STRESS ANALYSIS FOR A THREE PHASE PLATE WITH A CONCENTRIC CIRCULAR INCLUSION BY DERIVING AN AIRY STRESS FUNCTION

by

ABHISHEK KUNCHALA

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ABSTRACT

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Abhishek Kunchala, MS

The University of Texas at Arlington, 2011

Supervising Professor: Dr. Seiichi Nomura

This thesis derives the Airy stress function for a three phase plate that consists of two concentric circular inclusions and a matrix phase. It incorporates the elastic property of each phase as well as the geometry of the inclusions. An Airy stress function has been derived that satisfies the continuity conditions of the displacement and traction across the phase interface precisely. The obtained results are new and could not have been possible without a computer algebra system (Mathematica).

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CHAPTER 1

INTRODUCTION

1.1 Overview of elasticity

Elasticity is a subject appertained with the determination of stresses and displacements in a body as a result of applied mechanical or thermal loads. Elastic behavior is governed by Hooke's law. It is characterized by the conditions that stress is a unique function of strain and the material has the property for complete recovery to natural shape upon the removal of applied loads. These conditions ascertain the use of linear superposition and a wide range of transformation techniques to solve the problems associated with elasticity. Elasticity attempts to develop the solution directly and meticulously from the principles of Newton's laws of motion and Hooke's law. Continuum mechanics and partial differential field equations are used in this theory to solve the mathematical problems associated with elasticity.

In general, direct solutions to the equations of equilibrium of an elastic body pose difficulties. The theory of Complex variable methods is imparted in solving the boundary value problems associated with elasticity. Plane elasticity problems reduce to the solution of Navier's displacement equations of equilibrium when subjected to certain boundary conditions. This formulation then allows many powerful mathematical techniques available from the complex variable theory to be applied to the elasticity problem. By the implementation of complex variable notations, elastic displacement is derived as functions of complex variable potentials. These potentials must be made to satisfy the boundary conditions on the surface of the body.

Analytical closed form solutions for three-dimensional problems are reduced to two-dimensional axisymmetry problems. This is done to lessen the complexity involved in solving the elasticity field equations. There are numerous solutions to plane stress and plane strain problems. These solutions can be established by the implementation of a distinct stress function techniques.

The method of Airy stress function reduces the general formulation to a single governing equation in terms of a single unknown [3]. Several techniques such as fourier methods, integral transforms, finite differences, finite elements etc., can be used to obtain many analytical solutions to the problems emerging from the resulting governing equations. The basic idea of developing a stress field is to form a single governing equation that satisfies the equilibrium and compatibility equations.

This research stands out from the rest as the single governing Airy stress function has been derived from certain combinations of complex variable potentials. Until now, theories have been proposed on infinite plates with circular holes, inclusions and discs separately using the Airy stress function. However the Airy stress function for a three phase plate could not be worked on extensively due to lack of algebraic software to solve the simultaneous equations and verify the compatibility and equilibrium equations. This thesis used algebraic software (Mathematica) and successfully derived the Airy stress function for a three phase plate with concentric circular inclusions and a matrix enclosing it. The previous studies of plates with circular holes and inclusions are a special case of this topic.

1.2 Use of symbolic software

Software packages such as MATLAB, MAPLE, *MATHEMATICA* are now available because of the immense development in hardware and software of computers. Older packages such as

Macsyma which was one of the very first general-purpose symbolic computations systems were written in LISP [6]. However, new ones such as *Mathematica* are written in C language and its variations and is one of the most widely available symbolic systems.

One can evaluate mathematical expressions analytically without any approximation using symbolic algebra systems. The major features of symbolic algebra systems include differentiations, integrations, expansions and solving equations. Most of the symbolic algebra systems have been used by mathematicians and theoretical physicists [9]. One of the most powerful feature of this system is its ability to deal with both symbolic formulae and numbers. It is this feature which makes it possible to do both algebra and calculus. It has been demonstrated that in certain circumstances the widely held view that one can always dramatically improve on the CPU time required for lengthy computations by using compiled C or Fortran code instead of advanced quantitative programming environments such as *Mathematica*, MATLAB etc… is wrong. A well written C program can be expected to outperform *Mathematica*, R, S-Plus or MATLAB [8] but, if the C program is not efficiently programmed using the best possible algorithm then in fact it may take longer than using a symbolic software byte-code compiler.

The advantage of using Mathematica lies in its built in functions. The code editor provides many features to speed and improve applications development. It gives the largest collection of algorithms, covering areas such as numerical computation, symbolic computation, graphics, statistics, and data analysis. Mathematica can derive closed-form solutions for beams with circular, elliptical, equilateral-triangular, and rectangular cross sections [6]. Symbolic software also addresses the finite element method and is useful in finding shape functions, creating different types of meshes and can solve problems different materials. They are also useful in the kinematic modeling of fully constrained systems [6].

CHAPTER 2

THEORY OF AIRY STRESS FUNCTION

2.1 Airy Stress Function

Various techniques employ the Airy stress function to reduce the governing equations with solvable unknowns for the plane stress and plane strain problems. The equilibrium equations are obtained by considering a small rectangular block of edges a, b and unity.

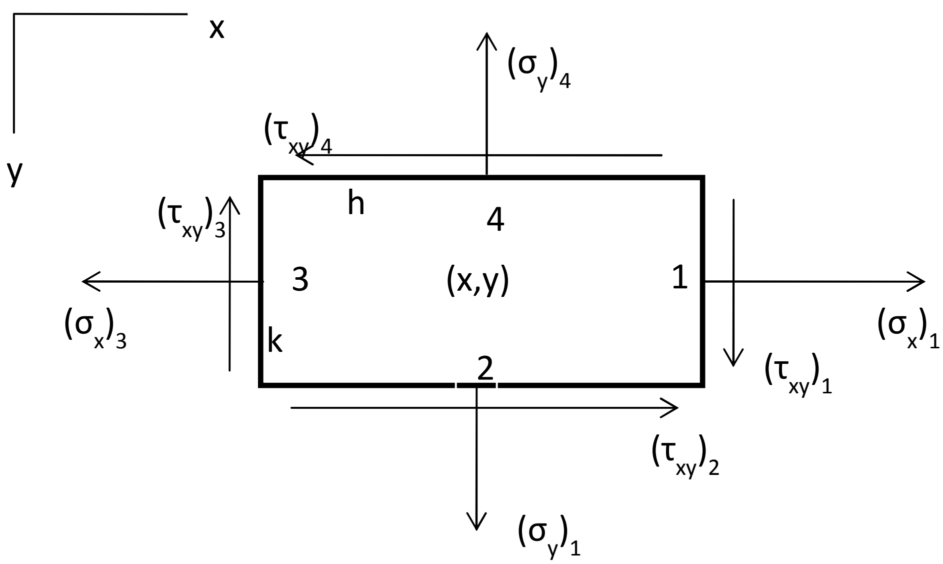

Figure 2.1 *Equilibrium of a rectangular block* [1]

The forces on the faces are determined by multiplying the stress components with the corresponding areas they act upon. In plane elasticity, boundary value problems can be considered by neglecting the body forces.

$$(\sigma x)1a - (\sigma x)3a + (\tau xy)2b - (\tau xy)3b = 0 \qquad (2.1.1)$$

Dividing the above equation by the area ab,

$$\frac{(\sigma x)1 - (\sigma x)3}{b} + \frac{(\tau xy)2 - (\tau xy)3}{a} = 0 \quad (2.1.2)$$

When the rectangular block is considered to be very small, then $a -> 0$ and b->0.

$$\lim_{b \to 0} [\frac{(\sigma x)1 - (\sigma x)3}{b}] = \frac{\partial \sigma x}{\partial x}$$

Similarly, (2.1.3)

$$\lim_{a \to 0} [\frac{(\tau xy)2 - (\tau xy)3}{a}] = \frac{\partial \tau xy}{\partial y}$$

Hence the equations of equilibrium in x and y direction take the following form.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (2.1.4)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \quad (2.1.5)$$

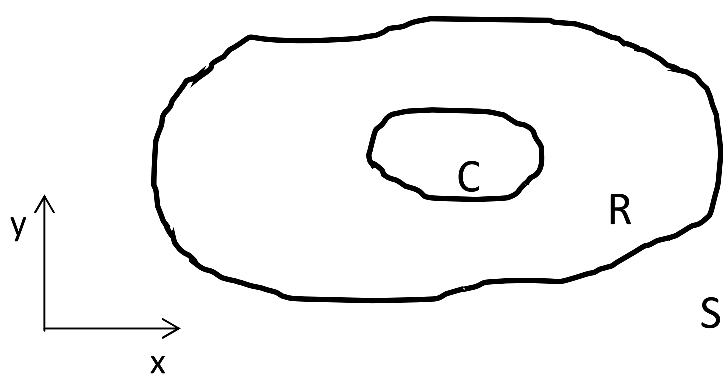

Figure 2.2 *Sample Domain*

A smooth contour can be represented by the functions which have continuous derivative throughout the domain. The equilibrium equations will be satisfied by choosing the representation [3].

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} \tag{2.1.6}$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

Where $\phi = \phi\,(x, y)$ is the Airy stress function.

This method provides a variety of solutions to the equations of equilibrium but the perfect solution would be the one which satisfies the compatibility equation given below [1].

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y\right) = 0 \tag{2.1.7}$$

$$\nabla^2\left(\sigma_x + \sigma_y\right) = 0 \tag{2.1.8}$$

Where ∇is the Laplace operator.

Substituting the airy stress function relations in the compatibility equation, we get [1].

$$\frac{\partial^4 \phi}{\partial x^4} + \frac{2\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial^4 y} = \nabla^4 \phi = 0 \tag{2.1.9}$$

The above expression is called a biharmonic equation and every solution to this equation is termed as a biharmonic function. The problem of elasticity is now reduced to a plane elasticity problem with a single equation in terms of the Airy stress function,ϕ. The solution is determined in the region R enclosed by the boundary S as shown in Figure 2.1. Appropriate boundary conditions are applied over the boundary S and the resulting solutions satisfy the compatibility equation.

2.2 Equations in polar coordinates

Polar coordinate system is often chosen to represent curved surfaces. Many plane problems in elasticity are solved by developing the equations in polar coordinate system. The solution to the governing equations in plane stress and plane strain problems involves the determination of the displacements and stress in the plane corresponding to the region R subjected to the boundary conditions S.

