A SNAPSHOT OF ADVANCED HIGH SCHOOL STUDENTS' UNDERSTANDING OF CONTINUITY

by

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ABSTRACT

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We report on a study of sixteen high school calculus and seven precalculus students' concept image and concept definition of continuity after one-trimester of instruction at a large suburban high school in the southwestern United States. The researchers developed a questionnaire based upon the work of Tall and Vinner (1981) to determine if calculus students had developed a more sophisticated concept image and concept definition of continuity than students in pre-calculus after a typical treatment in both courses. Using data from the written assessment it was not evident that calculus students demonstrated a more sophisticated concept image and concept definition than pre-calculus students. However, findings suggest that a weak concept image or concept definition of continuity reflects practices in precalculus and calculus instruction.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Figure Page

LIST OF TABLES

Table Page

CHAPTER 1

INTRODUCTION

David Tall and Shlomo Vinner first studied students' concept image of continuity in 1981. This idea of the student's concept image and its relationship to the concept definition has since been studied in other areas of mathematics. This relationship helps educators to understand how to help students reconcile the conflicts within their own concept image and as their concept images relate to the concept definition. In recent years, there also been a strong push in mathematics education to emphasize conceptual understanding instead of mathematical procedures or processes. Students need to be taught how to reason and how to problem solve in mathematics courses. When teachers move from a simple procedure- based curriculum to a problem-based curriculum, these conflicts in students' concept images become more apparent. If teachers know what some common misconceptions among students are, they can create situations in which the students are forced to confront these misconceptions and hopefully begin to create a concept image that reflects the concept definition.

I looked at three studies on concept image, each of which studied the concept image of college students. Since I teach an AP Calculus course, I was interested to see how the results of advanced high school students would compare those results in the studies focusing on college students. Thus my research question is, "Do AP Calculus students have a better (more formal mathematically elegant) understanding of continuity than Pre AP Precalculus students?" Continuity is first introduced in the Precalculus course; however, it is studied more in depth in the Calculus course. In the Calculus course, the students are expected to apply their understanding of continuity to problems involving The Mean Value Theorem, The Intermediate Value Theorem and The Definition of a Derivative.

Students in both courses were taught continuity according to their respective curriculum. Weeks after the concept of continuity had been taught, students were given a questionnaire over continuity, similar to that used by Tall and Vinner in their 1981 study. The results of the questionnaire were analyzed and common flaws in students' concept image were discussed.

CHAPTER 2

LITERATURE REVIEW

2.1 How Do We Measure Student Understanding?

One of the goals of teaching advanced mathematics is to teach students how to think mathematically. We want our students to understand the concepts involved in the subject, not just the processes. We want them to be able to apply these concepts and to problem solve. With the wide spread use of calculators and computers that can do the processing, it should allow us to put more emphasis on understanding the concepts behind these processes. The very foundation for calculus is the complex concept of a limit. The definition of the derivative and the definition of the integral both involve limits. Thus, the need for students to understand such abstract concepts is crucial for success in calculus. White and Mitchelmore studied students' understanding of related rates. Their study involved forty first year full-time university mathematics students. They found that students struggle more with questions in which they have to interpret the calculus; the students have to abstract knowledge from the problem. White and Mitchelmore define abstract concepts as "concepts formed by the process of abstracting." This process of "abstracting" requires students to not only have procedural knowledge, but conceptual knowledge as well. According to Hiebert and Leferve (1986) as cited in White and Mitchelmore, conceptual knowledge is defined as understanding relationships between mathematical objects. Hiebert and Leferve believe that students need to know how concepts, symbols, and procedures relate. For example, when discussing continuity, students need to know how the concepts of limits and domain relate. Dawkins and Epperson studied student progress in problem solving. Their study involved two hundred two first semester university calculus students. Dawkins and Epperson (2007) describe procedural knowledge as "procedures learned by cues," and conceptual knowledge as "grasping the relationships between mathematical objects in context."

Tall and Vinner studied the relationship between students' concept image and the concept definition as applied to limits and continuous functions. They gathered results from forty-one university mathematics students. They define concept definition "to be a form of words used to specify that concept." A concept definition for continuity at $x = c$ is "The function $f(x)$ is continuous $x = c$ if the following three conditions hold: 1.) $f(c)$ exists for c in the domain. 2.) $\lim_{x \to c} f(x)$ exists. 3.) $\lim_{x \to c} f(x) = f(c)$. They defined concept image to "describe the total cognitive structure that is associated with the concept." An example of a concept image students have on continuity is that continuous functions have no gaps. This concept image poses a problem when students are confronted with continuous functions whose domain is not the set of all real numbers. This struggle with a function being a continuous function even if it has a "break" could be a conflict of intuition. Dreyfus and Eisenberg (1982) studied the effects of student intuition and mathematical understanding. They gathered results from four hundred ninety-three students in grades six through nine in twelve different schools. Dreyfus and Eisenberg define *intuition* as "the mental representations of facts that seem self-evident." These intuitions, like a student's concept image, are based on personal experience and may not be the same from student to student. They found that students' intuition of functional relationships does grow as they progress through school.

For this study, I will be looking at students' concept image of continuity and measuring that against the concept definition: "A function is a continuous function if it is continuous at every point in its domain." in order to gain more insight into students' conceptual knowledge of continuity.

2.2 Students' Understanding of Continuity

According to research, many students' concept image of continuous functions requires the graph of the given function to have no "gaps". In Tall and Vinner's (1981) study "Concept

Image and Concept Definition in Mathematics with particular reference to Limits and Continuity" they found that 35 out of 41 students (approximately 85%) believed the function $f(x) = \frac{1}{x}, x \neq 0$ is not continuous. Several of the subjects justified their claim of discontinuous by stating in some form that the graph had a "gap". Takaci, Pesic, and Tatar performed a similar study in 2006, they found that 76.2% of subjects who were "talented in mathematics" 66.7% of subjects whose majors were science and mathematics believed $f(x) = \frac{1}{x}, x \neq 0$ is not continuous. Again, students stated justifications mentioning "$f(x)$ is not defined at $x = 0$." It is important to note in both studies the graph of $f(x) = \frac{1}{x}, x \neq 0$ was provided. Takaci, Pesic, and Tatar also asked if $f(x) = \frac{1}{x}, x \neq 0$ is continuous without providing a graph. They found that the results were actually a little better when no graph was provided. It seems that the visual representation of a function, its graph, greatly influences the students' opinion on the continuity of the function. In a 1989 study by Vinner and Dreyfus, they found that students may be able to quote a specific definition; however, they could not use that definition to solve a problem or answer a particular question. This is a phenomenon known as the compartmentalization phenomenon (Vinner, Hershkowitz, Bruckheimer 1981).

Walter and Hart (2009) studied student motivation in mathematics. They gathered results from students enrolled in honor calculus courses at a large private university. They looked specifically at six students enrolled in one of the first semester classes. Their findings lead them to believe that students will remain unmotivated until either a mathematical need or a social-personal need arises. Perhaps students are aware of some conflicts within their concept image of continuity, but until they see a need to reconcile the two they may simply continue on in the same manner.

2.3 Students' Understanding of Other Calculus Topics

Continuity is not the only area of difficulty for calculus students. Research has found that while students are successful with the rote processes of calculus, unfortunately, many students struggle with conceptual knowledge. For example students can easily find a derivative, however they do not really understand what a derivative is. They do not understand that a derivative is an instantaneous rate of change, the slope of the tangent line, etc. In an article by White and Mitchelmore (1996), “Conceptual Knowledge in Introductory Calculus,” the authors studied this phenomenon as it applies to related rates. They found that students struggled more with the problems in which they have to interpret the calculus from the problem. Students showed much more success on test items in which the notation was already laid out for them. In other words, they could do the rote calculations, but understanding how to set up the problem, how the variables related to one another, was the problem. It is important to look at these studies as it relates to students understanding. Could students struggle with the concept of continuous functions because they struggle with the concept of domain? We define continuous functions as “a function that is continuous at every point on its domain.” If students struggle with domain, then determining whether a function is continuous or not would be difficult. Again, students may know the definition, but they struggle with the application.

In a study by Bingolbali and Monaghan (2008), the authors looked at the effects of departmental affiliation on concept image. This study compares fifty Mathematical Engineering (ME) students and thirty-two Mathematics (M) students at a university in Turkey. Students are each enrolled in their respective departments’ calculus course. The calculus courses cover functions and derivative only. The study found that at the end of the course, ME students had a deeper concept image of derivative as it relates to rates of change and M students had a deeper concept image of derivative as it relates to tangent lines. A look at how the courses were taught shows that this difference in concept image could be related to how the courses were taught. The ME course focused on rates of change examples and the M course focused on tangent line examples. The M course takes a graphical approach using left and right hand limits and discusses the relationship between differentiability and continuity. Bingolbali and

Monaghan also raise the question if department affiliation alone can seep into one's concept image.

CHAPTER 3

METHODOLOGY

3.1 Setting

This study took place at a large (1,981 students) suburban high school in the Southwest. The school runs on a trimester schedule. This study took place during the second trimester. There are two groups involved in this study. The Precalculus students are enrolled in the second half of the Precalculus course, Pre AP/IB Precalculus B, during the second trimester. The AP Calculus students are enrolled in Calculus for all three trimesters. The second trimester is integral calculus. The students in AP Calculus have successfully completed the Pre AP/IB Precalculus course during the previous year.

3.2 Participants

The majority of the students in the Pre AP/IB Precalculus course are juniors; however, there are a few seniors. The majority of these students have completed Pre AP Algebra II and Pre AP Geometry. These students will go on to take AP Calculus, AP Statistics, or an IB math course their senior year. The AP Calculus students are all seniors. They have completed Pre AP Geometry, Pre AP Algebra II, and Pre AP/IB Precalculus. The AP Calculus students are preparing for the AP Calculus exam in May 2011 and will be attending university in the fall 2011. The students in both classes range from students with strong mathematical ability, back ground, and work ethic to those who were once strong mathematical students, but are struggling in their current course. The textbook used in the Pre AP/IB Precalculus course is Precalculus with Limits, Larson and Hostetler Texas edition (2007). The textbook used in the calculus course is Calculus of a Single Variable, Larson, Hostetler and Edwards 4th edition (2006).

The researcher is currently teaching the Precalculus students; however, they were taught by one of the three different Precalculus teachers for the first half of the course when the unit on continuity is taught. The Calculus students had been taught continuity in Precalculus by one of the three different Precalculus teachers and again as the first unit in AP Calculus by the researcher. See table for demographics.

Table 3.1 Study Demographics

Group	**# of Students**	**Male**	**Female**	**White**	**African American**	**Hispanic**	**Asian-Pacific Islander**
Calculus	16	15	1	14	1	1	2
Pre-Calculus	7	4	3	5		1	1

3.3 Instructional Method

3.3.1 Precalculus Instruction

The unit on continuity is covered along with increasing/decreasing functions, even/odd functions and symmetric functions. Since limits have not been covered, students are taught continuity from a graphical perspective. Prior to introducing continuity, students review finding domain and range of a given function. When introducing the topic, I provide several graphs of functions, some of which are continuous and some are not. We then discuss what they think of when they hear the term “continuous function.” We investigate continuity on the real numbers and on open (and closed) intervals. Then we discuss how to determine whether a function is a *continuous function* and how the domain plays a role in that determination. For each of the examples, we have discussed, we go back and find the intervals on which the function is continuous. Then we find the domain of the function. We look at the definition of a continuous function as a function that is continuous at every point in its domain. Then we look at the intervals on which the function is continuous and compare that to the intervals they found for the domain. If they are the same, then we can say the function is a continuous function. They are then given a worksheet for homework answering similar questions. If a graph is not provided, then students are expected to graph the function before answering the questions. Having them graph these functions also serves as a review of graphing functions (without a calculator).

3.3.2 Calculus Instruction

Continuity is reviewed in Calculus in the first unit. We begin the Calculus course by looking at limits. Students work with limits in Precalculus, so this is mostly review. Then we review the concept of continuity as it was taught in Precalculus. Then through a series of investigations and questioning we bring the concept of limits into continuity as seen below.

A function f is continuous at a point c if all the following conditions hold:

1.) $f(c)$ exists for some c in the domain of f.

2.) $\lim_{x\to c} f(x)$ exists for some c.

3.) $\lim_{x\to c} f(x) = f(c)$ for some c in the domain of f.

Then we recall that "a function is a continuous function if it is continuous at every point in its domain." Thus, again we review domain. I often ask the students the following three questions, "Where are the points of discontinuity on the graph located?" "Are they in the domain?" and "So, is it a continuous function?" I want to try and highlight the relationship between domain and continuity as much as possible.

After we look at how limits and continuity relate, we study the limit definition of the derivative. Once we have a good understanding of this relationship, we discuss the relationship between the continuity of a function and its differentiability. The two are related through the concept of limits. We discuss the fact that differentiability of a function implies continuity, however, the implication in the other direction does not hold. I use the absolute value function to as an example. We also look at the three graphical conditions that destroy differentiability: corner, cusp, or vertical tangent.

3.4 Questionnaire

Both groups were given a continuity questionnaire during the second trimester of their respective course. The first three questions and the last question were the same on both questionnaires. However, since the Calculus students have a more in depth study on continuity, their questionnaire had questions that were not on the Precalculus questionnaire. I called the Calculus questionnaire "Continuity Questionnaire C" and the Precalculus questionnaire "Continuity Questionnaire P."

1. Explain what it means for a function to be continuous on $(-\infty,\infty)$.

Figure 3.1 Continuity Questionnaire C and P Question 1.

I chose this question because I wanted to see what the students' general understanding of continuity was. I wanted to get some insight as to their concept image without a function, a graph, or an interval to distract them.

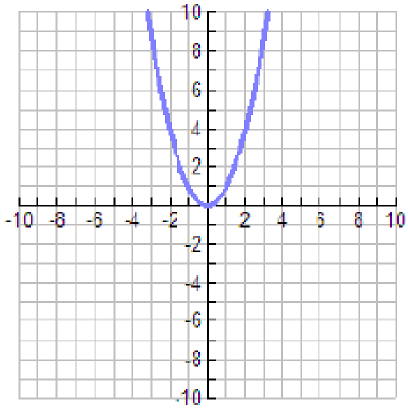
2. Explain what it means for a function to be continuous on the interval $[0,1]$.

Figure 3.2 Continuity Questionnaire C and P Question 2.

