THE EQUIVALENCE AND GENERALIZATION

OF OPTIMIZATION CRITERIA

by

SURACHAI CHAROENSRI

Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

August 2011

ACKNOWLEDGEMENTS

Many people have assisted and supported in the completion of my doctoral program. I extend my sincere gratitude to you all.

In particular, I would like to express my deep appreciation to my supervising professor, Dr. H.W. Corley. This dissertation would not have been completed without his guidance, suggestions, and help. His scholarly advice and continual encouragement were invaluable in directing me academically and personally throughout my years at UT Arlington. He contributed enormously to my personal and intellectual development. I am also grateful to Dr. Victoria Chen, both the Director of COSMOS and a committee member, who funded, supervised, and advised me as research assistant on the CREST project during the first year of my doctoral program. In addition, I am thankful to the other members of my committee, Dr. Jay Rosenberger and Dr. Erick Jones, for their time and suggestions. These professors are but a few of the teachers who have instilled in me their knowledge and encouraged me to develop my skills. I am most grateful to you all.

I acknowledge with appreciation many friends at COSMOS, including Bancha, Poovich, Pin, Narakorn, Passakorn, Aera, Diana, Nadia, and Chatabush, for our valuable discussions on many things. A special note of gratitude goes to Chotiya Sukchotrat, who has helped to make my everyday life comfortable.

I also appreciate the assistance of the support staff at the Industrial Manufacturing and Systems Engineering, in particular, to Richard Zercher, Julie Estill, and Kim Williams, whose assistance I often required.

Many Thai people in the USA – including Kittipat, Kannanat, Jiraphan, Chutima, Janewit, Usanee, Nongnuch, and Ketwalee – gave me their friendship. Thank you all my friends, these and others, for providing warmth and happiness.

For my two years in Gainesville, Florida, I thank Pa Pam, Loong Mor, Michael Funk and Dr. Vasana Maneerattana for their help in various ways and Sunee Snyder for her encouragement.

Most importantly, I express my sincere appreciation to my family for their love, support, and understanding - especially my mother, Sukanya Charoensri; my father, Yindee Charoensri; and my sister Phakarat Charoensri. My gratitude extends to all other family members in Thailand. Finally, I most graciously acknowledge my uncle and aunt-in-law, Bumroong Wongbundhit and Dr. Yuwadee Wongbundhit, for providing the inspiration and opportunity to study for a Ph.D. in the USA.

At last, I made it. I love you all.

July 14, 2011

ABSTRACT

THE EQUIVALENCE AND GENERALIZATION

OF OPTIMIZATION CRITERIA

Surachai Charoensri, PhD

The University of Texas at Arlington, 2011

Supervising Professor: H.W. Corley

In this dissertation we first show that existing optimization criteria are equivalent to the maximization of a real-valued function in a one-dimensional Euclidean space. The criteria are said to be scalar equivalent. All solutions and only solutions to an optimization problem involving the original criterion can be obtained by scalarization without the typical convexity or concavity assumptions on the original objective functions and feasible region. Examples include Pareto (including the scalar case), satisficing, maximin, and cone-ordered optimization, as well as the more general notion of set-valued optimization in abstract spaces. Moreover, equivalences between various different optimization criteria are also established directly. As a consequence, any problem stated as one criterion can be solved as another.

Second, we axiomatize and generalize the definition of an optimization criterion definition to include the existing standard criteria as special cases. We discuss our choices of axioms and explain why other possible axioms are excluded from our formalization. We then propose an equivalent scalarization of a general optimization criterion problem. In other words, we can obtain solutions of a problem involving any criterion satisfied our definition by simply solving scalar maximization problems. We present examples of new optimization criteria and apply them in practical decision-making situations. In addition, to provide insight into the scope of our work, we give a decision rule that is not a criterion within our framework.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

CHAPTER 1

INTRODUCTION

The research of this dissertation considers the concept of an optimization criterion, which is effectively the way one makes a best decision according to some interpretation of the term "best." For example, a business may try to maximize its profit, so the optimization criterion is to maximize the amount of money made. On the other hand, a fire station might be built to serve a particular population area so as to minimize the maximum distance a fire truck would have to travel. The decision on where to build the fire station would thus be decided based a maximin criterion. Moreover, a person might aspire to a certain salary in finding a job. In fact, any job meeting the salary goal would be deemed acceptable, so the decision would be based on other factors than salary. This criterion is called satisficing. As a final example, legislators want to meet energy demands without depleting natural resources. Tradeoffs are required. Various optimization criteria consider such multiple objectives, including the well-known Pareto criterion.

We show here that all standard optimization criteria can be scalarizable; i.e., a solution of the problem can be achieved as the maximization or minimization of a real-valued objective function subject to certain constraints. No matter what the criterion of the original problem is, we can obtain its solutions by finding the largest or smallest scalar number via a real-valued maximization or scalar minimization problem. Furthermore, we show that each existing optimization criterion can be solved as the scalarization of any other criteria. In other words, all existing optimization criteria may be called scalar equivalent, and any problem involving one criterion can be formulated as a problem involving any other.

This realization motivates us to define a more general definition of optimization criteria to include all existing optimization criteria as special cases. Thus we give an axiomatic mathematical definition of an optimization criterion to state consistent rules for calling something "the best." Next we develop an equivalent scalarization of an optimization problem involving a general criterion in the following sense. All solutions to the original problem and only solutions to it can be obtained via the maximization of a related real-valued function that is a scalarization of the original problem.

Finally we construct two new optimization criteria. One of these criteria interprets "optimize" as "compromise." Such a compromising criterion appears useful for multi-objective optimization in general and for game theory in particular.

The organization of the dissertation is as follows. In chapter 2 we review the notions of maximin, satisficing (goal programming), and cone-ordered optimization (including the including Pareto and set-valued cases). We also summarize such concepts as cones in finite-dimensional real vector spaces, as well as the orders induced by such cones.

In chapter 3, we present an equivalent scalarization of the standard Pareto, satisficing, maximin, and cone-ordered optimization criteria, as well as the more general notion of set-valued optimization in abstract spaces. As an example, we establish the scalar equivalence between the maximin and Pareto criteria. In addition, the equivalence of various standard criteria is established directly without resorting to scalarization. In other words, any problem involving one criterion can be restated as an equivalent problem involving another criterion in the sense of obtaining all solutions and only solutions to the original problem. Scalar equivalence thus follows. We illustrate the direct equivalence between the standard optimization criteria with the cases of maximin and Pareto maximization, Pareto maximization and lexicographic maximization, goal programming and Pareto maximization, as well as set-valued maximization and cone-ordered maximization.

In chapter 4, an axiomatization and generalization of optimization criteria are presented. We discuss our choice of axioms and explain why other possibilities are excluded. We then show that existing optimization criteria satisfy the axioms.

In chapter 5, we define the new optimization criteria of "compromising" and give applications in multi-objective optimization and game theory. We next show that the notion of "randomizing" is formally an optimization criterion in the situation where any action can be taken but some decision is required. We then present two group decision-making schemes for voting that do not conform to our definition of a general optimization criterion.

Finally, in chapter 6, we discuss the contributions of this research and discuss possible future work.

CHAPTER 2

PRELIMINARIES

In this chapter the notions of maximin, satisficing (goal programming), and cone-ordered (including Pareto and set-valued) optimization are presented. We also summarize such concepts as cones in *n*-dimensional Euclidean space, as well as the orders induced by such cones.

2.1. Notation

The following notion will be used throughout the dissertation.

- Vectors are represented by boldface lowercase Roman letter such as **x** and **y**.
- $\mathbf{x}^{\mathbf{t}}$ denotes the transpose of vector **x**. Thus if **x** is a column vector, then $\mathbf{x}^{\mathbf{t}}$ is a row vector and vice versa.
- x_i denotes the component i^{th} of vector **x**.
- Scalar values are denoted by lower case Roman and Greek letter such as $c, \alpha,$ and λ.
- The *n*-dimensional Euclidean space is the set of all real vectors containing *n* components. It is denoted by R^n.

2.2. Maximin Problem

Let $f : R^n \times R^m \to R$ be a real-valued function. For each $\mathbf{x} \in A \subset R^n$, define the set $B(\mathbf{x}) \subset R^m$ to be a nonempty feasible region. Assume that the function $g(\mathbf{x}) = \min_{\mathbf{y} \in B(\mathbf{x}) \subset R^m} f(\mathbf{x}, \mathbf{y})$ is well-defined for all $\mathbf{x} \in A$. Referring to [1], the **general maximin problem** can be stated as

$$\max_{\mathbf{x} \in A \subset R^n} \min_{\mathbf{y} \in B(\mathbf{x}) \subset R^m} f(\mathbf{x}, \mathbf{y}).$$

Note that for different $\mathbf{x}_1, \mathbf{x}_2 \in A \subset R^n$, the associated feasible regions $B(\mathbf{x}_1)$ and $B(\mathbf{x}_2)$ are not necessarily identical. In another words, this formulation restricts the feasible choices of $\mathbf{y}$ depending on the certain choices of $\mathbf{x}$. If $B(\mathbf{x}) = B$ for all $\mathbf{x} \in A \subset R^n$, the above problem takes the more familiar form

$$\max_{\mathbf{x} \in A \subset R^n} \min_{\mathbf{y} \in B \subset R^m} f(\mathbf{x}, \mathbf{y}).$$

In particular, if $B = \{1, ..., n\}$ for some given positive integer n, the problem becomes the discrete maximin problem

$$\max_{\mathbf{x} \in A \subset R^n} \min\{ f(\mathbf{x}, 1), ..., f(\mathbf{x}, n)\}.$$

Example 2.2.1. Let $A = [1,9] \subset R$ and $B(x) = \{y \in [1, x] : x - y \geq y - 1\}$ for each $x \in A$.

Define $f(x, y) = \dfrac{x}{y}$ for $x \in A, y \in B(x)$, and consider the maximin problem

$$\max_{x \in [1,9]} \min_{y \in B(x)} \frac{x}{y}.$$

In this example, the feasible region of variable y in minimization depends on the value of variable x given. For example, we have that $B(5)=[1,3]$ while $B(7)=[1,4]$. The dotted area in Figure 2.1 represents the feasible region of this general maximin problem.

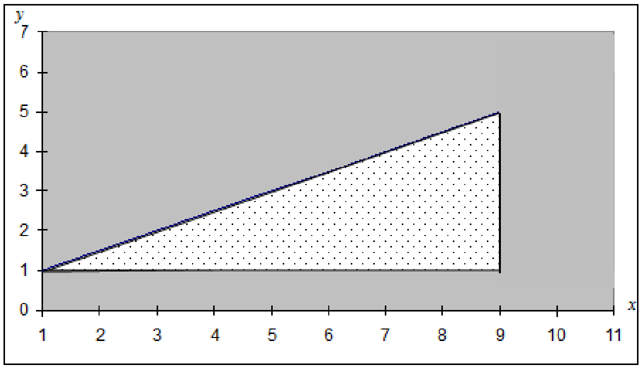

Figure 2.1 The feasible region for Example 2.2.1.

2.3. Pareto Optimization

Let $A \subset R^m$ be a set of feasible solutions and $f: R^m \to R^n$ be the n-dimensional objective function. The objective function value can also be represented as $f(\mathbf{x}) = (f_1(\mathbf{x}),..., f_n(\mathbf{x}))$ for all $\mathbf{x} \in A$, where $f_i : R^m \to R$ is defined to be the i^{th} objective function of the problem for each $i = 1,..,n$. Then **Pareto maximization,** or **vector maximization**, can be stated as

$$\underset{\mathbf{x}\in A}{\text{Vmax}}(f_1(\mathbf{x}),..., f_n(\mathbf{x})).$$

A feasible solution $\mathbf{x} \in A$ is called a **Pareto maximum** or **efficient point** if there is no $\mathbf{y} \in A$ such that $f_i(\mathbf{x}) \le f_i(\mathbf{y})$ for all $i = 1,..,m$ and $f_j(\mathbf{x}) < f_j(\mathbf{y})$ at least one index

j. The set $\{f(\mathbf{x}) \in R^n : \mathbf{x} \text{ are Pareto maxima}\}$ is called the **Pareto frontier or efficient frontier**.

2.4. Goal Programming

Goal programming is usually stated written as a scalar maximization or minimization of a function involving only the deviational variables. However, we present here the more general definition as given in [2] in which it formulated as a Pareto optimization.

Let $f_i : R^m \to R \text{ for } i = 1,\dots,n$ be the goal functions and $b_1,\dots,b_n$ represent the associated aspiration levels for objective 1 to n, respectively. Then the **goal programming** problem can be stated

$$\left\{\begin{array}{ll} \min\limits_{\mathbf{x},s^+,s^-} & (s_1^- \text{ or } s_1^+, \dots, s_n^- \text{ or } s_n^+) \\ \text{s.t.} & f_1(x) + s_1^- - s_1^+ = b_1 \\ & \vdots \\ & f_n(x) + s_n^- - s_n^+ = b_n \\ & s_i^- \cdot s_i^+ = 0 \\ & s_i^-, s_i^+ \geq 0, \mathbf{x} \in A \end{array}\right\}.$$

The objective is to minimize the deviations s_i^-, s_i^+ to obtain a feasible **x** making the goal functions as close to the aspiration levels b_i as possible. For more details, see [3] and [4].

2.5. Cones, Orders, and Dual cones

The concepts of an order induced by a cone in a vector space, as well as its dual cone, are next defined.

Definition 2.5.1. A nonempty set $C \subset R^n$ is called a **cone** if $\lambda \mathbf{c} \in C$ for all $\mathbf{c} \in C$ and $\lambda \geq 0$. A cone C is **pointed** if the set $C \cap -C$ contains only the vector of zero. Moreover, a **convex cone** C is a cone such that $\lambda_1 \mathbf{c}_1 + \lambda_2 \mathbf{c}_2 \in C$ for all $\mathbf{c}_1, \mathbf{c}_2 \in C$ and $\lambda_1, \lambda_2 \geq 0$.

Example 2.5.2. The left drawing below in figure 2.2 shows a nonconvex cone in two-dimensional Euclidean space while the right picture represents an important convex cone in the space. We usually call the convex cone in the right picture as the **nonnegative orthant** in R^2 and denote it as $R^2_{\geq} = \{(x, y) : x, y \geq 0\}$. Notice that both cones are pointed.

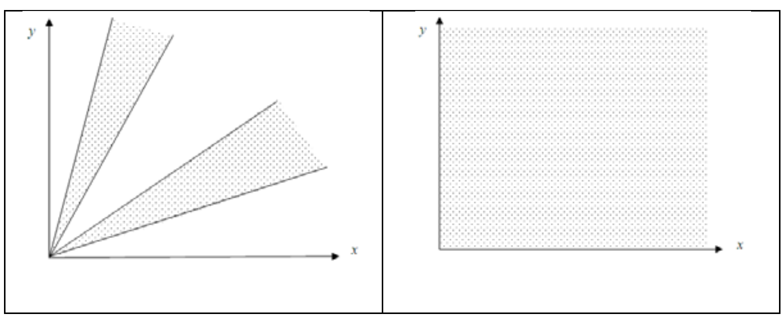

Figure 2.2 Examples of cones in R^2.

Example 2.5.3. Another important cone is called the lexicographic cone [5] used to define lexicographic optimization [6], where individual goals are ordered by priority so that any higher level preempts a lower level one. For example, in R^2, the lexicographic cone is defined as

$$L = \{(x, y) \in R^2 : \text{either } x > 0 \text{ or else } x = 0 \text{ and } y > 0\}.$$

Notice that the lexicographic cone is a pointed and convex. Below, we graph lexicographic cone in two-dimensional Euclidean space. Note that the line $\{(x_1, x_2) \in R^2 : x_1 = 0, x_2 < 0\}$ is missing from the cone of figure 2.3.

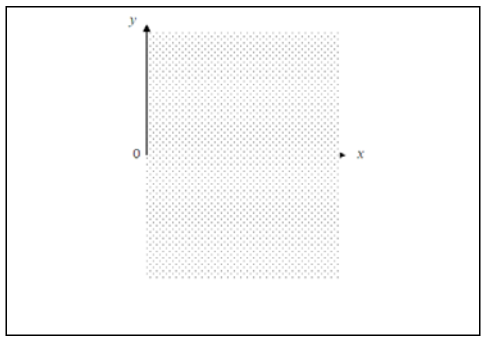

Figure 2.3 The lexicographic cone in Euclidean 2-space.

Definition 2.5.4. Let C be a pointed convex cone in R^n and define a relation **order** $\leq_C$ on R^n as follows. For any $\mathbf{y}_1, \mathbf{y}_2 \in R^n$, we say that $\mathbf{y}_1 \leq_C \mathbf{y}_2$ if $\mathbf{y}_2 - \mathbf{y}_1 \in C$. Define $\mathbf{y}_1 <_C \mathbf{y}_2$ if $\mathbf{y}_1 \leq_C \mathbf{y}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$. In particular, we say that $\mathbf{y}_2$ **dominates** $\mathbf{y}_1$ if $\mathbf{y}_1 \leq_C \mathbf{y}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$. A vector $\mathbf{y}_1 \in B \subset R^n$ is said to be **non-dominated** in B if there is no $\mathbf{y}_2 \in B$ such that $\mathbf{y}_1 \leq_C \mathbf{y}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$. Denote the set $\max_C B$ as the set containing all non-dominated vectors in B with respect to the cone C.

Proposition 2.5.5. Let C be a cone in R^n. If $\mathbf{a} \leq_C \mathbf{b}$ then $\mathbf{a}+\mathbf{d} \leq_C \mathbf{b}+\mathbf{d}$ for any $\mathbf{d} \in R^n$.

Proof. Let C be a cone in R^n and assume that $\mathbf{a} \leq_C \mathbf{b}$. By definition, we have $\mathbf{b} = \mathbf{a}+\mathbf{c}$ for some $\mathbf{c} \in C$. Then it follows that $(\mathbf{b}+\mathbf{d}) = (\mathbf{a}+\mathbf{d})+\mathbf{c}$; i.e., $\mathbf{a}+\mathbf{d} \leq_C \mathbf{b}+\mathbf{d}$ for any $\mathbf{d} \in R^n$. ■

Example 2.5.6 For the lexicographic cone of Example 2.5.3., we construct the order induced by it. Let $B = \{(0,0),(0,1),(1,0),(1,1)\} \subset R^2$ and L be the lexicographic cone in R^2. Then

$$(0,0) \leq_L (0,1), (0,1) \leq_L (1,0), \text{and } (1,0) \leq_L (1,1).$$

Definition 2.5.7. A relation order $\preceq$ on $A \subset R^n$ is said to be a **partial order** if it satisfies the following 3 properties.

1. **Reflexive** property: $\mathbf{x} \preceq \mathbf{x}$ for all $\mathbf{x} \in A$.
2. **Antisymmetric** property: If $\mathbf{x} \preceq \mathbf{y}$ and $\mathbf{y} \preceq \mathbf{x}$ for any $\mathbf{x}, \mathbf{y} \in A$, then $\mathbf{x} = \mathbf{y}$.
3. **Transitive** property: If $\mathbf{x} \preceq \mathbf{y}$ and $\mathbf{y} \preceq \mathbf{z}$ for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A$, then $\mathbf{x} \preceq \mathbf{z}$.

If $\preceq$ is reflexive and transitive, then we say that $\preceq$ is a **preorder**. A partial order implies a preorder, but the converse is not true.

Definition 2.5.8. A partial order $\preceq$ on $A \subset R^n$ is a **total order** if $\mathbf{x} \preceq \mathbf{y}$ or $\mathbf{y} \preceq \mathbf{x}$ for any $\mathbf{x}, \mathbf{y} \in A$. A set $B \subset A$ of totally ordered elements is called a **total ordered set** or **a chain**.

Definition 2.5.9. Let $A \subset R^n$ with a partial order $\leqq$. A vector $\mathbf{x} \in A$ is said to be a **maximal element** of A if $\mathbf{x} \leqq \mathbf{z}$ implies $\mathbf{x} = \mathbf{z}$ for any $\mathbf{z} \in A$.. For a subset of B of A, a vector $\mathbf{y} \in A$ is said to be an **upper bound** of B if $\mathbf{x} \leqq \mathbf{y}$ for all $\mathbf{x} \in A$.

Definition 2.5.10. If a partial order $\leqq$ on $A \subset R^n$ has no a maximal element, we say that A is **unbounded** from above.

Lemma 2.5.11 Zorn's Lemma [7]. A partial order $\leqq$ has a maximal element on any $A \subset R^n$ in which every chain has an upper bound.

Definition 2.5.12 [8]. Let $(R^n, \leqq)$ be a preordered set. We say that the preorder $\leqq$ is **order separable in the sense of Cantor** if there exists a countable subset $Z \subset R^n$ such that whenever $\mathbf{x} \prec \mathbf{y}$, there exists $\mathbf{z} \in Z$ such that $\mathbf{y} \prec \mathbf{z} \prec \mathbf{x}$.

Theorem 2.5.13 [8]. Let $(R^n, \leqq)$ be a partially ordered set that is order separable in the sense of Cantor. Then there is a real-valued function f on R^n such that $\mathbf{y}_1 \prec \mathbf{y}_2$ implies $f(\mathbf{y}_1) < f(\mathbf{y}_2)$. Such a real-valued function f is called a **strictly monotone functional** on $(R^n, \leqq)$.

Remark 2.5.14 [9]. The order $\leq_C$ induced by a cone C in R^n is a partial order if and only if C is a pointed and convex cone.

Definition 2.5.15. Let C be a pointed cone in R^n. A **linear functional** l is a function mapping R^n into R, which satisfies the following property:

$$l(\alpha_1 \mathbf{y}_1 + \alpha_2 \mathbf{y}_2) = \alpha_1 l(\mathbf{y}_1) + \alpha_2 l(\mathbf{y}_2) \text{ for all } \alpha_1, \alpha_2 \in R \text{ and } \mathbf{y}_1, \mathbf{y}_2 \in R^n.$$

Moreover, a linear functional l is said to be **strictly positive** on C if $l(\mathbf{c}) > 0$ for all non-zero vectors $\mathbf{c} \in C$. The **dual cone** associated with C is defined as the collection of all strictly positive linear functionals on C and denoted by

$$C^+ = \left\{\text{Any linear functional } l : R^n \to R \text{ such that } l(\mathbf{c}) > 0 \text{ for all non-zero } \mathbf{c} \in C\right\}.$$

Example 2.5.16. Consider R^2 equipped with the order induced by the nonnegative orthant cone $R^2_{\geq} = \{(x, y) : x, y \geq 0\}$. We construct a linear functional $l : R^2 \to R$ given by $l(x, y) = x + y$ for all $x, y \in R$. Then, it follows that l is a linear functional such that $l(x, y) = x + y > 0$ for all non-zero $(x, y) \in R^2_{\geq}$. The existence of this linear functional shows that the dual cone $(R^2_{\geq})^+ \neq \phi$.

An important standard property of a strictly linear functional l on a pointed cone C is given in the next lemma, which is proved. It is followed by a well-known existence theorem for strictly linear functionals on C. In particular, the "pointed" property of a cone is required for a strictly positive linear functional on C to exist.

Lemma 2.5.17. Let C is a pointed cone in R^n and assume that $C^+ \neq \phi$. If $\mathbf{x}_0 <_C \mathbf{x}_1$ then $l(\mathbf{x}_0) < l(\mathbf{x}_1)$ for any $l \in C^+$.

Proof. Assume that $\mathbf{x}_0 <_C \mathbf{x}_1$. By definition it follows that $\mathbf{0} \neq \mathbf{x}_1 - \mathbf{x}_0 \in C$, and consequently we have $\mathbf{x}_0 - \mathbf{x}_1 \in -C$. Let $l \in C^+$. Thus we obtain $l(\mathbf{x}_1 - \mathbf{x}_0) > 0$, implying $-l(\mathbf{x}_1 - \mathbf{x}_0) < 0$. From the linear property of l, we get the following

$$l(\mathbf{x}_0) - l(\mathbf{x}_1) = l(\mathbf{x}_0 - \mathbf{x}_1) = l(-(\mathbf{x}_1 - \mathbf{x}_0)) = -l(\mathbf{x}_1 - \mathbf{x}_0) < 0,$$

which leads to the condition $l(\mathbf{x}_0) - l(\mathbf{x}_1) < 0$, i.e., $l(\mathbf{x}_0) < l(\mathbf{x}_1)$. ■

Theorem 2.5.18 (cone separation theorem [10]). Assume S_1, S_2 are closed convex cones in R^n such that $S_1 \bigcap S_2 = \{\mathbf{0}\}$, and denote the topological dual of R^n by $(R^n)'$. Suppose that the dual cone S_1^+ has nonempty interior in some topology τ which provides R^n as the dual of $(R^n)'$. Then there exists $s^+ \in (S_1^+)^0$ such that $-s^+ \in S_1^{\ +}$ and $s^+(\mathbf{s}_1) > 0$ for all non-zero vector $\mathbf{s}_1 \in S_1$.

Remark 2.5.19. If C is not a pointed cone, the set C^+ is empty.

Proof. Assume that C is not a pointed cone in R^n. Then, we have $C \bigcap -C \neq \{\mathbf{0}\}$. To obtain a contradiction, suppose that $C^+ \neq \phi$. Let $l \in C^+$ and a non-zero vector $\mathbf{c} \in C \bigcap -C$. Since $\mathbf{c} \in C$, we have $l(\mathbf{c}) > 0$. In addition, since $\mathbf{c} \in -C$, we obtain that $-\mathbf{c} \in C$ and $l(-\mathbf{c}) > 0$. But $-l(\mathbf{c}) = l(-\mathbf{c}) > 0$. It follows that $l(\mathbf{c}) < 0$, contradicting with $l \in C^+$. ■

According to Remark 2.5.19, the pointed cone is a necessary condition for existence of a strictly linear functional on *C*. (If $C^+ \neq \phi$, then *C* is a pointed cone.)

Remark 2.5.20. $L^+ = \phi$ where *L* is a lexicographic cone in R^n.

Proof. It suffices to prove for the case of $n = 2$. To obtain a contradiction, suppose that there exists a strictly linear functional on the lexicographic cone in R^2. We call that existing strictly linear functional as *f*. Since $(0,1), (1,0) \in L$, we must have $f(0,1), f(1,0) > 0$. Let $\alpha = f(1,0) > 0, \beta = f(0,1) > 0$. Then, we have that $(\frac{1}{\alpha}, \frac{1}{-\beta}) \in L$, thus by definition of a strictly linear functional, we obtain $f(\frac{1}{\alpha}, \frac{1}{-\beta}) > 0$. However, the linearity of *f* provides that

$$f(\frac{1}{\alpha}, \frac{1}{-\beta}) = \frac{1}{\alpha} f(1,0) - \frac{1}{\beta} f(0,1) = 1 - 1 = 0.$$

This contradicts the previous inequality. ■

Note that even though the lexicographic cone is a pointed convex cone, the associated dual cone is still an empty set. However, lexicographic optimization still has a scalar equivalence to be presented in Example 3.2.4.3.

2.6. Cone-Ordered Maximization

Definition 2.6.1. Let C be a pointed convex cone in R^n and $f : R^m \to R^n$. Suppose $A \subset R^m$ is a feasible region. Then **cone-ordered maximization,** or ***C*-maximization,** can be written as

$$C \max_{\mathbf{x}\in A} f(\mathbf{x})$$

The problem is to find all $\mathbf{x} \in A$ for which $f(\mathbf{x}) \in \max_C f(A)$, for $f(A) = \bigcup_{\mathbf{x}\in A} f(\mathbf{x})$ and $\max_C f(A) = \{\text{All non-dominated } f(\mathbf{x}) \text{ in } R^{\mathrm{n}} \text{ for } \mathbf{x} \in A\}$. Thus the problem is to find non-dominated f ($\mathbf{x}$) for all feasible solution $\mathbf{x} \in A$. General optimality conditions are found in [11].

Note that if a cone C is specified to be the **nonnegative orthant** $R^n_{\geq} = \{(c_1,...,c_n) : c_i \geq 0 \text{ for } i = 1,...,n\}$ for a given positive integer n, C-maximization becomes Pareto maximization with n objective functions. Pareto maximization is thus a special case of cone-ordered maximization with respect to the nonnegative orthant cone in R^n.

Example 2.6.2. The lexicographic cone in Example 2.5.3 can be used to define a certain cone-ordered maximization to be called lexicographic maximization. Recall that in Example 2.5.6, the set $B = \{(0,0),(0,1),(1,0),(1,1)\} \subset R^2$. If we define the objective function f to be the identity map on set B, the cone-ordered maximization with respect to the cone L becomes the lexicographic maximization

$$\underset{\mathbf{x}\in B}{\text{Leximax}}\, f(\mathbf{x})$$

The problem now is to find a feasible solution $\mathbf{x} \in B$ for which there is no other vector $\mathbf{y} \in B$ such that $\mathbf{x} <_L \mathbf{y}$. Notice that $(1,1)$ is the only non-dominated vector in B and therefore the solution to the lexicographic maximization.

