ON THE EXISTENCE OF TOTALLY REFLEXIVE MODULES

by

KRISTEN ANN BECK

Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

May 2011

You can't direct the wind, but you can adjust the sails.

– Anonymous

To my teachers, for adjusting the sails.

ACKNOWLEDGEMENTS

It is my pleasure, first and foremost, to thank my thesis advisor, Dave Jorgensen, for his continued support and guidance during my years at UT Arlington. He has not only trained me to be the mathematician that I am, but he has also instilled in me the self-confidence I needed to make it this far.

I am grateful to the entire Math Department at UT Arlington for its much-needed help along the way. In particular, I owe thanks to the many professors who have served as inspiration. I am especially grateful, though, to the members of my committee: Minerva Cordero, Ruth Gornet, Barbara Shipman, and Michaela Vancliff. Over the years, they have each taught me invaluable lessons, both mathematical and otherwise.

For the many friends and family members that have supported my efforts to continually push against the grain, I am so very thankful. Their words of encouragement will never be forgotten. Most of all, however, I am indebted to Tanya for her love, support, and super-human patience at the end of each day.

April 11, 2011

ABSTRACT

ON THE EXISTENCE OF TOTALLY REFLEXIVE MODULES

KRISTEN ANN BECK, Ph.D.

The University of Texas at Arlington, 2011

Supervising Professor: David A. Jorgensen

In this manuscript, we investigate the existence of non-free totally reflexive modules over two classes of commutative local (Noetherian) rings.

First, we demonstrate existence over a class of local rings which are defined by Gorenstein homomorphisms. Among the corollaries to this result, we recover a theorem of Avramov, Gasharov, and Peeva [12] concerning the existence of non-free totally reflexive modules over local rings with embedded deformations. We also give a general construction for a class of local rings which satisfy the hypotheses of our theorem, and we show it is able to produce rings without embedded deformations.

The second focus of this work is to give necessary conditions for the existence of a non-free totally reflexive module with a Koszul syzygy over a local ring for which the fourth power of the maximal ideal vanishes. We characterize the Hilbert series of such a ring in terms of the Betti sequence of the module. These characterizations extend similar results of Yoshino [50] concerning the same existence question over local rings for which the cube of the maximal ideal is zero. In particular, we consider necessary conditions for the existence of certain asymmetric complete resolutions, which are known to exist by work of Jorgensen and Şega [31].

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Auslander and Bridger first introduced totally reflexive modules in the late 1960s as a tool for defining Gorenstein dimension [2, 5], referring to such modules as having G-dimension zero. It was not until 2002 that Avramov and Martsinkovsky [13] coined the modern name. Indeed, by calling the modules 'totally reflexive,' one emphasizes the fact that these modules form a certain subclass of the reflexive modules. Totally reflexive modules have important applications even away from their connection to Gorenstein dimension. However, in order to sufficiently understand how totally reflexive modules fit into the bigger picture, it is wise to first view them in light of this connection.

As it turns out, Gorenstein dimension is a natural generalization of projective dimension. That is, the G-dimension of any module is always bounded above by its projective dimension, with equality holding only for modules of finite projective dimension. This bound, which is easily justified by the fact that every finitely generated module is (trivially) totally reflexive, allows for the generalization of two famous results.

First of all, recall that the Auslander-Buchsbaum formula [7, Theorem 3.7] characterizes the (finite) projective dimension of a finitely generated module over a local ring as the difference between the depth of the ring and that of the module. More general than this, the Auslander-Bridger formula [5, Theorem 4.13] provides a refinement which moreover strengthens the Auslander-Buchsbaum formula: it asserts that the (finite) G-dimension of a module is characterized as the same difference.

Another generalization afforded by Gorenstein dimension is through the classification of Gorenstein local rings. Specifically, a local ring is Gorenstein if and only if each of its finitely generated modules has finite G-dimension; in fact, it is enough to know that the residue field possesses this property. This result, due to Auslander [2], extends the famous characterization of regular local rings by Auslander and Buchsbaum [6] and Serre [42], which is given analogously in terms of projective dimension.

These interesting generalizations might cause one to ask whether the results of classical homological algebra can be stated analogously in terms of Gorenstein dimension; that is, by using resolutions by totally reflexive modules in place of those by projective modules. This question was first asked by Holm in [26]. The area of research which employs this technique — wherein projective resolutions are replaced by resolutions by 'more general' modules — is known as relative homological algebra, and was pioneered by Enochs and Jenda [21, 22].

Aside from their connection to Gorenstein dimension, totally reflexive modules are also significantly useful in their own right. Take, for example, the fact that any non-trivial totally reflexive R-module M is naturally equipped with a 'doubly-infinite minimal free resolution', call it $\mathbf{C}$ (cf. Section 2.4 for details). By using such a resolution in place of an arbitrary minimal free resolution of a module, one can construct the so-called Tate (co)homology groups.

$$\widehat{\mathrm{Ext}}^i_R(M,N) := \mathrm{H}^i\left(\mathrm{Hom}_R(\mathbf{C},N)\right)$$
$$\widehat{\mathrm{Tor}}^i_R(M,N) := \mathrm{H}_i\left(\mathbf{C}\otimes_R N\right)$$

As is clear from the construction, Tate (co)homology provides 'twice as much' information as does absolute (co)homology, in the sense that the resulting groups are $\mathbb{Z}$-graded (cf. [13] for details).

Quite possibly the most influential application of totally reflexive modules in current research is in regards to characterization of simple singularities among arbitrary complete local rings. Auslander proved in [3] that every complete Cohen-Macaulay local ring which has only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules is an isolated singularity. Furthermore, this approach has been generalized to the level of Gorenstein representation type: in [19], Christensen, Piepmeyer, Striuli, and Takahashi demonstrate that if a complete local ring has a finite (but positive) number of isomorphism classes of non-free indecomposable totally reflexive modules, then the ring is a simple singularity. In fact, their result generalizes the results of Auslander, Huneke, Leuschke, and Wiegand for rings of finite Cohen-Macaulay type [4, 28, 34], and this generalization reinforces the similarity in behavior between a totally reflexive module over an arbitrary local ring and a maximal Cohen-Macaulay module over a Gorenstein local ring.

One very transparent class of totally reflexive modules is the class of those that arise in the wake of an exact pair of zero divisors, cf. [24]. While such totally reflexive modules may seem trite, their existence implies the existence of infinite families of non-isomorphic non-free indecomposable totally reflexive modules. This fact was recently shown by Holm [27] in the case that the exact pair of zero divisors can be extended to a regular sequence of length three, and it was then significantly extended by Christensen, Jorgensen, Rahmati, Striuli, and Wiegand [18], who show that such infinite families of totally reflexive modules can be constructed so as to contain, for any $n \in \mathbb{N}$, a module which is minimally generated by n elements. They furthermore demonstrate that over an algebraically closed residue field, one can construct, for a fixed $n \in \mathbb{N}$, an infinite family of pairwise non-isomorphic non-free totally reflexive modules which are each minimally n-generated.

Of course, a much broader question would be the following: Does the existence of one non-trivial totally reflexive module necessarily imply the existence of infinitely many isomorphism classes of non-projective indecomposable totally reflexive modules? Certainly, an answer to this question has more than existential significance, as seen in [19]. In particular, Takahashi obtains an affirmative answer to this question in [46], given the existence of a sufficient prime ideal, and he furthermore shows that the number of resulting isomorphism classes is uncountable. Moreover, the authors of [19] also give an affirmative answer over a complete local ring. However, the proofs contained in [46] and [19] are not constructive; the only constructive results at this point are from [27] and [18].

Indeed, the above discussion points to the sufficiency and importance of characterizing rings which admit *at least* one non-trivial totally reflexive module. This question is quickly answered over non-regular Gorenstein rings; in this case, the totally reflexive and maximal Cohen-Macaulay modules coincide. It is away from this landscape of Gorenstein rings that non-trivial totally reflexive modules are more difficult to pin down. In fact, Avramov and Martsinkovsky show in [13] that over a non-Gorenstein ring which is Golod, every totally reflexive module is trivial; specifically, this includes the class of Cohen-Macaulay rings of minimal multiplicity. Moreover, it is a simple exercise to show that local rings for which the square of the maximal ideal is zero share the same property.

However, the literature also contains positive results as to the existence of non-trivial totally reflexive modules over non-Gorenstein rings. In some sense, the structurally simplest Artinian local rings which satisfy this property are characterized by the vanishing of the cube of the maximal ideal. In [50], Yoshino characterizes the Hilbert series, among other invariants, of these rings. Specifically, he shows that the admittance of a non-free totally reflexive module over such a local ring necessarily implies that the ring is a Koszul algebra. Furthermore, Takahashi and Watanabe

[47] use a geometric approach to construct totally reflexive modules from smooth projective curves of genus at least two. But perhaps the largest class of (possibly non-Gorenstein) rings which admit non-trivial totally reflexive modules is the class of local rings which have an embedded deformation. These rings, which can be realized as the quotient of local rings by regular sequences contained in the square of the maximal ideal, are actually shown to more generally admit modules of finite complete intersection dimension by Avramov, Gasharov, and Peeva in [12]. In particular, the results of [12] and [50] are motivating factors for the results herein.

This thesis is a compilation of two of my papers [14, 15]; therefore, its contents are divided accordingly so. Chapter 2 serves as a basic review of concepts and definitions that will be used throughout. Excellent sources for additional background material are [16], [37], and [49]. Results and examples are contained in the remaining chapters. It should furthermore be noted that the results in Chapter 4 do not rely on those of Chapter 3.

In Chapter 3, we discuss the existence of totally reflexive modules via certain Gorenstein homomorphisms. Our main result of the chapter is motivated by a result of Avramov, Gasharov, and Peeva [12, Theorem 3.2], which establishes the existence of totally reflexive modules over rings with embedded deformations; such rings are defined and discussed in Section 3.1. In Section 3.2, we establish the main result, stated in Theorem 3.2.3, and list its corollaries, among which we demonstrate the ability to recover [12, Theorem 3.2]. We furthermore demonstrate the novelty of our result by concluding the chapter with a general construction, in Section 3.3, of rings which admit non-trivial totally reflexive modules by virtue of Theorem 3.2.3, but do not have embedded deformations. Section 3.4 concludes the chapter by providing methods which detect whether a local ring has an embedded deformation.

Finally, Chapter 4 is concerned with necessary conditions for the existence of totally reflexive modules over so-called short local rings — that is, local rings for which the fourth power of the maximal ideal is zero. The results are motivated not only by the work of Yoshino in [50], but also by an example produced by Jorgensen and Şega [31] which illustrates the existence of asymmetric complete resolutions over short local rings. Since, in both of these works, the respective totally reflexive modules are found to have linear complete resolutions, we begin the chapter with a discussion, in Section 4.2, of the Poincaré series of finitely generated modules with linear resolutions which possess a certain duality property. The characterization of the Poincaré series provided in this section makes up the framework for the results in the sequel of the chapter. Specifically, we are able to characterize, in Section 4.4, the Hilbert series of a short local ring which admits a finitely generated module, possessing the aforementioned duality property, and admitting a linear resolution. We furthermore give conditions for the existence of certain acyclicity in Section 4.4.3.

CHAPTER 2

PRELIMINARY CONCEPTS

Throughout, R shall denote a commutative Noetherian ring and M a finitely generated R-module. The notation $(R, \mathfrak{m}, k)$ will indicate that R is also local, with unique maximal ideal $\mathfrak{m}$ and residue class field $k := R/\mathfrak{m}$.

2.1 Homological Techniques

The study of totally reflexive modules is rich in the utilization of homological algebra. In this section, we will outline merely the concepts needed in the sequel. For a more thorough treatment of the subject, one should consult [41] or [49].

2.1.1 Free Resolutions

Recall that a sequence of R-module homomorphisms

$$\mathbf{C}: \quad \cdots \to C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots$$

is called a *chain complex* if $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i$ for each $i \in \mathbb{Z}$. With the convention that $C^i = C_{-i}$ for each $i \in \mathbb{Z}$, one can analogously define the concept of a *cochain complex*

$$\mathbf{C}: \quad \cdots \to C^{n-2} \xrightarrow{\partial^{n-2}} C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \xrightarrow{\partial^{n+1}} C^{n+2} \to \cdots$$

of R-modules. Note that any chain complex can be viewed as a cochain complex, and vice versa. Except for certain cases, we shall work with chain complexes.

Whenever $\operatorname{im} \partial_{i+1} = \ker \partial_i$ for each $i \in \mathbb{Z}$, the comlplex $\mathbf{C}$ is said to be *acyclic*, or an *exact sequence*. In general, however, the failure of the complex $\mathbf{C}$ to be exact at

C_i is captured by the non-vanishing of the R-module $\ker \partial_i / \operatorname{im} \partial_{i+1}$. This fact leads to the concept of *(co)homology*. Specifically, the homology (resp. cohomology) of $\mathbf{C}$ is defined to be the graded R-module given by

$$\begin{aligned}\mathrm{H}(\mathbf{C}) &= \bigoplus_{i\in\mathbb{Z}} \mathrm{H}_i(\mathbf{C}) := \bigoplus_{i\in\mathbb{Z}} \ker \partial_i / \operatorname{im} \partial_{i+1} \\ &= \bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(\mathbf{C}) := \bigoplus_{i\in\mathbb{Z}} \ker \partial^{i-1} / \operatorname{im} \partial^{i}\end{aligned}$$

respectively. Clearly, $\mathbf{C}$ is exact if and only if $\mathrm{H}(\mathbf{C}) = 0$. Further, $\mathbf{C}$ is said to be *exact at n* if $\mathrm{H}_n(\mathbf{C}) = 0$ (resp. if $\mathrm{H}^n(\mathbf{C}) = 0$).

Of all the various uses of exact sequences in the study of homological algebra, perhaps the most fundamental is the utility of resolutions. In particular, given a finitely generated R-module M, an exact sequence of R-modules of the form

$$\mathbf{F}: \quad \cdots \to F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

where F_i is free for each $i \in \mathbb{N}$, is called a *free resolution* of M over R and is often denoted (F_i, ∂_i). For reasons that will become apparent later, we often omit the module M when expressing its free resolution; such a resolution is then called *deleted*. Clearly, such a resolution of M is not right exact; in fact, its homology is concentrated in degree zero and is isomorphic to M. When the notation $\mathbf{F} \to M \to 0$ is used, it shall be assumed that $\mathbf{F}$ is a free resolution of M.

Remark 2.1.1.1. One can more generally define a *projective* resolution of a module. However, the results contained in this manuscript are specific to Noetherian local rings: a setting in which projective and free modules coincide. We therefore choose to work with the more user-friendly of the two.

It is easy to see that every R-module possesses a free resolution. Moreover, Nakayama's Lemma (cf. [37, Theorem 2.2]) guarantees that over a local ring, one can always choose such a resolution in a 'minimal' way. In fact, the concept of minimality applies in more generality to complexes of free modules.

Definition 2.1.1.2. Suppose that

$$\mathbf{C}: \quad \cdots \to C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots$$

is a complex of free modules over a local ring $(R, \mathfrak{m})$. If $\partial_i(C_{i+1}) \subseteq \mathfrak{m}C_i$ for each $i \in \mathbb{Z}$, then $\mathbf{C}$ is said to be *minimal*. Equivalently, $\mathbf{C}$ is minimal if and only if, for each $i \in \mathbb{Z}$, there are no units among the entries of the matrix representing ∂_i.

Given a free resolution $\mathbf{F} = (F_i, \partial_i)$ of an R-module M, it is easy to see that, for each $n \in \mathbb{N}$, the map ∂_n naturally factors through the kernel of ∂_{n-1}. In fact, assuming that $\mathbf{F}$ is also minimal, the R-module defined by

$$\Omega_n(M) := \ker \partial_{n-1} = \operatorname{im} \partial_n$$

is called the nth *syzygy module* of M, and is simply written Ω_n when the module is understood. With this definition, notice that $\Omega_0(M) = M$. Furthermore, we may now represent $\mathbf{F}$ in the following manner.

$$\begin{tikzcd}
\mathbf{F}: & \cdots \arrow[r] & F_3 \arrow[rr, "\partial_3"] \arrow[dr] & & F_2 \arrow[rr, "\partial_2"] \arrow[dr] & & F_1 \arrow[rr, "\partial_1"] \arrow[dr] & & F_0 \arrow[r] & M \arrow[r] & 0 \\
& & & \Omega_3 \arrow[ur, hook] & & \Omega_2 \arrow[ur, hook] & & \Omega_1 \arrow[ur, hook] & & &
\end{tikzcd}$$

Remark 2.1.1.3. The concept of a syzygy module is not unique to free resolutions, but may analogously be defined for acyclic complexes. The nth syzygy module of an acyclic complex $\mathbf{C}$ is often denoted $\Omega_n(\mathbf{C})$.

In essence, the process of constructing a minimal free resolution of a module yields information about how far away the module is from being free. If it is possible to resolve an R-module M in terms of free modules in a finite number of steps, then we say that M has finite projective dimension, and write $\operatorname{pd}_R M < \infty$. In general, however, the *projective dimension* of an R-module M with minimal free resolution $\mathbf{F}$ is given by

$$\operatorname{pd}_R M = \sup\{n \in \mathbb{N} \mid F_n \neq 0\}$$

Since the length of a minimal free resolution of an R-module M is unique, the above quantity is well-defined. It is easy to see that any free R-module has projective dimension zero. Furthermore, $\operatorname{pd}_R M$ is finite if and only if there exists $n \in \mathbb{N}$ such that $\Omega_n(M)$ is a (non-zero) free module. In this case, $\operatorname{pd}_R M = n$.

Over a local ring, the minimal free resolution of a finitely generated module is unique up to isomorphism of chain complexes (see [49] for details). One important consequence of this fact that is that the ranks of the free modules in any minimal free resolution are well-defined. Specifically, for a finitely generated module M over a local ring R with minimal free resolution $\mathbf{F} \to M \to 0$, we define

$$\beta_n^R(M) := \operatorname{rank} F_n$$

to be the nth *Betti number* of M. When there is no risk of confusion as to the ring (or the module), we write $\beta_n(M)$ (or simply β_n). It is common to express the sequence of Betti numbers of an R-module M in the form of a power series

$$P_M^R(t) := \sum_{i \in \mathbb{N}} \beta_i^R(M)\, t^i \in \mathbb{Z}[\![t]\!]$$

called the *Poincaré series* of M. Furthermore, one often speaks of the Poincaré series of the residue field of a local ring $(R, \mathfrak{m}, k)$ as being the Poincaré series of the ring itself. That is,

$$P_R(t) := P_k^R(t).$$

In general, the Poincaré series of a module with finite projective dimension is merely a polynomial. For an arbitrary R-module M, however, it is not necessarily even true that $P_M^R(t)$ is a rational function; cf. [9, 4.3.10]. For our uses, the Poincaré series will be utilized extensively in Chapter 4.

2.1.2 Ext and Tor

In order to study the properties of a given module over a Noetherian ring via homological methods, one 'replaces' the module with its free resolution. This approach is advantageous for two reasons. First of all, free modules are often more desirable objects with with to work. Furthermore, by representing an R-module with its free resolution, we are able to consider the action of the derived functors Ext and Tor on the given module.

To this end let L, M and N be R-modules, and suppose that $\mathbf{F} = (F_i, \partial_i)$ is a deleted free resolution of M. Recall that we have induced complexes of R-modules

$$\operatorname{Hom}_R(\mathbf{F}, N): \quad 0 \to \operatorname{Hom}_R(F_0, N) \xrightarrow{\partial_1^*} \operatorname{Hom}_R(F_1, N) \xrightarrow{\partial_2^*} \operatorname{Hom}_R(F_2, N) \to \cdots$$

where $\partial_i^*(f) = f \circ \partial_i$ for each $f : F_{i-1} \to N$ and $i \in \mathbb{N}$, and

$$L \otimes_R \mathbf{F}: \quad \cdots \to L \otimes_R F_2 \xrightarrow{1 \otimes \partial_2} L \otimes_R F_1 \xrightarrow{1 \otimes \partial_1} L \otimes_R F_0 \to 0$$

where $1 \otimes \partial_i$ acts by $\ell \otimes x \mapsto \ell \otimes \partial_i(x)$ for each $i \in \mathbb{N}$.

Since $\mathbf{F}$ is not exact (recall that it is deleted), there is no reason to believe that either of the above complexes will be (right or left) exact. Indeed, this is often the case, and we define the following derived functors to quantify the respective inexactness.