The transformation of the stress components from Cartesian to polar coordinate system is as follows:

$$\sigma_r = \sigma_x cos^2\theta + \sigma_y sin^2\theta + 2\tau_{xy}\, sin\theta\, cos\theta$$

$$\sigma_\theta = \sigma_x sin^2\theta + \sigma_y cos^2\theta + 2\tau_{xy}\, sin\theta\, cos\theta \qquad (2.2.1)$$

$$\tau_{r\theta} = (\sigma_y - \sigma_x)\, sin\theta\, cos\theta + \tau_{xy}\,(cos^2\theta - sin^2\theta)$$

The two dimensional problems can be solved by the Airy stress function representation in the relations (2.1.6) with the transformation of stress components into polar coordinates as shown below:

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \qquad (2.2.2)$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}$$

The biharmonic equation in terms of polar coordinates can be written as

$$\nabla^2(\sigma_x + \sigma_y) = 0 \qquad (2.2.3)$$

The above equation takes the following form in polar coordinate system

$$\nabla^4\phi = \left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0 \qquad (2.2.4)$$

From various solutions of this biharmonic equation formulated in terms of Airy stress function solutions can be obtained for plane elastic problems subjected to discrete boundary conditions in polar coordinates.

2.3 Complex Variable Theory

The general notation for a complex variable z with real variables x and y is

$$z = x + i\,y \qquad (2.3.1)$$

where x is the real part and y is the imaginary part of the complex number.

The polar form expression for a complex number is

Figure

$$z = r(Cos\,\theta + i\,Sin\,\theta) = r\,e^{i\theta} \qquad (2.3.2)$$

where$r$ is the modulus of z given by

$$r = \sqrt{x^2 + y^2}, \qquad (2.3.3)$$

θ is the argument of z given as

$$\theta = Tan^{-1}\left(\frac{y}{x}\right)$$

(2.3.4)

$$\bar{z} = x - i\,y = r\,e^{-i\theta}$$

Where $\bar{z}$ is the conjugate of the complex number z.

The following relations are developed by an ordinary coordinate transformation

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{z\,\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad (2.3.5)$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

2.3.1 Complex variables in elasticity

The method of Complex variables is applied to resolve many problems associated with elasticity. These not only include torsion and plane problems but also cases anisotropic and thermo elastic materials. The plane problems are attributed with cases of plane stress and plane strain for which the formulation of equations remain the same. The implementation of complex variables in plane problems of elasticity reduces to the solution of Navier's displacement equations of equilibrium when subjected to certain boundary conditions.

The expression for stresses in terms of displacements is given by the following equations

$$\sigma_x = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_y = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y} \qquad (2.3.6)$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Where λ is the Lame's constant and μ is the shear modulus of elasticity.

Implementing the complex variable theory in the Airy stress function transforms the plane problem into a complex variable equation in polar coordinates. Taking advantage of putting down the complex terms z and $\bar{z}$ in terms of variables x and y, we can write the Airy stress function as $\emptyset = \emptyset(z, \bar{z})$. Applying the differential operators defined in the equations (2.3.5) results in the following equation [3].

$$\Delta^2(\,) = 4\frac{\partial^2(\,)}{\partial z \partial \bar{z}}$$

(2.3.7)

$$\Delta^4(\,) = 16\frac{\partial^4(\,)}{\partial z \partial \bar{z}}$$

Hence the biharmonic equations in elasticity can be expressed as

$$\frac{\partial^4(\emptyset)}{\partial z \partial \bar{z}} = 0 \quad (2.3.8)$$

Integrating the above equation yields

$$\emptyset(z, \bar{z}) = \frac{1}{2}\left(z\overline{\gamma(z)} + \bar{z}\gamma(z) + \chi(z) + \overline{\chi(z)}\right)$$

(2.3.9)

$$= Re\left(\bar{z}\gamma(z) + \chi(z)\right)$$

Where γ and χ are arbitrary functions of the indicated variables and ϕ must be real. This shows the formulation of Airy stress function in terms of two complex potentials.

Considering the Navier equation [3],

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla(\nabla . \mathbf{u}) = 0 \quad (2.3.10)$$

Where ∇^2 is the Laplacian operator.

Introducing the complex variable for the displacement $U = u + i\,v$in the Navier equation given above, we get

$$(\lambda + \mu)\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial z} + \frac{\partial \overline{U}}{\partial z}\right) + 2\mu\left(\frac{\partial^2 U}{\partial z\, \partial \bar{z}}\right) = 0 \quad (2.3.11)$$

Integrating the above expression yields a complex solution for displacement

$$2\mu U = \kappa\,\gamma(z) - z\overline{\gamma'(z)} - \overline{\psi(z)} \quad (2.3.12)$$

Where $\gamma(z)$ and $\psi(z) = \chi'(z)$ are functions of an arbitrary complex variable and the parameter κ depends on the Poisson's ratio ν.

$$\kappa = \begin{cases} \frac{\lambda+3\mu}{\lambda+\mu} = 3 - 4\nu \text{ , Plane strain} \\ \frac{5\lambda+6\mu}{3\lambda+2\mu} = \frac{3-\nu}{1+\nu} \text{ , Plane stress} \end{cases} \tag{2.3.13}$$

Using the relations for stresses implementing the Airy stress function in equations (2.1.6), (2.1.7) and the integrated result of the biharmonic function in equation (2.3.12)

$$\sigma_x + \sigma_y = 2\left(\gamma'(z) + \overline{\gamma'(z)}\right)$$

(2.3.14)

$$\sigma_x - \sigma_y + 2i\tau_{xy} = 2\left(\bar{z}\gamma''(z) + \psi'(z)\right)$$

Simplifying the above equations using standard transformations, the relation for stresses, displacements in polar coordinates and Cartesian coordinates can be written as

$$\sigma_r + \sigma_\theta = \sigma_x + \sigma_y$$

$$\sigma_\theta - \sigma_r + 2i\tau^{r\theta} = \left(\sigma_y - \sigma_x + 2i\tau_{xy}\right)e^{2i\theta} \tag{2.3.15}$$

$$u_r + i\,u_\theta = (u + i\,v)e^{-i\theta}$$

2.4 Complex Potentials

The formulation of Airy stress function involves the determination of the complex potentials $\gamma(z)$ and $\psi(z)$. They are analytic functions which can be determined by applying certain stress and displacement conditions. The representation of the function depends on the domain of the problem under study. The different domains include finite simply connected, finite multiply connected and infinite multiply connected domains.

2.4.1 Finite Simply Connected Domain

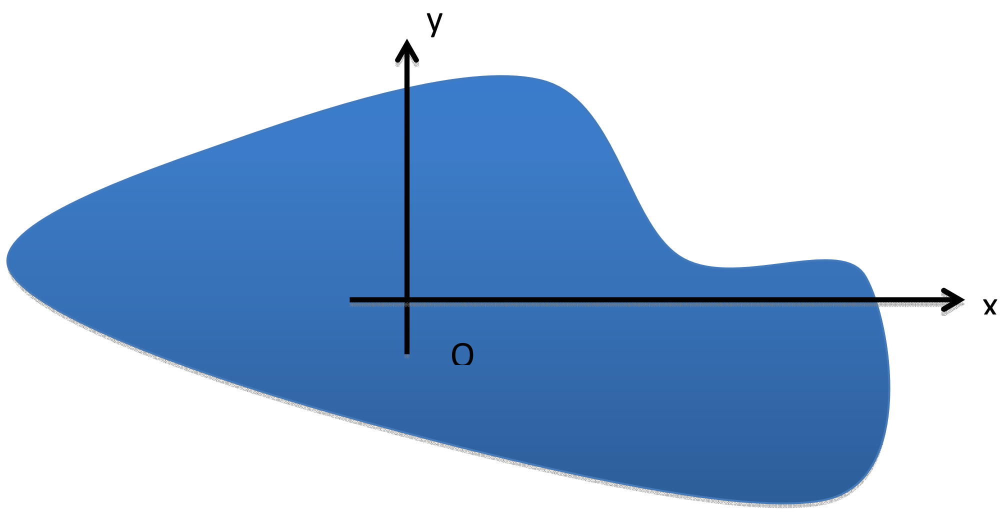

Figure 2.3 *Finite Simply Connected Domain*

Consider a finite simply connected domain R bounded by a contour C as shown in the figure above. For this case, the single valued analytic functions have the power series representation as

$$\gamma(z) = \sum_{n=0}^{\infty} a_n z^n$$

(2.4.1)

$$\psi(z) = \sum_{n=0}^{\infty} b_n z^n$$

Where a_n and b_n are the constants which can be determined by applying the boundary conditions.

2.4.2 Finite Multiply Connected Domain

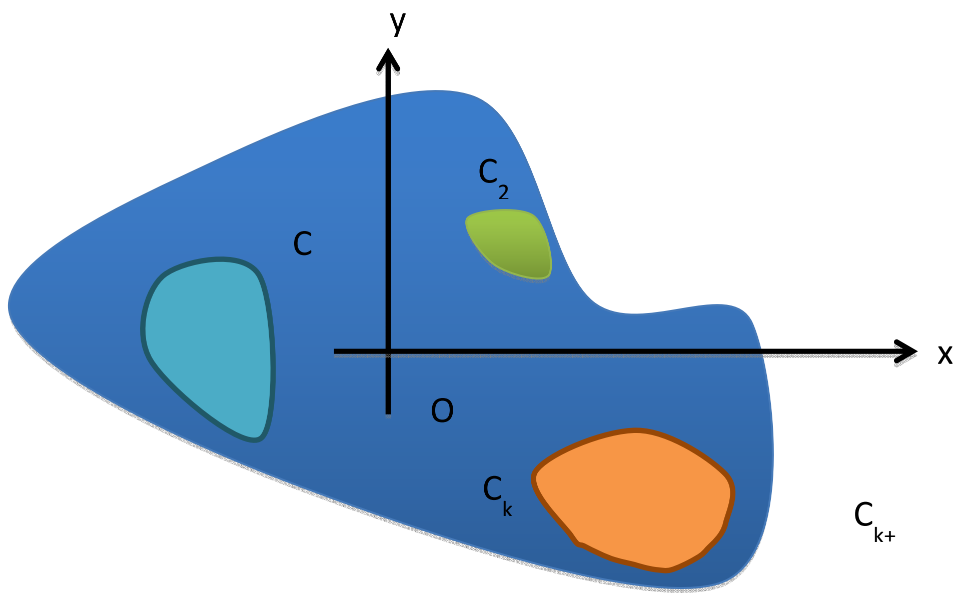

Figure 2.4 *Finite Multiply Connected Domain*

If R is a finite multiply connected domain bounded by the exterior contour C_{m+1} and by the m interior contours C_k (k = 1, 2, 3, . . . , m) as shown in the figure below, if the displacements and stresses arte single valued functions throughout R, then γ and ψ have the following structures:

$$\gamma(z) = \sum_{k=0}^{\infty} \frac{-F_k}{2\pi(1+\kappa)} \log(z - z_k) + \gamma^*(z)$$

(2.4.2)

$$\psi(z) = \sum_{k=0}^{\infty} \frac{\kappa \overline{F_k}}{2\pi(1+\kappa)} \log\,(z - z_k) + \psi^*(z)$$

whereF$_k$ is the resultant vector of external forces applied to thecontourC$_k$,$z_k$is an arbitrary point within the contour C$_k$in the simple connected region R. The functions $\gamma^*(z)$ and $\psi^*(z)$are arbitrary analytic functions in R, and κ is the material constant.