2.7. Set-Valued Optimization

Definition 2.7.1. Let $F : R^m \to 2^{R^n}$ be a point-to-set map. An order in R^n is induced by a pointed convex cone C in R^n. We define a s**et-valued maximization** over a subset A of R^m as $\max_{\mathbf{x}\in A} F(\mathbf{x})$ as the problem of finding all feasible vector $\mathbf{x} \in A \subset R^m$ such that $F(\mathbf{x}) \bigcap \max_C F(A) \neq \phi,$ where $F(A) = \bigcup_{\mathbf{x}\in A} F(\mathbf{x}).$ Stated differently, the problem is to find all feasible $\mathbf{x}$ for which there exists $\mathbf{y} \in F(\mathbf{x})$ and $\mathbf{y} \in \max_C F(A).$ If F is indicated to be a point mapping to a singleton set, then set-valued maximization becomes cone-ordered maximization. Set-valued optimization was defined in [12], where general optimality conditions were given.

Example 2.7.2. Let $A = \{(x_1, x_2) \in R^2 : x_1 + x_2 \leq 1, x_1, x_2 \geq 0\} \subset R^2$, and $C = R^2_{\geq}$. Define $F(x_1, x_2) = [0, x_1] \times [0, x_2] \subset R^2$ for all $x_1, x_2 \in [0,1]$. Notice that the function F is a point-to-set map, and the problem $\max_{\mathbf{x}\in A} F(\mathbf{x})$ is a set-valued maximization. The set of solutions the set $\{(x_1, x_2) \in R^2 : x_1 + x_2 = 1, x_1, x_2 \geq 0\}$.

CHAPTER 3

EQUIVALENCE OF STANDARD OPTIMIZATION CRITERIA

The scalar equivalence of the standard optimization criteria of chapter 2 are now established. Equivalence proofs are given, and some examples are presented.

3.1. Background and Motivation

A multiple-objective optimization problem is typically solved by transforming the original problem into the scalar maximization of a real-valued function in which certain parameters are varied to give alternate solutions to the original multiple-objective problem. See [2], [3], [6], [13], and [14] for more details. However, the most frequently used such scalarizations of Pareto optimization require assumptions about the convexity or concavity of functions to guarantee that a scalarization exists and yields all solutions to the original Pareto problem. Because of this limitation, we say that a non-scalar optimization problem is scalarizable if and only if all solutions and only solutions of the non-scalar problem can be obtained by a possibly parameterized scalar maximization problem called its **equivalent scalarization**. In that case, the scalarization is said to be scalar equivalent to the original non-scalar problem. More generally, any two optimization problems are said to be **criteria equivalent** if all solutions and only solutions to one optimization problem are obtained as the solutions to the other, despite different notions of optimality. In another words, the set of solutions of one problem is the set of solutions to the other.

The notion of scalar equivalence stems then work of Corley [15] (see also [2] and [6]) in cone-ordered optimization, which includes Pareto and scalar optimization. This equivalent scalarization involves no more effort to solve than scalarizations requiring various convexity or concavity assumptions on the original problem. It is now known as a hybrid method [2] from its relation to the Corley hybrid fixed point theorems of [16].

In this chapter we show that any optimization problem has an equivalent scalarization (i.e., can be reduced to real-valued maximization) and that all standard optimization problems are criteria equivalent. In other words, a maximin problem is criteria equivalent to, say, a satisficing or lexicographic or Pareto problem. Any one type of problem can be solved as any other type directly or by the other's scalarization.

3.2. Equivalent Scalarizations of Standard Optimization Criteria

3.2.1. Maximin

In this section, a scalarization equivalence of a given maximin problem is presented. We denote $A1$ below as a given maximin problem, where $g(\mathbf{x}) = \min_{\mathbf{y}\in B(\mathbf{x})\subset R^m} f(\mathbf{x},\mathbf{y})$ for all $\mathbf{x} \in A$. The problem $A2$ is an obvious equivalence of $A1$ after introducing a real-value decision variable v to be the value of $g(\mathbf{x})$. We prove that $A3$ is a scalar equivalence of the given maximin $A1$. We note that in $A3$ the variable $\mathbf{y}$ in the set of constraints is not a decision variable but relates the constraints of $A2$ to the set $B(\mathbf{x})$ for each feasible point $\mathbf{x}$.

$$A1: \max_{\mathbf{x}\in A\subset R^n} g(\mathbf{x}) \qquad A2: \left\{ \begin{array}{lc} \max\limits_{\mathbf{x},v} & v \\ \text{s.t.} & v = g(\mathbf{x}) \\ & \mathbf{x}\in A\subset R^n, v\in R \end{array} \right\} \qquad A3: \left\{ \begin{array}{lc} \max\limits_{\mathbf{x},v} & v \\ \text{s.t.} & v \le f(\mathbf{x},\mathbf{y}), \forall \mathbf{y}\in B(\mathbf{x}) \\ & \mathbf{x}\in A\subset R^n, v\in R \end{array} \right\}$$

Lemma 3.2.1.1. If $(v_3^*, \mathbf{x}_3^*)$ is a solution to $A3$, then $v_3{}^* = f(\mathbf{x}_3{}^*, \mathbf{y}^*)$ for some $\mathbf{y}^* \in B(\mathbf{x}_3^*)$. That is, $v_3{}^* = g(\mathbf{x}_3{}^*)$ and $(v_3^*, \mathbf{x}_3^*)$ is also a feasible solution to $A2$.

Proof. Assume that $(v_3^*, \mathbf{x}_3^*)$ is a solution to $A3$. With the feasibility, we observe that $v_3^* \le f(\mathbf{x}_3^*, \mathbf{y})$ for all $\mathbf{y} \in B(\mathbf{x}_3^*)$. To obtain a contradiction, suppose $v_3^* < f(\mathbf{x}_3^*, \mathbf{y})$ for all $\mathbf{y} \in B(\mathbf{x}_3^*)$. By the assumption that $g(\mathbf{x}_3^*)$ exists, we have that $g(\mathbf{x}_3^*) = \min\limits_{\bar{\mathbf{y}}\in B(\mathbf{x}_3^*)} f(\mathbf{x}_3^*, \bar{\mathbf{y}})$ is a finite real number. Then, it follows that $\dfrac{v_3^* + \min\limits_{\bar{\mathbf{y}}\in B(\mathbf{x}_3^*)} f(\mathbf{x}_3^*, \bar{\mathbf{y}})}{2} \le f(\mathbf{x}_3^*, \mathbf{y})$ for all $\mathbf{y} \in B(\mathbf{x}_3^*)$, which implies that $\left(\dfrac{v_3^* + \min\limits_{\mathbf{y}\in B(\mathbf{x}_3^*)} f(\mathbf{x}_3^*, \bar{\mathbf{y}})}{2}, \mathbf{x}_3^*\right)$ is a feasible solution of $A3$. However, we also have that $v_3^* < \dfrac{v_3^* + \min\limits_{\mathbf{y}\in B} f(\mathbf{x}_3^*, \bar{\mathbf{y}})}{2}$, contradicting that v_3^* is the optimal objective value of $A3$. Thus, we can conclude that $v_3^* = f(\mathbf{x}_3^*, \mathbf{y}^*)$ for some $\mathbf{y}^* \in B(\mathbf{x}_3^*)$ and $v_3^* \le f(\mathbf{x}_3^*, \mathbf{y})$ for $\mathbf{y} \ne \mathbf{y}^*$, i.e., $v_3^* = g(\mathbf{x}_3^*)$. ■

Theorem 3.2.1.2. The point $(v^*, \mathbf{x}^*)$ is a solution to $A2$ if and only if $(v^*, \mathbf{x}^*)$ is a solution to $A3$.

Proof. Suppose $(v^*, \mathbf{x}^*)$ is an optimal solution to $A2$. By the definition of the function g, we have that $(v^*, \mathbf{x}^*)$ is a feasible solution to $A3$ as well. To obtain a contradiction, suppose that $(v^*, \mathbf{x}^*)$ is not an optimal solution to $A3$. Then there is another feasible solution $(v_3^*, \mathbf{x}_3^*)$ of $A3$ such that $v^* < v_3^* \le f(\mathbf{x}_3^*, \mathbf{y}), \forall \mathbf{y} \in B(\mathbf{x}_3^*)$.

Case 1: $v_3^* = f(\mathbf{x}_3^*, \mathbf{y}^*)$ for some $\mathbf{y}^* \in B(\mathbf{x}_3^*)$. In this case $v_3^* = g(\mathbf{x}_3^*)$, and hence $(v_3^*, \mathbf{x}_3^*)$ is a feasible solution to $A2$. However, we have that $v^* < v_3^*$, contradicting that $(v_2^*, \mathbf{x}_2^*)$ is an optimal solution to $A2$.

Case 2: $v_3^* < f(\mathbf{x}_3^*, \mathbf{y}), \forall \mathbf{y} \in B(\mathbf{x}_3^*)$. Since $g(\mathbf{x}) = \min\limits_{\mathbf{y} \in B(\mathbf{x}) \subset R^m} f(\mathbf{x}, \mathbf{y})$ is well-defined for all $\mathbf{x} \in R^n$, let $\hat{v} = g(\mathbf{x}_3^*)$. By the construction, we have that $(\hat{v}, \mathbf{x}_3^*)$ is a feasible solution to $A2$. However, we also obtain the condition $v^* < v_3^* < \hat{v}$, contradicting that $(v^*, \mathbf{x}^*)$ is an optimal solution to $A2$. Thus we conclude that $(v^*, \mathbf{x}^*)$ is an optimal solution to $A3$.

To establish the reverse implication, suppose $(v^*, \mathbf{x}^*)$ is an optimal solution to $A3$. By Lemma 3.2.1.1, we have that $(v^*, \mathbf{x}^*)$ is a feasible solution to $A2$. To obtain a contradiction, suppose that $(v^*, \mathbf{x}^*)$ is not an optimal to $A2$. Then there is another feasible solution $(v_2^*, \mathbf{x}_2^*)$ of $A2$ such that $v^* < v_2^*$. Since the feasible region of $A2$ is a subset of the feasible region of $A3$, it follows that $(v_2^*, \mathbf{x}_2^*)$ is a feasible solution to $A3$ such that $v^* < v_2^*$. This inequality is a contradiction because $(v^*, \mathbf{x}^*)$ is an optimal solution to $A3$. Thus we obtain that $(v^*, \mathbf{x}^*)$ is an optimal solution to $A2$. ■

The next two corollaries follow immediately.

Corollary 3.2.1.3. For $f : A \times B \to R$, an equivalent scalarization for the maximin problem $\max\limits_{\mathbf{x} \in A \subset R^n} \min\limits_{\mathbf{y} \in B \subset R^m} f(\mathbf{x}, \mathbf{y})$ is

$$\left\{ \begin{array}{ll} \max\limits_{\mathbf{x}, v} & v \\ \text{s.t.} & v \leq f(\mathbf{x}, \mathbf{y}), \forall \mathbf{y} \in B \\ & \mathbf{x} \in A \subset R^n, v \in R \end{array} \right\}.$$

Corollary 3.2.1.4. For $f_i : A \to R$ for all $i \in 1, \ldots, n$ for a fixed positive integer n, an equivalent scalarization for the discrete maximin problem, $\max\limits_{\mathbf{x} \in A \subset R^n} \min\limits_{i \in \{1, \ldots, n\}} \{ f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}) \}$ is

$$\left\{ \begin{array}{ll} \max\limits_{\mathbf{x}, v} & v \\ \text{s.t.} & v \leq f_1(\mathbf{x}) \\ & \vdots \\ & v \leq f_n(\mathbf{x}) \\ & \mathbf{x} \in A \subset R^n, v \in R \end{array} \right\}.$$

It should be noted that the scalar equivalence for the discrete maximin of Corollary 3.2.1.4 has been used extensively and referred to in [6], [17], and [18], among numerous places, with either no valid reference or else by referring to the proof of

Dantzig [19] for the linear case using the duality theory of linear programming. Proofs for the nonlinear and general maximin cases have not been found in the literature.

Example 3.2.1.5. Consider the following maximin problem.

$$\max_{x \in R} \min\{f_1(x) = x,\ f_2(x) = -x\}$$

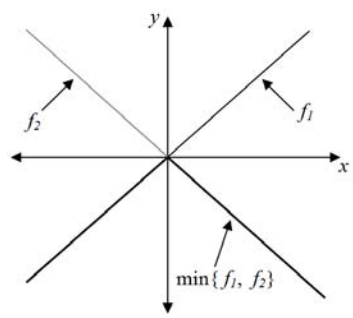

Figure 3.1 The graph of maximin Example 3.2.1.5.

It is analytically difficult to solve such a maximin problem directly as a maximization problem with a discontinuous objective function. Algorithms to do so have been developed in [17], [20], and [21]). However, a graphical interpretation of Figure 3.1 shows that $x^* = 0$ is the unique solution, as does the scalar equivalence

$$\left\{\begin{array}{ll} \max\limits_{v,x} & v \\ \text{s.t.} & v \le x \\ & v \le -x \\ & v, x \in R \end{array}\right\}.$$

3.2.2. Pareto Maximization

For Pareto optimization with *m*-dimensional objective functions, where *m* is a positive integer, Corley [15] provided a scalar equivalence to the problem without assumptions such as convexity or concavity. The Corley method, as it is called in [2] and [6], allows us to obtain all solutions and only solutions for a given Pareto via solving a family of parameterized scalar problems. We restate the scalar equivalence as follows.

$\left\{\begin{array}{ll}\max\limits_{\mathbf{x}\in A\subset R^n} & \lambda\cdot(f_1(\mathbf{x}),\dots,f_m(\mathbf{x})) \\ \text{s.t.} & (f_1(\mathbf{x}),\dots,f_m(\mathbf{x}))-\mathbf{y}\in C\end{array}\right\}$ for all $\mathbf{y}\in R^m$, where C is a pointed convex cone in R^m, and $\lambda\in C^+=\{\lambda\in R^m : \lambda\cdot\mathbf{c}>0, \forall\mathbf{c}\in C\setminus\{\mathbf{0}\}\}$ for given positive integers *n*, and *m*.

Example 3.2.2.1. Consider the following Pareto problem

$$\left\{\begin{array}{ll}\operatorname{Vmax}\limits_{x_1,x_2} & (x_1,x_2) \\ \text{s.t.} & x_1+x_2\le 1 \\ & x_1,x_2\ge 0\end{array}\right\}.$$

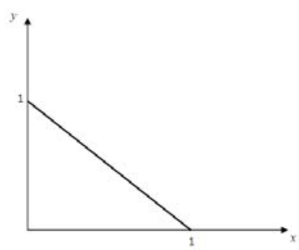

Figure 3.2 Pareto frontier of Example 3.2.2.1.

Figure 3.2 shows the set of all solutions for the Pareto problem as well as the Pareto frontier. Again, our approach is to solve the Pareto maximization by solving its equivalent scalarization

$$P(y_1, y_2): \left\{ \begin{array}{ll} \max\limits_{x_1, x_2} & x_1 + x_2 \\ \text{s.t.} & x_1 \geq y_1 \\ & x_2 \geq y_2 \\ & x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array} \right\}, \text{ for all } y_1, y_2 \in R.$$

To illustrate the parameterization, choose $y_1 = \frac{1}{2}$, and $y_2 = \frac{1}{2}$. Then solving the problem $P(\frac{1}{2}, \frac{1}{2})$ gives $(x^*, y^*) = (\frac{1}{2}, \frac{1}{2})$. In theory we can similarly obtain all solutions of the Pareto problem, by solving $P(y_1, y_2)$ for all feasible choices of y_1 and y_2. In practice, a reasonable number of such solutions will approximate the Pareto frontier.

Remark 3.2.2.2. Refer to Example 3.2.2.1, where an optimal solution of the scalarization problem $P(y_1, y_2)$ for parameters y_1 and y_2 is only one solution of the given Pareto maximization. Any other solution of the Pareto maximin problem can be also achieved by solving $P(y_1, y_2)$ for suitable parameters y_1 and y_2. In summary, we theoretically obtain all solutions and only solutions for the Pareto maximization problem by solving a collection of the problems $P(y_1, y_2)$ for all possible values of y_1 and y_2.

3.2.3. Set-Valued Maximization

We next establish a scalar equivalence for set-valued maximization. Denote the general set-valued maximization as $B1$. A scalar equivalence is presented in $B2$ for a convex, pointed cone $C \subset R^n$.

$$B1: \quad \max_{\mathbf{x} \in A \subset R^m} F(\mathbf{x}) \qquad B2(\mathbf{w}): \left\{ \begin{array}{ll} \max\limits_{\mathbf{x},\mathbf{t}} & l(\mathbf{t}) \\ & \mathbf{t} \in F(\mathbf{x}) \\ & \mathbf{t} - \mathbf{w} \in C \\ & \mathbf{x} \in A, \mathbf{t} \in R^n \end{array} \right\} \text{ for } l \in C^+ \text{and all } \mathbf{w} \in R^n.$$

To ensure the existence of a linear functional l in the dual cone C^+, we usually assume that C is pointed and satisfies the conditions of Theorem 2.5.18 because of Remark 2.5.19.

Lemma 3.2.3.1. If the problem $B2(\mathbf{w})$ has a solution for some $\mathbf{w} \in R^n$, the problem $B1$ has a solution as well.

Proof. Suppose the problem $B2(\mathbf{w})$, where $\mathbf{w} \in R^n$, has a solution. Let $(\mathbf{x}_2, \mathbf{t}_2)$ be a solution of $B2(\mathbf{w})$. By feasibility, we have $\mathbf{t}_2 \in F(\mathbf{x}_2)$ and $\mathbf{w} \leq_C \mathbf{t}_2$. To obtain a contradiction, suppose that the set $\max F(A)$ is an empty set. Then there exists $\mathbf{x}_1 \in A$ and $\mathbf{t}_1 \in F(\mathbf{x}_1)$ for which $\mathbf{t}_2 <_C \mathbf{t}_1$, otherwise $\mathbf{t}_2 \in \max F(A)$. From the convexity of C, we have that $\mathbf{w} \leq_C \mathbf{t}_2$ and $\mathbf{t}_2 <_C \mathbf{t}_1$ implies $\mathbf{w} \leq_C \mathbf{t}_1$, so $(\mathbf{x}_1, \mathbf{t}_1)$ is feasible to $B2(\mathbf{w})$. However, since $\mathbf{t}_2 <_C \mathbf{t}_1$, by Lemma 2.5.17 we have $l(\mathbf{t}_2) < l(\mathbf{t}_1)$ in contradiction to the optimality of $(\mathbf{x}_2, \mathbf{t}_2)$. ■

Theorem 3.2.3.2. If $\mathbf{x}_1$ solves $B1$, then $(\mathbf{x}_1, \mathbf{t}_1)$ is a solution of $B2(\mathbf{w})$ for $\mathbf{w} = \mathbf{t}_1 \in F(\mathbf{x}_1) \cap \max F(A).$

Proof. Assume that $\mathbf{x}_1$ solves $P1$. Then, there exists $\mathbf{t}_1 \in F(\mathbf{x}_1) \cap \max_C F(A).$ We observe that $(\mathbf{x}_1, \mathbf{t}_1)$ is a feasible solution of $B2(\mathbf{t}_1).$ Now let $(\mathbf{x}_2, \mathbf{t}_2)$ be any feasible solution to $B2(\mathbf{t}_1).$ Therefore it follows that $\mathbf{t}_2 \in F(\mathbf{x}_2) \subset F(A)$ and $\mathbf{t}_2 - \mathbf{t}_1 \in C.$ However, this conclusion contradicts with $\mathbf{t}_1 \in \max_C F(A)$ unless $\mathbf{t}_2 = \mathbf{t}_1.$ Thus, every feasible solution of $B2(\mathbf{t}_1)$ is also a solution. Since $(\mathbf{x}_1, \mathbf{t}_1)$ is a feasible solution of $B2(\mathbf{t}_1),$ then, it solves $B2(\mathbf{t}_1).$ ■

Theorem 3.2.3.3. If $(\mathbf{x}_2, \mathbf{t}_2)$ solves $B2(\mathbf{w})$ for $\mathbf{w} \in R^n$, then $\mathbf{x}_2$ is a solution of $B1$.

Proof. Assume that $(\mathbf{x}_2, \mathbf{t}_2)$ solves $B2(\mathbf{w})$ for $\mathbf{w} \in R^n.$ To obtain a contradiction, suppose that $\mathbf{x}_2$ does not solve $B1$, i.e., $F(\mathbf{x}_2) \cap \max F(A) = \phi.$ By Lemma 3.2.1.1, there exist a solution $\mathbf{x}_1$ of $B1$ and a vector $\mathbf{t}_1 \in F(\mathbf{x}_1)$ such that $\mathbf{t}_1 - \mathbf{t}_2 \in C \setminus \{\mathbf{0}\}.$ Since $(\mathbf{x}_2, \mathbf{t}_2)$ is feasible to $B2(\mathbf{w}),$ we have $\mathbf{t}_2 - \mathbf{w} \in C.$ It follows that $\mathbf{t}_1 - \mathbf{w} \in C$ because of the convexity of C, so $(\mathbf{x}_1, \mathbf{t}_1)$ is feasible to $B2(\mathbf{w}).$ However, by Lemma 2.5.17, $l(\mathbf{t}_2) < l(\mathbf{t}_1)$ in contradiction to the optimality of $(\mathbf{x}_2, \mathbf{t}_2).$ ■

Example 3.2.3.4. Recall the set-valued maximization problem in Example 2.7.2. with the problem

$$\max_{\mathbf{x} \in A} F(\mathbf{x}), \text{ where } F(x_1, x_2) = [0, x_1] \times [0, x_2] \subset R^2 \text{ for } x_1, x_2 \in [0,1],$$

$$A = \{(x_1, x_2) : x_1 + x_2 \le 1, x_1, x_2 \ge 0\} \subset R^2, \text{ and } C = R^2_{\ge}.$$

The equivalent scalarization for this problem is

$$B(w_1, w_2) : \left\{ \begin{array}{ll} \max\limits_{x_1, x_2, t_1, t_2} & l(\mathbf{t}) = t_1 + t_2 \\ \text{s.t.} & (t_1, t_2) \in F(x_1, x_2) \\ & t_1 \ge w_1 \\ & t_2 \ge w_2 \\ & x_1 + x_2 \le 1 \\ & t_1, t_2, x_1, x_2 \in R \end{array} \right\} \text{ for all } w_1, w_2 \in R.$$

In order to obtain all solutions and only solutions of the set-valued maximization, we can theoretically solve the problem $B(w_1, w_2)$ for all feasible choices of w_1, w_2. For $w_1 = \frac{1}{3}, w_2 = \frac{2}{3}$, the problem $P(\frac{1}{3}, \frac{2}{3})$ gives that $(x_1^* = \frac{1}{3}, x_2^* = \frac{2}{3}, t_1^* = \frac{1}{3}, t_2^* = \frac{2}{3})$ is a solution for the set-valued problem. Again, in practice a large number of such solutions will approximate the Pareto frontier.

Remark 3.2.3.5. An alternate scalarization for set-valued maximization has been proposed in [22]. However, the approach there requires assumptions regarding convexity and concavity. In addition, another scalarization to set-valued optimization is proposed in [23], but only certain solutions can be obtained.

3.2.4. Cone-Ordered Maximization

Let C be a convex cone in R^n. The cone-ordered maximization is stated as $C1$. We propose a scalar equivalence to $C1$ and denote it as $C2$.

$$C1:\quad C\max_{\mathbf{x}\in A\subset R^n} f(\mathbf{x}) \qquad\qquad C2(\mathbf{w}):\ \begin{Bmatrix} \max\limits_{\mathbf{x}} & l(f(\mathbf{x})) \\ \text{s.t.} & f(\mathbf{x})-\mathbf{w}\in C \\ & \mathbf{x}\in A\subset R^n \end{Bmatrix}, \text{where } l\in C^{+} \text{ for all } \mathbf{w}\in R^n$$

Here again, in addition to the assumption that the cone C is pointed and convex, we must usually assume that the cone C satisfies the conditions to Theorem 2.5.18 to ensure the existence of a linear functional l in the dual cone C^{+}.

Theorem 3.2.4.1. If $\mathbf{x}_1$ is a solution of $C1$, then $\mathbf{x}_1$ solves $C2(\mathbf{w})$ for $\mathbf{w} = f(\mathbf{x}_1)$.

Proof. Assume that $\mathbf{x}_1$ solves $C1$. By the choice $\mathbf{w} = f(\mathbf{x}_1)$, we know that $\mathbf{x}_1$ is a feasible solution to $C2(f(\mathbf{x}_1))$. Let $\mathbf{x}_2$ be any feasible solution to $C2(f(\mathbf{x}_1))$. We therefore have $f(\mathbf{x}_2) - f(\mathbf{x}_1) \in C$. Since $\mathbf{x}_1$ solves $C1$, the only possibility is that $f(\mathbf{x}_2) = f(\mathbf{x}_1)$, so every feasible point of $C2(f(\mathbf{x}_1))$ is a solution as well. Since $\mathbf{x}_1$ is a feasible to $C2(f(\mathbf{x}_1))$, it solves $C2(f(\mathbf{x}_1))$. ■

Theorem 3.2.4.2. If $\mathbf{x}_2$ solves $C2(\mathbf{w})$ for $\mathbf{w}\in R^n$, then $\mathbf{x}_2$ is a solution of $C1$.

Proof. Assume that $\mathbf{x}_2$ solves $C2$ for some $\mathbf{w}$. To obtain a contradiction, suppose that $\mathbf{x}_2$ does not solve $C1$. Then there exists $\mathbf{x}_1 \in A$ such that $f(\mathbf{x}_2) <_C f(\mathbf{x}_1)$, i.e., $f(\mathbf{x}_1) - f(\mathbf{x}_2) \in C\setminus\{\mathbf{0}\}$. It follows that $\mathbf{x}_1$ is a feasible solution of $C2(\mathbf{w})$. Since l is a strictly positive linear functional on C, we have $l(f(\mathbf{x}_1) - f(\mathbf{x}_2)) > 0$. The linearity of l now yields that $l(f(\mathbf{x}_1)) - l(f(\mathbf{x}_2)) = l(f(\mathbf{x}_1) - f(\mathbf{x}_2)) > 0$. Thus $l(f(\mathbf{x}_1)) > l(f(\mathbf{x}_2))$ in contradiction to the optimality of $\mathbf{x}_2$. ■

As mentioned in Remark 2.5.20, the dual cone $L^{+} = \phi$ for the lexicographic cone L in R^{n}. Thus we cannot we cannot use Theorems 3.2.4.1 and 3.2.4.2 to construct an equivalent scalarization for lexicographic optimization. However, lexicographic maximization can be scalarizable via another way as illustrated in the following example.

Example 3.2.4.3 (Scalarization for lexicographic maximization).

Consider the lexicographic maximization

$$\left\{\begin{array}{ll} \operatorname*{Leximax}_{\mathbf{x}} & (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in A \subset R^n \end{array}\right\} \text{ where } f_i : R^n \to R \text{ for } i = 1,2,3.$$

This problem can be solved in stages corresponding to the objective functions.

Step 1: Solve $\max_{\mathbf{x} \in A} f_1(\mathbf{x})$ and denote f_1^* the optimal objective value of this problem.

Step 2: Solve $\left\{\begin{array}{ll} \max_{\mathbf{x}} & f_2(\mathbf{x}) \\ \text{s.t.} & f_1(\mathbf{x}) = f_1^* \\ & \mathbf{x} \in A \end{array}\right\}$ and denote f_2^* the optimal objective value of this problem.

Step 3: Solve $\left\{\begin{array}{ll} \max_{\mathbf{x}} & f_3(\mathbf{x}) \\ \text{s.t.} & f_1(\mathbf{x}) = f_1^* \\ & f_2(\mathbf{x}) = f_2^* \\ & \mathbf{x} \in A \end{array}\right\}$.

Solutions from the scalar problem in Step 3 are solutions for the given lexicographic maximization and vice versa. Thus the maximization problem in Step 3 is an equivalent multiple-stage scalarization for the given lexicographic maximization. The sequence of steps is critical. While the above three steps involve real-valued maximizations, we have defined a scalar equivalence as a single-stage scalarization. In section 3.3.7, we collapse the above three stages into a Pareto maximization, which then yields a single-stage scalar equivalence for lexicographic maximization. Details about a more general lexicographic problem can be found in [2].

We summarize our previous results by noting that maximin problems, Pareto maximization, cone-ordered maximization, and set-valued maximization all have equivalent scalarizations. These results are summarized in Figure 3.3.

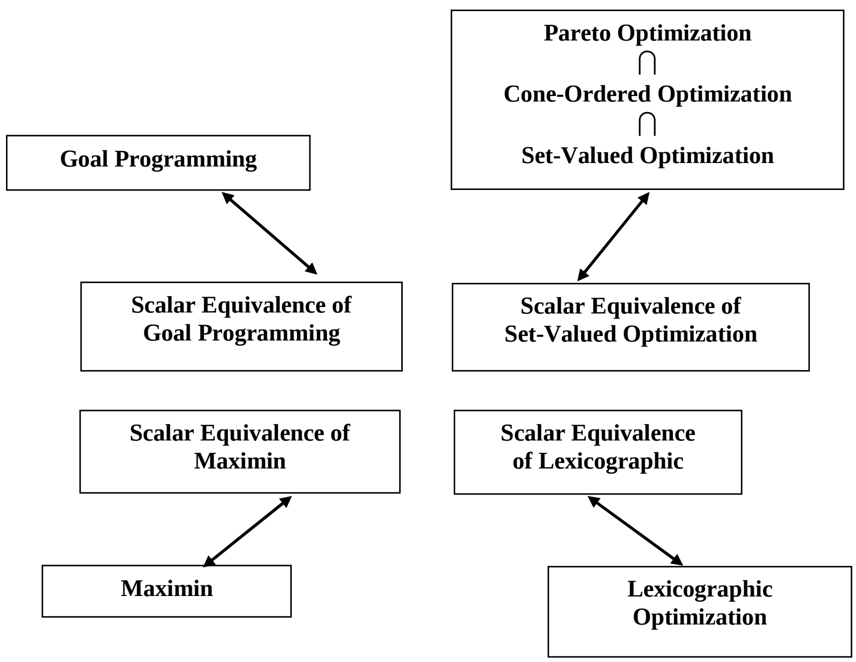

Figure 3.3 Scalar equivalence diagram.