Definitions 2.1.2.1. Let L, M and N be R-modules, and suppose that $\mathbf{F} \to M \to 0$ is a deleted projective resolution. Then we define the (co)homology functors

$$\operatorname{Ext}_R^i(M, N) := \mathrm{H}^i\left(\operatorname{Hom}_R(\mathbf{F}, N)\right)$$

$$\operatorname{Tor}_i^R(L, M) := \mathrm{H}_i\left(L \otimes_R \mathbf{F}\right)$$

from the category of R-modules to itself, for all $i \in \mathbb{N}$.

Remark 2.1.2.2. One can also calculate $\operatorname{Ext}^i_R(M,N)$ by taking the ith cohomology of the complex $\operatorname{Hom}_R(M,\mathbf{I})$, for any injective resolution $0 \to N \to \mathbf{I}$. Likewise, $\operatorname{Tor}^R_i(L,M)$ can be determined by the ith homology of the complex $\mathbf{G} \otimes_R M$ for any projective resolution $\mathbf{G} \to L \to 0$.

The utility of Ext and Tor for encoding information about rings and their modules is valuable, to say the least. We illustrate this fact by collecting several very basic properties of Ext and Tor.

Facts 2.1.2.3. *Let M be a finitely generated R-module.*

(1) $\operatorname{Ext}^0_R(M,N) \cong \operatorname{Hom}_R(M,N)$ *for any R-module N.*

(2) $\operatorname{Tor}^R_0(L,M) \cong L \otimes_R M$ *for any R-module L.*

(3) $\operatorname{Ext}^i_R(M,N) = 0$ *for all $i > 0$ and all R-modules N if M is projective.*

(4) $\operatorname{Tor}^R_i(L,M) = 0$ *for all $i > 0$ and all R-modules L if M is projective.*

(5) *If R is local,* $\operatorname{pd}_R M = \sup\{i \mid \operatorname{Ext}^i_R(M,k) \neq 0\} = \sup\{i \mid \operatorname{Tor}^R_i(M,k) \neq 0\}$

Remarks 2.1.2.4.

(1) We can say a bit more about (3) and (4) above. Specifically, it is also true that $\operatorname{Ext}^i_R(M,N) = 0$ (resp. $\operatorname{Tor}^R_i(L,M) = 0$) for all $i > 0$ and all R-modules M if N is injective (resp. if L is flat).

(2) The characterization of projective dimension in (5) above follows from the fact that, given any complex $\mathbf{C}$ of modules over a local ring R, the differentials of the induced complexes $\mathbf{C} \otimes_R k$ and $\operatorname{Hom}_R(\mathbf{C},k)$ are trivial. Therefore, letting $\mathbf{F} \to M \to 0$ be a minimal free resolution over R, we are able to alternately define the nth Betti number of M over R in either one of the following manners.

$$\beta_n(M) := \operatorname{rank} F_n = \dim_k \operatorname{Ext}^n_R(M,k) = \dim_k \operatorname{Tor}^R_n(M,k)$$

Many times, the usefulness of the Ext and Tor modules is manifested in their vanishing. Such is the case in this manuscript. As we shall see in Section 2.4, a totally reflexive R-module M is defined, in part, by the vanishing of $\operatorname{Ext}_R^i(M,R)$ for all $i>0$. Given a minimal free resolution $\mathbf{F}=(F_i,\partial_i)$ of M over R, these particular Ext modules measure the inexactness of the dual complex $\operatorname{Hom}_R(\mathbf{F},R)$. That is, since $\operatorname{Hom}_R(F,R)\cong F$ for any free R-module F, we have

$$\operatorname{Hom}_R(\mathbf{F},R): \quad 0 \to F_0 \xrightarrow{\partial_1^\intercal} F_1 \xrightarrow{\partial_2^\intercal} F_2 \xrightarrow{\partial_3^\intercal} F_3 \to \cdots$$

up to isomorphism of chain complexes, where we have slightly abused notation by writing $\partial_i^\intercal$ to represent the transpose of the matrix which represents ∂_i.

Another concept in which the vanishing of certain (co)homology modules plays a role is that of *flat dimension*. While this homological dimension is usually best understood as the shortest length of a resolution of a module by flat modules, it can also be defined in the following way.

$$\operatorname{fd}_R M = \sup\{n \in \mathbb{N} \mid \operatorname{Tor}_n^R(N,M) \neq 0 \text{ for all } R\text{-modules } N\}$$

Since projective (and therefore free) modules are necessarily flat, it follows that flat dimension is always bounded above by projective dimension. However, over a Noetherian ring, finitely generated projective and flat modules coincide, and so the respective dimensions do also.

2.2 Graded Rings

The results of Chapter 4 require a bit of knowledge concerning a certain invariant of graded rings: the Hilbert series. In this section, we discuss the concept of a Hilbert series for a local ring. Indeed, the fact that makes such a construction possible is that one can impose a grading upon an arbitrary ring by forming the so-called associated graded ring. We define such a ring, along with its modules and complexes, in the next section.

2.2.1 Associated Graded Objects

While associated graded objects can be defined with respect to any nonzero proper idea of a ring, we shall focus on such constructions with respect to the unique maximal ideal of a local ring $(R, \mathfrak{m}, k)$. In this setting, the construction of associated graded objects exploits the natural $\mathfrak{m}$-adic filtration of R.

$$R \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \mathfrak{m}^3 \supseteq \mathfrak{m}^4 \supseteq \cdots$$

Precisely, the *associated graded ring* of R with respect to $\mathfrak{m}$ is defined by

$$\operatorname{gr}_{\mathfrak{m}}(R) = \bigoplus_{i \in \mathbb{N}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

where we adopt the convention that $\mathfrak{m}^0 = R$. We notice that $\operatorname{gr}_{\mathfrak{m}}(R)_0 \cong k$ and that $\bigoplus_{i \geq 1} \operatorname{gr}_{\mathfrak{m}}(R)_i \cong \mathfrak{m}$, so that $\operatorname{gr}_{\mathfrak{m}}(R)$ defines a standard graded local ring. Since we have assumed that R is Noetherian, it follows that $\operatorname{gr}_{\mathfrak{m}}(R)$ also is. Furthermore, for any R-module M, we can define the *associated graded module* of M with respect to $\mathfrak{m}$ by

$$\operatorname{gr}_{\mathfrak{m}}(M) = \bigoplus_{i \in \mathbb{N}} \mathfrak{m}^i M/\mathfrak{m}^{i+1} M$$

which naturally has the structure of a graded $\operatorname{gr}_{\mathfrak{m}}(R)$-module.

Letting $\mathbf{C} = (C_i, \partial_i)_{i \in \mathbb{Z}}$ denote a minimal complex of R-modules, we can also define the *associated graded complex* $\operatorname{gr}_{\mathfrak{m}}(\mathbf{C})$ of $\mathbf{C}$ with respect to $\mathfrak{m}$. It is given by the family of subcomplexes $\{\operatorname{gr}_{\mathfrak{m}}(\mathbf{C})^j\}_{j \in \mathbb{Z}}$, where

$$\operatorname{gr}_{\mathfrak{m}}(\mathbf{C})^j : \quad \cdots \to \frac{\mathfrak{m}^{j-i-1} C_{i+1}}{\mathfrak{m}^{j-i} C_{i+1}} \xrightarrow{\delta^j_{i+i}} \frac{\mathfrak{m}^{j-i} C_i}{\mathfrak{m}^{j-i+1} C_i} \xrightarrow{\delta^j_i} \frac{\mathfrak{m}^{j-i+1} C_{i-1}}{\mathfrak{m}^{j-i+2} C_{i-1}} \to \cdots$$

such that, for each $\ell \in \mathbb{Z}$, δ_ℓ^j is the map induced on the restricted map $\mathfrak{m}^{j-\ell}C_\ell \to \mathfrak{m}^{j-\ell+1}C_{\ell-1}$, modulo $\mathfrak{m}^{j-\ell+1}C_\ell$. We furthermore maintain the convention that $\mathfrak{m}^n = R$ for $n \leq 0$. Such a construction is certainly possible by the minimality of $\mathbf{C}$.

Remark 2.2.1.1. An associated graded complex is often referred to as being the 'linear part' of a complex precisely because it filters the non-linear behavior from the differentials. As one would imagine, this sort of a construction does not always preserve the exactness of a complex. This concept is illustrated in the following example.

Example 2.2.1.2. Let k be a field and consider the local ring $R = k[x]/(x^3)$ with unique maximal ideal $\mathfrak{m} = (x)$, over which we have the following minimal complete resolution.

$$\mathbf{C}: \qquad \cdots \longrightarrow R \xrightarrow{x^2} R \xrightarrow{x} R \xrightarrow{x^2} R \xrightarrow{x} R \xrightarrow{x^2} R \xrightarrow{x} R \longrightarrow \cdots$$

Passing to the associated graded complex preserves only the linear differentials. Therefore, $\mathrm{gr}_{\mathfrak{m}}(\mathbf{C})$ is represented by the following graded complex

$$\begin{tikzcd}[row sep=tiny]
 & \vdots & 0 & & & & \\
 & \oplus & \oplus & & & & \\
\cdots \arrow[r, "0"] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r, "x"] & \mathfrak{m}^2/\mathfrak{m}^3 \arrow[r] & 0 & & & \\
 & \oplus & \oplus & \oplus & & & \\
0 \arrow[r] & k \arrow[r, "x"] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r, "0"] & \mathfrak{m}^2/\mathfrak{m}^3 \arrow[r] & 0 & & \\
 & \oplus & \oplus & \oplus & \oplus & & \\
 & 0 \arrow[r] & k \arrow[r, "0"] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r, "x"] & \mathfrak{m}^2/\mathfrak{m}^3 \arrow[r] & 0 & \\
 & & \oplus & \oplus & \oplus & \oplus & \\
 & & 0 \arrow[r] & k \arrow[r, "x"] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r, "0"] & \mathfrak{m}^2/\mathfrak{m}^3 \arrow[r] & 0 \\
 & & & \oplus & \oplus & \oplus & \\
 & & & 0 \arrow[r] & k \arrow[r, "0"] & \mathfrak{m}/\mathfrak{m}^2 \arrow[r, "x"] & \cdots \\
 & & & & \oplus & \oplus & \\
 & & & & 0 & \vdots &
\end{tikzcd}$$

which is clearly not exact. In fact, this complex is not even eventually exact. That is, there is no such $n \in \mathbb{Z}$ such that $\mathrm{H}_i(\mathrm{gr}_{\mathfrak{m}}(\mathbf{C})) = 0$ for all $|i| \geq n$.

In particular, if $\mathbf{F} \to M \to 0$ is a minimal free resolution of a module M over a local ring $(R, \mathfrak{m})$, the associated graded complex $\mathrm{gr}_{\mathfrak{m}}(\mathbf{F})$ is a minimal (graded) free resolution of $\mathrm{gr}_{\mathfrak{m}}(M)$ over $\mathrm{gr}_{\mathfrak{m}}(R)$. Modules M for which $\mathrm{gr}_{\mathfrak{m}}(\mathbf{F})$ is exact are the subject of the following definition.

Definitions 2.2.1.3. Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, k)$, and suppose that $\mathbf{F} \to M \to 0$ is a minimal free resolution. The *linearity defect* of M is given by the following quantity

$$\mathrm{ld}_R(M) := \{n \mid \mathrm{H}_i(\mathrm{gr}_{\mathfrak{m}}(\mathbf{F})) = 0 \text{ for all } i \geq n\}$$

whenever it is finite. If $\mathrm{ld}_R(M) = 0$, then $\mathbf{F}$ is said to be a *linear resolution* M, which is then called *Koszul*. However, if $0 < \mathrm{ld}_R(M) < \infty$, then $\mathbf{F}$ is called *eventually linear*. In this case, $\Omega_n(M)$ is a Koszul module. If it turns out that k is a Koszul R-module, then R is said to be a Koszul ring.

In the next section, we introduce a widely studied invariant of graded objects.

2.2.2 Hilbert Series

Let $V = \bigoplus_{i \in \mathbb{N}} V_i$ be a graded vector space over a field k. Then the formal power series given by

$$H_V(t) := \sum_{i \in \mathbb{N}} \dim_k V_i \, t^i$$

is called the *Hilbert series* of V.

In ring theory, the notion of a Hilbert series is certainly applicable to graded rings and their modules. In particular, we will be concerned with the Hilbert series of associated graded rings and modules. For a finitely generated module M over a local ring $(R, \mathfrak{m})$, we will often abuse terminology slightly by referring to the invariants $H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t)$ and $H_{\mathrm{gr}_{\mathfrak{m}}(M)}(t)$ as being the Hilbert series of R and M, respectively. This practice should not cause confusion, as the Hilbert series of a local ring (resp. its module) is widely regarded as the construction with respect to the associated graded ring (resp. module).

The following well-known fact gives a characterization of the Poincaré series of a Koszul ring.

Fact 2.2.2.1. *Let $(R, \mathfrak{m})$ be a local ring. If R is Koszul, then its Poincaré series is given by the following.*

$$P_R(t) = \frac{1}{H_{\mathrm{gr}_{\mathfrak{m}}(R)}(-t)}$$

2.3 Gorenstein Rings

In this section, we discuss concepts which serve to motivate the definition and properties of a Gorenstein local ring. With the exception of the section which immediately follows, we will suppose that all of our rings are local. It is important to note, however, that many of the definitions which follow can analogously be stated in the more general case. For a thorough treatment of such a subject, the reader should consult [16].

2.3.1 Systems of Parameters

For any prime ideal $\mathfrak{p}$ of R, we define the *height* of $\mathfrak{p}$, denoted $\operatorname{ht}\mathfrak{p}$, to be the supremum of the integers h such that there exists a chain

$$\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_2 \supsetneq \cdots \supsetneq \mathfrak{p}_h$$

of strict inclusions of prime ideals of R. The *Krull dimension*, which shall henceforth be referred to as simply the *dimension*, of the ring R is then defined by the following.

$$\dim R := \sup\{\operatorname{ht}\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal of } R\}$$

Likewise, if M is an R-module, we analogously define the dimension of M by

$$\dim_R M := \dim\left(R/\operatorname{Ann}_R M\right).$$

Clearly, $\dim M$ is always bounded above by $\dim R$.

Suppose that $(R, \mathfrak{m})$ is a local ring such that $\dim R = d$. Then a sequence $\mathbf{x} = x_1, \ldots, x_d \in \mathfrak{m}$ is called a *system of parameters* for R if $\dim\left(R/(\mathbf{x})\right) = 0$. (Notice that when $d = \dim R$, it is not possible for a sequence of fewer than d elements to possess such a property.) More generally, though, if $x_{n_1}, \ldots, x_{n_\ell}$ is part of a system of parameters for R, where $\ell \leq \dim R$, then we have $\dim\left(R/(x_{n_1}, \ldots, x_{n_\ell})\right) = \dim R - \ell$. Any ideal of R which is generated by a system of parameters is called a *parameter ideal* of R.

Indeed, if M is a finitely generated R-module with $\dim M = c$, we define a sequence $\mathbf{y} = y_1, \ldots, y_c \in \mathfrak{m}$ to be a system of parameters on M if $M/(\mathbf{y})M$ has finite length. Notice that when R is viewed as a module over itself, the ring- and module-theoretic definitions are equivalent.

The following equivalent conditions, which generalize Krull's principal ideal theorem [37], can be used to characterize a system of parameters for a local ring.

Fact 2.3.1.1. *Let $(R, \mathfrak{m})$ be a local ring of dimension d. The following are equivalent.*

(1) $\mathbf{x} = x_1, \ldots, x_d$ *is a system of parameters for R,*

(2) $\mathfrak{m}$ *is a minimal prime of* $(\mathbf{x})$*,*

(3) $\operatorname{Rad}(\mathbf{x}) = \mathfrak{m}$*,*

(4) $\mathfrak{m}^n \subseteq (\mathbf{x})$ *for some $n \in \mathbb{N}$, and*

(5) $(\mathbf{x})$ *is $\mathfrak{m}$-primary.*

Recall that, for a local ring $(R, \mathfrak{m})$, the minimal number of generators of $\mathfrak{m}$ is called the *embedding dimension* of R, and is denoted $\operatorname{embdim} R$. By virtue of the above Fact, it is straightforward to see that $\dim R \leq \operatorname{embdim} R$. The following definition addresses the case that arises when equality holds.

Definition 2.3.1.2. Let R be a local ring. If $\dim R = \operatorname{embdim} R$, then R is said to be a *regular local ring*.

The maximal ideal of a regular local ring is necessarily minimally generated by a (regular) system of parameters.

The simplest example of a regular local ring of dimension d is the power series ring $R = k[\![x_1, \ldots, x_d]\!]$ in d variables. In general, however, regular local rings can be thought of as precisely those local rings without zero-divisors. This fact is obvious since the zero ideal of such a ring is certainly not prime. Moreover, we generalize the concept of a regular local ring in the following section.

2.3.2 Regular Sequences

Let M be an R-module. A non-zero element $x \in R$ is said to be *M-regular* if it is not a zero-divisor on M. In the same spirit, we say that a sequence $\mathbf{x} = x_1, \ldots, x_n$ contained in R is *M-regular* if both

(1) x_i is not a zero-divisor on $M/(x_1, \ldots, x_{i-1})M$ for each $1 \leq i \leq n$, and

(2) $M/(\mathbf{x})M \neq 0$.

Indeed, by saying that a sequence is simply *regular*, we imply that it is regular on the ring as a module over itself.

Whenever the ring $(R, \mathfrak{m})$ is local, the concept of a regular sequence is made even more simple: not only is condition (2) above guaranteed by Nakayama's Lemma [37, Theorem 2.2] as long as $(\mathbf{x}) \subseteq \mathfrak{m}$, but it is also true that any permutation of a regular sequence is again regular.

A local ring which can be realized as the quotient of a regular local ring by an ideal generated by a regular sequence has special homological properties, which will be discussed in the sequel.

Definition 2.3.2.1. A local ring R is called a *complete intersection ring* if there exists a regular local ring S and an S-regular sequence $\mathbf{x}$ such that $\widehat{R} \cong S/(\mathbf{x})$.

In the more general case that M is a finitely generated module over a Noetherian ring R, one should note that any M-regular sequence $\mathbf{x}$ can always be extended to a maximal such sequence. Furthermore, the length of a maximal M-sequence is uniquely determined whenever M is finitely generated. Specifically, let I be a proper ideal of R such that $IM \neq M$. Then all of the maximal M-sequences contained in I have the same length, and this quantity is given by

$$n = \min\left\{ i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0 \right\}.$$

This invariant is used to define the concept of grade.

Let M and N be finitely generated R-modules. Then the *grade of N on M* is defined by

$$\operatorname{grade}_R(N, M) = \min\{i \mid \operatorname{Ext}^i_R(N, M) \neq 0\}.$$

Clearly, this quantity is simply the length of a maximal M-regular sequence contained in $\operatorname{Ann}_R N$. Often, we will speak of simply the *grade of M*, which is given by $\operatorname{grade}_R(M, R)$, and we will denote this quantity by $\operatorname{grade}_R M$.

If $(R, \mathfrak{m}, k)$ is local, then the length of a maximal M-regular sequence contained in $\mathfrak{m}$ is an invariant that has important application in the characterization of local rings. It is given by

$$\operatorname{depth}_R M = \operatorname{grade}_R(k, M)$$

and is called the *depth* of M in R. Since every M-regular sequence can be extended to a system of parameters of M, it must always be true that $\operatorname{depth}_R M \leq \dim M$. The situation arising when equality occurs is addressed in the next section.

2.3.3 Cohen-Macaulay Rings

One of the most widely studied classes of local rings are the so-called Cohen-Macaulay rings. Such rings can be thought of as having maximal depth, and they have special algebraic-geometric properties.

Definition 2.3.3.1. A finitely generated module M over a local ring R is said to be *Cohen-Macaulay* if $\operatorname{depth}_R M = \dim M$.

As one might guess, a ring which is Cohen-Macaulay as a module over itself is said to be a Cohen-Macaulay ring. In particular, complete intersection rings (and, therefore, regular local rings) are certainly Cohen-Macaulay. However, there are simple examples of Cohen-Macaulay local rings which are not complete intersections; such a ring is illustrated in the following example.

Example 2.3.3.2. Let $R = k[\![x, y]\!]/(x^2, xy, y^2)$. Certainly, $R = \widehat{R}$ cannot be written as the quotient of a regular local ring by a regular sequence — thus, it is not a complete intersection ring. However, notice that $\dim R = 0 = \operatorname{depth} R$, so that R is Cohen-Macaulay.

It is easy to modify the ring in the previous example to obtain a ring with the same depth, but higher dimension. This process yields the ring in the following example, which is commonly regarded as being the structurally simplest local ring which is not Cohen-Macaulay.