I chose this question to see if students understood the difference between continuity on **R** and continuity on an interval. Again I did not want a graph or a function to distract them.

3. Determine which of the following are continuous functions. Explain your reasoning.

a.) $f(x) = x^2$



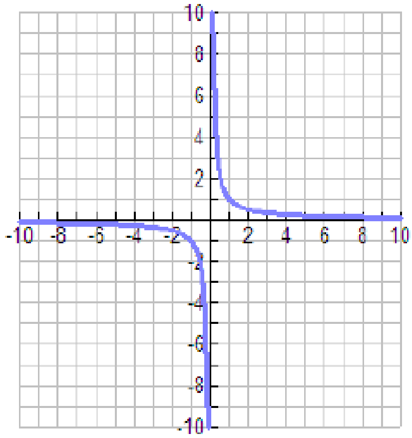
Is the function f continuous?

Explain your answer.

Figure 3.3 Continuity Questionnaire C and P Question 3a

I asked the students to explain their reasoning, so I could gain more insight into their concept definition of continuity. I was hoping the Precalculus students would refer to the relationship between the intervals on which the function was continuous and the domain, and that the calculus students would begin to use the limit definition of continuity at a point along with the domain. I assumed that most students (if not all) would say that $f(x)$ is continuous. I chose this function to see how they explained their answer. It was somewhat of a baseline question.

b.) $g(x) = \frac{1}{x}$



Is the function g continuous?

Explain your answer.

Figure 3.4 Continuity Questionnaire C and P Question 3b.

I assumed more students would have trouble with this function. I figured that several would have trouble with the asymptote at $x = 0$. I was hoping students would realize that since $x = 0$ is not in the domain, $g(x)$ is a continuous function.

c.) $h(x) = \begin{cases} 1, & x \le 1 \\ x, & x > 1 \end{cases}$

Is the function h continuous?

Explain your answer.

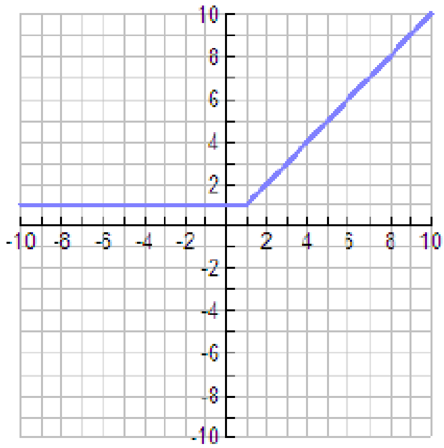


Figure 3.5 Continuity Questionnaire C and P Question 3c.

I was interested to see how students would handle this function. $h(x)$ is defined as a piecewise function, however, the graph does not contain any holes, jumps, or asymptotes and $\lim_{x\to 1^-} h(x) = \lim_{x\to 1^+} h(x) = 1 = h(1).$

d.) $j(x) = \begin{cases} -2, & x < 2 \\ 4, & x \geq 2 \end{cases}$

Is the function j continuous?

Explain your answer.

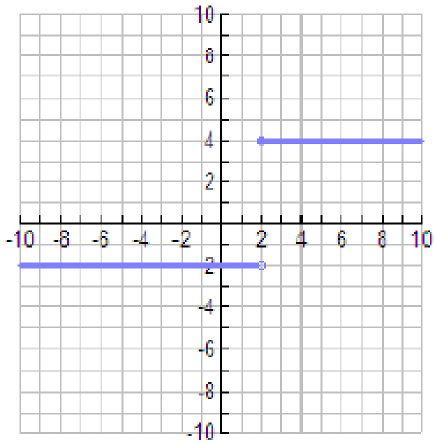


Figure 3.6 Continuity Questionnaire C and P Question 3d.

I assumed that most students would say $j(x)$ is discontinuous. I wanted to see what their reasoning was; would they mention the fact that $x = 2$ is in the domain? I was hoping my Calculus students would mention the fact that the limit at x approaches 2 does not exist.

e.) $k(x) = \begin{cases} x+3, & x < -1 \\ -x+1, & x > -1 \end{cases}$

Is the function k continuous?

Explain your answer.

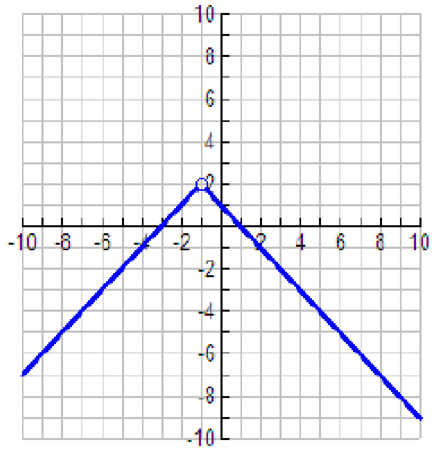


Figure 3.7 Continuity Questionnaire C and P Question 3e.

Again, I thought this function might give me more insight as to what holes my students have in their concept image of continuity. $k(x)$ is a piece-wise defined function; it has a hole at $x = -1$. However, I was again looking for the relationship between a single point of discontinuity and the domain.

f.) $m(x) = \begin{cases} x^3, & x \neq 2 \\ 6, & x = 2 \end{cases}$

Is the function m continuous?

Explain your answer.

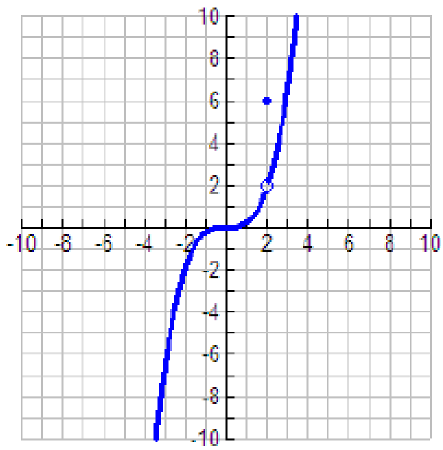


Figure 3.8 Continuity Questionnaire C and P Question 3f.

I chose this function to see if students had any conflicting thoughts as to the hole at $(2, 2)$ and the point at $(2, 6)$. I wanted to see how the students' explanations differed between 3e and 3d. The subtle differences in the functions and their graphs and how the students explain their belief of the functions' continuity will give me more insight into their concept image of continuity than if I chose only one or two of the functions to put on the questionnaire.

The following is question 9 from the Continuity Questionnaire C, however it was question 4 from the Continuity Questionnaire P.

9. *Explain the meaning* of the following statements:

a. A function g has domain $[3, \infty)$.

b. A function h has domain $x \geq 2$.

c. A function f has domain $\{x \in \mathbb{R} : x \geq 5\}$.

Figure 3.9 Continuity Questionnaire C Question 9 & Continuity Questionnaire P Question 4.

I chose this question to see what makes up the students' concept image of domain. In order to determine whether or not a given function is a continuous function, students must be able to find the domain of the function. I also wanted to see how well students understood the different mathematical notations.

As mentioned earlier, the Calculus questionnaire had several questions that were not on the Pre-calculus questionnaire. I had assumed that the Calculus students would have a better concept image of continuity and I wanted to see how deep their understanding was. I also wanted to see if they would use the limit definition instead of referring back to how we cover continuity in Precalculus. The following questions appeared on the Continuity Questionnaire C only.

For the next two items, circle the appropriate response.

4. Which of the following functions are continuous for all real numbers?

I. $f(x) = x^{3/4}$

II. $g(x) = e^{2x} + 1$

III. $h(x) = \csc x$

a) None b) I only c) II only d) I and II e) I and III

Figure 3.10 Continuity Questionnaire C Question 4

I chose this question to see if the students understood the difference between a continuous function (continuous on its domain) and a function that is continuous for all real numbers (continuous on $(-\infty, \infty)$). I did not put this question on the Precalculus questionnaire because we had not yet covered graphing trigonometric functions.

5. Let f be the function given by $f(x) = \frac{(x-2)(x^2-9)}{x^2-a}$. For which positive values of a is f continuous for all real numbers x?

a) None b) 1 only c) 2 only d) 9 only e) 2 and 9 only

Figure 3.11 Continuity Questionnaire C Question 5

This question was chosen to see if students would recognize that $x^2 - a$ would result in either an asymptote or a hole for any value of a. Therefore no value of a would make $f(x)$ continuous for all real numbers. It also allowed me to see if the students were able to differentiable between a continuous function (a function that is continuous at every point in its domain) and a function that is continuous for all real numbers.

6. Define "continuity of a function at a point." (Hint: you could begin with "a function f is continuous at a point $x = c$ if …")

Figure 3.12 Continuity Questionnaire C Question 6

This question was chosen so I could see if students understood what it meant for a function to be continuous at a single point. I was hoping the students would use the limit definition of continuity. If they struggle with this concept then of course they will struggle with the concept of "continuity over an interval" or "continuous function."

7. Let a be a real number and let $f(x) = \begin{cases} x^2, & x \le 2 \\ a(2-x), & x > 2 \end{cases}$. Determine, if possible, a value of a that makes f a continuous function. Justify your answer.

Figure 3.13 Continuity Questionnaire C Question 7

This question was chosen for the Calculus questionnaire because I wanted to see if students considered the value the limit of $f(x)$ as x approaches 2 from the right and from the left. I wanted to see if they could apply the definition they wrote in question 6.

8. Determine whether each of the following functions is continuous or discontinuous. If the function is discontinuous determine whether the discontinuity is removable or non-removable. Explain your reasoning.

a) $f(x) = \begin{cases} \sin x, & x \le 0 \\ x+1, & x > 0 \end{cases}$

b) $g(x) = \dfrac{x(x^2-4)}{x-2}$

Figure 3.14 Continuity Questionnaire C Question 8

This question was chosen to gain insight on how the graph of a function affected the students' concept image of continuity. I was curious to see if the students would graph the function as they did in Precalculus to determine the continuity or if they would use the limit definition and algebraically determine the function's continuity. I also wanted to see what level of understanding students had on removable and non-removable discontinuity.

3.5 Data Analysis

I used the general rubric shown below when scoring each of the questionnaires which I adapted from Chan (2011). Each questionnaire was graded with this rubric. I then compared the Precalculus students as a whole with the Calculus students as a whole to see if there was any overall growth in the understanding on continuous functions from one year to the next. I then looked at each individual student's results to see if he/she was having the same flaw in his/her concept image. For example, if a student said that a function $f(x) = x^2$ is continuous because "every *x* value has a *y* value." Then I looked to see how that student reconciled that with the continuity of the function $(x) = \begin{cases} -2, & x < 2 \\ 4, & x \geq 2 \end{cases}$. I looked for the similar patterns with students who mention left and right hand limits but did not check to see if the value x was approaching was in the domain. I also looked for patterns in student responses regarding differentiability. I was looking to see if those students understood that the implication "differentiability implies continuity" does not hold in the other direction.

Conceptual Understanding Rubric

5	Student's explanation shows complete understanding of appropriate mathematical concepts and principles relative to the problem; use of mathematical terminology and notations are appropriate and complete
4	Student's explanation shows nearly complete understanding of appropriate mathematical concepts and principles relative to the problem; use of mathematical terminology and notations are appropriate and nearly complete.
3	Student's explanation shows partially complete understanding of mathematical concepts and principles relative to the problem; use of mathematical terminology and notations is partially displayed.
2	Student's explanation show limited or underdeveloped understanding of appropriate mathematical concepts and principles relative to the problem; inappropriate, sketchy, or nonexistent use of mathematical terminology and notations.
1	Student's explanations shows completely inappropriate or no understanding of mathematical concepts and principles relative to the problem; inappropriate or nonexistent use of mathematical terminology and notations.

Figure 3.15 Conceptual Understanding Rubric

Accuracy Rubric

3	If the problem requires a YES/NO response, then the student's given answer is correct. If the problem requires a numerical response, then the student's provided value is correct. If the problem requires a multiple choice answer, then the student's given answer is correct.
2	If the problem requires a numerical response; then the student's provided value is incorrect because of a minor calculation error. If the problem requires a multi-part multiple choice answer, then the student's provided response is partly correct.
1	If the problem requires a YES/NO response, then the student's given answer is incorrect. If the problem requires a numerical response, then the student's provided value is incorrect. If the problem requires a multiple choice answer, then the student's provided answer is incorrect.

Figure 3.16 Accuracy Rubric

Based upon this rubric, each individual question was examined to determine what type of evidence would be required to obtain scores in each of the rubric categories. The following figures contain examples for each question for which the rubric was used.

Question 1

5	The function is continuous at every point on $(-\infty, \infty)$. Mentions all of the following: a.) f(c) is defined, so that c is in $(-\infty, \infty)$. b.) $\lim_{x \to c} f(x)$ exists for c in $(-\infty, \infty)$. c.) $\lim_{x \to c} f(x) = f(c)$
4	Can be drawn without lifting pencil on $(-\infty, \infty)$.
3	Mentions b from above only
2	There are no holes, asymptotes, corners or cusps Differentiable on $(-\infty, \infty)$
1	Every x value has a y value There are no corners or cups

Figure 3.17 Question 1 Rubric

Question 2

5	The function is continuous at every point on [0, 1]. Mentions all of the following: a.) f(c) is defined, so that c is in [0, 1]. b.) $\lim_{x \to c} f(x)$ exists for c in [0, 1]. c.) $\lim_{x \to c} f(x) = f(c)$
4	Can be drawn without lifting pencil on [0, 1].
3	Mentions b from above only.
2	There are no holes, asymptotes, jumps, corners or cusps Differentiable on [0, 1]
1	Every x value has a y value on [0, 1]. There are no corners or cusps.

