$$

$$\begin{aligned}
&\partial(f_1\otimes f_2\otimes g_1\otimes g_2)\\
&\qquad= \partial^{\mathbf{F}_1\otimes_k\mathbf{F}_2}(f_1\otimes f_2)\otimes(g_1\otimes g_2) + (-1)^{|f_1\otimes f_2|}(f_1\otimes f_2)\otimes\partial^{\mathbf{G}_1\otimes_k\mathbf{G}_2}(g_1\otimes g_2)\\
&\qquad= \big(\partial^{\mathbf{F}_1}(f_1)\otimes f_2 + (-1)^{|f_1|} f_1\otimes\partial^{\mathbf{F}_2}(f_2)\big)\otimes(g_1\otimes g_2)\\
&\qquad\quad +(-1)^{|f_1\otimes f_2|}(f_1\otimes f_2)\otimes\big(\partial^{\mathbf{G}_1}(g_1)\otimes g_2 + (-1)^{|g_1|} g_1\otimes\partial^{\mathbf{G}_2}(g_2)\big).
\end{aligned}$$

After expanding terms linearly, the previous expression becomes:

$$\partial^{\mathbf{F}_1}(f_1)\otimes f_2\otimes g_1\otimes g_2 + (-1)^p f_1\otimes\partial^{\mathbf{F}_2}(f_2)\otimes g_1\otimes g_2$$

$$+(-1)^{p+q} f_1 \otimes f_2 \otimes \partial^{\mathbf{G}_1}(g_1) \otimes g_2 + (-1)^{p+q+r} f_1 \otimes f_2 \otimes g_1 \otimes \partial^{\mathbf{G}_2}(g_2).$$

Apply Φ (termwise) to the previous expression:

$$\Phi(\partial^{\mathbf{F}_1}(f_1) \otimes f_2 \otimes g_1 \otimes g_2) = (-1)^{qr} \partial^{\mathbf{F}_1}(f_1) \otimes g_1 \otimes f_2 \otimes g_2;$$

$$\Phi((-1)^p f_1 \otimes \partial^{\mathbf{F}_2}(f_2) \otimes g_1 \otimes g_2) = (-1)^{p+(q-1)r} f_1 \otimes g_1 \otimes \partial^{\mathbf{F}_2}(f_2) \otimes g_2;$$

$$\Phi((-1)^{p+q} f_1 \otimes f_2 \otimes \partial^{\mathbf{G}_1}(g_1) \otimes g_2) = (-1)^{p+q+q(r-1)} f_1 \otimes \partial^{\mathbf{G}_1}(g_1) \otimes f_2 \otimes g_2;$$

$$\Phi((-1)^{p+q+r} f_1 \otimes f_2 \otimes g_1 \otimes \partial^{\mathbf{G}_2}(g_2)) = (-1)^{p+q+r+qr} f_1 \otimes g_1 \otimes f_2 \otimes \partial^{\mathbf{G}_2}(g_2).$$

The sequence of operations $\partial' \circ \Phi$ will be evaluated on $f_1 \otimes f_2 \otimes g_1 \otimes g_2$:

$\Phi(f_1 \otimes f_2 \otimes g_1 \otimes g_2) = (-1)^{qr} f_1 \otimes g_1 \otimes f_2 \otimes g_2;$

$$\begin{aligned}
&\partial' \left((-1)^{qr} f_1 \otimes g_1 \otimes f_2 \otimes g_2\right) \\
&\quad = (-1)^{qr} \left(\partial'^{\mathbf{F}_1 \otimes_{R_1} \mathbf{G}_1}(f_1 \otimes g_1) \otimes (f_2 \otimes g_2)\right) \\
&\qquad +(-1)^{qr} \left((-1)^{|f_1 \otimes g_1|}(f_1 \otimes g_1) \otimes \partial'^{\mathbf{F}_2 \otimes_{R_1} \mathbf{G}_2}(f_2 \otimes g_2)\right) \\
&\quad = (-1)^{qr} \left(\partial'^{\mathbf{F}_1}(f_1) \otimes g_1 + (-1)^{|f_1|} f_1 \otimes \partial'^{\mathbf{G}_1}(g_1)\right) \otimes (f_2 \otimes g_2) \\
&\qquad +(-1)^{|f_1 \otimes g_1| + qr}(f_1 \otimes g_1) \otimes \left(\partial'^{\mathbf{F}_2}(f_2) \otimes g_2 + (-1)^{|f_2|} f_2 \otimes \partial'^{\mathbf{G}_2}(g_2)\right) \\
&\quad = (-1)^{qr} \partial'^{\mathbf{F}_1}(f_1) \otimes g_1 \otimes f_2 \otimes g_2 + (-1)^{qr+p} f_1 \otimes \partial'^{\mathbf{G}_1}(g_1) \otimes f_2 \otimes g_2 \\
&\qquad +(-1)^{qr+p+r} f_1 \otimes g_1 \otimes \partial'^{\mathbf{F}_2}(f_2) \otimes g_2 + (-1)^{qr+p+r+q} f_1 \otimes g_1 \otimes f_2 \otimes \partial'^{\mathbf{G}_1}(g_2).
\end{aligned}$$

Comparing terms from the two operations on the same element indicates that

$(\Phi \circ \partial)(f_1 \otimes f_2 \otimes g_1 \otimes g_2) = (\partial' \circ \Phi)(f_1 \otimes f_2 \otimes g_1 \otimes g_2)$ since

$(-1)^{p+(q-1)r} f_1 \otimes g_1 \otimes \partial^{\mathbf{F}_2}(f_2) \otimes g_2 = (-1)^{qr+p+r} f_1 \otimes g_1 \otimes \partial'^{\mathbf{F}_2}(f_2) \otimes g_2$

and

$(-1)^{p+q+q(r-1)} f_1 \otimes \partial^{\mathbf{G}_1}(g_1) \otimes f_2 \otimes g_2 = (-1)^{qr+p} f_1 \otimes \partial'^{\mathbf{G}_1}(g_1) \otimes f_2 \otimes g_2.$

The proof that $\Psi \circ \partial' = \partial \circ \Psi$ is similar. Q.E.D.

2.3.3 Φ and Ψ are Isomorphisms of DG Algebras

At this stage all that has been demonstrated is the commutativity of the isomorphisms (of the complexes) with the differentials. It remains to be shown that these are isomorphisms of differential graded algebras. To accomplish this the differentials ∂ and ∂' will be shown to be $R_1 \otimes_k R_2$-linear. The map Φ will also be shown to commute with multiplication in the complex. The demonstration that Ψ commutes with multiplication is similar and will not be given. All of the above will suffice to show that the maps Φ and Ψ are both differential graded algebra homomorphisms with respect to $R_1 \otimes_k R_2$ and k coefficient actions.

2.3.4 The $R_1 \otimes_k R_2$-Linearity of the Differentials

PROPOSITION (2.13): *∂ is $R_1 \otimes_k R_2$-linear.*

PROOF: Consider the arbitrary simple tensor $f_1 \otimes f_2 \otimes g_1 \otimes g_2$ of the complex $\mathbf{F}_1 \otimes_k \mathbf{F}_2 \otimes_{R_1 \otimes_k R_2} \mathbf{G}_1 \otimes_k \mathbf{G}_2$.