The results of sections 3.2.1 - 3.2.4 demonstrate that all standard non-scalar optimization criteria can be scalarizable. We next claim that the equivalent scalarization of a standard criterion can be formulated in terms of the equivalent scalarization of any other criterion. Rather than confirm all the cases of this claim, we illustrate the proofs in section 3.2.5 by showing the equivalent scalarization of maximin is equivalent to the equivalent scalarization of Pareto maximization.

3.2.5. An Example of the Scalar Equivalence of Criteria

We now indicate how the equivalence between two different criteria can be established via their equivalent scalarizations. Again, however, we show this fact only for the equivalent scalarizations of maximin problems and Pareto maximization.

3.2.5.1. Maximin Scalarization as Pareto Scalarization

Let the problem $D1$ below be the equivalent scalarization to a given maximin problem.

$$D1: \left\{ \begin{array}{ll} \max\limits_{\mathbf{x},v} & v \\ \text{s.t.} & v \le f_1(\mathbf{x}) \\ & \vdots \\ & v \le f_n(\mathbf{x}) \\ & \mathbf{x} \in A \subset R^n, v \in R \end{array} \right\} \text{ where } f_i : R^n \to R \text{ for } i = 1,\dots,n .$$

We write $D1$ as the equivalent scalarization $D2$ below of Pareto maximization. For $i = 1,\dots,n,$ let $g_i(\mathbf{x},v) = \dfrac{v}{\lambda_i}$ for all $\mathbf{x} \in A \subset R^n$ and $v \in R,$ where $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i = 1.$

Define $A_1 = \{(\mathbf{x},v) \in R^{n+1} : (\mathbf{x},v) \text{ is a feasible solution to } D1\}$, so the set A_1 is exactly the feasible region of $D1$. Now an equivalent scalarization for Pareto maximization of the n-objective function of $(g_1,\dots,g_n)$ is given below as $D2$.

$$
D2(y_1,\ldots,y_n):\left\{\begin{array}{ll}\max\limits_{\mathbf{x},v} & \lambda_1 g_1(\mathbf{x},v)+\ldots+\lambda_n g_n(\mathbf{x},v)\\ \text{s.t.} & g_1(\mathbf{x},v)=\dfrac{v}{\lambda_1}\ge y_1\\ & \vdots\\ & g_n(\mathbf{x},v)=\dfrac{v}{\lambda_n}\ge y_n\\ & (\mathbf{x},v)\in A_1\end{array}\right\}\text{ for all } y_1,\ldots,y_n\in R.
$$

Theorem 3.2.5.1.1. If $(\mathbf{x}^*,v^*)$ solves $D1$, then $(\mathbf{x}^*,v^*)$ solves $D2$ for $y_i=\dfrac{v^*}{n\lambda_i}$ for all $i=1,\ldots,n.$

Proof. Assume that $(\mathbf{x}^*,v^*)$ solves $D1$. According to the feasibility of $(\mathbf{x}^*,v^*)$, we also have $(\mathbf{x}^*,v^*)\in A_1$. Moreover, we have $g_i(\mathbf{x}^*,v^*)=\dfrac{v^*}{n\lambda}=y_i$ for all $i=1,\ldots,n.$ This conclusion implies that $(\mathbf{x}^*,v^*)$ is a feasible solution to $D2(y_1,\ldots,y_n)$. To obtain a contradiction, suppose that $(\mathbf{x}^*,v^*)$ does not solve $D2(y_1,\ldots,y_n)$. Then there exists another feasible solution $(\mathbf{x}_1,\mathbf{v}_1)$ to $D2(y_1,\ldots,y_n)$ such that

$$
\lambda_1 g_1(\mathbf{x}_1,v_1)+\ldots+\lambda_n g_n(\mathbf{x}_1,v_1)=\lambda_1\cdot\frac{v_1}{n\lambda_1}+\ldots+\lambda_n\cdot\frac{v_1}{n\lambda_n}=v_1>v^*.
$$

Since $(\mathbf{x}_1,v_1)\in A_1$, then $(\mathbf{x}_1,v_1)$ is feasible to $D1$. But this contradicts that $(\mathbf{x}^*,v^*)$ is an optimal solution of $D1$. ■

Theorem 3.2.5.1.2. If $(\mathbf{x}^*,v^*)$ solves $D2(y_1,\ldots,y_n)$ for parameters $y_1,\ldots,y_n\in R$, then $(\mathbf{x}^*,v^*)$ solves $D1$.

Proof. Assume that $(\mathbf{x}^*, v^*)$ solves $D2(y_1,..., y_n)$. As a member of A_1, the solution $(\mathbf{x}^*, v^*)$ of $D2(y_1,..., y_n)$ is also a feasible solution to $D1$. To obtain a contradiction, suppose that $(\mathbf{x}^*, v^*)$ does not solve $D1$. Then there exists another feasible solution $(\mathbf{x}_1, \mathbf{v}_1)$ to $D1$ such that $v^* < v^1$. Next we will show that $(\mathbf{x}_1, \mathbf{v}_1)$ is a feasible solution to $D2(y_1,..., y_n)$. With the feasibility to $D1$, we have $(\mathbf{x}_1, v_1) \in A_1$. In addition, because $v^* < v^1$ the conditions

$$\begin{cases} \dfrac{v_1}{n\lambda_1} \geq \dfrac{v^*}{n\lambda_1} \geq y_1 \\ \quad\vdots \\ \dfrac{v_1}{n\lambda_n} \geq \dfrac{v^*}{n\lambda_n} \geq y_n \end{cases}$$

hold. Thus $(\mathbf{x}_1, \mathbf{v}_1)$ is a feasible solution to $D2(y_1,..., y_n)$. However, we also have

$$\lambda_1 g_1(\mathbf{x}_1, v_1) + ... + \lambda_n g_n(\mathbf{x}_1, v_1) = \lambda_1 \cdot \frac{v_1}{n\lambda_1} + ... + \lambda_n \cdot \frac{v_1}{n\lambda_n} = v_1 > v^*$$

in contradiction to the optimality of $(\mathbf{x}^*, v^*)$. ■

3.2.5.2 Pareto Scalarization as Maximin Scalarization

Let $f_i : R^m \to R$ for $i = 1,..., n,$ where n is a positive integer. We write $E1$ below as the equivalent scalarization of [15] for Pareto maximization.

$$E1(y_1,..., y_n): \left\{ \begin{array}{ll} \max\limits_{\mathbf{x}} & \lambda_1 f_1(\mathbf{x}) + ... + \lambda_n f_n(\mathbf{x}) \\ \text{s.t.} & f_1(\mathbf{x}) \geq y_1 \\ & \vdots \\ & f_n(\mathbf{x}) \geq y_n \\ & \mathbf{x} \in A_1 \subset R^m \end{array} \right\} \text{ for all } y_1,..., y_n \in R.$$

Define $A_2(y_1,...,y_n) = \{\mathbf{x} \in A_1 : \mathbf{x} \text{ is a feasible solution to } E1(y_1,...,y_n)\}$ for $y_1,...,y_n \in R$. Obviously, the set $A_2(y_1,...,y_n)$ is the set of feasible solutions of $E1(y_1,...,y_n)$ for $y_1,...,y_n \in R$. Consider the following n functions

$$g_1(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i f_i(\mathbf{x}),\ g_2(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i f_i(\mathbf{x}) + 2,\ \ldots,\ g_n(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i f_i(\mathbf{x}) + n,$$

$$\text{for all } \mathbf{x} \in A(y_1,...,y_n).$$

Notice that $g_1(\mathbf{x}) < g_2(\mathbf{x}) < ... < g_{n-1}(\mathbf{x}) < g_n(\mathbf{x})$ for all $\mathbf{x} \in A(y_1,...,y_n)$. Write the equivalent scalarization $E2(y_1,...,y_n)$ below of the maximin of the $g_1(\mathbf{x}),...,g_n(\mathbf{x})$.

$$E2(y_1,...,y_n): \left\{ \begin{array}{lc} \max\limits_{\mathbf{x},v} & v \\ \text{s.t.} & v \le g_1(\mathbf{x}) \\ & \vdots \\ & v \le g_n(\mathbf{x}) \\ & \mathbf{x} \in A_2(y_1,...,y_n), v \in R \end{array} \right\} \text{ for all } y_1,...,y_n \in R$$

The following result is true by definition.

Theorem 3.2.5.2.1. If $\mathbf{x}^*$ solves $E1(y_1,...,y_n)$ for parameters $y_1,...,y_n \in R$, then $(\mathbf{x}^*, v^* = g_1(\mathbf{x}^*))$ solves $E2(y_1,...,y_n)$. Moreover, if (x^*, v^*) solves $E2(y_1,...,y_n)$ for parameters $y_1,...,y_n \in R$, then $\mathbf{x}^*$ solves $E1(y_1,...,y_n)$.

In the following Sections 3.3, we directly establish equivalences between the standard optimization criteria.

3.3. Direct Equivalence between Two Different Criteria

We establish the equivalence between discrete maximin problem and Pareto maximization problem, continuous maximin problem and Pareto maximization, goal programming and Pareto maximization, Lexicographic maximization and Pareto maximization as well as set-valued maximization and cone-ordered maximization.

3.3.1. Maximin as Pareto Maximization

Let $H1$ denote a given maximin problem, where $g(\mathbf{x}) = \min\{f_1(\mathbf{x}), f_2(\mathbf{x}),..., f_n(\mathbf{x})\}$ for all $\mathbf{x} \in R^m$ and $f_i : R^m \to R$ for all $i = 1,...,n$ for a given positive integer n.

$$H1\text{: } \max_{\mathbf{x}\in A\subset R^m} g(\mathbf{x}) \qquad H2\text{: }\left\{\begin{array}{ll} \max\limits_{v,\mathbf{x}} & v \\ \text{s.t.} & v = g(\mathbf{x}) \\ & \mathbf{x} \in A, v \in R \end{array}\right\} \qquad H3\text{: }\left\{\begin{array}{ll} \operatorname{Vmax}\limits_{\mathbf{x},v} & (v,v,...,v) \\ \text{s.t.} & v \le f_1(\mathbf{x}) \\ & \vdots \\ & v \le f_n(\mathbf{x}) \\ & \mathbf{x} \in A, v \in R \end{array}\right\}$$

The problem $H2$ is obviously equivalent to $H1$. Moreover, $H3$ is obviously equivalent to $H2$ because the objective function of $H3$ is just a replication of the objective function of $H2$. Obviously any single optimization of a real-valued function can be transformed an equivalent Pareto optimization in this way.

Example 3.3.1.1. Recall the maximin problem in Example 3.2.15. It was $\max\limits_{x\in R} \min\{ f_1(x) = x, f_2(x) = -x\}$, with solution is $x^* = 0$. We can solve this same problem as the Pareto problem

$$\begin{Bmatrix} \underset{x,v}{\text{Vmax}} & (v,v) \\ \text{s.t.} & v \le x \\ & v \le -x \\ & x, v \in R \end{Bmatrix}$$

to obtain $x^* = 0$ again.

3.3.2. Pareto Maximization as Maximin

Consider the following problems $K1$ and $K2$:

$$K1\text{: } \underset{\mathbf{x} \in A \subset R^m}{\text{Vmax}} \quad (f_1(\mathbf{x}), \cdots, f_n(\mathbf{x})) \qquad K2\text{:} \begin{Bmatrix} \underset{\mathbf{x}}{\max} & f_1(\mathbf{x}) + \ldots + f_n(\mathbf{x}) \\ \text{s.t.} & f_1(\mathbf{x}) \ge y_1 \\ & \vdots \\ & f_n(\mathbf{x}) \ge y_n \\ & \mathbf{x} \in A \end{Bmatrix}.$$

for all $y_1, y_2, \ldots, y_n \in R$.

$K1$ is a given Pareto maximization problem, and the problem $K2$ represents an equivalent scalarization as in [15]. Consider now the maximin equivalence $K3$ of $K2$

$$K3\text{:} \begin{Bmatrix} \underset{\mathbf{x}}{\max} \min & \left\{ \sum_{j=1}^{n} f_j(\mathbf{x}), \sum_{j=1}^{n} f_j(\mathbf{x}) + 1, \ldots, \sum_{j=1}^{n} f_j(\mathbf{x}) + (n-1) \right\} \\ \text{s.t.} & f_1(\mathbf{x}) \ge y_1 \\ & \vdots \\ & f_n(\mathbf{x}) \ge y_n \\ & \mathbf{x} \in A \end{Bmatrix} \text{ for all } y_1, y_2, \ldots, y_n \in R.$$

Since the value of $\min\left\{ \sum_{j=1}^{n} f_j(\mathbf{x}), \sum_{j=1}^{n} f_j(\mathbf{x}) + 2, \ldots, \sum_{j=1}^{n} f_j(\mathbf{x}) + n \right\} = \sum_{j=1}^{n} f_j(\mathbf{x})$, problem $K3$ is obviously equivalent to $K2$. Therefore, we can solve $K3$ instead of $K2$. Thus Pareto maximization and maximin problem are equivalent. In Section 3.3.4. and 3.3.5., we consider the more general maximin formulation.

Example 3.3.2.1. Recall the Pareto maximization in Example 3.2.2.1.

$$\left\{\begin{array}{ll}\operatorname{Vmax}\limits_{x_1,x_2} & (x_1,x_2)\\ \text{s.t.} & x_1+x_2\le 1\\ & x_1,x_2\ge 0\end{array}\right\}.$$

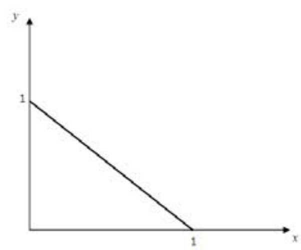

Figure 3.4 The set of Pareto maxima.

Figure 3.4 shows the set of all solutions of the Pareto maximization. By the above construction, we are also able to solve the Pareto maximization with the maximin problem

$$K(y_1,y_2):\left\{\begin{array}{ll}\max\limits_{x_1,x_2}\min & \{x_1+x_2,(x_1+x_2)+1\}\\ \text{s.t.} & x_1\ge y_1\\ & x_2\ge y_2\\ & x_1+x_2\le 1\\ & x_1,x_2\ge 0\end{array}\right\}, \text{ for all } y_1,y_2\in R.$$

For example, select $y_1=\frac{1}{3}, y_2=\frac{2}{3}$ and solve the associated problem corresponding to these parameters. Then $(x_1{*}=\frac{1}{3}, x_2{*}=\frac{2}{3})$ solves the problem. This solution is only a single solution of the Pareto maximization problem. To obtain all solutions and only solutions of the Pareto problem, we can theoretically solve the maximin problem $K(y_1, y_2)$ for all choices of y_1 and y_2. In practice, again, we need only solve a sufficient number to illustrate the Pareto frontier.

3.3.3. General Maximin as Pareto Maximization

Consider the problems $L1$, $L2$, and $L3$, where

$$L1\text{: } \max_{\mathbf{x}\in A\subset R^n} g(\mathbf{x}) \qquad L2\text{: }\left\{\begin{array}{lc} \max\limits_{v,\mathbf{x}} & v \\ \text{s.t.} & v = g(\mathbf{x}) \\ & \mathbf{x}\in A, v\in R \end{array}\right\} \qquad L3\text{: }\left\{\begin{array}{lc} \operatorname{Vmax}\limits_{v,\mathbf{x}} & (v,v) \\ \text{s.t.} & v \le f(\mathbf{x},\mathbf{y}), \forall \mathbf{y}\in B(x) \\ & \mathbf{x}\in A, v\in R \end{array}\right\}.$$

Here $L1$ is the general maximin problem, $g(\mathbf{x}) = \min\limits_{\mathbf{y}\in B(\mathbf{x})\subset Y} f(\mathbf{x},\mathbf{y})$ for all $\mathbf{x}\in R^n$, and $f: R^n \times R^m \to R.$ Problem $L2$ is obviously equivalent to $L1$. But $L3$ is also obviously equivalent to $L2$ because the objective function of $L3$ is just duplicating the objective function of $L2$ into a two-objective function Pareto maximization.

3.3.4. Pareto Maximization as General Maximin

For the Pareto problem $M1$, $M2$ represents its scalar equivalence from [15].

$$M1\text{: } \operatorname{Vmax}_{\mathbf{x}\in A\subset R^n} \ (f_1(\mathbf{x}),\cdots, f_n(\mathbf{x})) \qquad M2\text{: }\left\{\begin{array}{lc} \max\limits_{\mathbf{x}} & f_1(\mathbf{x}) + \ldots + f_n(\mathbf{x}) \\ \text{s.t.} & f_1(\mathbf{x}) \ge y_1 \\ & \vdots \\ & f_n(\mathbf{x}) \ge y_n \\ & \mathbf{x}\in A\subset R^n \end{array}\right\} \text{for } y_1, y_2, \ldots, y_n \in R.$$

Define $R(\mathbf{x}) = R$ for all $\mathbf{x} \in A$ and $g(\mathbf{x}, y) = \sum_{j=1}^{n} f_j(\mathbf{x}) + y^2$ for $x \in A$ and $y \in R(\mathbf{x}) = R$. Then the maximin problem $M3$ below is equivalent to $M2$.

$$M3: \left\{ \begin{array}{ll} \max\limits_{\mathbf{x} \in A} \min\limits_{y \in R(\mathbf{x})} & g(\mathbf{x}, y) \\ \text{s.t.} & f_1(\mathbf{x}) \geq y_1 \\ & \vdots \\ & f_n(\mathbf{x}) \geq y_n \\ & \mathbf{x} \in A \end{array} \right\} \text{ for all } y_1, y_2, ..., y_n \in R.$$

It is obvious that for each $\mathbf{x} \in A$, $\min\limits_{y \in R}(\sum_{j=1}^{n} f_j(\mathbf{x}) + y^2) = \sum_{j=1}^{n} f_j(\mathbf{x})$, so the equivalence follows. We thus conclude that Pareto maximization and general maximin optimization are equivalent.

3.3.5. Goal Programming as Pareto Maximization

It suffices to show that we can solve any given Pareto maximization with a two-objective function by solving a goal programming. For a Pareto maximization with three or more objective functions, the same technique applies as we show by example.

The problem $N1$ below denotes a given Pareto maximization with two objective functions and $N2$ the equivalence of $N1$ in term of goal programming.

$$N1: \ \operatorname*{Vmax}_{\mathbf{x} \in A}(f_1(\mathbf{x}), f_2(\mathbf{x}))$$

$$N2:\left\{\begin{array}{ll}\underset{\mathbf{x},s_1^+,s_2^-,s_3^+,s_4^-}{\text{Vmin}} & (s_1^-,s_1^+,s_2^-,s_2^+)\\ \text{s.t.} & g_1(\mathbf{x})-s_1^+=0\\ & g_2(\mathbf{x})+s_2^-=0\\ & g_3(\mathbf{x})-s_3^+=0\\ & g_4(\mathbf{x})+s_4^-=0\\ & s_1^+,s_2^-,s_3^+,s_4^-\geq 0\\ & \mathbf{x}\in A\end{array}\right\}\text{ where}$$

$g_1(\mathbf{x})=\begin{cases}\dfrac{1}{f_1(\mathbf{x})} & \text{if } f_1(\mathbf{x})>0\\ 0 & \text{otherwise}\end{cases}$	$g_2(\mathbf{x})=\begin{cases}f_1(\mathbf{x}) & \text{if } f_1(\mathbf{x})\leq 0\\ 0 & \text{otherwise}\end{cases}$
$g_3(\mathbf{x})=\begin{cases}\dfrac{1}{f_2(\mathbf{x})} & \text{if } f_2(\mathbf{x})>0\\ 0 & \text{otherwise}\end{cases}$	$g_4(x)=\begin{cases}f_2(\mathbf{x}) & \text{if} f_2(\mathbf{x})\leq 0\\ 0 & \text{otherwise.}\end{cases}$

Theorem 3.3.5.1. If $\mathbf{x}*$ solves $N1$, then $(\mathbf{x}^*,s_1^+,s_2^-,s_3^+,s_4^-)$ solve $N2$, where $s_1^+=g_1(\mathbf{x}^*)$, $s_2^-=-g_2(\mathbf{x}^*), s_3^+=g_3(\mathbf{x}^*),$ and $s_4^-=-g_4(\mathbf{x}^*)$.

Proof. Let $\mathbf{x}*$ solves $N1$. To obtain a contradiction, suppose $(\mathbf{x}^*,s_1^+,s_2^-,s_3^+,s_4^-)$ does not solve $N2$. By construction, we have $(\mathbf{x}^*,s_1^-,s_2^-,s_2^+)$ is feasible to $N2$. There exists a feasible solution $(\hat{\mathbf{x}},\hat{s}_1^+,\hat{s}_2^-,\hat{s}_3^+,\hat{s}_4^-)$ to $N2$ such that $\hat{s}_1^+\leq s_1^+,\hat{s}_2^-\leq s_2^-,\hat{s}_3^+\leq s_3^+,\hat{s}_4^-\leq s_4^-$ and the strictly less than sign holds for at least one of them. Since $\hat{\mathbf{x}}\in A,$ it is a feasible solution to $N1$. Hence the following four conditions are satisfied with at least one of them holding with a strict inequality:

$$\begin{Bmatrix} (1): & \dfrac{1}{f_1(\hat{\mathbf{x}})} = g_1(\hat{\mathbf{x}}) = \hat{s}_1^+ \le s_1^+ = g_1(\mathbf{x}^*) = \dfrac{1}{f_1(\mathbf{x}^*)} \\ (2): & -f_1(\hat{\mathbf{x}}) = -g_2(\hat{\mathbf{x}}) = \hat{s}_2^- \le s_2^- = -g_2(\mathbf{x}^*) = -f_1(\mathbf{x}^*) \\ (3): & \dfrac{1}{f_2(\hat{\mathbf{x}})} = g_3(\hat{\mathbf{x}}) = \hat{s}_3^+ \le s_3^+ = g_3(\mathbf{x}^*) = \dfrac{1}{f_2(\mathbf{x}^*)} \\ (4): & -f_2(\hat{\mathbf{x}}) = -g_4(\hat{\mathbf{x}}) = \hat{s}_4^- \le s_4^- = -g_4(\mathbf{x}^*) = -f_2(\mathbf{x}^*) \end{Bmatrix}.$$

They are equivalent to the following conditions with at least one of them holding for a strict inequality:

$$\begin{Bmatrix} (1): & f_1(\hat{\mathbf{x}}) \ge f_1(\hat{\mathbf{x}}^*) \\ (2): & f_2(\hat{\mathbf{x}}) \ge f_2(\hat{\mathbf{x}}^*) \end{Bmatrix}$$

in contradiction to $\mathbf{x}^*$ solving $N1$, so the proof is complete. ■

Theorem 3.3.5.2. If $(\mathbf{x}^*, s_1^+, s_2^-, s_3^+, s_4^-)$ solves $N2$, then $\mathbf{x}^*$ solves $N1$.

Proof. Assume that $(\mathbf{x}^*, s_1^+, s_2^-, s_3^+, s_4^-)$ solve $N2$. To obtain a contradiction, suppose that $\mathbf{x}^*$ does not solve $N1$. Note that $\mathbf{x}^*$ is a feasible solution to $N1$ because $\mathbf{x}^* \in A$. Then there exists $\hat{\mathbf{x}} \in A$ such that $f_1(\hat{\mathbf{x}}) \ge f_1(\mathbf{x}^*)$ and $f_2(\hat{\mathbf{x}}) \ge f_2(\mathbf{x}^*)$ where $f_1(\hat{\mathbf{x}}) > f_1(\mathbf{x}^*)$ or $f_2(\hat{\mathbf{x}}) > f_2(\mathbf{x}^*)$.

- $s_1^+ = g_1(\hat{\mathbf{x}}) = \begin{cases} \dfrac{1}{f_1(\hat{\mathbf{x}})} & \text{if } f_1(\hat{\mathbf{x}}) > 0 \\ 0 & \text{otherwise} \end{cases}$ and $s_2^- = -g_2(\hat{\mathbf{x}}) = \begin{cases} -f_1(\hat{\mathbf{x}}) & \text{if } f_1(\hat{\mathbf{x}}) \le 0 \\ 0 & \text{otherwise} \end{cases}$

- $s_3^+ = g_3(\hat{\mathbf{x}}) = \begin{cases} \dfrac{1}{f_2(\hat{\mathbf{x}})} & \text{if } f_2(\hat{\mathbf{x}}) > 0 \\ 0 & \text{otherwise} \end{cases}$ and $s_4^- = -g_4(\hat{\mathbf{x}}) = \begin{cases} -f_2(\hat{\mathbf{x}}) & \text{if } f_2(\hat{\mathbf{x}}) \le 0 \\ 0 & \text{otherwise} \end{cases}$

o $s_1^{+*} = g_1(\mathbf{x}^*) = \begin{cases} \dfrac{1}{f_1(\mathbf{x}^*)} & \text{if } f_1(\mathbf{x}^*) > 0 \\ 0 & \text{otherwise} \end{cases}$ and

$$s_2^{-*} = -g_2(x^*) = \begin{cases} -f_1(x^*) & \text{if } f_1(x^*) \le 0 \\ 0 & \text{otherwise} \end{cases}$$

o $s_3^{+*} = g_3(\mathbf{x}^*) = \begin{cases} \dfrac{1}{f_2(\mathbf{x}^*)} & \text{if } f_2(\mathbf{x}^*) > 0 \\ 0 & \text{otherwise} \end{cases}$ and

$$s_4^{-*} = -g_4(\mathbf{x}^*) = \begin{cases} -f_2(\mathbf{x}^*) & \text{if } f_2(\mathbf{x}^*) \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

We now have $\begin{Bmatrix} s_1^+ \le s_1^{+*} \\ s_2^- \le s_2^{-*} \\ s_3^+ \le s_3^{+*} \\ s_4^- \le s_4^{-*} \end{Bmatrix}$, where strict inequality holds at least once, so a contradiction is obtained to the fact that $(\mathbf{x}^*, s_1^+, s_2^-, s_3^+, s_4^-)$ solves $N2$. ■

Example 3.3.5.3. Consider the following Pareto maximization with three objective functions

$$\begin{Bmatrix} \underset{x_1,x_2}{\text{Vmax}} & (x_1x_2, x_2 - x_1, x_1x_3) \\ \text{s.t.} & x_1 + x_2 + x_3 \le 1 \\ & x_1, x_2, x_3 \ge 0 \end{Bmatrix}.$$

By our approach, we can solve the Pareto maximization with the following goal programming equivalence.

$$\left\{\begin{array}{ll} \underset{s_1^+, s_2^-, s_3^+, s_4^-, s_5^+, s_6^-, x_1, x_2, x_3}{\text{Vmin}} & (s_1^+, s_2^-, s_3^+, s_4^-, s_5^+, s_6^-) \\ \text{s.t.} & g_1(x_1, x_2, x_3) - s_1^+ = 0 \\ & g_2(x_1, x_2, x_3) + s_2^- = 0 \\ & g_3(x_1, x_2, x_3) - s_3^+ = 0 \\ & g_4(x_1, x_2, x_3) + s_4^- = 0 \\ & g_5(x_1, x_2, x_3) - s_5^+ = 0 \\ & g_6(x_1, x_2, x_3) + s_6^- = 0 \\ & x_1 + x_2 + x_3 \le 1 \\ & s_1^+, s_2^-, s_3^+, s_4^-, s_5^+, s_6^- \ge 0 \\ & x_1, x_2, x_3 \ge 0 \end{array}\right\} \text{ where}$$

$g_1(x_1,x_2,x_3) = \begin{cases} \frac{1}{x_1 x_2} & \text{if } x_1 x_2 \le 0 \\ 0 & \text{otherwise} \end{cases}$	$g_2(x_1,x_2,x_3) = \begin{cases} x_1 x_2 & \text{if } x_1 x_2 \le 0 \\ 0 & \text{otherwise} \end{cases}$
$g_3(x_1,x_2,x_3) = \begin{cases} \frac{1}{x_2 - x_1} & \text{if } x_2 - x_1 > 0 \\ 0 & \text{otherwise} \end{cases}$	$g_4(x_1,x_2,x_3) = \begin{cases} x_2 - x_1 & \text{if } x_2 - x_1 \le 0 \\ 0 & \text{otherwise} \end{cases}$
$g_5(x_1,x_2,x_3) = \begin{cases} \frac{1}{x_1 x_3} & \text{if } x_1 x_3 \le 0 \\ 0 & \text{otherwise} \end{cases}$	$g_6(x_1,x_2,x_3) = \begin{cases} x_1 x_3 & \text{if } x_1 x_3 \le 0 \\ 0 & \text{otherwise.} \end{cases}$

3.3.6. *Goal Programming as Pareto Maximization*

Since we have defined a goal programming problem as a Pareto minimization problem, it can be solved by the Pareto maximization of the negative of the objective function in Pareto minimization. It thus follows that Pareto maximization and goal programming are equivalent.

