CONDITIONAL CONFIDENCE INTERVALS OF PROCESS CAPABILITY INDICES FOLLOWING REJECTION OF PRELIMINARY TESTS

by

JIANCHUN ZHANG

Presented to the Faculty of the Graduate School of
The University of Texas at Arlington in Partial Fulfillment
of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON
May 2010

To my family

My wife Qizhi Ye and daughter Jane Zhang

ACKNOWLEDGMENTS

I would like to give my deepest appreciation to my supervising professor Dr. Chien-Pai Han, without his support and advices, all of these works are not possible to be finished. His constant patience and motivation encouraged me throughout the entire period of my Ph.D. study. I also would like to say thanks to Dr. D.L. Hawkins, Dr. Danny Dyer, Dr. Shan Sun-Mitchell, and Dr. Andrzej Korzeniowski to serve as my committee. I received lots of academic helps form them for the past several years, especially from Dr. Hawkins, without his helps on SAS programming, some of the works are also not possible to be done. Also many thanks to all the other teachers: Dr. Carole King Krueger, Dr. Hua Shan, Dr. Ruth Gornet, Dr. Hristo Kojouharov, Dr David liu, etc.. I will never forget the financial support I got from the Department of Mathematics, without the GTA support from the Math Department, it's hardly for me to concentrate on my Ph.D. study, so, many thanks to the Department of Mathematics of UTA, and the department chair Dr. Jianping Zhu. I also want to say thanks to the department graduate adviser Dr. Jianzhong Su for his valued advices. Finally, I want to say thanks to my family, my wife Qizhi, my daughter Jane and my parents, thank them for trusting me always. Special thanks to my brothers Jianguo Zhang and Jianbing Zhang, thanks to them for taking care of all the things I left behind in China, especially for taking care of my parents.

April 9, 2010

ABSTRACT

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JIANCHUN ZHANG, Ph.D.

The University of Texas at Arlington, 2010

Supervising Professor: Chien-Pai Han

Finding an ordinary confidence interval of an unknown parameter is well known, but finding a conditional confidence interval following rejection of a preliminary test is not so noted, especially for finding a conditional confidence interval of the process capability indices C_p or C_{pk} following rejection of some preliminary tests. This dissertation will provide some basic theories and computational methods for finding such conditional confidence intervals of the two process capability indices. The most basic method used in this dissertation is the general method for finding a confidence interval of an unknown parameter. Numerical methods are also used for finding the values of these conditional confidence limits. The conditional confidence intervals of the process capability index C_p and C_{pk} are obtained. Computational programming code and other useful information and methods are provided.

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CHAPTER 1

INTRODUCTION

1.1 Overview

The process capability analysis has been proved to be a very useful tool in product quality control. There are many process capability indices (PCIs) which are used currently in industry, to name a few, C_p, C_{pk}, C_{pm} and C_{pmk}, etc., but the most commonly used two indices are C_p and C_{pk}. Some books have been published related to the PCIs (refer to: Kotz & Lovelace (1998), Kotz & Johnson (1993), etc.), they discussed the basic inferences about the PCIs, including point estimators, confidence intervals and testing hypotheses. In some practical situations, the investigators may have prior information about the unknown parameters, but he/she is uncertain about the information. A preliminary test can be used to resolve the uncertainty. We consider the conditional confidence intervals of the PCIs after rejecting the null hypotheses of the preliminary tests. For the case that some null hypotheses have been rejected, the original unconditional confidence interval is no longer valid. So, to find a conditional confidence interval is a strict two-stage inference procedure: first, test the hypotheses of preliminary tests, if the tests result in rejecting some of null hypotheses, then go to the second stage to obtain a conditional confidence interval following rejection of the null hypotheses. In this paper, we will mainly discuss this type of problems, the conditional confidence intervals of the process capability indices.

1.2 The process capability indices C_p and C_{pk}

The process capability indices are measures of the quality of a manufacturing process, they have been popular in industry for more than 40 years. In 1920s, Bell Laboratories, a leader in the use of statistics to control and improve quality, began the first serious investigations into the application of statistic theory to lot sampling and the use of significance theory in process control. Shewhart, along with Dr. J.M. Juran and Dr. W. Edwards Deming, developed most of the early theories and concepts of statistical quality control.

The concept of process capability index was first introduced by the Japanese. At the beginning of 1970s, there were only five process capability indices, known as the original Japanese process capability indices, and these include the two most common process capability indices C_p and C_{pk}. The process capability index C_p was the most original process capability index, which was introduced by Juran *et al.* (1974). This process capability index only counts for the process variation, and ignore the process mean. So, the process capability index C_p measures the potential capability, which is defined only by the actual process spread, it does not reflect the impact of shifting the process mean on the process capability to produce qualified products. To overcome the weakness of C_p, the process capability index C_{pk} was created in Japan in late 1970s, and it takes both the process variation and the process mean into consideration.

1.3 Literature Review

Before I started this dissertation, a literature review of related topics, which include some books and lots of papers, had been completed. The basic information about the process capability indices, which include the history and some basic analysis of the process capability indices, can be found in the two books written by Kotz & Lovelace (1998) and Kotz & Johnson (1993). Since the capability indices C_p and C_{pk} involves two process parameters: the process variance σ^2 and the process mean μ, therefore, we will discuss several papers about the conditional confidence intervals of these parameters following rejection of preliminary tests. The first and also the most important paper is written by Meeks & D'Agostino (1983), they discussed the conditional confidence interval of the normal mean μ following rejection of a one-sided test. In order to apply the general method for finding a confidence interval of an unknown parameter, the book written by Bain & Engelhardt (1992) was carefully examined. Since the preliminary tests in this dissertation also involve some two-sided tests, so the paper written by Arabatzis, Gregoire and Reynolds (1989) was also reviewed, in this paper, they discussed some aspects of the conditional confidence interval of the normal mean following rejection of a two-sided test.

Some other papers related to the conditional confidence intervals were also published, these include the papers written by Chiou & Han (1994), in this paper, they discussed the conditional confidence Interval of the exponential scale parameter following rejection of a preliminary test. Chiou & Han (1995) discussed the conditional confidence interval of the exponential location parameter following rejection of a pre-test. In Chiou & Han (1999), they give the conditional interval estimation of the ratio of variance components following rejection of a pre-test.

CHAPTER 2

CONDITIONAL CONFIDENCE INTERVALS OF C_p

2.1 The Process Capability Index C_p

Consider a measurement X of a product, the manufacture usually requires that the product must meet some specifications for the measurement X. If we set a lower specification limit (LSL) and an upper specification limit (USL) for X, then the values of X outside these limits will be termed 'nonconforming' (NC) (Kotz & Johnson, 1993). An indirect measure of the potential ability (capability) to meet the requirement (LSL ≤ X ≤ USL) is the process capability index C_p, which is defined as

$$C_p = \frac{USL - LSL}{6\sigma}$$

Suppose X follows a normal distribution with mean μ and variance σ^2, if the expected value of X is equal to the mid-point of the specified interval, i.e., $\mu = \frac{1}{2}$(LSL + USL), let $d = \frac{1}{2}$(USL - LSL), then the expected proportion of NC product is $2\Phi(-d/\sigma)$, in terms of C_p , it is $2\Phi(-3C_p)$.

From above we see that the bigger the value of C_p, the smaller proportion of NC product. The relationship between C_p and the proportion of NC product is so straight forward, therefore, the process capability index C_p is widely used in product quality control.

2.2 Basic Inferences about C_p

2.2.1 Point Estimator

Suppose we have a random sample $X_1, X_2, \ldots, X_n$ taken from $N(\mu, \sigma^2)$, then the most commonly used estimator of μ is the sample mean $\overline{X}$, and the most commonly used estimator of σ is the sample standard deviation S, i.e.,

$$\hat{\mu} = \overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

and

$$\hat{\sigma} = S = \left\{ \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \overline{X})^2 \right\}^{\frac{1}{2}}$$

Since $C_p = \frac{USL - LSL}{6\sigma} = \frac{d}{3\sigma}$, the only parameter need to be estimated here is σ, therefore

$$\hat{C}_p = \frac{d}{3\hat{\sigma}} = \frac{d}{3S}$$

is a point estimator of C_p.

Since it is well known that $E[\frac{1}{S}] \neq \frac{1}{\sigma}$, so we must have, $E[\hat{C}_p] \neq C_p$. Therefore, this is a biased estimator. The bias will be given in section 2.3.

2.2.2 Confidence Interval

The (unconditional) confidence interval of C_p can be derived directly from the (unconditional) confidence interval of σ. Since the random sample $X_1, X_2, \dots, X_n$ is taken from $N(\mu, \sigma^2)$, therefore, $(n-1)\frac{S^2}{\sigma^2}$ is distributed as χ^2_{n-1}. This leads to the result that a 100(1-α)% confidence interval for σ^2 is

$$\left(\frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}}\right)$$

where s^2 is the observed value of S^2.

Since $\hat{C}_p^{-1} = \frac{3\hat{\sigma}}{d}$, therefore, a 100(1-α)% confidence interval of C_p^{-1} is

$$\left(\frac{3(n-1)^{\frac{1}{2}}s}{d\sqrt{\chi^2_{n-1,1-\alpha/2}}}, \frac{3(n-1)^{\frac{1}{2}}s}{d\sqrt{\chi^2_{n-1,\alpha/2}}}\right)$$

and consequently, a 100(1-α)% confidence interval of C_p is

$$\left(\frac{d\sqrt{\chi^2_{n-1,\alpha/2}}}{3s(n-1)^{\frac{1}{2}}}, \frac{d\sqrt{\chi^2_{n-1,1-\alpha/2}}}{3s(n-1)^{\frac{1}{2}}}\right)$$

$$\equiv \left(\frac{\sqrt{\chi^2_{n-1,\alpha/2}}}{(n-1)^{\frac{1}{2}}}\hat{C}_p, \frac{\sqrt{\chi^2_{n-1,1-\alpha/2}}}{(n-1)^{\frac{1}{2}}}\hat{C}_p\right)$$

2.3 Sampling Distribution of $\hat{C}_p$

Since our goal is to investigate the conditional confidence interval of C_p, and a point estimator of C_p is $\hat{C}_p$, so we first need to find the probability density function (pdf) and the cumulative distribution function (CDF) of $\hat{C}_p$.