2.4.3 Infinite Domain

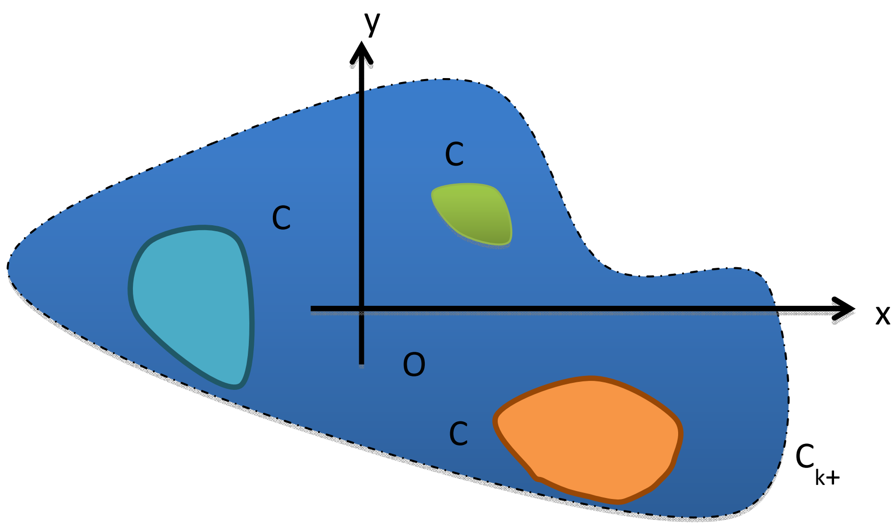

Figure 2.5 *Infinite Domain*

For an infinite region R, bounded by several simple closed contours C_k(k = 1, 2, 3, . . . ,m) and if the stress components are bounded in the neighborhood of the point at infinity, then for sufficiently large |z|,

$$\gamma(z) = \sum_{k=0}^{\infty} \frac{-F_k}{2\pi(1+\kappa)} \log(z - z_k) + \frac{\sigma_x^{\infty} - \sigma_y^{\infty}}{4} z + \gamma^{**}(z)$$

(2.4.3)

$$\psi(z) = \sum_{k=0}^{\infty} \frac{\kappa \overline{F_k}}{2\pi(1+\kappa)} \log(z - z_k) + \frac{\sigma_y^{\infty} - \sigma_x^{\infty} + 2i\tau_{xy}^{\infty}}{4} z + \psi^{**}(z)$$

Whereσ_x^{∞}, σ_y^{∞}, τ_{xy}^{∞} are the stresses at infinity, $\gamma^{**}(z)$ and $\psi^{**}(z)$ are arbitrary analytic functions outside the region enclosing all contours. They are represented using the power series notation as follows:

$$\gamma^{**}(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$$

$$\psi^{**}(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^n} \tag{2.4.4}$$

The displacements at infinity would indicate unbounded behavior as even abounded strain over an infinite length will produce infinite displacements. Therefore the case of the above region is obtained by dropping the summation terms.

CHAPTER 3

APPLICATIONS OF AIRY STRESS FUNCTION

3.1 Finite plate with a hole subjected to tensile loading

Applying the concept of finite multiply connected domain as discussed in the previous chapter to a plate with finite boundaries subjected to a tensile loading as shown below.

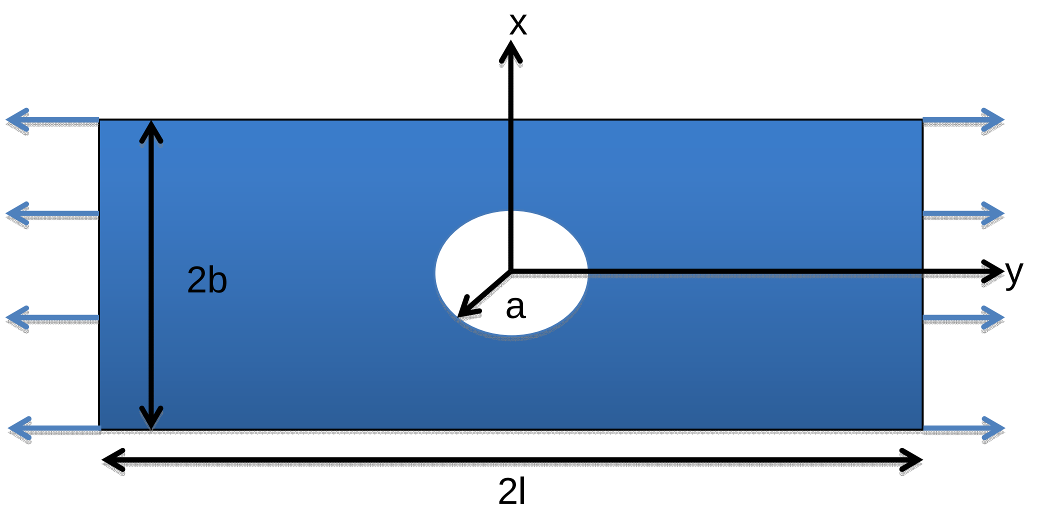

Figure 3.1 *Finite Plate With A Hole Subjected To Tensile Loading*

Assuming a rectangular plate of length $2l$ and width $2b$ with a hole of radius 'a' at the center of the plate subjected to uniform tension S acting along the X-axis. Applying the complex potentials as given in the equations … for a finite multiply connected domain and integrating the second complex potential since $\psi(z) = \chi'(z)$, we get

$$\chi(z) = \int \left(\sum_{k=0}^{\infty} \frac{\kappa \overline{F_k}}{2\pi(1+\kappa)} \log(z - z_k) + \psi^*(z) \right) dz \,, \qquad (3.1.1)$$

Where k = 1 since it has only 1 internal boundary and the center of the circle is taken as the origin $(0,0)$ which makes $z_k = 0$. The arbitrary analytic functions $\gamma^*(z)$ and $\psi^*(z)$ can be defined as

$$\gamma^*(z) = \sum_{n=0}^{\infty} a_n z^n ,$$

(3.1.2)

$$\psi^*(z) = \sum_{n=0}^{\infty} b_n z^n .$$

Substituting the above equations in the complex potential functions and solving for the Airy stress function defined in equation (2.3.9) we get

$$\emptyset(z,\bar{z}) = Re[\bar{z}\sum_{n=0}^{\infty} a_n z^n + \int \sum_{n=0}^{\infty} b_n z^n] \quad (3.1.3)$$

Applying the boundary conditions for the plate

$$\sigma_x(\underline{+}l, y) = S,$$

$$\sigma_y(x, \underline{+}b) = 0, \quad (3.1.4)$$

$$\tau_{xy}(\underline{+}l, y) = \tau_{xy}(x, \underline{+}b) = 0$$

$$\sigma_r(\underline{+}a, \underline{+}a) - \tau_{r\theta}(\underline{+}a, \underline{+}a) = 0$$

Solving the Airy stress function we get

$$\emptyset(x,y) = \frac{S(x^2 - y^2)(-6a^2 + x^2 + y^2) + 12b_0 x(a^2 - y^2) + 12a_0 x(a^2 - y^2)}{12(a^2 - y^2)} \quad (3.1.5)$$

3.2 Infinite plate with a hole subjected to tensile loading

Consider an infinite plate with a central hole subjected to uniform tensile loading $\sigma_x^{\infty} = S$ in the x direction.

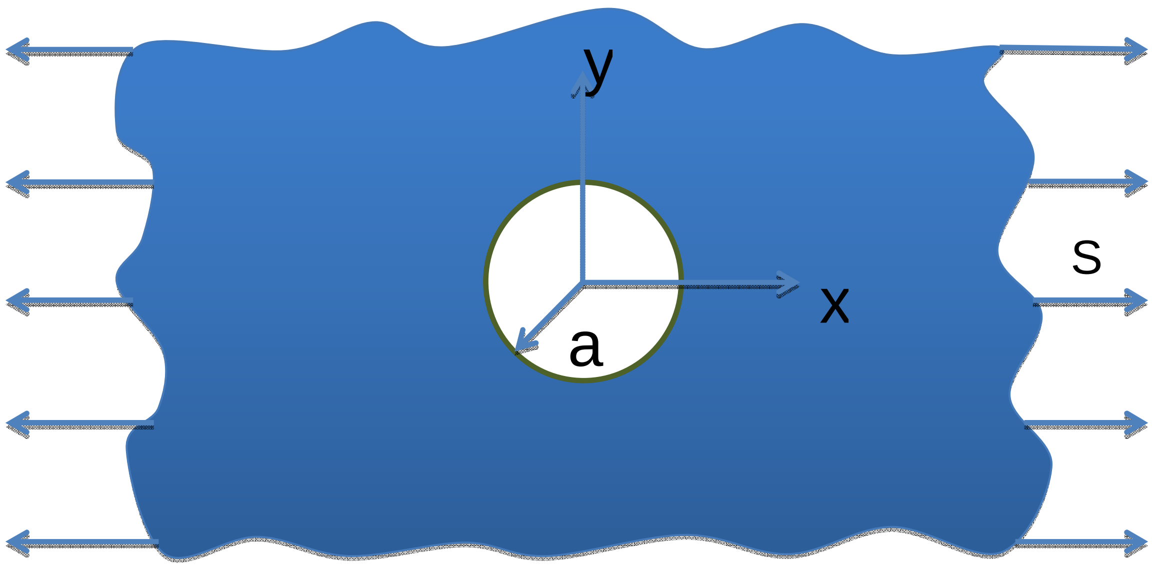

Figure 3.2 *Infinite Plate With A Hole Subjected To Tensile Loading*

The Airy stress function for an infinite plate with a circular hole can be derived by applying the complex potential equations given in (2.4.3). The logarithmic part of the equation is neglected as it corresponds to discontinuities in the displacement which does not exist in this case as it is an elastic material. Substituting $z = re^{i\theta}$, applying the boundary conditions $\sigma_x^{\infty} = S\,, \sigma_y^{\infty} = \tau_{xy}^{\infty} = 0$and solving for the unknown constants in the complex potential functions assumed for an infinite domain, we get

$$\emptyset(r,\theta) = \frac{-2a^2(a^2-3r^2)S\ \cos 2\theta - 3a^2Sr^2 \log r^2 + 6r^4\ S\ sin^2\theta}{12r^2} \qquad (3.2.1)$$

3.3 Two Dimensional Circular Inclusion

In the previous sections, we have seen the derivation for an Airy stress function for finite and infinite plates with a circular hole. This section deals with an infinite plate having a circular inclusion. Research has been done on this topic and the Airy stress function has been derived for this geometry by my peer.