3.3.7. Lexicographic Maximization as Pareto Maximization

Let $Q1$ denote a given Lexicographic maximization.

$$Q1:\left\{\begin{array}{ll}\underset{\mathbf{x}}{\text{Leximax}} & (f_1(\mathbf{x}),\ldots,f_n(\mathbf{x})) \\ \text{s.t.} & \mathbf{x}\in A\subset R^n\end{array}\right\},\ \text{where } f_i: R^n \to R \text{ for } i=1,2,3.$$

$$Q2:\left\{\begin{array}{ll}\max & f_n(\mathbf{x}) \\ \text{s.t.} & f_1(\mathbf{x})=f_1^* \\ & \vdots \\ & f_{n-1}(\mathbf{x})=f_{n-1}^* \\ & \mathbf{x}\in A\end{array}\right\},\qquad Q3:\left\{\begin{array}{ll}\text{Vmax} & (f_n(\mathbf{x}),f_n(\mathbf{x})) \\ \text{s.t.} & f_1(\mathbf{x})=f_1^* \\ & \vdots \\ & f_{n-1}(\mathbf{x})=f_{n-1}^* \\ & \mathbf{x}\in A\end{array}\right\},$$

where $f_1^* = \max\{f_1(\mathbf{x}):\mathbf{x}\in A\}$ and $f_k^* = \max\{f_k(\mathbf{x}): f_1(\mathbf{x})=f_1^*,\ldots,f_{k-1}(\mathbf{x})=f_{k-1}^*\}$ for $k=2,\ldots,n-1$.

The problem $Q2$ is obviously equivalent to $Q1$ by definition. Moreover, $Q3$ is obviously equivalent to $Q2$ because the objective function of $Q3$ is just replication of the objective function of $Q2$.

3.3.8. Pareto Maximization as Lexicographic Maximization

Let $R1$ denote a given Pareto maximization problem, and the problem $R2$ represent an equivalent scalarization as in [15].

$$R1:\underset{\mathbf{x}\in A\subset R^m}{\text{Vmax}}\quad (f_1(\mathbf{x}),\cdots,f_n(\mathbf{x}))\qquad R2:\left\{\begin{array}{ll}\underset{\mathbf{x}}{\max} & f_1(\mathbf{x})+\ldots+f_n(\mathbf{x}) \\ \text{s.t.} & f_1(\mathbf{x})\ge y_1 \\ & \vdots \\ & f_n(\mathbf{x})\ge y_n \\ & \mathbf{x}\in A\end{array}\right\}\ \text{for all}\ y_1,y_2,\ldots,y_n\in R.$$

Consider now the maximin equivalence $R3$ of $R2$

$$R3:\left\{\begin{array}{cc} \underset{\mathbf{x}}{\text{Leximax}} & (c,\ f_1(\mathbf{x})+\ldots+f_n(\mathbf{x})) \\ \text{s.t.} & f_1(\mathbf{x})\geq y_1 \\ & \vdots \\ & f_n(\mathbf{x})\geq y_n \\ & \mathbf{x}\in A \end{array}\right\} \text{ for all } y_1,y_2,\ldots,y_n\in R.$$

Problem $R3$ is obviously equivalent to $R2$ by definition. It follows that we can solve $R3$ instead of $R2$. Thus Pareto maximization and lexicographic maximization problems are equivalent.

3.3.9. Set-Valued Maximization as Cone-Ordered Maximization

Let problem $P1$ denote a set-valued maximization. We show that $P1$ is equivalent to the cone-ordered maximization $P2$, where C is a convex cone in R^n.

$$P1: \max_{\mathbf{x}\in A} F(\mathbf{x}) \qquad P2: \left\{\begin{array}{cc} C\max\limits_{\mathbf{x}\in A} & f(\mathbf{x},\mathbf{y})=\mathbf{y} \\ \text{s.t.} & f(\mathbf{x},\mathbf{y})=\mathbf{y}\geq_C \mathbf{w} \end{array}\right\} \text{for all } \mathbf{w}\in R^n,$$

where $f(\mathbf{x},\mathbf{y})=\mathbf{y}$ for all $\mathbf{x}\in A, \mathbf{y}\in F(\mathbf{x})$.

Theorem 3.3.9.1. If $\mathbf{x}_1$ solves $P1$ then we have that $(\mathbf{x}_1,\mathbf{y}_1)$ solves $P2$ for $\mathbf{w}=\mathbf{y}_1$ and $\mathbf{y}_1\in F(\mathbf{x}_1)\cap\max F(A)$.

Proof. Suppose $\mathbf{x}_1$ solves $P1$. Then there exists $\mathbf{y}_1\in F(\mathbf{x}_1)\cap\max F(A)$. Let $(\mathbf{x}_2,\mathbf{y}_2)$ be any feasible solution of $P2$ where $\mathbf{w}=\mathbf{y}_1$. Then we have $f(\mathbf{x}_2,\mathbf{y}_2)=\mathbf{y}_2\geq_C\mathbf{y}_1$, which contradicts the fact that $\mathbf{y}_1$ is non-dominated unless $\mathbf{y}_2 = \mathbf{y}_1$. Thus any feasible solution of $P2(\mathbf{y}_1)$ is also a solution of $P2(\mathbf{y}_1)$. Since $(\mathbf{x}_1, \mathbf{y}_1)$ is feasible to $P2(\mathbf{y}_1)$, it solves $P2(\mathbf{y}_1)$. ■

Theorem 3.3.9.2. If $(\mathbf{x}_2, \mathbf{y}_2)$ solves $P2$ for $\mathbf{w} \in R^n$, then $\mathbf{x}_2$ is a solution to $P1$ and $\mathbf{y}_2 \in F(\mathbf{x}_2) \bigcap \max F(A)$.

Proof. Assume that $(\mathbf{x}_2, \mathbf{y}_2)$ solves $P2$ for some $\mathbf{w} \in R^n$. To obtain a contradiction, suppose that $\mathbf{x}_2$ does not solve $P1$, i.e., $F(\mathbf{x}_2) \bigcap \max F(A) = \phi$. Hence there must exist $\mathbf{y}_1 \in \max F(A) = \max \bigcup_{\mathbf{x} \in A} F(\mathbf{x})$ such that $\mathbf{y}_2 <_C \mathbf{y}_1$. In particular, there is an element $\mathbf{x}_1$ in A such that $\mathbf{y}_1 \in F(\mathbf{x}_1)$. Since $\mathbf{w} \leq_C \mathbf{y}_2 <_C \mathbf{y}_1$ and $\mathbf{x}_1 \in A$, with convexity of C, we have $(\mathbf{x}_1, \mathbf{y}_1)$ is feasible to $P2(\mathbf{w})$. However, since $\mathbf{y}_2 <_C \mathbf{y}_1$, then $(\mathbf{x}_2, \mathbf{y}_2)$ does not solve $P2(\mathbf{w})$, contradicting the optimality of $(\mathbf{x}_2, \mathbf{y}_2)$. ■

3.3.10. Cone-Ordered Maximization as Set-Valued Maximization

To solve the given cone-ordered maximization as a set-valued maximization, we simply define the objective value of the set-valued maximization to be a singleton set containing only the objective value of the given cone-order maximization. Thus we conclude that cone-ordered maximization and set-valued maximization are equivalent.

Example 3.3.10.1. Given any cone optimization $C\max_{\mathbf{x} \in A \subset R^m} f(\mathbf{x})$, where R^m, R^n are real vector spaces and $f : A \subset R^m \to R^n$, define $F : A \to 2^{R^n}$ by $F(\mathbf{x}) = \{f(\mathbf{x})\}$ for all $\mathbf{x} \in A$.

Then the set-valued optimization $\max_{\mathbf{x}\in A}{}_C\, F(\mathbf{x})$ is equivalent to the cone maximization, $C\max_{\mathbf{x}\in A\subset R^m} f(\mathbf{x})$.

3.4. Summary of all equivalent results

Having shown that different optimization criteria are directly equivalent, we summarize our previous results.

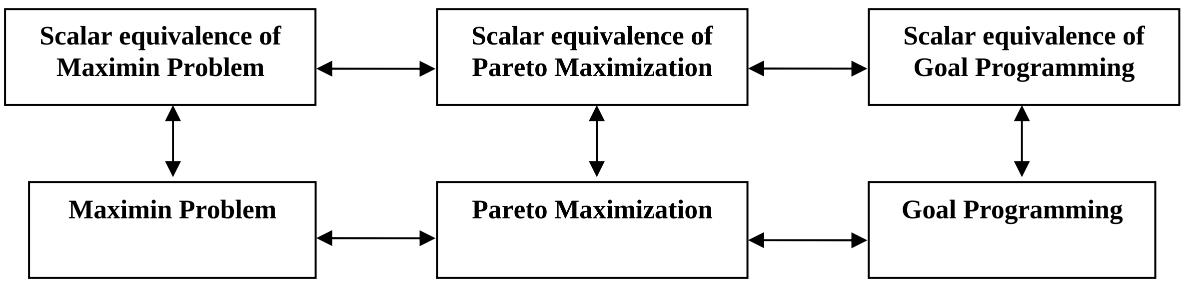

Figure 3.5 Equivalent scalarization summary.

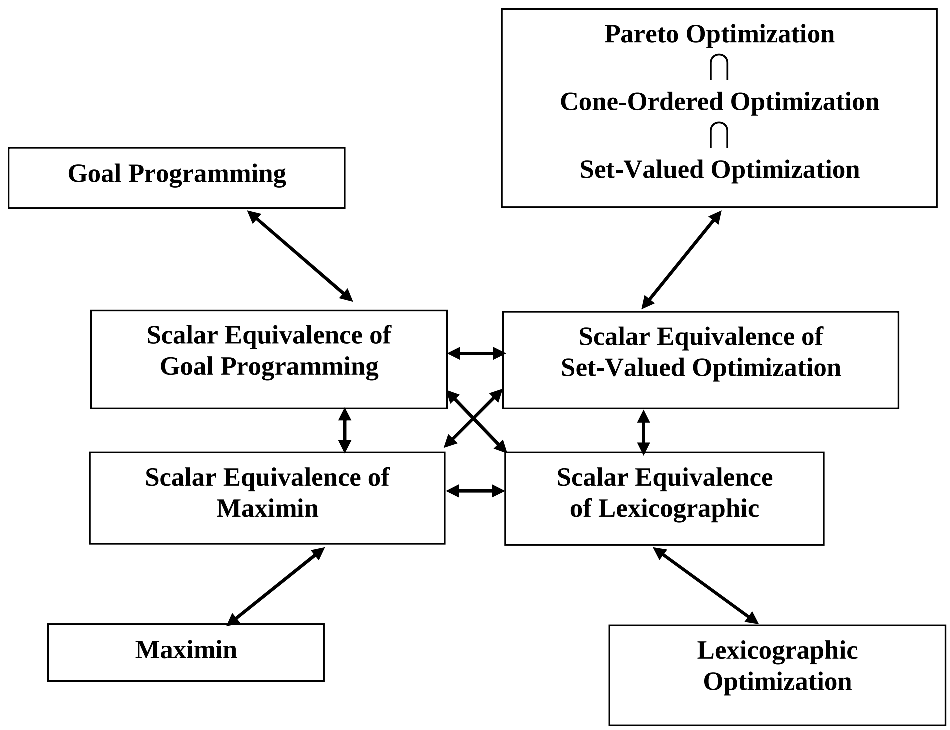

Figure 3.6 Criterion equivalence summary.

Figure 3.6 depicts our result that all standard non-scalar optimization criteria can be scalarized. Moreover, the equivalent scalarization of any such criterion can be formulated as the equivalent scalarization of any other. All solutions and only solutions of any optimization problem involving a standard criterion can be obtained by solving an optimization problem involving any other criterion.

CHAPTER 4

GENERALIZATION AND AXIOMATIZATION OF OPTIMIZATION CRITERIA

In this chapter an abstract definition of optimization problem is given under a more general concept of preferences. Moreover, a set of axioms for general optimization criteria is proposed, and an equivalent scalarization of a general optimization criterion is presented. Examples of optimization criteria are then provided. These include both the standard optimization criteria, as well as two new optimization criteria with applications. Finally, a counterexample is presented, an example of decision rule that does not satisfy our axioms.

4.1. Preference Orders

To justify one's preference in quantitative intuition, typically one uses the notion of an order. In the previous chapters, we defined orders involving existing standard optimization problems by using cones in vector spaces, for example. Now, without using cones, we invoke more general orders that subsume all previous ones as special cases.

4.1.1. Preferences for Vectors

Define a binary relation strictly order $\prec$ on R^n with the requirement:

$$\text{Not}\, \mathbf{y} \prec \mathbf{y}, \forall \mathbf{y} \in R^n.$$

Next define an order relation $\preceqq$ such that

$$\mathbf{y}_1 \preceqq \mathbf{y}_2 \text{ where } \mathbf{y}_1, \mathbf{y}_2 \in R^n \text{ if either } \mathbf{y}_1 \prec \mathbf{y}_2 \text{ or } \mathbf{y}_1 = \mathbf{y}_2.$$

The order $\preceqq$ is called a **preference order**. In this definition, the strict relation $\mathbf{y}_1 \prec \mathbf{y}_2$ may not exist. However $\mathbf{y}_1 \preceqq \mathbf{y}_2$ can be defined whenever $\mathbf{y}_1 = \mathbf{y}_2$. We say that $\mathbf{y}_2$ is more preferred than or equal to $\mathbf{y}_1$ whenever $\mathbf{y}_1 \preceqq \mathbf{y}_2$. Moreover, if $\mathbf{y}_2$ is more preferred than $\mathbf{y}_1$, this fact is represented by $\mathbf{y}_1 \prec \mathbf{y}_2$.

4.1.2. Preferences for Sets

We now extend the concept of preference from a comparison of vectors to one of sets. Let $A, B \subset R^n$. We consider three different types of comparison between sets A and B.

1. $A \preceqq^u B$ if and only if $\forall \mathbf{a} \in A, \exists \mathbf{b} \in B, \mathbf{a} \preceqq \mathbf{b}$.

2. $A \preceqq^l B$ if and only if $\forall \mathbf{b} \in B, \exists \mathbf{a} \in A, \mathbf{a} \preceqq \mathbf{b}$.

3. $A \preceqq B$ if and only if $\forall \mathbf{a} \in A, \forall \mathbf{b} \in B, \mathbf{a} \preceqq \mathbf{b}$.

Notice that the order involving the preference relations here refers to the preference order $\preceqq$, which is more general than an order induced by a cone. For further information on ordering sets by cones, see [24 - 27].

4.2. General Optimization Criteria (GOC)

In this section, we define a general optimization problem on a preference order as the problem of seeking all feasible variables for which there are no more preferable choices of objective function values among the feasible variables

4.2.1. Optimality Notion

Given a preference order $\preceqq$ on R^n and $\mathbf{y}_1, \mathbf{y}_2 \in R^n$, we say that $\mathbf{y}_2$ **dominates** $\mathbf{y}_1$ if $\mathbf{y}_1 \preceqq \mathbf{y}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$. A vector $\mathbf{y}_1 \in A \subset R^n$ is said to be **non-dominated** in A if there is no $\mathbf{y}_2 \in A$ such that $\mathbf{y}_1 \preceqq \mathbf{y}_2$ and $\mathbf{y}_1 \neq \mathbf{y}_2$. Denote the set $\operatorname{opt} A$ as the set containing all **non-dominated** vectors in A with respect to the preference order $\preceqq$.

A subset A of R^n is said to be **partially bounded** if it contains at least one non-dominated vector. A subset A of R^n is said to be **totally bounded** if all chains in A containing any vector $\mathbf{y} \in A \subset R^n$ have non-dominated vectors. Notice that **totally bounded** implies **partially bounded**, but the converse is not necessarily true. In addition, if $A \subset B \subset R^n$ and B is totally bounded then $A \preceqq^u B$ and $A \preceqq^l B$ are true by definition.

4.2.2. Problem Statement

Consider the following general optimization problem.

$$\left\{\begin{array}{l}\underset{\mathbf{x}}{\text{opt}}\ f(\mathbf{x}) \\ \text{s.t.}\ \mathbf{x} \in A \subset R^m\end{array}\right\},$$

where $f: R^m \to R^n$. Let R^n have a preference order $\preceq$. The function f is called the objective function of the problem. We seek a vector $\mathbf{x}^* \in A \subset R^m$ for which there is no vector $\mathbf{x} \in A$ such that $f(\mathbf{x}^*) \prec f(\mathbf{x})$, or equivalently that $f(\mathbf{x}^*) \preceq f(\mathbf{x})$ and $f(\mathbf{x}^*) \neq f(\mathbf{x})$. Such an $\mathbf{x}^* \in A \subset R^m$ is called an optimal solution to the problem. Denote $\text{opt}\ f(A)$ as the set of all optimal objective values, which could be empty.

Example 4.2.2.1. The cone-ordered maximization, $C\max_{\mathbf{x}\in A} f(\mathbf{x})$, where $f: R^m \to R^n$ and $A \subset R^n$, is a special case of the general optimization problem where the preference order is $\leq_C$.

4.2.3. Axioms for General Optimization Criteria

Given any optimization problem in R^n, "opt" is considered to be a ($\preceq$) optimization criterion on R^n if the problem satisfies the following two axioms.

Axiom 1: Axiom of Partial Order (APO).

The preference order $\preceq$ is a partial order.

Axiom 2: Axiom of Scalarizability Property (ASP).

Any such optimization problem has an equivalent scalarization.

4.2.4. Discussion of Axioms

Reasons for choosing Axioms 1 and 2 are now given. It should be noted that the goal of these axioms is to provide a consistent framework for making best decisions. In practice, people may make preference decisions using methods not satisfying our axioms. However, such methods will not regarded as optimization criteria according to our general definition. The goal here is to provide a consistent but flexible decision making framework that yields identical optimal decisions in identical situations for a large class of applications.

4.2.4.1. Axiom of Partial Order (APO)

No decision choice should be preferred more or less than itself; i.e., $\mathbf{x} \prec \mathbf{x}$. In other words, the preference order for a decision should have the reflexive property of a partial order. As for the antisymmetric and transitive properties, the following examples illustrate the difficulty of making a reasonable choice without them.

Example 4.2.4.1.1. Consider the relation order $\trianglelefteq$ on the set {3, 5} with 3=3, 5=5, $3 \prec 5$ and $5 \prec 3$. This order lacks the antisymmetric property because 3 does not identically equal 5. Also, it is not logical to have $3 \prec 5$ and $5 \prec 3$ in the same time for a decision maker. Moreover, there is no "best" value or values to choose, though each value is compared to each value. Hence, the antisymmetric property seems to be a reasonable requirement.

Example 4.2.4.1.2. Consider the relation order $\preceq$ on the set of {5, 8, 10} with 5=5, 8=8, 10=10, $5 \prec 8$, $8 \prec 10$, and $10 \prec 5$. This order lacks transitivity because 5 is "better" than 10. Again, there is no best value or values to choose. In this case, the reason is that 8 is "better" than 5, 10 is "better" than 8, but 5 is "better" than 10. A decision maker using such a preference order would be inconsistent. Actually such intransitivity can occur in elections. A voter may prefer candidate A to B, B to C, and C to A. The difficulty is that if a selection were conducted by successive pairwise comparison, then a different "best" candidate would be chosen for different pairwise comparisons. In other words, the simultaneous comparisons of candidates should give the same result as sequential pairwise comparisons in a decision framework that purports to select a "best" solution. So transitivity is needed for a preference order in an optimization criterion. Of course, decisions can be made without this property, but the term "optimal" cannot be applied according to our framework.

4.2.4.2. Axiom of Scalarizability Property (ASP)

ASP is reasonable since one can always define a utility function on a set of choices. Furthermore, all standard criteria are scalarizable, so ASP seems a natural extension. The determining reason, though, was that we were unable to construct a problem of finding all maximal elements and only maximal elements of a constrained function with respect to a partial order in R^n for which it could be proved that no scalar equivalence exists. On the other hand, we did construct such a problem for which no single equivalent scalarization was found.

Recall that lexicographic maximization has both a multiple-stage scalarization and a single-stage one. Analogously, we constructed a problem in example 4.2.4.2.1 below where multiple scalarizations could obtain all maximal elements and only maximal elements of a constrained function. However, no single real-valued maximization problem was discovered. To maintain the appealing requirement of an equivalent scalarization for a general criterion, ASP requires one. It remains an open question, though, whether there exists a constrained function in R^n without an equivalent scalarization with respect to some partial order.

Example 4.2.4.2.1. (A family of maximizations with different objective functions).

Let $\preceq$ be a partial order relation on R^2. For each $\mathbf{x} \in R^2$, we define a subset of R^2 to have $\mathbf{x}$ as the first element, $I(\mathbf{x}) = [\mathbf{x}, \rightarrow) =: \{\mathbf{y} \in R^2 : \mathbf{x} \preceq \mathbf{y}\}$. For each $\mathbf{x} \in R^2$, we create a collection $C(\mathbf{x})$ containing all chains in $I(\mathbf{x})$ to have $\mathbf{x}$ as the first element as follows.

Let $C(\mathbf{x}) = \{ P_i(\mathbf{x}) \subset I(\mathbf{x}) : i \in \wedge_{\mathbf{x}} \}$, where $\wedge_{\mathbf{x}}$ is an index set, and $P_i(\mathbf{x})$ has the following properties:

1. $\forall \mathbf{y}_1, \mathbf{y}_2 \in P_i(\mathbf{x}), \mathbf{y}_1 \leqq \mathbf{y}_2$ or $\mathbf{y}_2 \leqq \mathbf{y}_1$,
2. $\mathbf{x} \in P_i(\mathbf{x})$.

According to Lemma 4.2.4.2.2 below, we have $(R^2,\preceq) = (\bigcup_{\substack{\mathbf{x}\in R^2 \\ P_i(\mathbf{x})\in C(\mathbf{x})}} P_i(\mathbf{x}),\preceq)$. In other words, $(R^2,\preceq)$ can be decomposed into an uncountable union of chains.

Lemma 4.2.4.2.2. $(R^2,\leqq) = (\bigcup_{\substack{\mathbf{x}\in R^2 \\ P_i(\mathbf{x})\in C(\mathbf{x}) \\ i\in\wedge_{\mathbf{x}}}} P_i(\mathbf{x}),\leqq)$.

Proof. By the above construction, $P_i(\mathbf{x}) \subset I(\mathbf{x}) \subset R^2$ for all $\mathbf{x}\in R^2$, $i\in\wedge_{\mathbf{x}}$, and $(\bigcup_{\substack{\mathbf{x}\in R^2 \\ P_i(\mathbf{x})\in C(\mathbf{x}) \\ i\in\wedge_{\mathbf{x}}}} P_i(\mathbf{x}),\leqq) \subset (R^2,\leqq)$. For the converse, let $\mathbf{y}\in R^2$. Since $\mathbf{y}\in P_i(\mathbf{y})$ for all $i\in\wedge_{\mathbf{y}}$, we have that $\mathbf{y}\in P_i(\mathbf{y}) \subset \bigcup_{\substack{\mathbf{x}\in R^2 \\ P_i(\mathbf{x})\in C(\mathbf{x}) \\ i\in\wedge_{\mathbf{x}}}} P_i(\mathbf{x})$. We have $R^2 \subset \bigcup_{\substack{\mathbf{x}\in R^2 \\ P_i(\mathbf{x})\in C(\mathbf{x})}} P_i(\mathbf{x})$. Let $(\mathbf{x},\mathbf{y})\in(R^2,\leq)$; i.e., $\mathbf{x}\leqq\mathbf{y}$. We have $(\mathbf{x},\mathbf{y})\in(P_i(\mathbf{x}),\leq)$ for some $i\in\wedge_{\mathbf{x}}$ by definition. The conclusion now follows that $(R^2,\leqq) \subset (\bigcup_{\substack{\mathbf{x}\in R^2 \\ P_i(\mathbf{x})\in C(\mathbf{x}) \\ i\in\wedge_{\mathbf{x}}}} P_i(\mathbf{x}),\leqq)$. ■

Consider the following general optimization $A1$, for which "opt" may not represent an axiomatically formal optimization criterion.

$A1: \left\{ \begin{array}{l} \underset{\mathbf{x}}{\text{opt}}\ f(\mathbf{x}) \\ \text{s.t.}\ \mathbf{x}\in A\subset R^m \end{array} \right\}$, where $f: R^m \to R^2$, and R^2 has a partial order $\preceq$.

We construct a family of uncountable number of scalar maximizations as follows.

$$A2: \begin{cases} \max\limits_{\mathbf{x}} & l_{\mathbf{w}}^i(f(\mathbf{x})) \\ \text{s.t.} & f(\mathbf{x}) \in P_i(\mathbf{w}) \\ & \mathbf{x} \in A \end{cases}$$ for all $\mathbf{w} \in R^2, i \in \wedge_{\mathbf{w}}$, and $l_{\mathbf{w}}^i$ is a real-valued function mapping from R^2 with the property that if $f(\mathbf{x}) \prec f(\mathbf{y})$ for $f(\mathbf{x}), f(\mathbf{y}) \in P_i(\mathbf{w})$ for each $\mathbf{w} \in R^2, i \in \wedge_{\mathbf{w}}$, then $l_{\mathbf{w}}^i(f(\mathbf{x})) < l_{\mathbf{w}}^i(f(\mathbf{y}))$.

Theorem 4.2.4.2.3. If $\mathbf{x}_0$ solves $A1$ then $\mathbf{x}_0$ solves $A2$ for $\mathbf{w} = f(\mathbf{x}_0)$ and for all $i \in \wedge_{\mathbf{w}}$.

Proof. Assume that $\mathbf{x}_0$ solves $A1$. By the choice $\mathbf{w} = f(\mathbf{x}_0)$ we know that $\mathbf{x}_0$ is a feasible solution to $A2$ for $\mathbf{w} = f(\mathbf{x}_0)$ and any $i \in \wedge_{\mathbf{w}}$. Let $\mathbf{x}_1$ be any feasible solution to $A2$ for $\mathbf{w} = f(\mathbf{x}_0)$ and $i \in \wedge_{\mathbf{w}}$. Since $\mathbf{x}_0$ solves $A1$, the only possibility is that $f(\mathbf{x}_1) = f(\mathbf{x}_0)$, so every feasible point of $A2(f(\mathbf{x}_0), i)$ is a solution as well. Since $\mathbf{x}_0$ is a feasible to $S2(f(\mathbf{x}_0), i)$, it solves $S2(f(\mathbf{x}_0), i)$. ■

Theorem 4.2.4.2.4. If $\mathbf{x}_0$ solves $A2$ for $\mathbf{w} \in R^n$ and any $i \in \wedge_{\mathbf{w}}$, then $\mathbf{x}_0$ solves $A1$.

Proof. Assume that $\mathbf{x}_0$ solves $A2$ for $\mathbf{w} \in R^n$ and $i \in \wedge_{\mathbf{w}}$. To obtain a contradiction, suppose that $f(\mathbf{x}_0) \notin \text{opt } f(A)$. Then there exists $\mathbf{x}_1 \in A$ such that $f(\mathbf{x}_0) \prec f(\mathbf{x}_1)$, otherwise $f(\mathbf{x}_0) \in \text{opt } f(A)$. Since $f(\mathbf{x}_0) \in P_i(\mathbf{w})$ and by the definition of $P_i(\mathbf{w})$, we also have that $f(\mathbf{x}_1) \in P_i(\mathbf{w})$. In another word, $\mathbf{x}_1$ is feasible to $A2$. Since $f(\mathbf{x}_0) \prec f(\mathbf{x}_1)$, we have that $l_{\mathbf{w}}^i(f(\mathbf{x}_0)) < l_{\mathbf{w}}^i(f(\mathbf{x}_1))$ in contradiction to the optimality of $\mathbf{x}_0$. ■

Under the existence of $l_{\mathbf{w}}^{i}$ for all $i \in \wedge_{\mathbf{w}}$ and $\mathbf{w} \in R^2$, problems $A2$ and $A1$ are equivalent according to Theorems 4.2.4.2.3 and 4.2.4.2.4. All solutions and only solutions of $A1$ can be theoretically obtained by $A2$ and vice versa. The separability in the sense of Cantor of all chains $P_i(\mathbf{w})$ in R^2 guarantees the existence of a strictly monotone function $l_{\mathbf{w}}^{i}$. However, the objective function $l_{\mathbf{w}}^{i}(f(\mathbf{x}))$ may obviously be different from $l_{\mathbf{w}}^{j}(f(\mathbf{x}))$ where $i, j \in \wedge_{\mathbf{w}}$, or different from $l_{\mathbf{y}}^{i}(f(\mathbf{x}))$ where $i \in \wedge_{\mathbf{y}}$ for $\mathbf{w}, \mathbf{y}, \mathbf{z} \in R^2$. Therefore $A2$ is not considered as an equivalent scalarization of $A1$ since there is no common objective function for the family.