Example 2.3.3.3. Let $R = k[\![x,y]\!]/(x^2, xy)$. Clearly, $\dim R = 1$ by virtue of the (maximal) chain of proper containments $\mathfrak{m} = (x,y) \supsetneq (y)$. Furthermore, notice that since $\mathfrak{m}$ does not contain a regular sequence, $\operatorname{depth} R = 0$, and therefore R is not Cohen-Macaulay.

Indeed, there exists a special class of Cohen-Macaulay local rings which properly contains the complete intersection rings. This particular class of rings, defined below, will prove to be an integral part of our discussion of the existence of totally reflexive modules in the sequel.

Definition 2.3.3.4. Let R be a Cohen-Macaulay local ring. Then R is said to be *Gorenstein* if every parameter ideal $(\mathbf{x})$ of R is indecomposable; that is, if there do not exist ideals I, J of R which properly contain $(\mathbf{x})$, such that $(\mathbf{x}) = I \cap J$.

As it turns out, Gorenstein rings fit into the following chain of containments of local rings.

$$\text{regular} \quad \subseteq \quad \text{complete intersection} \quad \subseteq \quad \text{Gorenstein} \quad \subseteq \quad \text{Cohen-Macaulay}$$

Example 2.3.3.5. Let R be the one-dimensional Cohen-Macaulay local ring given by $R = k[\![x,y,z]\!]/(x^2, xy, y^2)$. Notice that the parameter ideal (z) can be written $(x,z) \cap (y,z)$, which implies that it is not irreducible. Therefore, R is not Gorenstein.

If R is a zero-dimensional Cohen-Macaulay local ring, we can use an alternate approach to check whether R is Gorenstein. To this end, recall that the *socle* of a local ring $(R, \mathfrak{m}, k)$ is defined by

$$\operatorname{Soc} R := \operatorname{Ann}_R \mathfrak{m}.$$

Certainly, if R is zero-dimensional, then $\operatorname{Soc} R$ is non-trivial. If R is moreover Cohen-Macaulay, then it is Gorenstein precisely when $\dim_k \operatorname{Soc} R = 1$. In other words, a zero-dimensional Gorenstein ring is simply a Cohen-Macaulay ring with a 'minimal' socle.

Remark 2.3.3.6. The above characterization of zero-dimensional Gorenstein rings is equivalent to Definition 2.3.3.4, assuming the restriction on dimension. Indeed, since a zero-dimensional local ring R does not have a system of parameters, one can regard the zero ideal as a parameter ideal of R. If this ideal is indecomposable, then R is Gorenstein.

Example 2.3.3.7. Let R be the zero-dimensional Cohen-Macaulay local ring given by $R = k[\![x, y]\!]/(x^2, xy, y^2)$. Since $\operatorname{Soc} R = (x, y)$ has k-vector space dimension two, R is not Gorenstein. This fact can also been seen by realizing that $0 = (x) \cap (y)$ is not indecomposable.

Remark 2.3.3.8. Indeed, it is possible to check whether a local ring is Gorenstein with any *a priori* knowledge of it being Cohen-Macaulay. In fact, a local ring is generally defined to be Gorenstein if it has finite injective dimension as a module over itself [16, Definition 3.1.18]. However, in order to avoid making definitions regarding injective modules and their properties, we have chosen to define Gorenstein rings in the present setting.

Gorenstein local rings are commonly associated with their nice symmetry properties. We illustrate two of these now.

(1) Suppose that M is a finitely generated module of finite projective dimension over a Gorenstein local ring R. Then the minimal R-free resolution of M is symmetric in regards to the ranks of its free modules. That is,

$$\beta_i(M) = \beta_{n-i}(M)$$

for all $0 \leq i \leq \frac{n}{2}$, where $n = \operatorname{pd}_R M$.

(2) Let $(R, \mathfrak{m}, k)$ be a zero-dimensional Gorenstein ring such that $\mathfrak{m}^\ell = 0$ for some $\ell \in \mathbb{N}$ (note that this condition is necessary in order for R to have dimension zero). Then the Hilbert series of R is a polynomial which is symmetric in its coefficients. That is,

$$\dim_k \left(\mathfrak{m}^i/\mathfrak{m}^{i+1}\right) = \dim_k \left(\mathfrak{m}^{\ell-i}/\mathfrak{m}^{\ell-i+1}\right)$$

for all $0 \leq i \leq \ell$.

We conclude our investigation of Gorenstein rings with a discussion of a certain class of local ring homomorphisms which produce rings which, in some sense, 'behave like' Cohen-Macaulay (resp. Gorenstein) rings.

2.3.4 Cohen-Macaulay Homomorphisms

Definitions 2.3.4.1. A finitely generated module M over a local ring Q is called *perfect* if $\operatorname{pd}_Q M = \operatorname{grade}_Q M$. Moreover, if $\varphi : Q \to R$ is a surjective local ring homomorphism such that R is perfect as a Q-module, then we say that φ is *Cohen-Macaulay.*

Indeed, recalling the characterizations of projective dimension and grade in terms of vanishing of certain Ext modules, the previous definitions imply that M is a perfect Q-module precisely when the following holds.

$$\min\{n \mid \operatorname{Ext}^n_Q(M,Q) \neq 0\} = \operatorname{grade}_Q M = \operatorname{pd}_Q M = \max\{n \mid \operatorname{Ext}^n_Q(M,Q) \neq 0\}$$

Thus, in some sense, we can say that M is a perfect Q-module if the cohomology $\operatorname{Ext}^*_Q(M,Q)$ is concentrated in one degree.

Definitions 2.3.4.2. An ideal I of a local ring Q is called *Gorenstein* whenever

(1) Q/I is a perfect Q-module, and

(2) $\operatorname{Ext}^g_Q(Q/I, Q) \cong Q$

for $g = \operatorname{pd}_Q Q/I = \operatorname{grade}_Q Q/I$. Furthermore, a surjective local ring homomorphism $\varphi : Q \to R$ is called *Gorenstein* if it is Cohen-Macaulay and if $\ker \varphi$ is a Gorenstein ideal of Q.

By the above discussion regarding perfect modules, one can conclude that a surjective local ring homomorphism $\varphi : Q \to R$ is Gorenstein precisely when $\operatorname{Ext}^*_Q(R, Q) \cong R$. In particular, this condition implies that the last non-zero free module in a Q-free resolution of R has rank one.

Remark 2.3.4.3. Cohen-Macaulay and Gorenstein homomorphisms can be used to obtain rings which generalize Cohen-Macaulay and Gorenstein rings, respectively. We shall discuss this further in Chapter 3. However, notice that if $\varphi : Q \to R$ is a Cohen-Macaulay (resp. Gorenstein) homomorphism, R is a Cohen-Macaulay (resp. Gorenstein) ring exactly when the same is true of Q.

2.4 Total Reflexivity

In this section, we give a careful introduction to the modules with which this work is concerned. However, before offering definitions and examples, we first give a basic overview of Gorenstein dimension in order to highlight totally reflexive modules within the bigger picture.

2.4.1 Gorenstein Dimension

The following definitions are first due to Auslander and Bridger in [5].

Definitions 2.4.1.1. Let M be a finitely generated R-module. A *Gorenstein reso-*

lution of M is defined to be an exact sequence

$$\mathbf{G}: \quad \cdots \to G_3 \xrightarrow{\partial_3} G_2 \xrightarrow{\partial_2} G_1 \xrightarrow{\partial_1} G_0 \to M \to 0$$

of R-modules such that each G_i is totally reflexive. Furthermore, we define the *Gorenstein dimension* (or *G-dimension*) of M by

$$\operatorname{G-dim}_R(M) := \min\{n \in \mathbb{N} \mid G_n = 0 \text{ for all Gorenstein } \mathbf{G} \to M \to 0\}$$

whenever this quantity is defined. Otherwise, we have that $\operatorname{G-dim}_R M = \infty$.

Clearly, totally reflexive modules are simply those modules of G-dimension zero. In fact, such terminology was precisely the way by which totally reflexive modules were first identified in [5].

We shall see in the next section that finitely generated projective modules are always totally reflexive. As such, $\operatorname{G-dim}_R M \leq \operatorname{pd}_R M$, with equality holding only when $\operatorname{pd}_R M < \infty$. This fact was shown in [5], and it naturally implies that modules of finite projective dimension have finite (but positive) Gorenstein dimension, and are thus not totally reflexive.

Indeed, the close analogy between G-dimension and projective dimension offers nice extensions of some famous classical results. First of all, G-dimension can be used to characterize Gorenstein local rings in precisely the same way that projective dimension characterizes regular local rings [6, 42].

Theorem 2.4.1.2. [2, Section 3.2] *Let $(R, \mathfrak{m}, k)$ be a local ring. The following are equivalent.*

1. *R is Gorenstein.*
2. $\operatorname{G-dim}_R M < \infty$ *for every finitely generated R-module M.*
3. $\operatorname{G-dim}_R k < \infty$.

Furthermore, the following characterization, which has come to be known as the Auslander-Bridger formula, is a natural refinement of the famous Auslander-Buchsbaum formula [7, Theorem 3.7].

Theorem 2.4.1.3. [5, Theorem 4.13] *Let R be a local ring and M a finitely generated R-module. If* $\operatorname{G-dim}_R M < \infty$*, then* $\operatorname{G-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$.

2.4.2 Definitions

In words, totally reflexive modules are homological extensions of reflexive modules. This is made clear in the following definition.

Definition 2.4.2.1. The R-module M is said to be *totally reflexive* if each of the following conditions hold.

(1) The canonical map $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), R)$ is an isomorphism.

(2) $\operatorname{Ext}^i_R(M, R) = 0$ for all $i > 0$.

(3) $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M, R), R) = 0$ for all $i > 0$.

The R-module $\operatorname{Hom}_R(M, R)$ is commonly called the *algebraic dual* of M, and shall be denoted M^* in what follows. With this notation, notice that the reflexivity condition $M \cong M^{**}$ is among the conditions for total reflexivity. Indeed, vanishing of the $\operatorname{Ext}^i_R(M, R)$ and $\operatorname{Ext}^i_R(M^*, R)$ for $i > 0$ provides a homological analog of reflexivity in the sense that it defines an isomorphism between the minimal free resolution of M and its 'double dual'.

By their definition, totally reflexive modules possess a sort of 'doubly-infinite minimal free resolution'. In order to construct such a resolution, we first consider minimal free resolutions $\mathbf{F} \to M \to 0$ and $\mathbf{G} \to M^* \to 0$. The vanishing of $\operatorname{Ext}^i_R(M^*, R)$

implies that the sequence $0 \to M^{**} \to \mathbf{G}^*$ is exact; thus we obtain an acyclic complex

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[rr] \arrow[dr] & & G_0^* \arrow[r] & G_1^* \arrow[r] & G_2^* \arrow[r] & \cdots \\
& & & & M \arrow[ur, hook] & & & &
\end{tikzcd}$$

with the obvious maps, which is called the *complete resolution* of M, and denoted $\mathbf{F}|\mathbf{G}^*$. Any complete resolution is a *totally acyclic complex* — that is, its dual $\mathbf{G}|\mathbf{F}^*$ is also exact. Conversely, any syzygy module of a totally acyclic complex is totally reflexive. Therefore, totally reflexive modules and totally acyclic complexes are really one and the same concept.

Remarks 2.4.2.2.

(1) Any finitely generated projective module is totally reflexive, and trivially so. Therefore, it is standard to refer to non-projective totally reflexive modules as being *non-trivial*.

(2) Any finitely generated module of (positive) finite projective dimension is not totally reflexive. To see why, recall that the projective dimension of an R-module M can be realized as the supremum of the values of $i \in \mathbb{N}$ such that $\operatorname{Ext}^i_R(M, R) \neq 0$.

(3) By virtue of (2), one should note that the complete resolution of a non-trivial totally reflexive R-module M is necessarily doubly-infinite. That is, it must be true that $\operatorname{pd}_R M = \infty = \operatorname{pd}_R M^*$.

2.4.3 Examples

Below, we illustrate the above definitions with a few examples of non-trivial totally reflexive modules and their respective complete resolutions. Before giving these examples, however, it is first useful to note that totally reflexive modules are ubiquitous over Gorenstein local rings. This is made precise in the following fact.

Fact 2.4.3.1. *Over a (non-regular) Gorenstein local ring, a (non-free) finitely generated module is totally reflexive if and only if it is maximal Cohen-Macaulay.*

Example 2.4.3.2. Let $(R, \mathfrak{m})$ be local. If $a, b \in \mathfrak{m}$ are such that $\operatorname{Ann}_R(a) = (b)$ and $\operatorname{Ann}_R(b) = (a)$, then $M = R/(a)$ and $N = R/(b)$ are both totally reflexive R-modules, and their complete resolutions are given by the following.

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r, "b"] & R \arrow[r, "a"] & R \arrow[rr, "b"] \arrow[dr] & & R \arrow[rr, "a"] \arrow[dr] & & R \arrow[r, "b"] & R \arrow[r, "a"] & \cdots \\
& & & M \arrow[ur, hook] & & N \arrow[ur, hook] & & &
\end{tikzcd}$$

Remark 2.4.3.3. The pair a, b exhibited in Example 2.4.3.2 is called an *exact pair of zero divisors*. The Hilbert series of short local rings which admit such elements is discussed at length in [24].

Example 2.4.3.4. Let $R = k[\![x, y]\!]/(x^2, y^2)$. Then by virtue of the previous Example, the modules $R/(x)$ and $R/(y)$ are non-trivial totally reflexive modules. Furthermore, it is also true that k is a totally reflexive R-module with complete resolution given by the following.

$$\begin{tikzcd}
\cdots \arrow[r] & R^3 \arrow[r, "\begin{bmatrix} x & 0 & y \\ 0 & y & -x \end{bmatrix}"] & R^2 \arrow[r, "\begin{bmatrix} x & y \end{bmatrix}"] & R \arrow[rr, "xy"] \arrow[dr] & & R \arrow[r, "\begin{bmatrix} x \\ y \end{bmatrix}"] & R^2 \arrow[r, "\begin{bmatrix} x & 0 \\ 0 & y \\ y & -x \end{bmatrix}"] & R^3 \arrow[r] & \cdots \\
& & & & k \arrow[ur, hook] & & & &
\end{tikzcd}$$

In order to gain a complete understanding of what totally reflexive modules look like, it is perhaps wise to also see examples of finitely generated modules with are *not* totally reflexive. First, we illustrate such a module with infinite projective dimension.

Example 2.4.3.5. Let $R = [\![x, y]\!]/(x^2, xy)$, and consider the R-module $M = R/(y)$.

Then a deleted minimal free resolution of M over R can be written

$$\mathbf{F}: \quad \cdots \to R \xrightarrow{x} R \xrightarrow{y} R \to M \to 0$$

which implies that

$$\operatorname{Hom}_R(\mathbf{F}, R): \quad 0 \to R \xrightarrow{y} R \xrightarrow{x} R \to \cdots$$

upon dualization. Since $\operatorname{Ext}^1_R(M, R) = \operatorname{H}^1(\operatorname{Hom}_R(\mathbf{F}, R)) \cong (x) \neq 0$, we see that M is not a totally reflexive R-module.

Remark 2.4.3.6. Recall from Example 2.3.3.3 that the ring R illustrated in the previous example is not Cohen-Macaulay (and therefore non-Gorenstein). In fact, since R has codimension one, we also know that it is Golod [43]. These facts together imply, by work of Avramov and Martsinkovsky [13], that the only totally reflexive modules over R are the free modules.

The following example serves to illustrate an abundance of finitely generated modules which are reflexive, but not totally reflexive.

Example 2.4.3.7. For some $n \geq 3$, let $R = k[\![x_1, x_2, \ldots, x_n]\!]$, and consider the presentation matrix of k over R.

$$d = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

By definition, $\Omega_2(k) = \ker d$. It is easy for one to check that $(\ker d)^* \cong \operatorname{coker} d^\intercal$, and furthermore that $(\operatorname{coker} d^\intercal)^* \cong \ker d$. Therefore, $\Omega_2(k)$ is a reflexive R-module, and it cannot be totally reflexive since $\operatorname{pd}_R \Omega_2(k) = n - 2 > 0$; cf. Remark 2.4.2.2(2).

Remark 2.4.3.8. The previous example serves to illustrate the necessity of conditions (2) and (3), regarding the vanishing of Ext, in Definition 2.4.2.1. However, the overall independence of conditions (1), (2) and (3) is not yet completely understood over an arbitrary local ring. Indeed, it is known that if R is Gorenstein, any module which satisfies condition (2) is totally reflexive. In fact, Yoshino [51] has studied this problem in more generality. Furthermore, Jorgensen and Şega demonstrated the independence of conditions (2) and (3) in [32]. In particular, they construct a local ring R which admits a finitely generated reflexive R-module satisfying $\mathrm{Ext}_R^i(M,R) = 0 \neq \mathrm{Ext}_R^i(M^*,R)$ for all $i > 0$. Finally, it is still an open question whether there exists a finitely generated module over a local ring which satisfies conditions (2) and (3), but is not reflexive.

CHAPTER 3

EXISTENCE VIA GORENSTEIN HOMOMORPHISMS

The framework for the main result of this chapter is motivated by the phenomenon apparent in one of the most general class of rings previously known to always admit non-trivial totally reflexive modules: local rings having an embedded deformation.

3.1 Embedded Deformations

Definition 3.1.1. A local ring R is said to *have an embedded deformation* if there exists a local ring S and an S-regular sequence $\mathbf{x} \subseteq \mathfrak{m}_S^2$ such that $R \cong S/(\mathbf{x})$. The natural projection $S \to S/(\mathbf{x}) \cong R$ is referred to as the *embedded deformation*.

The existence of non-trivial totally reflexive modules over rings with embedded deformations was established by Avramov, Gasharov, and Peeva in [12, Theorem 3.2]. In fact, the result in [12] is much more general; it demonstrates the existence of modules of finite complete intersection dimension over such rings. Thus, the theorem which follows is actually corollary to the result in [12]. We shall not include the constructive proof of its authors, but rather show in the following section that that their result is a direct consequence of our main theorem.

Theorem 3.1.2. [12, Theorem 3.2] *Let R be a local ring which has an embedded deformation. Then there exist non-trivial totally reflexive modules over R.*

In order to gain an understanding for the justification of this result, notice that whenever $S \to S/I$ is an embedded deformation, the ring $R = S/I$ carries some of the traits of a complete intersection ring; this fact follows because the relations defining I are regular on the ambient ring S. However, since we only need S to be local, it is entirely possible to construct such an R which is not a complete intersection, and therefore is not Gorenstein. Such a situation is illustrated below.

Example 3.1.3. Let $R = k[\![x, y, z]\!]/(x^2, xy, z^2)$. To see that R has an embedded deformation, let $S = k[\![x, y, z]\!]/(x^2, xy)$, and notice that $R \cong S/(z^2)$. Noticing that $\dim R = 1 \neq 0 = \operatorname{depth} R$, it follows that that R is not Cohen-Macaulay and therefore not Gorenstein. Moreover, we have that $(z) = \operatorname{Ann}_R(z)$, whence $R/(z)$ is a non-trivial totally reflexive R-module.

3.2 Results

Our goal for this section is to demonstrate a method for generalizing the class of local rings with embedded deformations in such a way that we preserve the existence of non-trivial totally reflexive modules. To this end, consider the class of local rings which are defined by a Gorenstein ideal.

Fact 3.2.1. *Any embedded deformation is necessarily a Gorenstein homomorphism.*

Proof. Let S be a local ring over which $x_1, \dots, x_n \subseteq \mathfrak{m}_S^2$ is an S-regular sequence. Furthermore, define $R = S/I$, where $I = (x_1, \dots, x_n)$. Clearly, we have that $\operatorname{grade}_S R = n = \operatorname{pd}_S R$. Furthermore, the minimal free resolution of R over S is simply the Koszul complex (cf. [20, Chapter 17]) on $x_1, \dots, x_n$. This is enough to imply that the rank of the last free module in the resolution must be one, and therefore $\operatorname{Ext}_S^i(R, S) \cong R$. $\square$

Hence, our main result will focus on the existence of non-trivial totally reflexive modules over the image of a Gorenstein homomorphism. However, we should not expect to extend a known result to a landscape endowed with less structure for free. In order to ensure the existence of totally reflexive modules in the setting of a Gorenstein homomorphism, we shall moreover assume that the Gorenstein homomorphism can be lifted to a ring which does, in fact, admit non-trivial totally reflexive modules: a non-regular Gorenstein ring. In order to obtain the desired result under this setting, we need the following basic fact concerning the descent of totally reflexive modules along homomorphisms of finite flat dimension, cf. [17, 5.6(b)].

