

REGULAR ALGEBRAS RELATED TO REGULAR GRADED SKEW
CLIFFORD ALGEBRAS OF LOW GLOBAL DIMENSION

by

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ABSTRACT

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M. Artin, W. Schelter, J. Tate, and M. Van den Bergh introduced the notion of non-commutative regular algebras, and classified regular algebras of global dimension 3 on degree-one generators by using geometry (i.e., point schemes) in the late 1980s. Recently, T. Cassidy and M. Vancliff generalized the notion of a graded Clifford algebra and called it a graded skew Clifford algebra.

In this thesis, we prove that all classes of quadratic regular algebras of global dimension 3 contain graded skew Clifford algebras or Ore extensions of graded skew Clifford algebras of global dimension 2. We also prove that some regular algebras of global dimension 4 can be obtained from Ore extensions of regular graded skew Clifford algebras of global dimension 3. We also show that a certain subalgebra R of a regular graded skew Clifford algebra A is a twist of the polynomial ring if A is a twist of a regular graded Clifford algebra B . We have an example that demonstrates that this can fail when A is not a twist of B .

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CHAPTER 1

INTRODUCTION

M. Artin, W. Schelter, J. Tate, and M. Van den Bergh introduced the notion of non-commutative regular algebras and invented new methods in algebraic geometry in the late 1980s to study them ([2], [3], [4]). Such algebras are viewed as non-commutative analogues of polynomial rings; indeed, polynomial rings are examples of regular algebras.

By the 1980s, a lot of algebras had arisen in quantum physics, specifically quantum groups, and many traditional algebraic techniques failed on these new algebras. In physics, quantum groups are viewed as algebras of non-commuting functions acting on some “non-commutative space” ([6]). In the early 1980s, E. K. Sklyanin, a physicist, constructed a family of graded algebras on four generators ([16]). These algebras were later proved to depend on an elliptic curve and an automorphism ([8]). By the late 1980s, it was known that many of the algebras in quantum physics are regular algebras; in particular, the family of algebras constructed by Sklyanin consists of regular algebras.

The main results in [2], [3], and [4] concern the classification of regular algebras of global dimension 3 on degree-one generators. M. Artin, J. Tate, and M. Van den Bergh also defined the notion of twisting an algebra by an automorphism, and they proved that regularity and GK-dimension are preserved under such twisting ([4, §8]).

The quadratic regular algebras of global dimension 3 can be described using geometry, i.e. the point scheme $E \subseteq \mathbb{P}^2$. These algebras, where E contains a line as well as those that are “generic”, are given in [3], and [4], and entail: \mathbb{P}^2 , elliptic curve, conic union a line, triangle, (triple) line, a union of n lines where $n \in \{2, 3\}$ with one intersection point. It should be noted that the cases where E is a nodal cubic curve or a cuspidal cubic curve are not discussed in [3] or [4] as such algebras are not generic.

Classifying the regular algebras of global dimension 4 is still an open problem. In fact, even the *quadratic* regular algebras of global dimension 4 are still unclassified.

T. Cassidy and M. Vancliff introduced a class of algebras that provide an “easy” way to write down some quadratic regular algebras of global dimension n where $n \in \mathbb{N}$ ([5]). In fact, they generalized the notion of a graded Clifford algebra and called it a graded skew Clifford algebra (see Definition 2.2.1). It is hoped that graded skew Clifford algebras might be useful in the attempted classification of the regular algebras of global dimension 4.

This thesis has three main objectives as follows: to see how many point schemes of regular graded algebras of global dimension 3 can be obtained from graded skew Clifford algebras; to see how many known examples of regular algebras of global dimension 4 can be obtained from graded skew Clifford algebras; and to determine if a certain subalgebra of a regular graded skew Clifford algebra A is a twist of the polynomial ring whenever A is a twist of a graded Clifford algebra. The thesis is outlined as follows.

In Chapter 2, we define regular algebras (see Definition 2.1.13), graded skew Clifford algebras (see Definition 2.2.1), and the quadric system associated to it (see Definition 2.2.3).

In Chapter 3, we show that the point schemes of some quadratic regular algebras of global dimension 3 can be obtained by using only regular graded skew Clifford algebras. For the remaining point schemes, we use Ore extensions of regular graded skew Clifford algebras of global dimension 2. Consequently, we show in Chapter 3 that all classes of quadratic regular algebras of global dimension 3 contain either a regular graded skew Clifford algebra or an Ore extension of a regular graded skew Clifford algebra of global dimension 2. The work in this chapter led to my paper [14] with M. Vancliff and Jun Zhang, in which we prove that all quadratic regular algebras of global dimension 3 are related in some way to a regular graded skew Clifford algebra.

In Chapter 4, we consider various known quadratic regular algebras of global dimension 4 and try to relate them to graded skew Clifford algebras. In particular, we prove that the regular algebras of global dimension 4 in the first half of [18] can be obtained from Ore extensions of regular graded skew Clifford algebras of global dimension 3. Some of these algebras arise in quantum physics such as the algebra in Proposition 4.2. However, the Sklyanin algebras on 4 generators, which are regular algebras of global dimension 4, appear not to be directly related, in the sense of Chapter 3, to any graded skew Clifford algebra, although they could perhaps be weakly related in some way (c.f., [14, Remark 4.4]).

In Chapter 5, we take A to be a regular graded skew Clifford algebra of global dimension n and study the subalgebra R of A generated by the y_i (see Definition

2.2.1). In Theorem 5.7, we prove that if A is a twist (in the sense of [4, §8]) of a regular graded Clifford algebra by an automorphism, then R is a twist of a polynomial ring by an automorphism, and is a skew polynomial ring. We thank S. P. Smith (University of Washington) for the suggestion to study the algebra R . We give an example that demonstrates that Theorem 5.7 can fail when A is not a twist of a regular graded Clifford algebra.

CHAPTER 2

GRADED SKEW CLIFFORD ALGEBRAS OF GLOBAL DIMENSION n

Throughout the thesis, \mathbb{K} denotes an algebraically closed field, $\text{char}(\mathbb{K}) \neq 2$, and \mathbb{K}^\times denotes $\mathbb{K} \setminus \{0\}$.

2.1 Definitions

2.1.1 Definition of Graded Algebras [3]

In this thesis, a \mathbb{K} -algebra A is called a graded algebra if:

- (1) $A = \bigoplus_{i \geq 0} A_i$ where the A_i are vector spaces over \mathbb{K} ,
- (2) $\dim A_1 < \infty$,
- (3) $A_i A_j \subseteq A_{i+j}$ for all i, j ,
- (4) $A_0 = \mathbb{K}$,
- (5) A generated by A_1 only.

For each i , A_i is the span of the homogeneous elements of degree i .

2.1.2 Examples

- (1) The polynomial ring $A = \mathbb{K}[x_1, \dots, x_d]$ where x_1, \dots, x_d have degree 1.

Here,

$$A_1 = \mathbb{K}x_1 \oplus \mathbb{K}x_2 \oplus \dots \oplus \mathbb{K}x_d,$$

and

$$\dim_{\mathbb{K}} A_i = \binom{i+d-1}{d-1} \quad \text{for all } i \quad (\text{c.f., [13]}).$$

- (2) The free algebra $A = \mathbb{K}\langle x_1, \dots, x_d \rangle$ where x_i , for all i , have degree $n_i \in \mathbb{Z}$.

Here, A is a non-commutative analogue of the algebra A in (1).

2.1.3 Nonexamples

(1) The algebra

$$A = \frac{\mathbb{K}[x, y]}{\langle x^2 - y \rangle},$$

where x and y have degree 1, is not graded. The relation $x^2 = y$ in A is not homogeneous and so $A_1 \cap A_2 \neq \{0\}$ which violates (1) in Definition 2.1.1.

(2) The algebra

$$A = \frac{\mathbb{K}[x, y]}{\langle x^2 - y \rangle},$$

where x has degree 1 and y has degree 2, is graded but not generated by A_1 since $y \in A_2$.

2.1.4 Definition of Quadratic \mathbb{K} -Algebra

A \mathbb{K} -algebra A is called quadratic if:

- (1) A is graded (as defined above),
- (2) A is a quotient of the free algebra by homogeneous relations of degree 2.

2.1.5 Example

The algebra

$$\mathbb{K}[x_1, \dots, x_d] = \frac{\mathbb{K}\langle x_1, \dots, x_d \rangle}{\langle x_i x_j - x_j x_i; 1 \leq i, j \leq d \rangle}, \quad \deg(x_i) = 1 \quad \text{for all } i$$

is quadratic.

2.1.6 Nonexample

The algebra

$$A = \frac{\mathbb{K}[x]}{\langle x^3 \rangle}, \quad \text{where } x \text{ has degree 1,}$$

is graded but is not quadratic. The relation $x^3 = 0$ has degree 3.

In order to define a regular algebra, we first need the concepts of polynomial growth, global dimension, and Gorenstein, which we now define.

2.1.7 Global Dimension

The algebra A has global dimension $d < \infty$ if every A -module M has projective dimension $\leq d$ and there exists at least one module M with projective dimension d .

2.1.8 Example

The polynomial ring, $\mathbb{K}[x_1, \dots, x_d]$, has global dimension d by Hilbert's syzygy theorem (c.f., [15]).

2.1.9 Definition of Polynomial Growth (c.f.,[13])

A graded algebra A , as above, is said to have polynomial growth if there exists positive real numbers c, δ such that

$$\dim_{\mathbb{K}} A_n \leq cn^\delta \quad \text{for all } n \gg 0.$$

For all known quadratic regular algebras of global dimension d , the minimal such δ is $d - 1$ ([3, §2]).

2.1.10 Example

Let $A = \mathbb{K}[x_1, x_2]$, then

$$\dim_{\mathbb{K}} A_n = \binom{n+1}{1} = n+1 \leq n^{1+\epsilon},$$

for all $\epsilon > 0$ where $n \gg 0$. Thus A has polynomial growth.

2.1.11 Definition of Gorenstein [2]

By [3, §2], for a graded algebra A as in Definition 2.1.1, the global dimension of A equals the projective dimension of the graded left module ${}_A\mathbb{K}$ (and projective dimension of the right module \mathbb{K}_A).

The algebra A is Gorenstein if

- (1) the projective modules P^i appearing in a minimal resolution

$$0 \rightarrow P^d \rightarrow \dots \rightarrow P^1 \rightarrow P^0 \rightarrow {}_A\mathbb{K} \rightarrow 0$$

of ${}_A\mathbb{K}$ are finitely generated, and if

- (2) applying the functor

$$M \rightsquigarrow M^* := \text{Hom}_A(M, A) = \{\text{graded homomorphisms : } M \rightarrow A\}$$

to the resolution in (1) yields a projective resolution

$$0 \rightarrow P^{0*} \rightarrow P^{1*} \rightarrow \dots \rightarrow P^{d*} \rightarrow \mathbb{K}_A \rightarrow 0$$

of the graded right A -module \mathbb{K}_A .

2.1.12 Example

The algebra

$$A = \frac{\mathbb{K}\langle x, y \rangle}{\langle xy - qyx \rangle}, \quad \text{where } q \in \mathbb{K}^\times,$$

is Gorenstein ([2, §0]).

2.1.13 Definition of Regular Algebras [3]

A graded \mathbb{K} -algebra A is called a regular algebra if

- (1) A has polynomial growth,
- (2) A has finite global dimension,
- (3) A is Gorenstein.

2.1.14 Definition of Normalizing Sequence

A sequence a_1, \dots, a_n of elements of a ring R with identity is called a normalizing sequence if a_1 is normal element in R (i.e. $a_1R = Ra_1$) and for each $j \in \{1, \dots, n-1\}$, a_{j+1} is a normal element in $R/\sum_{i=1}^j a_iR$ and also $\sum_{i=1}^n a_iR \neq R$.

2.2 Graded Skew Clifford Algebras

T. Cassidy and M. Vancliff defined a class of algebras in [5] that provide an “easy” way to write down some quadratic regular algebras of global dimension d for all $d \in \mathbb{N}$.

2.2.1 Definition of Graded Skew Clifford Algebras [5]

For $\{i, j\} \subset \{1, \dots, n\}$, let $\mu_{ij} \in \mathbb{K}^\times$ satisfy $\mu_{ij}\mu_{ji} = 1$ for all $i \neq j$, and write $\mu = (\mu_{ij}) \in M(n, \mathbb{K})$. A matrix $M \in M(n, \mathbb{K})$ is called μ -symmetric if $M_{ij} = \mu_{ij}M_{ji}$ for all $i, j = 1, \dots, n$.