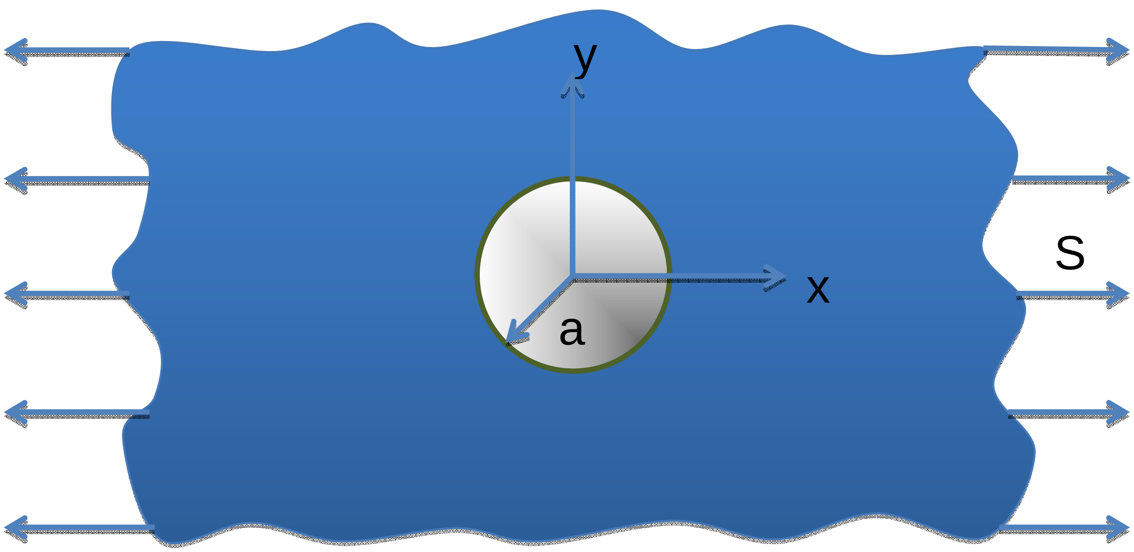

Figure 3.3 *Two Dimensional Circular Inclusion*

We can consider the above figure to be an infinite plate with a circular disc embedded in it instead of a hole. The infinite plate with a hole can be grouped with the finite simply connected domain by maintaining the equilibrium and continuity of stresses and displacements at the boundary of the two phases.

3.3.1 Stress Field Inside The Two Dimensional Circular Inclusion

Considering the circular inclusion embedded within the infinite plate to be similar to a finite simply connected domain. Hence the complex potentials are assumed to be

$$\gamma(z) = \sum_{n=0}^{2} c_n z^n,$$

(3.3.1)

$$\psi(z) = \sum_{n=0}^{2} d_n z^n.$$

Similar boundary conditions are applied to the disc and the infinite plate matrix surrounding it. Considering the first three terms of the power series, we have

$$\gamma(z) = c_0 + c_1 z + c_2 z^2,$$
$$\psi(z) = d_0 + d_1 z + d_2 z^2, \qquad (3.3.2)$$
$$\chi(z) = d_0 z + d_1 \frac{z^2}{2} + d_2 \frac{z^3}{3}.$$

Expressing the Airy stress function in the polar coordinate system using $z = r\, e^{i\theta}$, we get

$$\phi(r,\theta) = c_0 \mathrm{r} \cos\theta + c_1 \mathrm{r}^2 \cos^2\theta + c_1 \mathrm{r}^2 \sin^2\theta + c_2 \mathrm{r}^3 \cos\theta \cos 2\theta +$$
$$c_2 \mathrm{r}^3 \sin\theta \sin 2\theta + d_0 \mathrm{r} \cos\theta \cos 2\theta + d_0 \mathrm{r} \sin\theta \sin 2\theta + \qquad (3.3.3)$$
$$\frac{1}{2} d_1 r^2 \cos\theta \cos 3\theta + \frac{1}{2} d_1 r^2 \sin\theta \sin 3\theta$$
$$\frac{1}{2} d_2 r^3 \cos\theta \cos 4\theta + \frac{1}{2} d_2 r^3 \sin\theta \sin 4\theta$$

Applying the Cartesian to polar coordinate transformation as in equation (2.2.1)

$$\sigma_{rr}^{in} = 2c_1 + 2c_2\, r\, \cos\theta - d_1 \cos 2\theta - 2d_2\, r\, \cos 3\theta\, ,$$

$$\sigma_{\theta\theta}^{in} = 2c_1 + 6c_2\, r\, \cos\theta - d_1 \cos 2\theta - 2d_2\, r\, \cos 3\theta\, , \qquad (3.3.4)$$

$$\tau_{\theta\theta}^{in} = 2c_2\, r\, \sin\theta - d_1 \sin 2\theta - 2d_2\, r\, \sin 3\theta\, .$$

Determining the displacements using the equation (2.3.12)

$$u_r^{in} = \frac{1}{2\mu}\,(\,\kappa\, c_0 \cos\theta - c_1\, r + \kappa\, c_1\, r - 2c_2 r^2 \cos\theta + \kappa\, c_2 r^2 \cos\theta -$$

$$d_0 \cos\theta - d_1\, r\, \cos 2\theta - d_2 r^2 \cos 2\theta) \qquad (3.3.5)$$

$$u_\theta^{in} = \frac{1}{2\mu}\,(-\,\kappa\, c_0 \sin\theta + 2c_2 r^2 \sin\theta + \kappa c_2 r^2 \sin\theta +$$

$$d_0 \sin\theta - d_1\, r\, \sin 2\theta - d_2 r^2 \sin 3\theta\ .$$

The stresses and displacements are now evaluated for the boundary of the circular inclusion with the material constants μ and κ by substituting $r = a$ in the equations (3.3.4) and (3.3.5).

3.3.2 Stress Field For Infinite Plate Surrounding The Disc

The stresses and displacements for the infinite matrix surrounding the circular inclusion with material constant κ_1 and shear modulus μ_1 at radius $r = a$ can be derived from the equations (3.3.4) and (3.3.5).

$$\sigma_{rr}^{out} = \frac{1}{2a^4}\,(\,a^4 S + a^4 S\, \cos 2\theta + 2a^2 b_1 + 4ab_2 \cos\theta$$

$$+\, 6b_3 \cos 2\theta - 8a^2 a_1 \cos 2\theta - 20aa_2 \cos 3\theta)\, ,$$

$$\sigma_{\theta\theta}^{out} = \frac{1}{2a^4}\,(\,a^4 S - a^4 S\, \cos 2\theta\, - 2a^2 b_1 - 4ab_2 \cos\theta \qquad (3.3.6)$$

$$-\, 6b_3 \cos 2\theta + 4aa_2 \cos 3\theta)\, ,$$

$$\tau_{\theta\theta}^{out} = \frac{1}{2a^4}\,(-\,a^4 S\, \sin 2\theta + 4ab_2 \sin\theta + 6b_3 \sin 2\theta$$

$$- 4a^2 a_1 \sin 2\theta - 12 a a_2 \sin 3\theta) ,$$

$$u_r^{out} = \frac{1}{8a^3\mu_1} (- a^4 S + a^4 S\, \kappa_1 - 2a^4 S \cos 2\theta - 4a^2 b_1$$

$$- 4ab_2 \cos\theta - 4b_3 \cos 2\theta + 4a^2 a_1 \cos 2\theta$$

$$+ 4a^2 \kappa_1 a_1 \cos 2\theta + 8aa_2 \cos 3\theta + 4a\kappa_1 a_2 \cos 3\theta) , \quad (3.3.7)$$

$$u_\theta^{out} = \frac{1}{4a^3\mu_1} (- a^4 S \sin 2\theta - 2ab_2 \sin\theta - 2b_3 \sin 2\theta + 2a^2 a_1 \sin 2\theta$$

$$- 2a^2 \kappa_1 a_1 \sin 2\theta + 4a_2 \sin 3\theta - 2a\kappa_1 a_2 \sin 3\theta).$$

If the traction force and displacements are satisfied at the boundary of the circular disc and infinite plate interface, then the continuity equation can be satisfied.

Equating (3.3.4) to (3.3.6) and (3.3.5) to (3.3.7) and solving for the constants $a_0, a_1, a_2, b_0, b_1, b_2$ by equating the coefficients of $\cos\theta, \cos 2\theta, \cos 3\theta, \sin\theta, \sin 2\theta, \sin 3\theta$ and making the constants zero.

$$a_1 = -\frac{a^2(S\mu - S\mu_1)}{2(\kappa_1\mu + \mu_1)}, \qquad a_2 = 0 \qquad a_3 = 0$$

$$b_1 = \frac{a(-aS\mu + aS\kappa_1\mu + aS\mu_1 - aS\kappa\mu_1)}{2(2\mu - \mu_1 + \kappa\mu_1)}, \; b_2 = 0, \qquad b_3 = -\frac{a^4(S\mu - S\mu_1)}{2(\kappa_1\mu + \mu_1)}, \quad (3.3.8)$$

$$c_0 = \frac{d_0}{\kappa}, \qquad c_1 = \frac{\mu(S + S\kappa_1)}{4(2\mu - \mu_1 + \kappa\mu_1)}, \qquad c_2 = 0$$

$$d_0 = \kappa c_0, \qquad d_1 = \frac{\mu(S + S\kappa_1)}{2(\kappa_1\mu + \mu_1)}, \qquad d_2 = 0$$

Substituting the solutions into the equations (3.3.4), (3.3.5) and (3.3.6), (3.3.7) to obtain the final stress and displacement equations for the circular inclusion and the infinite plate surrounding it respectively. The validity of these equations can be verified by substituting them in the equations of equilibrium given by equation (2.2.2).

Hence the Airy stress function chosen for this geometry is valid and it can be utilized to determine the stresses and displacements for a two dimensional plate with a circular inclusion.

3.4 Three Phase Plate With A Concentric Circular Inclusion

Considering a three phase plate with a concentric circular inclusion embedded in an infinite plate. This can be assumed as an extension to the previously discussed circular inclusion problem by embedding another circular disc within the previous circular inclusion.