Here, we'll use the transformation method to derive the pdf of $\hat{C}_p$. Since the random sample $X_1, X_2, \ldots, X_n$ is taken from $N(\mu, \sigma^2)$, we have that $(n-1)\frac{S^2}{\sigma^2}$ is distributed as χ^2_{n-1}. Since $\hat{C}_p = \frac{d}{3S} = \frac{d}{3\sigma}\frac{\sigma}{S}$, therefore, $\hat{C}_p$ is distributed as $C_p\sqrt{\frac{n-1}{\chi^2_{n-1}}}$. Using transformation method, we can derive the pdf of $\hat{C}_p$.

Let $X \sim \chi^2_{n-1}$ and $Y = f(X) = C_p\sqrt{\frac{n-1}{X}}$. Since $X > 0$, the transformation $X = g(Y) = (n-1)\left(\frac{C_p}{Y}\right)^2$ is one-to one, the Jacobian of the transformation is

$$J = \frac{d[g(Y)]}{dy}$$

$$= (n-1)\cdot 2\left(\frac{C_p}{Y}\right)\cdot\left(-\frac{C_p}{Y^2}\right)$$

Since $f_X(x) = \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}\cdot x^{\frac{n-1}{2}-1}\cdot e^{-\frac{x}{2}}$, $x > 0$,

therefore, we have

$$f_Y(y) = f_X[g(Y)]\cdot|J|$$

$$= \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})} \cdot [(n-1)(\frac{C_p}{y})^2]^{\frac{n-1}{2}-1} \cdot e^{-\frac{(n-1)(\frac{C_p}{y})^2}{2}} \cdot \left|(n-1)\cdot 2(\frac{C_p}{y})(-\frac{C_p}{y^2})\right|$$

Simplify the above equation, we get the (unconditional) pdf of $\hat{C}_p$:

$$f_{\hat{c}_p}(y) = \frac{(n-1)^{\frac{n-1}{2}} c_p^{n-1}}{2^{\frac{n-3}{2}}\Gamma(\frac{n-1}{2})} \cdot y^{-n} \cdot e^{-\frac{(n-1)}{2}c_p^2 y^{-2}}, \quad y > 0$$

where n is the sample size.

Once we have the pdf of $\hat{C}_p$, we can use it to calculate the mean and the variance of $\hat{C}_p$ by first carrying out its rth moment about the origin. To simplify the calculations, here we take the advantage of the Chi-Square distribution of X to calculate the rth moments of $\hat{C}_p$ about the origin:

$$E(\hat{C}_p^r) = E[(\frac{d}{3S})^r]$$

$$= (n-1)^{\frac{r}{2}} C_p^r \; E[(\frac{(n-1)S^2}{\sigma^2})^{-\frac{r}{2}}]$$

$$= (n-1)^{\frac{r}{2}} C_p^r \int_0^\infty x^{-\frac{r}{2}} \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})} \cdot x^{\frac{n-1}{2}-1} \cdot e^{-\frac{x}{2}} dx$$

$$= (\frac{n-1}{2})^{\frac{r}{2}} C_p^r \frac{\Gamma(\frac{n-r-1}{2})}{\Gamma(\frac{n-1}{2})}$$

Therefore, we obtain the rth moments of $\hat{C}_p$ about the origin:

$$E[\hat{C}_p^r] = (\frac{n-1}{2})^{\frac{r}{2}} \cdot \frac{\Gamma(\frac{1}{2}(n-1-r))}{\Gamma(\frac{1}{2}(n-1))} \cdot C_p^r$$

So, the mean of $\hat{C}_p$:

$$E[\hat{C}_p] = (\frac{n-1}{2})^{\frac{1}{2}} \cdot \frac{\Gamma(\frac{1}{2}(n-2))}{\Gamma(\frac{1}{2}(n-1))} \cdot C_p = \frac{1}{b_f} \cdot C_p$$

and the variance:

$$Var(\hat{C}_p) = (\frac{n-1}{n-3} - b_f^{-2})C_p^2$$

where b_f is an unbiased factor given by

$$b_f = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\frac{n-1}{2}} \cdot \Gamma(\frac{n-2}{2})}$$

Thus, the bias of $\hat{C}_p$ is

$$b(\hat{C}_p) = E[\hat{C}_p] - C_p$$

$$= (\frac{1}{b_f} - 1)C_p$$

Some of the above formulas can also be found in Kotz & Lovelace (1998), page 38-39.

2.4 Conditional Confidence Interval of C_p

The most common test hypothesis in process capability analysis is to test H_o: the process is not capable vs. H_1: the process is capable. As we already knew, large C_p value results in small proportion of NC product. Therefore, for the C_p index, this leads to the hypothesis H_o: $C_p \le c_o$ vs. H_1: $C_p > c_o$.

For each value of c_o, there corresponds another value σ_o, such that $c_o = d/3\sigma_o$. To test H_o: $C_p \le c_o$ is equivalent to test H_o: $\sigma \ge \sigma_o$.

If the null hypothesis is not rejected, which means the process is not capable, then we stop. A not capable process means a large proportion of non-conforming products will happen, in this case, the most important and necessary thing we need to do is to find out the weakness of the process and try to improve it. Therefore, no further statistical analyses need to be done before we improve the process. If the null hypothesis has been rejected, which means the current C_p value is within our acceptable region, in this case, a confidence interval is needed to give more information about the possible region of C_p. This confidence interval is different from the previous unconditional confidence interval (UCI), it is the conditional confidence interval (CCI) following rejection of the null hypothesis H_o: $C_p \le c_o$.

We'll reject H_o at level α if the test statistic $V = (n-1)\frac{s^2}{\sigma_o^2} \le \chi^2_{n-1;\alpha}$, i.e.

$$\hat{C}_p \ge c_o\sqrt{\frac{n-1}{\chi^2_{n-1;\alpha}}} = c_1$$

2.4.1 General Method for Constructing CCI

In section 2.3, we have derived the unconditional pdf of $\hat{C}_p$:

$$f_{\hat{c}_p}(y) = \frac{(n-1)^{\frac{n-1}{2}} c_p^{n-1}}{2^{\frac{n-3}{2}} \Gamma(\frac{n-1}{2})} \cdot y^{-n} \cdot e^{-\frac{(n-1)}{2} C_p^2 y^{-2}}, \quad y > 0$$

The conditional pdf of $\hat{C}_p$ following rejecting the null hypothesis H$_o$: $C_p \le c_o$ is the truncated distribution of the above pdf, where the test statistic (for significance level α) $\nu_o = (n-1)\frac{s^2}{\sigma_o^2}$ must not exceed the critical value $\chi^2_{n-1;\alpha}$, i.e., $s \le \sqrt{\frac{\chi^2_{n-1;\alpha}}{n-1}}\sigma_o$, and this is equivalent to $\hat{C}_p \ge \frac{d}{3\sigma_o}\sqrt{\frac{n-1}{\chi^2_{n-1;\alpha}}} = c_o\sqrt{\frac{n-1}{\chi^2_{n-1;\alpha}}} = c_1$.

The conditional pdf of $\hat{C}_p$ following rejecting the null hypothesis H$_o$: $C_p \le c_o$. is given by

$$g_{\hat{C}_p}(c) = \frac{\frac{(n-1)^{(n-1)/2} c_p^{n-1}}{2^{(n-3)/2}\Gamma(\frac{n-1}{2})} \cdot c^{-n} \cdot e^{-\frac{(n-1)}{2} c_p^2 c^{-2}}}{\int_{c_1}^{\infty} \frac{(n-1)^{(n-1)/2} c_p^{n-1}}{2^{(n-3)/2}\Gamma(\frac{n-1}{2})} \cdot y^{-n} \cdot e^{-\frac{(n-1)}{2} C_p^2 y^{-2}} dy}$$

$$= \frac{c^{-n} \cdot e^{-\frac{(n-1)}{2} c_p^2 c^{-2}}}{\int_{c_1}^{\infty} y^{-n} \cdot e^{-\frac{(n-1)}{2} C_p^2 y^{-2}} dy}, \qquad c \ge c_1 = c_o\sqrt{\frac{n-1}{\chi^2_\alpha(n-1)}}$$

The CDF of the conditional distribution of $\hat{C}_p$ following rejecting the null hypothesis H$_o$: $C_p \le c_o$ is given by

$$G_{\hat{c}_p}(c) = \frac{\int_{c_1}^{c} y^{-n} \cdot e^{-\frac{(n-1)}{2} c_p^2 y^{-2}} dy}{\int_{c_1}^{\infty} y^{-n} \cdot e^{-\frac{(n-1)}{2} c_p^2 y^{-2}} dy} , \quad c \geq c_1$$

[Result 2.4.1] Let G(c; c_p) denote the conditional CDF of the $\hat{C}_p$ distribution following rejecting the null hypothesis H_o: $C_p \leq c_o$, and suppose $h_1(c_p)$ and $h_2(c_p)$ are functions that satisfy

$$G(h_1(c_p); c_p) = \alpha_1$$

and

$$G(h_2(c_p); c_p) = 1 - \alpha_2$$

for each $c_p \in \Omega$ (Ω is the parameter space of c_p), where $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ with $\alpha_1 + \alpha_2 < 1$. Let c be an observed value of $\hat{C}_p$, then we have the following

1. If $h_1(c_p)$ and $h_2(c_p)$ are increasing functions of c_p, then the solutions of
   $$G(c;\ c_p^L) = 1 - \alpha_2 \quad \text{and} \quad G(c;\ c_p^U) = \alpha_1$$
   construct a $100(1- \alpha_1 - \alpha_2)$% conditional confidence interval of c_p.