Figure 3.18 Question 2 Rubric

Question 3a

5	Polynomials (or a quadratic in this case) are continuous everywhere Mentions all of the following: a.) $f(x_0)$ is defined, so that x_0 is in the domain of f b.) $\lim_{x \to x_0} f(x)$ exists for x in the domain of f c.) $\lim_{x \to x_0} f(x) = f(x_0)$
4	Can be drawn without lifting pencil Uses a term other than polynomial, such as "parabola"
3	In regards to a, b, and c above, the student either mentions b or c alone, or any combination of 2 of a, b, or c. Or Restricts a, b, c to only a particular point "Differentiable at all points"
2	Uses factual criteria about the function to show that the function is not continuous. Examples of answers receiving a 2: -Every x value has a y value -includes all real numbers
1	States that the function is continuous without any further justification Uses nonfactual criteria or simply states that the function is not continuous

Figure 3.19 Question 3a. Rubric

Question 3b

5	Mentions all of the following: a.) $f(x_0)$ is defined, so that x_0 is in the domain of f b.) $\lim_{x \to x_0} f(x)$ exists for x in the domain of f c.) $\lim_{x \to x_0} f(x) = f(x_0)$ "Undefined at x=0, but domain is $(-\infty,0) \cup (0,\infty)$ so it's continuous"-The only discontinuity of the function is outside of the given domain
4	Summarizes the above a, b, and c with minor error.
3	In regards to a, b, c from above, the student either mentions b or c alone, or any combination of 2 of a, b, or c Or Restricts a, b, c to only a particular point Mentions only that the function specifically does not include 0 "Two parts continuous on own domain" "differentiable" "domain is never 0"
2	Uses vague reference to left and right hand limits Uses factual criteria about the function to show that the function is not continuous
1	"Continuous on domain" States that the function is continuous without any further justification Uses nonfactual criteria or simply states that the function is not continuous

Figure 3.20 Question 3b. Rubric

Question 3c

5	Joining of two polynomials; polynomials are continuous everywhere Joining of two continuous functions Mentions all of the following: a.) $f(x_0)$ is defined, so that x_0 is in the domain of f b.) $\lim_{x \to x_0} f(x)$ exists for x in the domain of f c.) $\lim_{x \to x_0} f(x) = f(x_0)$
4	Can be drawn without lifting pencil No gaps, breaks, holes, discontinuities Both [functions] are continuous, but doesn't say why
3	In regards to a, b, c from above, the student either mentions b or c alone, or any combination of 2 of a, b, or c Or Restricts a, b, c from above to only a particular point L/R hand limits as x approaches 0 are equal/the same (uses reference to left and right hand limits but only in regards to the "problem point")
2	Uses vague reference to left and right hand limits "domain at all real #s" "same value when function changes" "h(x) exist for all x" Uses factual criteria about the function to show that the function is not continuous
1	Mentions the presence of a sudden change or a sharp corner Continuous in (-∞,∞) States that the function is continuous on the interval without any further justification Uses nonfactual criteria or simply states that the function is not continuous

Figure 3.21 Question 3c. Rubric

Question 3d

5	From the following: a.) $f(x_0)$ is defined, so that x_0 is in the domain of f b.) $\lim_{x \to x_0} f(x)$ exists for x in the domain of f c.) $\lim_{x \to x_0} f(x) = f(x_0)$ shows that (b) does not hold because left and right hand limits of j(x) as x approaches 2 are not the same, thus the limit does not exist there There is a break at and is in the domain. Thus the function is discontinuous.
4	"Not continuous at x=2" Mentions a discontinuity at x=2
3	Mentions a discontinuity on the interval, not specifying where Mentions that L/R hand limits are not equal (not specifying where)
2	Example: "limit do not equal each other" (no L/R specification) Uses factual criteria about the function to show that the function is continuous
1	Uses the wrong point as the "problem point" States that the function is not continuous on the interval without any further justification Uses nonfactual criteria or simply states that the function is continuous everywhere/on the interval

Figure 3.22 Question 3d. Rubric

Question 3e

5	Combination of polynomial functions, which are continuous everywhere Mentions all of the following: a.) $f(x_0)$ is defined, so that x_0 is in the domain of f b.) $\lim_{x \to x_0} f(x)$ exists for x in the domain of f c.) $\lim_{x \to x_0} f(x) = f(x_0)$
4	No breaks/holes/jumps/discontinuities "hole, break at -1 but not in domain" "differentiable at all points"
3	In regards to a, b, c from above, the student either mentions b or c alone, or any combination of 2 of a, b, or c Or Restricts a, b, c from above to only a particular point Or L/R hand limits as x approaches -1 are equal/the same (uses reference to left and right hand limits but only in regards to the "problem point")
2	"Values are equal when the function changes" "Value of f(-1) can be found using limits" Uses factual criteria about the function to show that the function is not continuous
1	States that the function is continuous on the interval without any further justification Uses nonfactual criteria or simply states that the function is not continuous

Figure 3.23 Question 3e. Rubric

Question 3f

5	Mentions all of the following: a.) $f(x_0)$ is defined, so that x_0 is in the domain of f b.) $\lim_{x \to x_0} f(x)$ exists for x in the domain of f c.) $\lim_{x \to x_0} f(x) = f(x_0)$ shows that c does not hold There is a jump at and is in the domain. Thus the function is discontinuous.
4	Mentions the above with minor notation error
3	Breaks, holes, jumps, discontinuities at x = 2
2	Uses the correct point as well as an incorrect point as the "problem points" Uses factual criteria about the function to show that the function is continuous
1	Uses the wrong point as the "problem point" States that the function is not continuous on the interval without any further justification Uses nonfactual criteria or simply states that the function is continuous everywhere/on the interval

Figure 3.24 Question 3f. Rubric

Question 9a

5	The function g is defined for all values of x in [3, inf)
4	g exists on [3, inf)
3	The x-values go from [3, inf)
2	
1	The function g is continuous on [3, inf)

Figure 3.25 Question Questionnaire P 4a. & Continuity Questionnaire C 9a. Rubric

Question 9b

5	The function g is defined for all values of x in [2, inf)
4	g exists on [2, inf)
3	The x-values go from [2, inf)
2	
1	The function g is continuous on [2, inf)

Figure 3.26 Question Questionnaire P 4b. & Continuity Questionnaire C 9b. Rubric

Question 9c

5	The function g is defined for all values of x in [5, inf)
4	g exists on [5, inf)
3	The x-values go from [5, inf)
2	
1	The function g is continuous on [5, inf)

Figure 3.27 Question Questionnaire P 4c. & Continuity Questionnaire C 9c. Rubric

Question 4

3	Chose answer choice c
2	Chose answer choice d
1	Chose answer choice a, b, or e

Figure 3.28 Question Questionnaire C 4 Rubric

Question 5

3	Chose answer choice a
2	
1	Chose answer choice b, c, d, or e

Figure 3.29 Question 5 Rubric

Question 6

5	A function f is continuous at a point x = c if each of the following hold: a.) f(c) is defined, so that c is in the domain. b.) $\lim_{x \to c} f(x)$ exists for c in the domain. c.) $\lim_{x \to c} f(x) = f(c)$
4	Mentions all of the above with a minor notation error
3	Mentions b or c only or any 2 of a, b, or c
2	
1	Closed circle Has a corresponding y value

Figure 3.30 Question 6 Rubric

Question 7

5	Not possible because $\lim_{x \to 2^-} f(x) = 4 \neq \lim_{x \to 2^+} f(x) = 0$ for all values of a.
4	Mentions above with minor notation error
3	Mentions only part of above
2	
1	Found a value of a Not possible with no explanation

Figure 3.31 Question 7 Rubric

Question 8a

5	From the following: a.) $f(x_0)$ is defined, so that x_0 is in the domain of f b.) $\lim_{x \to x_0} f(x)$ exists for x in the domain of f c.) $\lim_{x \to x_0} f(x) = f(x_0)$ shows that (b) does not hold because left and right hand limits of j(x) as x approaches 0 are not the same, thus the limit does not exist there
4	Cannot be drawn without lifting pencil Mentions a discontinuity at x=0
3	Mentions a discontinuity on the interval States that the y-values jump
2	Uses vague reference to L/R hand limits States that x=0 is undefined Uses factual criteria about the function to show that the function is continuous
1	Uses the wrong point as the "problem point" States that the function is not continuous on the interval without any further justification Uses nonfactual criteria or simply states that the function is continuous everywhere/on the interval

Figure 3.32 Question 8a. Rubric

Question 8b

5	Continuous because each of the following hold: a.) $f(x_0)$ is defined, so that x_0 is in the domain of f b.) $\lim_{x \to x_0} f(x)$ exists for x in the domain of f c.) $\lim_{x \to x_0} f(x) = f(x_0)$ There is a hole at x = 2, but 2 is not in the domain, thus the function is continuous
4	States above with minor notation error.
3	Uses 2 of a, b, or c
2	Uses factual information to say that the function is continuous
1	Uses factual information to say that the function is discontinuous States the function is continuous with no explanation

Figure 3.33 Question 8b. Rubric

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CHAPTER 4

RESULTS

There were sixteen subjects in the calculus group and seven subjects in the precalculus group. First we will look at each groups' response to each question on their respective questionnaire. Then I have provided some individual student responses to particular questions for later discussion.

4.1 Group Results

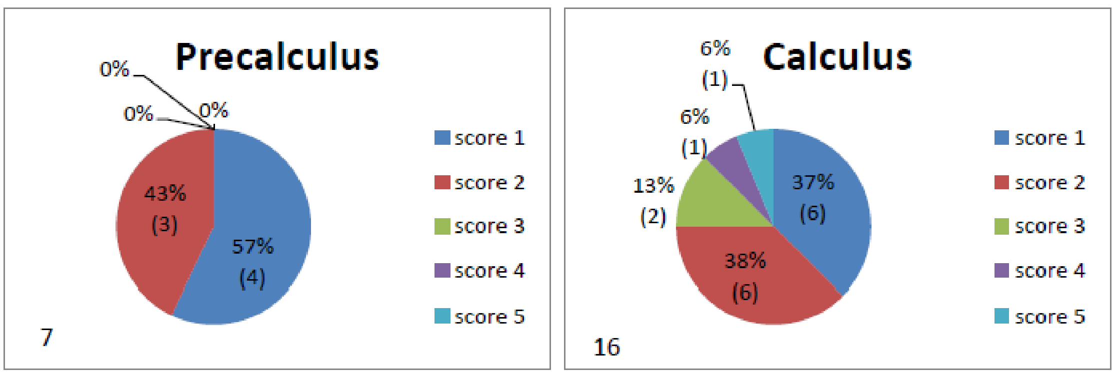


Figure 4.1 Question 1 Conceptual Understanding Results

For question 1, no precalculus student was able to fully explain what it means for a function to be "continuous on $(-\infty, \infty)$. Only one calculus student was able to fully explain it; receiving a score of 5. Three calculus students mentioned that there were no holes, or vertical asymptotes. Four precalculus students and three calculus students mentioned that the function is defined for every x. One calculus student wrote, "The function does not have any holes, jumps, cusps, or vertical asymptotes on $(-\infty, \infty)$." Three other calculus students mentioned cusps or corners and two other calculus students said the function is differentiable on $(-\infty, \infty)$.

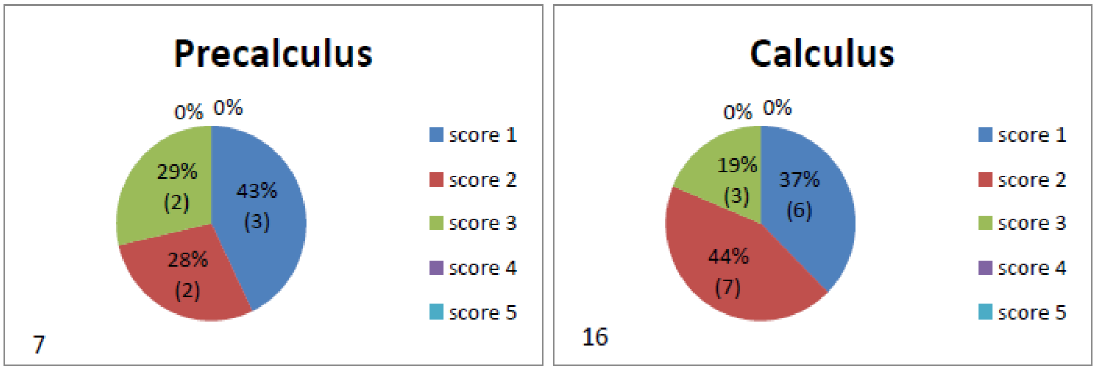


Figure 4.2 Question 2 Conceptual Understanding Results

In question 2, only one precalculus student mentioned domain in his/her explanation, while six calculus students gave an answer similar to “the function exists for all number on the interval [0, 1].” The six calculus students that mentioned corners, cusps, and differentiability in their response to question one did so again in question 2.

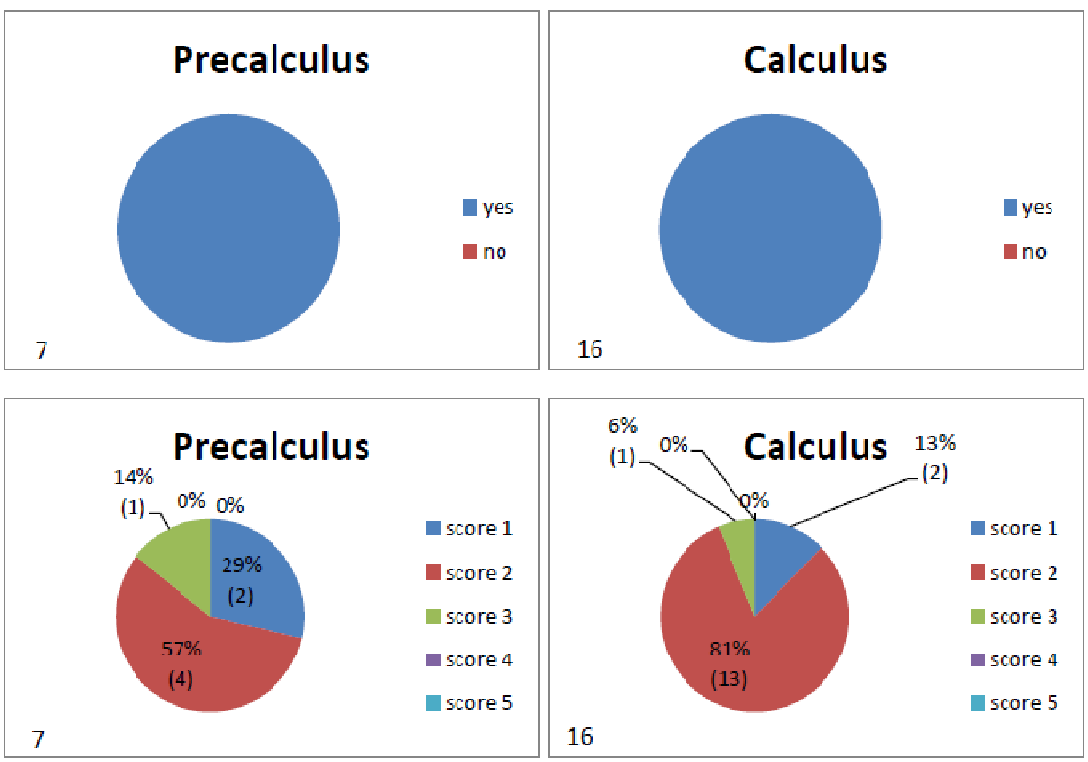


Figure 4.3 Question 3a. Answers and Conceptual Understanding Results

As you can see 100% of both groups agreed that the given function is a continuous function. However, there were some discrepancies in their explanations. Four of the calculus students wrote an explanation similar to "all the *x*-values are defined." Four precalculus students and two calculus students mention at one or two of the following, "There are no holes, jumps, or vertical asymptotes." Three calculus students mention that there are no corners or cusps. One calculus students says, "There are no vertical asymptotes and the function is differentiable."