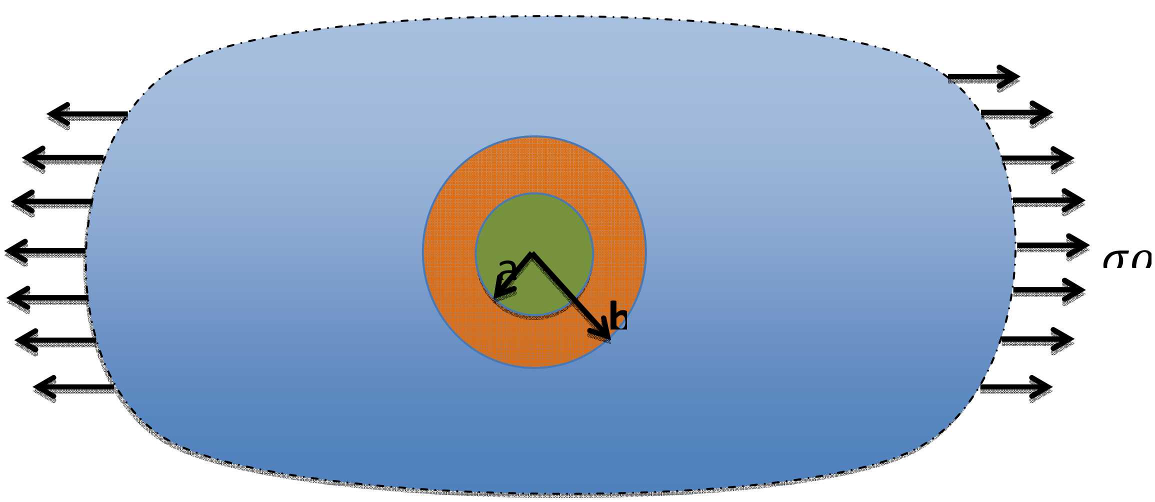

Figure 3.4 *Three Phase Plate With A Concentric Circular Inclusion*

The Inner circular inclusion can be considered as a finite simple connected domain, the outer circular disc as a finite multiply connected domain and the infinite matrix surrounding them to be an infinite domain.

3.4.1 Formulation Of Equations For Stresses And Displacements

The stresses for an Airy stress function ϕ can be derived using the equations below.

$$\sigma_r + \sigma_\theta = 2\left(\gamma'(z) + \overline{\gamma'(z)}\right)$$

(3.4.1)

$$\sigma_r - \sigma_\theta + 2i\tau_{r\theta} = 2\left(\bar{z}\gamma''(z) + \psi'(z)\right)$$

The displacement equations can be given as follows

$$2\mu U = \kappa\,\gamma(z) - z\overline{\gamma'(z)} - \overline{\psi(z)}$$

(3.4.2)

$$u_r + i\,u_\theta = (u + i\,v)e^{-i\theta}$$

Where $U = u + i\,v$.

3.4.2 Stress Field Inside The Inner Circular Inclusion

The complex potentials for the inner most circular inclusion can be given as

$$\gamma_{in} = (a_1 + ib_1)z + (a_2 + ib_2)z^3,$$

(3.4.3)

$$\psi_{in} = (a_3 + ib_3)z\,.$$

Substituting the complex potentials in the equations (3.4.1), (3.4.2) and solving for stresses and displacements, we get

$$\sigma_{rrin} = 2a_1 - a_3\cos 2\theta + b_3\sin 2\theta,$$

$$\sigma_{\theta\theta in} = 2a_1 + 12r^2a_2\cos 2\theta + a_3\cos 2\theta - 12r^2b_2\sin 2\theta - b_3\sin 2\theta, \qquad (3.4.4)$$

$$\tau_{r\theta in} = 6r^2a_2\sin 2\theta + a_3\sin 2\theta + 6r^2b_2\cos 2\theta + b_3\cos 2\theta.$$

$$U_{rin} = \frac{1}{2\mu_{in}}r(\,(-1+\kappa_{in})a_1 + r^3a_2(-3+\kappa_{in})\cos 2\theta - a_3\cos 2\theta +$$

$$3r^3b_2\sin 2\theta - r^3b_2\kappa_{in}\sin 2\theta + rb_3\sin 2\theta)$$

$$U_{\theta in} = \frac{1}{2\mu_{in}} r(r^2 a_2(3 + \kappa_{in}) \sin 2\theta + a_3 r \sin 2\theta + r b_1 (1 + \kappa_{in}) + \tag{3.4.5}$$

$$3r^3 b_2 \cos 2\theta + r^3 b_2 \kappa_{in} \cos 2\theta + r b_3 \cos 2\theta)$$

3.4.3 Stress Field For Concentric Circular Inclusion

Consider the outer circular inclusion in the infinite plate for which the complex potentials can be defined as

$$\gamma = (a_4 + ib_4)\frac{1}{z} + (a_5 + ib_5)z + (a_6 + ib_6)z^3$$

(3.4.6)

$$\psi = (a_7 + ib_7)\frac{1}{z^3} + (a_8 + ib_8)\frac{1}{z} + (a_9 + ib_9)z$$

Substituting the complex potentials in the equations (3.4.1) we have

$$\sigma_{rr} = -\frac{4a_4 \cos 2\theta}{r^2} + 2a_5 + \frac{3a_7 \cos 2\theta}{r^4} + \frac{a_8}{r^2} - a_9 \cos 2\theta -$$

$$\frac{4b_4 \sin 2\theta}{r^2} + \frac{3b_7 \sin 2\theta}{r^4} + b_9 \sin 2\theta$$

$$\sigma_{\theta\theta} = 2a_5 + 12r^2 a_6 \cos 2\theta - \frac{3a_7 \cos 2\theta}{r^4} - \frac{a_8}{r^2} + a_9 \cos 2\theta -$$

$$12r^2 b_6 \sin 2\theta - \frac{3b_7 \sin 2\theta}{r^4} - b_9 \sin 2\theta \tag{3.4.7}$$

$$\tau_{r\theta} = -\frac{2a_4 \sin 2\theta}{r^2} - 6r^2 a_6 \sin 2\theta + \frac{3a_7 \sin 2\theta}{r^4} + a_9 \sin 2\theta + \frac{2b_4 \cos 2\theta}{r^2} +$$

$$6r^2 b_6 \cos 2\theta - \frac{3b_7 \cos 2\theta}{r^4} - \frac{b_8}{r^2} + b_9 \cos 2\theta$$

The displacements at the boundary of the outer circular inclusion are obtained by the equations (3.4.2).

$$U_r = \frac{1}{2r^3\mu}(a_4 r^2 \cos 2\theta\,(1+\kappa) - r^4 a_5 (1-\kappa) - r^6 a_6 \cos 2\theta\,(3-\kappa) - a_7 \cos 2\theta - r^2 a_8$$

$$-r^4 a_9 \cos 2\theta + r^2 b_4 \sin 2\theta\,(1+\kappa) + r^6 b_6 \sin 2\theta\,(3-\kappa) - b_7 \sin 2\theta + r^4 b_9 \sin 2\theta$$

(3.4.8)

$$U_\theta = \frac{1}{2r^3\mu}(a_4 r^2 \sin 2\theta\,(1-\kappa) + r^6 a_6 \sin 2\theta\,(3+\kappa) - a_7 \sin 2\theta + r^4 a_9 \sin 2\theta$$

$$-b_4 r^2 \cos 2\theta\,(1-\kappa) + r^4 b_5 (1+\kappa) + r^6 b_6 \cos 2\theta\,(3+\kappa) + b_7 \cos 2\theta +$$

$$r^2 b_8 + r^4 b_9 \cos 2\theta$$

3.4.4 Stress Field For The Infinite Matrix Surrounding Circular Inclusion

The complex potential functions for the infinite matrix have been chosen as given below

$$\gamma_{out} = \left(\frac{\sigma o}{4}\right) z + (a_{10} + ib_{10})\frac{1}{z}$$

(3.4.9)

$$\psi_{out} = \left(\frac{\sigma o}{2}\right) z + (a_{11} + ib_{11})\frac{1}{z} + (a_{12} + ib_{12})\frac{1}{z^3}$$

Substituting the above complex potentials in the equations and (3.4.1) we have the stress components as

$$\sigma_{rrout} = -\frac{-r^4\sigma o + r^4 \sigma o \cos 2\theta + 8r^2 \cos 2\theta\, a_{10} - 2r^2 a_{11} - 6\cos 2\theta\, a_{12}}{2r^4}$$

$$+\frac{8r^2 \sin 2\theta\, b_{10} - 6 \sin 2\theta\, b_{12}}{2r^4}$$

$$\sigma_{\theta\theta out} = \frac{\sigma o}{2} + \frac{1}{2}\sigma o \cos 2\theta - \frac{a_{11}}{r^2} - \frac{3\cos 2\theta a_{12}}{r^4} - \frac{3\sin 2\theta b_{12}}{r^4} \quad (3.4.10)$$

$$\tau_{r\theta out} = -\frac{-r^4\sigma o \sin 2\theta + 4r^2 \sin 2\theta\, a_{10} - 6\sin 2\theta\, a_{12}}{2r^4} -$$

$$\frac{4r^2\cos 2\theta\, b_{10} - 2r^2 b_{11} - 6\cos 2\theta\, b_{12}}{2r^4}$$

The displacement functions are:

$$U_{rout} = \frac{1}{8r^3\mu_{out}}(-r^4\sigma o(1-\kappa_{out}) - 2r^4\sigma o\cos 2\theta + 4a_{10}r^2\cos 2\theta\,(1+\kappa_{out}) - 4r^2 a_{11} -$$

$$4a_{12}\cos 2\theta + 4r^2 b_{10}\sin 2\theta\,(1+\kappa_{out}) - 4b_{12}\sin 2\theta)$$

(3.4.11)

$$U_{\theta out} = \frac{1}{4r^3\mu_{out}}\left(r^4\sigma o \sin 2\theta + 2a_{10}r^2\sin 2\theta\,(1-\kappa_{out}) - 2a_{12}\sin 2\theta - 2b_{10}r^2\cos 2\theta\,(1 -\right.$$

$\kappa out+2r2b11+2\cos 2\theta b12$

3.4.5 Continuity Equations

In the previous section, we have defined the stress and displacement field functions for the complete geometry. For the Airy stress function to hold valid throughout the plate, it has to satisfy the continuity equations at the interface of the circular inclusions boundary and the infinite matrix surrounding it.