2. If $h_1(c_p)$ and $h_2(c_p)$ are decreasing functions of c_p, then the solutions of
   $$G(c;\ c_p^L) = \alpha_1 \quad \text{and} \quad G(c;\ c_p^U) = 1 - \alpha_2$$
   construct a $100(1- \alpha_1 - \alpha_2)$% conditional confidence interval of c_p.

The above result follows directly from the general method theorems given in Bain & Engelhardt (1992). The general method states that, if a statistic for a parameter exists with a distribution that depends on this parameter but not on any other nuisance parameters, then we can use the general method to find a confidence interval of this parameter. We prefer this statistic to be a sufficient statistic or some reasonable estimators such as an MLE, but this is not required.

In our case, the conditional CDF of $\hat{c}_P$ depends only on C_p but not on any other nuisance parameters, so we can use the general method to find a CCI of C_p. Since we can not solve for $h_1(c_p)$ and $h_2(c_p)$ explicitly in this case, it's hard for us to prove in theory whether the two functions $h_1(c_p)$ and $h_2(c_p)$ are increasing or decreasing, therefore we state the above result with the two possible cases.

Result 2.4.1 shows theoretically that we can find a conditional confidence interval of C_p using the conditional CDF of $\hat{c}_P$. But in practice, this procedure is somewhat complicated, since the conditional CDF of $\hat{c}_P$ is not a commonly used distribution function, there is no existing computer programs which can be used directly to calculate the probabilities of this distribution. So next, we'll use another method to find a conditional confidence interval of C_p.

2.4.2 Constructing CCI of C_p through first Constructing CCI of σ^2

There are two main advantages for doing this way. First, since the estimator of σ^2 is S^2, and $(n-1)S^2/\sigma^2$ follows a χ^2_{n-1} distribution, so we can use the cumulative distribution function H(z ; υ) of the chi-square distribution with υ degrees of freedom to give the conditional CDF of S^2 distribution, as well as the two implicit functions which contain the upper (or lower) limits of σ^2 and the observed statistic value t for S^2. This might help a lot later on when we calculate the two conditional confidence limits for given examples using mathematical software such as IMSL numerical library. Second, once we obtain the CCI of σ^2, we also obtain the CCI of 1/ σ^2, as well as the CCI of $(\frac{1}{\sigma})^r$ for any r ≥1. This result may have some more applications for other statistical analysis related to the conditional confidence interval of $(\frac{1}{\sigma})^r$.

For the random sample $X_1, X_2, \ldots, X_n$ drawn from $N(\mu, \sigma^2)$, a level α test for testing H_o: $\sigma^2 \geq \sigma_o^2$ vs. H_1: $\sigma^2 < \sigma_o^2$ has the critical region

$$K = \{ S^2 : (n-1) S^2/\sigma_o^2 < \chi^2_{n-1;\alpha} \}$$

The null hypothesis is rejected only if $S^2 \in K$, and a conditional confidence interval of σ^2 is computed only if the null hypothesis has been rejected. The conditional pdf of S^2 can be expressed in the following way

$$f_c(s^2) = \begin{cases} f(s^2)/D, & if \dfrac{(n-1)s^2}{\sigma_o^2} < \chi^2_{n-1;\alpha} \\ 0, & otherwise \end{cases}$$

where $f(s^2)$ is the unconditional pdf which is determined by $S^2 \sim \frac{\sigma^2}{n-1}\chi^2_{n-1}$, D is the power of the test given by

$$D = P((n-1)\frac{s^2}{\sigma_o^2} < \chi^2_{n-1;\alpha} | \sigma)$$

$$= P((n-1)\frac{s^2}{\sigma^2} < \frac{\sigma_o^2}{\sigma^2} \cdot \chi^2_{n-1;\alpha})$$

$$= H(\psi \cdot \chi^2_{n-1;\alpha}),$$

where $\psi = \frac{\sigma_o^2}{\sigma^2}$ and H(·) is the CDF of chi-square distribution with (n-1) degrees of freedom. Under H_o, max{D} = α . When $\sigma \to 0$, the value $\psi \to \infty$, so the power D approaches 1.

The (unconditional) distribution of S^2 is similar to the χ^2_{n-1} distribution except it has an unknown parameter σ^2. To express the unconditional CDF of S^2 in terms of H(·), we have

$$F(t) = P(S^2 \le t)$$

$$= P[\frac{(n-1)S^2}{\sigma^2} \le \frac{n-1}{\sigma^2} \cdot t]$$

$$= H(\frac{n-1}{\sigma^2} \cdot t), \qquad t > 0$$

Therefore, the conditional CDF of S^2 following rejecting the null hypothesis H_o: $\sigma^2 \ge \sigma_o^2$ is given by

$$F_c(t) = \frac{F(t)}{D}$$

$$= \frac{H(\frac{n-1}{\sigma^2} \cdot t)}{H(\psi \cdot \chi^2_{n-1;\alpha})}, \qquad \text{with } t < \frac{\sigma_o^2}{n-1} \chi^2_{n-1;\alpha}$$

[Result 2.4.2] Let $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ with $\alpha_1 + \alpha_2 < 1$, and t be an observed value of S^2, let H(·) denote the CDF of chi-square distribution with (n-1) degrees of freedom. If the observed value t results in rejecting the null hypothesis H_o: $\sigma^2 \ge \sigma_o^2$, then the solutions of

$$\frac{H(\frac{n-1}{\sigma_U^2}\cdot t)}{H(\frac{\sigma_o^2}{\sigma_U^2}\cdot \chi^2_{n-1;\alpha})} = \alpha_1 \qquad (2.4.1)$$

and

$$\frac{H(\frac{n-1}{\sigma_L^2}\cdot t)}{H(\frac{\sigma_o^2}{\sigma_L^2}\cdot \chi^2_{n-1;\alpha})} = 1 - \alpha_2 \qquad (2.4.2)$$

construct a 100(1- α_1 - α_2)% conditional confidence interval of σ^2.

We use the general method theorem to show this result. Let F(t; σ^2) denote the conditional CDF of the S^2 distribution, and suppose $h_1(\sigma^2)$ and $h_2(\sigma^2)$ are functions that satisfy $F(h_1(\sigma^2);\sigma^2) = \alpha_1$ and $F(h_2(\sigma^2);\sigma^2) = 1 - \alpha_2$ for each $\sigma^2 \in \Omega$, where $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ and $\alpha_1 + \alpha_2 < 1$, let t be an observed value of S^2. It's easy to check in this case that both $h_1(\sigma^2)$ and $h_2(\sigma^2)$ are increasing functions of σ^2 (this is similar to the situation for constructing CCI of the normal mean μ provided by Meeks and D'Agostino in 1983). In fact, for small fixed values α_1 or α_2, if σ^2 increase, then both $h_1(\sigma^2)$ and $h_2(\sigma^2)$ increase, and the rates of changing for $h_1(\sigma^2)$ and $h_2(\sigma^2)$ are always smaller than that of σ^2. This result is supported both analytically and numerically. Following the theorems given by Bain & Engelhardt (1992), we can obtain a conditional confidence interval of σ^2 by solving the two equations F(t; σ_U^2) = α_1 and F(t; σ_L^2) = 1 - α_2. This gives the above result.

Once we obtain a conditional confidence interval of σ^2 as (σ_L^2, σ_U^2), a conditional confidence interval of 1/σ is followed as $(1/\sigma_U, 1/\sigma_L)$. Thus, a conditional confidence interval of C_p is given by

$$(\frac{d}{3\sigma_U}, \frac{d}{3\sigma_L})$$

2.5 Examples

2.5.1 Example 1, Simulated Data

Suppose a certain product manufactured from a factory has the following specifications for the measurement: The specified lower limit is 84.25, the specified upper limit is 85.25. Suppose the process mean is known as 84.75, and the machines used in the factory can manufacture the product that is normally distributed with mean 84.75 and standard deviation 0.08.

According to the above situation, we can use a simple SAS program to simulate the manufacturing procedure and obtain a certain number of simulated observations

```
data simu;
do  k=1 to 48;
    y= 84.75 + 0.08*normal(184321);
    output;
end;
proc print; title 'simulated data';
run;
```

The result of the above SAS program gives us the following data set with total 48 observations

84.7333 84.5799 84.7818 84.8443 84.6155 84.7086
84.6563 84.8193 84.5705 84.8573 84.8465 84.7184

84.7627 84.6206 84.5076 84.6921 84.6780 84.7024
84.6625 84.7599 84.8669 84.8663 84.7446 84.7608

84.7252 84.7575 84.8299 84.8315 84.7452 84.6923
84.5008 84.7812 84.8034 84.7837 84.8399 84.7555
84.6795 84.7289 84.7087 84.7007 84.5365 84.7320
84.7210 84.7034 84.7995 84.8414 84.6743 84.7081

For the above data set, after we did some basic analysis, we get the sample mean $\overline{X} \cong$ 84.73 and the sample standard deviation S $\cong$ 0.0915. Since in this case d = (USL - LSL)/2 = 1.0*0.5 = 0.5, so the point estimate of C_p is

$$\hat{C}_p = \frac{d}{3s} = \frac{0.5}{3*0.0915} = 1.82$$

A 95% unconditional confidence interval of σ^2 is

$$\left(\frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}}\right)$$

$$= \left(\frac{(48-1)*0.0915^2}{\chi^2_{47,0.975}}, \frac{(48-1)*0.0915^2}{\chi^2_{47,0.025}}\right)$$

$$= (0.0058, 0.0131)$$

Therefore, a 95% unconditional confidence interval of C_p is

$$\left(\frac{\sqrt{\chi^2_{n-1,\alpha/2}}}{(n-1)^{\frac{1}{2}}}\hat{C}_p, \frac{\sqrt{\chi^2_{n-1,1-\alpha/2}}}{(n-1)^{\frac{1}{2}}}\hat{C}_p\right)$$

$$= \left(\frac{\sqrt{\chi^2_{47,0.025}}}{\sqrt{47}} \hat{C}_p, \frac{\sqrt{\chi^2_{47,0.975}}}{\sqrt{47}} \hat{C}_p \right)$$

$$= (1.45, 2.19)$$

For reasonable values of C_p, Montgomery (1985) recommended minimum values for C_p as 1.33 for an existing process, and 1.50 for a new process. For those processes related to essential safety, for example, in manufacturing of bolts which are used in bridge construction, a minimum value of 1.50 is recommended for an existing process and 1.67 is recommended for a new process.