Henceforth, suppose $\mu_{ii} = 1$ for all i , and fix μ -symmetric matrices $M_1, \dots, M_n \in M(n, \mathbb{K})$. A graded skew Clifford algebra associated to μ and M_1, \dots, M_n is a graded \mathbb{K} -algebra on degree-one generators x_1, \dots, x_n and on degree-two generators y_1, \dots, y_n with defining relations given by:

- (a) $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^n (M_k)_{ij} y_k$ for all $i, j = 1, \dots, n$, and
- (b) the existence of a normalizing sequence $\{r_1, \dots, r_n\}$ of homogeneous elements that span $\mathbb{K}y_1 + \dots + \mathbb{K}y_n$.

2.2.2 Example

Let $\mu_{21}, \lambda \in \mathbb{K}^\times$. If

$$M_1 = \begin{bmatrix} 0 & 1 \\ \mu_{21} & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2\lambda \end{bmatrix},$$

then any graded skew Clifford algebra A associated to M_1, M_2 satisfies

$$\frac{\mathbb{K}\langle x_1, x_2 \rangle}{\langle x_2^2 - \lambda x_1^2 \rangle} \twoheadrightarrow A$$

since

$$x_1 x_2 + \mu_{12} x_2 x_1 = y_1, \quad y_2 = x_1^2, \quad \lambda y_2 = x_2^2.$$

2.2.3 Definition of Quadric System [5]

Let S be the \mathbb{K} -algebra on generators z_1, \dots, z_n with defining relations

$$z_j z_i = \mu_{ij} z_i z_j, \quad \text{for all } i, j$$

and let

$$q_k := \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix} M_k \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in S.$$

We say $\{q_1, \dots, q_n\}$ is a quadric system.

2.2.4 Example

For the algebra A in Example 2.2.2, we have

$$S = \frac{\mathbb{K}\langle z_1, z_2 \rangle}{\langle z_2 z_1 - \mu_{12} z_1 z_2 \rangle}.$$

Moreover,

$$q_1 = 2z_1 z_2, \quad q_2 = 2z_1^2 + 2\lambda z_2^2.$$

However, since $\text{char}(\mathbb{K}) \neq 2$, we consider:

$$q_1 = z_1 z_2, \quad q_2 = z_1^2 + \lambda z_2^2.$$

2.2.5 Definition of Normalizing Quadric System

A quadric system $\{q_1, \dots, q_n\}$ is normalizing if $\sum_{k=1}^n \mathbb{K}q_k \subset S$ is spanned by a normalizing sequence of S .

2.2.6 Example

Referring to Example 2.2.4, in S , z_i is normal for all i , and

$$q_1 z_1 = \mu_{12} z_1 q_1, \quad q_1 z_2 = \mu_{21} z_2 q_1.$$

Therefore q_1 is normal in S .

In $\frac{S}{\langle q_1 \rangle}$, we have

$$q_2 z_1 = z_1(z_1^2 + \lambda \mu_{12}^2 z_2^2), \quad q_2 z_2 = \mu_{21}^2 z_2(z_1^2 + \lambda \mu_{12}^2 z_2^2).$$

So q_2 is normal in $\frac{S}{\langle q_1 \rangle}$ if $\lambda = 0$ or if $\lambda \neq 0$ and $\mu_{12}^2 = 1$.

2.2.7 Definition of Zero Locus [5]

Suppose $A = \mathbb{K}\langle x_1, \dots, x_n \rangle$ and $f \in A_2$. We define the zero locus $\mathcal{V}(f)$ of f to be

$$\mathcal{V}(f) = \{p \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} : f(p) = 0\},$$

where \mathbb{P}^{n-1} is identified with $\mathbb{P}(A_1^*)$.

Similarly if $f_1, \dots, f_m \in A_2$, then

$$\mathcal{V}(f_1, \dots, f_m) = \{p \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} : f_i(p) = 0 \text{ for all } i\}.$$

2.2.8 Definition of Base-Point Free [5]

Let Z be the zero locus in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ of the defining relations of S , i.e.

$$Z = \bigcap_{i,j} \mathcal{V}(z_j z_i - \mu_{ij} z_i z_j) \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}.$$

The quadric system $\{q_1, \dots, q_n\}$ is said to be base-point free (BPF) if $Z \cap \mathcal{V}(q_1, \dots, q_n)$ is empty.

2.2.9 Example

Referring to Example 2.2.4, let

$$p = ((\alpha_1, \alpha_2), (\beta_1, \beta_2)) \in \mathbb{P}^1 \times \mathbb{P}^1,$$

and let

$$(z_2 z_1 - \mu_{12} z_1 z_2)(p) = 0.$$

Therefore, we have

$$\alpha_2 \beta_1 - \mu_{12} \alpha_1 \beta_2 = 0.$$

If $\alpha_2 = 0$, then $\beta_2 = 0$. So $((1, 0), (1, 0)) \in \mathbb{P}^1 \times \mathbb{P}^1$.

If $\alpha_2 \neq 0$, i.e., $\alpha_2 = 1$, then $\beta_1 = \mu_{12} \alpha_1 \beta_2$. So, $((\alpha_1, 1), (\mu_{12} \alpha_1, 1)) \in \mathbb{P}^1 \times \mathbb{P}^1$.

Therefore,

$$Z = \{((\alpha_1, \alpha_2), (\mu_{12} \alpha_1, \alpha_2)) : (\alpha_1, \alpha_2) \in \mathbb{P}^1\}.$$

Let $p \in Z$. We have

$$0 = q_1(p) = \alpha_1 \alpha_2, \quad 0 = q_2(p) = \mu_{12} \alpha_1^2 + \lambda \alpha_2^2.$$

Thus $\alpha_1 = \alpha_2 = 0$ which is contradiction. Therefore $\{q_1, q_2\}$ is BPF.

CHAPTER 3

REGULAR GRADED SKEW CLIFFORD ALGEBRAS OF GLOBAL DIMENSION 3

The quadratic regular algebras of global dimension 3 can be described using geometry, i.e., the point scheme $E \subseteq \mathbb{P}^2$ ([3]). These algebras, where E contains a line as well as those that are “generic”, are given in [3], and [4], and entail: \mathbb{P}^2 , elliptic curve, conic union a line, triangle, (triple) line, a union of n lines where $n \in \{2, 3\}$ with one intersection point.

It should be noted that the cases where E is a nodal cubic curve or a cuspidal cubic curve are not discussed in [3] or [4] as such algebras are not generic. In this chapter, we prove that all classes of quadratic regular algebras of global dimension 3 contain either a regular graded skew Clifford algebra or an Ore extension of a regular graded skew Clifford algebra of global dimension 2.

In order to compare quadratic regular algebras in [3] with regular graded skew Clifford algebras, we first recall a result from [5] that identifies when a graded skew Clifford algebra is a quadratic and regular.

3.1 Theorem [5]

Let μ be as in Definition 2.2.1, and let M_1, \dots, M_n be μ -symmetric $n \times n$ matrices. A graded skew Clifford algebra A associated to μ and M_1, \dots, M_n is quadratic, regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series $\frac{1}{(1-t)^n}$ if and only if the quadrics in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ determined by the M_k are BPF and

form a normalizing quadric system. In this case, A is unique up to isomorphism, noetherian and has no zero divisors.

3.2 First Family of Examples

This subsection is devoted to one particular family of algebras that are defined as follows.

Let $\mu_{ij} \in \mathbb{K}^\times$ satisfy $\mu_{ij}\mu_{ji} = 1$ for all $i \neq j$, $\mu_{ii} = 1$ for all i , and $\lambda_i \in \mathbb{K}$ for all i .

The matrices

$$M_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & \lambda_3 \\ 0 & \mu_{32}\lambda_3 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & \lambda_2 \\ 0 & 2 & 0 \\ \mu_{31}\lambda_2 & 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & \lambda_1 & 0 \\ \mu_{21}\lambda_1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

are μ -symmetric.

The graded skew Clifford algebra A defined by these three matrices will have three degree-2 relations and possibly more relations, e.g.,

$$x_1x_2 + \mu_{12}x_2x_1 = \lambda_1y_1, \quad x_1^2 = y_1, \quad \text{etc.}$$

So we have

$$\frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle g_1, g_2, g_3 \rangle} \twoheadrightarrow A$$

where

$$g_1 = x_1x_2 + \mu_{12}x_2x_1 - \lambda_1x_3^2,$$

$$g_2 = x_1x_3 + \mu_{13}x_3x_1 - \lambda_2x_2^2,$$

$$g_3 = x_2x_3 + \mu_{23}x_3x_2 - \lambda_3x_1^2.$$

From Definition 2.2.3, we have

$$q_1 = 2z_1^2 + \mu_{32}\lambda_3z_3z_2 + \lambda_3z_2z_3 = 2(\lambda_3z_2z_3 + z_1^2),$$

$$q_2 = 2z_2^2 + \mu_{31}\lambda_2z_3z_1 + \lambda_2z_1z_3 = 2(\lambda_2z_1z_3 + z_2^2),$$

$$q_3 = 2z_3^2 + \mu_{21}\lambda_1z_2z_1 + \lambda_1z_1z_2 = 2(\lambda_1z_1z_2 + z_3^2).$$

However, since $\text{char}(\mathbb{K}) \neq 2$, we consider:

$$q_1 = \lambda_3z_2z_3 + z_1^2, \quad q_2 = \lambda_2z_1z_3 + z_2^2, \quad q_3 = \lambda_1z_1z_2 + z_3^2$$

and

$$S = \frac{\mathbb{K}\langle z_1, z_2, z_3 \rangle}{\langle s_1, s_2, s_3 \rangle}$$

where

$$s_1 = z_2z_1 - \mu_{12}z_1z_2, \quad s_2 = z_3z_1 - \mu_{13}z_1z_3, \quad s_3 = z_3z_2 - \mu_{23}z_2z_3.$$

3.2.1 Lemma

If $Z =$ zero locus in $\mathbb{P}^2 \times \mathbb{P}^2$ of the defining relations of S , i.e., $Z = \cap_{i,j} \mathcal{V}(z_jz_i - \mu_{ij}z_iz_j)$, then

- (1) $Z = \{((a_1, a_2, a_3), (a_1, \mu_{21}a_2, \mu_{31}a_3)) : (a_1, a_2, a_3) \in \mathbb{P}^2\}$ if and only if $\mu_{13} = \mu_{12}\mu_{23}$, or
- (2) $Z = P_1 \cup P_2 \cup P_3$ if and only if $\mu_{13} \neq \mu_{12}\mu_{23}$, where

$$P_1 = \{((0, a_2, a_3), (0, a_2, \mu_{32}a_3)) : (a_2, a_3) \in \mathbb{P}^1\},$$

$$P_2 = \{((a_1, 0, a_3), (a_1, 0, \mu_{31}a_3)) : (a_1, a_3) \in \mathbb{P}^1\},$$

$$P_3 = \{((a_1, a_2, 0), (a_1, \mu_{21}a_2, 0)) : (a_1, a_2) \in \mathbb{P}^1\}.$$

Proof:

We have

$$z_2z_1 = \mu_{12}z_1z_2, \quad z_3z_1 = \mu_{13}z_1z_3, \quad z_3z_2 = \mu_{23}z_2z_3.$$

Therefore, to find Z , we must solve the system of equations:

$$0 = (z_2z_1 - \mu_{12}z_1z_2)((a_1, a_2, a_3), (b_1, b_2, b_3)) = a_2b_1 - \mu_{12}a_1b_2,$$

$$0 = (z_3 z_1 - \mu_{13} z_1 z_3)((a_1, a_2, a_3), (b_1, b_2, b_3)) = a_3 b_1 - \mu_{13} a_1 b_3,$$

$$0 = (z_3 z_2 - \mu_{23} z_2 z_3)((a_1, a_2, a_3), (b_1, b_2, b_3)) = a_3 b_2 - \mu_{23} a_2 b_3,$$

which yields

$$\begin{bmatrix} a_2 & -\mu_{12} a_1 & 0 \\ a_3 & 0 & -\mu_{13} a_1 \\ 0 & a_3 & -\mu_{23} a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In order to have a solution $(b_1, b_2, b_3) \in \mathbb{P}^2$, the determinant of the first matrix must be zero. So,

$$a_1 a_2 a_3 (\mu_{13} - \mu_{12} \mu_{23}) = 0$$

which implies two cases:

- (1) $\mu_{13} = \mu_{12} \mu_{23}$, or
- (2) $\mu_{13} \neq \mu_{12} \mu_{23}$ and $a_1 a_2 a_3 = 0$.