4.2.5. Elimination of Axioms

To find appropriate axioms for General Optimization Criteria (GOC), we investigated many potential properties of the standard optimization of chapter 2. Two ultimately eliminated but potential axioms are discussed in this section. One reasonable property is the domination property, in which a rational decision maker cannot gain less benefit with more choices. Another is the triangular inequality for optimization, stated below. We explain why such properties are not general enough to be axioms.

4.2.5.1. Domination property

We show that Axiom 1 (APO) implies the domination property.

Lemma 4.2.5.1.1. Let $\preceqq$ be a partial order in R^n and $f: R^m \to R^n$ be an objective function for the general optimization problem $\underset{\mathbf{x}\in A\subset R^m}{\text{opt}} f(\mathbf{x})$. Then for any $\mathbf{y} \in f(A)$ either

1. $\mathbf{y} \leq_C f(\mathbf{x}^*)$ for some optimal solution $\mathbf{x}^*$, or
2. $\mathbf{y}$ is in some unbounded chain in $f(A)$.

Proof. It suffices to show that if (2) is not true, then (1) is true. Assume that $\mathbf{y}$ is in a bounded chain. Since the chain is bounded from above, the maximal element exists according to Lemma 2.5.11 (Zorn's Lemma). Then by the definition of optimality, that maximal element equals $f(\mathbf{x}^*)$ for some optimal solution $\mathbf{x}^*$. ■

Property 4.2.5.1.2 (domination property). Let A and B be subsets of R^m such that $A \subset B$ and $f: R^m \to R^n$. Assume that $\underset{\mathbf{x}\in A}{\text{opt}} f(x) \neq \phi$ and $\underset{\mathbf{x}\in B}{\text{opt}} f(B) \neq \phi$. Then the following two statements are true.

1. If $f(B)$ is totally bounded with respect to $\preceqq$ in R^n then $\underset{\mathbf{x}\in A}{\text{opt}} f(x) \preceqq^u \underset{\mathbf{x}\in B}{\text{opt}} f(B)$.
2. If $f(B)$ is not totally bounded with respect to $\preceqq$ in R^n then $\underset{\mathbf{x}\in A}{\text{opt}} f(x) \preceqq^u f(B)$.

Proof. Let A and B are subsets of R^m such that $A \subset B$. Assume that $\underset{\mathbf{x}\in A}{\text{opt}} f(x) \neq \phi$ and $\underset{\mathbf{x}\in B}{\text{opt}} f(B) \neq \phi$ where $f: R^m \to R^n$. Consider the following 2 cases.

Case 1: $f(B)$ is totally bounded. Let $f(\mathbf{x}) \in \text{opt } f(A)$. Since A is a subset of B, we have $\text{opt } f(A) \subset f(A) \subset f(B)$. Therefore $f(\mathbf{x}) \in f(B)$. By Lemma 4.2.5.1.1, there exists some optimal solution $\mathbf{x}^* \in B$ such that $f(\mathbf{x}) \preceqq f(\mathbf{x}^*)$. Thus by definition $\text{opt } f(A) \preceqq^u \text{opt } f(B)$.

Case 2: $f(B)$ is not totally bounded. Since $\text{opt } f(A) \subset f(A) \subset f(B)$, then again by definition $\text{opt } f(A) \preceqq^u f(B)$. ■

4.2.5.2. Triangle inequality

It is next shown that domination property for cone-ordered optimizations implies the triangle inequality. These properties are first stated in the cone-ordered setting.

Property 4.2.5.2.1. (domination property for cone-ordered maximization).

$$\underset{\mathbf{x} \in A \subset R^n}{C\text{max}} f(\mathbf{x}) \leq_C^u \underset{\mathbf{x} \in B \subset R^n}{C\text{max}} f(\mathbf{x}), \text{ where } f : R^m \to R^n \text{ and } A \subset B.$$

Property 4.2.5.2.2. (triangle inequality for cone-ordered maximization).

$$\underset{\mathbf{x} \in A \subset R^m}{\text{Cmax}} (f+g)(\mathbf{x}) \leq_C^u \underset{\mathbf{x} \in A \subset R^m}{\text{Cmax}} f(\mathbf{x}) + \underset{\mathbf{x} \in A \subset R^m}{\text{Cmax}} g(\mathbf{x}), \text{ where } f, g : R^m \to R^n.$$

Lemma 4.2.5.2.3. Let $f, g : R^m \to R^n$ and $S \subset R^m$ then

$$\underset{(\mathbf{x},\mathbf{y})\in S\times S}{\text{Cmax}}[f(\mathbf{x})+g(\mathbf{y})] \leq_C^u \underset{\mathbf{x}\in S}{\text{Cmax}}\, f(\mathbf{x}) + \underset{\mathbf{y}\in S}{\text{Cmax}}\, g(\mathbf{y}).$$

Proof. Let $f, g : R^m \to R^n$ and $S \subset R^m$. Assume that $(\mathbf{x}^*, \mathbf{y}^*)$ solves $\underset{(\mathbf{x},\mathbf{y})\in S\times S}{C\text{max}}[f(\mathbf{x})+g(\mathbf{y})]$, i.e., that $f(\mathbf{x}^*)+g(\mathbf{y}^*) \in \underset{(\mathbf{x},\mathbf{y})\in S\times S}{C\text{max}}[f(\mathbf{x})+g(\mathbf{y})]$. We claim that the following two statements are true.

$$(1).\ \ f(\mathbf{x}^*) \in \underset{\mathbf{x}\in S}{C\text{max}}\, f(\mathbf{x})$$

$$(2).\ \ g(\mathbf{y}^*) \in \underset{\mathbf{x}\in S}{C\text{max}}\, g(\mathbf{x}).$$

To obtain (1) by a contradiction, suppose $\exists \tilde{\mathbf{x}} \in S, f(\mathbf{x}^*) <_C f(\tilde{\mathbf{x}})$. Then $f(\mathbf{x}^*)+\mathbf{c}_1 = f(\tilde{\mathbf{x}})$ for some $\mathbf{c}_1 \in C \setminus \{\mathbf{0}\}$. It now follows that

$$f(\mathbf{x}^*)+g(\mathbf{y}^*)+\mathbf{c}_1 = f(\tilde{\mathbf{x}})+g(\mathbf{y}^*).$$

In other words, $f(\mathbf{x}^*)+g(\mathbf{y}^*) <_C f(\tilde{\mathbf{x}})+g(\mathbf{y}^*)$, an inequality that contradicts the optimality of $(\mathbf{x}^*, \mathbf{y}^*)$. Thus $f(\mathbf{x}^*) \in \underset{\mathbf{x}\in S}{C\text{max}}\, f(\mathbf{x})$.

To obtain (2) by a contradiction, suppose $\exists \tilde{\mathbf{y}} \in S, g(\mathbf{y}^*) <_C g(\tilde{\mathbf{y}})$. Then $g(\mathbf{y}^*)+\mathbf{c}_2 = g(\tilde{\mathbf{y}})$ for some $\mathbf{c}_2 \in C \setminus \{\mathbf{0}\}$. It follows that

$$f(\mathbf{x}^*)+g(\mathbf{y}^*)+\mathbf{c}_2 = f(\mathbf{x}^*)+g(\tilde{\mathbf{y}}).$$

Therefore $f(\mathbf{x}^*)+g(\mathbf{y}^*) <_C f(\mathbf{x}^*)+g(\tilde{\mathbf{y}})$, an inequality that contradicts the optimality of $(\mathbf{x}^*, \mathbf{y}^*)$. Thus $g(\mathbf{x}^*) \in \underset{\mathbf{x}\in S}{C\text{max}}\, g(\mathbf{x})$.

From (1) and (2), we conclude that $f(\mathbf{x}^*)+g(\mathbf{y}^*) \in \underset{\mathbf{x}\in S}{\text{Cmax}}\, f(\mathbf{x}) + \underset{\mathbf{y}\in S}{\text{Cmax}}\, g(\mathbf{y})$. ■

Theorem 4.2.5.2.4. For a cone-ordered maximization, the domination property implies the triangle inequality.

Proof. Assume the domination property holds. Let $f, g : R^m \to R^n$ and $S \subset R^m$ be the feasible region of $C\max_{\mathbf{x}\in S} f(\mathbf{x})$ and $C\max_{\mathbf{x}\in S} g(\mathbf{x})$. Define $h : S \times S \to R$ by

$$h(\mathbf{x},\mathbf{y}) = f(\mathbf{x}) + g(\mathbf{y}) \text{ for all } \mathbf{x},\mathbf{y} \in S.$$

Let $L = \{(\mathbf{x},\mathbf{y}) \in S \times S \text{ such that } \mathbf{x} = \mathbf{y}\}$. Equivalently, $L = \bigcup_{\mathbf{x}\in S}\{(\mathbf{x},\mathbf{x}) \in S \times S\}$. It follows that $L \subset S \times S$. Then by the domination property,

$$C\max_{(\mathbf{x},\mathbf{y})\in L} h(\mathbf{x},\mathbf{y}) \leq_C^u C\max_{(\mathbf{x},\mathbf{y})\in S\times S} h(\mathbf{x},\mathbf{y}). \tag{1}$$

Since $\mathbf{x} = \mathbf{y}$ for any $(\mathbf{x},\mathbf{y}) \in L$, $C\max_{(\mathbf{x},\mathbf{y})\in L}[f(\mathbf{x}) + g(\mathbf{y})] = C\max_{\mathbf{x}\in S}[f(\mathbf{x}) + g(\mathbf{x})]$. Therefore

$$C\max_{\mathbf{x}\in S}[f(\mathbf{x}) + g(\mathbf{x})] = C\max_{(\mathbf{x},\mathbf{y})\in L}[f(\mathbf{x}) + g(\mathbf{y})] = C\max_{(\mathbf{x},\mathbf{y})\in L} h(\mathbf{x},\mathbf{y}). \tag{2}$$

But by definition

$$\mathrm{C}\max_{(\mathbf{x},\mathbf{y})\in S\times S} h(\mathbf{x},\mathbf{y}) = \mathrm{C}\max_{(\mathbf{x},\mathbf{y})\in S\times S}[f(\mathbf{x}) + g(\mathbf{y})]. \tag{3}$$

Lemma 4.2.5.2.3 now gives

$$\mathrm{C}\max_{(\mathbf{x},\mathbf{y})\in S\times S}[f(\mathbf{x}) + g(\mathbf{y})] \leq_C^u \mathrm{C}\max_{\mathbf{x}\in S} f(\mathbf{x}) + \mathrm{C}\max_{\mathbf{y}\in S} g(\mathbf{y}). \tag{4}$$

Combining from (1-4) yields

$$C\max_{\mathbf{x}\in S}[f(\mathbf{x}) + g(\mathbf{x})] = C\max_{(\mathbf{x},\mathbf{y})\in L} h(\mathbf{x},\mathbf{y}) \leq_C^u C\max_{(\mathbf{x},\mathbf{y})\in S\times S} h(\mathbf{x},\mathbf{y}) \leq_C^u C\max_{\mathbf{x}\in S} f(\mathbf{x}) + C\max_{\mathbf{y}\in S} g(\mathbf{y}).$$

Hence $C\max_{\mathbf{x}\in S}[f(\mathbf{x}) + g(\mathbf{x})] \leq_C^u C\max_{\mathbf{x}\in S} f(\mathbf{x}) + C\max_{\mathbf{y}\in S} g(\mathbf{y})$. ■

4.3. An Equivalent Scalarization for General Optimization Criteria

In this section we present an equivalent scalarization for the general optimization problem. To summarize the previous development, an equivalent scalarization for an optimization problem is described again as follows. All solutions and only solutions to an optimization problem involving the original criterion can be obtained by certain scalar maximization problems and vice versa. These scalar maximization problems must be either (a) a single real-valued maximization subject to constraints or (b) a collection of such scalar maximization problems with a common real-valued objective function but with parameters in the constraints. In (b) a different set of parameters yields a different set of constraints for the common objective function.

In chapter 3, we developed equivalent scalarizations for standard optimization problem such as Pareto maximization and lexicographic maximization. To extend scalarizability in a general optimization framework, we present here two methods of scalarization. The first scalarization is Corley's Method (CM) [2, p.63] with transformations for solving a general cone-ordered optimization problem. The transformation process is explained in section 4.3.1.1. with various examples. Since we consider only a partial order, according to the result of Remark 2.5.14 it suffices to consider only a cone-ordered optimization for which a cone is pointed and convex. The second scalarization is the Lexicographic Hybrid Method (LHM), which incorporates features of both Corley's Method (CM) and the equivalent scalarization for lexicographic maximization presented in Example 3.2.4.3. The Lexicographic Hybrid Method (LHM) can be considered as an equivalent scalarization for a general optimization problem. LHM is applicable for both cone-ordered optimization and non cone-ordered optimization under appropriate assumptions.

4.3.1. CM with Transformations

CM has been introduced in section 3.2.4. It plays a central role as an equivalent scalarization for Pareto maximization without requiring any assumption on both the objective function and the set of constraints. CM is an equivalent scalarization to cone-ordered optimization for which the cone is pointed and convex and for which a strictly positive linear functional exists.

As described in section 3.2.4, the crucial requirement for CM is the existence of a strictly positive linear functional for converting the objective function of the original problem into a scalar function. For a general cone it is not always easy to construct such a strictly positive linear functional except in the case of Pareto maximization. For example, one may need to apply Theorem 2.5.18 (cone separation theorem). This inconvenience prompts us to create a concept of transformation of a cone-ordered maximization into another equivalent cone-ordered maximization for which the strictly linear functional is readily available. Moreover, this technique can be applied to transform certain cone-order optimizations where a strictly linear functional does not exist (lexicographic optimization, for example) into a known scalarizable problem.

We focus only an optimization problem with a pointed and convex cone because of Axiom 1 (APO). Two distinguished types of pointed and convex cones in cone-ordered optimization are considered as follows.

I. $\overline{C}$ is pointed.

II. $\overline{C}$ is not pointed.

$\overline{C}$ denotes the closure of *C*, i.e., the smallest closed superset of *C*.

Useful transformation techniques to obtain an equivalent cone-ordered optimization for cones of Type I and II are presented in 4.3.1.1 and 4.3.1.2.

4.3.1.1. Type I Transformation

Assume that the cone $\overline{C}$ is pointed. In addition, the cone *C* must satisfy the following properties:

- There exists a basis $\{\mathbf{b}_1,...,\mathbf{b}_n\} \subset \overline{C}$ of R^n.
- The cone *C* can be represented in the following manner: $C = \{\alpha_1\mathbf{b}_1 + ... + \alpha_n\mathbf{b}_n : \alpha_i \geq 0, \alpha_j > 0, i \in I, j \in J\}$, where $I \subset \{1,...,n\}$ is an index set indicating nonnegative coefficient and $J \subset \{1,...,n\}$ is an index set indicating positive coefficient. Notice that $\overline{C} = \{\alpha_1\mathbf{b}_1 + ... + \alpha_n\mathbf{b}_n : \alpha_i \geq 0\}$.

The set $\{\mathbf{b}_1,...,\mathbf{b}_n\}$ is a basis of R^n means the following two statements:

1. $\{\alpha_1\mathbf{b}_1 + ... + \alpha_n\mathbf{b}_n : \alpha_i \in R\} = R^n$, and
2. If $\alpha_1\mathbf{b}_1 + ... + \alpha_n\mathbf{b}_n = \mathbf{0}$ then $\alpha_1 = ... = \alpha_n = 0$.

In other words, any $\mathbf{c} \in \overline{C}$ must be uniquely written as the non-negative linear combination of the basis vectors $\mathbf{b}_i$, where $i = 1,..., n$.

We apply transformation to *B*1 and obtain *B*2 as an equivalent problem to *B*1.

$B1$: $C_1 \max\limits_{\mathbf{x}\in A\subset R^m}(f_1(\mathbf{x}),..., f_n(\mathbf{x}))$ where $f_i : R^m \to R,$ and C_1 has the properties

- There exists a basis $\{\mathbf{b}_1,...,\mathbf{b}_n\} \subset \overline{C_1}$ of R^n.
- The cone $C_1 = \{\alpha_1\mathbf{b}_1 + ... + \alpha_n\mathbf{b}_n : \alpha_i \geq 0, \alpha_j > 0, i \in I, j \in J\}$, where $I, J \subset \{1,...,n\}$ are index sets indicating nonnegative and positive coefficient respectively. Notice that $\overline{C_1} = \{\alpha_1\mathbf{b}_1 + ... + \alpha_n\mathbf{b}_n : \alpha_i \geq 0\}$.

$$B2: \left\{\begin{array}{lc} C_2\max & (\alpha_1(\mathbf{x}),...,\alpha_n(\mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in A \\ & (f_1(\mathbf{x}),..., f_n(\mathbf{x})) = \alpha_1(\mathbf{x})\mathbf{b}_1 + ... + \alpha_n(\mathbf{x})\mathbf{b}_n \end{array}\right\}, \text{ where}$$

$C_2 = \{\alpha_1\mathbf{e}_1 + ... + \alpha_i\mathbf{e}_i + ... + \alpha_n\mathbf{e}_n : \alpha_i \geq 0, \alpha_j > 0,\ i \in I, j \in J\}$ and $\mathbf{e}_i = (0,...,0,\overset{i^{\text{th}}}{1},0...,0).$ Here $I, J \subset \{1,...,n\}$ are index sets indicating nonnegative and positive coefficients of C_1. Thus C_2 is a transformation of C_1 by simply replacing $\mathbf{b}_i$ with $\mathbf{e}_i$ for all $i = 1,...,n.$ If $J = \phi$ then C_2 is the Pareto cone in R^n. Note that $\overline{C_2}$ is the Pareto cone in R^n. The next two theorems establish the equivalence between $B1$ and $B2$.

Theorem 4.3.1.1.1. If $\mathbf{x}_0$ solves $B1$, then $\mathbf{x}_0$ solves $B2$.

Proof. Assume that $\mathbf{x}_0$ solves $B1$. It is then a feasible to $B2$. We claim that $\mathbf{x}_0$ solves $B2$. To obtain a contradiction, suppose $\mathbf{x}_0$ does not solve $B2$. Then there exists $\mathbf{x}_1 \in A$ such that $(f_1(\mathbf{x}_1),..., f_n(\mathbf{x}_1)) = \alpha_1(\mathbf{x}_1)\mathbf{b}_1 + ... + \alpha_n(\mathbf{x}_1)\mathbf{b}_n$ where $\alpha_i(\mathbf{x}_1) \geq \alpha_i(\mathbf{x}_0)$ for all $i = 1,...,n$ and $\alpha_j(\mathbf{x}_1) > \alpha_j(\mathbf{x}_0)$ for some j. Therefore

$$(f_1(\mathbf{x}_1),...,(f_n(\mathbf{x}_1)) - (f_1(\mathbf{x}_0),...,(f_n(\mathbf{x}_0))$$

$$= [\alpha_1(\mathbf{x}_1) - \alpha_1(\mathbf{x}_0)]\mathbf{b}_1 + ... + [\alpha_n(\mathbf{x}_1) - \alpha_n(\mathbf{x}_0)]\mathbf{b}_2 \in C_1 \setminus \{\mathbf{0}\}.$$

Thus we get $f(\mathbf{x}_1) >_{C_1} f(\mathbf{x}_0)$, which contradicts the optimality of $\mathbf{x}_0$. ■

Theorem 4.3.1.1.2. If $\mathbf{x}_0$ solves $B2$, then $\mathbf{x}_0$ solves $B1$.

Proof. Assume that $\mathbf{x}_0$ solves $B2$, so it is feasible to $B1$. We claim that $\mathbf{x}_0$ solves $B1$. To obtain a contradiction, suppose that $\mathbf{x}_0$ does not solve $B1$. Then there exists $\mathbf{x}_1 \in A$ such that $(f_1(\mathbf{x}_0),..., f_n(\mathbf{x}_0)) <_{C_1} (f_1(\mathbf{x}_1),..., f_n(\mathbf{x}_1))$. Therefore by definition, we obtain

$$(f_1(\mathbf{x}_1) - f_1(\mathbf{x}_0),..., f_n(\mathbf{x}_1) - f_n(\mathbf{x}_0)) = [\alpha_1(\mathbf{x}_1) - \alpha_1(\mathbf{x}_0)]\mathbf{b}_1 + [\alpha_2(\mathbf{x}_1) - \alpha_2(\mathbf{x}_0)]\mathbf{b}_2 \in C_1 \setminus \{\mathbf{0}\}.$$

Thus $\alpha_i(\mathbf{x}_1) \geq \alpha_i(\mathbf{x}_0)$ for all $i = 1,...,n$ and $\alpha_j(\mathbf{x}_1) > \alpha_j(\mathbf{x}_0)$ for some *j*, a contradiction to the optimality of $\mathbf{x}_0$. ■

Since we already have a strictly linear functional on C_2 for the problem $B2$, i.e., $l(x_1,...,x_n) = x_1 + ... + x_n$ for $x_1,...,x_n \in R$, we have the following $B3$ as an equivalent scalarization of $B2$.

$$B3: \left\{ \begin{array}{ll} \max\limits_{\mathbf{x} \in A} & \alpha_1(\mathbf{x}) + ... + \alpha_n(\mathbf{x}) \\ \text{s.t.} & (\alpha_1(\mathbf{x}),...,\alpha_n(\mathbf{x})) \geq_{C_2} (y_1,...,y_n) \\ & (f_1(\mathbf{x}),..., f_2(\mathbf{x})) = \alpha_1(\mathbf{x})\mathbf{b}_1 + \alpha_2(\mathbf{x})\mathbf{b}_2 \\ & \mathbf{x} \in A \end{array} \right\} \text{ for all } y_1,...,y_n \in R.$$

Notice that the equivalent scalarization $B3$ is CM with a transformation. This transformation is actually a change of basis of R^n. See [28, p.384] for details about changing the basis of finite dimensional vector spaces.

Example 4.3.1.1.3. Consider the following cone in R^2.

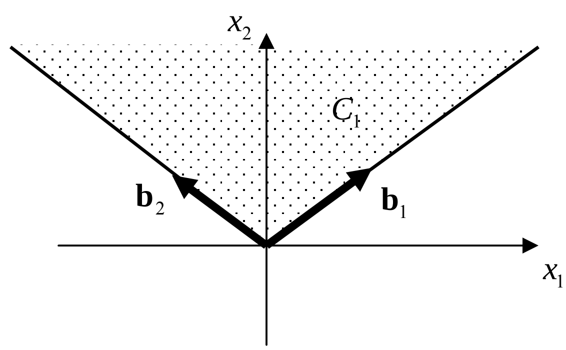

Figure 4.1 The cone C_1 for problem $D1$.

The cone-ordered maximization is denoted by $D1$.

$$D1:\left\{\begin{array}{ll} C_1 \max & f(\mathbf{x}) = (x_1, x_2) \\ \text{s.t.} & x_1^2 + x_2^2 = 1 \\ & x_1, x_2 \geq 0 \end{array}\right\}.$$

We represent the cone C_1 as follows.

$$C_1 = \{\alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\} \text{ where } B = \{\mathbf{b}_1, \mathbf{b}_2\} = \{(1,1), (-1,1)\} \subset \overline{C_1} = C_1.$$

Now we will express $(f_1(\mathbf{x}), f_2(\mathbf{x}))$ in term of the nonnegative linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$.

- $\begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1(\mathbf{x}) \\ \alpha_2(\mathbf{x}) \end{bmatrix} = \alpha_1(\mathbf{x})\mathbf{b}_1 + \alpha_2(\mathbf{x})\mathbf{b}_2$, where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and

  $\alpha_1, \alpha_2 \in R.$

- Solve $\alpha_1(\mathbf{x}), \alpha_2(\mathbf{x})$ in the following system of equations.

  $$\begin{bmatrix} \alpha_1(\mathbf{x}) \\ \alpha_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix}.$$

- We have $\alpha_1(\mathbf{x}) = 0.5 f_1(\mathbf{x}) + 0.5 f_2(\mathbf{x})$ and $\alpha_2(\mathbf{x}) = -0.5 f_1(\mathbf{x}) + 0.5 f_2(\mathbf{x})$.

- Therefore, $(f_1(\mathbf{x}), f_2(\mathbf{x})) = (0.5 f_1(\mathbf{x}) + 0.5 f_2(\mathbf{x}))\cdot \mathbf{b}_1 + (-0.5 f_1(\mathbf{x}) + 0.5 f_2(\mathbf{x}))\cdot \mathbf{b}_2$.

An equivalent problem of $D1$ can now be formulated as $D2$ as follows.

$$D2: \left\{ \begin{array}{ll} C_2 \max & (0.5x_1 + 0.5x_2, -0.5x_1 + 0.5x_2) \\ \text{s.t.} & x_1^{\,2} + x_2^{\,2} = 1 \\ & (f_1(x), f_2(x)) = (x_1, x_2) = (0.5x_1 + 0.5x_2)\cdot \mathbf{b}_1 + (-0.5x_1 + 0.5x_2)\cdot \mathbf{b}_2 \\ & x_1 \in R \\ & x_2 \in R \end{array} \right\},$$

where $C_2 = \{\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}, \mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)$. Note that the cone C_2 is the Pareto Cone in R^2. Therefore, an equivalent scalarization of $P2$ can be stated below as $D3$.

$$D3: \left\{ \begin{array}{ll} \max & (0.5x_1 + 0.5x_2) + (-0.5x_1 + 0.5x_2) \\ \text{s.t.} & x_1^{\,2} + x_2^{\,2} = 1 \\ & 0.5x_1 + 0.5x_2 \geq y_1 \\ & -0.5x_1 + 0.5x_2 \geq y_2 \\ & (f_1(x), f_2(x)) = (x_1, x_2) = (0.5x_1 + 0.5x_2)\cdot b_1 + (-0.5x_1 + 0.5x_2)\cdot b_2 \\ & x_1, x_2 \in R \end{array} \right\},$$

for all $y_1, y_2 \in R$.

4.3.1.1.1. Examples in R^2.

To illustrate the transformation process, five examples of cone-ordered maximization in R^2 are now presented. For $R^n, n \geq 3$, the transformation process is similar.

Example 4.3.1.1.1.1. Consider the following cone-ordered maximization $E1$.

$E1$: $\underset{\mathbf{x}\in A\subset R^m}{C_1 \max}(f_1(\mathbf{x}), f_2(\mathbf{x}))$, where $f_i : R^m \to R, i = 1,2,$ and

$$C_1 = \{\alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}.$$

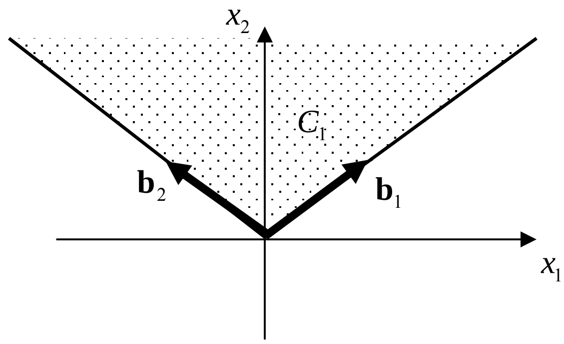

Figure 4.2 The cone $C1$ for problem $E1$.

We formulate an equivalent cone-ordered optimization by using the transformation technique type I and denote it as $E2$.

$$E2 : \left\{ \begin{array}{ll} \underset{\mathbf{x}\in A\subset R^m}{C_2 \max} & (\alpha_1(\mathbf{x}), \alpha_2(\mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in A \\ & (f_1(\mathbf{x}), f_2(\mathbf{x})) = \alpha_1(\mathbf{x})b_1 + \alpha_2(\mathbf{x})b_2 \end{array} \right\}, \text{where}$$

$$C_2 = \{\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 : \mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1), \alpha_1 \geq 0, \alpha_2 \geq 0\}.$$

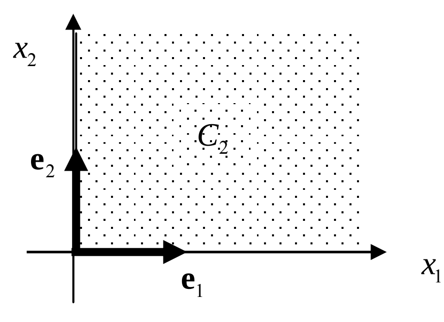

Figure 4.3 The cone $C2$ for problem $E2$.

Notice that $E2$ is Pareto maximization and $E3$ below is Corley's Method (CM) of $E2$.

$$E3:\left\{\begin{array}{ll}\max\limits_{x\in R^m} & \alpha_1(\mathbf{x})+\alpha_2(\mathbf{x}) \\ \text{s.t.} & \alpha_1(\mathbf{x})\ge y_1 \\ & \alpha_2(\mathbf{x})\ge y_2 \\ & \mathbf{x}\in A \\ & (f_1(\mathbf{x}),\, f_2(\mathbf{x}))=\alpha_1(\mathbf{x})b_1+\alpha_2(\mathbf{x})b_2\end{array}\right\},\text{ for all } y_1,y_2\in R.$$

Example 4.3.1.1.1.2. Consider the following cone-order maximization $F1$.