In our case, suppose we require a minimum value of the C_p as 1.33. Then we need to construct a testing hypothesis as H_o: $C_p \leq 1.33$ vs. H_1: $C_p > 1.33$.

For the testing hypothesis H_o: $C_p \leq 1.33$ vs. H_1: $C_p > 1.33$, it is equivalent to the hypothesis H_o: $\sigma \geq \sigma_o$ vs. H_1: $\sigma < \sigma_o$, where $\sigma_o = d/3c_o$ = 0.5/(3*1.33) = 0.1253. We'll reject H_o at level $\alpha = 0.05$ if the test statistic V = (n-1)$\frac{s^2}{\sigma_o^2} \leq \chi^2_{47,0.05}$ = 32.27.

Now the test statistic:

$$V = (n-1)\frac{s^2}{\sigma_o^2}$$

$$= 47*(0.0915/0.1253)^2$$

$$= 25.06$$

therefore, we reject the null hypothesis H_o: $C_p \leq 1.33$ (or H_o: $\sigma \geq 0.1253$) at the level $\alpha = 0.05$.

According to result 2.4.2, a 97.5% conditional upper confidence limit of σ^2 following rejection of the null hypothesis H_o: $C_p \leq 1.33$ (or H_o: $\sigma \geq 0.1253$) at level $\alpha = 0.05$ can be obtained by solving the equation

$$\frac{H(\frac{n-1}{\sigma_U^2} \cdot t)}{H(\frac{\sigma_o^2}{\sigma_U^2} \cdot \chi^2_{n-1;\alpha})} = \alpha_1 = 0.025$$

where $t = 0.0915^2$ is the observed value of S^2, and $\sigma_o = 0.1253$ is the value of σ under the null hypothesis. The above equation can be solved by using IMSL numerical library (refer to the attached FORTRAN code in Appendix A). The solution of the above equation is

$$\sigma_U^2 = 0.02357$$

Similarly, a 97.5% conditional lower confidence limit of σ^2 following rejection of the null hypothesis H_o: $C_p \leq 1.33$ (or H_o: $\sigma \geq 0.1253$) at level $\alpha = 0.05$ can be obtained by solving the equation

$$\frac{H(\frac{n-1}{\sigma_L^2} \cdot t)}{H(\frac{\sigma_o^2}{\sigma_L^2} \cdot \chi^2_{n-1;\alpha})} = 1 - \alpha_2 = 0.975$$

The solution of the above equation is

$$\sigma_L^2 = 0.005808$$

Thus, a 95% conditional confidence interval of σ^2 following rejection of the null hypothesis H_o: $C_p \leq 1.33$ is given by:

$$(\sigma_L^2, \sigma_U^2) = (0.005808, 0.02357)$$

and consequently a 95% conditional confidence interval of C_p following rejection of the null hypothesis H_o: $C_p \leq 1.33$ can be determined as

$$(C_p^L, C_p^L) = (\frac{d}{3\sigma_U}, \frac{d}{3\sigma_L})$$

$$= (1.09, 2.19)$$

Compare the above conditional confidence interval of C_p to the unconditional confidence interval of C_p we found previously, we can see that the length of the conditional confidence interval of C_p is relatively longer, and it covers the whole length of the unconditional confidence interval of C_p. The most interesting thing is, the conditional confidence interval of C_p covers back some region of the C_p values which has been rejected by the hypothesis test previously. The reason for this kind of result is not quite sure, this can be due to the type I errors.

Further analysis (see section 2.6 and section 2.7) shows that this is a common situation for the conditional confidence interval of C_p following rejection of a one-sided test. That is, the conditional confidence interval of C_p covers the unconditional conditional confidence interval of C_p, and in most of the cases, the conditional confidence interval of C_p covers back some values of C_p which have been rejected by the null hypothesis of the test. This situation is quite similar to the one discussed by Meeks & D'Agostino (1983). In their case, they discussed the conditional confidence interval of the normal mean following rejection of a one-sided test.

2.5.2 Example 2, Real Data

The following data set consists of weight measurements (in ounces) for 60 major league baseballs (see in Bain & Engelhardt (1992), page 169, problem 24).

5.09 5.08 5.21 5.17 5.07 5.24 5.12 5.16 5.18 5.19
5.26 5.10 5.28 5.29 5.27 5.09 5.24 5.26 5.17 5.13
5.27 5.26 5.17 5.19 5.28 5.28 5.18 5.27 5.25 5.26

5.26 5.18 5.13 5.08 5.25 5.17 5.09 5.16 5.24 5.23
5.28 5.24 5.23 5.23 5.27 5.22 5.26 5.27 5.24 5.27
5.25 5.28 5.24 5.26 5.24 5.24 5.27 5.26 5.22 5.09

Suppose the process mean of this product is known as 5.25, the specified upper limit and lower limit are 5.45 and 4.85 respectively (It's reasonable that, for this product, we allow more deviation from the lower side of the process mean. Therefore, d = (USL - LSL)/2 = 0.60*0.5 = 0.30. After doing some basic analysis to the above data set, we obtain the mean and the standard deviation as $\overline{X}$ = 5.2110 and S = 0.0649, so the point estimate of C_p is

$$\hat{C}_p = \frac{d}{3s} = \frac{0.30}{3*0.0649} = 1.54$$

A 95% unconditional confidence interval of σ^2 can be determined by

$$\left(\frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}} \right)$$

$$= \left(\frac{(60-1)*0.0649^2}{\chi^2_{59,0.975}}, \frac{(60-1)*0.0649^2}{\chi^2_{59,0.025}} \right)$$

$$= (0.0030, 0.0063)$$

A 95% unconditional confidence interval of C_p is

$$\left(\frac{\sqrt{\chi^2_{n-1,\alpha/2}}}{(n-1)^{\frac{1}{2}}} \hat{C}_p, \frac{\sqrt{\chi^2_{n-1,1-\alpha/2}}}{(n-1)^{\frac{1}{2}}} \hat{C}_p \right)$$

$$= \left(\frac{\sqrt{\chi^2_{59,0.025}}}{\sqrt{59}} \hat{C}_p, \frac{\sqrt{\chi^2_{59,0.975}}}{\sqrt{59}} \hat{C}_p \right)$$

$$= (1.26, 1.82)$$

Suppose the minimum required value of the process capability index C_p for this process is 1.00, then a test hypothesis for testing the value of C_p can be constructed as H_o: $C_p \le 1.00$ vs. H_1: $C_p > 1.00$.

For the test hypothesis H_o: $C_p \le 1.00$ vs. H_1: $C_p > 1.00$, It is equivalent to the hypothesis H_o: $\sigma \ge \sigma_o$ vs. H_1: $\sigma < \sigma_o$, where $\sigma_o = d/3C_o = 0.30/(3*1.00) = 0.1000$. We'll reject H_o at level $\alpha = 0.05$ if the test statistic $V = (n-1)\frac{s^2}{\sigma_o^2} \le \chi^2_{59,0.05} = 42.3393$. Now the test statistic:

$$V = (n-1)\frac{s^2}{\sigma_o^2} = 59*(0.0649/0.1000)^2$$

$$= 24.85$$

Therefore, we reject the null hypothesis H_o: $C_p \le 1.00$ (or H_o: $\sigma \ge 0.1000$) at level $\alpha = 0.05$.

Next, we will construct a 95% conditional confidence interval of C_p following the rejection of the null hypothesis H_o: $C_p \le 1.00$.

According to result 2.4.2, a 97.5% conditional upper confidence limit of σ^2 following rejection of the null hypothesis H_o: $C_p \le 1.00$ (or H_o: $\sigma \ge 0.1000$) at level $\alpha = 0.05$ can be obtained by solving the equation

$$\frac{H(\frac{n-1}{\sigma_U^2} \cdot t)}{H(\frac{\sigma_o^2}{\sigma_U^2} \cdot \chi^2_{n-1;\alpha})} = \alpha_1 = 0.025$$

where $t = 0.0649^2$ is the observed value of S^2, $\sigma_o = 0.1000$, $n = 60$, and $\chi^2_{n-1,\alpha} = 42.34$. The above equation can also be solved by using IMSL numerical library . The solution of the equation is

$$\sigma_U^2 = 0.006427$$

For the solution of a 97.5% conditional lower confidence limit of σ^2 following rejection of the null hypothesis H_o: $C_p \le 1.00$ (or H_o: $\sigma \ge 0.1000$) at level $\alpha = 0.05$, according to result 2.4.2, we can find the solution by solving the equation

$$\frac{H(\frac{n-1}{\sigma_L^2} \cdot t)}{H(\frac{\sigma_o^2}{\sigma_L^2} \cdot \chi^2_{n-1;\alpha})} = 1 - \alpha_2 = 0.975$$

But this time the IMSL numerical library is unable to reach a solution due to floating errors. Further analysis (see in section 2.6) shows this lower limit do exist, and the ratio of the unconditional lower limit over conditional lower limit equal to 1.0. Therefore, according to the results in section 2.6, we obtain the conditional lower confidence limit of σ^2 as

$$\sigma_L^2 = 0.0030$$

Finally, a 95% conditional confidence interval of σ^2 following rejection of the null hypothesis H_o: $C_p \leq 1.00$ is given by:

$$(\sigma_L^2, \sigma_U^2) = (0.0030,\ 0.006427)$$

and consequently a 95% conditional confidence interval of C_p following rejection of the null hypothesis H_o: $C_p \leq 1.00$ is given by

$$(C_p^L, C_p^U) = \left(\frac{d}{3\sigma_U}, \frac{d}{3\sigma_L}\right)$$

$$= (1.25,\ 1.82)$$

2.6 Comparisons of the Lengths for UCI and CCI

In order to compare the lengths of the conditional confidence intervals of C_p to the lengths of the unconditional confidence intervals of C_p, we will derive formulas for the ratios of the conditional confidence limits of C_p over the unconditional confidence limits of C_p.