Addressing (1) we find the solutions are all points in \mathbb{P}^2 as stated in part (1) of the result. Addressing (2), if $a_3 = 0$, then the zero locus is given by Example 2.2.9. Similarly, if $a_1 = 0$ or if $a_2 = 0$. ■

Associated to M_1, M_2, M_3 , we have the quadric system

$$\{q_1 = z_1^2 + \lambda_3 z_2 z_3, \quad q_2 = z_2^2 + \lambda_2 z_1 z_3, \quad q_3 = z_3^2 + \lambda_1 z_1 z_2\},$$

which is a normalizing sequence in S if and only if q_1 is normal in S , q_2 is normal in $\frac{S}{\langle q_1 \rangle}$, and q_3 is normal in $\frac{S}{\langle q_1, q_2 \rangle}$ (c.f., Definition 2.1.14).

3.2.2 Proposition

The quadric system $\{q_1, q_2, q_3\}$ is BPF if and only if either

- (1) $\mu_{13} = \mu_{12} \mu_{23}$ and $\lambda_1 \lambda_2 \lambda_3 + \mu_{13} \neq 0$, or

$$(2) \quad \mu_{13} \neq \mu_{12}\mu_{23}.$$

Proof:

We want to find $\mathcal{V}(q_1, q_2, q_3) \cap Z$.

If $\mu_{13} = \mu_{12}\mu_{23}$, then, by Lemma 3.2.1(1),

$$Z = \{((a_1, a_2, a_3), (a_1, \mu_{21}a_2, \mu_{31}a_3)) : (a_1, a_2, a_3) \in \mathbb{P}^2\} \subseteq \mathbb{P}^2 \times \mathbb{P}^2.$$

Let

$$p = ((a_1, a_2, a_3), (a_1, \mu_{21}a_2, \mu_{31}a_3)) \in Z.$$

We must solve the system of equations

$$0 = q_1(p) = a_1^2 + \lambda_3 a_2 \mu_{31} a_3,$$

$$0 = q_2(p) = \mu_{21} a_2^2 + \lambda_2 \mu_{31} a_1 a_3,$$

$$0 = q_3(p) = \mu_{31} a_3^2 + \lambda_1 \mu_{21} a_1 a_2.$$

Thus, if $a_1 = 0$, then $a_2 = a_3 = 0$, which is contradiction. Similarly, if $a_2 = 0$ or if $a_3 = 0$. Hence, we may assume $a_1 a_2 a_3 \neq 0$, $\lambda_1 \lambda_2 \lambda_3 \neq 0$, and $a_1 = 1$. So

$$1 + \lambda_3 \mu_{31} a_2 a_3 = 0, \quad (1)$$

$$\mu_{21} a_2^2 + \lambda_2 \mu_{31} a_3 = 0, \quad (2)$$

$$\mu_{31} a_3^2 + \lambda_1 \mu_{21} a_2 = 0. \quad (3)$$

Therefore, by (3), $a_2 = \frac{-\mu_{32} a_3^2}{\lambda_1}$, and by substituting a_2 in (1), we have

$$1 + \lambda_3 \mu_{31} \left(\frac{-\mu_{32} a_3^3}{\lambda_1} \right) = 0.$$

Consequently, $a_3^3 = \frac{\lambda_1}{\lambda_3 \mu_{31} \mu_{32}}$. By substituting for a_2 and a_3 in (2), we have

$$\lambda_1 \lambda_2 \lambda_3 + \mu_{13} = 0.$$

Thus $\{q_1, q_2, q_3\}$ is BPF if $\mu_{13} = \mu_{12}\mu_{23}$ and $\lambda_1 \lambda_2 \lambda_3 + \mu_{13} \neq 0$.

If $\mu_{13} \neq \mu_{12}\mu_{23}$, then Z is given by Lemma 3.2.1(2). Let

$$p = ((0, a_2, a_3), (0, a_2, \mu_{32}a_3)) \in Z.$$

As before, we solve

$$0 = q_1(p) = \lambda_3 a_2 \mu_{32} a_3,$$

$$0 = q_2(p) = a_2^2,$$

$$0 = q_3(p) = \mu_{32} a_3^2.$$

Thus $a_2 = 0 = a_3$ which is contradiction. Similarly if

$$p = ((a_1, 0, a_3), (a_1, 0, \mu_{31}a_3)) \in Z \quad \text{or} \quad p = ((a_1, a_2, 0), (a_1, \mu_{21}a_2, 0)) \in Z.$$

Hence $\{q_1, q_2, q_3\}$ is BPF if $\mu_{13} \neq \mu_{12}\mu_{23}$. ■

To find out if the algebra A in §3.2 is regular, we need to prove that the quadric system associated to A is normalizing.

Henceforth, condition (*) will denote the case $\mu_{13} = \mu_{12}\mu_{23}$.

3.2.3 Proposition

The sequence $\{q_1, q_2, q_3\}$ is a normalizing sequence in S if and only if either

- (1) $\lambda_1 = \lambda_2 = \lambda_3 = 0$, or
- (2) $\lambda_2 = \lambda_3 = 0 \neq \lambda_1$, $\mu_{32} = \mu_{13}$, or
- (3) $\lambda_1 = \lambda_2 = 0 \neq \lambda_3$, $\mu_{13} = \mu_{21}$, $\mu_{12}^2 = \mu_{32}$ (which implies (*)), or
- (3') $\lambda_1 = \lambda_3 = 0 \neq \lambda_2$, $\mu_{23} = \mu_{12}$, $\mu_{13} = \mu_{12}^2$ (which implies (*)), or
- (4) $\lambda_1 = 0$, $\lambda_2 \neq 0 \neq \lambda_3$, and $\mu_{13} = \mu_{21} = \mu_{32} = \mu_{12}^2$ (which implies (*)), or
- (4') $\lambda_2 = 0$, $\lambda_3 \neq 0 \neq \lambda_1$, and $\mu_{13} = \mu_{21} = \mu_{32} = \mu_{12}^2$ (which implies (*)), or
- (5) $\lambda_3 = 0$, $\lambda_1 \neq 0 \neq \lambda_2$, and $\mu_{13} = \mu_{32} = \mu_{21} = \mu_{12}^2$ (which implies (*)), or
- (6) $\lambda_i \neq 0$ for all i and $\mu_{13} = \mu_{21} = \mu_{32}$, $\mu_{12}^3 = 1$ (which implies (*)).

Proof:

We have

$$S = \frac{\mathbb{K}\langle z_1, z_2, z_3 \rangle}{\langle s_1, s_2, s_3 \rangle}$$

where

$$s_1 = z_2 z_1 - \mu_{12} z_1 z_2, \quad s_2 = z_3 z_1 - \mu_{13} z_1 z_3, \quad s_3 = z_3 z_2 - \mu_{23} z_2 z_3,$$

therefore z_i is normal in S for all i . Moreover,

$$q_1 z_1 = z_1(z_1^2 + \lambda_3 \mu_{13} \mu_{12} z_2 z_3) \quad (\text{i})$$

$$q_1 z_2 = \mu_{21}^2 z_2(z_1^2 + \lambda_3 \mu_{12}^2 \mu_{23} z_2 z_3) \quad (\text{ii})$$

$$q_3 z_3 = \mu_{31}^2 z_3(z_1^2 + \lambda_3 \mu_{13}^2 \mu_{32} z_2 z_3) \quad (\text{iii})$$

If $\lambda_3 = 0$, then $q_1 = z_1^2$ is normal in S . If $\lambda_3 \neq 0$, then, by (i), (ii), (iii), q_1 is normal in S if

$$\mu_{13} = \mu_{21}, \quad \mu_{12}^2 \mu_{23} = 1, \quad \mu_{13}^2 \mu_{32} = 1.$$

Similarly, q_2 is normal in $\frac{S}{\langle q_1 \rangle}$ if $\lambda_2 = 0$ or if

$$\lambda_2 \neq 0 \quad \text{and} \quad \mu_{23} \mu_{21} = 1 = \mu_{21}^2 \mu_{13} = \mu_{23}^2 \mu_{31} = 1,$$

and q_3 is normal in $\frac{S}{\langle q_1, q_2 \rangle}$ if $\lambda_1 = 0$ or if

$$\lambda_1 \neq 0 \quad \text{and} \quad \mu_{32} = \mu_{13}.$$

Analysis of the possibilities yields the result. ■

By Theorem 3.1, when the λ_k and the μ_{ij} satisfy Propositions 3.2.2 and 3.2.3, the graded skew Clifford algebra A associated to μ and M_1, M_2, M_3 (defined at the start of §3.2) is unique up to isomorphism and quadratic and regular.

Our next result yields the point scheme of A in these cases.

3.2.4 Proposition

If

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle g_1, g_2, g_3 \rangle}$$

where

$$g_1 = x_1x_2 + \mu_{12}x_2x_1 - \lambda_1x_3^2,$$

$$g_2 = x_1x_3 + \mu_{13}x_3x_1 - \lambda_2x_2^2,$$

$$g_3 = x_2x_3 + \mu_{23}x_3x_2 - \lambda_3x_1^2,$$

then the point scheme \mathcal{P} of A is given by one of the following:

- (1a) $\mathcal{P} = \mathbb{P}^2$ if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\mu_{13} + \mu_{12}\mu_{23} = 0$, or
- (1b) $\mathcal{P} = \mathcal{V}(x_1) \cup \mathcal{V}(x_2) \cup \mathcal{V}(x_3)$ (i.e., triangle, see Figure 3.1) if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and $\mu_{13} + \mu_{12}\mu_{23} \neq 0$, or
- (2a) $\mathcal{P} = \mathcal{V}(x_3) \cup \mathcal{V}((\mu_{13} + \mu_{12}\mu_{23})x_1x_2 + \lambda_1x_3^2)$ (i.e., conic union line, two intersection points, see Figure 3.2) if and only if $\lambda_2 = \lambda_3 = 0 \neq \lambda_1$ and $\mu_{32} = \mu_{13}$, and $\mu_{13} + \mu_{12}\mu_{23} \neq 0$, or
- (2b) $\mathcal{P} = \mathcal{V}(x_3^3)$ (i.e., triple line, see Figure 3.3) if and only if $\lambda_2 = \lambda_3 = 0 \neq \lambda_1$, $\mu_{32} = \mu_{13}$, and $\mu_{13} + \mu_{12}\mu_{23} = 0$, or
- (3) $\mathcal{P} = \mathcal{V}(x_1) \cup \mathcal{V}(2\mu_{13}x_2x_3 + \lambda_3x_1^2)$ (i.e., conic union line, two intersection points, see Figure 3.2) if and only if $\lambda_1 = \lambda_2 = 0 \neq \lambda_3$ and $\mu_{21} = \mu_{13}$, $\mu_{12}^2 = \mu_{32}$, or
- (4) $\mathcal{P} = \mathcal{V}(\mu_{12}\lambda_2x_2^3 + 2\mu_{13}x_1x_2x_3 + \lambda_3x_1^2)$ (i.e., nodal cubic curve in \mathbb{P}^2 with one singular point (node) at $(0, 0, 1)$, see Figure 3.4) if and only if $\lambda_1 = 0$, $\lambda_2 \neq 0 \neq \lambda_3$, and $\mu_{13} = \mu_{21} = \mu_{32} = \mu_{12}^2$, or
- (5) $\mathcal{P} = \mathcal{V}(\mu_{12}\lambda_2x_2^3 + 2\mu_{13}x_1x_2x_3 + \lambda_1x_3^3)$ (i.e., nodal cubic curve in \mathbb{P}^2 with one singular point (node) at $(1, 0, 0)$, see Figure 3.4) if and only if $\lambda_3 = 0$, $\lambda_2 \neq 0 \neq \lambda_1$ and $\mu_{13} = \mu_{32} = \mu_{21} = \mu_{12}^2$, or

- (6) $\mathcal{P} = \mathcal{V}(\mu_{12}\lambda_2x_2^3 + (2\mu_{13} - \lambda_1\lambda_2\lambda_3)x_1x_2x_3 + \lambda_3x_1^3 + \lambda_1x_3^3)$ if and only if $\lambda_i \neq 0$ for all i and $\mu_{13} = \mu_{21} = \mu_{32}$, and $\mu_{12}^3 = 1$ (i.e., an elliptic curve in \mathbb{P}^2 if and only if $\lambda_1\lambda_2\lambda_3 \neq 8\mu_{13}$, see Figure 3.5).



Figure 3.1. Depiction of the Point Scheme in Proposition 3.2.4(1b).