$F1: C_1\max\limits_{\mathbf{x}\in A\subset R^m}(f_1(\mathbf{x}),\, f_2(\mathbf{x}))$, where $f_i: R^2\to R, i=1,2,$ and

$$C_1=\{\alpha_1\mathbf{b}_1+\alpha_2\mathbf{b}_2:\alpha_1\ge 0,\alpha_2>0\}\cup\{(0,0)\}.$$

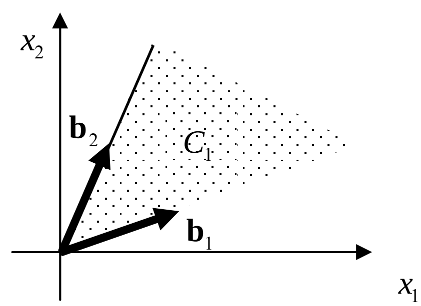

Figure 4.4 The cone $C1$ for problem $F1$.

We formulate an equivalent cone-ordered maximization by using the transformation technique and denote it as $F2$.

$$F2:\left\{\begin{array}{ll}C_2\max\limits_{\mathbf{x}\in A\subset R^m} & (\alpha_1(\mathbf{x}),\alpha_2(\mathbf{x})) \\ \text{s.t.} & \mathbf{x}\in A \\ & (f_1(\mathbf{x}),\, f_2(\mathbf{x}))=\alpha_1(\mathbf{x})b_1+\alpha_2(\mathbf{x})b_2\end{array}\right\},\text{ where}$$

$C_2=\{\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2:\alpha_1\ge 0,\alpha_2>0\}\cup\{(0,0)\}$. Notice that $\overline{C_2}$ is Pareto cone in R^2.

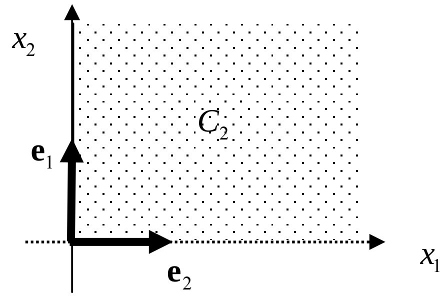

Figure 4.5 The cone $C2$ for problem $F2$.

Since we know that $l(\alpha_1(\mathbf{x}),\alpha_2(\mathbf{x})) = \alpha_1(\mathbf{x}) + \alpha_2(\mathbf{x}) \in C_2^+$, i.e., l is the required strictly positive functional for CM, we will have the equivalent scalarization *F3* below.

$$F3:\left\{\begin{array}{ll} \max\limits_{\mathbf{x}} & \alpha_1(\mathbf{x})+\alpha_2(\mathbf{x}) \\ \text{s.t.} & \alpha_1(\mathbf{x}) \geq y_1 \\ & \alpha_2(\mathbf{x}) > y_2 \\ & \mathbf{x} \in A \\ & (f_1(\mathbf{x}), f_2(\mathbf{x})) = \alpha_1(\mathbf{x})b_1 + \alpha_2(\mathbf{x})b_2 \end{array}\right\} \text{for all } y_1, y_2 \in R.$$

Example 4.3.1.1.1.3. Consider the following cone-order optimization $G1$. $G1: C_1 \max\limits_{\mathbf{x}\in A\subset R^2}(f_1(\mathbf{x}), f_2(\mathbf{x}))$, where $f_i : R^2 \to R, i = 1,2$, and

$$C_1 = \{\alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 : \alpha_1 > 0, \alpha_2 \geq 0\} \cup \{(0,0)\}.$$

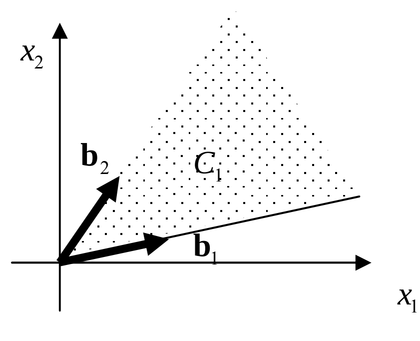

Figure 4.6 The cone $C1$ for problem $G1$.

We formulate an equivalent cone-ordered optimization by using the transformation technique and denote it as $G2$.

$$G2:\left\{\begin{array}{lc} C_2\max\limits_{\mathbf{x}} & (\alpha_1(\mathbf{x}),\alpha_2(\mathbf{x})) \\ \text{s.t.} & \mathbf{x}\in A \\ & (f_1(\mathbf{x}), f_2(\mathbf{x}))=\alpha_1(\mathbf{x})b_1+\alpha_2(\mathbf{x})b_2 \end{array}\right\}, \text{where}$$

$C_2=\{\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2:\mathbf{e}_1=(1,0),\mathbf{e}_2=(0,1),\alpha_1>0,\alpha_2\geq 0\}\bigcup\{(0,0)\}$. Notice that $\overline{C_2}$ is a Pareto cone in R^2.

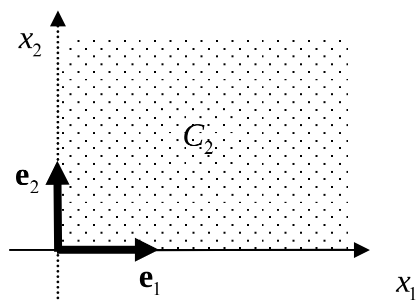

Figure 4.7 The cone $C2$ for problem $G2$.

Since we know that $l(\alpha_1(\mathbf{x}),\alpha_2(\mathbf{x}))=\alpha_1(\mathbf{x})+\alpha_2(\mathbf{x})\in C_2^+$, according to Corley's Method (CM), we will have an equivalent scalarization $G3$ below.

$$G3:\left\{\begin{array}{lc} \max & \alpha_1(\mathbf{x})+\alpha_2(\mathbf{x}) \\ \text{s.t.} & \alpha_1(\mathbf{x})>y_1 \\ & \alpha_2(\mathbf{x})\geq y_2 \\ & \mathbf{x}\in A \\ & (f_1(\mathbf{x}), f_2(\mathbf{x}))=\alpha_1(\mathbf{x})b_1+\alpha_2(\mathbf{x})b_2 \end{array}\right\} \text{for all } y_1, y_2\in R.$$

4.3.1.2 Type II Transformation

We present the second transformation for a cone-ordered maximization where the closure $\overline{C}$ is not pointed. In addition, the cone C must satisfy the followings:

- $\overline{C}$ is the closed half space of R^n such that $\overline{C} = \{\mathbf{x} : \mathbf{p}^t\mathbf{x} \ge 0\}$ where $\mathbf{p}$ is a nonzero vector in R^n.

We illustrate the Type II transformation with the following two cone-ordered examples in R^2.

4.3.1.2.1 Examples in R^2

We here present 2 examples of type II transformation in R^2. For a case in R^n, where $n \ge 3$, the similar transformation process can be applied.

Example 4.3.1.2.1.1.

$H1: C_1 \max\limits_{\mathbf{x} \in A \subset R^2}(f_1(\mathbf{x}), f_2(\mathbf{x}))$ where $f_i : R^2 \to R, i = 1,2,$ and

$$C_1 = \{\alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 : \text{either } \alpha_1 > 0, \alpha_2 \in R \text{ or } \alpha_1 = 0, \alpha_2 > 0\} \cup \{(0,0)\}.$$

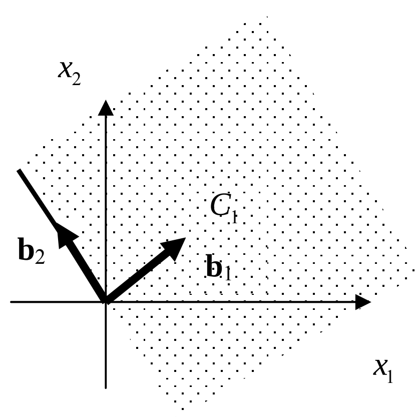

Figure 4.8 The cone $C1$ for problem $H1$.

$$H2: \left\{ \begin{array}{lc} C_2\max\limits_{\mathbf{x}} & (\alpha_1(\mathbf{x}), \alpha_2(\mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in A \\ & (f_1(\mathbf{x}), f_2(\mathbf{x})) = \alpha_1(\mathbf{x})b_1 + \alpha_2(\mathbf{x})b_2 \end{array} \right\} \text{where}$$

$$C_2 = \{\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 : \text{either } \alpha_1 > 0, \alpha_2 \in R \text{ or } \alpha_1 = 0, \alpha_2 > 0, \text{ where } \mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)\}.$$

The cone $C2$ can be obtained from $C1$ by replacing $\mathbf{b}_1$,$\mathbf{b}_2$ with $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. Notice that $C2$ is the lexicographic cone in R^2. Therefore, problem $H2$ can be scalarizable by the previous multiple-stage lexicographic scalarization.

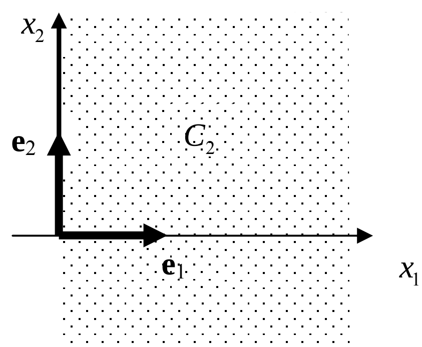

Figure 4.9 The cone $C2$ for problem $H2$.

Since lexicographic maximization and Pareto maximization are equivalent, $H2$ is also scalarizable by CM as follows. Recall that

$$C_2 = \{\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 : \text{either } \alpha_1 > 0, \alpha_2 \in R \text{ or } \alpha_1 = 0, \alpha_2 > 0, \mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)\}.$$

Let $C_3 = \{\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}$ and $C_4 = \{\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 : \alpha_1 \geq 0, \alpha_2 < 0\}$. We have now $C_2 = C_3 \bigcup C_4$, where C_3 is the Pareto cone in R^2, so there is an equivalent Pareto maximization of $H2$ denoted as $H3$ below.

$$H3:\begin{cases} C_3 \max\limits_{\mathbf{x}} & (\alpha_1(\mathbf{x}), \alpha_2(\mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in \arg C_4 \max\limits_{z \in A} (\alpha_1(z), \alpha_2(z)) \\ & \mathbf{x} \in A \\ & (f_1(\mathbf{x}), f_2(\mathbf{x})) = \alpha_1(\mathbf{x}) b_1 + \alpha_2(\mathbf{x}) b_2 \end{cases}.$$

It is significant that $H3$ is a Pareto maximization. The problem $C_4 \max$ is effectively embedded in the constraints of H3, which can be solved by the Type I transformation of the previous section. The general proof of equivalence between problems such $H2$ and $H3$ is given in Theorems 4.3.1.2.1.2. and 4.3.1.2.1.3 below.

Let $f: R^m \to R^n$ and a cone C in R^n which can be written as union of cones C_1 and C_2, i.e., $C = C_1 \bigcup C_2$. Consider the following problems $K1$ and $K2$.

$$K1:\begin{cases} C \max\limits_{\mathbf{x}} & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in A \subset R^n \end{cases} \qquad K2:\begin{cases} C_1 \max\limits_{\mathbf{x}} & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in \arg C_2 \max\limits_{\mathbf{z} \in A} f(\mathbf{z}) \\ & \mathbf{x} \in A \end{cases}.$$

We show that $K1$ and $K2$ are equivalent.

Theorem 4.3.1.2.1.2. If $\mathbf{x}_0$ solves $K1$, then $\mathbf{x}_0$ solves $K2$.

Proof. Assume that $\mathbf{x}_0$ solves $K1$. Then, by definition, there is no $\mathbf{x} \in A, f(\mathbf{x}_0) <_C f(\mathbf{x})$. Since $C_2 \subset C$, there must be no $\mathbf{x} \in A, f(\mathbf{x}_0) <_{C_2} f(\mathbf{x})$. Then $\mathbf{x}_0 \in \arg C_2 \max\limits_{z \in A} f(z)$, i.e., $\mathbf{x}_0$ is feasible to $K2$. Since $C_1 \subset C$, there is also no $\mathbf{x} \in A, f(\mathbf{x}_0) <_{C_1} f(\mathbf{x})$. Thus, $\mathbf{x}_0$ solves $K2$.■

Theorem 4.3.1.2.1.3. If $\mathbf{x}_0$ solves $K2$, then $\mathbf{x}_0$ solves $K1$.

Proof. Assume that $\mathbf{x}_0$ solves $P2$. By its feasibility we have $\mathbf{x}_0 \in A$ but no other $\mathbf{x} \in A$, with $f(\mathbf{x}_0) <_{C_2} f(\mathbf{x})$. By its optimality, there is no $\mathbf{x} \in f(A), f(\mathbf{x}_0) <_{C_1} f(\mathbf{x})$. Since $C = C_1 \bigcup C_2$, there is no $\mathbf{x} \in A, f(\mathbf{x}_0) <_C f(\mathbf{x})$. Thus, $\mathbf{x}_0$ solves $K1$.■

Example 4.3.1.2.1.4.

$L1: C_1 \max\limits_{\mathbf{x} \in A \subset R^2} (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where $f_i : R^2 \to R, i = 1,2,$ and

$$C_1 = \{\alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 : \alpha_1 > 0, \alpha_2 \in R\} \bigcup \{(0,0)\}.$$

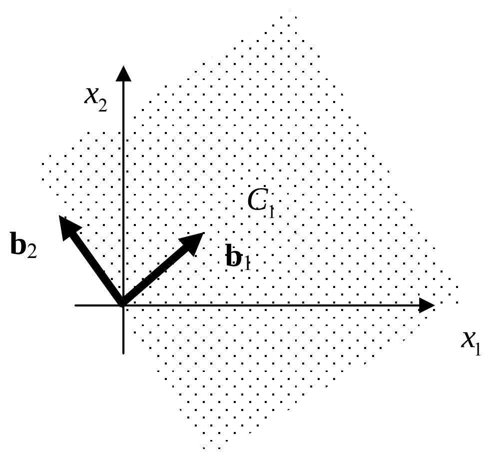

Figure 4.10 The cone $C1$ for problem $L1$.

$$L2: \left\{ \begin{array}{ll} C_2 \max & (\alpha_1(\mathbf{x}), \alpha_2(\mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in A \\ & (f_1(\mathbf{x}), f_2(\mathbf{x})) = \alpha_1(\mathbf{x}) b_1 + \alpha_2(\mathbf{x}) b_2 \end{array} \right\}, \text{ where}$$

$$C_2 = \{\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 : \alpha_1 > 0, \alpha_2 \in R, \mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)\} \bigcup \{(0,0)\}.$$

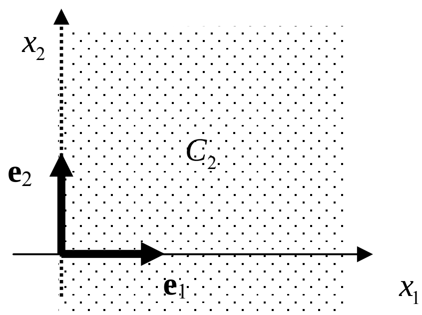

Figure 4.11 The cone $C2$ for problem $L2$.

We have $l(x_1, x_2) = x_1$ for $x_1, x_2 \in R$ is a strictly monotone functional because $l(\mathbf{c} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2) = \alpha_1 > 0$ for any $\mathbf{c} \in C_2$. Therefore CM applies to $L2$ and yields $L3$ below.

$$L3: \left\{ \begin{array}{ll} \max\limits_{\mathbf{x}} & \alpha_1(\mathbf{x}) \\ \text{s.t.} & (\alpha_1(\mathbf{x}), \alpha_2(\mathbf{x})) \geq_{C_2} \mathbf{y} \\ & \mathbf{x} \in A \\ & (f_1(\mathbf{x}), f_2(\mathbf{x})) = \alpha_1(\mathbf{x})\mathbf{b}_1 + \alpha_2(\mathbf{x})\mathbf{b}_2 \end{array} \right\} \text{ for all } \mathbf{y} \in R^2.$$

4.3.2. Lexicographic Hybrid Method

We now incorporate features from the equivalent scalarization *H3* above of lexicographic optimization as well as CM to formulate an equivalent scalarization for a general optimization problem. The beneficial feature in CM is its parameterization technique for obtaining all solutions and only solutions of the original problem. The advantage of the equivalent scalarization for lexicographic optimization is the common objective function for the family of parameterized maximizations when the existence of a common strictly monotone function is not guaranteed.

Motivated by the lexicographic scalarization *H3*, we now develop the idea of having a strictly monotone real-valued function corresponding to each of n components when the others are fixed. Since we are dealing with a general partial order in R^n rather than one induced by a cone, we initially construct n induced orders corresponding to each component of R^n and utilize the separability in the sense of Cantor to provide a strictly monotone function [Theorem 2.5.13] corresponding to each component in R^n with the other components held fixed.

4.3.2.1. Component Orders

Consider a partial order $\leqq$ in R^n. For each $1 \le m \le n,$ define an induced order $\leqq^m$ on R corresponding to the m^{th} component of vectors in R^n as follows. Define

$$a_m \leqq^m b_m \text{ if and only if } (0,...,a_m,...,0) \leqq (0,...,b_m,...,0) \text{ for } a_m, b_m \in R.$$

We first show that the induced order $\leqq^m$ is partially ordered.

Theorem 4.3.2.1.1. The induced order $\leqq^m$ is a partial order in R for any $1 \le m \le n.$

Proof. Let $m \in \{1,...,n\}$. We show that $\leqq^m$ is reflexive, antisymmetric, and transitive.

(Reflexive). Let $a_m \in R.$ Since $\leqq$ is a reflexive in R^n, we must have

$(0,...,a_m,...,0) \leqq (0,...,a_m,...,0).$ Then, by definition, we obtain $a_m \leqq^m a_m.$

(Antisymmetric). Let $a_m, b_m \in R$ such that $a_m \preceqq^m b_m$ and $b_m \preceqq^m a_m$. By definition, we have that $(0,...,a_m,...,0) \preceqq (0,...,b_m,...,0)$ and $(0,...,b_m,...,0) \preceqq (0,...,a_m,...,0)$. Since $\preceqq$ is antisymmetric, we obtain $(0,...,a_m,...,0) = (0,...,b_m,...,0)$. It then follows that $a_m = b_m$.

(Transitive). Let $a_m, b_m, c_m \in R$ such that $a_m \preceqq^m b_m$ and $b_m \preceqq^m c_m$. We also have $(0,...,a_m,...,0) \preceqq (0,...,b_m,...,0)$ and $(0,...,b_m,...,0) \preceqq (0,...,c_m,...,0)$ by definition. Since $\preceqq$ is transitive, we have $(0,...,a_m,...,0) \preceqq (0,...,c_m,...,0)$. Thus $a_m \preceqq^m c_m$.

It follows that $\preceqq^m$ is a partial order in R for any m such that $1 \le m \le n$. ■

Now we can utilize Theorem 2.5.13. There exists a strictly monotone function $l^m : R \to R$ with respect to the order $\preceqq^m$ for any $1 \le m \le n$, because $(R, \preceqq^m)$ is separable in the sense of Cantor, whereas R^n is not.

4.3.2.2. Assumption and Formulation

We present the Lexicographic Hybrid Method (LHM) to be an equivalent scalarization of a general optimization for which the partial order $\preceqq$ satisfies Assumption 4.3.2.2.1 below.

Assumption 4.3.2.2.1. Let $\preceqq$ be a partial order in R^n and $\preceqq^m$ be an induced component order as defined in section 4.3.2.1 for all $1 \le m \le n$. Then the following statements are true.

If $(a_1, a_2, ..., a_n) \preceqq (b_1, b_2, ..., b_n)$, then $a_1 \preceqq^1 b_1$.

If $(c_1, a_2, ..., a_n) \preceqq (c_1, b_2, ..., b_n)$ for some $c_1 \in R$, then $a_2 \preceqq^2 b_2$.

If $(c_1, c_2, a_3, ..., a_n) \preceqq (c_1, c_2, b_3, ..., b_n)$ for some $c_1, c_2 \in R$, then $a_3 \preceqq^3 b_3$.

$$\vdots$$

If $(c_1, c_2, c_3, ..., c_{n-1}, a_n) \preceqq (c_1, c_2, c_3, ..., c_{n-1}, b_n)$ for some $c_1, c_2, ..., c_{n-1} \in R$, then $a_n \preceqq^n b_n$.

Proposition 4.3.2.2.2. The orders induced by the Pareto and lexicographic cones satisfy Assumption 4.3.2.2.1.

As a consequence of Proposition 4.3.2.2.2, LHM will be an equivalent scalarization of a cone-ordered optimization that is equivalent to Pareto maximization or lexicographic maximization.

$M1$ below denotes a general optimization problem where the partial order $\preceqq$ satisfies Assumption 4.3.2.2.1. The problem $M2$ is an equivalent LHM scalarization for $M1$.

$$M1\text{: } \underset{x \in A}{\text{opt}}(f_1(\mathbf{x}), ..., f_n(\mathbf{x})) \text{ where } f_i : R^m \to R, i = 1, ..., n, A \subset R^m.$$

$$
M2(\mathbf{y}):\left\{\begin{array}{ll}
\max\limits_{\mathbf{x}} & l^n(f_n(\mathbf{x})) = (0,\ldots,0,1)^T \cdot (l^1(f_1(\mathbf{x})),\ldots,l^n(f_n(\mathbf{x}))) \\
\text{s.t.} & f(\mathbf{x}) \geqq \mathbf{y} \\
& l^1(f_1(\mathbf{x})) = a_1(\mathbf{y}) \\
& l^2(f_2(\mathbf{x})) = a_2(\mathbf{y}) \\
& \vdots \\
& l^{n-2}(f_{n-2}(\mathbf{x})) = a_{n-2}(\mathbf{y}) \\
& l^{n-1}(f_{n-1}(\mathbf{x})) = a_{n-1}(\mathbf{y}) \\
& \mathbf{x} \in A
\end{array}\right\} \text{ for all } \mathbf{y} \in f(A),
$$

where $a_m(\mathbf{y}), m = 1,\ldots,n-1,$ is the optimal objective values of the following problems.

$$
M(\mathbf{y},1):\left\{\begin{array}{ll}
\max & l^1(f_1(\mathbf{x})) \\
\text{s.t.} & f(\mathbf{x}) \geqq \mathbf{y} \\
& \mathbf{x} \in A
\end{array}\right\} \text{ for } m\text{=1 and}
$$

$$
M(\mathbf{y},m):\left\{\begin{array}{ll}
\max\limits_{\mathbf{x}} & l^m(f_m(\mathbf{x})) \\
\text{s.t.} & l^1(f_1(\mathbf{x})) = a_1(\mathbf{y}) \\
& \\
& l^{m-1}(f_{m-1}(\mathbf{x})) = a_{m-1}(\mathbf{y}) \\
& f(\mathbf{x}) \geqq \mathbf{y} \\
& \mathbf{x} \in A
\end{array}\right\} \text{ for } 2 \le m \le n-1.
$$

Lemma 4.3.2.2.3. If $\mathbf{x}_0$ solves $M1$, then $\mathbf{x}_0$ is feasible to $M2(\mathbf{y})$ for $\mathbf{y} = f(\mathbf{x}_0)$.

Proof. Let $\mathbf{x}_0$ solve $M1$. By the optimality of $\mathbf{x}_0$, if $f(\mathbf{x}) \geqq f(\mathbf{x}_0)$ for $\mathbf{x} \in A$, we must have $f(\mathbf{x}) = f(\mathbf{x}_0)$. Thus $\mathbf{x}_0$ solves $M(\mathbf{y},1),\ldots,M(\mathbf{y},n-1)$, where $\mathbf{y} = f(\mathbf{x}_0)$. We conclude that $l^k(f_k(\mathbf{x}_0)) = a_k(\mathbf{y} = f(\mathbf{x}_0))$ for any $1 \le k \le n-1$. It is obvious that $f(\mathbf{x}_0) \geqq f(\mathbf{x}_0)$, so $\mathbf{x}_0$ is feasible to $M2(\mathbf{y})$ for $\mathbf{y} = f(\mathbf{x}_0)$. ■

Theorem 4.3.2.2.4. If $\mathbf{x}_0$ solves $M1$, then $\mathbf{x}_0$ solves *M2*(**y**) for $\mathbf{y} = f(\mathbf{x}_0)$.

Proof. Assume $\mathbf{x}_0$ solves $M1$. By Lemma 4.3.2.2.3, $\mathbf{x}_0$ is feasible to $M2(f(\mathbf{x}_0))$; i.e., $l^1(f_1(\mathbf{x}_0)) = a_1,..., l^{n-1}(f_{n-1}(\mathbf{x}_0)) = a_{n-1}$, and $f(\mathbf{x}_0) \geqq f(\mathbf{x}_0)$. To obtain a contradiction, suppose $\mathbf{x}_0$ does not solve $M2 f(\mathbf{x}_0)$. Then there exists a feasible solution $\mathbf{x}_1 \in A$ such that $l^n(f_n(\mathbf{x}_1)) > l^n(f_n(\mathbf{x}_0))$. Since $\mathbf{x}_1$ is feasible to $M2 f(\mathbf{x}_0)$, we have $f(\mathbf{x}_1) \succeq f(\mathbf{x}_0)$. Since $l^n(f_n(\mathbf{x}_1)) > l^n(f_n(\mathbf{x}_0))$, it follows that $f_n(\mathbf{x}_1)$ does not equal to $f_n(\mathbf{x}_0)$; i.e., $f(\mathbf{x}_1) \neq f(\mathbf{x}_0)$. Then $f(\mathbf{x}_1) \succ f(\mathbf{x}_0)$, an inequality contradicting the optimality of $\mathbf{x}_0$. ■

Theorem 4.3.2.2.5. If $\mathbf{x}_0$ solves *M2*(**y**) for $\mathbf{y} \in f(A)$, then $\mathbf{x}_0$ solves $M1$.

Proof. Assume that $\mathbf{x}_0$ solves *M2*(**y**) for $\mathbf{y} \in f(A)$. Then $\mathbf{x}_0$ is feasible to $M1$ and $\mathbf{y} \preceq (f_1(\mathbf{x}_0),..., f_n(\mathbf{x}_0))$. Let $\mathbf{x}_1$ be any feasible solution to $M1$ such that

$$\mathbf{y} \preceq (f_1(\mathbf{x}_0),..., f_n(\mathbf{x}_0)) \preceq (f_1(\mathbf{x}_1),..., f_n(\mathbf{x}_1)).$$

By Assumption 4.3.2.2.1, we have that $f_1(\mathbf{x}_0) \preceq^1 f_1(\mathbf{x}_1)$. Since

$$l^1(f_1(\mathbf{x}_0)) = \max\{l^1(f_1(\mathbf{x})) : f(\mathbf{x}) \succeq \mathbf{y}, \mathbf{x} \in A\},$$

we obtain $f_1(\mathbf{x}_0) = f_1(\mathbf{x}_1)$. Again by Assumption 4.3.2.2.1, $f_2(\mathbf{x}_0) \preceq^2 f_2(\mathbf{x}_1)$. Since

$$l^2(f_2(\mathbf{x}_0)) = \max\{l^2(f_2(\mathbf{x})) : f(\mathbf{x}) \succeq \mathbf{y}, l^1(f(\mathbf{x})) = l^1(f(\mathbf{x}_0)), \mathbf{x} \in A\},$$

it follows that $f_2(\mathbf{x}_0) = f_2(\mathbf{x}_1)$. Applying a similar argument sequentially, we finally get $f_3(\mathbf{x}_0) = f_3(\mathbf{x}_1),..., f_n(\mathbf{x}_0) = f_n(\mathbf{x}_1)$, respectively. Thus $\mathbf{x}_0$ solves $M1$. ■

Example 4.3.2.2.6

Consider the following Pareto maximization problem.

$\operatorname{V}\max_{x\in A}(x_1^{\ 2}, x_2^{\ 2})$ s.t. $A = \{(x_1, x_2) : x_1^{\ 2} + x_2^{\ 2} \leq 1, 0 \leq x_1, x_2 \leq 1\} \subset R^2$.

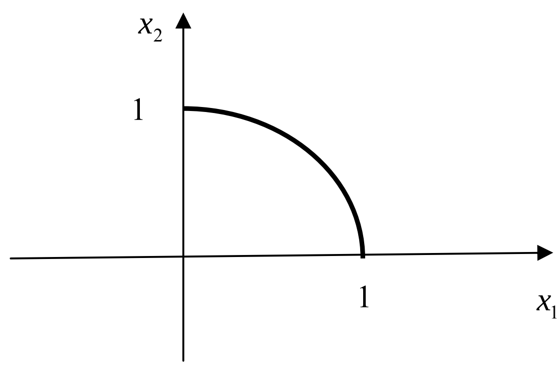

Figure 4.12 Pareto frontier of example 4.3.2.2.6.

We solve the Pareto maximization with LHM as follows. Define the induced orders on each component by $x_1 \leq^1_{Pareto} y_1 \leftrightarrow (x_1, 0) \leq^1_{Pareto} (y_1, 0)$ for $x_1, y_1 \in R,$ and

$$x_2 \leq^2_{Pareto} y_2 \leftrightarrow (0, x_2) \leq^1_{Pareto} (0, y_2) \text{ for } x_2, y_2 \in R.$$

Formulate LHM scalar equivalence as

$$N1(y_1, y_2) : \left\{ \begin{array}{c} \max\limits_{x_1, x_2} \quad (0,1)^T \cdot (x_1, x_2) = (x_2) \\ x_1 \geq y_1 \\ x_2 \geq y_2 \\ x_1 = a_1(y_1, y_2) \\ x_1^{\ 2} + x_2^{\ 2} \leq 1 \\ 0 \leq x_1, x_2 \leq 1 \end{array} \right\} \text{ for all } (y_1, y_2) \in f(A),$$

where $f(A) = \{(x_1, x_2) : x_1^{\ 2} + x_2^{\ 2} \leq 1, 0 \leq x_1, x_2 \leq 1\}$.