Lemma 3.2.2. *Let Q be a non-regular Gorenstein ring and $\varphi : Q \to R$ a local homomorphism of finite flat dimension. Then R admits non-trivial totally reflexive modules.*

Proof. Since Q is Gorenstein and non-regular, it admits non-trivial minimal totally acyclic complexes; let $\mathbf{C} = (C_i, \partial_i)_{i \in \mathbb{Z}}$ be such a complex, and consider the totally reflexive module $\Omega_j \mathbf{C}$ for some $j \in \mathbb{Z}$. If $\operatorname{fd}_Q R = n$, it follows that $\operatorname{Tor}_i^Q(\Omega_j \mathbf{C}, R)$ vanishes for all $i > n$. Letting j vary implies that $\mathbf{C} \otimes_Q R$ is exact over R. Furthermore, [30, Proposition 2.3] provides the isomorphism

$$\operatorname{Hom}_R(\mathbf{C} \otimes_Q R, R) \cong \operatorname{Hom}_Q(\mathbf{C}, Q) \otimes_Q R$$

which in turn implies that one can use the same argument to show that $(\mathbf{C} \otimes_Q R)^* = \operatorname{Hom}_R(\mathbf{C} \otimes_Q R, R)$ is also exact over R. Therefore, $\mathbf{C} \otimes_Q R$ is a totally acyclic complex over R, and it is non-trivial and minimal since $\mathbf{C}$ was. So, for any $j \in \mathbb{Z}$, $\Omega_j(\mathbf{C} \otimes_Q R)$ is a non-trivial totally reflexive R-module. □

With this background established, we are now equipped to state and prove our main theorem.

Theorem 3.2.3. *Let $\varphi : Q \to R$ be a Gorenstein homomorphism of local rings whose kernel is contained in $\mathfrak{m}_Q^2$. Suppose that there exists a Gorenstein local ring P and* *homomorphism $\psi : P \to Q$ of finite flat dimension. Furthermore let $S = P/I$ be such that*

(1) $0 \neq I \subseteq \mathfrak{m}_P^2$,

(2) $S \otimes_P Q \cong R$,

(3) $\operatorname{Tor}_i^P(S,Q) = 0$ *for all* $i > 0$*, and*

(4) $\operatorname{grade}_P(S,P) \geq \operatorname{grade}_P(S,Q)$.

Then there exist non-trivial totally reflexive modules over R.

Proof. Consider the natural projection $\varphi' : P \to S = P/I$, and the natural map $\psi' : S \to S \otimes_P Q \cong R$ which acts by $s \mapsto s \otimes 1$. Then we have the following commutative diagram of local ring homomorphisms.

$$\begin{tikzcd} & P \arrow[dl, "\varphi'"'] \arrow[dr, "\psi"] & \\ S \arrow[dr, "\psi'"'] & & Q \arrow[dl, "\varphi"] \\ & R & \end{tikzcd} \tag{3.2.3.1}$$

We will show that S is a non-regular Gorenstein ring and that $\operatorname{fd}\psi' < \infty$, whence the result will follow from Lemma 3.2.2.

To this end, let $\mathbf{F}$ be a minimal free resolution of S over P. Since $\operatorname{Tor}_i^P(S,Q)$ vanishes for each $i > 0$, a minimal free resolution of R over Q is given by $\mathbf{F} \otimes_P Q$; in particular, $\operatorname{pd}_P S = \operatorname{pd}_Q R < \infty$. Also by the vanishing of $\operatorname{Tor}_i^P(S,Q)$ for each $i > 0$, we have isomorphisms

$$\operatorname{Ext}_P^i(S,Q) \cong \operatorname{Ext}_Q^i(R,Q)$$

for all i; cf. [30, 2.1(1)]. Now by definition, $\operatorname{grade}_P(S,Q) = \operatorname{grade}_Q(R,Q)$. This and the grade hypothesis now yield the following inequality.

$$\begin{aligned}\operatorname{grade}_P(S,P) &\geq \operatorname{grade}_P(S,Q)\\ &= \operatorname{grade}_Q(R,Q)\\ &= \operatorname{pd}_Q R\\ &= \operatorname{pd}_P S\end{aligned}$$

Since the reverse inequality holds trivially, we have that $\operatorname{grade}_P(S,P) = \operatorname{pd}_P S$, and so φ' is Cohen-Macaulay. Furthermore, as P is assumed to be a Cohen-Macaulay ring, it follows that S has the same property.

To show that S is a Gorenstein ring, it is sufficient to show that the rank of the last nonzero free module in the minimal free resolution $\mathbf{F}$ of S over P is one. But this follows from the fact that $\mathbf{F} \otimes_P Q$ is a minimal free resolution of R over Q whose last nonzero free module has rank one by assumption.

Next we justify that S is non-regular. By assumption, $I = \ker \varphi'$ is contained in $\mathfrak{m}_P^2$; therefore we have $\mathfrak{m}_S/\mathfrak{m}_S^2 = \mathfrak{m}_P/(\mathfrak{m}_P^2 + I) = \mathfrak{m}_P/\mathfrak{m}_P^2$. Using the Cohen-Macaulayness of S along with the Auslander-Buchsbaum formula, we obtain $\dim P - \dim S = \dim P - \operatorname{depth}_P S = \operatorname{pd}_P S > 0$. Therefore,

$$\mu_S(\mathfrak{m}_S) = \mu_P(\mathfrak{m}_P) \geq \dim P > \dim S$$

which certainly implies that S is non-regular.

Finally we show that the map ψ' has finite flat dimension. Let M be any S-module. By the vanishing of $\operatorname{Tor}_i^P(S,Q)$ for each $i > 0$, we have isomorphisms $\operatorname{Tor}_i^S(R,M) \cong \operatorname{Tor}_i^P(Q,M)$ for all i. Since Q has finite flat dimension as a P-module, this homology eventually vanishes. The finite flat dimension of R over S is immediate, and the result now follows from Lemma 3.2.2. $\square$

Remarks 3.2.4.

(1) The hypotheses given in parts (2) and (3) of Theorem 3.2.3 specify the conditions necessary for the so-called *lifting* of R to P via Q. Furthermore, it is common to refer to the P-modules S and Q as being *Tor-independent*.

(2) Theorem 3.2.3 may be regarded as a sort of 'ascent' analogue of [48, Theorem 3.1(2)]. Both results use a similar factorization involving Gorenstein homomorphisms, whereas the point of the theorem from [48] is that the hypotheses placed on φ' descend to φ.

We list as corollaries several cases in which Theorem 3.2.3 applies to establish the existence of non-trivial totally reflexive modules over R. We shall begin by showing that our result recovers the class of local rings which have embedded deformations. Our proof of this result, as well as that of Corollary 3.2.5 below, use standard constructions of ring homomorphisms, as, for example, seen in [48].

Proof of Theorem 3.1.2. Let $\varphi : Q \to Q/(x_1, \ldots, x_n) \cong R$ define an embedded deformation for R, so that $x_1, \ldots, x_n$ is a Q-regular sequence contained in $\mathfrak{m}_Q^2$. Also choose a minimal system of generators $z_1, \ldots, z_e$ of $\mathfrak{m}_Q$ and, for $1 \leq i \leq n$, write $x_i = \sum_{j=1}^{e} r_{ij} z_j$ for some $r_{ij} \in \mathfrak{m}_Q$.

Now let ρ_{ij}, ζ_j for $1 \leq i \leq n$ and $1 \leq j \leq e$ be indeterminates over $\mathbb{Z}$. If $p = \operatorname{char} Q/\mathfrak{m}_Q$, set

$$P = \mathbb{Z}[\rho_{ij}, \zeta_j]_{(p, \rho_{ij}, \zeta_j)}$$

and consider the local ring homomorphism $\psi : P \to Q$ defined by $\rho_{ij} \mapsto r_{ij}$, and $\zeta_j \mapsto z_j$ for all respective i, j. Furthermore, we define $\chi_i = \sum_{j=1}^{e} \rho_{ij} \zeta_j$ for each $1 \leq i \leq n$ and let

$$S = P/\left(\chi_1, \ldots, \chi_n\right).$$

It now suffices to show that S satisfies the hypotheses of Theorem 3.2.3.

First of all, notice that by way of construction, S is a lifting of R to P via Q. To see this more explicitly, notice that

$$P/(\chi_1, \ldots, \chi_n) \otimes_P Q \xrightarrow{\cong} P \otimes_P Q/(x_1, \ldots, x_n)$$

as P-modules via the mapping given by $\overline{a} \otimes b \mapsto 1 \otimes \overline{\psi(a)b}$, whence $S \otimes_P Q \cong R$. Furthermore, as $x_1, \ldots, x_n$ is Q-regular and $\chi_1, \ldots, \chi_n$ is P-regular, if $\mathbf{F}$ is a P-free resolution of S, then $\mathbf{F} \otimes_P Q$ yields a Q-free resolution of R. Thus, we obtain Tor-independence and the desired lifting condition follows.

The required grade inequality is actually a grade equality in this case,

$$\mathrm{grade}_P(S, P) = n = \mathrm{grade}_P(S, Q).$$

The remaining hypotheses are obvious, and the result follows. $\square$

Corollary 3.2.5. *Suppose that $\varphi : Q \to R$ is a Gorenstein homomorphism of grade three. Then there exist non-trivial totally reflexive R-modules.*

Proof. By the Buchsbaum-Eisenbud structure theorem (cf. [16, Theorem 3.4.1(b)]), there exists a (deleted) minimal free resolution of R over Q given by

$$\mathbf{F}: \qquad 0 \to Q \xrightarrow{\beta} Q^d \xrightarrow{\alpha} Q^d \xrightarrow{\beta^*} Q \to 0$$

with d odd, where $\alpha = (a_{ij})$ is skew-symmetric and $\beta = (b_j)$, such that b_j is determined by the Pfaffian of the matrix obtained by the deletion of the jth row and column of α. Notice that β is non-trivial since d is assumed to be odd.

Through a similar process as that in the previous proof, we define a regular local ring

$$P = \mathbb{Z}[T]_{(p;T)}$$

where $T = \{t_{ij} \,|\, 1 \le i < j \le d\}$ is a set of indeterminates over $\mathbb{Z}$ and $p = \mathrm{char}\, Q/\mathfrak{m}_Q$, and a local homomorphism $\psi : P \to Q$ which acts by $t_{ij} \mapsto a_{ij}$ for each i and j.

Now consider the $d \times d$ matrix over P given by $\tau = (t_{ij})$ where $t_{ii} = 0$ and $t_{ji} = t_{ij}$ for all $i < j$. Let $\sigma = (s_j)$ be the $d \times 1$ matrix over P such that s_j is defined by the Pfaffian of the matrix obtained by deleting the jth row and column of τ. If S is the P-module defined by the cokernel of σ^*, the Buchsbaum-Eisenbud structure theorem implies that

$$\mathbf{G}: \qquad 0 \to P \xrightarrow{\sigma} P^d \xrightarrow{\tau} P^d \xrightarrow{\sigma^*} P \to 0$$

is a (deleted) minimal free resolution of S over P. With these defined, notice that $\mathbf{G} \otimes_P Q \cong \mathbf{F}$, which implies that R lifts to P via Q. We can now use Theorem 3.2.3 to obtain the desired result. □

The next result addresses the case that Q is a Cohen-Macaulay ring. Its proof uses a corollary to Robert's 'new intersection theorem' (cf. [40]), which we state as a lemma.

Lemma 3.2.6. [40] *Let R be a local ring, and suppose that M and N are finitely generated* Tor*-independent R-modules of finite projective dimension over R. If $M \otimes_R N$ is Cohen-Macaulay, then both M and N are Cohen-Macaulay.*

Proof. Note that the vanishing of $\operatorname{Tor}_i^R(M, N)$ for all $i > 0$ implies that

$$\operatorname{pd}_R(M \otimes_R N) = \operatorname{pd}_R M + \operatorname{pd}_R N.$$

By using [16, Corollary 9.4.6] along with the Auslander-Buchsbaum formula, we obtain the following

$$\begin{aligned}\dim_R N &\leq \operatorname{pd}_R M + \dim_R(M \otimes_R N)\\ &= \operatorname{pd}_R(M \otimes_R N) - \operatorname{pd}_R N + \operatorname{depth}_R(M \otimes_R N)\\ &= \dim R - \operatorname{pd}_R N\\ &= \operatorname{depth}_R N\end{aligned}$$

which implies that $\dim_R N = \operatorname{depth}_R N$. The same proof shows that M is also Cohen-Macaulay. $\square$

Corollary 3.2.7. *Let $\varphi : Q \to R$ be a Gorenstein homomorphism of local rings whose kernel is contained in $\mathfrak{m}_Q^2$. Suppose that there exists a Gorenstein local ring P and Cohen-Macaulay homomorphism $\psi : P \to Q$ whose kernel is contained in $\mathfrak{m}_P^2$. If R lifts to P via Q, then there exist non-trivial totally reflexive R-modules.*

Proof. If P is non-regular, then by virtue of the fact that $\operatorname{pd}_P Q$ and $\operatorname{pd}_Q R$ are both finite, we have that $\operatorname{pd}_P R$ is finite as well. Thus Lemma 3.2.2 shows that non-trivial totally reflexive modules exist over R. The rest of the proof addresses the case that P is regular.

Let S be a lifting of R to P. First, we want to show that S is a quotient ring of P, equivalently a cyclic P-module. We have that the induced map $P/\mathfrak{m}_P \to Q/\mathfrak{m}_Q$ is an isomorphism, and therefore

$$\begin{aligned} R/\mathfrak{m}_Q R &\cong R \otimes_Q Q/\mathfrak{m}_Q \\ &\cong (S \otimes_P Q) \otimes_Q Q/\mathfrak{m}_Q \\ &\cong S \otimes_P Q/\mathfrak{m}_Q \\ &\cong S \otimes_P P/\mathfrak{m}_P \\ &\cong S/\mathfrak{m}_P S \end{aligned}$$

which implies that $R/\mathfrak{m}_Q R \cong S/\mathfrak{m}_P S$ as vector spaces over $Q/\mathfrak{m}_Q$. It follows that

$$\mu_P(S) = \dim_{P/\mathfrak{m}_P} S/\mathfrak{m}_P S = \dim_{Q/\mathfrak{m}_Q} R/\mathfrak{m}_Q R = 1.$$

Now consider the natural projection $\varphi' : P \to S$, and the natural local ring homomorphism $\psi' : S \to S \otimes_P Q \cong R$ which acts by $s \mapsto s \otimes 1$. Since ψ being Cohen-Macaulay implies that $\operatorname{fd}_P Q < \infty$, it is sufficient to show that S satisfies the hypotheses of Theorem 3.2.3. To this end, we prove that $\operatorname{grade}_P(S,P) \geq \operatorname{grade}_P(S,Q)$ and that the kernel of φ' is contained in $\mathfrak{m}_P^2$.

In order to show the latter, let $x \in \ker \varphi'$. Then $\psi(x) \in \ker \varphi \subseteq \mathfrak{m}_Q^2$. Since the induced map $\mathfrak{m}_P/\mathfrak{m}_P^2 \to \mathfrak{m}_Q/\mathfrak{m}_Q^2$ is injective, it follows that $x \in \mathfrak{m}_P^2$. Now to show the former, note that Q is Cohen-Macaulay since it is perfect module over a Cohen-Macaulay ring. This fact, in turn, implies the same property for R. Recall that Lemma 3.2.6 shows that S must be Cohen-Macaulay as well. Since $\operatorname{pd}_P S < \infty$, it follows that φ' is Cohen-Macaulay. This fact and the lifting condition imply the following equalities.

$$\operatorname{grade}_Q(R,Q) = \operatorname{pd}_Q R = \operatorname{pd}_P S = \operatorname{grade}_P(S,P)$$

Recalling that vanishing of $\operatorname{Tor}_i^R(S,Q)$ for all $i > 0$ provides us with the equality $\operatorname{grade}_P(S,Q) = \operatorname{grade}_Q(R,Q)$ (cf. proof of Theorem 3.2.3), the desired grade condition is satisfied. We can now apply Theorem 3.2.3 to obtain the result. □

In the previous corollary, the assumption that $\ker \psi \subseteq \mathfrak{m}_P^2$ is essential in obtaining totally reflexive R-modules which are non-trivial. This fact is illustrated through the following example.

Example 3.2.8. Let k be a field and consider the local rings defined by:

$$P = k[\![x,y]\!] \quad Q = k[\![x]\!] \quad R = k[\![x]\!]/(x^2)$$

Furthermore, let $\psi : P \to k[\![x,y]\!]/(y-x^2) \cong Q$ and let $\varphi : Q \to R$ be the natural projection maps. Notice that $\varphi \circ \psi : P \to R$ can be alternately factored to obtain the following commutative diagram of local rings

$$\begin{tikzcd} & P \arrow[dl, "\varphi'"'] \arrow[dr, "\psi"] & \\ S \arrow[dr, "\psi'"'] & & Q \arrow[dl, "\varphi"] \\ & R & \end{tikzcd}$$

where $S = k[\![x,y]\!]/(y) \cong k[\![x]\!]$ and ψ' and φ' are the natural projection maps. Though S is clearly a lifting of R to P via Q, R does not fit the criteria for Theorem 3.2.3 as $\ker\psi \nsubseteq \mathfrak{m}_P^2$. The consequence lies in the fact that Q is regular, and thus all of its totally reflexive modules are in fact free. The induced R-modules will therefore be free as well.

Corollary 3.2.9. *Let $\varphi : Q \to R$ be a Gorenstein homomorphism of local rings whose kernel is contained in $\mathfrak{m}_Q^2$. Suppose that P is a Gorenstein local ring and $\psi : P \to Q$ a local homomorphism of finite flat dimension such that*

(1) *the induced map $\mathfrak{m}_P/\mathfrak{m}_P^2 \to \mathfrak{m}_Q/\mathfrak{m}_Q^2$ is injective and*

(2) *the induced map $P/\mathfrak{m}_P \to Q/\mathfrak{m}_Q$ is bijective.*

If there exists a Cohen-Macaulay lifting of R to P via Q, then there exist non-trivial totally reflexive R-modules.

Proof. Let S be such a lifting of R to P via Q. Since S is Cohen-Macaulay and the lifting of R implies that $\operatorname{pd}_P S$ is finite, we have that S is a perfect P-module. Using this fact, it is easy to follow the same steps as in the proof of Corollary 3.2.7 to verify that S satisfies the hypotheses necessary for the application of Theorem 3.2.3. $\square$

3.3 A Class of Examples

In this section we turn our attention to examples of rings which admit totally reflexive modules by virtue of Theorem 3.2.3. In fact, we are able to construct such rings with associated Gorenstein homomorphisms of arbitrary grade, which furthermore do not have embedded deformations.

Construction 3.3.1. Let k be a field over which we define indeterminates $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, \ldots, y_m\}$. Now let $P = k[\mathbf{x}, \mathbf{y}]_{\mathfrak{m}}$ such that $\mathfrak{m} = (\mathbf{x}, \mathbf{y})$ is the homogeneous maximal ideal over the polynomial ring $k[\mathbf{x}, \mathbf{y}]$. Furthermore, let $\mathfrak{m}_{\mathbf{x}} = (\mathbf{x})$ and $\mathfrak{m}_{\mathbf{y}} = (\mathbf{y})$ denote the homogeneous maximal ideals of the polynomial rings $k[\mathbf{x}]$ and $k[\mathbf{y}]$, respectively. Now let

$$f_i \in P_{\mathbf{x}} := k[\mathbf{x}]_{\mathfrak{m}_{\mathbf{x}}} \quad 1 \leq i \leq r$$
$$g_j \in P_{\mathbf{y}} := k[\mathbf{y}]_{\mathfrak{m}_{\mathbf{y}}} \quad 1 \leq j \leq s$$

each be contained in the square of the respective maximal ideals. If $(f_1, \ldots, f_r)P$ is a Gorenstein ideal of P and $(g_1, \ldots, g_s)P$ is a perfect ideal of P, then

$$R = P/(f_1, \ldots, f_r, g_1, \ldots, g_s)P$$

admits non-trivial totally reflexive modules. Moreover, if $(g_1, \ldots, g_s)P$ is chosen to be a non-Gorenstein ideal P, then R is a non-Gorenstein ring.

Proof. To justify these claims, let $S = P/(f_1, \ldots, f_r)P$ and $Q = P/(g_1, \ldots, g_s)P$, and notice that $R \cong S \otimes_P Q$. We need to show that S and Q are Tor-independent P-modules, that $\operatorname{grade}_P(S, P) \geq \operatorname{grade}_P(S, Q)$, and that the projection $Q \to R$ is a Gorenstein homomorphism. These facts are illustrated below. In order to make notation more concise, we let $(\mathbf{f})$ and $(\mathbf{g})$ denote the ideals $(f_1, \ldots, f_r)P_{\mathbf{x}}$ and

$(g_1, \ldots, g_s)P_{\mathbf{y}}$, respectively. Furthermore, unless otherwise stated, all tensor products are assumed to be taken over k.