Let the degrees of the elements of the tensor be as follows:

$|f_1| = p$; $|g_1| = r$; $|f_2| = q$; $|g_2| = s$. The following will be shown:

$\partial((r_1 \otimes r_2)(f_1 \otimes f_2 \otimes g_1 \otimes g_2)) = (r_1 \otimes r_2)\partial(f_1 \otimes f_2 \otimes g_1 \otimes g_2)$.

$$
\begin{aligned}
&\partial((r_1 \otimes r_2)(f_1 \otimes f_2 \otimes g_1 \otimes g_2)) = \partial(r_1 f_1 \otimes r_2 f_2 \otimes g_1 \otimes g_2) \\
&\quad = (\partial(r_1 f_1 \otimes r_2 f_2)) \otimes (g_1 \otimes g_2) + (-1)^{p+q}(r_1 f_1 \otimes r_2 f_2) \otimes (\partial(g_1 \otimes g_2)) \\
&\quad = (\partial(r_1 f_1) \otimes (r_2 f_2) \ + (-1)^p (r_1 f_1) \otimes \partial(r_2 f_2)) \otimes (g_1 \otimes g_2) \\
&\qquad + (-1)^{p+q}(r_1 f_1 \otimes r_2 f_2) \otimes ((\partial g_1) \otimes g_2 + (-1)^r g_1 \otimes (\partial g_2)) \\
&\quad = \partial(r_1 f_1) \otimes (r_2 f_2) \otimes (g_1 \otimes g_2) + (-1)^p (r_1 f_1) \otimes \partial(r_2 f_2) \otimes (g_1 \otimes g_2)
\end{aligned}
$$

$$+(-1)^{p+q}(r_1 f_1 \otimes r_2 f_2) \otimes ((\partial g_1) \otimes g_2) + (-1)^{p+q+r}(r_1 f_1 \otimes r_2 f_2) \otimes (g_1 \otimes (\partial g_2))$$

$$= r_1(\partial f_1) \otimes r_2(f_2) \otimes (g_1 \otimes g_2) \ + (-1)^p (r_1(f_1) \otimes r_2(\partial f_2)) \otimes (g_1 \otimes g_2)$$

$$+(-1)^{p+q} r_1(f_1) \otimes r_2(f_2) \otimes \partial(g_1) \otimes g_2 + (-1)^{p+q+r} r_1(f_1) \otimes r_2(f_2) \otimes (g_1 \otimes \partial(g_2)).$$

By the previous calculation :

$$\partial(f_1 \otimes f_2 \otimes g_1 \otimes g_2) = (\partial f_1) \otimes f_2 \otimes g_1 \otimes g_2 + (-1)^p f_1 \otimes (\partial f_2) \otimes g_1 \otimes g_2$$
$$+(-1)^{p+q} f_1 \otimes f_2 \otimes (\partial g_1) \otimes g_2 + (-1)^{p+q+r} f_1 \otimes f_2 \otimes g_1 \otimes (\partial g_2)$$

and by the defined $R_1 \otimes_k R_2$ coefficient action on $\mathbf{F}_1 \otimes_k \mathbf{F}_2 \otimes_{R_1 \otimes_k R_2} \mathbf{G}_1 \otimes_k \mathbf{G}_2$ the proposition has been proved. Thus ∂ is $R_1 \otimes_k R_2$-linear. Q.E.D.

PROPOSITION (2.14): *∂' is $R_1 \otimes_k R_2$-linear.*

PROOF: Consider an arbitrary tensor $f_1 \otimes g_1 \otimes f_2 \otimes g_2$ of the complex $\mathbf{F}_1 \otimes_{R_1} \mathbf{G}_1 \otimes_k \mathbf{F}_2 \otimes_{R_2} \mathbf{G}_2$. Let the degree of each element of the tensor be as in the previous theorem:

$|f_1| = p$; $|g_1| = r$; $|f_2| = q$; $|g_2| = s$.

$$\partial'((r_1 \otimes r_2)(f_1 \otimes g_1 \otimes f_2 \otimes g_2)) = \partial'((r_1 f_1 \otimes g_1) \otimes (r_2 f_2 \otimes g_2))$$

$$= (\partial'(r_1 f_1 \otimes g_1)) \otimes (r_2 f_2 \otimes g_2) + (-1)^{p+r}(r_1 f_1 \otimes g_1) \otimes (\partial'(r_2 f_2 \otimes g_2))$$

$$= (\partial'(r_1 f_1) \otimes g_1 + (-1)^p (r_1 f_1) \otimes (\partial' g_1)) \otimes (r_2 f_2 \otimes g_2)$$
$$+(-1)^{p+r}(r_1 f_1 \otimes g_1) \otimes (\partial'(r_2 f_2) \otimes g_2 + (-1)^q (r_2 f_2) \otimes (\partial' g_2))$$

$$= (\partial'(r_1 f_1) \otimes g_1) \otimes (r_2 f_2 \otimes g_2) + (-1)^p ((r_1 f_1) \otimes \partial' g_1) \otimes (r_2 f_2 \otimes g_2)$$
$$+(-1)^{p+r}(r_1 f_1 \otimes g_1 \otimes \partial'(r_2 f_2) \otimes g_2 + (-1)^{p+r+q}(r_1 f_1 \otimes g_1) \otimes (r_2 f_2 \otimes \partial' g_2).$$

By the previous calculation for the action of ∂':

$$\partial'(f_1 \otimes g_1 \otimes f_2 \otimes g_2)$$
$$= (\partial' f_1) \otimes g_1 \otimes f_2 \otimes g_2 + (-1)^p f_1 \otimes (\partial' g_1) \otimes f_2 \otimes g_2$$
$$+(-1)^{p+r} f_1 \otimes g_1 \otimes (\partial' f_2) \otimes g_2 + (-1)^{p+r+q} f_1 \otimes g_1 \otimes f_2 \otimes (\partial' g_2)$$

and the defined $R_1 \otimes_k R_2$ coefficient action, the following has been shown:

$\partial'((r_1 \otimes r_2)(f_1 \otimes g_1 \otimes f_2 \otimes g_2) = (r_1 \otimes r_2)\partial'(f_1 \otimes g_1 \otimes f_2 \otimes g_2)$.

Hence, ∂' is $R_1 \otimes_k R_2$-linear. Q.E.D.