In addition, $a_1(y_1, y_2)$ is the optimal objective value of the following problem.

$$N2(y_1, y_2): \left\{ \begin{array}{ll} \max & x_1 \\ \text{s.t.} & x_1 \geq y_1 \\ & x_2 \geq y_2 \\ & x_1^2 + x_2^2 \leq 1 \\ & 0 \leq x_1, x_2 \leq 1 \end{array} \right\}.$$

We select $(y_1, y_2) = (0.5, 0.5)$. By solving $N2(0.5, 0.5)$, we obtain $a_1(0.5, 0.5) = 0.866$. Then we solve $N1(0.5, 0.5)$ and obtain the optimal solution $(x_1^*, x_2^*) = (0.866, 0.5)$. Notice that (0.866,0.5) is on the Pareto frontier in figure 4.12 above. To obtain all solutions, we would need to solve $S(y_1, y_2)$ for all values of y_1 and y_2.

To show a case where LHM cannot be used to solve an optimization problem, we present a non-cone optimization in the next example.

Example 4.3.2.2.8.

Consider the following Hasse diagram (a diagram representing a partial order relation) on the next page.

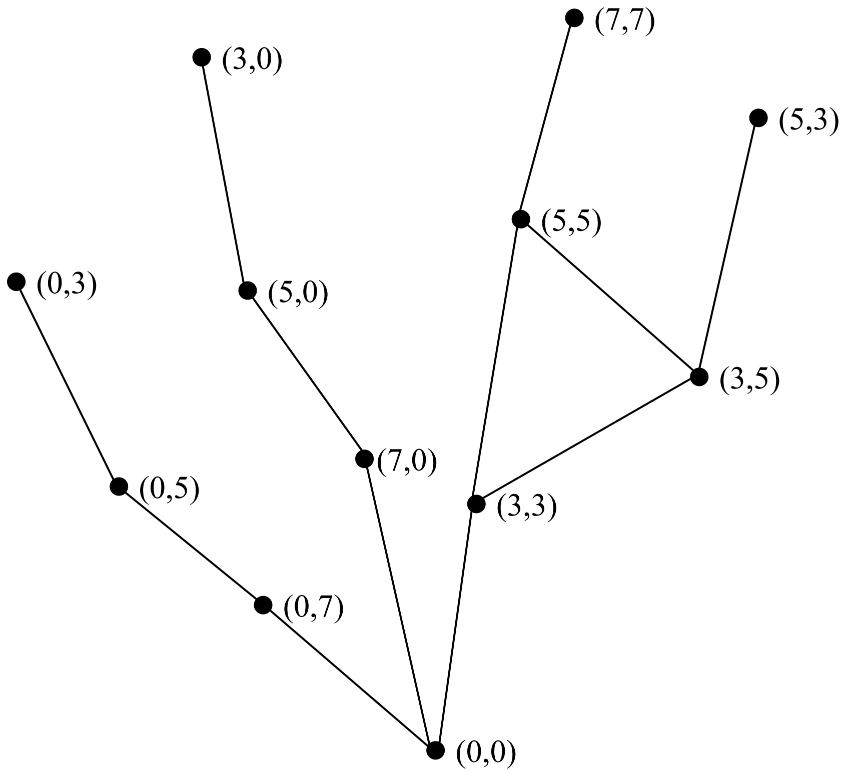

Figure 4.13 Hasse diagram.

In the diagram, if (x_1, y_1) is below and connected to (x_2, y_2), we write $(x_1, y_1) \preceq (x_2, y_2)$ so $(0,0) \preceq (7,0)$ and $(3,3) \preceq (3,5)$, for example. This construction is a partial order but not induced by a cone. From the diagram, we have the following relations.

1. $(0,0) \preceq (0,7) \preceq (0,5) \preceq (0,3)$.
2. $(0,0) \preceq (7,0) \preceq (5,0) \preceq (3,0)$.
3. $(0,0) \preceq (3,3) \preceq (5,5) \preceq (7,7)$.
4. $(0,0) \preceq (3,3) \preceq (3,5) \preceq (5,5) \preceq (7,7)$.

5. $(0,0) \preceqq (3,3) \preceqq (3,5) \preceqq (5,3)$.

Now, let $A = \{(0,0),(0,7),(0,5),(0,3),(7,0),(5,0),(3,0),(3,3),(3,5),(5,3),(5,5),(7,7)\} \subset R^2$. Define $f : A \to A$ by $f(x_1, x_2) = (x_1, x_2)$ for all $(x_1, x_2) \in A$. Therefore we have $f(0,7) = (0,7)$, for example. Now consider $P1$ below.

$$P1\text{: } \text{opt}\{f(x_1, x_2) : (x_1, x_2) \in A\}.$$

We now attempt to find the optimal solutions (non-dominated solutions) by LHM. Define $a_1 \preceqq^{(1)} b_1 \leftrightarrow (a_1, 0) \preceqq (b_1, 0)$ and $a_2 \preceqq^{(2)} b_2 \leftrightarrow (0, a_2) \preceqq (0, b_2)$. Therefore

$0 \preceqq^{(1)} 7 \preceqq^{(1)} 5 \preceqq^{(1)} 3$ and $0 \preceqq^{(2)} 7 \preceqq^{(2)} 5 \preceqq^{(2)} 3$.

LHM next yields $P2$ with identity functions for the strictly monotone functions l^1, l^2

$$P2(\mathbf{y}) : \left\{ \begin{array}{ll} \max & l^2(f_2(x_1, x_2)) = f_2(x_1, x_2) = x_2 \\ \text{s.t.} & x_1 = a_1(\mathbf{y}) \\ & f(x_1, x_2) \succeqq \mathbf{y} \\ & (x_1, x_2) \in A \end{array} \right\} \text{ for all } \mathbf{y} \in f(A),$$

where $a_1(\mathbf{y}) = \max\{l^1(f_1(x_1, x_2)) = f_1(x_1, x_2) = x_1 : \mathbf{x} \in A, \mathbf{x} \succeqq \mathbf{y}\}$.

Notice that $\max_{(1)}$ is maximization with respect to $\preceqq^{(1)}$ and $\max_{(2)}$ is maximization with respect to $\preceqq^{(2)}$. Recall that $0 \preceqq^{(1)} 7 \preceqq^{(1)} 5 \preceqq^{(1)} 3$ and $0 \preceqq^{(2)} 7 \preceqq^{(2)} 5 \preceqq^{(2)} 3$. Let $\mathbf{y} = (3,3)$. We then obtain $a_1((3,3)) = 3$, by solving $\max_{(1)}\{f_1(x_1, x_2) = x_1 : x \in A, (x_1, x_2) \succeqq (3,3)\} = 3$. Then $P2(3,3)$ becomes $a_1((3,3)) = 3$,

$$P2(3,3):\left\{\begin{array}{ll}\max\limits_{x_1,x_2} & x_2 \\ \text{s.t.} & x_1 = 3 \\ & f(x_1,x_2) \geqq (3,3) \\ & (x_1,x_2) \in A\end{array}\right\}.$$

We obtain the optimal solution $(x_1*, x_2*) = (3,3)$. However, $(3,3)$ is not an optimal solution to the original problem because $(3,3) \preceqq (3,5) \preceqq (5,5) \preceqq (7,7)$. Thus LHM provides a non-optimal solution to the original optimization problem. ■

4.3.3 Summary and Diagrams of Equivalent Scalarizations

We summarize the equivalent scalarizations of a general optimization below.

1. Corley's Method (CM) is an equivalent scalarization for cone-ordered optimization where the cone is pointed and convex and transformation techniques are applicable.
2. The Lexicographic Hybrid Method (LHM) is an equivalent scalarization for cone-ordered optimization where the cone is pointed and convex, Assumption 4.3.2.2.1 is satisfied and separability in the sense of Cantor is available for each component order. Any cone-ordered optimization that is scalarizable by CM can be solved by LHM.
3. LHM is also an equivalent scalarization for non-cone optimization where Assumption 4.3.2.2.1 is satisfied.
4. A general optimization under a non-cone partial order does not use LHM according to example 4.3.2.2.8. However, the problem in example 4.3.2.2.8 is scalarizable by other approaches.

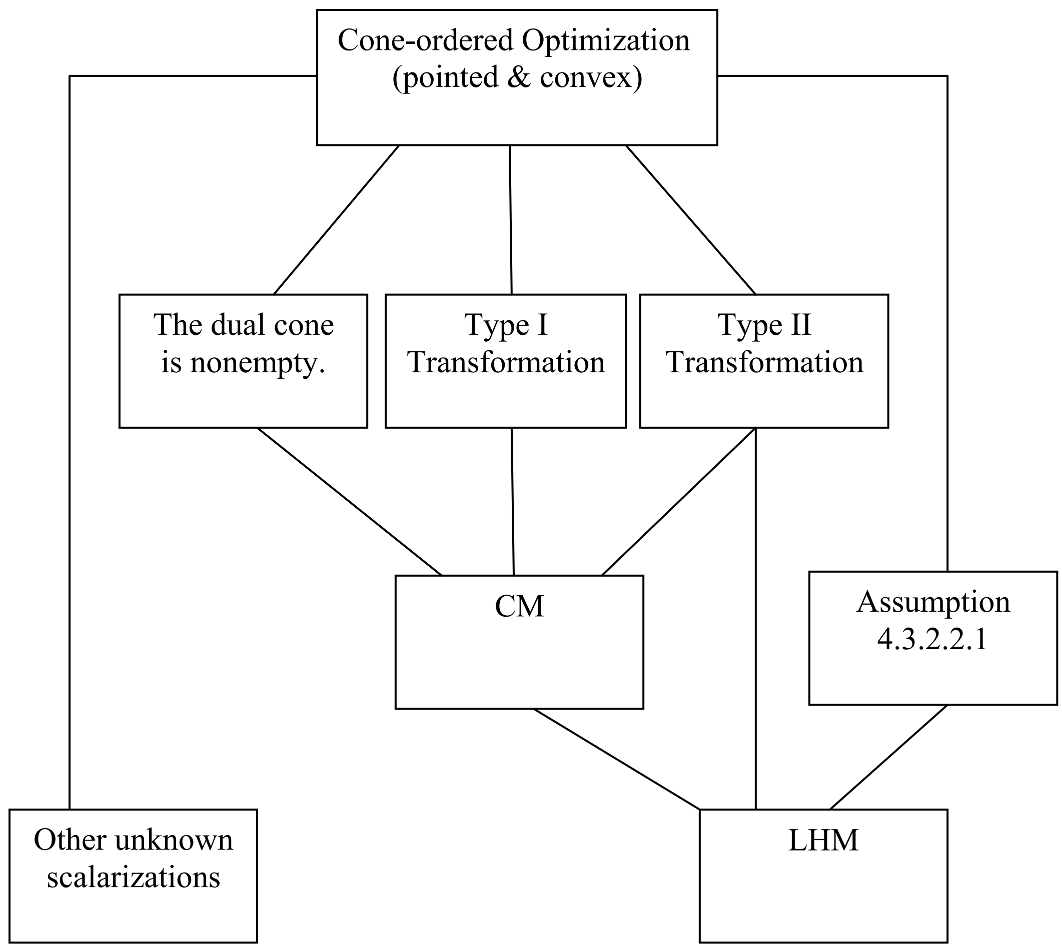

Figure 4.14 Scalarization diagram for cone optimization.

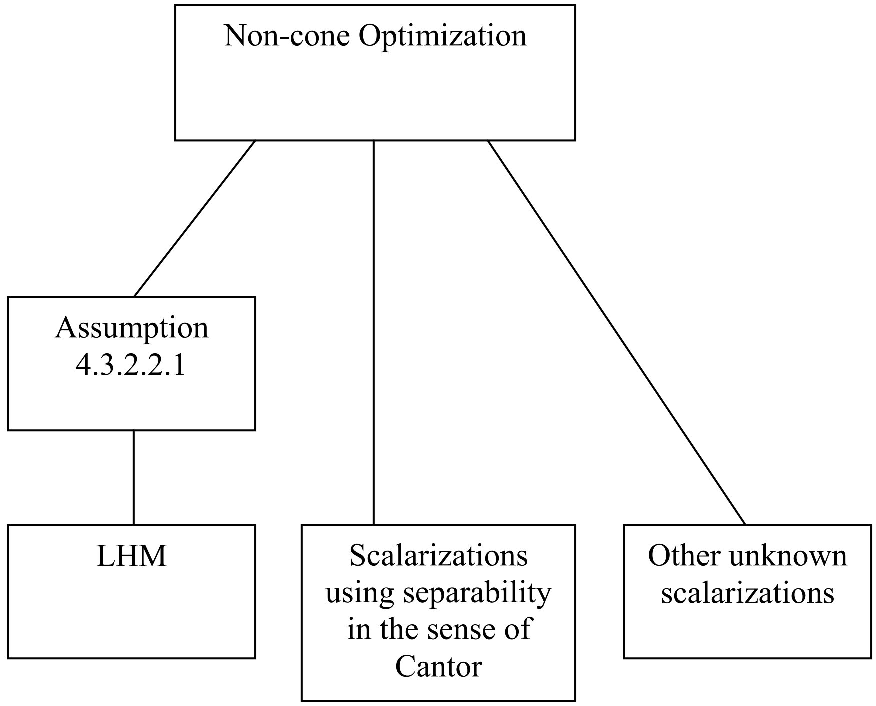

Figure 4.15 Scalarization diagram for non-cone optimization.

4.4. Examples of Optimization Criteria

We now verify that standard optimization criteria satisfy the requirements for a general optimization criterion.

4.4.1. Standard Optimization Problems

Standard optimizations include a cone-ordered optimization, set-valued optimization, goal programming, and maximin problem. Lexicographic optimization, Pareto optimization, and scalar optimization are special cases of cone-ordered optimization.

4.4.1.1. Cone-Ordered maximization

If a cone is pointed and convex then the order induced the cone is a partial order according to Remark 2.5.14. Therefore Axiom 1 (APO) satisfies for the pointed and convex cone. In addition, if any cone-ordered optimization is scalarizable, then Axiom 2 (ASP) is satisfied. From the results in Chapter 3 and the scalarization diagram in figure 4.14, an optimization problem with cones such as lexicographic optimization, Pareto optimization including scalar case are general optimization criteria. Any scalarizable coned-order optimization will be an optimization criterion.

4.4.1.2. Set-Valued Maximization

Consider the following set-valued maximization.

$\max_{\mathbf{x}\in A} F(\mathbf{x})$, where $F : R^m \to 2^{R^n}$ is a point-to-set map, where the order in R^n is induced by a pointed convex cone C in R^n. According to Remark 2.5.10, the order induced by pointed convex cone is a partial order. Therefore Axiom APO satisfies. Moreover, the problem is scalarizable according to section 3.2.3. Thus set-valued maximization is an optimization criterion.

4.4.1.3. Maximin Problem

A maximin optimization is a scalar maximization where the objective function is a minimization. Thus it is an optimization criterion.

4.4.1.4. Goal Programming

Goal programming can be considered as a Pareto maximization which is an optimization criterion.

CHAPTER 5

NEW OPTIMIZATION CRITERIA WITH APPLICATIONS

In this chapter we develop new optimization criteria including compromise problem and randomization.

5.1. Compromise Criterion

Let $f : R^m \to R^n$ be a nonnegative objective function for the Pareto maximization $\operatorname*{Vmax}_{\mathbf{x} \in A \subset R^m} (f_1(\mathbf{x}),..., f_n(\mathbf{x}))$. Assume that $-\infty < \min_{\mathbf{x} \in A} f_i(\mathbf{x})$ and $\max_{\mathbf{x} \in A} f_i(\mathbf{x}) < \infty$ for all *i*. Denote $M_i = \max_{\mathbf{x} \in A} f_i(\mathbf{x})$ and $m_i = \min_{\mathbf{x} \in A} f_i(\mathbf{x})$. Now define $T_{Compr} : f(A) \to R$ by

$$T_{Compr}(f_1(\mathbf{x}),..., f_n(\mathbf{x})) = [(\frac{f_1(\mathbf{x}) - m_1 + 1}{M_1 - m_1 + 1}) \times ... \times (\frac{f_n(\mathbf{x}) - m_n + 1}{M_n - m_n + 1})], \text{ for all } \mathbf{x} \in A.$$

Define a strictly compromise order on $f(A)$ as follows.

For any $f(\mathbf{x}_1), f(\mathbf{x}_2) \in f(A)$, $f(\mathbf{x}_1) <_{Compr} f(\mathbf{x}_2)$ if and only if either (1) or (2), where

1. If $f(\mathbf{x}_1)$ and $f(\mathbf{x}_2)$ are comparable according to the Pareto order, then $f(\mathbf{x}_1) <_{Pareto} f(\mathbf{x}_2)$.
2. If $f(\mathbf{x}_1)$ and $f(\mathbf{x}_2)$ are not comparable according to the Pareto order, then $T_{Compr}(f(\mathbf{x}_1)) < T_{Compr}(f(\mathbf{x}_2))$.

If $f(\mathbf{x}_1)$ and $f(\mathbf{x}_2)$ are not comparable according to Pareto order and $T_{Compr}(f(\mathbf{x}_1)) = T_{Compr}(f(\mathbf{x}_2))$, we say that $f(\mathbf{x}_1)$ and $f(\mathbf{x}_2)$ are not comparable with respect to the compromise order. Next, define the compromise order $\leq_{Compr}$ by

$$f(\mathbf{x}_1) \leq_{Compr} f(\mathbf{x}_2) \text{ if and only if } f(\mathbf{x}_1) <_{Compr} f(\mathbf{x}_2) \text{ or } f(\mathbf{x}_1) = f(\mathbf{x}_2).$$

A compromising problem can be written as $\underset{\mathbf{x}\in A}{\text{Compromise}}\, f(\mathbf{x})$ or $\underset{\mathbf{x}\in A}{\text{Opt}}\, f(\mathbf{x})$ with respect to $\leq_{Compr}$. The problem is to find a vector $\mathbf{x}^* \in A \subset X$ for which there is no vector $\mathbf{x} \in A$ such that $f(\mathbf{x}^*) <_{Compr} f(\mathbf{x})$, or equivalently that $f(\mathbf{x}^*) \leq_{Compr} f(\mathbf{x})$ and $f(\mathbf{x}^*) \neq f(\mathbf{x})$.

Lemma 5.1.1. For any $f(\mathbf{x}), f(\mathbf{y}) \in f(A)$, if $f(\mathbf{x}) <_{Pareto} f(\mathbf{y})$ then

$$T_{Compr}(f(\mathbf{x})) < T_{Compr}(f(\mathbf{y})).$$

Proof. Let $f(\mathbf{x}), f(\mathbf{y}) \in f(A)$, such that $f(\mathbf{x}) <_{Pareto} f(\mathbf{y})$. Then, $0 \leq f_i(\mathbf{x}) \leq f_i(\mathbf{y})$ for all $i = 1,\dots,n$ and $0 \leq f_j(\mathbf{x}) < f_j(\mathbf{y})$ for some index *j*. Since all elements in $f(A)$ are nonnegative and definition of m_i and M_i, we have

$$0 \leq \frac{f_i(\mathbf{x}) - m_i + 1}{M_i - m_i + 1} \leq \frac{f_i(\mathbf{y}) - m_i + 1}{M_i - m_i + 1}, \text{ for all } i = 1,\dots,n, \text{ and}$$

$$0 \leq \frac{f_j(\mathbf{x}) - m_j + 1}{M_j - m_j + 1} < \frac{b_j - m_j + 1}{M_j - m_j + 1} \text{ for some index } j.$$

It follows that

$$T_{Compr}(f(\mathbf{x})) = \prod_{i=1}^{n} \frac{f_i(\mathbf{x}) - m_i + 1}{M_i - m_i + 1} < \prod_{i=1}^{n} \frac{f_i(\mathbf{y}) - m_i + 1}{M_i - m_i + 1} = T_{Compr}(f(\mathbf{y})).\ \blacksquare$$

Lemma 5.1.2. If $f(\mathbf{x}) \in \text{Compromise } f(A)$, then $f(\mathbf{x}) \in \text{Vmax } f(A)$.

Proof. Assume that $f(\mathbf{x}) \in \text{Compromise } f(A)$. To obtain a contradiction, suppose that $f(\mathbf{x}) \notin \text{Vmax } f(A)$. Then there exist $f(\mathbf{y}) \in f(A)$ such that $f(\mathbf{x}) <_{Pareto} f(\mathbf{y})$. It follows that $f(\mathbf{x}) <_{Compr} f(\mathbf{y})$ which contradicts with optimality of $f(\mathbf{x})$. We conclude that $f(\mathbf{x}) \in \text{Vmax } f(A)$. ■

Lemma 5.1.3. Compromise $f(A) \subset \text{Vmax } f(A)$.

Proof. It follows directly from Lemma 5.1.2. ■

Theorem 5.1.4. The preference order $\leq_{Compr}$ is a partial order on $f(A)$.

Proof. We show that $\leq_{Compr}$ is reflexive, transitive, and antisymmetric.

(Reflexive). Since $f(\mathbf{x}) = f(\mathbf{x})$, we have $f(\mathbf{x}) \leq_{Compr} f(\mathbf{x})$ for any $f(\mathbf{x}) \in f(A)$.

(Transitive).Let $f(\mathbf{x}) \leq_{Compr} f(\mathbf{y})$ and $f(\mathbf{y}) \leq_{Compr} f(\mathbf{z})$ for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in A$.

Case 1: $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$ and $f(\mathbf{y}) \leq_{Pareto} f(\mathbf{z})$.

Since Pareto order is transitive, we have that $f(\mathbf{x})$ comparable to $f(\mathbf{z})$ and in particular $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{z})$. Therefore, $f(\mathbf{x}) \leq_{Compr} f(\mathbf{z})$.

Case 2: $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$ and $f(\mathbf{y})$ are not Pareto comparable with $f(\mathbf{z})$ with $T_{Compr}(f(\mathbf{y})) < T_{Compr}(f(\mathbf{z}))$.

Case 2.1: $f(\mathbf{x})$ is Pareto comparable with $f(\mathbf{z})$.

We claim that $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{z})$. Suppose that $f(\mathbf{z}) <_{Pareto} f(\mathbf{x})$. By Lemma 5.1.1., we have $T_{Compr}(f(\mathbf{z})) < T_{Compr}(f(\mathbf{x}))$. Since $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$ and by Lemma 5.1.1, we have $T_{Compr}(f(\mathbf{x})) < T_{Compr}(f(\mathbf{y}))$. Therefore we obtain $T_{Compr}(f(\mathbf{z})) < T_{Compr}(f(\mathbf{y}))$ in contradiction to the assumption that $T_{Compr}(f(\mathbf{y})) < T_{Compr}(f(\mathbf{z}))$. We conclude that $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{z})$. Thus $f(\mathbf{x}) \leq_{Compr} f(\mathbf{z})$.

Case 2.2: $f(\mathbf{x})$ is not Pareto comparable with $f(\mathbf{z})$.

Since $f(\mathbf{x}) \leq_{Pareto} f(\mathbf{y})$ by Lemma 5.1.1, we have $T_{Compr}(f(\mathbf{x})) \leq T_{Compr}(f(\mathbf{y}))$. Combining with $T_{Compr}(f(\mathbf{y})) < T_{Compr}(f(\mathbf{z}))$, we obtain $T_{Compr}(f(\mathbf{x})) \leq T_{Compr}(f(\mathbf{z}))$, i.e., $f(\mathbf{x}) \leq_{Com} f(\mathbf{z})$.

Case 2.3: $f(\mathbf{y}) \leq_{Pareto} f(\mathbf{z})$ and $f(\mathbf{x})$ are not comparable with $f(\mathbf{y})$ with $T_{Compr}(f(\mathbf{x})) < T_{Compr}(f(\mathbf{y}))$. The proof is similar to Case 2.1.

From Case 1 and 2, we obtain $f(\mathbf{x}) \leq_{Compr} f(\mathbf{z})$.

(Anti-Symmetric). Let $f(\mathbf{x}) \leq_{Compr} f(\mathbf{y})$ and $f(\mathbf{y}) \leq_{Compr} f(\mathbf{x})$. We must have $f(\mathbf{x}) = f(\mathbf{y})$. To obtain a contradiction, suppose that $f(\mathbf{x}) \neq f(\mathbf{y})$. Immediately we have $f(\mathbf{x}) <_{Compr} f(\mathbf{y})$ and $f(\mathbf{y}) <_{Compr} f(\mathbf{x})$.

Case 1: $f(\mathbf{x})$ is Pareto comparable to $f(\mathbf{y})$.

Since $f(\mathbf{x}) <_{Compr} f(\mathbf{y})$, we obtain $f(\mathbf{x}) <_{Pareto} f(\mathbf{y})$. Since $f(\mathbf{y}) <_{Compr} f(\mathbf{x})$, we obtain $f(\mathbf{y}) <_{Pareto} f(\mathbf{x})$, which contradicts the previous conclusion.

Case 2: $f(\mathbf{x})$ is not comparable to $f(\mathbf{y})$.

Since $f(\mathbf{x}) <_{Compr} f(\mathbf{y})$, we have $T_{Compr}(f(\mathbf{x})) < T_{Compr}(f(\mathbf{y}))$. Also, since $f(\mathbf{y}) <_{Compr} f(\mathbf{x})$, we have $T_{Compr}(f(\mathbf{y})) < T_{Compr}(f(\mathbf{x}))$, contradicting the above fact that $T_{Compr}(f(\mathbf{x})) < T_{Compr}(f(\mathbf{y}))$.

From Case 1 and 2, we conclude that $f(\mathbf{x}) = f(\mathbf{y})$.

Thus we conclude that $\leq_{Compr}$ is a partial order on $f(A)$. ■

An equivalent scalarization of the compromise optimization problem is

$$\begin{Bmatrix} \max\limits_{\mathbf{x}\in A} & T_{Compr}(f(\mathbf{x})) \\ \text{s.t.} & \mathbf{x} \in A \end{Bmatrix}.$$

5.1.1. An Application in Multi-objective Optimization

In a Pareto maximization problem, a decision maker often selects as a solution to the problem a non-dominated point satisfying some secondary criteria. More generally, after a multi-objective optimization, a secondary criteria is invoked - such as choosing the largest summation of the objective function values. The secondary criterion here will be to select a solution that attempts to equitably distribute the benefit among all objectives. Indeed, the compromise solution applied to the objective function can accomplish both the primary and secondary criteria because of Lemma 5.1.2.

Consider the following Pareto maximization

$$\left\{\begin{array}{lc} \text{Vmax} & (f_1(x_1,x_2), f_2(x_1,x_2)) = (x_1,x_2) \\ \text{s.t.} & x_1^2 + x_2^2 \le 1 \\ & x_1, x_2 \ge 0 \end{array}\right\}.$$

The Pareto frontier is shown in Figure 5.1 below.

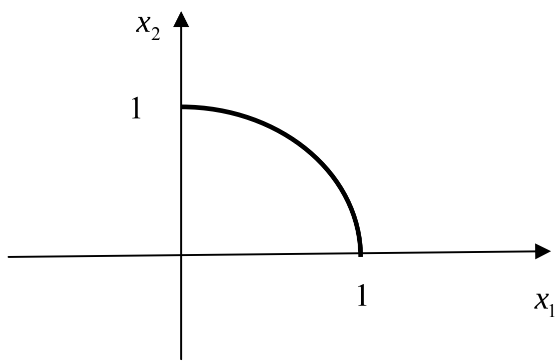

Figure 5.1 Pareto frontier.

Let $A = \{(x_1, x_2) : x_1^2 + x_2^2 \le 1, x_1, x_2 \ge 0\}$. We find $M_i = \max_{\mathbf{x}\in A} f_i(\mathbf{x})$ and $m_i = \min_{\mathbf{x}\in A} f_i(\mathbf{x})$ for $i = 1,2$.

- $M_1 = [\max_{\mathbf{x}\in A}(f_1(\mathbf{x}) = x_1)] = 1$, $M_2 = [\max_{\mathbf{x}\in A}(f_2(\mathbf{x}) = x_2)] = 1$.
- $m_1 = [\min_{\mathbf{x}\in A}(f_1(\mathbf{x}) = x_1)] = 0$, $m_2 = [\min_{\mathbf{x}\in A}(f_2(\mathbf{x}) = x_2)] = 0$.
- The compromise transformation function is as follows.

$$\left\{\begin{array}{lcc} T_{Compr}(f_1(\mathbf{x}), f_2(\mathbf{x})) & = & [(\frac{f_1(\mathbf{x}) - m_1 + 1}{M_1 - m_1 + 1}) \cdot (\frac{f_2(\mathbf{x}) - m_2 + 1}{M_2 - m_2 + 1})] \\ & = & [(\frac{x_1 + 1}{2}) \cdot (\frac{x_2 + 1}{2})] \end{array}\right\} \text{for } x \in A.$$

The compromise problem, with the order $\leq_{Compr}$, can be formulated as follows

$$\left\{\begin{array}{ll} \text{Compromise} & (f_1(x_1,x_2), f_2(x_1,x_2)) = (x_1,x_2) \\ \quad\text{s.t.} & x_1^2 + x_2^2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}\right\}.$$

An equivalent scalarization is as follows.

$$\left\{\begin{array}{ll} \max & T_{Compr}(f(\mathbf{x})) = (\frac{x_1+1}{2}) \cdot (\frac{x_2+1}{2}) \\ \text{s.t.} & x_1^2 + x_2^2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}\right\}.$$

The solution is $(x_1^*, x_2^*) = (0.707, 0.707)$ with objective value of 0.729.