2.6.1 Ratio of the Lower Limits

The 100(1- α_1)% unconditional upper limit of σ^2, denoted by σ_U^2, is formed by solving the equation

$$H(\frac{n-1}{\sigma_U^2}\cdot t)=\alpha_1 \qquad (2.6.1)$$

The 100(1- α_1)% conditional upper limit of σ^2, denoted by σ_U^2, following rejection of the null hypothesis H_o: $C_p \le c_o$ vs. H_1: $C_p > c_o$ is obtained by solving the equation

$$\frac{H(\frac{n-1}{\sigma_U^{*2}}\cdot t)}{H(\frac{\sigma_o^2}{\sigma_U^{*2}}\cdot \chi_\alpha^2)}=\alpha_1 \qquad (2.6.2)$$

From (2.6.1), we have:

$$\frac{n-1}{\sigma_U^2}\cdot t=\chi_{\alpha_1}^2 \quad \Rightarrow t=\frac{\sigma_U^2}{n-1}\cdot \chi_{\alpha_1}^2 \qquad (2.6.3)$$

plug (2.6.3) into (2.6.2), we get

$$\alpha_1=\frac{H(\frac{n-1}{\sigma_U^{*2}}\cdot\frac{\sigma_U^2}{n-1}\cdot\chi_{\alpha_1}^2)}{H(\frac{\sigma_U^2}{\sigma_U^{*2}}\cdot\frac{\sigma_o^2}{\sigma_U^2}\cdot\chi_\alpha^2)}$$

Let $\Lambda_L^2=\sigma_U^2/\sigma_U^{*2}$ and substituting $\sigma_U^2=\dfrac{(n-1)t}{\chi_{\alpha_1}^2}$ in the denominator, we have

$$\alpha_1 = \frac{H(\Lambda_L^2 \cdot \chi_{\alpha_1}^2)}{H(\Lambda_L^2 \cdot \lambda \cdot \chi_{\alpha_1}^2)} \qquad (2.6.4)$$

where $\lambda = \chi_\alpha^2 / V$, and V= (n-1)$\frac{s^2}{\sigma_o^2}$ = (n-1)$\frac{t}{\sigma_o^2}$

Let C_p^L and $C_p^{L^*}$ be the unconditional and conditional 100(1- α_1)% lower confidence limits of C_p respectively, then

$$\Lambda_L^2 = \sigma_U^2 / \sigma_U^{*2} = (C_p^{L^*} / C_p^L)^2$$

From equation (2.6.4), we see that the ratio of the conditional 100(1- α_1)% lower confidence limit of C_p over the unconditional 100(1- α_1)% lower confidence limit of C_p only depends on the value of the specified significance level α_1, the sample size *n* and the ratio of the critical value over the observed chi-square test statistic of the preliminary test for testing H_o: $C_p \le C_0$ (or H_o: $\sigma \ge \sigma_o$).

For the given example 2.5.1 in section 2.5, we have α_1= 0.025, n=48; so, $\chi_{47,0.025}^2 = 29.96$, $\lambda = \chi_\alpha^2 / V$ = 32.27/25.06 = 1.2877; therefore, use IMSL numerical library to solve the equation (2.6.4), we obtain the ratio of the conditional 97.5% lower confidence limit of C_p over the unconditional 97.5% lower confidence limit of C_p in this case as

$$\Lambda_L = C_p^{L^*} / C_p^L = 0.7468$$

This result shows that in the given example, the conditional 97.5% lower confidence limit of C_p is relatively smaller than the unconditional 97.5% lower confidence limit of C_p. The above result matches the result we obtained in section 2.5.

For the example 2.5.2, we have α_1= 0.025, n=60; so, $\chi^2_{59,0.025} = 39.66$, $\lambda = \chi^2_\alpha / V$ = 42.34/25.85 = 1.6379, therefore, use IMSL numerical library to solve the equation (2.6.4), we get the ratio of the conditional 97.5% lower confidence limit of C_p over the unconditional 97.5% lower confidence limit of C_p in this case as

$$\Lambda_L = C_p^{L^*} / C_p^L = 0.9812$$

This result also matches the result we obtained in example 2.5.2. In the example, we got the unconditional 97.5% lower confidence limit of C_p as 1.25 and the conditional 97.5% lower confidence limit of C_p as 1.26, which lead to the ratio of the conditional upper confidence limit over the unconditional upper confidence limit to 0.9921. (a tiny difference occurs between the two results just because of the rounding errors).

Table 2.1 gives the relationship between Λ_L and λ for different sample size *n* and significance level α_1 combinations. Figures 2.1 and Figure 2.2 are the SAS plots showing the relationship.

It's quite clear from the table and the two graphs that the ratio of $\Lambda_L = C_p^{L^*} / C_p^L$ is always less than or equal to 1, which means the conditional lower confidence limit of C_p is always smaller than or equal to the unconditional lower confidence limit of C_p. When the sample size *n* and the ratio $\lambda = \chi^2_\alpha / V$ get bigger and bigger, Λ_L approaches 1. Λ_L becomes bigger too when the significance level α_1 gets bigger.

Table 2.1 Relationship between Λ_L and λ for different n and α_1 combinations

Λ_L \ λ		1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	3.0	4.0
α_1 = 0.005	n=10												0.6748
	n=20								0.3410	0.5392	0.6515	0.9516	0.9914
	n=40				0.5804	0.7497	0.8375	0.8903	0.9244	0.9473	0.9633	1.0000	1.0000
	n=80		0.5326	0.8307	0.9150	0.9544	0.9750	0.9864	0.9927	0.9962	0.9980	1.0000	1.0000
	n=160	0.5772	0.9001	0.9636	0.9862	0.9951	0.9984	0.9995	0.9998	1.0000	1.0000	1.0000	1.0000
	n=320	0.8829	0.9759	0.9948	0.9991	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
α_1 = 0.025	n=10											0.7742	0.9274
	n=20					0.2852	0.5718	0.7031	0.7833	0.8371	0.8753	0.9981	0.9989
	n=40			0.6560	0.8125	0.8871	0.9289	0.9543	0.9703	0.9807	0.9875	0.9999	1.0000
	n=80		0.8013	0.9167	0.9605	0.9808	0.9907	0.9957	0.9980	0.9992	0.9996	1.0000	1.0000
	n=160	0.7896	0.9467	0.9828	0.9946	0.9985	0.9996	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
	n=320	0.9319	0.9881	0.9981	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