2.3.5 Multiplicative Structure of the Complexes

PROPOSITION (2.15): *The chain maps Φ and Ψ commute with multiplication.*

PROOF: Let the tensors $f_1 \otimes f_2 \otimes g_1 \otimes g_2$ and $f_1' \otimes f_2' \otimes g_1' \otimes g_2'$ be elements of $\mathbf{F}_1 \otimes_k \mathbf{F}_2 \otimes_{R_1 \otimes_k R_2} \mathbf{G}_1 \otimes_k \mathbf{G}_2$ with the following degrees:

$|f_1| = p_1$; $|f_2| = q_1$; $|g_1| = r_1$; $|g_2| = s_1$;

$|f_1'| = p_2$; $|f_2'| = q_2$; $|g_1'| = r_2$; $|g_2'| = s_2$;

$$\begin{aligned}
&\Phi\left((f_1 \otimes f_2 \otimes g_1 \otimes g_2)(f_1' \otimes f_2' \otimes g_1' \otimes g_2')\right) \\
&\qquad = (-1)^{(r_1+s_1)(p_2+q_2)}\Phi\left(((f_1 \otimes f_2)(f_1' \otimes f_2')) \otimes ((g_1 \otimes g_2)(g_1' \otimes g_2'))\right) \\
&\qquad = (-1)^{(r_1+s_1)(p_2+q_2)+p_2q_1+r_2s_1}\Phi\left((f_1f_1') \otimes (f_2f_2') \otimes (g_1g_1') \otimes (g_2g_2')\right) \\
&\qquad = (-1)^{(r_1+s_1)(p_2+q_2)+p_2q_1+r_2s_1+(r_1+r_2)(q_1+q_2)}(f_1f_1') \otimes (g_1g_1') \otimes (f_2f_2') \otimes (g_2g_2') \\
&\qquad = (-1)^{r_1(p_2+q_1)+s_1(p_2+q_2+r_2)+q_1(p_2+r_2)+r_2q_2}\left((f_1f_1') \otimes (g_1g_1') \otimes (f_2f_2') \otimes (g_2g_2')\right).
\end{aligned}$$

$$\Phi(f_1 \otimes f_2 \otimes g_1 \otimes g_2) = (-1)^{r_1q_1} f_1 \otimes g_1 \otimes f_2 \otimes g_2;$$

$$\Phi(f_1' \otimes f_2' \otimes g_1' \otimes g_2') = (-1)^{r_2q_2} f_1' \otimes g_1' \otimes f_2' \otimes g_2';$$

$$\begin{aligned}
&\Phi\left(f_1 \otimes f_2 \otimes g_1 \otimes g_2\right)\Phi\left(f_1' \otimes f_2' \otimes g_1' \otimes g_2'\right) \\
&\qquad = (-1)^{r_1q_1+r_2q_2}(f_1 \otimes g_1 \otimes f_2 \otimes g_2)(f_1' \otimes g_1' \otimes f_2' \otimes g_2') \\
&\qquad = (-1)^{r_1q_1+r_2q_2+(p_2+r_2)(q_1+s_1)}\left(((f_1 \otimes g_1)(f_1' \otimes g_1')) \otimes ((f_2 \otimes g_2)(f_2' \otimes g_2'))\right) \\
&\qquad = (-1)^{r_1q_1+r_2q_2+(p_2+r_2)(q_1+s_1)+p_2r_1+s_1q_2}\left((f_1f_1') \otimes (g_1g_1') \otimes (f_2f_2') \otimes (g_2g_2')\right)
\end{aligned}$$

$$= (-1)^{r_1(p_2+q_1)+s_1(p_2+q_2+r_2)+q_1(p_2+r_2)+r_2q_2}\,((f_1f_1') \otimes (g_1g_1') \otimes (f_2f_2') \otimes (g_2g_2')).$$

Q.E.D.

The proof for Ψ commuting with multiplication is similar and will not be given.

2.4 The Homology Algebra of the Complex

The classification of rings by homology algebras will require extending the previous theorems by considering the tensor of a free module complex with another free module complex: $\mathbf{F}_1 \otimes_k \mathbf{F}_2$. This tensor of complexes will constructively be tensored again with another tensor of complexes: $\mathbf{G}_1 \otimes_k \mathbf{G}_2$. Homology will subsequently be defined on this "tensor of tensored complexes":
$(\mathbf{F}_1 \otimes_k \mathbf{F}_2) \otimes_{R_1 \otimes_k R_2} (\mathbf{G}_1 \otimes_k \mathbf{G}_2)$,
from which the definition of the homology algebra of
$(\mathbf{F}_1 \otimes_k \mathbf{F}_2) \otimes_{R_1 \otimes_k R_2} (\mathbf{G}_1 \otimes_k \mathbf{G}_2)$ and its isomorphism to the tensor of the homology algebras (of the complexes: $\mathbf{F}_1 \otimes_k \mathbf{F}_2$ and $\mathbf{G}_1 \otimes_k \mathbf{G}_2$) will be obtained by using the Kunneth relation.

PROPOSITION (2.16):

$$\mathrm{H}_n(\mathbf{F}_1 \otimes_k \mathbf{F}_2 \otimes_{R_1 \otimes_k R_2} \mathbf{G}_1 \otimes_k \mathbf{G}_2) \cong \bigoplus_{p+q=n} \mathrm{H}_p(\mathbf{F}_1 \otimes_{R_1} \mathbf{G}_1) \otimes_k \mathrm{H}_q(\mathbf{F}_2 \otimes_{R_2} \mathbf{G}_2).$$

PROOF: The isomorphism $\Phi : (F_1 \otimes_k F_2) \otimes_{R_1 \otimes_k R_2} (G_1 \otimes_k G_2) \xrightarrow{\cong} (F_1 \otimes_{R_1} G_1) \otimes_k (F_2 \otimes_{R_2} G_2)$ was established in section 2.3. Thus, for any specific degree n,
$\mathrm{H}_n\left((F_1 \otimes_k F_2) \otimes_{R_1 \otimes_k R_2} (G_1 \otimes_k G_2)\right) \xrightarrow{\cong} \mathrm{H}_n\left((F_1 \otimes_{R_1} G_1) \otimes_k (F_2 \otimes_{R_2} G_2)\right)$. The following

isomorphism is obtained from the definition of the total complex in section 2.1 by application of the Kunneth relation:

$$\mathrm{H}_n\left((F_1 \otimes_{R_1} G_1) \otimes_k (F_2 \otimes_{R_2} G_2)\right) \cong \bigoplus_{p+q=n} \mathrm{H}_p(F_1 \otimes_{R_1} G_1) \otimes_k \mathrm{H}_q(F_2 \otimes_{R_2} G_2).$$

Q.E.D.

PROOF of Corollary (2.7): Let Q_1, Q_2, I_1, I_2, R_1, R_2, m_1, and m_2 be as indicated in the statement of (2.7). The proof will be by substitution. Let F_1 denote the free resolution of k over Q_1. Let F_2 denote the free resolution of k over Q_2. Let G_1 denote the free resolution of R_1 over Q_1 and G_2 denote the free resolution of R_2 over Q_2. By Proposition (2.11) and the Kunneth relation, the following isomorphisms were established:

$$\begin{aligned}\mathrm{H}\left((F_1 \otimes_k F_2) \otimes_{Q_1 \otimes_k Q_2} (G_1 \otimes_k G_2)\right) &\cong \mathrm{H}\left((F_1 \otimes_{Q_1} G_1) \otimes_k (F_2 \otimes_{Q_2} G_2)\right) \\ &\cong \mathrm{H}(F_1 \otimes_{Q_1} G_1) \otimes_k \mathrm{H}(F_2 \otimes_{Q_2} G_2).\end{aligned}$$

By the balanced Tor functor:

$\mathrm{H}(F_1 \otimes_{Q_1} G_1) \otimes_k \mathrm{H}(F_2 \otimes_{Q_2} G_2) \cong \mathrm{H}(F_1 \otimes_{Q_1} R_1) \otimes_k \mathrm{H}(F_2 \otimes_{Q_2} R_2)$.