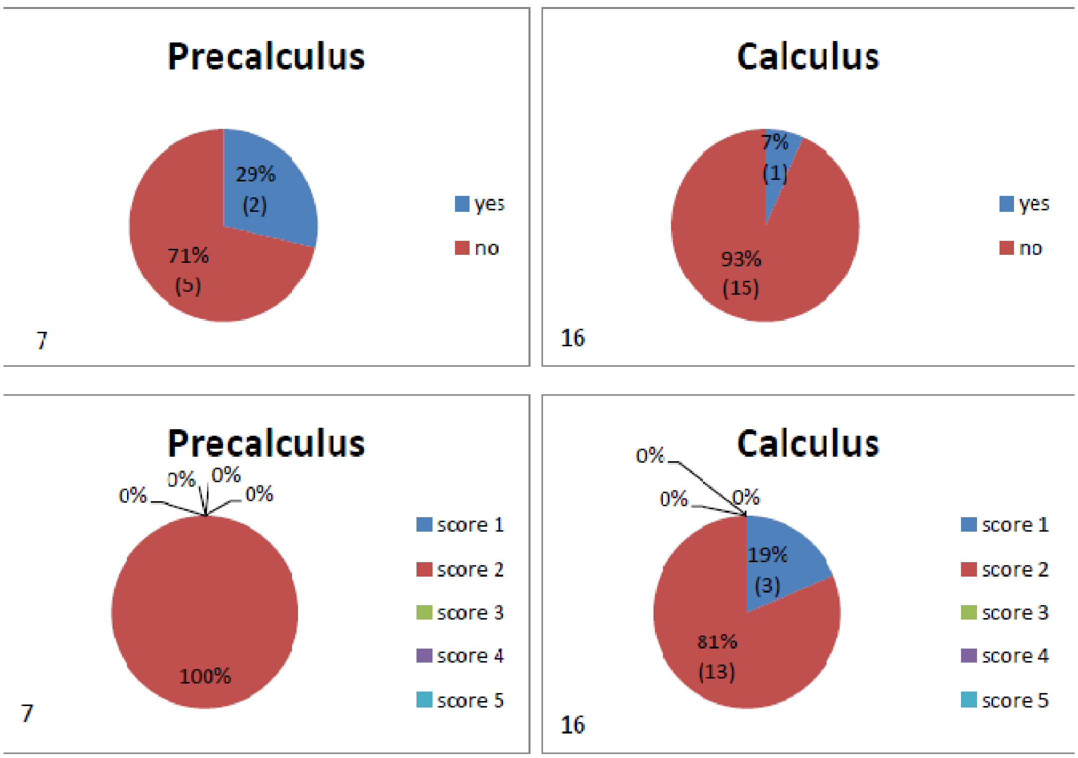


Figure 4.4 Question 3b. Answers and Conceptual Understanding Results

For question 3b, only two precalculus students and one calculus student correctly labeled the function $g(x) = \frac{1}{x}$ as continuous. However, none of them had correct explanations. The calculus student said the function is continuous because "no holes, asymptotes, cusps or corners." All of the other students said that the function was not a continuous function. Four precalculus students and six calculus students mentioned x being undefined at $x = 0$ in their explanation of why $g(x)$ is not continuous. One precalculus student and seven calculus students mentioned the fact that $g(x)$ has a vertical asymptote at $x = 0$ in their explanation.

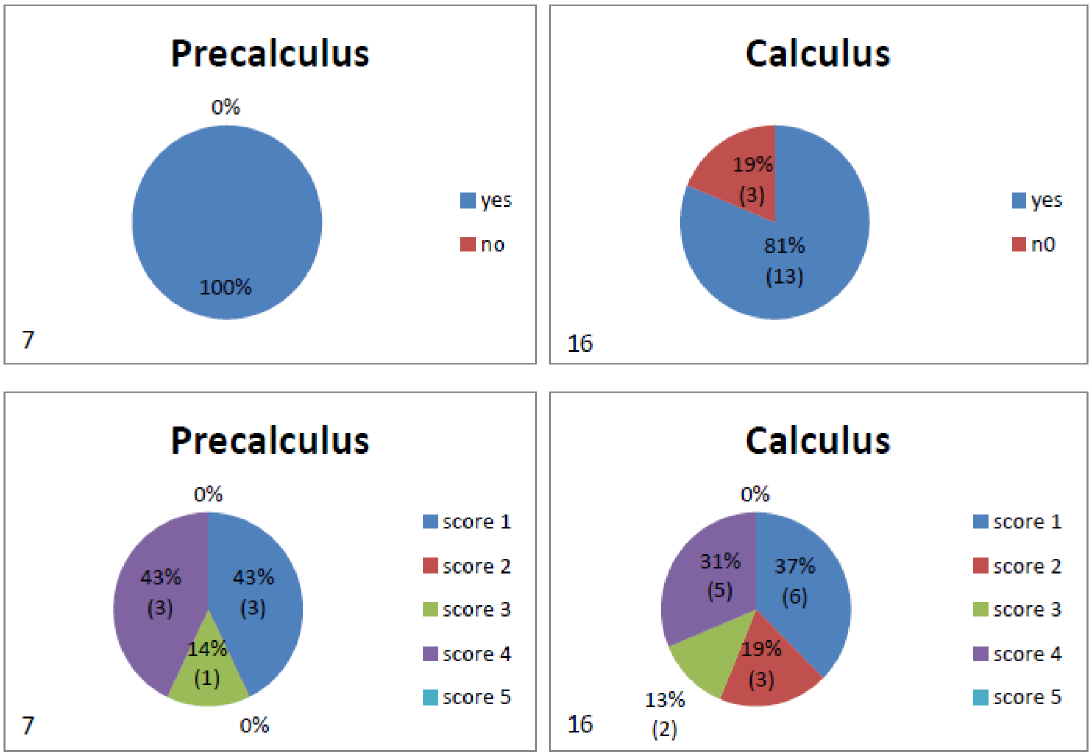


Figure 4.5 Question 3c. Answers and Conceptual Understanding Results

For question 3c, all of the precalculus students agree that $h(x)$ is a continuous function. Five of the seven precalculus students said that it was continuous because there are no holes or breaks in the graph. The two remaining precalculus students said that it was continuous because its domain is $(-\infty, \infty)$. Only two of the thirteen calculus students that said $h(x)$ is continuous mentioned limits in their explanation. However, each of them only had two of the following three necessary conditions listed in the rubric. Four of the thirteen calculus students justified their belief that $h(x)$ is continuous because the domain is $(-\infty, \infty)$. Three of the calculus students that believed $h(x)$ is continuous said so because there is a corner on the graph.

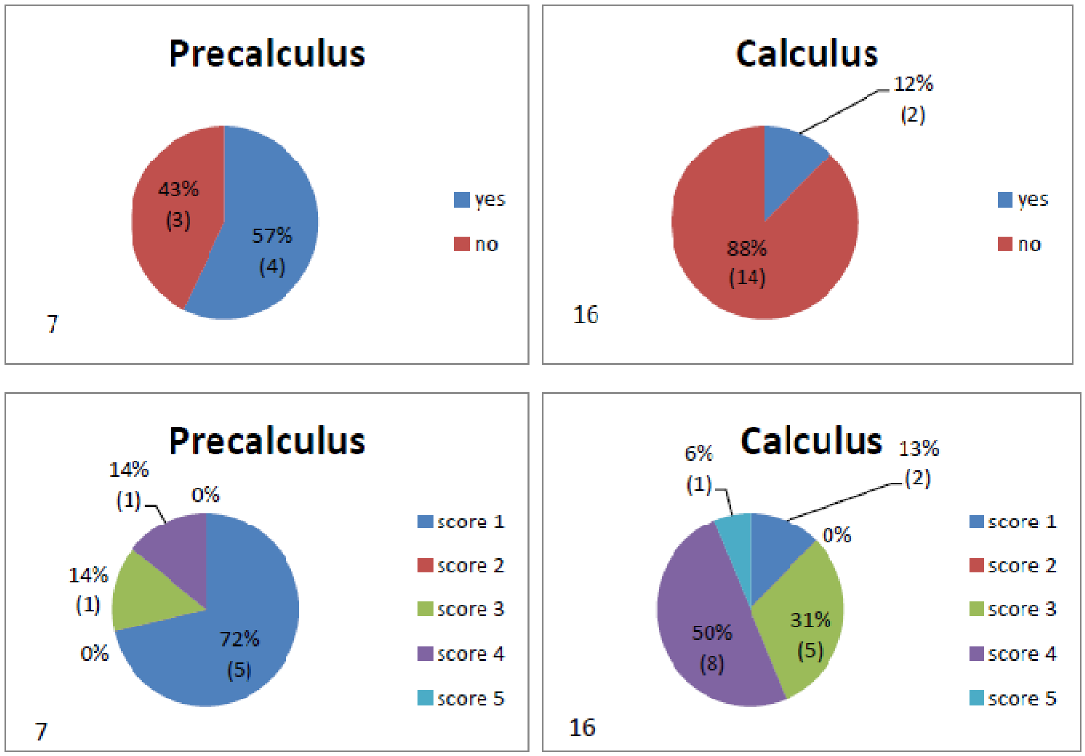


Figure 4.6 Question 3d. Answers and Conceptual Understanding Results

For question 3d, four of the seven precalculus students said that $j(x)$ is continuous. Each of them used the fact that $x = 2$ is in the domain of $j(x)$ in their justification. The three precalculus students that said $j(x)$ is not continuous used the fact that there is a jump, or an "open hole" in the graph as their justification. However, they did not mention that the jump or "open hole" occurs at $x = 2$ and that $x = 2$ is in the domain. Only two of the sixteen calculus students said that $j(x)$ is continuous. One of the two students did not write an explanation, the other student said the function was continuous because "there is a y-value for every x-value." Of the fifteen calculus students who believed $j(x)$ is continuous, seven of them said it was continuous because there was a jump at $x = 2,$ but say did not say anything about whether or not $x = 2$ is in the domain. Five of the fifteen students mentioned a jump but did not mention where the jump occurred.

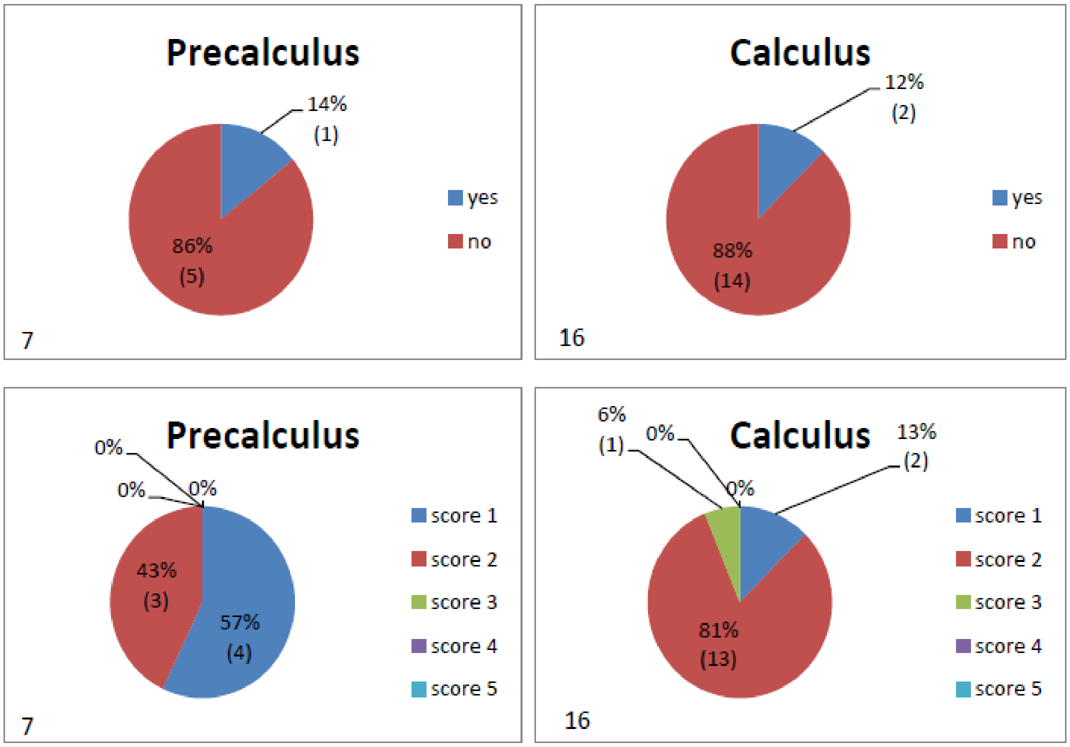


Figure 4.7 Question 3e. Answers and Conceptual Understanding Results

For question 3e, six of the seven precalculus students said $k(x)$ is not continuous. Each of them mentioned either a hole or discontinuity at $x = -1$ in their reasoning. The one precalculus student that said the function was continuous wrote, "there isn't a hole in the graph, it just stops at $x = -1.$" Fourteen of the sixteen calculus students believed $k(x)$ is not continuous. Thirteen of those students mentioned either a hole at $x = -1$ or "no point" at $x = -1$. Of the two calculus students that believed $k(x)$ to be continuous, one of them said is was continuous because there was a removable continuity. The remaining calculus student said $k(x)$ is continuous because $\lim_{x\to 1} k(x) = 2$.