First we show that $\operatorname{Tor}_i^R(S, Q)$ vanishes for positive i. Take $\mathbf{F} \to P_{\mathbf{x}}/(\mathbf{f}) \to 0$ and $\mathbf{G} \to P_{\mathbf{y}}/(\mathbf{g}) \to 0$ to be a free resolutions over $P_{\mathbf{x}}$ and $P_{\mathbf{y}}$, respectively. Then $(\mathbf{F} \otimes P_{\mathbf{y}})_{\widetilde{\mathfrak{m}}}$ and $(P_{\mathbf{x}} \otimes \mathbf{G})_{\widetilde{\mathfrak{m}}}$ are free resolutions of S and Q, respectively, over P, where we define $\widetilde{\mathfrak{m}} = \mathfrak{m}_{\mathbf{x}} \otimes P_{\mathbf{y}} + P_{\mathbf{x}} \otimes \mathfrak{m}_{\mathbf{y}}$. To see that S and Q are Tor-independent over P, notice that

$$\operatorname{Tor}_i^P(S, Q) = \mathrm{H}_i\left((\mathbf{F} \otimes P_{\mathbf{y}})_{\widetilde{\mathfrak{m}}} \otimes_P (P_{\mathbf{x}} \otimes \mathbf{G})_{\widetilde{\mathfrak{m}}}\right) \cong \mathrm{H}_i(\mathbf{F} \otimes \mathbf{G})_{\widetilde{\mathfrak{m}}}$$

where the isomorphism is obtained from [29, 2.2]. Since this homology is isomorphic to R for $i = 0$ and vanishes otherwise, we have the lifting of R to P via Q.

Next we establish the grade inequality. As $S = P/(\mathbf{f})P$ is Gorenstein and $\mathbf{y}$ is regular on S, we have that

$$S/(\mathbf{y})S \cong P_{\mathbf{x}}/(\mathbf{f})$$

is also Gorenstein; in particular, $P_{\mathbf{x}}/(\mathbf{f})$ is perfect as a module over $P_{\mathbf{x}}$. This fact implies the last of the following equalities.

$$\operatorname{grade}_P(S, P) = \operatorname{pd}_P S = \operatorname{pd}_{P_{\mathbf{x}}} P_{\mathbf{x}}/(\mathbf{f}) = \operatorname{grade}_{P_{\mathbf{x}}}(P_{\mathbf{x}}/(\mathbf{f}), P_{\mathbf{x}})$$

Furthermore, note the following isomorphisms of complexes

$$\begin{aligned} \operatorname{Hom}_P((\mathbf{F} \otimes P_{\mathbf{y}})_{\widetilde{\mathfrak{m}}}, Q) &\cong \operatorname{Hom}_{P_{\mathbf{x}} \otimes P_{\mathbf{y}}}(\mathbf{F} \otimes P_{\mathbf{y}}, P_{\mathbf{x}} \otimes P_{\mathbf{y}}/(\mathbf{g}))_{\widetilde{\mathfrak{m}}} \\ &\cong (\operatorname{Hom}_{P_{\mathbf{x}}}(\mathbf{F}, P_{\mathbf{x}}) \otimes \operatorname{Hom}_{P_{\mathbf{y}}}(P_{\mathbf{y}}, P_{\mathbf{y}}/(\mathbf{g})))_{\widetilde{\mathfrak{m}}} \\ &\cong (\operatorname{Hom}_{P_{\mathbf{x}}}(\mathbf{F}, P_{\mathbf{x}}) \otimes P_{\mathbf{y}}/(\mathbf{g}))_{\widetilde{\mathfrak{m}}} \end{aligned} \tag{3.3.1.1}$$

the second of which is obtained from [29, Proof of Lemma 2.5(1)]. Now since

$$\begin{aligned} \mathrm{H}\left(\operatorname{Hom}_{P_{\mathbf{x}}}(\mathbf{F}, P_{\mathbf{x}}) \otimes P_{\mathbf{y}}/(\mathbf{g})\right) &= \operatorname{Ext}_{P_{\mathbf{x}}}^{\operatorname{pd}_P S}(P_{\mathbf{x}}/(\mathbf{f}), P_{\mathbf{x}}) \otimes P_{\mathbf{y}}/(\mathbf{g}) \\ &\cong P_{\mathbf{x}}/(\mathbf{f}) \otimes P_{\mathbf{y}}/(\mathbf{g}) \end{aligned}$$

is nonzero upon localizing at $\widetilde{\mathfrak{m}}$, we have $\operatorname{grade}_P(S, Q) = \operatorname{grade}_{P_{\mathbf{x}}}(P_{\mathbf{x}}/(\mathbf{f}), P_{\mathbf{x}})$, and therefore $\operatorname{grade}_P(S, P) = \operatorname{grade}_P(S, Q)$. In particular, the inequality holds.

Finally, we verify that $Q \to R$ is Gorenstein. To do this, recall that the vanishing of $\operatorname{Tor}_i^P(S, Q)$ for $i > 0$ implies that $\operatorname{Ext}_Q^i(R, Q) = \operatorname{Ext}_P^i(S, Q)$ for all $i \in \mathbb{N}$. Thus,

$$\operatorname{grade}_Q(R, Q) = \operatorname{grade}_P(S, Q) = \operatorname{grade}_P(S, P).$$

Since, furthermore, $\operatorname{pd}_Q R = \operatorname{pd}_P S$ due to the lifting of R, we have established that R is a perfect Q-module. To verify that $Q \to R$ is Gorenstein, it is enough check that $\operatorname{Ext}_Q^{\operatorname{pd}_Q R}(R, Q) \cong R$. However, this is equivalent to checking the same for $\operatorname{Ext}_P^{\operatorname{pd}_P S}(S, Q)$, which is obtained by taking cohomology of (3.3.1.1). To this end, we obtain the following isomorphism of P-modules.

$$\operatorname{Ext}_P^{\operatorname{pd}_P S}(S, Q) \cong (P_{\mathbf{x}}/(\mathbf{f}) \otimes P_{\mathbf{y}}/(\mathbf{g}))_{\widetilde{\mathfrak{m}}} \cong R$$

Therefore Theorem 3.2.3 establishes the existence of non-trivial totally reflexive R-modules. In order to check the validity of the final statement, notice that if $(\mathbf{g})P$ is not a Gorenstein ideal of P, then Q is Cohen-Macaulay but not Gorenstein. Thus, the rank of the last nonzero free module in $(P_{\mathbf{x}} \otimes \mathbf{G})_{\widetilde{\mathfrak{m}}}$ is greater than one. Furthermore,

$$S \otimes_P (P_{\mathbf{x}} \otimes \mathbf{G})_{\widetilde{\mathfrak{m}}} \to R \to 0$$

is a free resolution of R over S, and its last nonzero free module must also have rank greater than one. Since S is assumed to be Gorenstein, we have shown that R cannot be. $\square$

This section concludes with a specific example of a ring which demonstrates the previous construction, and which, as we shall show in the next section, has no embedded deformation.

Example 3.3.2. Let k be a field and define $P = k[x_1, \ldots, x_5, y_1, \ldots, y_4]_{\mathfrak{m}}$, where $\mathfrak{m} = (x_1, \ldots, x_5, y_1, \ldots, y_4)$ is the homogeneous maximal ideal over the polynomial ring $k[x_1, \ldots, x_5, y_1, \ldots, y_4]$. Now consider the local ring $R = P/I$, where I is defined by the following seventeen quadratic forms over P.

$$2x_1x_3 + x_2x_3,\ x_1x_4 + x_2x_4,\ x_3^2 + 2x_1x_5 - x_2x_5$$
$$x_4^2 + x_1x_5 - x_2x_5,\ x_1^2,\ x_2^2,\ x_3x_4,\ x_3x_5,\ x_4x_5,\ x_5^2$$
$$y_1^2,\ y_1y_2 - y_3^2,\ y_1y_3 - y_2y_4,\ y_1y_4,\ y_2^2 + y_3y_4,\ y_2y_3,\ y_4^2$$

We first notice that $R \cong S \otimes_P Q$, where $S = P/J$, $Q = P/K$, and J and K are the ideals generated by

$$2x_1x_3 + x_2x_3,\ x_1x_4 + x_2x_4,\ x_3^2 + 2x_1x_5 - x_2x_5$$
$$x_4^2 + x_1x_5 - x_2x_5,\ x_1^2,\ x_2^2,\ x_3x_4,\ x_3x_5,\ x_4x_5,\ x_5^2$$

and

$$y_1^2,\ y_1y_2 - y_3^2,\ y_1y_3 - y_2y_4,\ y_1y_4,\ y_2^2 + y_3y_4,\ y_2y_3,\ y_4^2$$

respectively, over P. This noted, it is clear from the discussion in the previous construction that R fits the criteria for Theorem 3.2.3, and so we are guaranteed that it admits non-trivial totally reflexive modules. However, as we will demonstrate in the subsequent section, R does not have an embedded deformation.

3.4 Embedded Deformations Revisited

The ultimate goal for this section is to develop machinery that will enable us to prove the following statement.

Proposition 3.4.0.1. *The local ring R defined in Example 3.3.2 does not have an embedded deformation.*

The goal is accomplished in the following section by utilizing the homotopy Lie algebra of the local ring R. Furthermore, in Section 3.4.2 we establish a relation between a local ring with an embedded deformation and the Poincaré series of its completion.

3.4.1 The Homotopy Lie Algebra

In this section, we investigate a method which uses the homotopy Lie algebra of a local ring to determine if it has an embedded deformation. General theory regarding such constructions can be found in [35].

Before giving results, we first review some related facts.

If $Q \to R$ is a surjective local homomorphism, there is an induced map $\pi^*(R) \to \pi^*(Q)$ on the respective graded homotopy Lie algebras. If, furthermore, $Q \to R$ is an embedded deformation, then the natural map $\pi^*(R) \to \pi^*(Q)$ is surjective and its kernel is comprised of the central elements of $\pi^2(R)$; for details, see [8, 6.1] For a local ring $(R, \mathfrak{m}, k)$ the universal enveloping algebra of $\pi^*(R)$ is precisely the graded k-algebra $\operatorname{Ext}_R^*(k,k)$. If R is moreover a Koszul algebra, then $\operatorname{Ext}_R^*(k,k)$ is generated as a k-algebra by $\operatorname{Ext}_R^1(k,k)$; for details, see [36, Theorem 1.2].

The following result, which characterizes the algebra generated by $\operatorname{Ext}_R^1(k,k)$ for a quadratic ring R, is a special case of a result of Löfwall in [36]. Its proof has been omitted, but details can be found in [36].

Theorem 3.4.1.1. [36, Corollary 1.3] *Let k be a field and define the quadratic k-algebra $R = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$, where*

$$f_i = \sum_{j \le \ell} a_{ij\ell} x_j x_\ell$$

with each $a_{ij\ell} \in k$, are homogenous for $1 \le i \le r$. Then the algebra generated by the degree one elements in $\operatorname{Ext}_R^*(k,k)$ is given by

$$\left[\operatorname{Ext}_R^1(k,k)\right] = k\langle T_1, \ldots, T_n\rangle/(\varphi_1, \ldots, \varphi_s)$$

where, for $1 \le i \le s$, we define

$$\varphi_i = \sum_{j \le \ell} c_{ij\ell}\,[T_j, T_\ell]$$

such that $c_{ij\ell} \in k$ and $[T_j, T_\ell] = T_j T_\ell + T_\ell T_j$ for all $j \le \ell$. Furthermore, the $(c_{ij\ell})_{j\ell}$ form a basis for the solution set to the system of linear equations given by

$$\sum_{j \le \ell} a_{ij\ell} x_{j\ell} = 0$$

for $1 \le i \le r$. That is, $(c_{ij\ell})_{j\ell}$ forms a basis for the nullspace of the matrix given by:

$$\begin{bmatrix} a_{111} & \cdots & a_{1nn} \\ \vdots & \ddots & \vdots \\ a_{r11} & \cdots & a_{rnn} \end{bmatrix}$$

As a consequence of Löfwall's result, we are able to consider degree two elements of the homotopy Lie algebra of a Koszul algebra R as quadratic forms in $[\mathrm{Ext}^1_R(k,k)]$. The following result illustrates this process by extending Theorem 3.4.1.1.

Lemma 3.4.1.2. *Let k be a field and consider the k-algebras given by*

$$Q = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

$$S = k[y_1, \ldots, y_m]/(g_1, \ldots, g_s)$$

where, for all $1 \le i \le r$ and $1 \le j \le s$, the f_i, g_j are homogeneous quadratic forms such that Q and S are finite dimensional and Koszul. If $R = Q \otimes_k S$, then R is local and

$$\pi^*(R) \cong \pi^*(Q) \times \pi^*(S).$$

In particular, $\pi^*(R)$ has nonzero central elements of degree two if and only if either $\pi^*(Q)$ or $\pi^*(S)$ does.

Proof. Since the polynomials which define Q and S are homogeneous quadratics, we can express them as

$$f_i = \sum_{j \leq \ell} a_{ij\ell} x_j x_\ell$$

for $1 \leq i \leq r$, and

$$g_i = \sum_{j \leq \ell} b_{ij\ell} y_j y_\ell$$

for $1 \leq i \leq s$. Further, since

$$R = Q \otimes_k S \cong k[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_r, g_1, \ldots, g_s)$$

we see that Theorem 3.4.1.1 implies

$$\left[\operatorname{Ext}^1_R(k,k)\right] = k\langle T_1, \ldots, T_n, U_1, \ldots, U_m\rangle/(\varphi_1, \ldots, \varphi_\alpha)$$

where $\alpha = (n+m)(n+m+1)/2 - (r+s)$. For simplicity in notation, we let $\beta = n(n+1)/2 - r$ and $\gamma = m(m+1)/2 - s$, so that $\alpha = \beta + \gamma + nm$. With this established, the first $\beta + \gamma$ of the φ_i are given by

$$\varphi_i = \begin{cases} \displaystyle\sum_{j \leq \ell} c_{ij\ell}[T_j, T_\ell] & 1 \leq i \leq \beta \\ \displaystyle\sum_{j \leq \ell} c_{ij\ell}[U_j, U_\ell] & \beta + 1 \leq i \leq \beta + \gamma \end{cases}$$

where $c_{ij\ell} \in k$ is defined in such a way that $(c_{ij\ell})_{j\ell}$ forms basis for the kernel of the $(r+s) \times (\beta + \gamma)$ matrix given by:

$$\begin{bmatrix} \begin{matrix} a_{111} & \cdots & a_{1nn} \\ \vdots & \ddots & \vdots \\ a_{r11} & \cdots & a_{rnn} \end{matrix} & \Large 0 \\ \Large 0 & \begin{matrix} b_{111} & \cdots & b_{1mm} \\ \vdots & \ddots & \vdots \\ b_{s11} & \cdots & b_{smm} \end{matrix} \end{bmatrix}$$

Furthermore, let

$$\sigma : \left\{(j,\ell) \in \mathbb{Z}^2 \,|\, 1 \leq j \leq n,\, 1 \leq \ell \leq m\right\} \to \{i \in \mathbb{Z} \,|\, 1 \leq i \leq nm\}$$

be a bijection; then the last nm of the φ_i are given by

$$\varphi_{\beta+\gamma+\sigma((j,\ell))} = [T_j, U_\ell]$$

for $1 \leq j \leq n$ and $1 \leq \ell \leq m$. Letting t_i (resp. u_i) denote the image in $\mathrm{Ext}^1_R(k,k)$ of T_i (resp. U_i), it follows that $[t_j, u_\ell] = [u_\ell, t_j] = 0$ for every $1 \leq j \leq n$ and $1 \leq \ell \leq m$. Now it follows that

$$[\mathrm{Ext}^1_R(k,k)] \cong k\langle T_1, \ldots, T_n\rangle/(\varphi_1, \ldots, \varphi_\beta) \otimes_k k\langle U_1, \ldots, U_m\rangle/(\varphi_{\beta+1}, \ldots, \varphi_{\beta+\gamma}).$$

Furthermore as R is assumed to be Koszul, $\pi^*(R)$ can be viewed as a linear subspace of the above expression via the natural inclusion $\pi^*(R) \hookrightarrow \mathrm{Ext}^*_R(k,k)$. This implies the result. $\square$

Remark 3.4.1.3. The statement of Lemma 3.4.1.2 holds even when Q, S, and R are local, but not necessarily Koszul. To see this, notice that we have an induced isomorphism $\mathrm{Tor}^*_R(k,k) \cong \mathrm{Tor}^*_Q(k,k) \otimes_k \mathrm{Tor}^*_S(k,k)$ of k-algebras which extends to an isomorphism of Hopf algebras with divided powers. Moreover, one can show that this isomorphism is equivalent to an isomorphism of homotopy Lie algebras by considering the equivalence of the respective categories. (For details, see [1], [38], [44].) Despite this more general fact, we have chosen to include the machinery of Theorem 3.4.1.1 and Lemma 3.4.1.2 so that we may justify the following result without needing the rigor which is required of Hopf algebras.

With these mechanisms established, we are now ready to prove the assertion at the beginning of the section.

Proof of Proposition 3.4.0.1. It suffices to show that $\pi^*(R)$ has no non-trivial central elements of degree 2. By Lemma 3.4.1.2, this condition is equivalent to neither $\pi^*(S)$ nor $\pi^*(Q)$ containing such elements. In [11, Example 2.1 & Section 3], Avramov, Gasharov, and Peeva prove this condition for $\pi^*(Q)$, so we only need to show the result for $\pi^*(S)$. We shall adopt the same approach as the authors of [11].

As a result of [36, Corollary 1.3], we know that the algebra generated by the universal enveloping algebra of $\pi^*(S)$ can be expressed as

$$\left[\operatorname{Ext}_S^1(k,k)\right] \cong k\langle T_1, T_2, T_3, T_4, T_5\rangle / I$$

where I is generated by the following relations.

$$T_1T_2 + T_2T_1, \quad (T_1T_3 + T_3T_1) - 2(T_2T_3 + T_3T_2), \quad (T_1T_4 + T_4T_1) - (T_2T_4 + T_4T_2)$$
$$T_3^2 + T_4^2 + (T_2T_5 + T_5T_2), \quad T_3^2 + (T_1T_5 + T_5T_1) + (T_2T_5 + T_5T_2)$$

So it follows that $\pi^*(S)$ is a graded Lie algebra on the variables t_1, t_2, t_3, t_4, t_5, each of degree one, which satisfies the following relations.

$$[t_1, t_2] = 0, \quad [t_1, t_3] = 2[t_2, t_3], \quad [t_1, t_4] = [t_2, t_4]$$
$$t_3^{(2)} + t_4^{(2)} = -[t_2, t_5], \quad 2t_3^{(2)} + t_4^{(2)} = [t_1, t_5]$$

It is straightforward to see that the following forms a basis of $\pi^2(S)$.

$$u_1 = t_1^{(2)} \qquad u_2 = t_2^{(2)} \qquad u_3 = t_3^{(2)} \qquad u_4 = t_4^{(2)} \qquad u_5 = t_5^{(2)}$$
$$u_6 = [t_1, t_3] \quad u_7 = [t_1, t_4] \quad u_8 = [t_3, t_4] \quad u_9 = [t_3, t_5] \quad u_{10} = [t_4, t_5]$$

Furthermore, we assert that a basis of $\pi^3(S)$ is given by the following.

$$v_i = \begin{cases} [u_{i+2}, t_1] & 1 \le i \le 8 \\ [u_{i-2}, t_3] & 9 \le i \le 12 \\ [u_{i-5}, t_4] & 13 \le i \le 14 \\ [u_{i-6}, t_5] & 15 \le i \le 16 \end{cases}$$

TABLE 3.1

MULTIPLICATION TABLE FOR $\pi^3(S)$

$[u_i, t_j]$	t_1	t_2	t_3	t_4	t_5
u_1	0	0	$-2v_4$	$-2v_5$	$-4v_1 + 2v_2$
u_2	0	0	$\frac{1}{2}v_4$	$2v_5$	$v_1 + v_2$
u_3	v_1	$\frac{1}{2}v_1$	0	$-2v_{10}$	$-2v_{11}$
u_4	v_2	$\frac{1}{2}v_2$	$-2v_{13}$	0	$-\frac{1}{2}v_3 + 4v_{11}$
u_5	v_3	$-v_3 + 4v_{11}$	$-2v_{15}$	$-2v_{16}$	0
u_6	v_4	$-\frac{1}{2}v_4$	$-\frac{1}{2}v_1$	$-v_9 - v_6$	$-v_7 + 2v_{13}$
u_7	v_5	$-v_5$	v_9	$-\frac{1}{2}v_2$	$-v_8 + 4v_{10}$
u_8	v_6	$\frac{1}{2}v_6 - \frac{1}{2}v_9$	v_{10}	v_{13}	$-v_{12} - v_{14}$
u_9	v_7	$\frac{1}{2}v_7 - 3v_{13}$	v_{11}	v_{14}	v_{15}
u_{10}	v_8	$v_8 - 6v_{10}$	v_{12}	$\frac{1}{4}v_3 - 2v_{11}$	v_{16}

For the reader's convenience, and in order to justify these claims, we include the above multiplication table for $\pi^3(S)$.