By definition: $\mathrm{H}(F_1 \otimes_{Q_1} R_1) = \mathrm{H}\left(K(x_1, \ldots, x_s)\right)$ and $\mathrm{H}(F_2 \otimes_{Q_2} R_2) = \mathrm{H}\left(K(y_1, \ldots, y_t)\right)$.

By Proposition (2.4) $F_1 \otimes_k F_2 \cong F$ where F is the free resolution of k with respect to the sequence $\{x_1 \otimes 1_{R_2}, \ldots, x_s \otimes 1_{R_2}, 1_{R_1} \otimes y_1, \ldots, 1_{R_1} \otimes y_t\}$; hence,

$\mathrm{H}\left((F_1 \otimes_k F_2) \otimes_{Q_1 \otimes_k Q_2} (G_1 \otimes_k G_2)\right) \cong \mathrm{H}\left(F \otimes_{Q_1 \otimes_k Q_2} (G_1 \otimes_k G_2)\right)$.

By the balanced Tor functor: $\mathrm{H}\left(F \otimes_{Q_1 \otimes_k Q_2} (G_1 \otimes_k G_2)\right) \cong \mathrm{H}\left(F \otimes_{Q_1 \otimes_k Q_2} (R_1 \otimes_k R_2)\right.$.

By definition of the Koszul complex and Proposition (2.4)

$\mathrm{H}\left(F \otimes_{Q_1 \otimes_k Q_2} (R_1 \otimes_k R_2)\right) = \mathrm{H}\left(K(x_1 \otimes 1_{R_2}, \ldots, x_s \otimes 1_{R_2}, 1_{R_1} \otimes y_1, \ldots, 1_{R_1} \otimes y_t)\right)$.

The conditions of the Corollary assume that $\mathrm{H}(F_1 \otimes_{Q_1} R_1) = \mathrm{H}\left(K(x_1, \ldots, x_s)\right)$ and $\mathrm{H}(F_2 \otimes_{Q_2} R_2) = \mathrm{H}\left(K(y_1, \ldots, y_t)\right)$ are both Koszul algebras. By Theorem 2 of Backelin and Fröberg (above), this implies that $\mathrm{H}\left(F \otimes_{Q_1 \otimes_k Q_2} (R_1 \otimes_k R_2)\right)$ $= \mathrm{H}\left(K(x_1 \otimes 1_{R_2}, \ldots, x_s \otimes 1_{R_2}, 1_{R_1} \otimes y_1, \ldots, 1_{R_1} \otimes y_t)\right)$ is also Koszul.

It remains to show that $R_1 \otimes_k R_2$ is formal. Recall the the quasi-isomorphism of the bi-complex:

$k \otimes_{Q_1 \otimes_k Q_2} G_1 \otimes_k G_2 \xleftarrow{\simeq} F \otimes_{Q_1 \otimes_k Q_2} (G_1 \otimes_k G_2) \xrightarrow{\simeq} F \otimes_{Q_1 \otimes_k Q_2} (R_1 \otimes_k R_2)$.

Since $G_1 \otimes_k G_2$ is tensored with the field k over $Q_1 \otimes_k Q_2$, therefore $k \otimes_{Q_1 \otimes_k Q_2} (G_1 \otimes_k G_2) \cong \mathrm{H}\left(k \otimes_{Q_1 \otimes_k Q_2} G_1 \otimes_k G_2\right)$ as in section 1.5. The following diagram summarizes the previous argument. The symbol $\mathcal{Q}$ will be used to denote the ring $Q_1 \otimes_k Q_2$ for purposes of economy of space.

$$\begin{array}{ccccc}
k \otimes_{\mathcal{Q}} (G_1 \otimes_k G_2) & \xleftarrow{\simeq} & F \otimes_{\mathcal{Q}} (G_1 \otimes_k G_2) & \xrightarrow{\simeq} & F \otimes_{\mathcal{Q}} (R_1 \otimes_k R_2) \\
\downarrow\cong & & \downarrow & & \downarrow \\
\mathrm{H}(k \otimes_{\mathcal{Q}} G_1 \otimes_k G_2) & \cong & \mathrm{H}\left(F \otimes_{\mathcal{Q}} (G_1 \otimes_k G_2)\right) & \cong & \mathrm{H}\left(F \otimes_{\mathcal{Q}} (R_1 \otimes_k R_2)\right)
\end{array}$$

The bottom row is a special case of the balanced Tor functor. Since the Koszul complex $F \otimes_{Q_1 \otimes_k Q_2} (R_1 \otimes_k R_2)$ is linked to the homology algebra $\mathrm{H}\left(F \otimes_{Q_1 \otimes_k Q_2} (R_1 \otimes_k R_2)\right)$ by a sequence of quasi-isomorphisms, and since $\mathrm{H}\left(F \otimes_{Q_1 \otimes_k Q_2} (R_1 \otimes_k R_2)\right)$ was shown to be a Koszul algebra, therefore $R_1 \otimes_k R_2$ has been shown to satisfy the Lemma (2.8) and is a PK ring. Q.E.D.

CHAPTER 3

COMMUTATIVE RINGS OF SMALL PROJECTIVE DIMENSION

3.1 Introduction

This section will classify some rings of small projective dimension as PK using previous work by L. Avramov [2, pp. 51-52] and Andrew R. Kustin and Matthew Miller [20, pp.348-349]. The archetype of PK rings are the local complete intersections. As a result of the classifications in this chapter it will be shown that PK rings are not restricted to the local complete intersections and, thus by the results of Chapter 2, arbitrary tensors of the rings classified in this chapter will yield other examples of PK rings. Throughout this section $R = Q/I$ where Q is a regular local ring, m_Q is the maximal ideal in Q, and $I \subseteq m_Q^2$. The classification proceeds according to the projective dimension of R over Q. Throughout H denotes the homology algebra of R with respect to a minimal set of generators of the maximal ideal of R. It is well-known that all rings R considered in this section have minimal free resolutions over Q which admit an algebra structure. Thus, by the results of Chapter 2, their Koszul complexes are formal, and to show that they are PK, it only must be shown that the their homology algebras H are Koszul algebras.

3.2 Classification of Projective Dimension 2 Rings

In the case that R has projective dimension two over Q, it is known that the homology algebra has trivial multiplication, and is therefore Koszul. Below is provided a different proof. It relies on the Hilbert-Burch Theorem (see D. Eisenbud: [14, pp.506-507]):

If the complex

$$\mathbf{F}: 0 \longrightarrow Q^n \xrightarrow{\phi_2} Q^{n+1} \xrightarrow{\phi_1} Q \longrightarrow R \longrightarrow 0$$

is exact, then there exists a nonzerodivisor a such that $I = aI_n(\phi_2)$ where $I_n(\phi_2)$ refers to the ideal generated by minors of size $n \times n$ in the matrix representing the map ϕ_2. The i^{th} entry of the matrix for ϕ_1 is $(-1)^i a$ times the minor obtained from ϕ_2 by leaving out the i^{th} row. The ideal $I_n(\phi_2)$ has depth exactly 2.