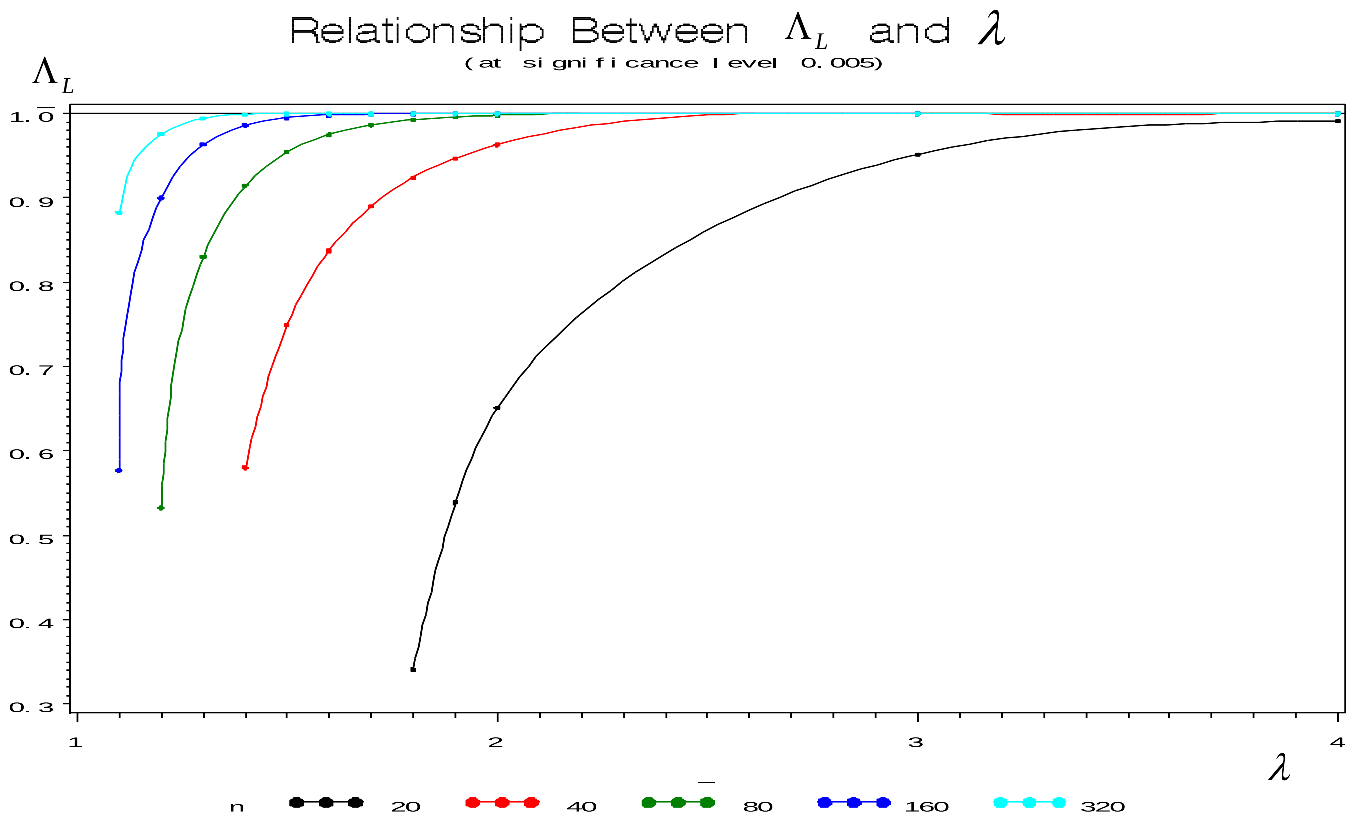

Figure 2.1 SAS plots for the relationship between $\Lambda_L = C_p^{L^*} / C_p^L$ and $\lambda = \chi_\alpha^2 / V$ at α_1= 0.005.

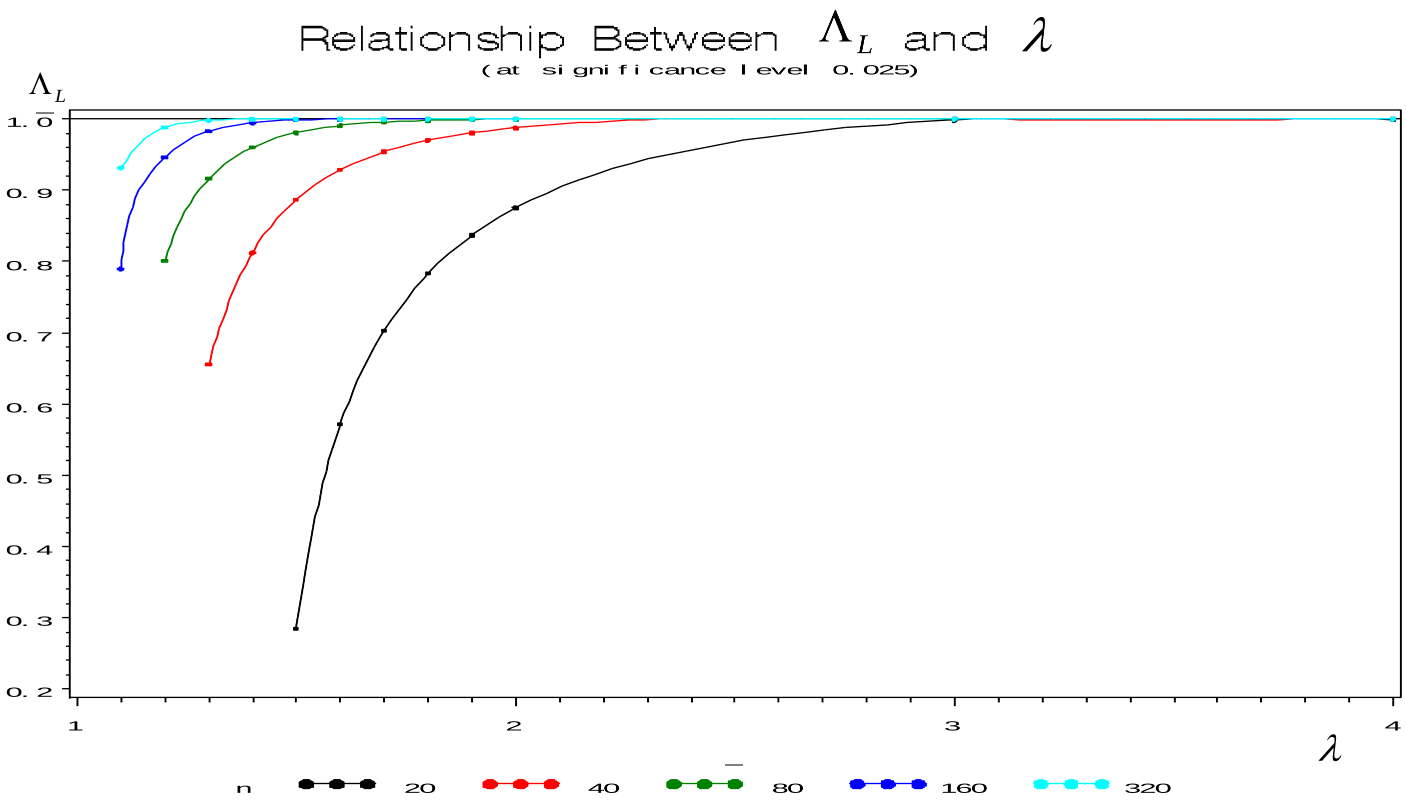

Figure 2.2 SAS plots for the relationship between $\Lambda_L = C_p^{L^*} / C_p^L$ and $\lambda = \chi_\alpha^2 / V$ at α_1= 0.025.

2.6.2 Ratio of the Upper Limits

The 100(1- α_2)% unconditional lower confidence limit of σ^2, denoted by σ_L^2, is formed by solving

$$H(\frac{n-1}{\sigma_L^2} \cdot t) = 1 - \alpha_2 \quad (2.6.5)$$

The 100(1- α_2)% conditional lower confidence limit of σ^2, denoted by σ_L^{*2}, following rejection of the null hypothesis H_o: $C_p \le c_o$ is given by solving the eqation

$$\frac{H(\frac{n-1}{\sigma_L^{*2}} \cdot t)}{H(\frac{\sigma_o^2}{\sigma_L^{*2}} \cdot \chi_\alpha^2)} = 1 - \alpha_2 \quad (2.6.6)$$

Similarly, after some algebraic manipulation, we have

$$1 - \alpha_2 = \frac{H(\Lambda_U^2 \cdot \chi_{1-\alpha_2}^2)}{H(\Lambda_U^2 \cdot \lambda \cdot \chi_{1-\alpha_2}^2)} \quad (2.6.7)$$

where

$$\lambda = \chi_\alpha^2 / V \quad \text{and} \quad \Lambda_U^2 = \sigma_L^2 / \sigma_L^{*2} = (C_p^{U^*} / C_p^U)^2$$

The above result shows that the ratio of unconditional 100(1-α_2)% upper confidence limit of C_p over the conditional 100(1-α_2)% upper confidence limit of C_p depends only on the value of the significance level α_2, the sample size *n* and the ratio of the critical value over the observed chi-square test statistic of the preliminary test.

For the example 2.5.1 in section 2.5, α_2 = 0.025, n = 48; therefore, $\chi^2_{47,0.975} = 67.82$, $\lambda = \chi^2_\alpha / V$ = 1.2877. Using IMSL, we obtain the ratio of the conditional 97.5% upper confidence limit of C_p over the unconditional 97.5% upper confidence limit of C_p for this case as

$$\Lambda_U = C_p^{U^*} / C_p^U = 0.9995$$

This result shows that in the given example, the conditional 97.5% upper confidence limit of C_p is almost unchanged compare to the unconditional 97.5% upper confidence limit of C_p. This result also consists with the result we obtained in example 2.5.1.

For the example 2.5.2, α_2 = 0.025, n = 60; therefore, $\chi^2_{59,0.975}$ = 82.12, $\lambda = \chi^2_\alpha / V$ = 1.6379. Hence, using IMSL, we obtain the ratio of the conditional 97.5% upper confidence limit of C_p over the unconditional 97.5% upper confidence limit of C_p as

$$\Lambda_U = C_p^{U^*} / C_p^U = 1.0000$$

Table 2.2 shows the relationship between Λ_U and λ for different sample size *n* and significance level α_2 combinations. Figures 2.3 and Figure 2.4 are the SAS plots showing the relationship.

From the table and plots, we see that the ratio of $\Lambda_U = C_p^{U^*} / C_p^U$ is still always less than or equal to 1, and in most of the cases, this ratio is equal (or almost equal) to 1. This result tells us that the conditional upper confidence limit of C_p is almost unchanged compare to the unconditional upper confidence limit of C_p in most of the cases. This situation is quite different from the ratio of the lower confidence limits. For the lower confidence limit case, the ratio varies a lot from case to case.

To summarize the previous results, we may conclude that, following rejection of a one-sided test, those ratios of Λ_U are all equal to or nearly equal to 1, therefore, the conditional upper confidence limits are almost unchanged compare to the unconditional upper confidence limits. But the ratios of Λ_L are always less than or equal to 1, so the conditional lower confidence limits are always less than or equal to the unconditional lower confidence limits. This lead to a common conclusion that, following this type of rejection, the length of the conditional confidence interval of C_p is always not less than that of unconditional confidence interval, and in most cases, the conditional confidence intervals of C_p covers the whole length of the unconditional confidence interval of C_p. This result also verified the results we obtained in examples 2.5.1 and 2.5.2. When the sample size n, the ratio λ and the values of α_1 and α_2 become bigger and bigger, the conditional confidence interval of C_p will become closer and closer to the unconditional confidence interval of C_p. It should be noted that the conditional coverage probability of the unconditional confidence interval is smaller than that of the conditional confidence interval, this is discussed in the next section.