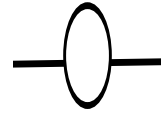


Figure 3.2. Depiction of the Point Scheme in Proposition 3.2.4(2a)&(3).

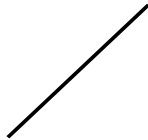


Figure 3.3. Depiction of the Point Scheme in Proposition 3.2.4(2b).

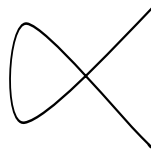


Figure 3.4. Depiction of the Point Scheme in Proposition 3.2.4(4)&(5).



Figure 3.5. Depiction of the Point Scheme in Proposition 3.2.4(6).

Proof:

Suppose

$$p = ((\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3)) \in \mathbb{P}^2 \times \mathbb{P}^2.$$

To find the point scheme \mathcal{P} of A , we solve

$$0 = g_1(p) = \alpha_1\beta_2 + \mu_{12}\alpha_2\beta_1 - \lambda_1\alpha_3\beta_3,$$

$$0 = g_2(p) = \alpha_1\beta_3 + \mu_{13}\alpha_3\beta_1 - \lambda_2\alpha_2\beta_2,$$

$$0 = g_3(p) = \alpha_2\beta_3 + \mu_{23}\alpha_3\beta_2 - \lambda_3\alpha_1\beta_1,$$

which yields

$$\begin{bmatrix} \mu_{12}\alpha_2 & \alpha_1 & -\lambda_1\alpha_3 \\ \mu_{13}\alpha_3 & -\lambda_2\alpha_2 & \alpha_1 \\ -\lambda_3\alpha_1 & \mu_{23}\alpha_3 & \alpha_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In particular, as in the proof of Lemma 3.2.1, we require the determinant of the first matrix to equal zero.

(1a) and (1b): We have $\lambda_1 = \lambda_2 = \lambda_3 = 0$, so

$$(\mu_{12}\mu_{23} + \mu_{13})\alpha_1\alpha_2\alpha_3 = 0.$$

If $\mu_{12}\mu_{23} + \mu_{13} = 0$, then $\mathcal{P} = \mathbb{P}^2$.

If $\mu_{12}\mu_{23} + \mu_{13} \neq 0$, then $\alpha_1\alpha_2\alpha_3 = 0$, so \mathcal{P} is a triangle.

(2a): Since $\lambda_2 = \lambda_3 = 0 \neq \lambda_1$ and $\mu_{32} = \mu_{13}$, we have

$$\alpha_3((\mu_{13} + \mu_{12}\mu_{23})\alpha_1\alpha_2 + \lambda_1\alpha_3^2) = 0.$$

So

$$\mathcal{P} = \mathcal{V}(x_3((\mu_{13} + \mu_{12}\mu_{23})x_1x_2 + \lambda_1x_3^2)).$$

In fact,

$$\begin{aligned} \mathcal{V}(g_1, g_2, g_3) = & \{((\alpha, \beta, 0), (\alpha, -\mu_{12}\beta, 0)) : (\alpha, \beta) \in \mathbb{P}^1\} \cup \\ & \{((\mu_{13} + \mu_{12}\mu_{23})\alpha^2, -\lambda_1\beta^2, (\mu_{13} + \mu_{12}\mu_{23})\alpha\beta), \\ & (-(1 + \mu_{12}\mu_{23}^2)\alpha^2, \mu_{13}\lambda_1\beta^2, (\mu_{13} + \mu_{12}\mu_{23})\alpha\beta)) : (\alpha, \beta) \in \mathbb{P}^1\}. \end{aligned}$$

Similarly for (2b), (3), and (4).

(5): Since $\lambda_1 \neq \lambda_3 = 0 \neq \lambda_2$ and

$$\mu_{12} = \mu_{23} = \mu_{31} = \mu_{32}\mu_{21} \quad \text{and} \quad \mu_{12}^3 = 1,$$

we have $\mu_{ij}^3 = 1$, for all i, j , so

$$\mu_{12}\lambda_2\alpha_2^3 + 2\mu_{13}\alpha_1\alpha_2\alpha_3 + \lambda_1\alpha_3^3 = 0.$$

In fact,

$$\begin{aligned} \mathcal{V}(g_1, g_2, g_3) = & \{((- \lambda_1\beta^3 - \lambda_2\mu_{12}\alpha^3, 2\mu_{13}\beta\alpha^2, 2\mu_{13}\beta^2\alpha), \\ & (-\mu_{13}\lambda_1\beta^3 + \lambda_2\alpha^3, 2\mu_{13}\beta\alpha^2, -2\beta^2\alpha)) : (\alpha, \beta) \in \mathbb{P}^1\}. \end{aligned}$$

(6): Since for all i , $\lambda_i \neq 0$, and

$$\mu_{13} = \mu_{21} = \mu_{32} = \mu_{12}\mu_{23}, \quad \mu_{12}^3 = 1,$$

which implies $\mu_{ij}^3 = 1$, for all i, j , so

$$\mu_{12}\lambda_2\alpha_2^3 + (2\mu_{13} - \lambda_1\lambda_2\lambda_3)\alpha_1\alpha_2\alpha_3 + \lambda_3\alpha_1^3 + \lambda_1\alpha_3^3 = 0 \quad (\dagger).$$

If $\lambda_1\lambda_2\lambda_3 = 8\mu_{13}$, then (\dagger) can be written as a product of two factors. In this case, the zero locus is not an elliptic curve. ■

3.3 Ore Extension of Graded Skew Clifford Algebras of Global Dimension 2

It remains to figure out which of the other types of quadratic regular algebras of global dimension 3 (i.e., those with point schemes not occurring in Proposition 3.2.4) are related to graded skew Clifford algebras. Such algebras have point schemes: a union of n lines where $n \in \{2, 3\}$; conic union line with one intersection point; and cuspidal cubic curve. To find such a relationship, we use the notion of Ore extension which uses certain types of derivations.

3.3.1 Definition of a σ -Derivation [9]

Let R be any ring with $1 \neq 0$ (possibly non-commutative), and let $\sigma \in \text{End}(R)$. A left (respectively, right) σ -derivation of R is an additive map $\delta : R \rightarrow R$ such that

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$$

(respectively, right σ -derivation

$$\delta(rs) = \delta(r)\sigma(s) + r\delta(s),$$

for all $r, s \in R$.

The definition of Ore extension is due to the following result.

3.3.2 Theorem (c.f., [9])

Let R be any ring with $1 \neq 0$ (possibly noncommutative). If $\sigma \in \text{End}(R)$ and if δ is a left σ -derivation, then there exists a ring A such that $R \subset A$ and there exists $y \in A \setminus R$ such that the elements of A can be expressed uniquely in the form

$$\sum_{i=0}^n r_i y^i \quad \text{where } r_i \in R \text{ for all } i,$$

and

$$yr = \sigma(r)y + \delta(r) \quad \text{for all } r \in R.$$

3.3.3 Definition of Ore Extension [9]

Let R be a ring with $1 \neq 0$ (possibly noncommutative). Let $\sigma \in \text{End}(R)$. By Theorem 3.3.2, the Ore extension $R[y; \sigma, \delta]$ is the ring obtained by giving the ring of polynomials

$$R[y] = \left\{ \sum_{i=1}^n y^i r_i : r_i \in R \right\}$$

a new multiplication, subject to the identity

$$yr = \sigma(r)y + \delta(r)$$

(respectively, $ry = y\sigma(r) + \delta(r)$) for all $r \in R$.

3.3.4 Theorem [5, Corollary 4.3]

If B is a quadratic regular algebra and if $\text{gldim}(B) \leq 2$, then B is a graded skew Clifford algebra.

We will look at Ore extensions of quadratic regular algebras of global dimension ≤ 3 . Such algebras are Auslander-regular ([7], [11], [12]). Auslander-regular algebras that have polynomial growth are regular ([11]). Hence, the next result implies that Ore extensions of quadratic regular algebras of global dimension ≤ 3 are regular algebras.

3.3.5 Examples [5]

(i) Quadratic regular algebras of global dimension 1 are isomorphic to $B = \mathbb{K}[x]$. We take $\mu = 1$ and $M_1 = 1$, then B is a regular graded skew Clifford algebra.

(ii) Up to isomorphism, there are exactly two types of quadratic regular algebras of global dimension 2:

(1) Let $\lambda \in \mathbb{K}^\times$, and let

$$B = \frac{\mathbb{K}\langle x_1, x_2 \rangle}{\langle x_1x_2 + \lambda x_2x_1 \rangle}.$$

If

$$M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix},$$

where $\mu_{12} = \lambda$, then B is a regular graded skew Clifford algebra.

(2) Let

$$B = \frac{\mathbb{K}\langle x_1, x_2 \rangle}{\langle x_1x_2 - x_2x_1 - x_1^2 \rangle}.$$

If

$$M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix},$$

where $\mu_{12} = -1$, then B is a regular graded skew Clifford algebra.

3.3.6 Theorem [7], [12]

Let R be a noetherian algebra and $S = R[y; \sigma, \delta]$ be an Ore extension of R where $\sigma \in \text{Aut}(R)$ and δ is a left σ -derivation. If R is an Auslander-regular algebra, then S is an Auslander-regular algebra.

3.3.7 Proposition

Let

$$B = \frac{\mathbb{K}\langle x_1, x_2 \rangle}{\langle x_1x_2 - x_2x_1 \rangle},$$

which is a regular graded skew Clifford algebra. Let

$$\sigma = \text{id}_B \in \text{Aut}(B),$$

and let $\delta : B \rightarrow B$ be the linear map such that

$$\delta(x_1) = x_1x_2 = \delta(x_2).$$

The map δ is a σ -derivation of B , and $A = B[x_3; \sigma, \delta]$ is a regular algebra. In fact, the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle g_1, g_2, g_3 \rangle}$$

where

$$g_1 = x_1x_2 - x_2x_1,$$

$$g_2 = x_3x_1 - x_1x_3 - x_1x_2,$$

$$g_3 = x_3x_2 - x_2x_3 - x_1x_2,$$

has point scheme $\mathcal{V}(x_1x_2(x_2 - x_1))$ given by the union of three lines L_1, L_2, L_3 such that $L_1 \cap L_2 \cap L_3 = \text{one point}$ (see Figure 3.6).

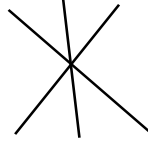


Figure 3.6. Depiction of the Point Scheme in Proposition 3.3.7.

Proof:

The algebra B is a regular graded skew Clifford algebra by Theorem 3.3.4. To prove δ is a left σ -derivation of B , we show that $\delta(0) = 0$ in B ; that is,

$$\begin{aligned}
 \delta(x_1x_2 - x_2x_1) &= \delta(x_1x_2) - \delta(x_2x_1) \\
 &= \sigma(x_1)\delta(x_2) + \delta(x_1)x_2 - \sigma(x_2)\delta(x_1) - \delta(x_2)x_1 \\
 &= x_1x_1x_2 + x_1x_2x_2 - x_2x_1x_2 - x_1x_2x_1 \\
 &= x_1(x_1x_2 - x_2x_1) + (x_1x_2 - x_2x_1)x_2 \\
 &= 0
 \end{aligned}$$

in B . Therefore, by Theorem 3.3.6, $A = B[x_3; \sigma, \delta]$ is a regular algebra. By definition of Ore extension, we have

$$x_3x_1 = \sigma(x_1)x_3 + \delta(x_1), \quad x_3x_2 = \sigma(x_2)x_3 + \delta(x_2),$$

which yields the relations g_2 and g_3 in the statement. ■

3.3.8 Proposition

Let

$$B = \frac{\mathbb{K}\langle x_1, x_2 \rangle}{\langle x_1x_2 - x_2x_1 \rangle},$$

which is a regular graded skew Clifford algebra. Let

$$\sigma = \text{id}_B \in \text{Aut}(B),$$

and let $\delta : B \rightarrow B$ be the linear map such that

$$\delta(x_1) = x_1x_2 \quad \text{and} \quad \delta(x_2) = 0.$$

The map δ is a σ -derivation of B , and $A = B[x_3; \sigma, \delta]$ is a regular algebra. In fact, the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle g_1, g_2, g_3 \rangle}$$

where

$$g_1 = x_3x_1 - x_1x_3 - x_1x_2,$$

$$g_2 = x_1x_2 - x_2x_1,$$

$$g_3 = x_3x_2 - x_2x_3,$$

has point scheme $\mathcal{V}(x_1x_2^2)$ given by the union of a line L_1 and a double line L_2 such that $L_1 \cap L_2 = \text{one point}$ (see Figure 3.7).