THEOREM (3.1): *If the local ring R has projective dimension 2 then it is a PK ring.*

PROOF: Let $\{x_1, \ldots, x_d\}$ be a minimal generating set for m_Q. Consider the free resolution of $R = Q/I$ over Q:

$$\mathbf{F}: 0 \longrightarrow Q^n \xrightarrow{\phi_2} Q^{n+1} \xrightarrow{\phi_1} Q \longrightarrow R \longrightarrow 0$$

The matrix representing the map ϕ_1 is comprised of the generators of I. By the Hilbert-Burch Theorem, ϕ_2 is represented by a $(n+1) \times n$ matrix M. We can assume that $n \geq 2$. Otherwise the ring will be a complete intersection and the homology algebra will be the exterior algebra, the case which is already known to be PK. The proof will be by considering two cases: Case (1): The nonzerodivisor a is not a unit; Case (2): The nonzerodivisor a is a unit. In both cases it will be demonstrated that the homology algebra is the trivial algebra, i.e., all products of homology generators are 0. The proof that projective dimension 2 rings are PK will follow from observing that the homology algebras are PBW algebras and are thus Koszul.

Case (1): Each generator in I is a nonzerodivisor a times the determinant of an $n \times n$ submatrix M_r, where the submatrix M_r is formed by eliminating the r^{th} row of M. Each generator of I therefore has a common factor of a and may all be written as products of a: $f_i = a \cdot f_i'$. The factor a may be written $a = \sum_{l=1}^{d} x_l \cdot p_l$ where p_l is a polynomial in the ring Q. A preimage $z_r \in Q \cdot e_1 \oplus \cdots \oplus Q \cdot e_d$ of a degree 1 homology generator $[z_r]$, that is associated with f_r, may be obtained by substituting an exterior algebra basis vector e_l for x_l in the terms of a. Denote this substitution by a'. Thus, the representation of z_r may be chosen such that $z_r = a' \cdot f_r'$. Then any two preimages z_r, z_s of homology generators of degree 1 of the homology algebra will have their product $z_r z_s = (a' \cdot f_r') \wedge (a' \cdot f_s') = (a' \wedge a') \cdot (f_r' f_s') = 0$ since $a' \wedge a' = 0$ in the exterior algebra. Hence, $[z_r][z_s] = 0$ in the homology algebra $H(K(\bar{x}_1, \ldots, \bar{x}_d))$, where ideal $(\bar{x}_1, \ldots, \bar{x}_d)$ is maximal in R.

Case (2): For a a unit, we let the elements $y_{i,j}$ of Q represent entries of M, the matrix representing ϕ_2 in the Hilbert-Burch theorem. In this case the Hilbert-Burch Theorem will be used to show that any product of degree 1 generators of the homology algebra will be zero. As in Case 1, the development will be done in Q with preimages of homology generators, and the final results will be obtained after tensoring the preimages in $Q \cdot e_1 \oplus \cdots \oplus Q \cdot e_d$ with R over Q to obtain the homology generators in $R \cdot e_1 \oplus \cdots \oplus R \cdot e_d$.

Before this is done, it will be shown that the choice of preimage of homology generator z_r is independent modulo boundaries. Two facts will be recalled:

1. The determinant of a $n \times n$ matrix of indeterminates $y_{i,j}$ (without repetition) is a homogeneous polynomial of degree n, with each term denoting a transversal of the

matrix;

2. The Koszul differential ∂_2 acting on an arbitrary degree 2 basis vector $e_p \wedge e_q$ in the exterior algebra $\Lambda^2 Q$ has form $\partial_2(e_p \wedge e_q) = x_p e_q - x_q e_p$.

LEMMA (3.2): *Let M_r be as stated in the Hilbert-Burch Theorem and let z_r be as stated above. Let F denote the Koszul resolution of the field $k = Q/m_Q$ over Q. Then the difference in any two representations of z_r lies in the* $\operatorname{Im}(\psi_2)$ *where ψ_2 is the second map in F. Thus, any two different representations of the same generator $\bar{z}_r$ are homologically equivalent in $F \otimes_Q R$.*

PROOF: A preimage z_r of degree 1 is obtained from a generator f_r of I by substituting an exterior algebra basis vector for a degree 1 factor in each term of f_r. Let $y_{i,j} = \sum_l p_l$ where $p_l \in Q$ are elements not all of degree < 2. There will be more than one way to factor the monomial terms of degree ≥ 2 in $y_{i,j}$ as products of degree 1 factors. Fix any row i in M_r and consider the substitution map: $\lambda^1_{i,1}: \; y_{i,1} \mapsto y'_{i,1}$ where $y'_{i,1} = \sum_{\alpha=1}^{d} e_\alpha \cdot p'_l$ (exterior algebra vector e_α is substituted for x_α). Similar substitution maps $\lambda^1_{i,2}, \ldots, \lambda^1_{i,n}$ may be defined for the remaining elements in row i of M_r. Let M'_r denote the new matrix after these substitutions. Now consider the substitution map: $\lambda^2_{i,1}: \; y_{i,1} \mapsto y''_{i,1}$ where $y''_{i,1} = \sum_{\beta=1}^{d} e_\beta \cdot p''_l$ (exterior algebra vector e_β is substituted for x_β where $x_\beta \neq x_\alpha$). Similar substitution maps $\lambda^2_{i,2}, \ldots, \lambda^2_{i,n}$ may be defined for the remaining elements in row i of M_r. Let M''_r denote the new matrix after these substitutions.

Thus two different representations of z_r: z'_r and z''_r may be obtained from $|M'_r|$ and $|M''_r|$ (the determinants of M'_r and M''_r). (Note: The substitution map $\lambda^2_{i,j}$ could similarly have been defined by fixing a column j and substituting $y'_{1,j}, \ldots, y'_{n,j}$ for $y_{1,j}, \ldots, y_{n,j}$ (resp.).

The problem of independence of representation of z_r therefore reduces to showing that the difference of any two representations of z_r: $z'_r - z''_r$ ($|M'_r| - |M''_r|$) lies in the image of ψ_2, and thus, after tensoring with R over Q, represents the same cycle in homology. The proof is immediate upon considering the following: Any two different cofactor evaluations of the determinant $|M_r|$ yield the same degree n polynomial. Thus, the two different representations are distinguished by different choices for degree 1 substitutions in each term of $|M_r|$. The map ψ_2 is comprised entirely of the differences of all pairs of such substitutions (as column vectors - refer to fact 2 above). Each term of $z'_r - z''_r$ will therefore occur in ψ_2 as polynomial combinations of column vectors in ψ_2, i.e., $z'_r - z''_r \in \mathrm{Im}(\psi_2)$. Q.E.D.