At $r = a$, we have the continuity equations for the concentric circular inclusions by equating (3.4.4) to (3.4.7) and (3.4.5) to (3.4.8).

$$\sigma_{rrin} - \sigma_{rr} = 0$$

$$2a_1 - a_3\cos 2\theta + b_3\sin 2\theta + \frac{4a_4\cos 2\theta}{a^2} - 2a_5 - \frac{3a_7\cos 2\theta}{a^4} - \frac{a_8}{a^2} + a_9\cos 2\theta + \quad (3.4.12)$$

$$\frac{4b_4\sin 2\theta}{a^2} - \frac{3b_7\sin 2\theta}{a^4} - b_9\sin 2\theta = 0$$

$$\tau_{r\theta in} - \tau_{r\theta} = 0$$

$$6a^2a_2 \sin 2\theta + a_3 \sin 2\theta + 6a^2b_2 \cos 2\theta + \frac{2a_4 \sin 2\theta}{a^2} - 6a^2a_6 \sin 2\theta - \frac{3a_7 \sin 2\theta}{a^4} \quad (3.4.13)$$

$$-a_9 \sin 2\theta - \frac{2b_4 \cos 2\theta}{a^2} - 6a^2b_6 \cos 2\theta + \frac{3b_7 \cos 2\theta}{a^4} + \frac{b_8}{a^2} - b_9 \cos 2\theta = 0$$

$$U_{rin} - U_r = 0$$

$$\frac{1}{2\mu_{in}} a((-1 + \kappa_{in})a_1 + a^2a_2(-3 + \kappa_{in}) \cos 2\theta - a_3 \cos 2\theta + 3a^2b_2 \sin 2\theta -$$

$$a^2b_2\kappa_{in} \sin 2\theta + b_3 \sin 2\theta) - \frac{1}{2r^3\mu}(a_4a^2 \cos 2\theta (1 + \kappa) - a^4a_5(1 - \kappa) \quad (3.4.14)$$

$$-a^6a_6 \cos 2\theta (3 - \kappa) - a_7 \cos 2\theta - a^2a_8 - a^4a_9 \cos 2\theta + a^2b_4 \sin 2\theta (1 + \kappa) +$$

$$a^6b_6 \sin 2\theta (3 - \kappa) - b_7 \sin 2\theta + a^4b_9 \sin 2\theta) = 0$$

$$U_{\theta in} - U_\theta = 0$$

$$\frac{1}{2\mu_{in}} a(a^2a_2(3 + \kappa_{in}) \sin 2\theta + a_3 \sin 2\theta + b_1(1 + \kappa_{in}) + 3a^2b_2 \cos 2\theta + a^2b_2\kappa_{in} \cos 2\theta$$

$$+b_3 \cos 2\theta) - \frac{1}{2r^3\mu}(a_4a^2 \sin 2\theta (1 - \kappa) + a^6a_6 \sin 2\theta (3 + \kappa) - a_7 \sin 2\theta + a^4a_9 \sin 2\theta \quad (3.4.15)$$

$$-b_4a^2 \cos 2\theta (1 - \kappa) + a^4b_5(1 + \kappa) + a^6b_6 \cos 2\theta (3 + \kappa) + b_7 \cos 2\theta + a^2b_8 + a^4b_9 \cos 2\theta = 0$$

Similarly, we have the continuity equations for the outer inclusion and the infinite matrix interface. Equating (3.4.7) and (3.4.10)

$$\sigma_{rr} - \sigma_{rrout} = 0$$

$$-\frac{4a_4 \cos 2\theta}{r^2} + 2a_5 + \frac{3a_7 \cos 2\theta}{r^4} + \frac{a_8}{r^2} - a_9 \cos 2\theta - \frac{4b_4 \sin 2\theta}{r^2} + \frac{3b_7 \sin 2\theta}{r^4} + b_9 \sin 2\theta + \quad (3.4.16)$$

$$\frac{-r^4\sigma o + r^4\sigma o \cos 2\theta + 8r^2 \cos 2\theta\, a_{10} - 2r^2a_{11} - 6 \cos 2\theta\, a_{12}}{2r^4} - \frac{8r^2 \sin 2\theta\, b_{10} - 6 \sin 2\theta\, b_{12}}{2r^4}$$

$$\tau_{r\theta} - \tau_{r\theta out} = 0$$

$$-\frac{2a_4 \sin 2\theta}{r^2} - 6r^2 a_6 \sin 2\theta + \frac{3a_7 \sin 2\theta}{r^4} + a_9 \sin 2\theta + \frac{2b_4 \cos 2\theta}{r^2} + 6r^2 b_6 \cos 2\theta -$$

$$\frac{3b_7 \cos 2\theta}{r^4} - \frac{b_8}{r^2} + b_9 \cos 2\theta + \frac{-r^4 \sigma o \sin 2\theta + 4r^2 \sin 2\theta a_{10} - 6 \sin 2\theta a_{12}}{2r^4} + \quad (3.4.17)$$

$$\frac{4r^2 \cos 2\theta\, b_{10} - 2r^2 b_{11} - 6 \cos 2\theta\, b_{12}}{2r^4}$$

$$U_r - U_{rout} = 0$$

$$\frac{1}{2b^3\mu}(a_4 b^2 \cos 2\theta\,(1+\kappa) - b^4 a_5 (1-\kappa) - b^6 a_6 \cos 2\theta\,(3-\kappa) - a_7 \cos 2\theta - b^2 a_8$$

$$-b^4 a_9 \cos 2\theta + b^2 b_4 \sin 2\theta\,(1+\kappa) + b^6 b_6 \sin 2\theta\,(3-\kappa) - b_7 \sin 2\theta + b^4 b_9 \sin 2\theta) - \quad (3.4.18)$$

$$\frac{1}{8r^3\mu_{out}}(-r^4 \sigma o (1-\kappa_{out}) - 2r^4 \sigma o \cos 2\theta + 4a_{10} r^2 \cos 2\theta\,(1+\kappa_{out}) - 4r^2 a_{11} - 4a_{12} \cos 2\theta +$$

$$4r^2 b_{10} \sin 2\theta\,(1+\kappa_{out}) - 4b_{12} \sin 2\theta)$$

$$U_\theta - U_{\theta out} = 0$$

$$\frac{1}{2r^3\mu}(a_4 r^2 \sin 2\theta\,(1-\kappa) + r^6 a_6 \sin 2\theta\,(3+\kappa) - a_7 \sin 2\theta + r^4 a_9 \sin 2\theta$$

$$b_4 r^2 \cos 2\theta\,(1-\kappa) + r^4 b_5 (1+\kappa) + r^6 b_6 \cos 2\theta\,(3+\kappa) + b_7 \cos 2\theta + r^2 b_8 \quad (3.4.19)$$

$$+r^4 b_9 \cos 2\theta - \frac{1}{4r^3\mu_{out}}(r^4 \sigma o \sin 2\theta + 2a_{10} r^2 \sin 2\theta\,(1-\kappa_{out}) - 2a_{12} \sin 2\theta -$$

$$2b_{10} r^2 \cos 2\theta\,(1-\kappa_{out}) + 2r^2 b_{11} + 2\cos 2\theta\, b_{12})$$

Equating the coefficients of $\cos\theta$, $\cos 2\theta$, $\sin\theta$, $\sin 2\theta$ and the constants in the above equations to zero, we get

$$2a_1 - 2a_5 - \frac{a_8}{a^2} = 0 \tag{3.4.20}$$

$$b_3 + \frac{4b_4}{a^2} - \frac{3b_7}{a^4} - b_9 = 0 \tag{3.4.21}$$

$$-a_3 + \frac{4a_4}{a^2} - \frac{3a_7}{a^4} + a_9 = 0 \tag{3.4.22}$$

$$-\frac{\sigma o}{2} + 2a_5 + \frac{a_8 - a_{11}}{b^2} = 0 \tag{3.4.23}$$

$$-\frac{4b_4}{b^2} + \frac{3b_7}{b^4} + b_9 + \frac{4b_{10}}{b^2} - \frac{3b_{12}}{b^4} = 0 \tag{3.4.24}$$

$$\frac{\sigma o}{2} - \frac{4a_4}{b^2} + \frac{3a_7}{b^4} - a_9 + \frac{4a_{10}}{b^2} - \frac{3a_{12}}{b^4} = 0 \tag{3.4.25}$$

$$\frac{b_8}{a^2} = 0 \tag{3.4.26}$$

$$6a^2a_2 + a_3 + \frac{2a_4}{a^2} - 6a^2a_6 - \frac{3a_7}{a^4} - a_9 = 0 \tag{3.4.27}$$

$$6a^2b_2 + b_3 - \frac{2b_4}{a^2} - 6a^2b_6 + \frac{3b_7}{a^4} - b_9 = 0 \tag{3.4.28}$$

$$\frac{-b_8 + b_{11}}{b^2} = 0 \tag{3.4.29}$$

$$-\frac{\sigma o}{2} - \frac{2a_4}{b^2} + 6b^2a_6 + \frac{3a_7}{b^4} + a_9 + \frac{2a_{10}}{b^2} - \frac{3a_{12}}{b^4} = 0 \tag{3.4.30}$$

$$\frac{2b_4}{b^2} + 6b^2 b_6 + \frac{3b_7}{b^4} + b_9 - \frac{2b_{10}}{b^2} + \frac{3b_{12}}{b^4} = 0 \quad (3.4.31)$$

$$\frac{a^2(-1+\kappa_{in})\mu a_1 + \mu_{in}\left(-a^2(-1+\kappa)a_5 + a_8\right)}{2a\mu\mu_{in}} = 0 \quad (3.4.32)$$

$$\frac{3a^3 b_2}{2\mu_{in}} - \frac{a^3\kappa_{in} b_2}{2\mu_{in}} + \frac{ab_3}{2\mu_{in}} - \frac{b_4}{2a\mu} - \frac{\kappa b_4}{2a\mu} - \frac{3a^3 b_6}{2\mu} + \frac{a^3\kappa b_6}{2\mu} + \frac{b_7}{2a^3\mu} - \frac{ab_9}{2\mu} = 0 \quad (3.4.33)$$

$$-\frac{3a^3 a_2}{2\mu_{in}} + \frac{a^3\kappa_{in} a_2}{2\mu_{in}} - \frac{aa_3}{2\mu_{in}} - \frac{a_4}{2a\mu} - \frac{\kappa a_4}{2a\mu} + \frac{3a^3 a_6}{2\mu} - \frac{a^3\kappa a_6}{2\mu} + \frac{a_7}{2a^3\mu} - \frac{aa_9}{2\mu} = 0 \quad (3.4.34)$$

$$\frac{4b^2(-1+\kappa)\mu_{out} a_5 - 4\mu_{out} a_8 + \mu\left(-b^2(-1+\kappa_{out})\sigma o + a_{11}\right)}{8b\mu\mu_{out}} = 0 \quad (3.4.35)$$

$$\frac{b\sigma o}{4\mu_{out}} + \frac{a_4}{2b\mu} + \frac{\kappa a_4}{2b\mu} - \frac{3b^3 a_6}{2\mu} + \frac{b^3\kappa a_6}{2\mu} - \frac{a_7}{2b^3\mu} - \frac{ba_9}{2\mu} - \frac{a_{10}}{2b\mu_{out}} - \frac{\kappa_{out} a_{10}}{2b\mu_{out}} + \frac{a_{12}}{2b^3\mu_{out}} = 0 \quad (3.4.36)$$