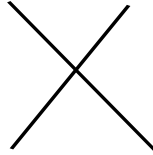


Figure 3.7. Depiction of the Point Scheme in Proposition 3.3.8.

Proof:

The algebra B is a regular graded skew Clifford algebra by Theorem 3.3.4. To prove

δ is a left σ -derivation of B , we show that $\delta(0) = 0$ in B ; that is,

$$\begin{aligned}
\delta(x_1x_2 - x_2x_1) &= \delta(x_1x_2) - \delta(x_2x_1) \\
&= \sigma(x_1)\delta(x_2) + \delta(x_1)x_2 - \sigma(x_2)\delta(x_1) - \delta(x_2)x_1 \\
&= 0 + x_1x_2x_2 - x_2x_1x_2 - 0 \\
&= (x_1x_2 - x_2x_1)x_2 \\
&= 0
\end{aligned}$$

in B . Therefore, by Theorem 3.3.6, $A = B[x_3; \sigma, \delta]$ is a regular algebra. By definition of Ore extension, we have

$$x_3x_1 = \sigma(x_1)x_3 + \delta(x_1), \quad x_3x_2 = \sigma(x_2)x_3 + \delta(x_2),$$

which yields the relations g_2 and g_3 in the statement. ■

3.3.9 Proposition

Let

$$B = \frac{\mathbb{K}\langle x_1, x_2 \rangle}{\langle x_1x_2 - x_2x_1 \rangle},$$

which is a regular graded skew Clifford algebra. Let $\sigma \in \text{Aut}(B)$ such that

$$\sigma(x_1) = x_1 \quad \text{and} \quad \sigma(x_2) = x_2 + \alpha x_1 \quad \text{where} \quad \alpha \in \mathbb{K}^\times,$$

and let $\delta : B \rightarrow B$ be the linear map such that

$$\delta(x_1) = 0 \quad \text{and} \quad \delta(x_2) = qx_2^2 \quad \text{where} \quad q \in \mathbb{K}^\times.$$

The map δ is a σ -derivation of B , and $A = B[x_3; \sigma, \delta]$ is a regular algebra. In fact, the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle g_1, g_2, g_3 \rangle}$$

where

$$\begin{aligned} g_1 &= x_1x_2 - x_2x_1, \\ g_2 &= x_3x_2 - x_2x_3 - \alpha x_1x_3 - qx_2^2, \\ g_3 &= x_3x_1 - x_1x_3, \end{aligned}$$

has point scheme

$$\mathcal{V}(x_1(qx_2^2 + \alpha x_1x_3))$$

which is the union of the conic $C = \mathcal{V}(qx_2^2 + \alpha x_1x_3)$ and the line $L = \mathcal{V}(x_1)$ such that $C \cap L =$ one point (see Figure 3.8).

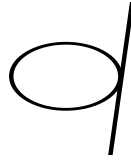


Figure 3.8. Depiction of the Point Scheme in Proposition 3.3.9.

Proof:

The algebra B is a regular graded skew Clifford algebra by Theorem 3.3.4. To prove δ is a left σ -derivation of B , we show that $\delta(0) = 0$ in B ; that is,

$$\begin{aligned} \delta(x_1x_2 - x_2x_1) &= \delta(x_1x_2) - \delta(x_2x_1) \\ &= \sigma(x_1)\delta(x_2) + \delta(x_1)x_2 - \sigma(x_2)\delta(x_1) - \delta(x_2)x_1 \\ &= x_1qx_2^2 + 0 - 0 - qx_2^2x_1 \\ &= q(x_1x_2^2 - x_2^2x_1) \\ &= qx_2(x_1x_2 - x_2x_1) \\ &= 0 \end{aligned}$$

in B . Therefore, by Theorem 3.3.6, $A = B[x_3; \sigma, \delta]$ is a regular algebra. By definition of Ore extension, we have

$$x_3x_1 = \sigma(x_1)x_3 + \delta(x_1), \quad x_3x_2 = \sigma(x_2)x_3 + \delta(x_2),$$

which yields the relations g_2 and g_3 in the statement. ■

3.3.10 Proposition

Let

$$B = \frac{\mathbb{K}\langle x_1, x_2 \rangle}{\langle x_2x_1 - x_1x_2 + x_1^2 \rangle},$$

which is a regular graded skew Clifford algebra. Let $\sigma \in \text{Aut}(B)$ such that

$$\sigma(x_1) = x_1 \quad \text{and} \quad \sigma(x_2) = x_2 - 2x_1,$$

and let $\delta : B \rightarrow B$ be the linear map such that

$$\delta(x_1) = 3x_2^2 + x_1^2 \quad \text{and} \quad \delta(x_2) = -3x_2^2 - 2x_1x_2.$$

The map δ is a σ -derivation of B , and $A = B[x_3; \sigma, \delta]$ is a regular algebra. In fact, the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle g_1, g_2, g_3 \rangle}$$

where

$$g_1 = x_2x_1 - x_1x_2 + x_1^2,$$

$$g_2 = x_3x_1 - 3x_2^2 - x_1x_3 - x_1^2,$$

$$g_3 = x_3x_2 - x_2x_3 + 3x_2^2 + 2x_1x_3 + 2x_1x_2,$$

has point scheme $\mathcal{V}(3(x_2^3 + x_1^2x_3))$ which is a cuspidal cubic curve if and only if $\text{char}(\mathbb{K}) \neq 3$ (see Figure 3.9).

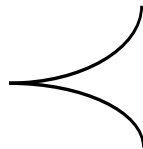


Figure 3.9. Depiction of the Point Scheme in Proposition 3.3.10.

Proof:

The algebra B is a regular graded skew Clifford algebra by Theorem 3.3.4. To prove δ is a left σ -derivation of B , we show that $\delta(0) = 0$ in B ; that is,

$$\begin{aligned}
\delta(x_2x_1 - x_1x_2 + x_1^2) &= \delta(x_2x_1) - \delta(x_1x_2) + \delta(x_1^2) \\
&= \sigma(x_2)\delta(x_1) + \delta(x_2)x_1 - \sigma(x_1)\delta(x_2) \\
&\quad - \delta(x_1)x_2 + \sigma(x_1)\delta(x_1) + \delta(x_1)x_1 \\
&= x_1^2x_2 + x_2x_1^2 - 2x_1x_2x_1 \\
&= x_1(x_2x_1 + x_1^2) + x_2x_1^2 - 2x_1x_2x_1 \\
&= x_1^3 + x_2x_1^2 - x_1x_2x_1 \\
&= (x_1^2) + x_2x_1 - x_1x_2)x_1 \\
&= 0
\end{aligned}$$

in B . Therefore, by Theorem 3.3.6, $A = B[x_3; \sigma, \delta]$ is a regular algebra. By definition of Ore extension, we have

$$x_3x_1 = \sigma(x_1)x_3 + \delta(x_1), \quad x_3x_2 = \sigma(x_2)x_3 + \delta(x_2),$$

which yields the relations g_2 and g_3 in the statement. ■

3.3.11 Theorem

All the point schemes of quadratic regular algebras of global dimension 3 can be obtained from either a regular graded skew Clifford algebra of global dimension 3 or from an Ore extension of a regular graded skew Clifford algebra of global dimension 2.

Proof:

The results follow by considering the first family of examples (§3.2), Proposition 3.3.7, Proposition 3.3.8, Proposition 3.3.9, and Proposition 3.3.10. ■

These results are extended in my paper “Classifying Quadratic Quantum \mathbb{P}^2 s By Using Graded Skew Clifford Algebras” with M. Vancliff, and Jun Zhang ([14]), in which we classify all quadratic regular algebras of global dimension 3 using regular graded skew Clifford algebras.

CHAPTER 4

REGULAR GRADED SKEW CLIFFORD ALGEBRAS OF GLOBAL DIMENSION 4

In this chapter, we prove that the regular algebras of global dimension 4 in [18] can be obtained from Ore extensions of graded skew Clifford algebras of global dimension 3.

4.1 Proposition

Suppose $q \in \mathbb{K}$, where $q^4 = 1$ but $q \neq 1$. Let

$$B = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle x_1x_2 - qx_2x_1, x_1x_3 - q^{-1}x_3x_1, x_2x_3 - qx_3x_2 \rangle},$$

which is a regular graded skew Clifford algebra. Let $\sigma \in \text{Aut}(B)$ such that

$$\sigma(x_i) = qx_i, \quad \text{for all } i = 1, 2, 3,$$

and let $\delta : B \rightarrow B$ be the linear map such that

$$\delta(x_1) = x_3^2, \quad \delta(x_2) = x_1^2 \quad \text{and} \quad \delta(x_3) = x_2^2.$$

The map δ is a σ -derivation of B , and $A = B[x_4; \sigma, \delta]$ is a regular algebra. In fact, the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, \dots, g_6 \rangle}$$

where

$$\begin{aligned} g_1 &= x_1x_2 - qx_2x_1, & g_2 &= x_2x_3 - qx_3x_2, \\ g_3 &= x_1x_3 - q^{-1}x_3x_1, & g_4 &= x_4x_1 - qx_1x_4 - x_3^2, \\ g_5 &= x_4x_2 - qx_2x_4 - x_1^2, & g_6 &= x_4x_3 - qx_3x_4 - x_2^2, \end{aligned}$$

has point scheme given by one point and appears in [19].

Proof:

The algebra B is a regular skew polynomial ring, and so is a regular graded skew Clifford algebra. The result now follows from [19, Lemma 3.2]. \blacksquare

4.2 Proposition

Suppose $q \in \mathbb{K}^\times$, and $q^2 \neq 1$. Let

$$B = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle x_1x_2 - q^{-1}x_2x_1, x_1x_3 - q^{-1}x_3x_1, x_2x_3 - x_3x_2 \rangle},$$

which is a regular graded skew Clifford algebra. Let $\sigma \in \text{Aut}(B)$ such that

$$\sigma(x_1) = x_1, \quad \sigma(x_i) = qx_i \quad \text{for } i = 2, 3,$$

and let $\delta : B \rightarrow B$ be the linear map such that

$$\delta(x_1) = (q - q^{-1})x_2x_3 \quad \text{and} \quad \delta(x_2) = 0 = \delta(x_3).$$

The map δ is a σ -derivation of B , and $A = B[x_4; \sigma, \delta]$ is a regular algebra. In fact, the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, \dots, g_6 \rangle}$$

where

$$\begin{aligned} g_1 &= x_2x_1 - qx_1x_2, & g_2 &= x_2x_3 - x_3x_2, \\ g_3 &= x_3x_1 - qx_1x_3, & g_4 &= x_4x_1 - x_1x_4 - (q - q^{-1})x_2x_3, \\ g_5 &= x_4x_2 - qx_2x_4, & g_6 &= x_4x_3 - qx_3x_4, \end{aligned}$$

has point scheme given by $\mathcal{V}(x_2, x_3) \cup \mathcal{V}(x_2x_3 - x_1x_4)$ (see Figure 4.1).

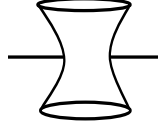


Figure 4.1. Depiction of the Point Scheme in Proposition 4.2.

Proof:

The algebra B is a regular graded skew Clifford algebra by §3.2. To prove δ is a left σ -derivation, we show that $\delta(0) = 0$ in B ; that is,

$$\begin{aligned}
 \delta(x_1x_2 - q^{-1}x_2x_1) &= \delta(x_1x_2) - q^{-1}\delta(x_2x_1) \\
 &= \sigma(x_1)\delta(x_2) + \delta(x_1)x_2 - q^{-1}\sigma(x_2)\delta(x_1) - q^{-1}\delta(x_2)x_1 \\
 &= 0 + (q - q^{-1})x_2x_3x_2 - q^{-1}qx_2(q - q^{-1})x_2x_3 \\
 &= (q - q^{-1})x_2(x_3x_2 - x_2x_3) \\
 &= 0
 \end{aligned}$$

in B . Similarly,

$$\delta(x_1x_3 - q^{-1}x_3x_1) = 0 \quad \text{and} \quad \delta(x_2x_3 - x_3x_2) = 0.$$

Therefore, by Theorem 3.3.6, $A = B[x_4; \sigma, \delta]$ is a regular algebra. By definition of Ore extension, we have

$$\begin{aligned}
 x_4x_1 &= \sigma(x_1)x_4 + \delta(x_1), \\
 x_4x_2 &= \sigma(x_2)x_4 + \delta(x_2), \\
 x_4x_3 &= \sigma(x_3)x_4 + \delta(x_3),
 \end{aligned}$$

which yields the relations g_4, g_5, g_6 in the statement. ■

4.3 Proposition

Suppose $\alpha \in \mathbb{K}^\times$. Let

$$B = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle x_1x_2 - x_2x_1, x_1x_3 - x_3x_1, x_2x_3 - x_3x_2 \rangle},$$

which is a regular graded skew Clifford algebra. Let $\sigma \in \text{Aut}(B)$ such that

$$\sigma(x_1) = x_1 - \alpha x_3, \quad \sigma(x_i) = x_i \quad \text{for } i = 2, 3,$$

and let $\delta : B \rightarrow B$ be the linear map such that

$$\delta(x_1) = \alpha x_1 x_2 \quad \text{and} \quad \delta(x_2) = 0 = \delta(x_3).$$

The map δ is a σ -derivation of B , and $A = B[x_4; \sigma, \delta]$ is a regular algebra. In fact, the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, \dots, g_6 \rangle}$$

where

$$\begin{aligned} g_1 &= x_1 x_2 - x_2 x_1, & g_2 &= x_3 x_2 - x_2 x_3, \\ g_3 &= x_1 x_3 - x_3 x_1, & g_4 &= x_4 x_1 - x_1 x_4 + \alpha(x_4 x_3 - x_1 x_2), \\ g_5 &= x_4 x_2 - x_2 x_4, & g_6 &= x_4 x_3 - x_3 x_4, \end{aligned}$$

has point scheme given by $\mathcal{V}(x_2(x_1 x_2 - x_3 x_4), x_3(x_1 x_2 - x_3 x_4))$ which contains the double line $\mathcal{V}(x_2, x_3)$ (see Figure 4.2).