EXAMPLE: Let $y_{i,j} \in Q = k[x_1, \ldots, x_6]$ be an element in ϕ_2; let term t in the determinant polynomial of M_r be $x_1x_3x_5x_6$; let $t'_1 = x_1e_3x_5x_6$ be the substituted term in the first representation of z; let $t'_2 = x_1x_3x_5e_6$ be the substituted term in the second representation of z. Then $t'_1 - t'_2 = (-x_1x_5)(x_6e_3 - x_3e_6) = (-x_1x_5)\cdot$ column 12 of ψ_2 (the second map in the Koszul resolution of k over Q).

Consider the matrices M_r, M_s, M_t which occur as $n \times n$ submatrices in M after removing row(s) r, s, t (resp.) from M. Any two of the three matrices will have all rows except one in common. Let $Row_{com}(\alpha, \beta)$ denote the set of common rows of matrices M_α and M_β. All rows of $Row_{com}(r, s)$ may be written with the same order as the first $n-1$ rows in both M_r and M_s, leaving row n in M_r and M_s to be the row where they differ. The assumptions on matrices M, M_r, M_s, M_t will force the submatrix M_t to have $n-2$ rows in $Row_{com}(r,s)$, with the n^{th} rows of M_r and M_s as the two remaining rows of M_t. Without loss of generality, M_t may be represented with the first $n-2$ rows from $Row_{com}(r,s)$ with row n from M_r and row n from M_s being rows $n-1$ and n of M_t (resp.).

By the previous lemma, the choice of representation of a homology generator preimage z is invariant. Therefore, without loss of generality, the exterior algebra substitutions $y_{i,j} \mapsto y'_{i,j}$ may be taken along row t, which is in $Row_{com}(r,s)$. Degree 1 preimages z_r, z_s will be obtained from $|M'_r|$ and $|M'_s|$ (resp.). The product of preimages $z_r \cdot z_s$ will be the product of the determinants $|M'_r| \cdot |M'_s|$. The following lemma will show that any degree 2 wedge product in $y'_{t,j_1} \wedge y'_{t,j_2}$ will have as a coefficient the product of the determinant of M_t with a polynomial, and thus will be in the ideal I. Hence, after tensoring with R over Q, $\bar{z}_r \bar{z}_s = 0$.

LEMMA (3.3): *Let M_r, M_s, M_t be matrices (as indicated above) such that any two of them differ in only one row. Let $m_*[i_1, \ldots, i_n | j_1, \ldots, j_m]$ denote the minor of M_* with rows $i_1, \ldots, i_n$ and columns $j_1, \ldots, j_m$ removed. Then for elements y_{t,j_1}, y_{t,j_2} in row t of M_r and M_s, the following is the case:*

$$m_r[t|j_1]m_s[t|j_2] - m_r[t|j_2]m_s[t|j_1] = m_t[n-1, n|j_1, j_2]|M_t|.$$

PROOF: Without loss of generality, it will be assumed that row t may be swapped to row 1 in both M_r and M_s and additionally, columns j_1 and j_2 will be swapped to columns 1 and 2 (resp.) in M_r, M_s and M_t. This will allow for the use of Laplacian cofactor expansions ([26, pp.10,11]) since the sum of the row and column transpositions, T, will be the same for both M_r and M_s. Thus, any corrective sign $(-1)^{T(mod)2}$ in evaluating the determinants M'_r and M'_s will cancel out in the product of the cofactors. The minors will be evaluated by cofactors. Elements in M_r, M_s, M_t will be denoted by $a_{i,j}, b_{i,j}, c_{i,j}$ (resp.).

$$\begin{aligned}
m_r[1|1] &= (-1)^{n+2}a_{n,2} \cdot m_r[1,n|1,2] + (-1)^{n+3}a_{n,3} \cdot m_r[1,n|1,3] + \dots \\
&\quad + (-1)^{n+n}a_{n,n} \cdot m_r[1,n|1,n]; \\
m_s[1|2] &= (-1)^{n+1}b_{n,1} \cdot m_s[1,n|1,2] + (-1)^{n+3}b_{n,3} \cdot m_s[1,n|2,3] + \dots \\
&\quad + (-1)^{n+n}b_{n,n} \cdot m_s[1,n|2,n]; \\
m_r[1|2] &= (-1)^{n+1}a_{n,1} \cdot m_r[1,n|1,2] + (-1)^{n+3}a_{n,3} \cdot m_r[1,n|2,3] + \dots \\
&\quad + (-1)^{n+n}a_{n,n} \cdot m_r[1,n|2,n]; \\
m_s[1|1] &= (-1)^{n+2}b_{n,2} \cdot m_s[1,n|1,2] + (-1)^{n+3}b_{n,3} \cdot m_s[1,n|1,3] + \dots \\
&\quad + (-1)^{n+n}b_{n,n} \cdot m_s[1,n|1,n];
\end{aligned}$$

By the way M_r, M_s, M_t were arranged, the following is also the case:

$$\begin{aligned}
m_r[1|1] &= (-1)^{n+2}c_{n-1,2} \cdot m_t[n-1,n|1,2] + (-1)^{n+3}c_{n-1,3} \cdot m_t[n-1,n|1,3] + \dots \\
&\quad + (-1)^{n+n}c_{n-1,n} \cdot m_t[n-1,n|1,n]; \\
m_s[1|2] &= (-1)^{n+1}c_{n,1} \cdot m_t[n-1,n|1,2] + (-1)^{n+3}c_{n,3} \cdot m_t[n-1,n|2,3] + \dots \\
&\quad + (-1)^{n+n}c_{n,n} \cdot m_t[n-1,n|2,n]; \\
m_r[1|2] &= (-1)^{n+1}c_{n-1,1} \cdot m_t[n-1,n|1,2] + (-1)^{n+3}c_{n-1,3} \cdot m_t[n-1,n|2,3] + \dots \\
&\quad + (-1)^{n+n}c_{n,n} \cdot m_t[n-1,n|2,n];
\end{aligned}$$

$$m_s[1|1] = (-1)^{n+2}c_{n,2}\cdot m_t[n-1,n|1,2] + (-1)^{n+3}c_{n,3}\cdot m_t[n-1,n|1,3] + \dots$$
$$+ (-1)^{n+n}c_{n,n}\cdot m_t[n-1,n|1,n];$$

Evaluating the expression and arranging the terms:

$$m_r[1|1]\cdot m_s[1|2] - m_r[1|2]\cdot m_s[1|1] =$$
$$(c_{n-1,1}c_{n,2} - c_{n,1}c_{n-1,2})m_t[n-1,n|1,2]\cdot m_t[n-1,n|1,2]$$
$$+(c_{n-1,1}c_{n,3} - c_{n,1}c_{n-1,3})m_t[n-1,n|1,2]\cdot m_t[n-1,n|1,3]$$
$$+(c_{n-1,1}c_{n,4} - c_{n,1}c_{n-1,4})m_t[n-1,n|1,2]\cdot m_t[n-1,n|1,4]$$
$$\dots$$
$$+(c_{n-1,n-1}c_{n,n} - c_{n,n-1}c_{n-1,n})m_t[n-1,n|1,2]\cdot m_t[n-1,n|n-1,n].$$

The evaluation shows that the final expression is the product of the maximal common cofactor $m_t[n-1,n|1,2]$ (found in $m_r[1|1]$, $m_s[1|2]$, $m_r[1|2]$, $m_s[1|1]$, and M_t) with the Laplacian cofactor expansion of $|M_t|$. Thus, any product of preimages of degree 1 homology generators $z_r z_s$ will have terms $y'_{i,j_1}\wedge y'_{i,j_2}$ such that all degree 2 wedge products have coefficients in the ideal I. Therefore, after tensoring with R over Q, $[z_r][z_s] = 0$. Q.E.D.