It is clear from this table that the elements $v_1, \ldots, v_{16}$ span $\pi^3(S)$. In order to justify their linear independence, we note that $\operatorname{rank}_k \pi^3(S) = \varepsilon_3(S)$, the third deviation of S, cf. [9, Theorem 10.2.1(2)]. This quantity can be calculated in terms of the Betti numbers of k over S as follows.

(3.4.1.3.1)
$$\begin{aligned} \varepsilon_1 &= b_1 \\ \varepsilon_2 &= b_2 - \binom{\varepsilon_1}{2} \\ \varepsilon_3 &= b_3 - \varepsilon_2\varepsilon_1 - \binom{\varepsilon_1}{3} \end{aligned}$$

(cf. [9, Section 7]). Calculating a minimal S-free resolution of k yields that

$$P_S k(t) = 1 + 5t + 20t^2 + 76t^3 + \cdots$$

which we use to evaluate the expressions in (3.4.1.3.1), and obtain $\varepsilon_3 = 16$. Thus, $v_1, \ldots, v_{16}$ is in fact a basis of $\pi^3(S)$.

Now suppose $u = \sum_{i=1}^{10} \alpha_i u_i$ is central in $\pi^2(S)$. Then $0 = [u, t_1] = \sum_{i=3}^{10} \alpha_i v_{i-3}$ implies that $u = \alpha_1 u_1 + \alpha_2 u_2$. Furthermore, using the above table yields

$$\begin{aligned} 0 &= [u, t_5] \\ &= \alpha_1 [u_1, t_5] + \alpha_2 [u_2, t_5] \\ &= (-4\alpha_1 + \alpha_2) v_1 + (2\alpha_1 + \alpha_2) v_2 \end{aligned}$$

which implies that $u = 0$. We have therefore proven that $\pi^2(S)$ does not contain nonzero central elements, and thus R does not have an embedded deformation. $\square$

Recalling that (Gorenstein) local rings of codimension at most 3 (resp. 4) have embedded deformations, we have that the ring defined in Example 3.3.2, in the way of codimension, the smallest such possible which satisfies the hypotheses of our main result, yet does not have an embedded deformation.

3.4.2 Completion

In this section, we demonstrate a relation between a local ring with an embedding deformation and the Poincaré series of its localization.

Proposition 3.4.2.1. *If R is an equicharacteristic local ring which has an embedded deformation, then -1 is a double root of $P_{\widehat{R}}(t)$.*

Proof. Let $\varphi : S \to R$ be a surjective homomorphism of local rings such that $\ker \varphi = (a) \subseteq \mathfrak{m}_S^2$ for some non-zero-divisor a of S. The Cohen structure theorem ensures that both $\widehat{R}$ and $\widehat{S}$ are the homomorphic images of regular local rings; say U and V, respectively. In particular, we can choose these rings so that $\operatorname{embdim} U = \operatorname{embdim} \widehat{R}$ and $\operatorname{embdim} V = \operatorname{embdim} \widehat{S}$. Since $a \in \mathfrak{m}_S^2$, it follows that $\operatorname{embdim} R = \operatorname{embdim} S$.

Furthermore, since $R/\mathfrak{m}_R \cong \widehat{R}/\mathfrak{m}_{\widehat{R}}$, we see that

$$\mathfrak{m}_R/\mathfrak{m}_R^2 \cong \left(\mathfrak{m}_R\widehat{R}\right) / \left(\mathfrak{m}_R^2\widehat{R}\right) \cong \mathfrak{m}_{\widehat{R}}/\mathfrak{m}_{\widehat{R}}^2$$

which, of course, implies that $\operatorname{embdim} R = \operatorname{embdim} \widehat{R}$. Similarly, we can show that $\operatorname{embdim} S = \operatorname{embdim} \widehat{S}$. Denote by d the common embedding dimension. Now, let k be a coefficient field for R. The fact that φ is a surjective local ring homomorphism implies that k is not only a coefficient field for S, but also for $\widehat{R}$ and $\widehat{S}$. Further, by the Cohen structure theorem, k is a coefficient field for U and V. The following isomorphism henceforth results

$$U \cong V \cong k[\![x_1, \ldots, x_d]\!]$$

which yields a commutative diagram of local ring homomorphisms

$$\begin{tikzcd} & U \arrow[dl, "\pi_S"'] \arrow[dr, "\pi_R"] & \\ \widehat{S} \arrow[rr, "\widehat{\varphi}"] & & \widehat{R} \\ S \arrow[u, "\iota_S"] \arrow[rr, "\varphi"] & & R \arrow[u, "\iota_R"'] \end{tikzcd}$$

where ι_S, ι_R, π_S, and π_R are the canonically defined maps.

Because completion is a faithfully flat functor, $\ker \widehat{\varphi} \cong (a) \otimes_S \widehat{S} \cong \widehat{(a)}$. Furthermore, as S is Noetherian, it embeds into $\widehat{S}$. Thus, there should be no confusion in writing $\ker \widehat{\varphi} = (a)$, where a is obviously viewed here as an element of $\widehat{S}$. Now, take $\tilde{a} \in U$ to be a preimage of a under π_S. By virtue of the fact that a is regular on $\widehat{S}$, we obtain the following isomorphism.

$$\widehat{R} \cong \widehat{S} \otimes_U (U/(\widetilde{a}))$$

Moreover, since $\widehat{S}$ and $U/(\widetilde{a})$ are Tor-independent over U, we have the following decomposition of the Poincaré series of $\widehat{R}$

$$P_{\widehat{R}}(t) = P_{\widehat{S}}(t) \cdot P_{U/(\widetilde{a})}(t)$$

(where each of the above Poincaré series is taken to be over U). Now, since $\widehat{S}$ and $U/(\widetilde{a})$ are both rank-zero U-modules, we know that $P_{\widehat{S}}(-1) = P_{U/(\widetilde{a})}(-1) = 0$. $\square$

Remark 3.4.2.2. As the ring R exhibited in Example 3.3.2 can be realized as the tensor product of Tor-independent modules, it will necessarily follow that -1 is a double root of $P_{\widehat{R}}(t) = P_R(t)$. Therefore, one cannot use Proposition 3.4.2.1 to show that R does not have an embedded deformation. However, it *is* possible to use the result to show that neither of the rings S and Q, defined in Example 3.3.2, has an embedded deformation.

CHAPTER 4

EXISTENCE OVER SHORT LOCAL RINGS

The results of this chapter are of a very different flavor than those of the previous chapter. One reason for this difference lies in the fact that the present chapter is concerned with a much broader question; that is, we are chiefly interested in the *necessary* conditions for the existence of totally reflexive modules over short local rings. Therefore, the established results will characterize the existence of a much more general class of modules. First, we consider the motivation for this work.

4.1 Motivation

The driving force for the work in this chapter is twofold. First, there is a motivation to better understand the necessary conditions for the existence of asymmetric complete resolutions. Such resolutions are known to exist, as illustrated by Jorgensen and Şega in [31]. For the reader's convenience, we reproduce their example below.

Example 4.1.1. [31, Proposition 2.2] Let k be a field which is not algebraically closed over a finite field, and choose $\alpha \in k$ to have infinite multiplicative order. Furthermore letting t, u, v, x, y, z be indeterminates over k, each of degree one, we define the quotient ring $R = k[t, u, v, x, y, z]/I$, where I is the ideal generated by the following fifteen quadratic forms.

$$z^2,\ uz - tx - \alpha uv,\ u^2,\ zy + vy,\ uy,\ y^2 - tx - (\alpha - 1)uv,$$

$$xz + \alpha vx,\ ux,\ xy,\ x^2 - tx - tv,$$

$$tz + ty + \alpha vx,\ tu,\ ty - vx + tv,\ t^2 + (\alpha + 1)uv - vy,\ v^2$$

Now let $d : R^3 \to R^2$ be given by the following matrix, with respect to the standard bases of R^3 and R^2.

$$\begin{bmatrix} v & y & 0 \\ x & z & tv \end{bmatrix}$$

In [31], the authors demonstrate that $M = \operatorname{coker} d$ is a totally reflexive R-module with a complete resolution $\mathbf{F}|\mathbf{G}^*$ having the property that $\{\operatorname{rank} F_i\}$ has exponential growth, whereas $\{\operatorname{rank} G_i\}$ is constant. Moreover, the authors demonstrate that R is a local Gorenstein ring with maximal ideal $\mathfrak{m} = (t, u, v, x, y, z)$ and is a Koszul algebra.

The local ring illustrated in the previous example has the property that the fourth power of the maximal ideal vanishes. Such local rings have been referred to in the literature [24] as being *short*, and we shall adopt the same terminology. Indeed, the existence of totally reflexive modules has been extensively studied over 'shorter' local rings. We outline the relevant results below.

Fact 4.1.2. [50, Proposition 2.4] *Let $(R, \mathfrak{m})$ be a non-Gorenstein local ring with $\mathfrak{m}^2 = 0$. Then every totally reflexive module is trivial.*

Theorem 4.1.3. [50, Theorem 3.1] *Let $(R, \mathfrak{m})$ be a non-Gorenstein local ring with $\mathfrak{m}^3 = 0$. If R admits a non-trivial totally reflexive module M, then the following hold.*

(1) *R has the structure of a standard graded ring, and its Hilbert series is balanced; that is, $H_R(-1) = 0$.*
(2) *R is a Koszul algebra.*
(3) *The Betti sequence of M is constant.*

Hence, our second source of motivation is to better understand the existence of non-trivial totally reflexive modules over a local ring $(R, \mathfrak{m})$ with $\mathfrak{m}^4 = 0$. Assuming the existence of such a module M over R, we ask the following questions.

(1) Can we characterize the Hilbert series of R (that is, of $\mathrm{gr}_{\mathfrak{m}}(R)$)?

(2) Is this Hilbert series necessarily balanced?

(3) What is the possible growth of the Betti sequence of M?

Guided by Theorem 4.1.3(2) and the fact that the ring exhibited in Example 4.1.1 is Koszul, we restrict our investigation to modules with (eventually) linear resolutions. The following section addresses the niceties of such modules; first, we set notation that will be used throughout the remainder of the chapter.

Notation 4.1.4. Let $(R, \mathfrak{m})$ be a local ring satisfying $\mathfrak{m}^4 = 0$. We let

$$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + ft^2 + gt^3$$

denote the Hilbert series of R, where $e = \mu(\mathfrak{m})$ is the embedding dimension of R, $f = \mu(\mathfrak{m}^2)$, and $g = \mu(\mathfrak{m}^3)$.

4.2 Linear Resolutions

One particular advantage of studying modules with eventually linear minimal free resolutions is that their Poincaré series are easy to compute; this is due to a result of Herzog and Iyengar in [25]. We shall state their result next, and furthermore include a proof for the reader's convenience.

Lemma 4.2.1. [25, 1.8] *Let M be a finitely generated module with an eventually linear minimal free resolution over a local ring $(R, \mathfrak{m})$. Then there exists $q \in \mathbb{Z}[t]$ such that the following holds.*

$$P^R_M(t) = \frac{q(t)}{H_{\mathrm{gr}_{\mathfrak{m}}(R)}(-t)(1+t)^{\dim R}}$$

Proof. Let $n = \mathrm{ld}_R(M)$, and notice that

$$P_M^R(t) = \sum_{i=0}^{n-1} \beta_i(M)t^i + \left(P^R_{\Omega_n(M)}(t)\right) t^n$$

where $\Omega_n(M)$ is necessarily Koszul. Since $P_M^R(t)$ has the desired form if and only if $P^R_{\Omega_n(M)}$ does, it suffices to assume that M is Koszul.

Now choose a minimal free resolution $\mathbf{F} \to M \to 0$. Since $\mathbf{F}$ is assumed to be linear, it follows that $\mathrm{gr}_\mathfrak{m}(\mathbf{F})$ is a (graded) minimal free resolution of $\mathrm{gr}_\mathfrak{m}(M)$ over $\mathrm{gr}_\mathfrak{m}(R)$. Therefore, we have that $P_M^R = P^{\mathrm{gr}_\mathfrak{m}(R)}_{\mathrm{gr}_\mathfrak{m}(M)}$. Furthermore, as Hilbert series are additive on short exact sequences, we have that

$$\begin{aligned} H_{\mathrm{gr}_\mathfrak{m}(M)}(t) &= H_{\mathrm{gr}_\mathfrak{m}(F_0)}(t) - H_{\mathrm{gr}_\mathfrak{m}(F_1)}(t) + H_{\mathrm{gr}_\mathfrak{m}(F_2)}(t) - H_{\mathrm{gr}_\mathfrak{m}(F_3)}(t) + \cdots \\ &= H_{\mathrm{gr}_\mathfrak{m}(R)}(t) P^{\mathrm{gr}_\mathfrak{m}(R)}_{\mathrm{gr}_\mathfrak{m}(M)}(-t) \\ &= H_{\mathrm{gr}_\mathfrak{m}(R)}(t) P^R_M(-t) \end{aligned}$$

where the second equality holds by virtue of the fact that $\mathrm{gr}_\mathfrak{m}(F_i)$ is a graded free $\mathrm{gr}_\mathfrak{m}(R)$-module, and the sum of the ranks of its components is equal to the ith (total) Betti number of $\mathrm{gr}_\mathfrak{m}(M)$, for each $i \in \mathbb{N}$. Therefore, we can write

$$P_M^R(t) = \frac{H_{\mathrm{gr}_\mathfrak{m}(M)}(-t)}{H_{\mathrm{gr}_\mathfrak{m}(R)}(-t)}.$$

Recalling that $H_{\mathrm{gr}_\mathfrak{m}(M)}(-t)$ is a rational function with a denominator of $(1-t)^{\dim R}$, the result now follows. $\square$

Moreover assuming that $\mathrm{Ext}^i_R(M,R)$ vanishes for $i > 0$, we obtain the following similar characterization of the Poincaré series of M in the variable $\frac{1}{t}$.

Proposition 4.2.2. *Let M be a finitely generated module with an eventually linear minimal free resolution over a local ring $(R, \mathfrak{m})$. If $\mathrm{Ext}^i_R(M,R) = 0$ for all $i > 0$, then there exists $q \in \mathbb{Z}[t]$ such that the following holds.*

$$P_M^R(\tfrac{1}{t}) = \frac{q(t)}{H_{\mathrm{gr}_\mathfrak{m}(R)}(-t)(1+t)^{\dim R}}$$

Proof. The proof is similar to that of Lemma 4.2.1; however, letting $\mathbf{F} \to M \to 0$ be a minimal free resolution, we apply the additivity of the Hilbert series to the exact sequence $\mathrm{Hom}_R(\mathbf{F}, R)$. $\square$

Obvious corollaries to the previous two results exist if we assume the ring to be zero-dimensional. Furthermore, as we will be working over short local rings, this choice is a natural one.

Corollary 4.2.3. *Let M be a finitely generated module with an eventually linear minimal free resolution over a zero-dimensional local ring $(R, \mathfrak{m})$. If $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > 0$, then the following hold.*

(1) $P^R_M(t) H_{\mathrm{gr}_\mathfrak{m}(R)}(-t) \in \mathbb{Z}[t]$

(2) $P^R_M(\frac{1}{t}) H_{\mathrm{gr}_\mathfrak{m}(R)}(-t) \in \mathbb{Z}[t]$

The desired consequence of Corollary 4.2.3 lies in the fact that if M is a finitely generated module with an eventually linear minimal free resolution over a short local ring $(R, \mathfrak{m})$, we can explicitly write down a recursion relation for its Betti sequence $\{b_i\}$ in terms of the Hilbert series $H_{\mathrm{gr}_\mathfrak{m}(R)}(t) = 1 + et + ft^2 + gt^3$ of R. That is,

$$b_{i+3} = b_{i+2}e - b_{i+1}f + b_i g \tag{4.2.3.1}$$

for all $i \gg 0$. If, in addition, $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > 0$, we also have that

$$b_i = b_{i+1}e - b_{i+2}f + b_{i+3}g \tag{4.2.3.2}$$

for all $i \gg 0$.

Remark 4.2.4. Notice that the recursion relation in (4.2.3.1) implies that any short local ring which admits a module with an eventually linear minimal free resolution has a Hilbert series whose general form is given by

$$H_{\mathrm{gr}_\mathfrak{m}(R)}(t) = 1 + et + ft^2 + \left(\frac{b_{i+1}}{b_i}f - \frac{b_{i+2}}{b_i}e + \frac{b_{i+3}}{b_i}\right)t^3$$

for $i \gg 0$, where $\{b_i\}$ is the Betti sequence of the module. However, by considering (4.2.3.1) and (4.2.3.2) simultaneously, we are able to do a bit better: we can write the entire Hilbert series of R in terms of the embedding dimension of R and the Betti sequence of M. The justification of this claim can be seen in the following fact.

Fact 4.2.5. *Let M be a finitely generated module with an eventually linear minimal free resolution over a short local ring $(R, \mathfrak{m})$. Moreover assume that $\operatorname{Ext}_R^i(M, R) = 0$ for all $i > 0$. Then the Hilbert series of R is given by*

$$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + ft^2 + gt^3$$

where

$$\begin{bmatrix} b_i & -b_{i+1} & b_{i+2} \\ 0 & -\dfrac{\Delta_1}{b_i} & \dfrac{\Delta_2}{b_i} \end{bmatrix} \begin{bmatrix} g \\ f \\ e \end{bmatrix} = \begin{bmatrix} b_{i+3} \\ \dfrac{\Delta_3}{b_i} \end{bmatrix}$$

and

$$\begin{aligned} \Delta_1(i,j) &= b_i b_{j+2} - b_{i+1} b_{j+3} \\ \Delta_2(i,j) &= b_i b_{j+1} - b_{i+2} b_{j+3} \qquad (4.2.5.1) \\ \Delta_3(i,j) &= b_i b_j - b_{i+3} b_{j+3} \end{aligned}$$

for all $i, j \gg 0$.

It is straightforward to see that the Δ_ℓ in Fact 4.2.5 are obtained by a simple row reduction of the system that is directly obtained from (4.2.3.1) and (4.2.3.2). Furthermore, a characterization of the Hilbert series of R is now entirely dependent on the possible vanishing of the Δ_ℓ, and in general, the growth of the Betti sequence of M. We address this topic in the following section.

4.3 The Betti Sequence

There is very little known about the asymptotic behavior of the Betti sequence of a finitely generated module of infinite projective dimension over an arbitrary local ring. In particular, an answer to the following question of Avramov remains unknown in general.

Question 4.3.0.1. [9, 4.3.3] If M is a finitely generated R-module with infinite projective dimension, is its Betti sequence eventually non-decreasing?

Of course, a negative answer to this question would introduce the possibility of eventual periodicity of the Betti sequence. We consider this question next.

4.3.1 Periodicity

Our main result for this section will show that Betti sequences associated with linear resolutions over short local rings cannot have 'small' periodicity. We make this statement precise in Theorem 4.3.1.3 below; first, however, we must consider the following fact.

Fact 4.3.1.1. *Let M be a finitely generated module over a short local ring $(R, \mathfrak{m})$, and denote the Betti sequence of M by $\{b_i\}$. Then for each $\ell \in \{1, 2, 3\}$, the quantity $\Delta_\ell(i, j)$, as defined in (4.2.5.1), vanishes for all $i, j \gg 0$ if and only if there exists a positive integer $n|\ell$ such that $\{b_i\}$ is eventually periodic of period n.*

Proof. Fix $\ell \in \{1, 2, 3\}$ and suppose that $b_i b_{j-\ell+3} = b_{i+\ell} b_{j+3}$ for all $i, j \gg 0$. Choosing $j = i + \ell - 3$, we see that $\{b_i\}$ eventually satisfies $b_i^2 = b_{i+\ell}^2$, which of course implies that $b_i = b_{i+\ell}$ for all $i \gg 0$. Thus, $\{b_i\}$ is periodic with period

$$n = \min\{m \mid b_i = b_{i+m} \text{ for all } i \gg 0\}$$

which clearly must divide ℓ.