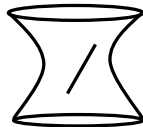


Figure 4.2. Depiction of the Point Scheme in Proposition 4.3.

Proof:

The algebra B is a regular graded skew Clifford algebra by §3.2. To prove δ is a left σ -derivation, we show that $\delta(0) = 0$ in B ; that is,

$$\delta(x_1 x_2 - x_2 x_1) = \delta(x_1 x_2) - \delta(x_2 x_1)$$

$$\begin{aligned}
&= \sigma(x_1)\delta(x_2) + \delta(x_1)x_2 - \sigma(x_2)\delta(x_1) - \delta(x_2)x_1 \\
&= 0 + \alpha x_1 x_2 x_2 - x_2 \alpha x_1 x_2 - 0 \\
&= \alpha(x_1 x_2 - x_2 x_1)x_2 \\
&= 0
\end{aligned}$$

in B . Similarly,

$$\delta(x_1 x_3 - x_3 x_1) = 0 \quad \text{and} \quad \delta(x_2 x_3 - x_3 x_2) = 0.$$

Therefore, by Theorem 3.3.6, $A = B[x_4; \sigma, \delta]$ is a regular algebra. By definition of Ore extension, we have

$$\begin{aligned}
x_4 x_1 &= \sigma(x_1)x_4 + \delta(x_1), \\
x_4 x_2 &= \sigma(x_2)x_4 + \delta(x_2), \\
x_4 x_3 &= \sigma(x_3)x_4 + \delta(x_3),
\end{aligned}$$

which yields the relations g_4, g_5, g_6 in the statement. ■

4.4 Proposition

Suppose $\alpha \in \mathbb{K}^\times \setminus \{-1\}$. Let

$$B = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle x_1 x_2 - x_2 x_1, x_1 x_3 - x_3 x_1, x_2 x_3 - x_3 x_2 \rangle},$$

which is a regular graded skew Clifford algebra. Let $\sigma \in \text{Aut}(B)$ such that

$$\sigma(x_i) = x_i \quad \text{for } i = 1, 3, \quad \sigma(x_2) = (1 + \alpha)x_2,$$

and let $\delta : B \rightarrow B$ be the linear map such that

$$\delta(x_1) = 0 = \delta(x_3) \quad \text{and} \quad \delta(x_2) = -\alpha x_1^2.$$

The map δ is a σ -derivation of B , and $A = B[x_4; \sigma, \delta]$ is a regular algebra. In fact, the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, \dots, g_6 \rangle}$$

where

$$\begin{aligned}
 g_1 &= x_1x_2 - x_2x_1, & g_2 &= x_2x_3 - x_3x_2, \\
 g_3 &= x_1x_3 - x_3x_1, & g_4 &= x_1x_4 - x_4x_1, \\
 g_5 &= x_2x_4 - x_4x_2 - \alpha(x_1^2 - x_2x_4), & g_6 &= x_4x_3 - x_3x_4,
 \end{aligned}$$

has point scheme given by $Q \cup L$ where $Q = \mathcal{V}(x_1^2 - x_2x_4)$ and $L = \mathcal{V}(x_1, x_3)$ (see Figure 4.3).

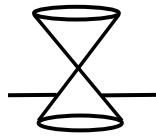


Figure 4.3. Depiction of the Point Scheme in Proposition 4.4.

Proof:

The algebra B is a regular graded skew Clifford algebra by §3.2. To prove δ is a left σ -derivation, we show that $\delta(0) = 0$ in B ; that is,

$$\begin{aligned}
 \delta(x_1x_2 - x_2x_1) &= \delta(x_1x_2) - \delta(x_2x_1) \\
 &= \sigma(x_1)\delta(x_2) + \delta(x_1)x_2 - \sigma(x_2)\delta(x_1) - \delta(x_2)x_1 \\
 &= x_1(-\alpha x_1^2) + 0 - 0 - (-\alpha x_1^2)x_1 \\
 &= -\alpha(x_1^3 - x_1^3) \\
 &= 0
 \end{aligned}$$

in B . Similarly,

$$\delta(x_1x_3 - x_3x_1) = 0 \quad \text{and} \quad \delta(x_2x_3 - x_3x_2) = 0.$$

Therefore, by Theorem 3.3.6, $A = B[x_4; \sigma, \delta]$ is a regular algebra. By definition of Ore extension, we have

$$x_4x_1 = \sigma(x_1)x_4 + \delta(x_1),$$

$$x_4x_2 = \sigma(x_2)x_4 + \delta(x_2),$$

$$x_4x_3 = \sigma(x_3)x_4 + \delta(x_3),$$

which yields the relations g_4, g_5, g_6 in the statement. ■

4.5 Proposition

Let

$$B = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle x_1x_2 - x_2x_1, x_1x_3 - x_3x_1, x_2x_3 - x_3x_2 \rangle},$$

which is a regular graded skew Clifford algebra. Let $\sigma \in \text{Aut}(B)$ such that

$$\sigma(x_1) = x_1 + x_3, \quad \sigma(x_i) = x_i, \quad \text{for } i = 2, 3,$$

and let $\delta : B \rightarrow B$ be the linear map such that

$$\delta(x_1) = -x_1^2 \quad \text{and} \quad \delta(x_2) = 0 = \delta(x_3).$$

The map δ is a σ -derivation of B , and $A = B[x_4; \sigma, \delta]$ is a regular algebra. In fact, the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, \dots, g_6 \rangle}$$

where

$$g_1 = x_1x_2 - x_2x_1, \quad g_2 = x_3x_2 - x_2x_3,$$

$$g_3 = x_1x_3 - x_3x_1, \quad g_4 = x_1x_4 - x_4x_1 - x_1^2 + x_4x_3,$$

$$g_5 = x_2x_4 - x_4x_2, \quad g_6 = x_3x_4 - x_4x_3,$$

has point scheme given by $Q \cup L$ where $Q = \mathcal{V}(x_2^2 - x_4x_3)$ and $L = \mathcal{V}(x_3, x_4)$ (so the line L is tangential to the quadric Q at a nonsingular point of Q)(see Figure 4.4).

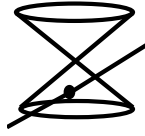


Figure 4.4. Depiction of the Point Scheme in Proposition 4.5.

Proof:

The algebra B is a regular graded skew Clifford algebra by §3.2. To prove δ is a left σ -derivation, we show that $\delta(0) = 0$ in B ; that is,

$$\begin{aligned}
 \delta(x_1x_2 - x_2x_1) &= \delta(x_1x_2) - \delta(x_2x_1) \\
 &= \sigma(x_1)\delta(x_2) + \delta(x_1)x_2 - \sigma(x_2)\delta(x_1) - \delta(x_2)x_1 \\
 &= 0 - x_1^2x_2 - x_2(-x_1^2) - 0 \\
 &= -x_1^2x_2 + x_1^2x_2 \\
 &= 0
 \end{aligned}$$

in B . Similarly,

$$\delta(x_1x_3 - x_3x_1) = 0 \quad \text{and} \quad \delta(x_2x_3 - x_3x_2) = 0.$$

Therefore, by Theorem 3.3.6, $A = B[x_4; \sigma, \delta]$ is a regular algebra. By definition of Ore extension, we have

$$\begin{aligned}
 x_4x_1 &= \sigma(x_1)x_4 + \delta(x_1), \\
 x_4x_2 &= \sigma(x_2)x_4 + \delta(x_2), \\
 x_4x_3 &= \sigma(x_3)x_4 + \delta(x_3),
 \end{aligned}$$

which yields the relations g_4, g_5, g_6 in the statement. ■

4.6 Proposition

Let

$$B = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle x_1x_2 - x_2x_1, x_1x_3 - x_3x_1, x_2x_3 - x_3x_2 \rangle},$$

which is a regular graded skew Clifford algebra. Let $\sigma \in \text{Aut}(B)$ such that

$$\sigma = \text{id}_B \in \text{Aut}(B),$$

and let $\delta : B \rightarrow B$ be the linear map such that

$$\delta(x_1) = -x_1^2 + x_2x_3 \quad \text{and} \quad \delta(x_2) = 0 = \delta(x_3).$$

The map δ is a σ -derivation of B , and $A = B[x_4; \sigma, \delta]$ is a regular algebra. In fact, the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, \dots, g_6 \rangle}$$

where

$$\begin{aligned} g_1 &= x_1x_2 - x_2x_1, & g_2 &= x_2x_3 - x_3x_2, \\ g_3 &= x_1x_3 - x_3x_1, & g_4 &= x_1x_4 - x_4x_1 - x_1^2 + x_2x_3, \\ g_5 &= x_2x_4 - x_4x_2, & g_6 &= x_3x_4 - x_4x_3, \end{aligned}$$

has point scheme given by $Q \cup L$ where $Q = \mathcal{V}(x_1^2 - x_2x_3)$ and $L = \mathcal{V}(x_2, x_3)$ (so the line L is tangential to the quadric Q at a singular point of Q)(see Figure 4.5).

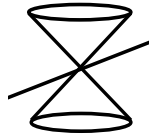


Figure 4.5. Depiction of the Point Scheme in Proposition 4.6.

Proof:

The algebra B is a regular graded skew Clifford algebra by §3.2. To prove δ is a left σ -derivation, we show that $\delta(0) = 0$ in B ; that is,

$$\delta(x_1x_2 - x_2x_1) = \delta(x_1x_2) - \delta(x_2x_1)$$

$$\begin{aligned}
&= \sigma(x_1)\delta(x_2) + \delta(x_1)x_2 - \sigma(x_2)\delta(x_1) - \delta(x_2)x_1 \\
&= 0 + (-x_1^2 + x_2x_3)x_2 - x_2(-x_1^2 + x_2x_3) - 0 \\
&= -x_1^2x_2 + x_2^2x_3 + x_1^2x_2 - x_2^2x_3 \\
&= 0
\end{aligned}$$

in B . Similarly,

$$\delta(x_1x_3 - x_3x_1) = 0 \quad \text{and} \quad \delta(x_2x_3 - x_3x_2) = 0.$$

Therefore, by Theorem 3.3.6, $A = B[x_4; \sigma, \delta]$ is a regular algebra. By definition of Ore extension, we have

$$x_4x_1 = \sigma(x_1)x_4 + \delta(x_1),$$

$$x_4x_2 = \sigma(x_2)x_4 + \delta(x_2),$$

$$x_4x_3 = \sigma(x_3)x_4 + \delta(x_3),$$

which yields the relations g_4, g_5, g_6 in the statement. ■

4.7 Proposition

Let

$$B = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle x_1x_2 - x_2x_1, x_1x_3 - x_3x_1, x_2x_3 - x_3x_2 \rangle},$$

which is a regular graded skew Clifford algebra. Let $\sigma \in \text{Aut}(B)$ such that

$$\sigma = \text{id}_B \in \text{Aut}(B),$$

and let $\delta : B \rightarrow B$ be the linear map such that

$$\delta(x_1) = 0 = \delta(x_2) \quad \text{and} \quad \delta(x_3) = -x_1^2 + x_2x_3.$$

The map δ is a σ -derivation of B , and $A = B[x_4; \sigma, \delta]$ is a regular algebra. In fact, the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{\langle g_1, \dots, g_6 \rangle}$$

where

$$\begin{aligned} g_1 &= x_1x_2 - x_2x_1, & g_2 &= x_2x_3 - x_3x_2, \\ g_3 &= x_1x_3 - x_3x_1, & g_4 &= x_1x_4 - x_4x_1, \\ g_5 &= x_2x_4 - x_4x_2, & g_6 &= x_3x_4 - x_4x_3 - x_1^2 + x_2x_3, \end{aligned}$$

has point scheme given by $\mathcal{V}(x_1(x_1^2 - x_2x_3), x_2(x_1^2 - x_2x_3))$, which contains the double line $V(x_1, x_2)$ (see Figure 4.6).