To complete the proof of Theorem (3.1) consider the following: Since the homology algebra is concerned only with the "word length" of a generator and not the homological degree, each degree 1 generator in the homology algebra of a projective dimension 2 ring is assigned word length 1. All products of degree 1 generators are quadratic monomial relations of the type $[z_r][z_s] = 0$ for all r, s. Thus, there exists a PBW basis on the algebra defined by these quadratic monomial relations consisting only of degree 1 “words”. This algebra is therefore a PBW algebra and is Koszul ([27, p.51]). The proof of Theorem (3.1) is now complete. Q.E.D.

3.3 Classification of Codimension 3 Rings

The classification of projective dimension 3 rings will depend upon the $M3$-$N5$ theorem in lattice theory ([12, pp.134-138]); (Note: Other texts list the Hasse diagrams from which this theorem derives as $N5$ and $M5$ [8, p.11]).

THEOREM 6.10 (M3-N5)[12, pp.134-138]: *Let L be a lattice. Then:*

(i) L is non-modular if and only if $N5 \hookrightarrow L$.

(ii) L is non-distributive if and only if $N5 \hookrightarrow L$ *or* $M3 \hookrightarrow L$.

Classifications of the rings B, G(r), H(p,q) will be immediate upon showing that their defining relations force all degree 3 monomials in their respective algebras to cancel to zero. The application of Backelin’s condition for Koszulness of an algebra will then be used to show that these algebras can not embed either the N5 or M3 Hasse diagram interpreted as a product ideal substructure.

3.4 Lattice criteria for Koszul Algebras

The lattice generated by $A_+^{\alpha_i} I^{\beta_i} A_+^{\gamma_i}$, $(\alpha_i, \beta_i, \gamma_i \in \mathbf{N}_{\geq 0})$ is a modular lattice [4, p.1]. Thus, with regard to the $N5$-$M3$ theorem, it is only necessary to show that $M3$ can not be embedded into the lattice. Let $M3$ be the Hasse lattice diagram that minimally violates distributivity of a lattice. Let $A_+^{\alpha_i} I^{\beta_i} A_+^{\gamma_i}$, $(\alpha_i, \beta_i, \gamma_i \in \mathbf{N}_{\geq 0})$ denote the product ideal of an algebra, where A_+ denotes the maximal ideal of a quadratic algebra A, and I denotes the ideal of generating relations of A.

LEMMA (3.4): *If the cubic monomials of a quadratic algebra all cancel to 0 then the algebra is distributive.*

PROOF: Failure of distributivity can only occur if M3 is embedded in the lattice structure of the algebra. If the lattice is generated by the (above) product ideal with inclusion as the partial order [4, p.12] then each of the nodes of M3 may be labeled by the "signature": $[\alpha_i, \beta_i, \gamma_i]$, where $\alpha_i, \beta_i, \gamma_i$ range over the positive integer powers of the ideals : A_+ and I in the product ideal $A_+^{\alpha_i} I^{\beta_i} A_+^{\gamma_i}$. Since the cubic monomials all cancel to 0, this implies $A_+^{\alpha_i} I^{\beta_i} A_+^{\gamma_i}$ is non-trivial iff $\alpha_i + \beta_i + \gamma_i < 3$. The only possible signatures for the product ideal are: $[1, 0, 0]$; $[2, 0, 0]$; $[0, 1, 0]$.
Inspection shows that: If the order relation is defined as inclusion, then M3 can not be realized as a substructure within the lattice. Q.E.D.

It therefore remains to show that the algebras: B, G(r), H(p,q) all have trivial cubic monomials. This is exhibited by the defining relations and in the multiplication table of the generators of each of the algebras. The relations of B, G(r), H(p,q) are given below, where the stronger condition of being a quadratic PBW algebra is proved. Since all of these algebras are quadratic, and since they are all distributive, they are Koszul ([4, p.23]).

3.5 Classification of PBW Algebras

The previous definitions given in Chapter 1 will allow the construction of PBW bases for three of the following four classes of homology algebras, with respect to minimal generating sets of the maximal ideal, taken from the paper by Avramov ([2, p.51]). The relations given below are for the non-trivial part of the homology algebras. Basis elements for the homology generators are denoted by: $e_1, \ldots, e_s$ for degree 1 homology; $f_1, \ldots, f_t$ for degree 2 homology; and $g_1, \ldots, g_u$ for degree 3 homology. Since the trivial extensions contribute no nontrivial relations to the total set of ideal relations in the homology algebra ideal by which additional PBW generators may be obtained, the following Proposition will establish that once a PBW basis has been found for the nontrivial part of the algebra, then this basis will serve the entire algebra.

PROPOSITION (3.5): *Let $A = B \oplus T$ be a graded k-algebra where B is a graded k-algebra, $BT = TB = 0$, and $T^2 = 0$, that is, A is a trivial extension of B by T. If B is a PBW k-algebra, then A is as well.*

PROOF: Let the word length 1 generators of B be the set $\{b_1, \ldots, b_n\}$ and the word length 1 generators of T be the set $\{t_1, \ldots, t_m\}$. Without loss of generality the ordering of the generators may be lexicographic with order as follows: $\{b_1, \ldots, b_n, t_1, \ldots, t_m\}$. Then for any b_i and any t_j the following is true: $b_i t_j = 0$, $t_j b_i = 0$, $t_i t_j = 0$ for all i, j. It follows that $A_p = B_p$ for $p \geq 2$, and the label sets $S^{(p)}(A)$ for A and $S^{(p)}(B)$

of B are the same for $p \geq 2$. Thus A is a PBW algebra if B is a PBW algebra. Q.E.D.

It should be recalled that the homology algebra is obtained from only the word length of the generators and does not consider their homological (internal) degree. Those generators that are not obtained or expressible as products of lower homological degree generators will be referred to in this section as *emergent generators.* Homological degree 1 generators and emergent generators will always have word length 1.

TYPE TE: (truncated exterior algebra) In this algebra $s = t = 3$; $u = 0$. The generators have the following non-trivial relations:

$e_1 e_2 = f_3$; $e_2 e_3 = f_1$; $e_3 e_1 = f_2$.

As these relations generate a cubic relation in the algebra:

$e_1 e_2 e_3 = 0$

this algebra may therefore be classified as **non − Koszul**.

TYPE B: In this algebra, $s = 2$; $t = 3$; $u = 1$. The generators for the PBW basis are: $e_1, e_2, f_1, f_2, f_3, g_1$. Let the generators have degrees and monomial ordering as given in the following table:

Table 3.1. TYPE B: Ordering of Generators

Generator	e_1	e_2	f_1	f_2	f_3	g_1
order	1	2	3	4	5	6
degree	1	1	2	2	2	3

The algebra is determined by the following four relations:

$e_1e_2 + e_2e_1$; $e_1f_1 - f_1e_1$; $e_2f_2 - f_2e_2$; $e_1f_1 - e_2f_2$.