Conversely, fix $\ell \in \{1, 2, 3\}$ and suppose that there exists some $n \in \mathbb{Z}^+$ with $n|\ell$ such that $\{b_i\}$ is periodic with period n. Then in particular $b_i = b_{i+\ell}$, and moreover $b_i b_{j-\ell+3} = b_{i+\ell} b_{j+3}$, for all $i, j \gg 0$. $\square$

Remarks 4.3.1.2.

(1) The assumption that $\Delta_\ell(i, j) = 0$ for *all* large i, j in the previous Fact is essential. Note that even the vanishing of $\Delta_\ell(i, j)$ for infinitely many choices of i, j does not imply periodicity of $\{b_i\}$, unless the sequence is additionally assumed to be eventually non-decreasing.

(2) The condition that a Betti sequence is not eventually constant is equivalent to the non-vanishing of $\Delta_1(i, j)$ for infinitely many i, j.

Theorem 4.3.1.3. *Let M be a finitely generated module with an eventually linear minimal free resolution over a short local ring $(R, \mathfrak{m})$. If $\operatorname{Ext}^i_R(M, R) = 0$ for all $i > 0$, then the Betti sequence of M is not eventually periodic of period two or three.*

Proof. Let the Betti sequence of M be denoted $\{b_i\}$. By Fact 4.3.1.1, it suffices to show that neither $\Delta_2(i, j)$ nor $\Delta_3(i, j)$, as defined in (4.2.5.1), can vanish for all large values of i and j. Since we can assume that $\{b_i\}$ is eventually non-constant, the set

$$I = \{(n, m) \in \mathbb{N}^2 \mid \Delta_1(n, m) \neq 0\}$$

has infinite cardinality.

First suppose that $\Delta_2(i, j) = 0$ for all $i, j \gg 0$, therefore implying that $\{b_i\}$ eventually has period two. We have

$$f = \frac{\Delta_3(n, m)}{\Delta_1(n, m)} = \frac{b_n b_m - b_{n+3} b_{m+3}}{b_{n+1} b_{m+3} - b_n b_{m+2}} = \frac{b_n b_m - b_{n+1} b_{m+1}}{b_{n+1} b_{m+1} - b_n b_m} = -1$$

for all $(n, m) \in I$, which is absurd.

Next suppose that $\Delta_3(i,j) = 0$ for all $i, j \gg 0$, implying that $\{b_i\}$ eventually has period three. Then

$$f = -\frac{\Delta_2(n,m)}{\Delta_1(n,m)}e = \frac{b_{n+2}b_{m+3} - b_n b_{m+1}}{b_{n+1}b_{m+3} - b_n b_{m+2}}e = \frac{b_{n+2}b_m - b_n b_{m+1}}{b_{n+1}b_m - b_n b_{m+2}}e$$

for all $(n,m) \in I$. If $(n,n) \in I$, then the above expression reduces to $f = -e$, which cannot be true. Otherwise, $(n,n) \notin I$ implies that

$$\Delta_1(n,n) = b_{n+1}b_{n+3} - b_n b_{n+2} = b_n(b_{n+1} - b_{n+2}) = 0$$

so that $b_{n+1} = b_{n+2}$. Because the cardinality of I must be infinite, it follows that $(n, n+1) = (n, n+2) \in I$. In particular, $\Delta_1(n, n+1) \neq 0$, which implies that

$$\Delta_1(n, n+1) = b_{n+1}b_{n+4} - b_n b_{n+3} = b_{n+1}^2 - b_n^2 \neq 0$$

whence $b_n \neq b_{n+1}$. Thus, the expression for h_2 simplifies to

$$f = \frac{b_{n+2}b_{n+1} - b_n b_{n+2}}{b_{n+1}^2 - b_n^2}e = \frac{b_{n+2}}{b_{n+1} + b_n}e.$$

However, since $b_{n+1} = b_{n+2}$, the above quantity is an integer if and only if $b_n = 0$, which is impossible. The result now follows. $\square$

4.3.2 Growth Rates

Among the mystery surrounding the Betti sequence of a finitely generated module over an arbitrary local ring is the following open question of Avramov.

Question 4.3.2.1. [9, 4.3.7] Does there exist a finitely generated module over a local ring whose Betti sequence grows subexponentially but superpolynomially?

In other words, it is not yet known whether the growth rate of a Betti sequence could be strictly bounded between polynomial and exponential rates. It is known that the Betti sequence of a module cannot grow superexponentially by work of Serre. Because of this, we cover all possible cases by making the following definitions.

Definition 4.3.2.2. Let M be a finitely generated R-module. Then the Betti sequence $\{b_i\}$ of M is said to have *polynomial growth* if there exists $n \in \mathbb{N}$ such that, for all $i \gg 0$,

$$\alpha i^n - \lambda_i \leq b_i \leq \alpha i^n + \lambda_i$$

for some $\alpha \in \mathbb{R}^+$ and some sequence $\{\lambda_i\}$ of real numbers satisfying $\lambda_i / i^n \to 0$.

Definition 4.3.2.3. Let M be a finitely generated R-module. Then the Betti sequence $\{b_i\}$ of M is said to have *exponential growth* (*of base* a) if there exists $1 < a \in \mathbb{R}^+$ such that, for all $i \gg 0$,

$$\beta a^i - \rho_i \leq b_i \leq \beta a^i + \rho_i$$

for some $\beta \in \mathbb{R}^+$ and some sequence $\{\rho_i\}$ of real numbers satisfying $\rho_i / a^i \to 0$.

Remarks 4.3.2.4.

(1) The literature often refers to such growth rates in the language of complexity and curvature; cf. [9, 4.2]. While these quantities specify a smallest upper bound for the asymptotic behavior certain Betti sequences, our definitions above provide a largest lower bound as well.

(2) Both of the above growth rates have been extensively studied. It is well-known that over a complete intersection ring, every finitely generated module has a Betti sequence which grows polynomially. Furthermore, exponential growth of Betti numbers has been demonstrated in a variety of settings, including over Golod rings [45], Cohen-Macaulay rings of small multiplicity [23, 39], and certain $\mathfrak{m}^3 = 0$ local rings [33].

Finally, we define a special type of linear growth which will be of importance to our results in the next section.

Definition 4.3.2.5. Let M be a finitely generated R-module. The Betti sequence $\{b_i\}$ of M is said to be *exceptional* if

$$b_{i+1} - b_i = b_{i+3} - b_{i+2}$$

for all $i \gg 0$.

Remark 4.3.2.6. Clearly, any Betti sequence which is either constant or exactly linear (ie. $b_{i+1} = b_i + \alpha$ for some $\alpha \in \mathbb{Z}^+$) is exceptional. However, a sequence such that

$$b_{i+1} - b_i = b_{i+3} - b_{i+2} \neq b_{i+2} - b_{i+1}$$

for all $i \gg 0$ is also exceptional. Though such growth may seem pathological, it has been discovered to occur, over certain codimension two complete intersections, by Avramov and Buchweitz [10]. The Betti sequence constructed by the authors of [10] is strictly increasing. It is unknown whether there exists a not eventually constant exceptional Betti sequence $\{b_i\}$ such that $b_{2i+1} \leq b_{2i}$ for all $i \gg 0$.

In the next section, we describe the Hilbert series of a short local ring in terms of the Betti sequence of its modules.

4.4 The Hilbert Series

The ultimate goal of this section is to investigate necessary conditions on the Hilbert series of a short local ring in order for the ring to admit certain asymmetric complete resolutions. However, there is much to be said about the Hilbert series of such a ring in the more general setting. That is, en route to the study of total reflexivity over a short local ring R, we shall first consider the existence of an R-module M having a linear resolution and satisfying the vanishing of $\mathrm{Ext}_R^i(M, R)$ for $i > 0$.

4.4.1 The General Form

Recall that the general form for the Hilbert series of a short local ring $(R, \mathfrak{m})$ which admits a finitely generated module M with an eventually linear minimal free resolution is given by

$$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + ft^2 + \left(\frac{b_{i+1}}{b_i}f - \frac{b_{i+2}}{b_i}e + \frac{b_{i+3}}{b_i}\right)t^3$$

for $i \gg 0$, where $\{b_i\}$ denotes the Betti sequence of M. If we furthermore assume that the Betti sequence of M has either polynomial or exponential growth, we obtain the following result.

Lemma 4.4.1.1. *Let M be a finitely generated module with an eventually minimal free resolution over a short local ring $(R, \mathfrak{m})$. The following hold.*

(1) *If the Betti sequence of M has polynomial growth, then*

$$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + ft^2 + (f - e + 1)t^3.$$

(2) *If the Betti sequence of M has exponential growth of base a, then*

$$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + ft^2 + (af - a^2e + a^3)t^3.$$

Proof. Let the Betti sequence of M be denoted by $\{b_i\}$. First we prove (1). By the hypothesis, we know that there exists $n \in \mathbb{N}$ such that, for all $i \gg 0$,

$$\alpha i^n - \lambda_i \leq b_i \leq \alpha i^n + \lambda_i$$

for some $\alpha \in \mathbb{R}^+$ and some sequence $\{\lambda_i\}$ satisfying $\lambda_i / i^n \to 0$. We therefore have the following bound.

$$\begin{aligned} g &= \frac{b_{i+1}e - b_{i+2}f + b_{i+3}}{b_i} \\ &\leq \frac{\left(\alpha(i+1)^n + \lambda_{i+1}\right)f - \left(\alpha(i+2)^n - \lambda_{i+2}\right)e + \left(\alpha(i+3)^n + \lambda_{i+3}\right)}{\alpha i^n - \lambda_i} \end{aligned}$$

Notice that this quantity can be made arbitrarily close to $f-e+1$ for $i \gg 0$, and we can similarly show that g is bounded from below by a quantity that asymptotically approaches $f-e+1$. Hence $H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + ft^2 + (f-e+1)t^3$, which is what was to be proved.

To show (2), notice that by the hypothesis, there exists an $1 < a \in \mathbb{R}^+$ such that, for all $i \gg 0$,

$$\beta a^i - \rho_i \leq b_i \leq \beta a^i + \rho_i$$

for some $\beta \in \mathbb{R}^+$ and some sequence $\{\rho_i\}$ satisfying $\rho_i/a^i \to 0$. As in the proof of (1), we proceed to bound g.

$$\begin{aligned} g &= \frac{b_{i+1}e - b_{i+2}f + b_{i+3}}{b_i} \\ &\leq \frac{(\beta a^{i+1} + \rho_{i+1})\, f - (\beta a^{i+2} - \rho_{i+2})\, e + (\beta a^{i+3} + \rho_{i+3})}{\beta a^i - \rho_i} \end{aligned}$$

Therefore, g is bounded from above by a quantity that can be made asymptotically close to $af - a^2e + a^3$ for $i \gg 0$. The same can be shown for a lower bound of g. Thus, $H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + ft^2 + (af - a^2e + a^3)t^3$, as claimed. □

We illustrate the application of the characterizations provided by Lemma 4.4.1.1 in the following examples.

Example 4.4.1.2. Let $R = k[\![w,x,y,z]\!]/(w^2, wx, x^2, y^2, z^2)$ with unique maximal ideal $\mathfrak{m} = (w,x,y,z)$, and consider the R-module $M = R/(w,x)$. One can use an inductive argument to show that the nth map of the minimal free resolution of M over R is represented by the block diagonal matrix

$$\begin{bmatrix} w & x & & & & & \\ & & w & x & & & \\ & & & & \ddots & & \\ & & & & & w & x \end{bmatrix}_{n \times 2n}$$

with respect to the standard bases of R^n and R^{2n}. It is easy to see that M has a linear minimal free resolution and that its Betti sequence has exponential growth of base 2. Thus, we can use Lemma 4.4.1.1(2) to recover the last coefficient of $H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t)$.

$$\begin{aligned} H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) &= 1 + 4t + 5t^2 + 2t^3 \\ &= 1 + 4t + 5t^2 + \left[(2)(5) - (2^2)(4) + (2^3)\right] t^3 \end{aligned}$$

The results of Lemma 4.4.1.1 do not assume the vanishing of $\mathrm{Ext}^i_R(M,R)$ for $i > 0$, and thus do not utilize the recursion relation in (4.2.3.2). In the next section, we shall investigate the additional restrictions on $H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t)$ which arise if we make this assumption.

4.4.2 Vanishing of $\mathrm{Ext}^*_R(M,R)$

The remaining results of this manuscript will heavily depend on the characterization in the following result.

Proposition 4.4.2.1. *Let M be a finitely generated module with an eventually linear minimal free resolution over a short local ring $(R,\mathfrak{m})$, and suppose that $\mathrm{Ext}^i_R(M,R) = 0$ for all $i > 0$. Choose $n \geq \mathrm{ld}_R(M)$, and let $b_n, \ldots, b_{n+3}$ be consecutive Betti numbers of M such that $b_{n+1}b_{n+3} \neq b_n b_{n+2}$. Then $H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + ft^2 + gt^3$, where the following hold.*

$$f = \frac{(b_{n+2}b_{n+3} - b_n b_{n+1})e - (b_{n+3}^2 - b_n^2)}{b_{n+1}b_{n+3} - b_n b_{n+2}}$$

$$g = \frac{(b_{n+2}^2 - b_{n+1}^2)e - (b_{n+2}b_{n+3} - b_n b_{n+1})}{b_{n+1}b_{n+3} - b_n b_{n+2}}$$

Proof. It suffices to solve the system of equations in Fact 4.2.5. Notice that the condition that $b_{n+1}b_{n+3} \neq b_n b_{n+2}$ is equivalent to assuming that $\Delta_1(n,n) \neq 0$. Therefore, we have

$$f = -\frac{\Delta_2(n,n)}{\Delta_1(n,n)}e + \frac{\Delta_3(n,n)}{\Delta_1(n,n)} = \frac{(b_{n+2}b_{n+3} - b_nb_{n+1})e - (b_{n+3}b_{n+3} - b_nb_n)}{b_{n+1}b_{n+3} - b_nb_{n+2}}$$

and

$$g = \frac{b_{n+1}}{b_n}f - \frac{b_{n+2}}{b_n}e + \frac{b_{n+3}}{b_n}.$$

which imply the result upon simplification. □

Example 4.4.2.2. Let $R = k[\![w,x,y,z]\!]/(w^2, wx, x^2, y^2, z^2)$ be as in the previous example, but consider the R-module $M = R/(x)$. Since $\operatorname{Ext}^1_R(M,R) \cong R/(w,x) \neq 0$, one would not expect Proposition 4.4.2.1 to recover the last two coefficients of $H_{\operatorname{gr}_{\mathfrak{m}}(R)}(t)$. Indeed,

$$\begin{aligned} f = 5 \neq \frac{19}{4} &= \frac{(b_3b_4 - b_1b_2)e - (b_4^2 - b_1^2)}{b_2b_4 - b_1b_3} \\ g = 2 \neq \frac{3}{2} &= \frac{(b_3^2 - b_2^2)e - (b_3b_4 - b_1b_2)}{b_2b_4 - b_1b_3}. \end{aligned}$$

We now consider the characterization in Proposition 4.4.2.1 given certain behavior in the Betti sequence. We begin with polynomial growth.

Theorem 4.4.2.3. *Let M be a finitely generated module with an eventually linear minimal free resolution over a short local ring $(R, \mathfrak{m})$, and suppose that $\operatorname{Ext}^i_R(M,R) = 0$ for all $i > 0$. If the Betti sequence of M has non-exceptional polynomial growth, then*

$$H_{\operatorname{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + et^2 + t^3.$$

Proof. By virtue of Lemma 4.4.1.1(1), the Hilbert series of R must take the form

$$H_{\operatorname{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + ft^2 + (f - e + 1)t^3.$$

We will use this fact, along with the statement of Proposition 4.4.2.1, to show that $f = e$, thus implying the result.

Let $\{b_i\}$ denote the Betti sequence of M. Then we have that

$$g = \frac{b_{i+1}}{b_i} f - \frac{b_{i+2}}{b_i} e + \frac{b_{i+3}}{b_i} = f - e + 1$$

for $i \gg 0$. Solving for f now yields

$$f = \left(\frac{b_{i+2} - b_i}{b_{i+1} - b_i} \right) e - \frac{b_{i+3} - b_i}{b_{i+1} - b_i} \tag{4.4.2.3.1}$$

for $i \gg 0$. Equating the expressions for f in Lemma 4.4.1.1 and (4.4.2.3.1) implies

$$\frac{(b_{i+2}b_{i+3} - b_i b_{i+1})e - (b_{i+3}^2 - b_i^2)}{b_{i+1}b_{i+3} - b_i b_{i+2}} = \frac{(b_{i+2} - b_i)e - (b_{i+3} - b_i)}{b_{i+1} - b_i}$$

whence we obtain the following equation, for all $i \gg 0$.

$$\begin{aligned}(b_i b_{i+2} - b_{i+2}^2 - b_{i+1}b_{i+3})e &- (b_i b_{i+2} - b_{i+2}b_{i+3} - b_{i+1}b_{i+3}) \\ &= (b_i b_{i+1} - b_{i+1}^2 - b_{i+2}b_{i+3})e - (b_i^2 - b_i b_{i+1} - b_{i+3}^2)\end{aligned}$$

One can now factor this equation and arrive at the following

$$(b_{i+2} - b_{i+1})(b_{i+1} + b_{i+2} - b_i - b_{i+3})e = (b_{i+3} - b_i)(b_{i+1} + b_{i+2} - b_i - b_{i+3})$$

which again must hold for all large values of i. Now, since we have assumed that $\{b_i\}$ is not exceptional, it follows that that $b_{i+1} + b_{i+2} - b_i - b_{i+3}$ does not vanish infinitely often. To make this precise, let

$$I = \{i \geq \mathrm{ld}_R(M) \mid b_{i+1} + b_{i+2} - b_i - b_{i+3} \neq 0\}.$$

Therefore,

$$e = \frac{b_{i+3} - b_i}{b_{i+2} - b_{i+1}}$$

for all $i \in I$. However, substituting this value for e into the the expression for f in

(4.4.2.3.1) and simplifying, we obtain

$$\begin{aligned} f &= \frac{b_{i+2}-b_i}{b_{i+1}-b_i}\left(\frac{b_{i+3}-b_i}{b_{i+2}-b_{i+1}}\right) - \frac{b_{i+3}-b_1}{b_{i+1}-b_i} \\ &= \frac{b_{i+3}-b_i}{b_{i+2}-b_{i+1}} \\ &= e \end{aligned}$$

for all $i \in I$. This, of course, implies the result. $\square$

Example 4.4.2.4. Let $R = k[\![x,y,z]\!]/(x^2,y^2,z^2)$. Then the residue field $k := R/(x,y,z)$ is a totally reflexive R-module; in particular, $\operatorname{Ext}^i_R(k,R) = 0$ for $i > 0$.

By virtue of the fact that R is a complete intersection ring, we know that the Betti sequence of k has polynomial growth. Even better than this, as R is Koszul, we can explicitly write down its Poincaré series as

$$P_R(t) = \frac{1}{H_{\operatorname{gr}_{\mathfrak{m}}(R)}(-t)} = \frac{1}{(1-t)^3} = \sum_{i\in\mathbb{N}} \tbinom{i+2}{2} t^i = \tfrac{1}{2}\textstyle\sum_{i\in\mathbb{N}} (i^2+3i+2)\, t^i.$$

Since the Betti sequence of k has quadratic growth, it is not exceptional. Furthermore, we know that the above resolution must be linear since R is Koszul. We also note that the Hilbert series of R is given by $H_{\operatorname{gr}_{\mathfrak{m}}(R)}(t) = 1+3t+3t^2+t^3$.

Remark 4.4.2.5. The hypothesis in Theorem 4.4.2.3 that $\{b_i\}$ is non-exceptional is sufficient to obtain a symmetric Hilbert series; however, it is not necessary. We shall illustrate this fact in the following examples.

Example 4.4.2.6. As in Example 4.4.2.4, let $R = [\![x,y,z]\!]/(x^2,y^2,z^2)$. Consider the totally reflexive R-module $M = R/(x)$ and its minimal free resolution given by

$$\cdots \to R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \to M \to 0$$

which is certainly linear. Furthermore, as the Betti sequence of M is constant, it is exceptional. Recall that the Hilbert series of R is symmetric.