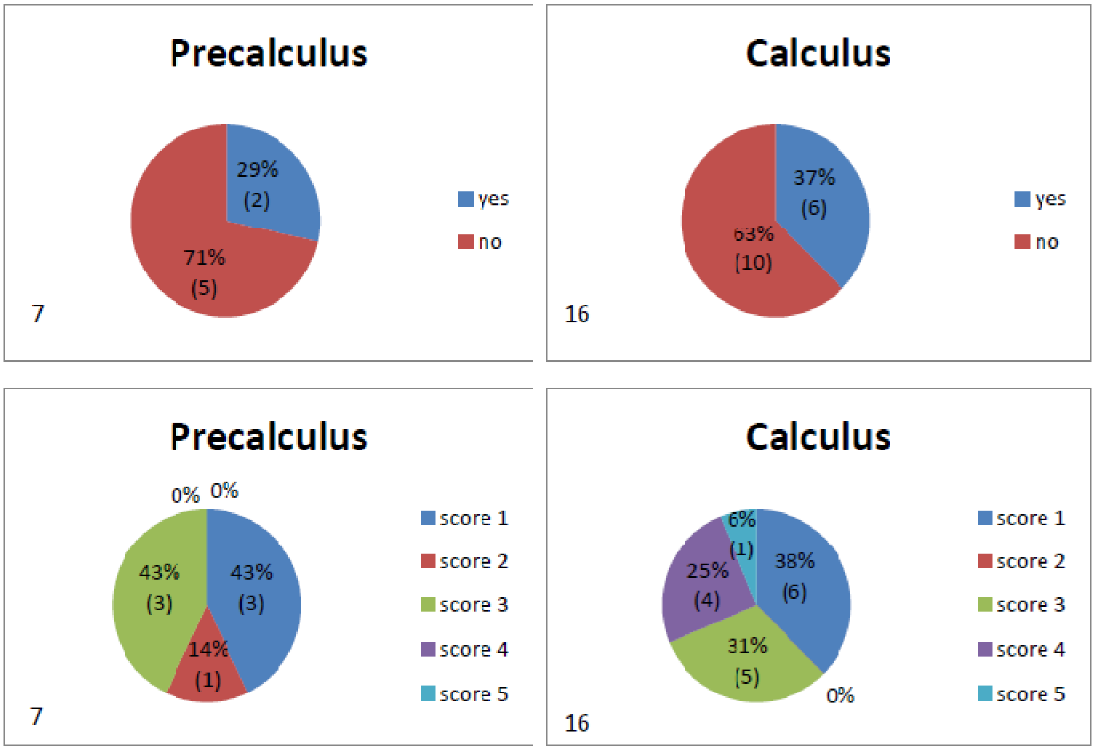


Figure 4.8 Question 3f. Answers and Conceptual Understanding Results

For question 3f, five of the precalculus students believe $m(x)$is not a continuous function. Four of those five use the fact that there is a hole or "break" in the graph as their justification. However, none of those students mention that the problem occurs at $x = 2$ or that $x = 2$ is in the domain. One of those five students says $m(x)$ is not continuous because "2 is not in the domain." Nine of the calculus students believe that $m(x)$ is not continuous because there is a hole in the graph. Only one calculus student correctly stated that $m(x)$ is not continuous because $\lim_{x\to 2} m(x) \neq m(2)$. Of the six calculus students that believe $m(x)$ to be continuous none of them have the same justification. One student says $m(x)$ is continuous because "every y value has an x value." Another student says $m(x)$ is continuous at $x = 2$ because "the point is defined and has a removable discontinuity." Yet another student says $m(x)$ is continuous because "it has a removable continuity." Another calculus student believes $m(x)$ is continuous because $\lim_{x\to 2} m(x) = 2$.

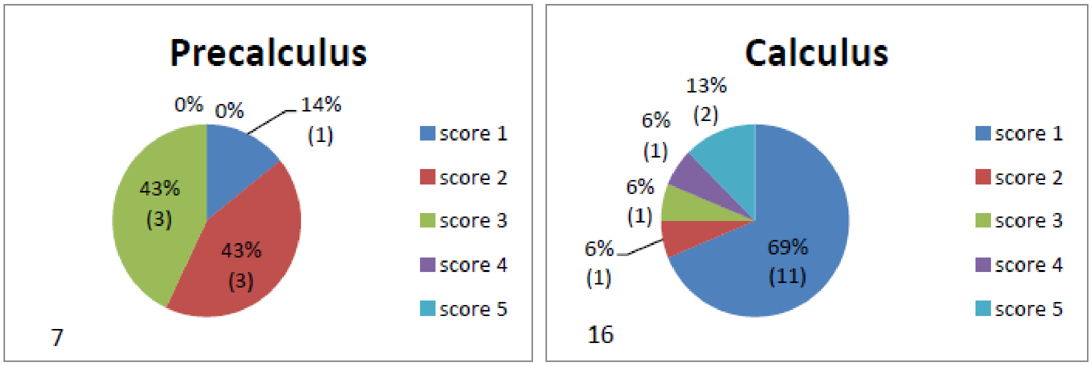


Figure 4.9 Question Questionnaire P 4a. & Questionnaire C 9a. Conceptual Understanding Results

The precalculus students had some variation of “the x-values start at 3 and go to infinity.” Six of the calculus students believe that g is continuous on $[3, \infty)$.

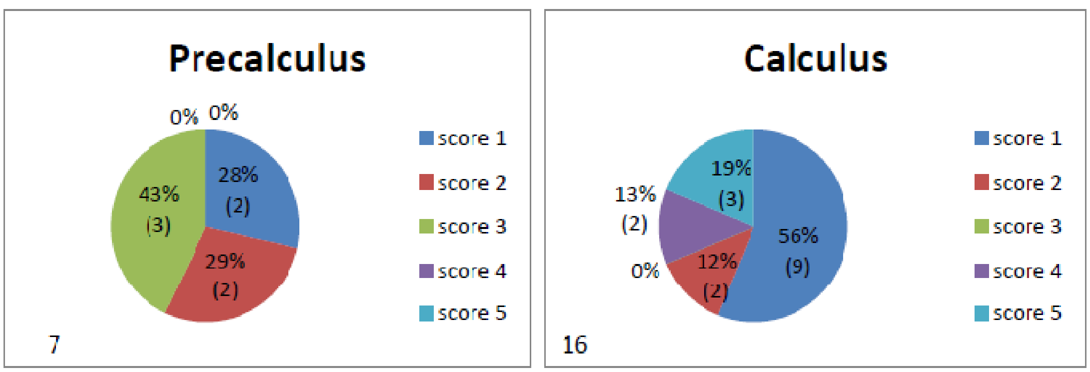


Figure 4.10 Question Questionnaire P 4b. & Questionnaire C 9b.Conceptual Understanding Results

For b. the precalculus students had some variation of “ it starts at 2 and goes to infinity.” Six of the calculus students believe that h is continuous when $x \geq 2$.

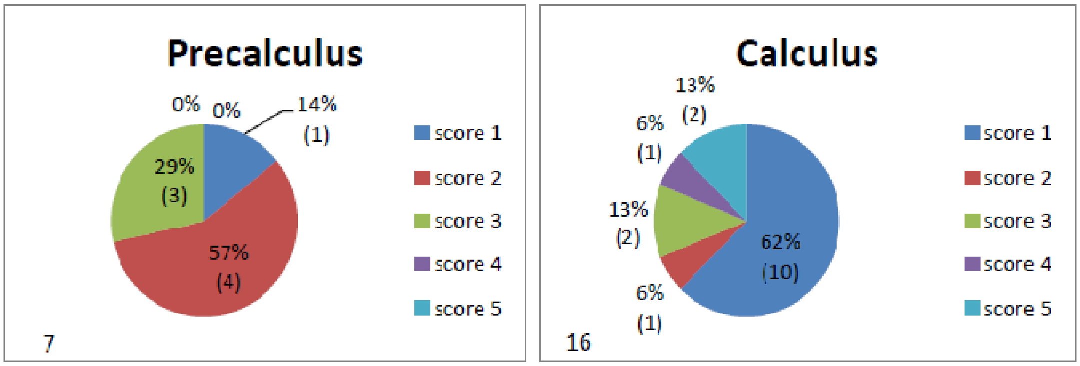


Figure 4.11 Question Questionnaire P 4c. & Questionnaire C 9c. Conceptual Understanding Results

For question 4c., five out of seven precalculus students added in the fact that" x is all real numbers" to their response. Only five of the calculus students mention continuity in their response to question 9c. Three of the calculus students make the same transition to x being a real number for *f*.

The remaining questions appeared on the Questionnaire C only.

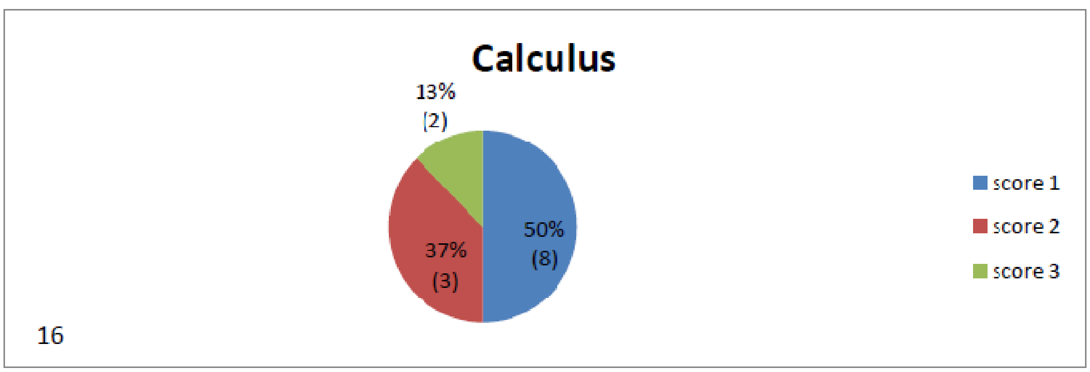


Figure 4.12 Question 4 Accuracy Results

No students showed work on this question.

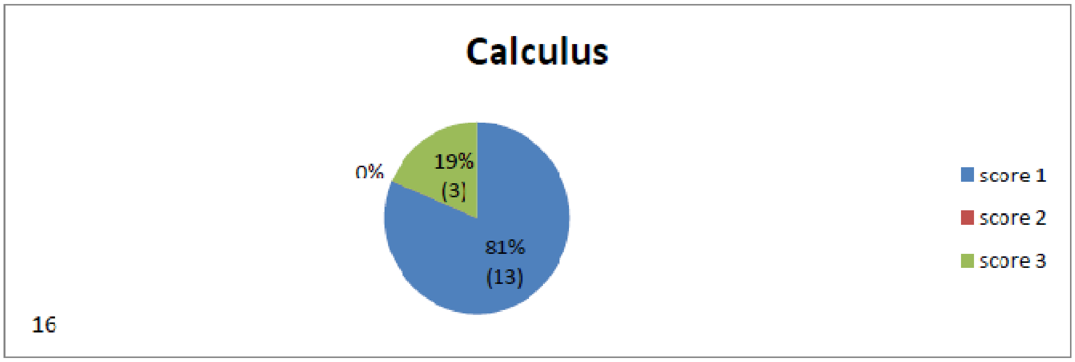


Figure 4.13 Question 5 Accuracy Results

Only one of the three students who answered question 5 correctly answered 4 correctly as well. Five of the calculus students chose d. Four students chose e.

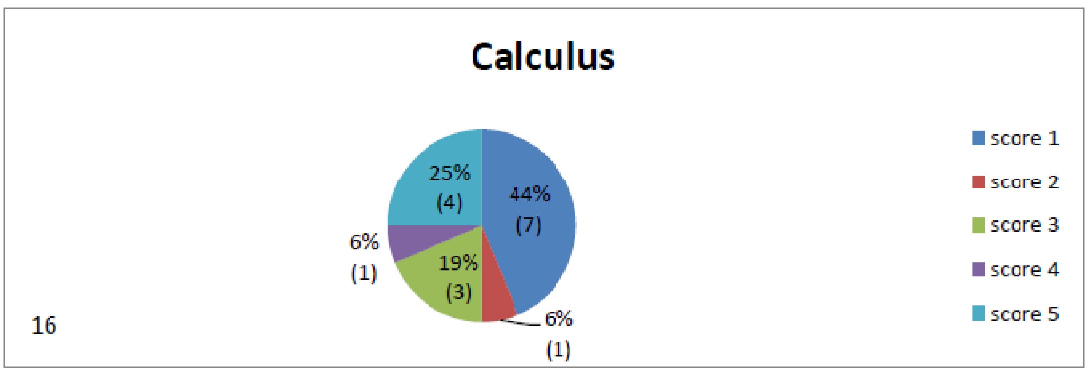


Figure 4.14 Question 6 Conceptual Understanding Results

For question 6, I only had three students answer this question fully: "A function *f* is continuous at a point $x = c$ if the $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$." I had two students only mention the left and right limit: "A function *f* is continuous at a point $x = c$ if the $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$." Two other students said "A function *f* is continuous at a point $x = c$ if the function is differentiable and defined at that point." Other answers included there not being a hole or jump, or that there is a *y*-value for every *x*-value.

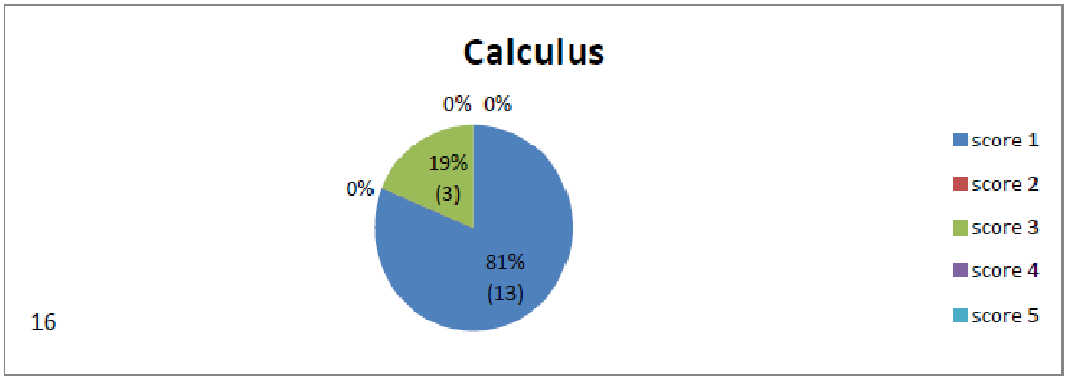


Figure 4.15 Question 7 Conceptual Understanding Results

Three of the eight students that said no value of *a* will make *f* a continuous function did not provide a justification. Three of the eight said that no such value of *a* exists because there is no value of *a* such that $\lim_{x\to 2^+} f(x) = 4 = \lim_{x\to 2^-} f(x)$. The remaining two mentioned a jump in the graph. The other eight students all found a value for *a.* One student who found *a* to be -4 wrote, "Since $f(2) = 4$ and when $a = -4$ the function $a(2-x) = 4$." There is no work to show why he/she thinks $-4(2-x) = 4$. However one other student found *a* to be -4 and three students said $a = 4$. None of these students showed their work in clear steps nor did they explain their answer.