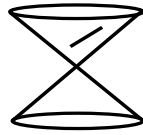


Figure 4.6. Depiction of the Point Scheme in Proposition 4.7.

Proof:

The algebra B is a regular graded skew Clifford algebra by §3.2. To prove δ is a left σ -derivation, we show that $\delta(0) = 0$ in B ; that is,

$$\begin{aligned} \delta(x_1x_2 - x_2x_1) &= \delta(x_1x_2) - \delta(x_2x_1) \\ &= \sigma(x_1)\delta(x_2) + \delta(x_1)x_2 - \sigma(x_2)\delta(x_1) - \delta(x_2)x_1 \\ &= 0 \end{aligned}$$

in B . Similarly,

$$\delta(x_1x_3 - x_3x_1) = 0 \quad \text{and} \quad \delta(x_2x_3 - x_3x_2) = 0.$$

Therefore, by Theorem 3.3.6, $A = B[x_4; \sigma, \delta]$ is a regular algebra. By definition of Ore extension, we have

$$\begin{aligned} x_4x_1 &= \sigma(x_1)x_4 + \delta(x_1), \\ x_4x_2 &= \sigma(x_2)x_4 + \delta(x_2), \\ x_4x_3 &= \sigma(x_3)x_4 + \delta(x_3), \end{aligned}$$

which yields the relations g_4, g_5, g_6 in the statement. ■

4.8 Remark

S. P. Smith and T. Stafford proved that the Sklyanin algebras on 4 generators (the family of algebras constructed by the physicist, E. K. Sklyanin [16]) are regular algebras of global dimension 4 [17]. However, they appear not to be directly related, in the sense of Chapter 3, to any graded skew Clifford algebra, although they could perhaps be weakly related in some way (c.f., [14, Remark 4.4]).

CHAPTER 5

TWISTING A REGULAR GRADED SKEW CLIFFORD ALGEBRA BY AN AUTOMORPHISM

In this chapter, we suppose A is a regular graded skew Clifford algebra that is a twist (in the sense of [4, §8]) of a regular graded Clifford algebra B by an automorphism. We prove in Theorem 5.7 that, under this hypothesis, the subalgebra R of A generated by the y_i (see Definition 2.2.1) is a twist of a polynomial ring by an automorphism, and is a skew polynomial ring. We also present an example that demonstrates that this can fail when A is not a twist of B (see Nonexample 5.3).

We thank S. P. Smith (University of Washington) for the suggestion to study the algebra R .

5.1 Definition of a Twist by an Automorphism [4, §8]

Let D denote a quadratic algebra, let D_1 denote the span of the homogeneous degree-one elements of D . Suppose τ is a graded degree-zero automorphism of D , that is, $\tau|_{D_i} : D_i \rightarrow D_i$ for all i . The twist D^τ of D by τ is a quadratic algebra that has the same underlying vector space as D , but has a new multiplication $*$ defined as follows:

$$\text{if } a, b \in D_1 = (D^\tau)_1, \text{ then } a * b = a\tau(b),$$

where the right-hand side is computed using the original multiplication in D .

In this chapter, a^τ means $\tau(a)$ for $a \in D_1 = (D^\tau)_1$. Also, we consider only automorphisms τ such that

$$\tau|_{D_i} : D_i \rightarrow D_i \quad \text{for all } i.$$

5.2 Example

Let $\lambda_1, \lambda_2 \in \mathbb{K}^\times$, and let

$$C = \frac{\mathbb{K}\langle X, Y \rangle}{\langle XY - YX \rangle}.$$

The map

$$\tau = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \in \text{Aut}(C),$$

so the algebra

$$A = \frac{\mathbb{K}\langle x, y \rangle}{\langle \lambda_1 \lambda_2^{-1} xy - yx \rangle}$$

is the twist of C by τ , since

$$\lambda_1 \lambda_2^{-1} x * y - y * x = XY - YX.$$

5.3 Nonexample

Let $\lambda \in \mathbb{K}$ and let

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle g_1, g_2, g_3 \rangle}$$

where

$$g_1 = x_1 x_2 + \mu_{12} x_2 x_1 - \lambda x_3^2,$$

$$g_2 = x_1 x_3 + \mu_{13} x_3 x_1,$$

$$g_3 = x_2 x_3 + \mu_{23} x_3 x_2,$$

where $0 \neq \mu_{ij} \in \mathbb{K}$ for $i, j = 1, 2, 3$ such that

$$\mu_{32} = \mu_{13} \neq \mu_{12} \mu_{23}.$$

By §3.2, A is a regular graded skew Clifford algebra. Let R be the \mathbb{K} -algebra generated by y_1, y_2, y_3 (see Definition 2.2.1). By Definition 2.2.1, $\deg(y_i) = 2$ for all i , and, in this algebra A , the y_i 's satisfy only two relations of degree 4, so R is not a skew polynomial ring, nor a twist of a polynomial ring by an automorphism.

The main result of this chapter, Theorem 5.7, proves that R is a twist of a polynomial ring if A is a twist of a graded Clifford algebra. Thus section 5.4 defines this concept and results useful in the proof of Theorem 5.7.

5.4 Definition of Graded Clifford Algebras [10]

Let $M_1, \dots, M_n \in M(n, \mathbb{K})$ denote symmetric matrices. A graded Clifford algebra B associated to M_1, \dots, M_n is a graded \mathbb{K} -algebra on degree-one generators X_1, \dots, X_n and on degree-two generators Y_1, \dots, Y_n with defining relations given by:

- (a) $X_i X_j + X_j X_i = \sum_{k=1}^n (M_k)_{ij} Y_k$ for all $i, j = 1, \dots, n$, and
- (b) Y_k central for all $k = 1, \dots, n$.

In Definition 2.2.1, if $\mu_{ij} = 1$ for all $i, j = 1, \dots, n$, and if r_i is central for all $i \in \{1, \dots, n\}$, then the graded skew Clifford algebra in that definition is a graded Clifford algebra.

5.4.1 Example

Let $\lambda \in \mathbb{K}^\times$. If

$$M_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2\lambda \end{bmatrix},$$

then the graded Clifford algebra B associated to M_1, M_2 is

$$\frac{\mathbb{K}\langle X_1, X_2 \rangle}{\langle X_2^2 - \lambda X_1^2 \rangle},$$

since

$$X_1X_2 + X_2X_1 = Y_1, \quad Y_2 = X_1^2, \quad \lambda Y_2 = X_2^2,$$

and Y_i is central for all i .

5.4.2 Definition of Quadric System

Let C be the \mathbb{K} -algebra on generators Z_1, \dots, Z_n with defining relations

$$Z_jZ_i = Z_iZ_j \quad \text{for all } i, j,$$

and let

$$Q_k := \begin{bmatrix} Z_1 & \cdots & Z_n \end{bmatrix} M_k \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \in C.$$

The collection $\{Q_1, \dots, Q_n\}$ is called a quadric system. In Definition 2.2.1, if $\mu_{ij} = 1$ for all $i, j = 1, \dots, n$, then the quadric system of a graded skew Clifford algebra is the quadric system of a graded Clifford algebra.

5.4.3 Example

For the algebra B in Example 5.4.1, we have

$$C = \frac{\mathbb{K}\langle Z_1, Z_2 \rangle}{\langle Z_2Z_1 - Z_1Z_2 \rangle}.$$

Moreover,

$$Q_1 = 2Z_1Z_2, \quad Q_2 = 2Z_1^2 + 2\lambda Z_2^2.$$

However, since $\text{char}(\mathbb{K}) \neq 2$, we consider:

$$Q_1 = Z_1Z_2, \quad Q_2 = Z_1^2 + \lambda Z_2^2.$$

5.4.4 Definition of Zero Locus

Suppose $C = \mathbb{K}[X_1, \dots, X_n]$ and $F \in C_2$. We define the zero locus $\Gamma(F)$ of F to be

$$\Gamma(F) = \{P \in \mathbb{P}^{n-1} : F(P) = 0\},$$

where \mathbb{P}^{n-1} is identified with $\mathbb{P}(C_1^*)$.

Similarly if $F_1, \dots, F_m \in C_2$, then

$$\Gamma(F_1, \dots, F_m) = \{P \in \mathbb{P}^{n-1} : F_i(P) = 0 \text{ for all } i\}.$$

5.4.5 Definition of Base-Point Free

The quadric system $\{Q_1, \dots, Q_n\} \subset C$ is said to be base-point free (BPF) if $\Gamma(Q_1, \dots, Q_n)$ is empty.

If $\mu_{ij} = 1$ for all $i, j = 1, \dots, n$, then Definition 5.4.5 is equivalent to Definition 2.2.8, since, in this case, Z is the graph of the identity map on \mathbb{P}^{n-1} .

5.4.6 Example

Let $P = (\alpha_1, \alpha_2) \in \mathbb{P}^1$. Referring to Example 5.4.3, we have

$$0 = Q_1(P) = \alpha_1\alpha_2, \quad 0 = Q_2(P) = \alpha_1^2 + \lambda\alpha_2^2.$$

Thus $\alpha_1 = \alpha_2 = 0$ which is contradiction. Therefore $\{Q_1, Q_2\}$ is BPF.

5.4.7 Theorem [1], [10]

Let M_1, \dots, M_n be symmetric $n \times n$ matrices. The graded Clifford algebra A associated to M_1, \dots, M_n is quadratic, regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series $\frac{1}{(1-t)^n}$ if and only if the quadric system

in \mathbb{P}^{n-1} determined by the M_k is BPF. In this case, A is noetherian and has no zero divisors.

Before returning to our main theorem, we first require some preliminary technical results.

5.5 Lemma

Let X_1, \dots, X_n and the Y_k be as in Definition 5.4. Let $D \subset B_2$ denote the homogeneous central elements in B of degree two. If $a, b \in B_1$, then $ab + ba \in D$.

Proof:

We may write

$$a = \sum_{m=1}^n \alpha_m X_m \quad \text{and} \quad b = \sum_{l=1}^n \beta_l X_l$$

with $\alpha_m, \beta_l \in \mathbb{K}$ for all m, l . It follows that

$$ab + ba = \sum_{m,l=1}^n \alpha_m \beta_l (X_m X_l + X_l X_m) = \sum_{m,l=1}^n \alpha_m \beta_l \left(\sum_{k=1}^n (M_k)_{ml} Y_k \right).$$

Therefore

$$ab + ba \in D \quad \text{for all} \quad a, b \in B_1,$$

since $Y_k \in D$ for all k . ■

5.6 Lemma

Let μ_{ij} be as defined in Definition 2.2.1 for all i, j , and let S be the skew polynomial ring on n generators defined in Definition 2.2.3. The algebra S is a twist of the polynomial ring

$$C = \mathbb{K}[Z_1, \dots, Z_n]$$

by an automorphism $\sigma \in \text{Aut}(C)$ if and only if

$$\mu_{ik} = \mu_{ij}\mu_{jk} \quad \text{for all } i, j, k;$$

in this case, $\sigma|_{C_1}$ is semisimple, and for all i, j , we have

$$\mu_{ij} = \frac{\rho_i}{\rho_j}, \quad \text{where } \rho_i \in \mathbb{K}^\times$$

and

$$\sigma(Z_i) = \rho_i Z_i \quad \text{for all } i.$$

Proof:

The first part follows from [3], since $\mu_{ik} = \mu_{ij}\mu_{jk}$ for all i, j, k if and only if the point scheme of S is \mathbb{P}^{n-1} , and the latter holds if and only if S is a twist of the polynomial ring on n variables by an automorphism.