$$\frac{b_4}{2b\mu} + \frac{\kappa b_4}{2b\mu} - \frac{3b^3 b_6}{2\mu} - \frac{b^3\kappa b_6}{2\mu} - \frac{b_7}{2b^3\mu} + \frac{bb_9}{2\mu} - \frac{b_{10}}{2b\mu_{out}} - \frac{\kappa_{out} b_{10}}{2b\mu_{out}} + \frac{b_{12}}{2b^3\mu_{out}} = 0 \quad (3.4.37)$$

$$\frac{a^2(1+\kappa_{in})\mu b_1 - \mu_{in}\left(-a^2(1+\kappa)b_5 + b_8\right)}{2a\mu\mu_{in}} = 0 \quad (3.4.38)$$

$$\frac{3a^3 b_2}{2\mu_{in}} + \frac{a^3\kappa_{in} b_2}{2\mu_{in}} + \frac{ab_3}{2\mu_{in}} + \frac{b_4}{2a\mu} - \frac{\kappa b_4}{2a\mu} - \frac{3a^3 b_6}{2\mu} - \frac{a^3\kappa b_6}{2\mu} - \frac{b_7}{2a^3\mu} - \frac{ab_9}{2\mu} = 0 \quad (3.4.39)$$

$$\frac{3a^3 a_2}{2\mu_{in}} + \frac{a^3\kappa_{in} a_2}{2\mu_{in}} + \frac{aa_3}{2\mu_{in}} - \frac{a_4}{2a\mu} + \frac{\kappa a_4}{2a\mu} - \frac{3a^3 a_6}{2\mu} - \frac{a^3\kappa a_6}{2\mu} + \frac{a_7}{2a^3\mu} - \frac{aa_9}{2\mu} = 0 \quad (3.4.40)$$

$$\frac{b^2(1+\kappa)\mu_{out}b_5+\mu_{out}b_8-\mu b_{11}}{2b\mu\mu_{out}}=0 \tag{3.4.41}$$

$$-\frac{b_4}{2b\mu}+\frac{\kappa b_4}{2b\mu}+\frac{3b^3b_6}{2\mu}+\frac{b^3\kappa b_6}{2\mu}+\frac{b_7}{2b^3\mu}+\frac{bb_9}{2\mu}+\frac{b_{10}}{2b\mu_{out}}-\frac{\kappa_{out}b_{10}}{2b\mu_{out}}-\frac{b_{12}}{2b^3\mu_{out}}=0 \tag{3.4.42}$$

$$-\frac{b\sigma o}{4\mu_{out}}+\frac{a_4}{2b\mu}-\frac{\kappa a_4}{2b\mu}+\frac{3b^3a_6}{2\mu}+\frac{b^3\kappa a_6}{2\mu}-\frac{a_7}{2b^3\mu}+\frac{ba_9}{2\mu}-\frac{a_{10}}{2b\mu_{out}}+\frac{\kappa_{out}a_{10}}{2b\mu_{out}}+\frac{a_{12}}{2b^3\mu_{out}}=0 \tag{3.4.43}$$

Solving the above equations for the unknowns using mathematica and simplifying the solution by applying material properties, we have

$$a_1 \to \frac{5b^2\sigma o}{a^2+28b^2}$$

$$a_2 \to \frac{24a^2(a^2-b^2)b^2\sigma o}{a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8}$$

$$a_3 \to \frac{5b^2(14a^6-9a^4b^2+75b^6)\sigma o}{-a^8+120a^6b^2-150a^4b^4+60a^2b^6+675b^8}$$

$$a_4 \to \frac{4a^2b^2(a^6+15b^6)\sigma o}{a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8}$$

$$a_5 \to \frac{14b^2\sigma o}{3a^2+84b^2}$$

$$a_6 \to \frac{20a^2(a^2-b^2)b^2\sigma o}{a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8}$$

$$a_7 \to \frac{4a^4b^4(a^4+15b^4)\sigma o}{a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8}$$

$$a_8 \to \frac{2a^2b^2\sigma o}{3a^2+84b^2}$$

$$a_9 \to -\frac{12(8a^6b^2-5a^4b^4+45b^8)\sigma o}{a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8}$$

$$a_{10} \to -\frac{9b^2(a^8 + 8a^6b^2 - 10a^4b^4 + 20a^2b^6 + 45b^8)\sigma o}{2(a^8 - 120a^6b^2 + 150a^4b^4 - 60a^2b^6 - 675b^8)}$$

$$a_{11} \to \frac{b^2(a^2 - 28b^2)\sigma o}{6(a^2 + 28b^2)}$$

$$a_{12} \to -\frac{b^4(9a^8 + 72a^6b^2 - 10a^4b^4 + 100a^2b^6 - 405b^8)\sigma o}{2(a^8 - 120a^6b^2 + 150a^4b^4 - 60a^2b^6 - 675b^8)}$$

$$b_1 \to 0, b_2 \to 0, b_3 \to 0, b_4 \to 0, b_5 \to 0, b_6 \to 0$$

$$b_7 \to 0, b_8 \to 0, b_9 \to 0, b_{10} \to 0, b_{11} \to 0, b_{12} \to 0 \qquad (3.4.44)$$

Substituting the above solutions into equations (3.4.4) and (3.4.5), we get the final stress equations for the inner circular inclusion as

$$\sigma_{rrin} = \frac{(2b^2\sigma o(5(a^8 - 120a^6b^2 + 150a^4b^4 - 60a^2b^6 - 675b^8)}{(a^2 + 28b^2)(a^8 - 120a^6b^2 + 150a^4b^4 - 60a^2b^6 - 675b^8)} +$$

$$\frac{4(14a^8 + 383a^6b^2 - 252a^4b^4 + 75a^2b^6 + 2100b^8)\cos 2\theta))}{(a^2 + 28b^2)(a^8 - 120a^6b^2 + 150a^4b^4 - 60a^2b^6 - 675b^8)}$$

$$\sigma_{\theta\theta in} = \frac{-(2b^2\sigma o(-5(a^8 - 120a^6b^2 + 150a^4b^4 - 60a^2b^6 - 675b^8)}{(a^2 + 28b^2)(a^8 - 120a^6b^2 + 150a^4b^4 - 60a^2b^6 - 675b^8)} +$$

$$\frac{4(a^2 + 28b^2)(14a^6 + 75a^6b^6 + 36a^2b^2r^2 - 9a^4(b^2 + 4r^2))(\cos 2\theta))}{(a^2 + 28b^2)(a^8 - 120a^6b^2 + 150a^4b^4 - 60a^2b^6 - 675b^8)}$$

$$\tau_{r\theta in} = -\frac{8b^2\left(14a^6+75a^6b^6+18a^2b^2r^2-9a^4\left(b^2+2r^2\right)\right)\sigma o \sin 2\theta}{a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8} \qquad (3.4.45)$$

Substituting the above solutions into equations (3.4.7) and (3.4.8), we get the final stress equations for the outer circular inclusion as

$$\sigma_{rr} = \frac{-(2b^2\sigma o(-(a^8 - 120a^6b^2 + 150a^4b^4 - 60a^2b^6 - 675b^8)r^2(a^2 + 14r^2)}{3(a^2 + 28b^2)(a^8 - 120a^6b^2 + 150a^4b^4 - 60a^2b^6 - 675b^8)} +$$

$$\frac{6(a^2 + 28b^2)(-60a^2b^6r^2 - 24a^6r^4 - 135b^6r^4 + a^8(3b^2 - 4r^2) + 15a^4b^2(3b^4 + r^4))\cos 2\theta))}{3(a^2 + 28b^2)(a^8 - 120a^6b^2 + 150a^4b^4 - 60a^2b^6 - 675b^8)}$$

$$\sigma_{\theta\theta}=\frac{(2b^2\sigma o(-(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^2(a^2-14r^2)}{3(a^2+28b^2)(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4}+$$

$$\frac{18(a^2+28b^2)(a^8b^2-8a^6r^4-45b^6r^4-20a^2b^2r^6+5a^4(3b^6+b^2r^4+4r^6))(\cos 2\theta))}{3(a^2+28b^2)(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4}$$

$$\tau_{r\theta}=-\frac{4b^2(24a^6r^4+13545b^6r^4+a^8(3b^2-2r^2)-30a^2b^2r^2(b^4-r^4)}{(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4}+$$

$$\frac{15a^4(3b^6-b^2r^4-2r^6)\sigma o\sin 2\theta}{(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4} \quad (3.4.46)$$

Substituting the above solutions into equations (3.3.10) and (3.3.11), we get the final stress equations for the infinite matrix surrounding the concentric circular inclusions as

$$\sigma_{rrout}=\frac{-(\sigma o((a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^2(-28b^2(b^2-3r^2)+a^2(b^2+3r^2))}{6(a^2+28b^2)(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4}$$

$$-\frac{3(a^2+28b^2)(135b^8(9b^4-12b^2r^2-5r^4)+60a^2b^6(5b^4-12b^2r^2-r^4)}{6(a^2+28b^2)(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4}+$$

$$\frac{a^8(27b^4-36b^2r^2+r^4)+24a^6(9b^6-12b^4r^2-5b^2r^4)-30a^4(b^8-12b^6r^2-5b^4r^4))\cos 2\theta))}{6(a^2+28b^2)(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4}$$

$$\sigma_{\theta\theta out}=\frac{(\sigma o(-(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^2(a^2(b^2-3r^2)-28(b^4+3b^2r^2))}{6(a^2+28b^2)(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4}$$

$$+\frac{3(a^2+28b^2)(135b^8(9b^4-5r^4)+60a^2b^6(5b^4-r^4)}{6(a^2+28b^2)(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4}+$$

$$\frac{a^8(27b^4+r^4)+24a^6(9b^6-5b^2r^4)-30a^4(b^8-5b^4r^4))\cos 2\theta))}{6(a^2+28b^2)(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4}$$

$$\tau_{r\theta out}=\frac{((-60a^2b^6(5b^4-6b^2r^2+r^4)+a^8(-27b^4+18b^2r^2+r^4)-135b^8(9b^4-6b^2r^2+5r^4)}{2(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4}$$

$$-\frac{24a^6(9b^6-6b^4r^2+5b^2r^4)+30a^4(b^8-6b^6r^2+5b^4r^4))\sigma o\sin 2\theta}{2(a^8-120a^6b^2+150a^4b^4-60a^2b^6-675b^8)r^4} \quad (3.4.47)$$

The above equations can be verified by substituting them into the equilibrium equations.

Example Problem:

Finding stresses for a thin infinite plate made of material M_1 with two concentric circular inclusions with the inner circular inclusion M_2 of radius $a = 2mm$ and the outer circular inclusion M_3 of radius $b = 4mm$ subjected to a far field tensile loading of 100 N/mm^2.