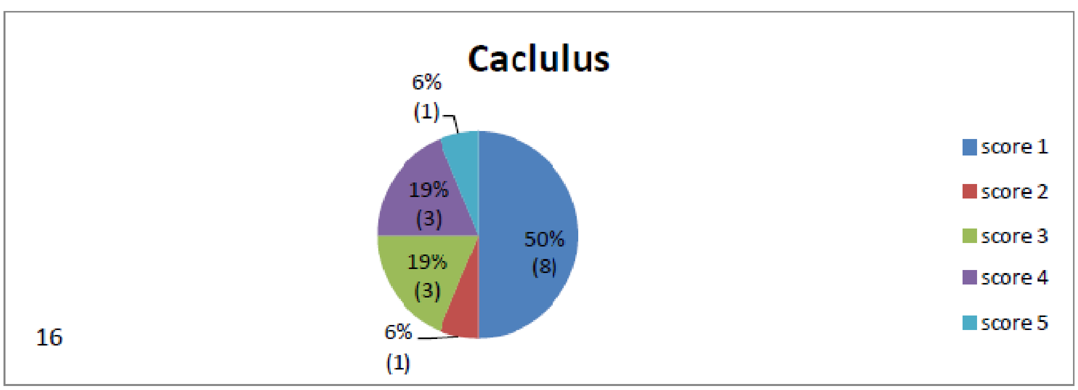


Figure 4.16 Question 8a.Conceptual Understanding Results

For question a., four calculus students said *f* is a continuous function. Each of them mention there being no break in the graph. Eight of the students said *f* is not continuous and of those eight six students said it had a non removable discontinuity. Three of those six students who believe *f* has a non removable discontinuity used the graph to justify their answer; stating a break or jump in the graph. The other three students used the fact that the limit as x approaches 0 from the left of *f(x)* does not equal the limit as *x* approaches 0 from the right.

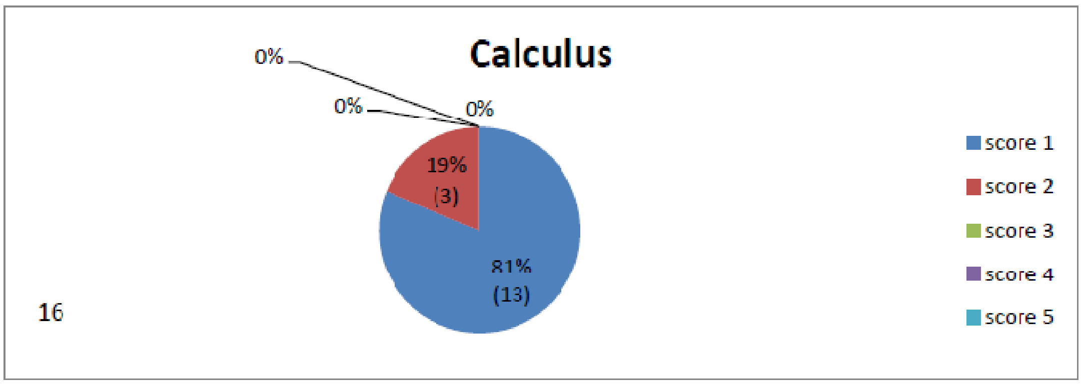


Figure 4.17 Question 8b. Conceptual Understanding Results

For question b., five students believe $g(x)$ is discontinuous. Each of their explanations is graphical. For example, "there is a hole." and "there is an $(x-2)$ on the top and bottom." Six students said the function was continuous. However, their justifications were not completely correct. One student said it was continuous because, "every *x* value has a *y* value." Other students mentioned there being no holes, vertical asymptotes, etc.

4.2 Selected Students' Results

Selected student results have been chosen to highlight interesting trends in results. These individual student responses will be given in this section. Since the students are to remain anonymous, I will refer to the individual calculus students as Student C1, C2, etc. and the Precalculus students as Student P1, P2, etc.

The first calculus student whose results will be reported is Student C1. Student C1's response to question one was "the function exists for all numbers." His responses to question 9a and 9b were "g exists on $[3, \infty)$" and "h exists on $[2, \infty)$" respectively. His response to question six was "A function is continuous at a point $x = c$ if $f(c)$ is a real number." His responses to question three are given in the figure below.

Student C1:

3a	Continuous	No explanation
3b	Not Continuous	Vertical asymptote at $x = 0$.
3c	Continuous	No explanation
3d	Continuous	No explanation
3e	Not Continuous	x does not equal 1
3f	Continuous	No explanation

Figure 4.18 Student C1's Question 3 Responses

The second calculus student whose responses will be discussed is Student C2. His response to question one was, "For a function to be continuous on $(-\infty, \infty)$ all real *x* values of $f(x)$ must be defined." His responses to question 9a, and 9b were "The function g is defined on all *x*-values on the interval $[3, \infty)$" and "the function *h* is defined on all *x*-values greater than or equal to 2" respectively. His response to question six was "a function f is continuous at a point c is $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$." His responses to question three are given in the figure below.

Student C2

3a	Continuous	All values of x are defined
3b	Not Continuous	$g(0)$ is undefined
3c	Continuous	All values of x are defined
3d	Not Continuous	There is a discontinuity at $x = 2$ in the form of a jump
3e	Not Continuous	There is a discontinuity at $x = -1$ in the form of a hole
3f	Not Continuous	There is a discontinuity at $x = 2$

Figure 4.19 Student C2's Question 3 Responses

Student C3's response to question one was "the function has 'y' value for every 'x' value on $(-\infty, \infty)$." His response to question 9c was "the graph of f has both an 'x' and a 'y' value on every 'x' value greater than or equal to x = 5." His response to question six was "A function *f* is continuous at a point x = c if there is a 'y' value at $x = c$." His responses to question three are given in the figure below.

Student C3

3a	Continuous	It has a 'y' value for every 'x' value
3b	Not Continuous	There is no 'y' value on $x = 0$
3c	Continuous	There is a 'y' value for every 'x' value
3d	Continuous	There is a 'y' value for every 'x' value
3e	Not Continuous	There is no 'y' value at $x = -1$
3f	Continuous	There is a 'y' value for every 'x' value.

Figure 4.20 Student C3's Question 3 Responses

Student C's response to question one was, "It does not have a corner, cusp or jump on $(-\infty, \infty)$." His response to question six was "*f* is continuous at point $x = c$if there is no hole, corner, asymptote, or cusp." His responses to question three are given in the table below.

Student C4

3a	Continuous	No holes, asymptotes
3b	Continuous	No holes, asymptotes, cusps or corners
3c	Not Continuous	There is a corner
3d	Not Continuous	It jumps
3e	Not Continuous	There is a corner
3f	Not Continuous	There is a hole

Figure 4.21 Student C4's Question 3 Responses

Student C5's response to question one was, "The function will continue forever and ever and when it does it will always be continuous with no breaks or cusps in the graph." His definition for continuity at a point was "a function f is continuous at a point $x = c$ if both sides of the point are continuous." His responses to question three are given in the figure below.

Student C5

3a	Continuous	$f(x)$ is continuous because there are no cusps or breaks
3b	Not Continuous	There is a break in the graph of $g(x)$ at $x = 0$ which is an undefined line.
3c	Continuous	The function $h(x)$ has no cusps, holes, or breaks in the graph.
3d	Not Continuous	There is a break in the graph of $j(x)$ at $x = 2$.
3e	Not Continuous	There is a hole in the graph of $k(x)$ at $(-1, 2)$
3f	Not Continuous	There is a hole in the graph at$(2, 2)$ and there is a removable point of continuity at $(2, 6)$.

Figure 4.22 Student C5's Question 3 Responses

Student C6's explanation of a function continuous on $(-\infty, \infty)$ was, "The function is differentiable from (-∞,∞) and the graph has no cusps, holes or other aspects that destroy differentiability." His response to question six was, "A function f is continuous at a point $x = c$ if the function is differentiable the point is defined." His responses to question three are given in the figure below.

Student C6

3a	Continuous	There are no holes, cusps, etc. The graph is never undefined on $(-\infty, \infty)$
3b	Not Continuous	The function is undefined at $x = 0$ leaving a hole breaking the continuity.
3c	Continuous	Neither of the equations are undefined on their given interval and they meet at the same point (1, 1).
3d	Not Continuous	The functions are not connected at $x = 2$.
3e	Not Continuous	Although both lines are continuous on their own and approach the point (1, -2) neither include the point making a hole.
3a	Continuous	There are no holes, cusps, etc. The graph is never undefined on $(-\infty, \infty)$

Figure 4.23 Student C6's Question 3 Responses

Student C8's response to question one was, "Every points between (-∞,∞) is continuous. It is differentiable on (-∞,∞)." His response to question six was "A function f is continuous at a point $x = c$ if $\lim_{x \to c} f(x) = f(c)$." His responses to question three are given in the figure below.

Student C8

3a	Continuous	There is no undefined and the domain is {x: x is R}
3b	Not Continuous	It is not continuous when $x = 0$
3c	Continuous	$\lim_{x \to 1} h(x) = 1$ and domain $\{x: x \in \boldsymbol{R}\}$
3d	Not Continuous	$\lim_{x \to 2} j(x)$ is undefined
3e	Continuous	$\lim_{x \to -1} k(x) = 2$
3f	Continuous	$\lim_{x \to 2} m(x) = 2$

Figure 4.24 Student C8's Question 3 Responses

The final calculus student whose responses have been select is Student C9. His response to question one was, "For every value of *x* there is a *y* solution, also $\lim_{x \to c^-} = \lim x \to c + = fc$. " His response to question six was, "A function is continuous at a point $x=c$ if $\lim_{x \to c^-} = \lim_{x \to c^+} = f(c)$." His responses to question three are given in the figure below.

Student C9

3a	Continuous	There are no vertical asymptotes and the function is differentiable
3b	Not Continuous	Because there is a vertical asymptote at $x = 0$
3c	Continuous	No vertical asymptotes
3d	Not Continuous	Because as you approach $x = 2$ from the right the corresponding 'y' value is 4, but from the left the 'y; value is -2, so it is not continuous.
3e	Not Continuous	At point (-1, 2) the point is undefined so it is not continuous
3f	Continuous	Since $x \neq 2$ the function cannot be defined at that point but it also states $x = 2$ has a value of 6 so every point along the function is continuous when $x \neq 2$.

Figure 4.25 Student C9's Question 3 Responses

The first Precalculus student whose responses will be reported is Student P1. His response to question one was, "It means it goes forever in a negative and positive direction." His responses to question 4a, 4b and 4c were "Its on the 3 for the x-axis and goes to infinity", "*h* starts at 2 and goes to infinity" and "*x* is all real #'s and x is on 5 and goes to infinity" respectively. His responses to question three are given in the chart below.

Student P1

3a	Continuous	It goes on forever in each direction
3b	Not Continuous	It never crosses 0
3c	Continuous	It is greater than 1 so it goes in a positive direction and also ≤ 1 so it will be at 1 and still go in an infinite direction in a negative direction.
3d	Continuous	Even though there is a break it's still $x \geq 2$ so it will go to infinity in both directions
3e	Not Continuous	It never goes on -1 it is either > or <
3f	Not Continuous	There is a hole in the graph.

Figure 4.26 Student P1's Question 3 Responses

The final student whose responses will be reported is Student P2. His response to question one was, "it means all values are used that are in the domain." His responses to question 4a, 4b and 4c were "the graph starts at x = 3 and goes on to infinity", "*f(h)*'s *x*-values are greater than or equal to two" and "the domain is all real numbers equal to or greater than five" respectively. His responses to question three are given in the figure below.

Student P2

3a	Continuous	There is no break in the domain and values in the interval never stop.
3b	Not Continuous	The vertical asymptote at $x = 0$ is in the interval but not in the domain.
3c	Continuous	The interval is included in the domain
3d	Continuous	The values in the interval are included in the domain.
3e	Not continuous	$x = -1$ is in the interval but not in the domain
3f	Not Continuous	You cannot have a split in the graph and replace it with a higher point.

Figure 4.27 Student P2's Question 3 Responses

CHAPTER 5

DISCUSSION

After reading the Tall and Vinner (1981) article on concept image and concept definition and the D. Takaci, D. Pesic and J. Tatar (2006) article on the continuity of functions, I was left with somewhat of a "now what?" feeling. Clearly students have flawed concept images of continuity, but I didn't know why. How can we as teachers help bridge the gap between their concept image and the concept definition? Since I did not have access to all of the actual student responses in these studies, I couldn't see what was causing the students to struggle in their concept image.

The small sample size of this study is certainly an issue in deriving far reaching conclusions from the data collected. However, based upon this small sample several areas for further investigation arise. After analyzing my students' results, I was able to identify three major areas of conflict within the students' concept image. These areas are the relationship between domain and continuous functions, differentiability and continuity, and the limit definition of continuity at a point.