Let S be a twist of the polynomial ring

$$C = \mathbb{K}[Z_1, \dots, Z_n]$$

by an automorphism $\sigma \in \text{Aut}(C)$. The relations in S are

$$z_j z_i = \mu_{ij} z_i z_j \quad \text{for all } i, j,$$

therefore, in C_1 we have

$$Z_j Z_i^\sigma = \mu_{ij} Z_i Z_j^\sigma \quad (**).$$

However, C is commutative and a unique factorization domain, and

$$\deg(Z_i) = 1 \quad \text{for all } i,$$

so Z_i is irreducible, and, for all $i \neq j$, $Z_i \nmid Z_j$. It therefore follows from (**) that $Z_i \mid Z_i^\sigma$ for all i . Since $\deg(Z_i^\sigma) = 1$, $Z_i^\sigma \in \mathbb{K}^\times Z_i$ for all i . Hence, $\sigma|_{C_1}$ is semisimple.

Writing

$$Z_i^\sigma = \rho_i Z_i \quad \text{for all } i,$$

where

$$\rho_i \in \mathbb{K}^\times \quad \text{for all } i,$$

and substituting into (**) completes the proof. ■

Recall B is a regular graded Clifford algebra and A is a regular graded skew Clifford algebra that is a twist of B by an automorphism $\tau \in \text{Aut}(B)$. From Definition 2.2.3, there is a skew polynomial ring S associated to A . By [5, Proposition 4.5], since A is a twist of B by τ , there exists a choice for S so that S is a twist of the polynomial ring C by τ^{-1} and conversely. By Lemma 5.6, $\tau|_{S_1}$ is semisimple; i.e. for each $i = 1, \dots, n$, we have

$$\tau(z_i) = \lambda_i z_i \quad \text{for some } \lambda_i \in \mathbb{K}^\times$$

and

$$\mu_{ij} = \frac{\lambda_j}{\lambda_i} \quad \text{for all } i, j.$$

(In the notation of Lemma 5.6,

$$\lambda_i = \rho_i^{-1} \quad \text{for all } i$$

since $\tau = \sigma^{-1}$.)

In the next result, R' is the subalgebra of B generated by the Y_i , so R' is the commutative polynomial ring $\mathbb{K}[Y_1, \dots, Y_n]$. The algebra R denotes the subalgebra of A generated by the y_i , and by Nonexample 5.3, this algebra is not, in general, a skew polynomial ring nor a twist of a polynomial ring.

5.7 Theorem

Suppose that A is a regular graded skew Clifford algebra on n degree-1 generators x_1, \dots, x_n (in the sense of Theorem 3.1), and R is the subalgebra of A generated by y_1, \dots, y_n . If A is a twist of a regular graded Clifford algebra B (in the sense of Theorem 5.4.7) by

$$\tau \in \text{Aut}(B),$$

then R is a twist of the polynomial ring R' on n variables and is a skew polynomial ring.

Proof:

By the preceding discussion

$$\mu_{ij} = \frac{\lambda_j}{\lambda_i} \quad \text{for all } i, j \quad (\dagger)$$

where $\lambda_i \in \mathbb{K}^\times$ and

$$\tau(z_i) = \lambda_i z_i \quad \text{for all } i.$$

Since $S_1 = C_1$, $\tau(Z_i) = \lambda_i Z_i$ for all i , so we may rechoose the X_k in B_1 so that the degree-two relations of B have the form given by Definition 5.3(a) (the M_k will also change) and so that $\{X_1, \dots, X_n\}$ is dual to the basis $\{Z_1, \dots, Z_n\}$ for C . With this choice, we have

$$X_i^\tau = \lambda_i X_i \quad \text{for all } i,$$

and the twist of X_i is x_i . For all i, j , we have

$$\begin{aligned} x_i x_j + \mu_{ij} x_j x_i &= x_i x_j + \frac{\lambda_j}{\lambda_i} x_j x_i \\ &= \frac{1}{\lambda_i} (\lambda_i x_i x_j + \lambda_j x_j x_i) \\ &= \frac{1}{\lambda_i} (x_i^\tau x_j + x_j^\tau x_i) \in \mathbb{K}^\times (x_i^\tau x_j + x_j^\tau x_i). \end{aligned}$$

For all i, j , let

$$n_{ij} = x_i^\tau x_j + x_j^\tau x_i.$$

By Definition 2.2.1, $n_{ij} \in R$ for all i, j , so

$$\mathbb{K}[n_{ij} : 1 \leq i, j \leq n] \subseteq R.$$

Since A is quadratic, each y_k is a function of the n_{ij} , and so

$$R = \mathbb{K}[n_{ij} : 1 \leq i, j \leq n].$$

Moreover, each n_{ij} is a normal element of A since, for all i, j, k , we have:

$$\begin{aligned} x_k n_{ij} &= x_k(x_i^\tau x_j + x_j^\tau x_i) \\ &= X_k(X_i^{\tau^2} X_j^{\tau^2} + X_j^{\tau^2} X_i^{\tau^2}) \\ &= \lambda_i^2 \lambda_j^2 X_k(X_i X_j + X_j X_i) \\ &= \lambda_i^2 \lambda_j^2 (X_i X_j + X_j X_i) X_k \\ &= \lambda_k^{-2} \lambda_i \lambda_j (x_i^\tau x_j + x_j^\tau x_i) x_k \\ &= \mu_{ki} \mu_{kj} n_{ij} x_k, \end{aligned}$$

where the fourth equality follows from Lemma 5.5. It follows that

$$n_{ij} n_{kr} = \mu_{ik} \mu_{jk} \mu_{ir} \mu_{jr} n_{kr} n_{ij} \quad \text{for all } i, j, k, r.$$

Hence, by (\dagger) , we have

$$n_{ij} n_{kr} = \mu_{ik}^2 \mu_{jr}^2 n_{kr} n_{ij} \quad \text{for all } i, j, k, r \quad (\dagger\dagger).$$

Therefore R is a skew polynomial ring. For all i, j, k, r , let

$$\nu_{ijk} = \mu_{ik}^2 \mu_{jr}^2.$$

It follows that

$$\nu_{ijk} \nu_{krab} = \nu_{ijab} \quad \text{for all } i, j, k, r, a, b,$$

so R is a twist of the polynomial ring R'' . For all i, j , let $N_{ij} \in B$ denote the element that twists to $n_{ij} \in A$. So

$$N_{ij} = X_i^\tau X_j^\tau + X_j^\tau X_i^\tau = \tau(X_i X_j + X_j X_i)$$

and, by $(\dagger\dagger)$, we have

$$N_{ij} N_{kr}^{\tau^2} = \nu_{ijk} N_{kr} N_{ij}^{\tau^2} \quad \text{for all } i, j, k, r.$$

In particular, R'' is the subalgebra of B generated by the Y_k , so $R'' = R'$. Defining

$$\tau' \in \text{Aut}(R')$$

by

$$\tau'(N_{ij}) = \lambda_i^2 \lambda_j^2 N_{ij} \quad \text{for all } i, j,$$

we find that R is the twist of R' by τ' . ■

5.8 Example

Let

$$B = \frac{\mathbb{K}\langle X_1, X_2, X_3 \rangle}{\langle f_1, f_2, f_3 \rangle}$$

where

$$f_1 = X_1 X_2 + X_2 X_1 - X_3^2,$$

$$f_2 = X_1 X_3 + X_3 X_1 - X_2^2,$$

$$f_3 = X_2 X_3 + X_3 X_2 - X_1^2$$

and let

$$\tau = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu_{12} & 0 \\ 0 & 0 & \mu_{13} \end{bmatrix} \in \text{Aut}(B).$$

Twisting B by τ yields the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3 \rangle}{\langle g_1, g_2, g_3 \rangle}$$

where

$$g_1 = x_1x_2 + \mu_{12}x_2x_1 - \mu_{32}x_3^2,$$

$$g_2 = x_1x_3 + \mu_{13}x_3x_1 - \mu_{23}x_2^2,$$

$$g_3 = x_2x_3 + \mu_{23}x_3x_2 - \mu_{13}x_1^2.$$

By Definition 5.3, B is a graded Clifford algebra, and by §3.2, A is a graded skew Clifford algebra. The subalgebra R of A generated by the y_i is the algebra

$$\frac{\mathbb{K}\langle y_1, y_2, y_3 \rangle}{\langle y_1y_2 - \mu_{12}y_2y_1, \quad y_2y_3 - \mu_{23}y_3y_2, \quad y_1y_3 - \mu_{13}y_3y_1 \rangle},$$

which is a skew polynomial ring and a twist of the polynomial ring $\mathbb{K}[Y_1, Y_2, Y_3] \subset B$.

REFERENCES

- [1] Aubry, M. and Lemaire, J.-M., *Zero Divisors in Enveloping Algebras of Graded Lie Algebras*, J. Pure and App. Algebra 38 (1985), 159-166.
- [2] Artin, M. and Schelter, W., *Graded Algebras of Global Dimension 3*, Adv. Math., 66 (1987), 171-216.
- [3] M. Artin, J. Tate and M. Van den Bergh, *Some Algebras Associated to Automorphisms of Elliptic Curves*, The Grothendieck Festschrift 1, Eds. P. Cartier et al. Birkhauser (1990), 33-85.
- [4] Artin, M., Tate, J., and Van den Bergh, M., *Modules Over Regular Algebras of Dimension 3*, Invent. Math., 106 (1991), 335-388.
- [5] Cassidy, T. and Vancliff, M., *Generalizations of Graded Clifford Algebras and of Complete Intersections*, Journal of the London Mathematical Society 81 (2010), 91-112.
- [6] Drinfel'd, V. G., *Quantum Groups*, Proc. Int. Cong. Math., Berkeley 1 (1986), 798-820.
- [7] Ekström, E. K., *The Auslander Condition on Graded and Filtered Noetherian Rings*, in: Séminaire Dubreil-Malliavin 1987-1988, Lecture Notes in Math., 1404, Springer, Berlin, 1989, 220-245.
- [8] Feigin, B. L., and Odesskii, A. B., *Elliptic Sklyanin Algebras*, Func. Anal. Appl. 23 (1989), 45-54.
- [9] Goodearl, K. R. and Warfield, R. B., *An Introduction to Noncommutative Noetherian Rings*, London Mathematical Society Student Texts, 2004.

- [10] Le Bruyn, L., *Central Singularities of Quantum Spaces*, J. Algebra 177 (1995) 142-153.
- [11] Levasseur, T., *Some Properties of Non-commutative Regular Graded Rings*, Glasgow Math. J. 34 (1992), 277-300.
- [12] Levasseur, T. and Stafford, J. T., *The Quantum Coordinate Ring of the Special Linear Group*, J. Pure Appl. Algebra 86 (1993), no. 2, 181-186.
- [13] McConnell, J. C. and Robson, J. C., *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics, American Mathematical Society, 2001.
- [14] Nafari M., Vancliff, M., and Zhang, Jun, *Classifying Quadratic Quantum \mathbb{P}^2 s by Using Graded Skew Clifford Algebras*, revised preprint June 2011, arXiv:1011.5279.
- [15] Rotman, Joseph J., *An Introduction to Homological Algebra*, Second edition, Universitext, Springer, New York, 2009, xiv+709 pp.
- [16] Sklyanin, E. K., *Some Algebraic Structures Connected to the Yang-Baxter Equation*, Func. Anal. Appl. 16 (1982), no. 4, 27-34.
- [17] Smith, S. P. and Stafford, J. T., *Regularity of the Four Dimensional Sklyanin Algebra*, Compositio Math. 83 (1992), no. 3, 259-289.
- [18] Shelton, B. and Vancliff, M., *Embedding a Quantum Rank Three Quadric in a Quantum \mathbb{P}^3* , Comm. Algebra 27 (1999), no. 6, 2877-2904.
- [19] Vancliff, M., Van Rompay, K. and Willaert, L., *Some Quantum \mathbb{P}^3 s with Finitely Many Points*, Comm. Algebra 26 (1988), no. 4, 1193-1208.

BIOGRAPHICAL STATEMENT

Manizheh Nafari earned her B.Sc. from Sharif University of Technology, Tehran, Iran, in 1998, and her M.Sc. from Tarbiat Modarres University, Tehran, Iran in 2001. She was a high-school part-time teacher for several years in Tehran, Iran. She also was the instructor of precalculus courses in The Hadi Institution of Higher Education in Tehran, Iran, for Winter and Spring 2004.

Manizheh Nafari started her Ph.D. under the supervision of Dr. M. Vancliff at the University of Texas at Arlington (UTA) in August 2007. While a graduate student at UTA, she has had the opportunity to teach, as the instructor of record, several undergraduate courses such as trigonometry, college algebra, and business calculus. Her research interest lies in the area of non-commutative algebra, (quadratic) regular algebras, and non-commutative algebraic geometry.