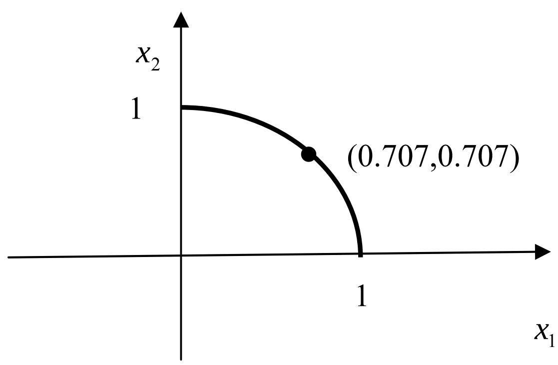

Figure 5.2 The compromise solution.

5.1.2. An Application in Game Theory

We next apply the compromise criterion to game theory. Our compromise solutions offer a new meaning of “best” in the sense that every player obtains a “fair” payoff. As a result, compromise solutions are more sustainable in certain situations than some commonly used game-theoretic equilibria. Such a set of strategies for the players will be call a Compromise Equilibrium (CE), which maximixes the scalar compromise transformation among all payoff combinations.

We now determine CEs for players in some games and compare the results to any Regret Equilibria (RE) – the well-known Nash equilibrium - and to the new Disappointment Equilibrium (DE) in the following six games. For more details of games like Chicken and Prisoner's Dilemma, as well as REs and DEs, refer to [29].

GAME 1: THREE-PERSON PAYOFF MATRIX

	γ_1		γ_2	
	β_1	β_2	β_1	β_2
α_1	(–110, 100, 10)	(–60, 40, 10)	(–110, 100, 10)	(–60, 40, 10)
α_2	(–110, 0, 100)	(–110, 10, 100)	(–30, 0, 10)	(–30, –10, 10)

Figure 5.3 Payoff matrix of game 1.

To be able to apply compromising transformation, we first shift all values to be positive numbers by adding (111,111,111) to all payoff values without affecting the RE, DE, or CE strategy combinations. The new payoff matrix becomes figure 5.4.

	γ_1		γ_2	
	β_1	β_2	β_1	β_2
α_1	(1, 211, 121)	(51, 151, 121)	(1, 211, 121)	(51,151, 121)
α_2	(1, 111, 211)	(1, 121, 211)	(81, 111, 121)	(81, 101, 121)

Figure 5.4 Shifted payoff matrix of game 1.

Calculate the $M_i = \max_{\substack{\alpha \in \{\alpha_1, \alpha_2\} \\ \beta \in \{\beta_1, \beta_2\} \\ \gamma \in \{\gamma_1, \gamma_2\}}} f_i(\alpha, \beta, \gamma)$ and $m_i = \min_{\substack{\alpha \in \{\alpha_1, \alpha_2\} \\ \beta \in \{\beta_1, \beta_2\} \\ \gamma \in \{\gamma_1, \gamma_2\}}} f_i(\alpha, \beta, \gamma)$ where $f_i(\alpha, \beta, \gamma)$ is the payoff value for player i^{th}, i =1, 2, 3 to give

- $M_1 = \max\{1,51,81\} = 81,\ m_1 = \min\{1,51,81\} = 1.$
- $M_2 = \max\{211,111,151,121,101\} = 211,\ m_2 = \min\{211,111,151,121,101\} = 101.$
- $M_3 = \max\{121,211\} = 211,\ m_3 = \min\{121,211\} = 121.$

Then we compute the compromise values using the following transformation

$$T_{Compr}(\alpha_i, \beta_j, \gamma_k) = [\frac{f_1(\alpha_i, \beta_j, \gamma_k) - m_1 + 1}{M_1 - m_1 + 1}] \times [\frac{f_2(\alpha_i, \beta_j, \gamma_k) - m_2 + 1}{M_2 - m_2 + 1}] \times [\frac{f_3(\alpha_i, \beta_j, \gamma_k) - m_3 + 1}{M_3 - m_3 + 1}]$$

for all $i, j, k = 1,2.$ The compromise values are shown in the Compromise Matrix (CM). We also calculate the Regret Matrix (RM) and the Disappointment Matrix (DM) as in [29].

COMPROMISE MATRIX

	γ_1			γ_2	
	β_1	β_2		β_1	β_2
α_1	0.00013	0.00317		0.000135	0.00317
α_2	0.0012	0.00233		0.00108	0.000009

Figure 5.5 Compromise matrix of game 1.

REGRET MATRIX

	γ_1		γ_2	
	β_1	β_2	β_1	β_2
α_1	(0, 0, 0)	(0, 60, 0)	(80, 0, 0)	(30, 60, 0)
α_2	(0, 10, 0)	(50, 0, 0)	(0, 0, 90)	(0, 10, 90)

Figure 5.6 Regret matrix of game 1.

DISAPPOINTMENT MATRIX

	γ_1		γ_2	
	β_1	β_2	β_1	β_2
α_1	(50, 0, 90)	(0, 0, 90)	(50, 0, 0)	(0, 0, 0)
α_2	(80, 100, 0)	(80, 30, 0)	(0, 100, 0)	(0, 50, 0)

Figure 5.7 Disappointment matrix of game 1.

Note: the strategy selection combination below does not equal the original payoff.

In summary,

- RE and DE at $(\alpha_1, \beta_1, \gamma_1) = (1, 211, 121)$.
- DE at $(\alpha_1, \beta_2, \gamma_2) = (51, 151, 121)$.
- CEs at $(\alpha_1, \beta_2, \gamma_1)$ and $(\alpha_1, \beta_2, \gamma_2)$ with a payoff of $(51, 151, 121)$.
- The DE is a CE, but both CEs are not DEs.
- No RE (Nash Equilibrium) is a CE.

- CEs are fairer than the RE. All players achieve a fair amount of payoff in the CE. In the RE, player I gain benefits only 1 unit while players II and III receive 211 and 121 units, respectively. In the CEs and DE, player I obtains 51 units in compromise solution while player II and III receive 151 and 121 units, respectively.

GAME 2: PAYOFF MATRIX

Player *II*

Player *I*		t_1	t_2	t_3
	s_1	(3,4)	(2,2)	(2,1)
	s_2	(2,3)	(7,1)	(7,4)
	s_3	(2,1)	(5,6)	(6,5)

Figure 5.8 Payoff matrix of game 2.

Calculate the $M_i = \max_{\substack{s\in\{s_1,s_2,s_3\}\\ t\in\{t_1,t_2,t_3\}}} f_i(s,t)$ and $m_i = \min_{\substack{s\in\{s_1,s_2,s_3\}\\ t\in\{t_1,t_2,t_3\}}} f_i(s,t)$ where $f_i(s,t)$ is the pay off value for player i^{th}, i =1, 2, as following.

- $M_1 = \max\{3,2,7,5,6\} = 7,\ m_1 = \max\{3,2,7,5,6\} = 2.$

- $M_2 = \max\{4,2,1,3,6,5\} = 6,\ m_2 = \min\{4,2,1,3,6,5\} = 1.$

We calculate Compromise values using the transformation,

$$T_{Compr}(s_i,t_j)=[\frac{f_1(s_i,t_j)-m_1+1}{M_1-m_1+1}]\times[\frac{f_2(s_i,t_j)-m_2+1}{M_2-m_2+1}] \text{ for all } i,j=1,2,3.$$

COMPROMISE MATRIX

Player *II*

		t_1	t_2	t_3
	s_1	0.2222	0.0555	0.0277
Player *I*	s_2	0.0833	0.1666	0.6666
	s_3	0.0277	0.6666	0.6944

Figure 5.9 Compromise matrix of game 2.

REGRET MATRIX

Player *II*

		t_1	t_2	t_3
	s_1	(0,0)	(5,2)	(5,3)
Player *I*	s_2	(1,1)	(0,3)	(0,0)
	s_3	(1,5)	(2,0)	(1,1)

Figure 5.10 Regret matrix of game 2.

DISAPPOINTMENT MATRIX

Player *II*

Player *I*		t_1	t_2	t_3
	s_1	(0,0)	(1,4)	(1,4)
	s_2	(5,1)	(0,5)	(0,1)
	s_3	(4,3)	(1,0)	(0,0)

Figure 5.11 Disappointment matrix of game 2.

The results are includes as follows.

- REs at $(s_1, t_1) = (3, 4)$ and $(s_2, t_3) = (7, 4)$.
- DEs at $(s_1, t_1) = (3, 4)$ and $(s_3, t_3) = (6, 5)$.
- CE at $(s_3, t_3) = (6, 5)$.
- No RE is a CE.
- The CE is a DE, but one DE is not a CE. Thus only some DEs are fair.

GAME3: PRISONER'S DILEMMA PAYOFF MATRIX

Player *II*

Player *I*		*Defect*	*Cooperate*
	Defect	(–3,–3)	(0,–5)
	Cooperate	(–5,0)	(–1,–1)

Figure 5.12 Payoff matrix of game 3.

We first shift all values to be positive numbers by adding all components by (6,6). The payoff matrix becomes as follows.

Player *II*

		Defect	*Cooperate*
Player *I*	*Defect*	(3,3)	(6,1)
	Cooperate	(1,6)	(5,5)

Figure 5.13 Shifted payoff matrix of game 3.

- $M_1 = \max\{3,1,6,5\} = 6,\ m_1 = \max\{3,1,6,5\} = 1.$
- $M_2 = \max\{3,61,5\} = 6,\ m_2 = \max\{3,61,5\} = 1.$

COMPROMISE MATRIX

Player *II*

		Defect	*Cooperate*
Player *I*	*Defect*	0.25	0.1666
	Cooperate	0.1666	0.6944

Figure 5.14 Compromise matrix of game 3.

REGRET MATRIX:

Player *II*

		Defect	*Cooperate*
Player *I*	*Defect*	(0,0)	(0,2)
	Cooperate	(2,0)	(1,1)

Figure 5.15 Regret matrix of game 3.

DISAPPOINTMENT MATRIX:

		Player *II*	
		Defect	*Cooperate*
Player *I*	*Defect*	(3,3)	(0,4)
	Cooperate	(4,0)	(0,0)

Figure 5.16 Disappointment matrix of game 3.

Note: the strategy selection combination below does not equal the original payoff.

We have the following results.

- RE at (*Defect*, *Defect*) with payoff (3,3).
- DE at (*Cooperate*, *Cooperate*) with payoff (5,5).
- CE at (*Cooperate*, *Cooperate*) with payoff value (5,5).
- No RE (Nash Equilibrium) is a CE.
- In Prisoner's Dilemma, the CE and DE are the same and thus fairer than the RE (Nash Equilibrium).

GAME 4: PAYOFF MATRIX WITH NO PURE EQUILBRIUM

		Player *II*		
		t_1	t_2	t_3
	s_1	(10,3)	(4,7)	(4,6)
Player *I*	s_2	(2,6)	(9,5)	(5,7)
	s_3	(4,8)	(5,6)	(7,5)

Figure 5.17 Payoff matrix of game 4.

- Calculate the $M_i = \max_{\substack{s \in \{s_1, s_2, s_3\} \\ t \in \{t_1, t_2, t_3\}}} f_i(s,t)$ and $m_i = \min_{\substack{s \in \{s_1, s_2, s_3\} \\ t \in \{t_1, t_2, t_3\}}} f_i(s,t)$ where $f_i(s,t)$ is the pay off value for player i^{th}, i =1, 2, as following.
- $M_1 = \max\{10,2,4,9,5,4,5,7\} = 10$, $m_1 = \min\{10,2,4,9,5,4,5,7\} = 2$.
- $M_2 = \max\{3,6,8,7,5,6\} = 8$, $m_2 = \min\{3,6,8,7,5,6\} = 3$.
- Calculate $T_{Compr}(s_i,t_j) = [\dfrac{f_1(s_i,t_j) - m_1 + 1}{M_1 - m_1 + 1}] \times [\dfrac{f_2(s_i,t_j) - m_2 + 1}{M_2 - m_2 + 1}]$ for $i, j = 1,2,3$.

COMPROMISE MATRIX

Player *II*

Player *I*		t_1	t_2	t_3
	s_1	0.1666	0.2777	0.2222
	s_2	0.0741	0.4444	0.3703
	s_3	0.3333	0.2963	0.3333

Figure 5.18 Compromise matrix of game 4.

REGRET MATRIX:

Player *II*

Player *I*		t_1	t_2	t_3
	s_1	(0,4)	(5,0)	(3,1)
	s_2	(8,1)	(0,2)	(2,0)
	s_3	(6,0)	(4,2)	(0,3)

Figure 5.19 Regret matrix of game 4.

DISAPPOINTMENT MATRIX:

Player *II*

Player *I*		t_1	t_2	t_3
	s_1	(0,5)	(6,0)	(6,1)
	s_2	(7,2)	(0,2)	(4,0)
	s_3	(3,0)	(2,1)	(0,2)

Figure 5.20 Disappointment matrix of game 4.

Results are listed below.

- No REs or DEs.
- CE at (s_2, t_2) with payoff of (9,5).
- There are many non-dominated strategy pairs yielding payoffs such as (10,3), (9,5), and (4,8), but the compromise solution (9,5) is the fairest non-dominated payoff.

GAME 5: CHICKEN'S GAME PAYOFF MATRIX

Player *II*

Player *I*		*Dare*	*Chicken out*
	Dare	(1,1)	(7,2)
	Chicken out	(2,7)	(6,6)

Figure 5.21 Payoff matrix of game 5.

COMPROMISE MATRIX :

Player *II*

Player *I*		*Dare*	*Chicken out*
	Dare	0.0204	0.2858
	Chicken out	0.2858	0.7347

Figure 5.22 Compromise matrix of game 5.

REGRET MATRIX:

Player *II*

Player *I*		*Dare*	*Chicken out*
	Dare	(1,1)	(0,0)
	Chicken out	(0,0)	(1,1)

Figure 5.23 Regret matrix of game 5.

DISAPPOINTMENT MATRIX

Player *II*

Player *I*		*Dare*	*Chicken out*
	Dare	(6,6)	(0,4)
	Chicken out	(4,0)	(0,0)

Figure 5.24 Disappointment matrix of game 5.

Results are explained as follows.

- RE at (*Chicken out*, *Dare*) and (*Dare*, *Chicken Out*) with payoffs of (2,7) and (7,2), respectively.
- DE at (*Chicken out*, *Chicken out*) with payoff of (6,6).
- CE at (*Chicken out*, *Chicken out*) with payoff of (6,6).
- The CE and DE are the same in this game and fairer solution than the RE (Nash Equilibrium).

GAME 6: PAYOFF MATRIX

Player *II*

Player *I*		t_1	t_2
	s_1	(1,1)	(2,7)
	s_2	(7,2)	(6,6)

Figure 5.25 Payoff matrix of game 6.

COMPROMISE MATRIX :

Player *II*

Player *I*		t_1	t_2
	s_1	0.0204	0.2857
	s_2	0.2857	0.7347

Figure 5.26 Compromise matrix of game 6.

REGRET MATRIX:

Player II

Player I

	t_1	t_2
s_1	(6,6)	(4,0)
s_2	(0,4)	(0,0)

Figure 5.27 Regret matrix of game 6.

DISAPPOINTMENT MATRIX:

Player II

Player I

	t_1	t_2
s_1	(1,1)	(0,0)
s_2	(0,0)	(1,1)

Figure 5.28 Disappointment matrix of game 6.

Results are listed below.

- RE at (s_2, t_2) with payoff value of (6, 6).
- DE at (s_2, t_1) and (s_1, t_2) with payoffs of (7,2) and (2,7), respectively.
- CE at (s_2, t_2) with payoff of (6, 6).
- RE is a CE.
- DE is not a CE.

5.2. Randomize Criterion

When simply making any decision choice is perceived as the best action, a randomization decision is optimal. Therefore randomization might be interpreted as an optimization criterion. We establish this fact below.

Consider a relation order on R^n defined by $\mathbf{x} \leq_{Random} \mathbf{y}$ if and only if $\mathbf{x} = \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in R^n$. This means that any vector $\mathbf{x}$ is comparable only to itself.

Theorem 5.2.1. The preference order $\leq_{Random}$ is a partial order on R^n.

Proof. We show that $\leq_{Random}$ is reflexive, transitivity, and antisymmetric.

(Reflexive). Since $\mathbf{x} = \mathbf{x}$ for any $\mathbf{x} \in R^n$, then $\mathbf{x} \leq_{Random} \mathbf{x}$.

(Transitive). Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^n$ such that $\mathbf{x} \leq_{Random} \mathbf{y}$ and $\mathbf{y} \leq_{Random} \mathbf{z}$. But $\mathbf{x} = \mathbf{y}$ and $\mathbf{y} = \mathbf{z}$. Thus it is clear that $\mathbf{x} = \mathbf{z}$.

(Antisymmetric). Let $\mathbf{x}, \mathbf{y} \in R^n$ such that $\mathbf{x} \leq_{Random} \mathbf{y}$ and $\mathbf{y} \leq_{Random} \mathbf{x}$. Then by definition, $\mathbf{x} = \mathbf{y}$.

Therefore, the order $\leq_{Random}$ is a partial order.■

Since no $\mathbf{x}$ is dominated by any other vector, the randomization criterion can be scalarized as the following maximization problem.

$$\left\{\begin{array}{ll} \max\limits_{\mathbf{x} \in A} & c \\ \text{s.t.} & \mathbf{x} \geq_{Random} \mathbf{y} \end{array}\right\}, \text{ for all } \mathbf{y} \in f(A), \text{ where } c \text{ is a constant number.}$$

5.3. Counterexample

Not every decision problem can be formalized with an optimization criterion. Group decision making with various voting schemes is often one that violates our framework of consistency. Two difficulties are explained below.

1. Intransitivity

Consider the well-known Condorcet Paradox [30] and [31] in voting.

Table 5.1 Condorcet paradox.

Individual	Preference order
Voter 1	A>B>C
Voter 2	B>C>A
Voter 3	C>A>B

In this voting three voters, 1, 2, and 3 are asked to consider three alternatives A, B, and C. As shown in Table 5.1, Voter 1 prefers A to B to C; Voter 2 prefers B to C to A; and Voter 3 prefer C to A to B. It is obvious that two people prefer A to B, two people prefer B to C, and two people prefer C to A. For any majority voting scheme, it immediately follows that $A < B$ and $B < C$, but $C < A$. Such a group preference order is intransitive, so it cannot be a partial order.

2. The preference order cannot be fixed in advance.

As another voting scheme that also violates our optimization criteria framework, consider three candidate, A, B and C, and five voters. Each voter has 10 points to distribute to the candidates in integer values among the candidates according to how well a candidate is rated by the voter. For example, a voter could award 5 points to A, 3 to B, and 2 to C.

- Let $(a_{\mathrm{A}}^n, a_{\mathrm{B}}^n, a_{\mathrm{C}}^n) \in R^3$, where $a_{\mathrm{A}}^n, a_{\mathrm{B}}^n, a_{\mathrm{C}}^n$ represent the distributed points of the n^{th} voter to candidate A, B, and C, respectively, for all $n = 1,\ldots,5$.
- Define the voting order in R^5 by

  $(a_i^1, a_i^2, a_i^3, a_i^4, a_i^5) \prec (a_j^1, a_j^2, a_j^3, a_j^4, a_j^5)$ iff $\sum_{k=1}^{5} a_i^k < \sum_{i=1}^{5} a_j^k$ for any $i, j \in \{\mathrm{A,B,C}\}$.
- We then define $(a_i^1, a_i^2, a_i^3, a_i^4, a_i^5) \preceqq (a_j^1, a_j^2, a_j^3, a_j^4, a_j^5)$ if and only if either $(a_i^1, a_i^2, a_i^3, a_i^4, a_i^5) \prec (a_j^1, a_j^2, a_j^3, a_j^4, a_j^5)$ or $(a_i^1, a_i^2, a_i^3, a_i^4, a_i^5) = (a_j^1, a_j^2, a_j^3, a_j^4, a_j^5)$.
- Define $f : \{\mathrm{A,B,C}\} \to R^5$ where $f(\mathrm{A})=$ the total score of candidate A from 5 voters, $f(\mathrm{B})=$ the total score of candidate B from 5 voters, and $f(\mathrm{C})=$ the total score of candidate C from 5 voters.
- The voting decision could be interpreted as $\operatorname{opt} f(x)$ subject to $x \in \{A, B, C\}$.

One difficulty is that voting depends on a preference order as defined by the decision process itself. There is no order until the decision is made, as opposed to the decision being made based on an existing order.

In addition, the domination property, a necessary condition of an optimization criterion, is not satisfied. The domination property says that more choices yield a better decision, or at least not worse. In voting, the choices are the candidates. But more candidates do not guarantee a winner as “good or better” even in the sense of overall voter appeal. For example, more candidates could split the vote. So obviously politics does not follow our framework for consistency in decision making.

CHAPTER 6

CONCLUSIONS

6.1. Summary

Scalar equivalences for all standard non-scalar optimization criteria have been presented without convexity or concavity requirements. In particular, equivalent scalarizations for maximin, Pareto optimization, goal programming, cone-ordered optimization, and set-valued optimization have been proposed. In addition, we have shown that the equivalences among standard non-scalar optimization criteria can be established directly. This result means that all standard optimization criteria are essentially equivalent since all solutions and only solutions of one can be directly obtained by solving an optimization problem involving any other standard criterion. Moreover, we have shown that the equivalent scalarizations of the standard criteria are equivalent in that all solutions and only solutions of one scalarization can be determined by solving a scalarization of any other standard criterion. Hence any criterion is equivalent to a scalar maximization problem in the following sense. All solutions and only solutions of a problem involving a criterion can be obtained, at least theoretically, as the solutions to a certain real-valued maximization problem of a type common to all criteria.

We have unified the notion of an optimization criterion within a general axiomatic framework to include all standard optimization criteria as special cases. Our choices of axioms have been explained, as well as the disqualification of other possibilities. One requirement for an optimization criterion is the scalarizability property. Hence all optimization criteria are equivalent to solving similar scalar maximization problems, and all are equivalent in a significant sense.

Two methods of scalarization have been proposed for optimization criteria. The first is Corley's Method for transforming a general cone-ordered optimization for which the cone is pointed and convex with appropriate assumptions. The second is the Lexicographic Hybrid Method for scalarizing a general optimization problem with any partial order.

Finally, we defined a "compromise" criterion for fairness as well as "randomize" for simply taking action. In particular, the compromise criterion was applied to game theory to obtain new results. Finally, the group decision making of two voting schemes was shown not to represent an optimization criterion in our formal framework.

6.2. Contributions of the Research

We have established a general framework of optimization criteria to cover all existing standard criteria into the same set of axioms. Thus all existing criteria are the same in the sense that any problem involving one criterion satisfying our definition can be formulated as a problem involving any other general criterion. In particular, all solutions and only solutions to one problem can be obtained by solving the other problem.

Since scalarizability is one requirement of optimization criteria and since all standard criteria are scalar equivalent, this work has effectively suggested the following hypothesis. People make most individual decisions based on numerical scales, regardless of the specific problem, to simplify the complexity of the real world. In other words, people devise heuristic metrics to rank choices.

6.3. Future Work

Future work should concentrate on two areas. First, the scalarizations presented here must be numerically solved to make actual decisions. Hence computational methods should be studied. Second, new optimization criteria should be developed to provide further models for decision making.

REFERENCES

[1] J. Bracken, J.E. Falk, and JM. McGill, Equivalence of two mathematical programs with optimization problems in the constraints, *Operations Research*, 22(1974), 1102-1104.

[2] Y. Collettte and P. Siarry, Multiobjective Optimization: Principles and Case Studies, Springer, Berlin, 2003.

[3] M. Ehrgott and X. Gandibleux, Multiple Criteria Optimization: State of the Art Annotated Bibliographic Survey, Vol. 52, International Series in Operations Research and Management Science, Kluwer, Boston, MA. 2002.

[4] J. P. Ignizio, A review of goal programming: a tool for multiobjective analysis, *Journal of the Operational Research Society*, 29(1978), 1109-1119.

[5] C.D. Aliprantis and R. Tourky, Cones and Duality, Graduate Studies in Mathematics 84, American Mathematical Society, Providence, 2007.

[6] M. Ehrgott, Multicriteria Optimization, 2nd Edition, Springer, Heidelberg, 2005.

[7] H. Behnke, F. Bachmann, K. Fladt, W. Suss, H. Kunle, and S.H. Gould, Fundamentals of Mathematics, Vol. 1, Foundations of Mathematics: The Real Number System and Algebra, The MIT press, Massachusetts, 1974.

[8] D.S. Bridges and G.B. Mehta, Representations of Preferences Orderings, Springer-Verlag, Berlin, 1995.

[9] G. Giogi, A. Guerraggio, and J. Thierfelder, Mathematics of Optimization: Smooth and Nonsmooth Case, Elsevier, Amsterdam, 2004.

[10] J. Borwein, Proper efficient points for maximizations with respect to cones, *SIAM J. Control Optimization*, 15(1977), 57–63.

[11] H.W. Corley, On optimality conditions for maximizations with respect to cones, *Journal of Optimization Theory and Applications*, 46(1985), 67-78.

[12] H.W. Corley, Optimality conditions for maximizations of set-valued functions, *Journal of Optimization Theory and Applications*, (58)1988, 1-10.

[13] K. Miettinen and M.M. Mäkelä, On scalarizing functions in multiobjective optimization, *OR Spectrum*, 24(2002), 193-213.

[14] K. Miettinen, Introduction to multiobjective optimization: noninteractive approaches, in Multiobjective Optimization: Interactive and Evolutionary Approach, Springer, Heidelberg, 2008.

[15] H.W. Corley, A new scalar equivalence for Pareto optimization, *IEEE Transactions on Automatic Control*, 25(1980), 829–830.

[16] H.W. Corley, Some hybrid fixed point theorems related to optimization, *Journal of Mathematical Analysis and Applications*, 120(1986), 528-532.

[17] A.D. Warren, L.S. Lasdon, and D.F. Suchman, Optimization in engineering design, *Proceedings of the IEEE*, 55(1967), 1885-1897.

[18] C. Charalambous and A. R. Conn, An efficient method to solve the minimax problem directly, *SIAM Journal on Numerical Analysis*, 15(1978), 162-187.

[19] G.B. Dantzig, A proof of the equivalence of the programming problem and game problem, *Activity Analysis of Production and Allocation*, Koopmans, T.C.(editor) 1951, John Wiley & Sons, New York, 330-335.

[20] V. F. Dem'yanov and V. N. Malozemov, On the theory of non-linear minimax problems, *Russian Mathematical Surveys*, 26(1971), 57-115.

[21] S. Xu, Smoothing method for minimax problems, *Computational Optimization and Applications*, 20(2001), 267-279.

[22] S.M. Guu, N.J. Huang, and J. Li, Scalarization approaches for set-valued vector optimization problems and vector variational inequalities, *Journal of Mathematical Analysis and Applications*, 356(2009), 564-576.

[23] Z. F. Li, Benson proper efficiency in the vector optimization of set-valued maps, *Journal of Optimization Theory and Applications*, 98(1998), 623–649.

[24] E. Hernández, L. Rodríguez-Marín, and M. Sama, Some equivalent problems in set optimization, *Operations Research Letters*, 37(2009), 61-64.

[25] D. Kuroiwa, T. Tanaka, and T.X.D. Ha, On cone convexity of set-valued maps, *Nonlinear Analysis*, 30(1997), 1487-1496.

[26] J. Jahn, Vector Optimization: Theory, Applications, and Extensions, Springer Verlag, Berlin, 2004.

[27] G.Y. Chen, X.Huang, and X. Yang, Vector Optimization: Set-valued and Variational Analysis, Lecture Notes in Economics and Mathematical Systems, Springer, Heidelberg, 2005.

[28] G. Strang, Introduction to Linear Algebra, 3rd Edition, Wellesley-Cambridge Press, 2003.

[29] P. Insuwan, New Equilibria for Noncooperative Games, (Doctoral dissertation, The University of Texas at Arlington, 2007).

[30] K.J. Arrow, Social Choice and Individual Values, 2nd Edition, Wiley, New York, 1963.

[31] M. Ehrgott, J.R. Figueira, and S. Greco, Trends in Multiple Criteria Decision Analysis, Springer, New York, 2010

BIOGRAPHICAL INFORMATION

Surachai Charoensri was born on March 4, 1980 in Chachoengsao, Thailand. He graduated with a B.S. and M.S. in Mathematics from Chulalongkorn University in 2002 and 2004, respectively. He received M.S. in Industrial and Systems Engineering from the University of Florida in 2008. He began his Ph.D. studies in the Department of Industrial and Manufacturing Systems Engineering at UT Arlington in Fall, 2008, where he improved his badminton.