Material properties M_1, M_2, M_3 are given below.

For material M_1, Shear modulus $\mu = 1$ and $\kappa = 1$.

For material M_2, Shear modulus $\mu = 4$ and $\kappa = 5$.

For material M_3, Shear modulus $\mu = 10$ and $\kappa = 10$.

Substituting the above values in the stress and displacement components derived above, we get

$$\sigma_{rrin} = 35.398 - 87.509\ \cos 2\theta$$

$$\sigma_{\theta\theta in} = 35.398 + (87.509 + 0.495r^2)\cos 2\theta$$

$$\tau_{r\theta in} = (87.509 + 0.247r^2)\sin 2\theta$$

$$\sigma_{rr} = \frac{9.439}{r^2} + 33.038 + \left(\frac{423.738}{r^4} - \frac{140.806}{r^2} - 78.79\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{9.439}{r^2} + 33.038 + \left(\frac{423.738}{r^4} + 78.79 + 0.412r^2\right)\cos 2\theta$$

$$\tau_{r\theta} = \left(\frac{423.738}{r^4} - \frac{70.033}{r^2} + 78.79 + 0.206r^2\right)\sin 2\theta$$

$$\sigma_{rrout} = 50 - \frac{261.94}{r^2} + \left(\frac{24223.655}{r^4} - \frac{2088.968}{r^2} - 50\right)\cos 2\theta$$

$$\sigma_{\theta\theta out} = 50 + \frac{261.94}{r^2} + \left(50 - \frac{24223.655}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta out} = \left(\frac{24223.655}{r^4} - \frac{1044.484}{r^2} + 50\right)\sin 2\theta$$

Converting the above equations into Cartesian coordinate system and plotting the graphs for various stress components.

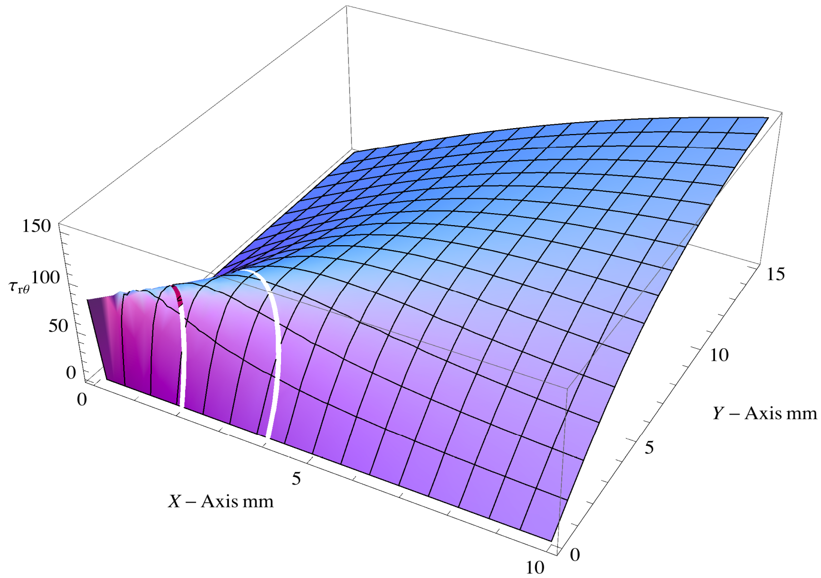

Figure 3.5 *Variation of shear stress along the plate, 1st quadrant view.*

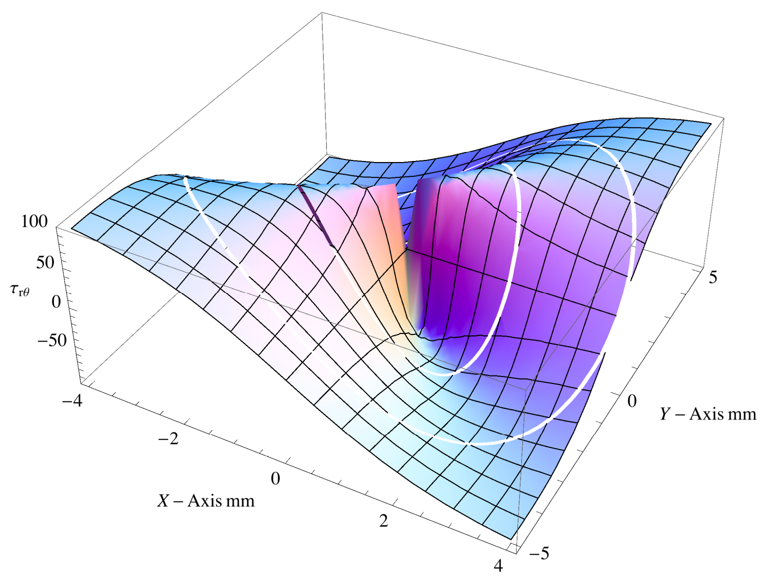

Figure 3.6 *Variation of shear stress along the plate*

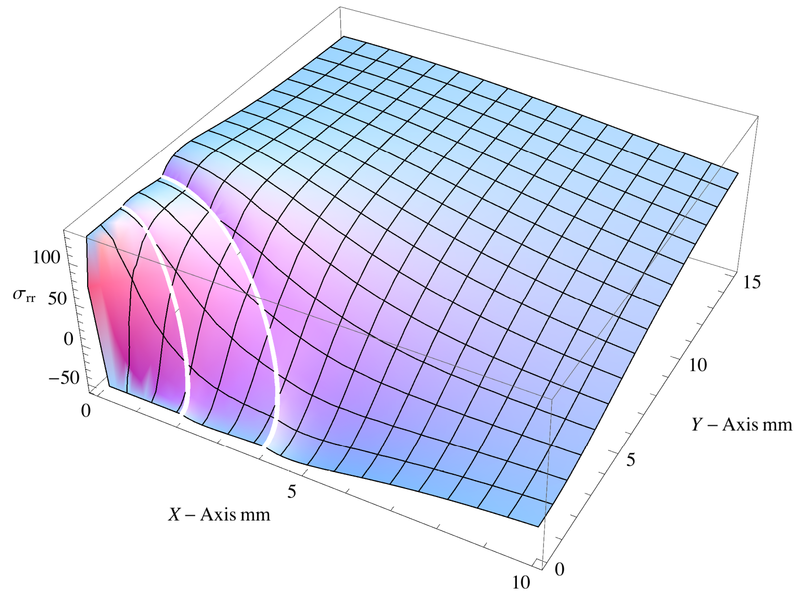

Figure 3.7 *Variation of stress along the plate, 1st quadrant view.*

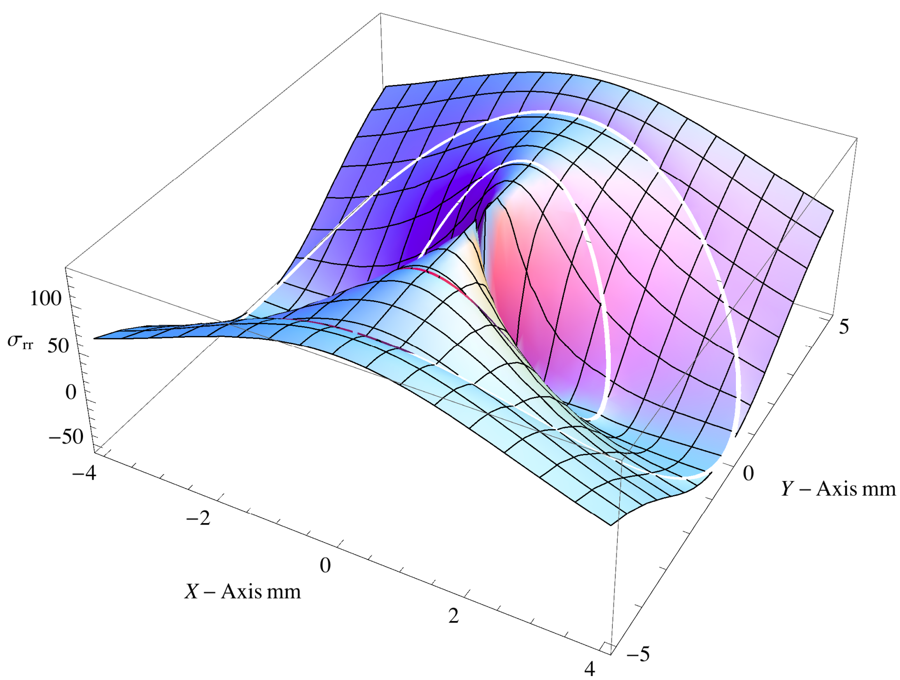

Figure 3.8 *Variation of stress along the plate*

CHAPTER 4

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

This research demonstrated the application of Airy stress function in solving elasticity problems. It demonstrated the methodology in determining the stress fields and displacements at any point on a two dimensional plate considering various domains subjected to different boundary conditions. Earlier, research has been carried out on derivation of airy stress function for a two dimensional plate with a circular inclusion subjected to far field tensile loading.

The thesis presented now derives an Airy stress function for a three phase plate with a concentric circular inclusion subjected to a tensile loading at far field and is successful in finding the stress fields and displacements at any point on the matrix. This work has been proposed earlier [Christen & Lo] but could not be computed due to lack of symbolic software. Using mathematica, the simultaneous system of equations with unknown variables has been solved successfully. Likewise the equations of equilibrium and continuity equations have been satisfied for the assumed airy stress function. This search stands out from the previous work done by other students till date. The earlier research work would be a special case of this thesis topic by equating the radii of the concentric circular inclusions.

From the graphical representation of the result, it can be seen that shear stress is zero along the x and y axes. It can also be observed that the shear stresses within the inclusions are constant respectively. The maximum tensile stress occurs at the interface of the concentric inclusion and the infinite matrix intersecting the y-axis and decreases along the y-axis further into the infinite matrix. The maximum compressive stress occurs at the boundary intersecting with the x-axis and decreases as it nears the y-axis along the interfacing boundary.

Recommended future work on this thesis topic would be

- Generalizing boundary conditions to include all the stress components at far field.
- Considering multi-layer circular inclusions with centers offset.
- Considering the shape of the inclusion to be elliptic.
- Concentric elliptical inclusions.

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BIOGRAPHICAL INFORMATION

Abhishek Kunchala was born in Hyderabad, Andhra Pradesh, India, in 1986. He received his Bachelor’s degree in Mechanical Engineering from Jawaharlal Nehru Technological University, Hyderabad, India, in 2008. He joined University of Texas at Arlington to pursue his graduate studies in Mechanial Engineering in the year 2009. His research interests include Solid Mechanics, Fracture Mechanics and Control Systems.