Example 4.4.2.7. Let $R = k[\![w,x,y,z]\!]/(w^2, wx, x^2, y^2, z^2)$, which has an embedded deformation given by $k[\![w,x,y,z]\!]/(w^2, x^2, wx) = S \to S/(y^2, z^2) \cong R$. If we define $M = R/(y,z)$, then one can check that $\operatorname{Ext}^i_R(M,R)$ vanishes for $i > 0$. Furthermore, the minimal free resolution of M over R is given by the following.

$$\cdots \to R^4 \xrightarrow{\begin{bmatrix} y & 0 & z & 0 \\ 0 & z & 0 & y \\ 0 & 0 & -y & -z \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} y & 0 & z \\ 0 & z & -y \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} y & z \end{bmatrix}} R \to M \to 0$$

At this point, the Poincaré series of M might be fairly obvious. However, to be thorough we set $P = k[\![w,x]\!]/(w^2, wx, x^2)$ and $Q = k[\![y,z]\!]/(y^2, z^2)$, and notice that $P \otimes_k Q \cong R$ and $P \cong M$ as k-algebras. Therefore, let $\mathbf{F} \to k \to 0$ be a minimal Q-free resolution of $k \cong Q/(y,z)$. The ranks of the free modules in $\mathbf{F}$ are well-understood since k is the residue field of Q; we demonstrate them in the following Poincaré series.

$$P_Q(t) = \frac{1}{H_{\operatorname{gr}_{\mathfrak{m}}(Q)}(-t)} = \frac{1}{(1-t)^2} = \sum_{i \in \mathbb{N}} (i+1)\, t^i \tag{4.4.2.7.1}$$

Furthermore, since $\operatorname{Tor}^k_i(P,Q) = 0$ for all $i > 0$ (cf. Example 3.3.2 for details), $\mathbf{F} \otimes_k P$ is a minimal free resolution of M over R. We can therefore conclude that the Poincaré series of M over R is the same as the one given in (4.4.2.7.1). In particular, the Betti sequence of M is exceptional. Furthermore, recall that the Hilbert series of R is not symmetric; in fact, we have $H_{\operatorname{gr}_{\mathfrak{m}}(R)}(t) = 1 + 4t + 5t^2 + 2t^3$.

We now consider the characterization of the Hilbert series of a short local ring whenever its modules have exponentially growing Betti numbers.

Theorem 4.4.2.8. *Let M be a finitely generated module with an eventually linear minimal free resolution over a short local ring $(R, \mathfrak{m})$, and suppose that $\operatorname{Ext}^i_R(M,R) = 0$ for all $i > 0$. If the Betti sequence of M has exponential growth of base a, then*

$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + ft^2 + gt^3$, *where*

$$f = \left(a + \frac{1}{a}\right)e - \left(a^2 + 1 + \frac{1}{a^2}\right)$$

$$g = e - \left(a + \frac{1}{a}\right).$$

Proof. Let $\{b_i\}$ denote the Betti sequence of M. By assumption, for all $i \gg 0$ we have

$$\beta a^i - \rho_i \leq b_i \leq \beta a^i + \rho_i \tag{4.4.2.8.1}$$

for some $\beta \in \mathbb{R}^+$ and some sequence $\{\rho_i\}$ of real numbers satisfying $\rho_i/a^i \to 0$.

We shall proceed by bounding the expressions for f and g found in Proposition 4.4.2.1 using (4.4.2.8.1). In the interest of space, we omit the tedious details which are analogous to those found in the proof of Lemma 4.4.1.1(2). Indeed, we obtain the following expressions

$$f = \frac{(a^5 - a)e - (a^6 - 1)}{a^4 - a^2}$$

$$g = \frac{(a^4 - a^2)e - (a^5 - a)}{a^4 - a^2}$$

which one can simplify to arrive at the result. □

An immediate corollary to the previous result is apparent if we consider the fact that the expressions for f and g must be positive integers.

Corollary 4.4.2.9. *Let M be a finitely generated module with an eventually linear minimal free resolution over a short local ring $(R, \mathfrak{m})$, and suppose that $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > 0$. If the Betti sequence of M has exponential growth of base a, then*

$$a = r + s\sqrt{\alpha}$$

for some non-zero $r, s \in \mathbb{Q}$ and $\alpha \in \mathbb{Z}^+$ such that $r^2 - \alpha s^2 = 1$. In particular, a must irrational.

Proof. Suppose that $a = \frac{p}{q}$, where $p, q \in \mathbb{Z}^+$ are relatively prime. Then

$$\frac{p}{q} + \frac{q}{p} = \frac{p^2 + q^2}{pq} = n$$

for some $n \in \mathbb{Z}^+$, which implies that $p^2 - npq + q^2 = 0$. Solving for p now yields

$$p = \frac{nq \pm \sqrt{n^2q^2 - 4q^2}}{2} = \frac{nq \pm q\sqrt{n^2 - 4}}{2}.$$

In order for this quantity to be an integer, it must be true that $n = 2$, which corresponds to the case that $p = q = 1$, a contradiction.

Therefore, let $r, s \in \mathbb{Q}$ and $\alpha \in \mathbb{R}^+ \setminus \mathbb{Q}$ be such that $a = r + s\sqrt{\alpha}$. (The fact that a can be written in this form is a direct consequence of Theorem 4.4.2.8.) One can quickly check that

$$a + \frac{1}{a} = \frac{r(r^2 - \alpha s^2 + 1) + (r^2 - \alpha s^2 - 1)\sqrt{\alpha}}{r^2 - \alpha s^2}$$

whence it follows that $r^2 - \alpha s^2 = 1$. $\square$

Remark 4.4.2.10. In light of Lemma 4.4.1.1, if one wishes to find a short local ring which admits linear resolutions and possesses a Hilbert series which is *not* balanced, it would be natural to expect the ring to only admit exponentially growing Betti sequences. In fact, Theorem 4.4.2.8 does not even guarantee that such a Hilbert series is balanced in the case that the module satisfies the vanishing of Ext condition. In the following example, we illustrate this scenario.

Example 4.4.2.11. Define local rings $S = k[\![x, y, z]\!]/(x^2 - y^2, x^2 - z^2, xy, xz, yz)$ and $Q = k[\![u, v]\!]/(u^2, uv, v^2)$, with maximal ideals $\mathfrak{m}_S = (x, y, z)$ and $\mathfrak{m}_Q = (u, v)$, respectively. Notice that Example 3.3.2 guarantees that the local ring

$$R := S \otimes_k Q \cong k[\![u, v, w, x, y, z]\!]/(u^2, uv, v^2, x^2 - y^2, x^2 - z^2, xy, xz, yz)$$

admits non-trivial totally reflexive modules; in particular, $M = R/(x, y, z)$ is one such module.

Now, one would assume that the Betti sequence of M over R would coincide with that of $k \cong S/\mathfrak{m}_S$ over S. To check this, first note that $M \cong Q$ as k-algebras. Further, let $\mathbf{F} \to k \to 0$ be a minimal S-free resolution. Since $\operatorname{Tor}_i^k(S, Q)$ vanishes for $i > 0$, it follows that $\mathbf{F} \otimes_k Q$ is a minimal free resolution of M over R. Since S is a Gorenstein ring satisfying $\mathfrak{m}_S^3 = 0$, the Betti sequence of the residue field k over S has exponential growth. Therefore, the same must be true of the Betti sequence of M over R.

Furthermore notice that one can easily check that the Hilbert series

$$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + 5t + 7t^2 + 2t^3$$

of R is clearly not balanced. Also, by using the statement of Theorem 4.4.2.8, we can recover the base a of the exponential growth of the Betti sequence of M. Indeed,

$$\begin{aligned} a &= \frac{e - g \pm \sqrt{((g-e)^2 - 4)}}{2} \\ &= \frac{3 \pm \sqrt{5}}{2} \end{aligned}$$

which implies that $a = \frac{3}{2} + \frac{1}{2}\sqrt{5} > 1$.

Remark 4.4.2.12. Indeed, we can generalize the previous example. To this end, let $S = k[\![x_1, \ldots, x_n]\!]/I$ and $Q = k[\![y_1, \ldots, y_m]\!]/J$ where I is generated over $k[\![x_1, \ldots, x_n]\!]$ by $x_1^2 - x_j^2$ and $x_i x_j$ for $0 \le i < j \le n$, and where J is generated over $k[\![y_1, \ldots, y_m]\!]$ by $y_i y_j$ for $1 \le i \le j \le n$. It is clear to see that S is a Gorenstein local ring with Hilbert series $H_{\mathrm{gr}_\mathfrak{m}(S)}(t) = 1+nt+t^3$, and that Q is a Cohen-Macaulay local ring with Hilbert series $H_{\mathrm{gr}_\mathfrak{m}(Q)}(t) = 1+mt$. Furthermore, as S and Q are Tor-independent over k, the Hilbert series of $R := S \otimes_k Q$ is given by

$$\begin{aligned} H_{\mathrm{gr}_\mathfrak{m}(R)}(t) &= H_{\mathrm{gr}_\mathfrak{m}(S)}(t) \cdot H_{\mathrm{gr}_\mathfrak{m}(Q)}(t) \\ &= (1+nt+t^2)(1+mt) \\ &= 1+(n+m)t+(1+nm)t^2+mt^3 \end{aligned}$$

which is only balanced if $m = 1$ or if $n = 2$.

We now turn our attention to the existence of R-modules M with linear resolutions which not only satisfy $\mathrm{Ext}^i_R(M,R) = 0$ for $i > 0$, but which are also totally reflexive.

4.4.3 Asymmetric Complete Resolutions

Our ultimate goal for this section is to investigate necessary conditions for a short local ring to admit certain asymmetric (linear) complete resolutions. However, our actual results are even more general than this: we only require the existence of two R-modules, M and N, such that both $\mathrm{Ext}^i_R(M,R)$ and $\mathrm{Ext}^i_R(N,R)$ vanish for $i > 0$. It is important for the reader to note that the added condition $N \cong M^*$ is not sufficient to obtain the total reflexivity of M. Indeed, it is also necessary to assume that M is reflexive, cf. Remark 2.4.3.8.

We begin by considering the sort of asymmetric growth of Betti numbers which is apparent in the example provided in [31]: polynomial vs. exponential growth.

Theorem 4.4.3.1. *Let M and N be finitely generated modules, each with an eventually linear minimal free resolution, over a short local ring $(R, \mathfrak{m})$. Suppose that the Betti sequence of M has polynomial growth, and the Betti sequence of N has* *exponential growth of base a. Furthermore assume that $\operatorname{Ext}_R^i(N,R) = 0$ for $i > 0$. Then*

$$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + et^2 + t^3.$$

Proof. By Lemma 4.4.1.1(1) we know that the Hilbert series of R must be balanced; that is, $H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) = 1 + et + ft^2 + (f - e + 1)t^3$. Furthermore, since $\operatorname{Ext}_R^i(N,R) = 0$ for $i > 0$, we can use the characterization of f in Theorem 4.4.2.8 to obtain

$$\begin{aligned} g &= f - e + 1 \\ &= \left(a + \frac{1}{a}\right) e - \left(a^2 + 1 + \frac{1}{a^2}\right) - e + 1 \\ &= \left(a - 1 + \frac{1}{a}\right) e - \left(a^2 + \frac{1}{a^2}\right). \end{aligned}$$

However, Theorem 4.4.2.8 provides $g = e - \left(a + \dfrac{1}{a}\right)$. Equating these expressions for g, we can solve for e to obtain

$$e = a + 1 + \frac{1}{a}.$$

With this value of e, it is now clear that the Hilbert series of R must also be symmetric.

□

Example 4.4.3.2. Let R and M be as in Example 4.1.1. Though the authors of [31] did not explicitly state the base a of the exponential growth of the Betti sequence of M, we are able to use Theorem 4.4.3.1 to recover its value. Indeed, recalling that the embedding dimension of R is six, we obtain the following quadratic equation in the variable a.

$$6 = a + 1 + \frac{1}{a} \qquad \Longrightarrow \qquad a^2 - 5a + 1 = 0$$

Solving now yields $a = \frac{5}{2} + \frac{1}{2}\sqrt{21}$.

The statement of Theorem 4.4.3.1 is quite interesting. One important application of it is summed up in the following remark.

Remark 4.4.3.3. In light of Theorem 4.4.3.1, it is impossible for the ring illustrated in Example 4.4.2.11 to admit Koszul modules with polynomially growing Betti sequences. Notice, in particular, that this implies the ring does not have an exact pair of zero divisors.

Finally, we state a result which essentially states that asymmetric complete resolutions with exponential vs. exponential growth cannot occur.

Theorem 4.4.3.4. *Let M and N be finitely generated modules, each with an eventually linear minimal free resolution, over a short local ring $(R, \mathfrak{m})$. Suppose that $\operatorname{Ext}^i_R(M,R) = 0 = \operatorname{Ext}^i_R(N,R)$ for all $i > 0$. If the Betti sequences of M and N have exponential growth of bases a and b, respectively, then $a = b$.*

Proof. Suppose the contrary. By Theorem 4.4.2.8 we have

$$g = e - \left(a + \frac{1}{a}\right) = e - \left(b + \frac{1}{b}\right)$$

which simplifies to yield $ab = 1$. Since both a and b must be larger than one, we have reached a contradiction. $\square$

REFERENCES

[1] M. André, *Hopf algebras with divided powers*, J. Algebra **18** (1971), 19–50. MR0277590

[2] M. Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des exposés de Maurice Auslander, Marquerite Mangeney, Christian Peskine et Lucien Szpiro. École Normale Supérieure de Jeunes Filles, Secrétariat mathématique, Paris, 1967. MR0225844

[3] ———, *Finite type implies isolated singularity*, Orders and their applications (Oberwolfach, 1984), 1985, pp. 1–4. MR812487

[4] ———, *Isolated singularities and existence of almost split sequences*, Representation theory, II (Ottawa, Ont., 1984), 1986, pp. 194–242. MR842486

[5] M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969. MR0269685

[6] M. Auslander and D. A. Buchsbaum, *Homological dimension in Noetherian rings*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 36–38. MR0075190

[7] ———, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390–405. MR0086822

[8] L. L. Avramov, *Modules of finite virtual projective dimension*, Invent. Math. **96** (1989), no. 1, 71–101. MR981738

[9] ———, *Infinite free resolutions*, Six lectures on commutative algebra, 2010, pp. 1–118. MR2641236

[10] L. L. Avramov and R.-O. Buchweitz, *Homological algebra modulo a regular sequence with special attention to codimension two*, J. Algebra **230** (2000), no. 1, 24–67. MR1774757

[11] L. L. Avramov, V. N. Gasharov, and I. V. Peeva, *A periodic module of infinite virtual projective dimension*, J. Pure Appl. Algebra **62** (1989), no. 1, 1–5. MR1026870

[12] ———, *Complete intersection dimension*, Inst. Hautes Études Sci. Publ. Math. **86** (1997), 67–114. MR1608565

[13] L. L. Avramov and A. Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. (3) **85** (2002), no. 2, 393–440. MR1912056

[14] K. A. Beck, *On the Hilbert series of an* $\mathfrak{m}^4 = 0$ *local ring admitting linear complete resolutions* (in preparation).

[15] ———, *Existence of totally reflexive modules via Gorenstein homomorphisms*, J. Commut. Algebra (to appear).

[16] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR1251956

[17] L. W. Christensen and H. Holm, *Ascent properties of Auslander categories*, Canad. J. Math. **61** (2009), no. 1, 76–108. MR2488450

[18] L. W. Christensen, D. A. Jorgensen, H. Rahmati, J. Striuli, and R. Wiegand, *Brauer-Thrall for totally reflexive modules* (submitted), available at `arXiv:1008.1737v1`.

[19] L. W. Christensen, G. Piepmeyer, J. Striuli, and R. Takahashi, *Finite Gorenstein representation type implies simple singularity*, Adv. Math. **218** (2008), no. 4, 1012–1026. MR2419377

[20] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. MR1322960

[21] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633. MR1363858

[22] ———, *Relative homological algebra*, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000. MR1753146

[23] V. N. Gasharov and I. V. Peeva, *Boundedness versus periodicity over commutative local rings*, Trans. Amer. Math. Soc. **320** (1990), no. 2, 569–580. MR967311

[24] I. B. Henriques and L. M. Şega, *Free resolutions over short Gorenstein local rings*, Math. Z. (to appear).

[25] J. Herzog and S. Iyengar, *Koszul modules*, J. Pure Appl. Algebra **201** (2005), no. 1-3, 154–188. MR2158753

[26] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 167–193. MR2038564

[27] ———, *Construction of totally reflexive modules from an exact pair of zero divisors*, Bull. London Math. Soc. (to appear).

[28] C. Huneke and G. J. Leuschke, *Two theorems about maximal Cohen-Macaulay modules*, Math. Ann. **324** (2002), no. 2, 391–404. MR1933863

[29] D. A. Jorgensen, *On tensor products of rings and extension conjectures*, J. Commut. Algebra **1** (2009), no. 4, 635–646. MR2575835

[30] D. A. Jorgensen and W. F. Moore, *Minimal intersections and vanishing (co)homology*, J. Commut. Algebra **1** (2009), no. 3, 507–536. MR2524864

[31] D. A. Jorgensen and L. M. Şega, *Asymmetric complete resolutions and vanishing of Ext over Gorenstein rings*, Int. Math. Res. Not. **56** (2005), 3459–3477. MR2200585

[32] ———, *Independence of the total reflexivity conditions for modules*, Algebr. Represent. Theory **9** (2006), no. 2, 217–226. MR2238367

[33] J. Lescot, *Asymptotic properties of Betti numbers of modules over certain rings*, J. Pure Appl. Algebra **38** (1985), no. 2-3, 287–298. MR814184

[34] G. J. Leuschke and R. Wiegand, *Ascent of finite Cohen-Macaulay type*, J. Algebra **228** (2000), no. 2, 674–681. MR1764587

[35] C. Löfwall, *On the homotopy Lie algebra of a local ring*, J. Pure Appl. Algebra **38** (1985), no. 2-3, 305–312. MR814186

[36] ———, *On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra*, Algebra, algebraic topology and their interactions (Stockholm, 1983), 1986, pp. 291–338. MR846457

[37] H. Matsumura, *Commutative ring theory*, Second, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid. MR1011461

[38] J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264. MR0174052

[39] I. Peeva, *Exponential growth of Betti numbers*, J. Pure Appl. Algebra **126** (1998), no. 1-3, 317–323. MR1600558

[40] P. Roberts, *Le théorème d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), no. 7, 177–180. MR880574

[41] J. J. Rotman, *An introduction to homological algebra*, Second, Universitext, Springer, New York, 2009. MR2455920

[42] J.-P. Serre, *Sur la dimension homologique des anneaux et des modules noethériens*, Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, 1956, pp. 175–189. MR0086071

[43] J. Shamash, *The Poincaré series of a local ring*, J. Algebra **12** (1969), 453–470. MR0241411

[44] G. Sjödin, *Hopf algebras and derivations*, J. Algebra **64** (1980), no. 1, 218–229. MR575792

[45] L.-C. Sun, *Growth of Betti numbers of modules over generalized Golod rings*, J. Algebra **199** (1998), no. 1, 88–93. MR1489355

[46] R. Takahashi, *An uncountably infinite number of indecomposable totally reflexive modules*, Nagoya Math. J. **187** (2007), 35–48. MR2354554

[47] R. Takahashi and K.-i. Watanabe, *Totally reflexive modules constructed from smooth projective curves of genus* $g \geq 2$, Arch. Math. (Basel) **89** (2007), no. 1, 60–67. MR2322781

[48] O. Veliche, *Construction of modules with finite homological dimensions*, J. Algebra **250** (2002), no. 2, 427–449. MR1899298

[49] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324

[50] Y. Yoshino, *Modules of G-dimension zero over local rings with the cube of maximal ideal being zero*, Commutative algebra, singularities and computer algebra (Sinaia, 2002), 2003, pp. 255–273. MR2030276

[51] ———, *A functorial approach to modules of G-dimension zero*, Illinois J. Math. **49** (2005), no. 2, 345–367 (electronic). MR2163939

BIOGRAPHICAL STATEMENT

Kristen Ann Beck was born in Lakeland, Florida on May 28, 1978 to Richard and Karen Beck. She was raised the eldest of four children, spending much of her childhood in the Dallas/Fort Worth metroplex. Kristen received her primary education from many public and private schools throughout the country, but was awarded a high school diploma from Covenant Christian Academy in Colleyville, Texas.

Kristen enrolled at the University of Texas at Arlington in the August of 1996 with the intention of studying Aerospace Engineering, but soon realized that she was more interested in theory than application. She earned a Bachelor of Science degree in Mathematics in May of 2002, and a Master of Science degree in Mathematics in August of 2005. Before beginning studies at UT Arlington that would lead to her Doctor of Philosophy degree, Kristen spent two years working as a Credit Risk Analyst for AmeriCredit Corporation in Fort Worth, Texas. In May 2011, she was awarded a Ph.D. in Mathematics under the direction of David Jorgensen.

Kristen's research interests lie in commutative algebra, with a particular focus on homological properties of commutative local (Noetherian) rings.