Table 2.2 Relationship between Λ_U and λ for different n and α_2 combinations

Λ_U \ λ		1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	3.0	4.0
α_2= 0.005	n=10	0.9644	0.9886	0.9958	0.9984	0.9994	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
	n=20	0.9822	0.9957	0.9988	0.9997	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	n=40	0.9922	0.9988	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	n=80	0.9972	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	n=160	0.9993	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	n=320	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
α_2= 0.025	n=10	0.9291	0.9735	0.9884	0.9947	0.9975	0.9989	0.9995	0.9998	0.9999	1.0000	1.0000	1.0000
	n=20	0.9651	0.9897	0.9966	0.9989	0.9996	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	n=40	0.9843	0.9968	0.9993	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	n=80	0.9939	0.9993	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	n=160	0.9982	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	n=320	0.9996	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

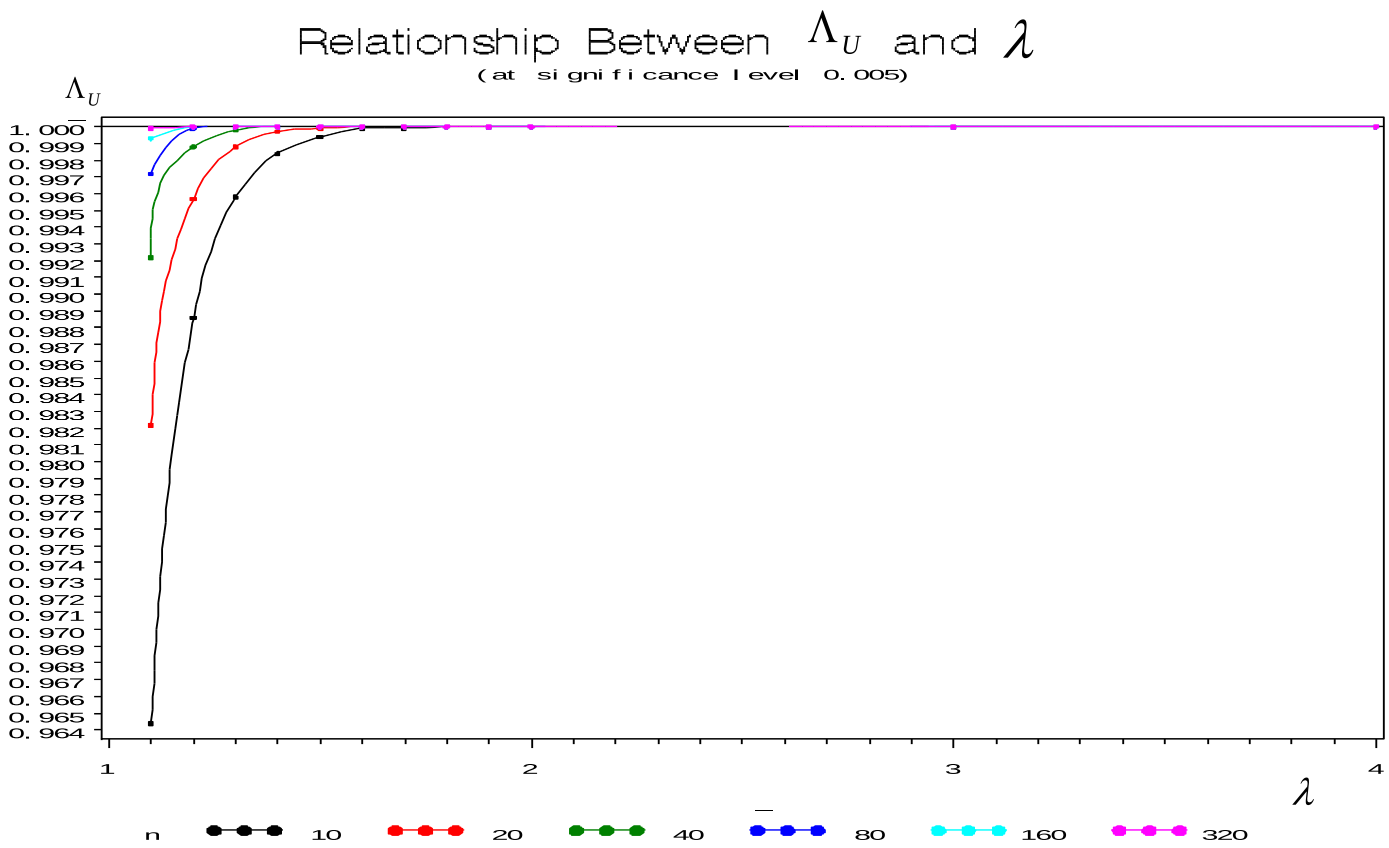

Figure 2.3 SAS plots for the relationship between $\Lambda_U = C_p^{U^*} / C_p^U$ and $\lambda = \chi_\alpha^2 / V$ at α_2= 0.005.

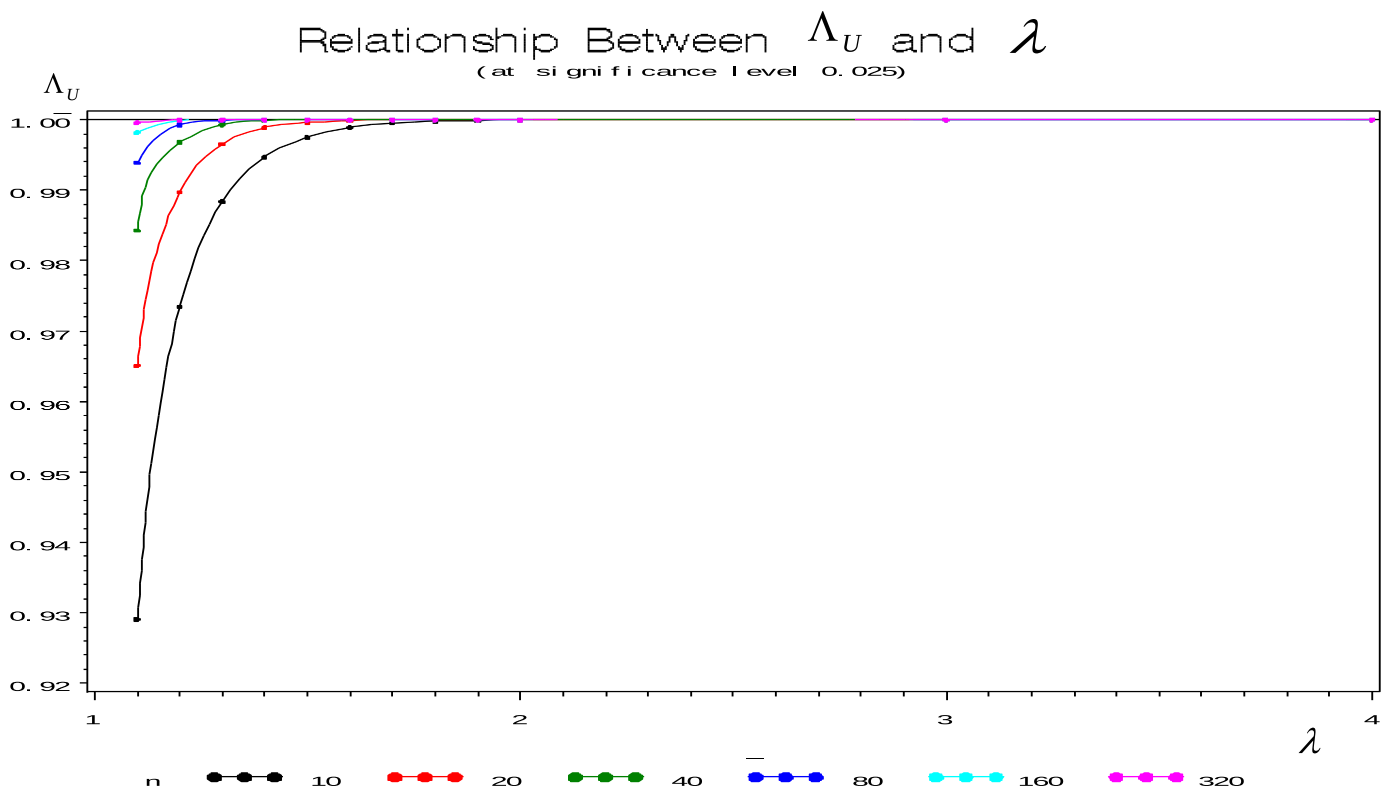

Figure 2.4 SAS plots for the relationship between $\Lambda_U = C_p^{U^*} / C_p^U$ and $\lambda = \chi_\alpha^2 / V$ at α_2= 0.025.

2.7 Conditional Coverage Probability of the UCI Analysis

In case the unconditional confidence intervals of C_p are used to approximate the conditional confidence intervals of C_p (this happens when we ignore the result of a preliminary test), it is necessary to give the actually coverage probability that is provided at the nominal (unconditional) (1 - p)*100 percent level.

From section 2.6, we know that the conditional lower confidence limit of C_p is relatively smaller than the unconditional lower confidence limit of C_p, and the conditional upper confidence limit of C_p almost keeps unchanged compare to the unconditional upper confidence limit of C_p. Therefore, the conditional confidence interval of C_p is always longer than the unconditional confidence interval of C_p, and the conditional confidence interval of C_p almost covers the whole length of the unconditional confidence interval of C_p. Thus, if we use the unconditional confidence intervals of C_p to approximate the conditional confidence interval of C_p, the actually coverage probability will always be less than or equal to the nominal level.

Since $C_p = \frac{d}{3\sigma}$, and actually we derived the conditional confidence intervals of C_p through the conditional confidence intervals of σ^2, therefore, when we use an unconditional confidence interval of C_p to approximate the conditional confidence interval of C_p, the actual coverage probability of the confidence interval of C_p is equal to the actual coverage probability of the corresponding confidence interval of σ^2 provided at the same nominal (unconditional) level. This gives the following result.