5.1 The Domain of a Function and the Continuity of a Function

The first area I would like to discuss is the relationship between domain and the continuity of a function. There were four precalculus students and three calculus students that used the concept of domain in there explanation of what it means for a function to be continuous on $(-\infty, \infty)$. However, either the concept was not used correctly or if it was used correctly, the student did not use it to provide a full explanation of continuity on$(-\infty, \infty)$. In order to gain more insight into their concept image, I looked at each of these students to see if this flaw in concept image was consistent across their questionnaire. The calculus students that I would like to discuss here are students C1, C2, and C3. Student C1's response to question one was "the function exists for all numbers." Assuming this student was referring to all real numbers, this is a true statement; a function must exist for all [real] numbers in order for it to be continuous on all real numbers, $(-\infty, \infty)$. However, simply having a domain of all real numbers is not enough for a function to be a *continuous* function. The function in question 3d. $j(x) = \begin{cases} -2, & x < 2 \\ 4, & x \geq 2 \end{cases}$ is an example of a function whose domain is $(-\infty, \infty)$ but, it is not a continuous function. For question two this Student C1 wrote that "the function exists for all numbers on the interval [0, 1]." When I first looked at this student's questionnaire, I did not think I was going to be able to gain much in sight into his/her concept image because he did not write an explanation on several of the questions. However, after looking closer I was surprised to see that there was an enormous amount of consistency in the responses. I believe the reason student C1 did not write an explanation for the functions that he believed to be continuous was because he had already done so in his answer to question one. Each of the functions that he said was continuous was defined for all real numbers; their domain was $(-\infty, \infty)$. The functions student C1 believed were not continuous were not defined for all real numbers, and he pointed that out in his explanation. This student's responses to question 9 were "*g* exists on [3,∞)" and "*h* exists on $[2, \infty)$." Student C1 defined continuity at a point by stating, "A function is continuous at a point $x = c$ if $f(c)$ is a real number." This student does seem to understand that there is a relationship between the domain of a function and its continuity. However, he does not seem to understand that there is a difference between the two.

The next calculus student, student C2, had a similar explanation to question one and two as student C1. He wrote, "For a function to be continuous on $(-\infty, \infty)$ all real *x* values of $f(x)$ must be defined." For question two he wrote, "For a function to be continuous on $[0, 1]$ all real *x* values on the interval [0, 1] must be defined." However, unlike student C1, student C2's responses were not consistent throughout the questionnaire. According to his answers to questions one and two, student C2 should have said the functions given in questions 3d and 3f were not continuous, he did not. Student C2's responses are also not consistent with his definition of continuity at a point. He defined continuity at a point by saying "A function is continuous at a point $x = c$ if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$." According to this definition, which does not include the important fact that these limits must equal $f(c)$, functions 3e and 3f would be continuous at $x = -1$ and $x = 2$ respectivley. I wish I could interview this student and ask him about this inconsistency. Like student C1, student C2 has a similar explanation in question 9 as his explanation in questions 1 and 2. This student also struggles with the relationship between domain and continuity. However, he does understand that they are different but I am still unsure as to how they are different in his concept image. Also, I am not sure what role his definition of continuity at a point plays into his concept image of a continuous function.

The third calculus student whose flaw in concept image of continuity seems to be tied to domain was student C3. His explanation of what it means for a function to be continuous on $(-\infty, \infty)$ was "the function has 'y' value for every 'x' value on $(-\infty, \infty)$." He has a similar response to question 2, "the function has a 'y' value for every 'x' value from $x = 0$ to $x = 1$ including $x = 0$ and $x = 1$." As students C1 and C2 did, student C3 is simply stating the definition of domain on $(-\infty, \infty)$ and [0, 1]. Like student C1, student C3's responses are consistent along question one, two and three. He defined continuity at a point by stating, "A function is continuous at $x = c$ if there is a 'y' value at $x = c$." Like student C1, student C3 does not understand the difference between the domain of a function and the continuity of the function. If I were not keeping the subjects anonymous in this study, I would like to interview students C1 and C3. My first question would be to ask them to explain the difference between a function that has a domain of all real numbers and a function that is continuous on all real numbers. I wonder if a question like this would force them to try and differentiate the two concepts and where that forced differentiation would take them.

Now I would like to look at the precalculus students who seemed to struggle with the relationship between domain and continuity. The first student that I will discuss is student P1. Student P1 responded to question 1 with "It means it goes forever in a negative and positive direction." When I first read this students response I wasn't sure if he was referring to the *x*-

values going on "forever" or the *y*-values. However, after looking over his responses to question 3, I feel confident he was referring to the *x*-values. Student P1 describes the meaning of domain of $[3, \infty)$ by saying, "it starts on the 3 for the *x* axis and goes to infinity." The interesting thing about this student is his response to question 3f. All of his responses are similar to those of students C1 and C3, indicating that he does not understand the difference between the domain of a function and its continuity except for question 3f. According to his other responses, he should have said the function given in 3f was continuous because it goes on forever in both a positive and negative direction. However, he said it was not continuous because there is a hole. However, he does not let the hole in the graphs of the function in 3d lead him to believe that the function is not continuous. Could that be because the "hole" (or filed in circle) is attached to a piece that goes on forever? This would be an interesting question to ask this student.

Now we will look at student P2's responses. His response to question one was "it means all values are used that are in the domain." His response to question two was just as confusing at first glance, "the graphs domain is used in the interval." As with the other students, his responses to the questions in three allow me to gain a better understanding of where this student's concept image is flawed. The student never mentioned what interval he is referring to. It could be the interval $(-\infty, \infty)$. I am not sure why the student believes the function in 3d is continuous because the values in the interval are included in the domain, but does not have the same response to the function in 3f. It could be because the "closed" circle is attached to the graph instead of a single point. Maybe the student is referring to intervals of defined points and he thinks the intervals on which the function in 3d is continuous is $(-\infty, 2) \cup [2, \infty)$. While this is incorrect, I often get responses like this on tests and quizzes in precalculus. Again, since the student did not make any mention of what the interval is I am simply trying to guess. This would have been an area of interest for me with this student if there had been an interview.

The remaining precalculus students had results similar to those of student P1 in that their responses were consistent throughout question 3. Also, their responses indicated that they did not have a strong distinction between continuity of a function and domain. I wonder if the "intuition" described by Dreyfus and Eisenberg plays a role in the student responses to question three. Based on how students intuitively think about the word "continuity" and how they use it in everyday language, they could intuitively think that a continuous function has "no breaks". If this is the case, then they would believe the functions in questions 3b, 3d, 3e, and 3f are not continuous functions regardless of their domain.

The concept of domain is in the very definition of a continuous function. "A function is continuous if it is continuous at every point in its domain." As, I mentioned in the methodology chapter, I try to tie the continuity of a function to its domain when teaching the concept in precalculus. However, students are not seeing a difference between the two. I am not sure if this is because they do not have a strong understanding of domain itself, or if in trying to highlight the relationship between the two I somehow blur the differences. I also wonder how the questionnaire itself affected the students' responses. For example, several of the students had a similar response to question 1 and question 4 (on Continuity Questionnaire P) and 9 (on Continuity Questionnaire C). I wonder how the responses would have differed if I had asked the students to define domain before asking them to define continuity on and interval. I also wonder if any responses would have been different if I would have asked students to find the domain for each function in question three.

<u>5.2 Differentiability implies Continuity</u>

The next area of interest was the relationship between continuity and differentiability. As I have stated before, in calculus we discuss the fact that differentiability implies continuity but continuity does *not* imply differentiability. I use the absolute value function as a typical example. In this same unit, a day or two later, we look at the graphical features of a function that destroy differentiability. These features are corners, cusps and vertical tangents. The results from this study make it apparent that this one directional implication causes conflict in students' concept image of continuity. There were six calculus students that mentioned some form of a corner, cusp or differentiability in their explanation of what it means for a function to be continuous on $(-\infty, \infty)$.

The first calculus student in this group that we will look at is student C4. In his response to question one, he wrote, "It does not have a corner, cusp or jump on $(-\infty, \infty)$." Notice there is no mention of vertical asymptotes. His response to question two was similar. Student C4 does not mention asymptotes in his explanation of question one; however he does in question 3a and 3b. It is also interesting that he says the function in 3b is continuous because there is no asymptote when there is a vertical asymptote at $x = 0$. Once could claim that maybe he meant that there is no vertical asymptote in the domain and thus the function is continuous. However, this claim is not consistent with his answer and response in question 3e. I believe that this student has connected the graphical features that destroy differentiability with those graphical features that destroy continuity. His response to question six gives further credence to this assumption. He defines continuity at a point by saying, "A function is continuous at a point $x = c$ if there is no hole, corner, asymptote or cusp."

The next calculus student that I would like to discuss is student C5. He answered question one by saying, "The function will continue forever and ever and when it does it will always be continuous with no breaks or cusps in the graph." His response to question two was similar, again mentioning "no break or cusp." To gain more insight as to what this student is referring to when he mentions a break or a cusp, I analyzed his responses to question 3. I have a feeling that student C5 does not know what a cusp is. This is the only term from the list of terms that we discuss in our section on conditions that destroy differentiability. I wish I had created a function for question three that had a cusp in the graph to gain more insight on this. One thing that is clear is that this student, like many others already mentioned, does understand the fact that if the hole or break occurs outside the function's domain then the function is a continuous function. This student also has a misconception that there is a "removable continuity." Since a removable discontinuity refers to those points of discontinuity that can be "fixed" by redefining one single point, I can understand why the term "removable continuity" would be used by this student. It could be another example of how intuition affects students' concept image. Student C5 defined continuity at a point by saying, "A function *f* is continuous at a point $x = c$ if both sides of the point are continuous." Another student had similar results to that of student C5, however, he believed the function in 3c was not continuous because "there is a cusp at $x = 1.$" Since there is actually a corner at $x = 1,$ I believe this student is also unsure of what a cusp is.

Students C6, C8 and C9 believe that in order for a function to be continuous on $(-\infty, \infty)$ the function must be differentiable on $(-\infty, \infty)$. While each of them are correct in saying the function in 3c is continuous, their answers to 3c are not consistent with their answers to question one. Since the graph of $h(x)$ has a corner at (1,1) it is not differentiable at (1, 1). Thus according to their answer in question one, each of them should have said $h(x)$ is not continuous because it is not differentiable. I am not sure if they made a connection between question one and question three. However, student C7 defined continuity at a point by saying, "A function *f* is continuous at a point $x = c$ if the function is differentiable and the point is defined." This leads me to believe that student C7 does not know how to graphically determine when a function is differentiable. Student C8 correctly defined continuity at a point using the limit conditions. Thus I believe he is lacking the fact that in order for a function to be a continuous function it must be continuous *at every point in its domain.* Again, the relationship of domain and continuity is apparent. I was confused when I first read student C9's response to question 3f. However, after reading his response to question six, I think I may be able to see where this answer is coming from. "A function is defined at a point $x = c$ if x is defined at value *c."* This response conflicts with his answer in 3d. Student C8 also struggles with the role of domain in his concept image of continuity.

5.3 The Limit Definition of Continuity at a Point

The final area of conflict that I noticed was with the limit definition of continuity. Only one student mentioned limits in his explanation for question one. Student C9 wrote, "For every value of *x* there is a *y* solution, also $\lim_{x \to c^-} = \lim_{x \to c^+} = f(c)$." While his notation is not correct, his explanation is correct. He also used this limit notation to define continuity at a point. Like many of the other students, student C9 is unclear of the fact that only points in the domain affect whether or not a function is a continuous function. Also, his reasoning for why the function given in 3a is continuous is incorrect. Based on this reasoning, the functions given in 3d and 3f would also be continuous. I was surprised to see that only students C8 and C10 used limits in their explanations in question three. Based on his response to question 3f, student C8 seemed to forget the third piece of the condition being $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$.

5.4 General Discussion

When I began this research, I was planning to compare the results between the precalculus student and the calculus students. I was hoping to see a growth in the responses from the calculus students. I believed the Calculus students would have a better understanding of continuity because they have covered the limit definition of continuity at a point. They have also studied the relationship between continuity and differentiability and applications of continuity. However, the results of this study indicate that AP Calculus students may not have a better understanding of continuity than Pre AP Precalculus students. The students in both groups seem to be struggling with the concept of the domain. The calculus students actually had more areas of conflict due to the new concepts introduced in the calculus course. The concept of differentiability and the limit definition of continuity at a point are not fully developed and cause gaps in the students' concept image of continuity.

In analyzing the students' responses, I cannot help but wonder how the questionnaire itself affected student responses. For example, I wonder if any of the responses to question three would have been different if I had asked the students to explain the meaning of domain first. I wonder if that would have triggered anything in their minds when they got to question three. I also wonder how the responses would have differed in question three if I had asked students to find the domain for each of the given functions.

Further insight into student thinking and understanding may have also surfaced in an interview setting. That is, after having students complete the questionnaire, a subset would be called in for interviews to explore their reasoning and ask follow up questions to probe their understanding. Being able to see the student responses, gives me more insight as to why or how the students' concept images of continuity are flawed. It also gives me the ability to address those specific areas. Clearly we need to do a better job of teaching continuity in precalculus. We need to clear up the lingering questions between domain and continuity. I wonder how many teachers are actually comfortable with this relationship. It would be interesting to study teachers' responses to this questionnaire. Once in calculus, I need to do a better job of seeing what the students' concept image of continuity from precalculus is before I tie it to limits and differentiation. After seeing these results, I know that I also need to do a better job of explaining and teaching the graphical features of a continuous function versus the graphical features of a differentiable function and tying these features to the unilateral implication: Differentiability implies continuity. As research has shown, students have a weak concept image of continuity, but now I have a better idea as to why [my students]. Thus we can begin to address those areas of conflict within the students' concept image.

I would like to see future study done on students' concept image of continuity that incorporates some of the problematic areas identified in this study. Students' concept image of continuity could be targeted by adapting questions from the questionnaires used in this study. For example, including a function to question three that is continuous on its domain, but has a cusp may give more insight as to whether students really believe a function that contains a cusp in its graph is not a continuous function. Participants could be required to determine the domain of each function provided in question three to help determine if students do not know what the domain of the given function is or if they simply do not realize that for the function to be a continuous function it must be continuous at every point in its domain. These are just a few ways in which my current findings can be used to construct an extended questionnaire or interview protocol which gives further insight into why students seem to struggle with the important concept of continuity.