The degree 1 generators include e_1, e_2 and the emergent degree 2 generators f_1, f_2. The degree 2 label set $S^{(2)}$ corresponds to the elements e_1e_2 e_1f_1, which form basis of H_2. Since there do not exist non-trivial cubic monomial generators in this algebra, this algebra represents a quadratic PBW algebra; hence, is **Koszul**.

TYPE G(r): In this algebra $s = t = r \geq 2$; $u = 1$. The defining relations are:

$e_if_i = g_1$, $1 \leq i \leq r$.

The label set $S^{(1)}$ corresponds to the homological degree 1 generators and the homological degree 2 generators (as the degree 2 generators are all emergent). After assigning a straight lexicographic ordering to these $2 \cdot r$ generators:

$o(e_i) = i$, $1 \leq i \leq r$; $o(f_i) = r + i$, $1 \leq i \leq r$; $o(g) = 2r + 1$

the degree 2 label set $S^{(2)}$ has only one corresponding generator: e_1f_1. There do not exist nontrivial cubic generators in this algebra. Hence, it is a quadratic PBW algebra, and thus **Koszul**.

TYPE H(p,q): In this algebra $s = p + 1$; $t = p + q$; $u = q$.

The defining relations are: $e_{p+1}e_i = f_i$, $1 \leq i \leq p$; $e_{p+1}f_{p+j} = g_j$, $1 \leq j \leq q$.

Let the variables have the order indicated:

Table 3.2. TYPE H(p,q): Ordering of Generators

Generator	e_i	e_{p+1}	f_{p+j}
order	i	p+1	p+j+1
degree	1	1	2

With this generator ordering, $S^{(1)}$ will correspond to the degree 1 generators and the emergent degree 2 generators: $\{e_i : 1 \leq i \leq p+1;\ f_j : p+1 \leq j \leq p+q\}$ in lexicographic order. The degree 2 label set $S^{(2)}$ corresponds to $\{e_i e_{p+1}\} \cup \{e_{p+1} f_{p+j}\}$, and this is a basis of H_2. Again, since there exist no non-trivial cubic monomials in this algebra, H(p,q) is a quadratic PBW algebra, hence is **Koszul**.

3.6 Classification of Codimension 4 Gorenstein Rings

The following classification of codimension 4 Gorenstein rings is derived from the classification relations and table found in the paper of Kustin and Miller [20, pp.348-349]. In this paper the following relations are given for four classes of homology algebras derived from codimension 4 Gorenstein rings:

THEOREM 2.2 [20, pp.348-349]: *Let R, m, k be a Gorenstein local ring in which 2 is a unit and assume that k has square roots. Let I be a grade four Gorenstein ideal in R, and $H = H(K(S))$ where S is a sequence of minimal generators of a maximal ideal of R. Then there are bases $\{x_1, ..., x_n\}$ for H_1, $\{y_1, ...y_{n-1}, y'_1, ..., y'_{n-1}\}$ for H_2, $\{z_1, ..., z_n\}$ for H_3, and $\{w\}$ for H_4 so that the multiplication $H_i \times H_{4-i} \rightarrow H_4 = k$ is given by $x_i z_j = \delta_{i,j} w$, $y_i y'_j = \delta_{i,j} w$, $y_i y_j = 0 = y'_i y'_j$ and other products in H_* are given by one of the following cases:*

(A) the ideal I is generated by a regular sequence, in which case H_* is the exterior algebra on the vector space $\bigoplus_{i=1}^{4} kx_i$.

(B) all products in H_1H_1 and H_1H_2 are zero.

(C) all products in H_1H_1 and H_1H_2 are zero except those indicated in the following

multiplication table(s):

Table 3.3. Kustin-Miller: Non-trivial Codim.4 Gorenstein Products

mult.	x_1	x_2	x_3	y'_1	y'_2	y'_3
x_1	0	y_3	$-y_2$	0	z_3	$-z_2$
x_2	$-y_3$	0	y_1	$-z_3$	0	z_1
x_3	y_2	$-y_1$	0	z_2	$-z_1$	0

(D) there is an integer p such that $x_{p+1}x_i = y_i$, $x_i y'_i = z_{p+1}$, $x_{p+1}y'_i = -z_i$ for $1 \leq i \leq p$
and all other products in H_1H_1 and H_1H_2 are zero.

Case (A): The homology algebra is the exterior algebra. This algebra is a complete intersection and is Koszul by being the dual of the free resolution of the field with respect to the regular ring, which is known to be **Koszul** (see R. Froberg [16, pp.3-4]).

Case (B): The homology algebra yields nontrivial products, all scalar multiples of each other, in the blocks associated with products H_iH_{4-i} for $1 \leq i \leq 3$. A PBW basis may be constructed using degree lexicographic monomial ordering of generators of H_* . The $S^{(2)}$ label set corresponds to only one generator: x_1z_1 . By the relations in ideal I and the product table, it is easily seen that the $S^{(3)}$ label set is zero. Hence, by either the Backelin lattice condition and/or the construction of a quadratic PBW basis, this algebra is **Koszul**.

Case (C): Cubic relations are immediately found in the product table blocks associated with H_1H_1 and H_1H_2 . This homology algebra is thus **non − Koszul**.

Case (D): Consider the lexicographic ordering across the degree 1 generators: $x_1, \ldots, x_{p+1}$ and emergent degree 2 generators $y'_1, \ldots, y'_p$ with $x_* < y'_*$. Then $S^{(1)}$ corresponds to the basis $\{x_1, \ldots, x_{p+1}, y'_1, \ldots, y'_p\}$ of H_1, $S^{(2)}$ corresponds the the basis $\{x_i x_{p+1} : 1 \le i \le p\ ,\ x_1 y'_1,\ x_{p+1} y'_i : 1 \le i \le p\}$ of H_2, and $S^{(3)}$ corresponds to the basis $\{x_1 x_{p+1} y'_1\}$ of H_3. There is only one word length 3 generator in $S^{(3)}$, therefore the elements of $S^{(3)}$ have no linear relations. There exists a PBW basis for the entire algebra (Theorem 2.1, [25, p.82]) and the relations in the homology ideal are quadratic. Therefore this is a **Koszul** algebra ([27, p.51]).

From the above analysis of relations, only the algebra TE in codimension 3 and case C in codimension 4 (Gorenstein) yield non-Koszul homology algebras. All other codimension 2, 3, and 4 rings considered in this section are PK rings and, by the previous section, any arbitrary tensors of these rings will also be PK rings.

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BIOGRAPHICAL STATEMENT

Paul Russell Stern was born in Dallas, Texas in 1954. He received his B.S. and B.A. degrees from Southern Methodist University in 1980. He attended Illinois State University as a fellowship student and received a M.A. degree 1987. From 1987 to 1988 he was a professor in the Department of Music at Radford University, Radford Virginia. Prior to attending the University of Texas at Arlington he taught music privately and made recordings of classical music. His research concerned topics in algebra and combinatorics.