APPENDIX A

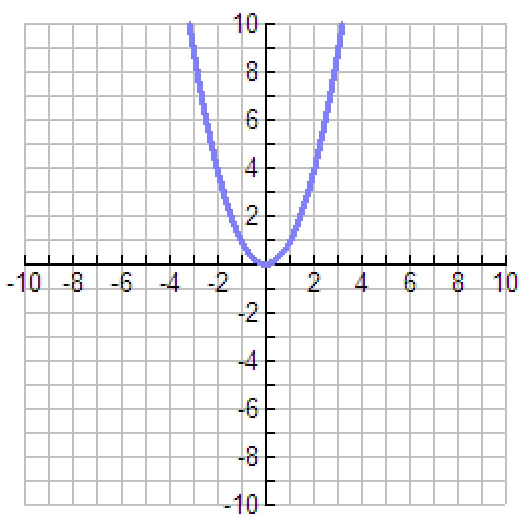
CONTINUITY QUESTIONNAIRE C

1. Explain what it means for a function to be continuous on $(-\infty, \infty)$.

2. Explain what it means for a function to be continuous on the interval $[0,1]$.

3. Determine which of the following are continuous functions. Explain your reasoning.

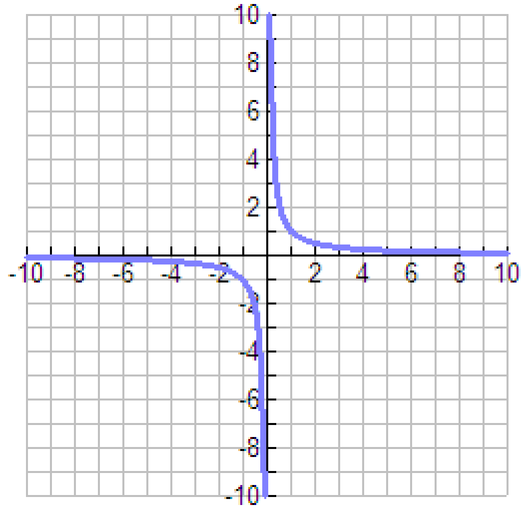
a.) $f(x) = x^2$



Is the function f continuous?

Explain your answer.

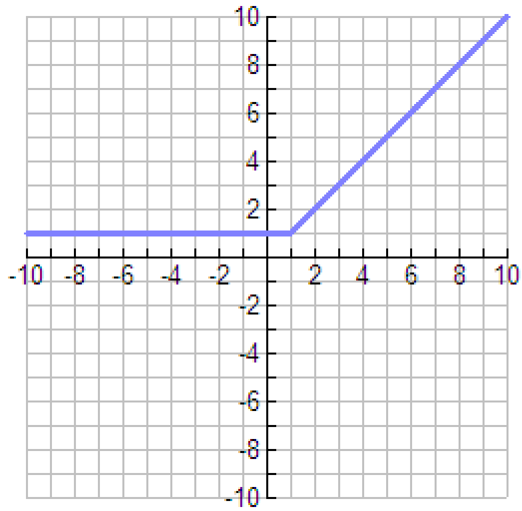
b.) $g(x) = \frac{1}{x}$



Is the function g continuous?

Explain your answer.

c.) $h(x) = \begin{cases} 1, & x \leq 1 \\ x, & x > 1 \end{cases}$



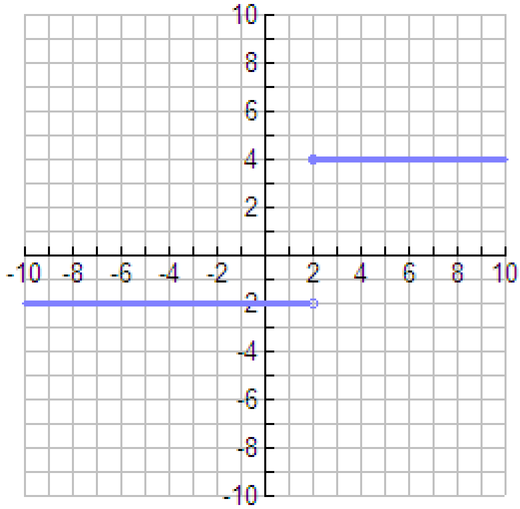
Is the function h continuous?

Explain your answer.

d.) $j(x) = \begin{cases} -2, & x < 2 \\ 4, & x \geq 2 \end{cases}$

Is the function *j* continuous?

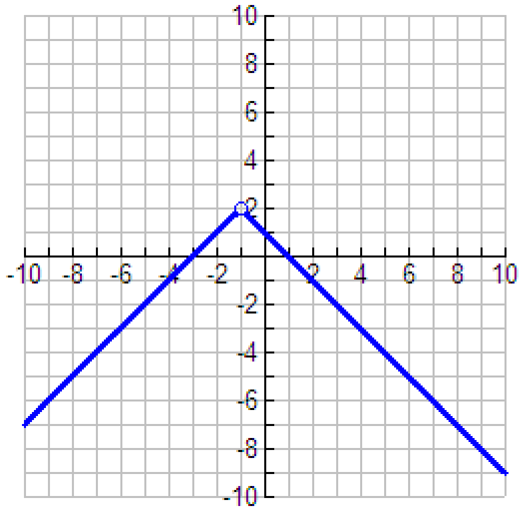
Explain your answer.



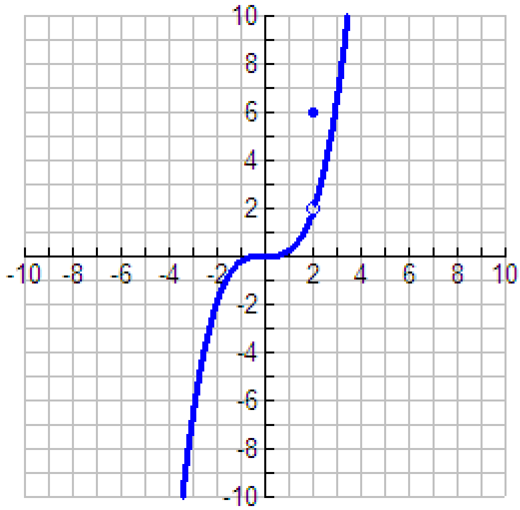
e.) $k(x) = \begin{cases} x + 3, & x < -1 \\ -x + 1, & x > -1 \end{cases}$

Is the function *k* continuous?

Explain your answer.



f.) $m(x) = \begin{cases} x^3, & x \neq 2 \\ 6, & x = 2 \end{cases}$



Is the function *m* continuous?

Explain your answer.

For the next two items, circle the appropriate response.

4. Which of the following functions are continuous for all real numbers?

   I. $f(x) = x^{3/4}$

   II. $g(x) = e^{2x} + 1$

   III. $h(x) = \csc x$

   a) None b) I only c) II only d) I and II e) I and III

5. Let f be the function given by $f(x) = \dfrac{(x-2)(x^2-9)}{x^2-a}$. For which positive values of a is f continuous for all real numbers x?

   a) None b) 1 only c) 2 only d) 9 only e) 2 and 9 only

6. Define “continuity of a function at a point.” (Hint: you could begin with “a function f is continuous at a point $x = c$ if …”)

7. Let a be a real number and let $f(x) = \begin{cases} x^2, & x \le 2 \\ a(2-x), & x > 2 \end{cases}$. Determine, if possible, a value of a that makes f a continuous function. Justify your answer.

8. Determine whether each of the following functions is continuous or discontinuous. If the function is discontinuous determine whether the discontinuity is removable or non-removable. Explain your reasoning.

a) $f(x) = \begin{cases} \sin x, & x \le 0 \\ x+1, & x > 0 \end{cases}$

b) $g(x) = \dfrac{x\left(x^2-4\right)}{x-2}$

9. *Explain the meaning* of the following statements:

   a. A function *g* has domain $[3, \infty)$.

   b. A function *h* has domain $x \ge 2$.

   c. A function *f* has domain $\left\{x \in \mathbb{R} : x \ge 5\right\}$.

APPENDIX B

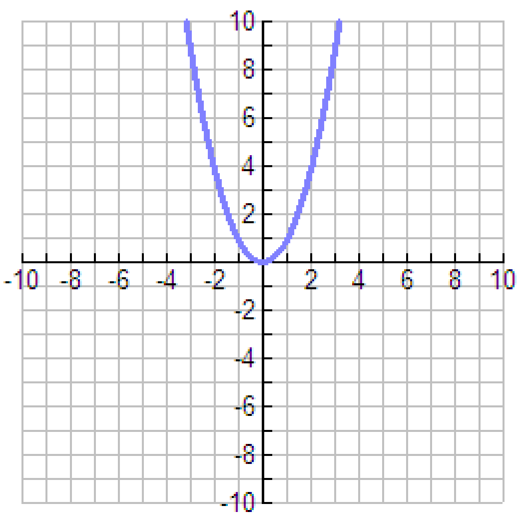
CONTINUITY QUESTIONNAIRE P

1. Explain what it means for a function to be continuous on $(-\infty, \infty)$.

2. Explain what it means for a function to be continuous on the interval $[0,1]$.

3. Determine which of the following are continuous functions. Explain your reasoning.

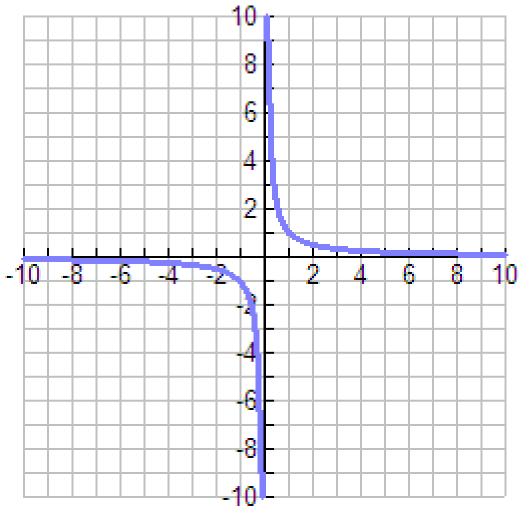
a.) $f(x) = x^2$



Is the function *f* continuous?

Explain your answer.

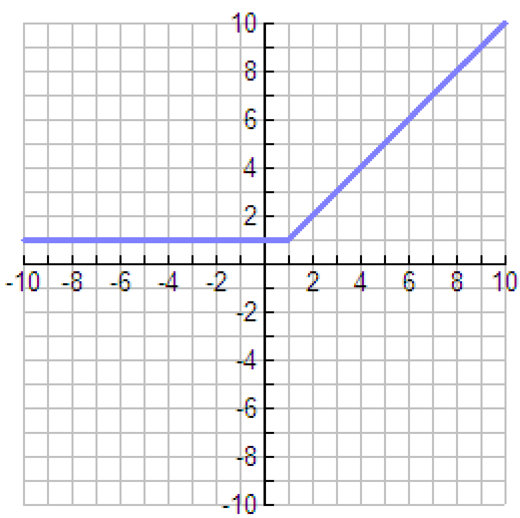
b.) $g(x) = \frac{1}{x}$



Is the function g continuous?

Explain your answer.

c.) $h(x) = \begin{cases} 1, & x \leq 1 \\ x, & x > 1 \end{cases}$



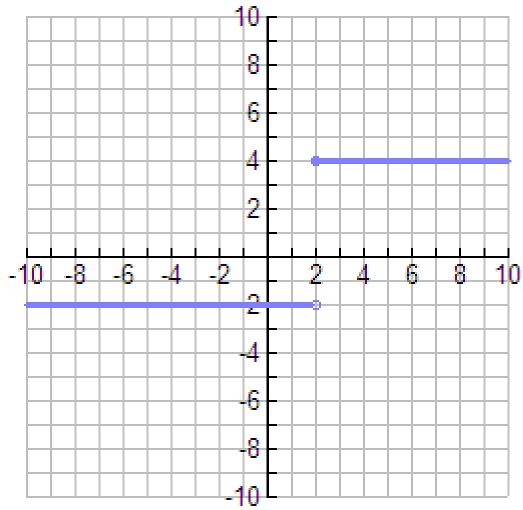
Is the function h continuous?

Explain your answer.

d.) $j(x) = \begin{cases} -2, & x < 2 \\ 4, & x \geq 2 \end{cases}$

Is the function *j* continuous?

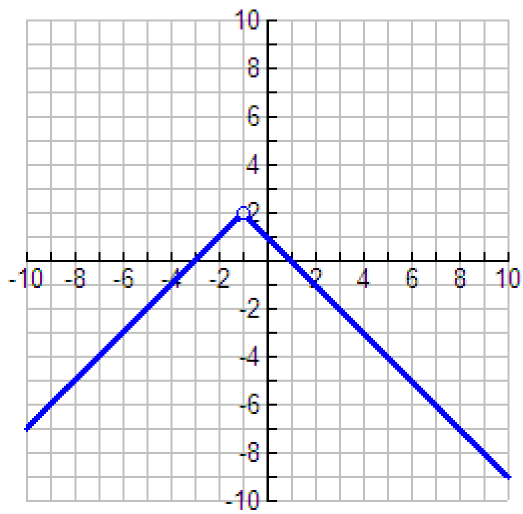
Explain your answer.



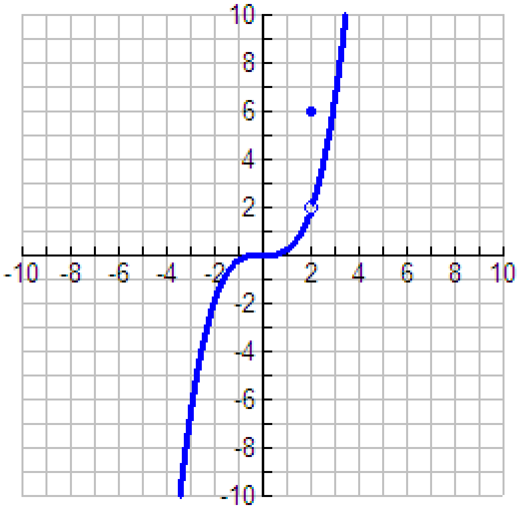
e.) $k(x) = \begin{cases} x + 3, & x < -1 \\ -x + 1, & x > -1 \end{cases}$

Is the function *k* continuous?

Explain your answer.



f.) $m(x) = \begin{cases} x^3, & x \neq 2 \\ 6, & x = 2 \end{cases}$



Is the function m continuous?

Explain your answer.

4. *Explain the meaning* of the following statements:

a. A function g has domain $[3, \infty)$.

b. A function h has domain $x \geq 2$.

c. A function f has domain $\{x \in \mathbb{R} : x \geq 5\}$.

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BIOGRAPHICAL INFORMATION

Melissa Jo Vela was born in Oklahoma City, Oklahoma on June 23, 1980. She graduated as salutatorian of her class in 1998 from Rockwall Christian Academy in Rowlett, Texas. In 2004 she graduated with a Bachelor of Arts degree in mathematics from Texas A&M University in College Station, Texas. She began teaching mathematics at Lawrence D. Bell High School in Hurst, Texas in 2005. She is in her sixth year of teaching secondary mathematics and her fifth year teaching AP Calculus at L.D. Bell. She married Jason Vela in 2006. She will graduate with a Master of Science in mathematics in May 2011 from the University of Texas at Arlington in Arlington, Texas.