[Result 2.7.1] For an equal tail (1 - p)*100 percent level unconditional confidence interval of C_p, the conditional coverage probability is given by

$$\frac{1-p/2}{H(\lambda\cdot\chi^2_{1-p/2})}-\frac{p/2}{H(\lambda\cdot\chi^2_{p/2})}$$

Proof: We only need to prove the above formula for the actual coverage probability of the equal tail unconditional confidence interval of σ^2. Since the equal tail (1 - p)*100 percent level unconditional confidence interval of σ^2 is

$$\left(\frac{(n-1)s^2}{\chi^2_{1-p/2}},\frac{(n-1)s^2}{\chi^2_{p/2}}\right)$$

Plug the above upper and lower limits into equations (2.4.1) and (2.4.2), we have

$$\alpha_1 = \frac{H(\frac{n-1}{\sigma_U^2}\cdot t)}{H(\frac{\sigma_o^2}{\sigma_U^2}\cdot\chi^2_{n-1;\alpha})}$$

$$= \frac{H(\frac{n-1}{(n-1)s^2/\chi^2_{p/2}}\cdot t)}{H(\frac{\sigma_o^2}{(n-1)s^2/\chi^2_{p/2}}\cdot\chi^2_{n-1;\alpha})}$$

$$= \frac{p/2}{H(\lambda\cdot\chi^2_{p/2})}$$

where $\lambda=\chi^2_\alpha/V$ is defined as same as in section 2.6, and

$$1-\alpha_2 = \frac{H(\frac{n-1}{\sigma_L^2}\cdot t)}{H(\frac{\sigma_o^2}{\sigma_L^2}\cdot\chi^2_{n-1;\alpha})}$$

$$= \frac{H\left(\frac{n-1}{(n-1)s^2/\chi^2_{1-p/2}}\cdot t\right)}{H\left(\frac{\sigma_o^2}{(n-1)s^2/\chi^2_{1-p/2}}\cdot \chi^2_{n-1;\alpha}\right)}$$

$$= \frac{1-p/2}{H(\lambda\cdot\chi^2_{1-p/2})}$$

Therefore, the actual conditional coverage probability is

$$1 - \alpha_1 - \alpha_2 = \frac{1-p/2}{H(\lambda\cdot\chi^2_{1-p/2})} - \frac{p/2}{H(\lambda\cdot\chi^2_{p/2})} \qquad (2.7.1)$$

Table 2.3 gives the actual conditional coverage probability of a nominal 90 percent confidence interval for different sample size *n* and ratio λ combinations. The result shows that when both the sample size n and ratio λ are small, the actual conditional coverage probability of a nominal 90 percent confidence interval is very small. When either the sample size *n* or the ratio λ or both of *n* and λ gets bigger and bigger, the actual conditional coverage probability of a nominal 90 percent confidence interval gets bigger and bigger, and eventually this actual coverage probability approaches 90 percent—the nominal percentage level.

Table 2.3 Actual coverage probability of 90 percent nominal confidence interval

n \ λ	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	3.0	4.0
10	0.2377	0.3970	0.5084	0.5889	0.6487	0.6942	0.7293	0.7570	0.7791	0.7970	0.8728	0.8912
20	0.3518	0.5446	0.6587	0.7303	0.7774	0.8095	0.8324	0.8483	0.8603	0.8693	0.8975	0.8999
40	0.4747	0.6745	0.7711	0.8225	0.8519	0.8696	0.8806	0.8877	0.8922	0.8952	0.9	0.9
80	0.5971	0.7759	0.8437	0.8734	0.8874	0.8943	0.8975	0.8990	0.8996	0.8999	0.9	0.9
160	0.7071	0.8440	0.8825	0.8949	0.8988	0.8998	0.9	0.9	0.9	0.9	0.9	0.9
320	0.7940	0.8819	0.8975	0.8998	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9

CHAPTER 3

CONDITIONAL CONFIDENCE INTERVALS OF C_{pk}

3.1. The Process Capability Index C_{pk}

At the beginning of Chapter 2, we discussed that if μ, the expected value of X (measurement of the product), or simply the process mean, is equal to the midpoint of the specified interval, then the expected proportion of NC product is equal to $2\Phi(-3C_p)$. But if the expected value of X is not equal to the midpoint of the specified interval, i.e., $\mu \neq \frac{1}{2}(LSL + USL)$, then the expected proportion of NC product will be bigger than $2\Phi(-3C_p)$. The above situation often happens in practice. For example, if we want to produce a pair of axle and bearing (bushing), then the actual measurement of diameter for the axle should be always less than the measurement of diameter for the bearing (bushing), otherwise, the axle can't be fitted in the bearing (bushing). Therefore, when we manufacture the axle, after we specified the target value of the diameter, normally we allow more deviation for the lower limit of the diameter. But for the bearing (bushing), it is just the opposite, that is, we allow more deviation for the upper limit of the diameter. In this case, it is not suitable for us to still use the process capability index C_p for either the process of manufacturing the axle or the process of manufacturing the bearing (bushing). Thus, we introduce another process capability index C_{pk} to overcome this drawback,

If we consider the effects of the value of the process mean μ, then the process capability index C_{pk} is defined as

$$C_{pk} = \frac{\min(USL - \mu, \mu - LSL)}{3\sigma}$$

Since min(a ,b) = $\frac{1}{2}$(|a + b|-|a - b|) for any a ≥ 0 and b ≥ 0, therefore

$$C_{pk} = \frac{\frac{1}{2}\left[(USL - LSL) - |USL + LSL - 2\mu|\right]}{3\sigma}$$

$$= \frac{d - \left|\mu - \frac{1}{2}(LSL + USL)\right|}{3\sigma}$$

$$= \left\{1 - \frac{\left|\mu - \frac{1}{2}(LSL + USL)\right|}{d}\right\} C_p$$

or simply

$$C_{pk} = \frac{d - |\mu - m|}{3\sigma}$$

$$= \left\{1 - \frac{|\mu - m|}{d}\right\} C_p$$

where $m = \frac{1}{2}(LSL + USL)$ is the midpoint of the specified interval.

Note, we assume that LSL ≤ μ ≤ USL in our discussion. If μ is outside of the specified interval, then by the initial definition of C_{pk}, the value of C_{pk} would be negative, and the process would clearly be inadequate for controlling the quality of the product.

Since we always have $\left\{1-\frac{|\mu-m|}{d}\right\}\leq 1$, therefore, $C_{pk} \leq C_p$, with equality if and only if μ = $\frac{1}{2}(LSL+USL)$. Similar to the index C_p, the smaller value of the C_{pk} corresponds to worse quality of the product.

The process capability index C_{pk} involves both the process mean and the process variance. When the two parameters are all unknown, and a random sample $X_1, X_2, \ldots, X_n$ is taken from $N(\mu, \sigma^2)$ distribution, then an estimator of μ is the sample mean $\overline{X}$, and an estimator of σ is the sample standard deviation S . Therefore, a point estimator of C_{pk} is given by

$$\hat{C}_{pk} = \frac{d-\left|\hat{\mu}-\frac{1}{2}(LSL+USL)\right|}{3\hat{\sigma}}$$

$$= \frac{d-\left|\overline{X}-\frac{1}{2}(LSL+USL)\right|}{3S}$$

$$= \frac{d-\left|\overline{X}-m\right|}{3S}$$

Since $\overline{X}$ and S are mutually independently distributed, it is still possible for us to calculate the mean and the variance of $\hat{C}_{pk}$ by first carrying out its rth moment about the origin, like we did in Chapter 2 for the index C_p (resulting formulas for C_{pk} refer to: Kotz & Lovelace (1998), page 55). But this procedure is much more complicated and it involves another type of distribution which is so called “folded” distribution.

If we consider both the mean μ and the variance σ^2 as unknown parameters, then the construction of (unconditional) confidence intervals of C_{pk}

is difficult due to the fact that the distribution of $\hat{C}_{pk}$ involves the joint distribution of two non-central t-distributed random variables. No single technique is considered best in practice at this time (Kotz & Lovelace (1998), page 57). Although the explicit expression of such a confidence interval is almost impossible, but theoretically, this confidence interval is still possible to be determined for particular problems. The idea is to extend the general method to the two parameters case. That is, first try to find the joint confidence region of the two parameters μ and σ, and then use this joint confidence region to obtain a confidence interval of C_{pk}, this method is discussed in section 3.4.

In the case that we have some uncertain prior information about the values of μ and σ, we will use preliminary tests for testing these two parameters. We will adopt two sequential tests for testing μ and σ separately, instead of testing μ and σ jointly. The conditional confidence interval of C_{pk} will be considered following rejection of any of the sequential tests.

In this chapter, we will discuss the conditional confidence intervals of C_{pk} for the following three different cases:

(1) The mean μ is known, the variance σ^2 is unknown

(2) The mean μ is unknown, the variance σ^2 is known

(3) Both the mean μ and the variance σ^2 are unknown

3.2 CCIs of C_{pk} When μ Is Known and σ^2 Is Unknown

If the process mean μ is known, then for the process capability index C_{pk}, there is only one unknown parameter σ. This situation is similar to the one for finding the conditional confidence interval of the process capability index C_p. In this case, the preliminary test should be constructed as H_o: $C_{pk} \le C_o$ vs. H_1:

$C_{pk} > C_o$, or, simply use the parameter σ: H_o: $\sigma \geq \sigma_o$ vs. H_1: $\sigma < \sigma_o$, where the value of σ_o can be determined by the formula

$$C_o = \frac{d - |\mu - m|}{3\sigma_o}$$

[Result 3.2.1] If a process has a known mean μ and an unknown variance σ^2, then a $100(1- \alpha_1 - \alpha_2)\%$ conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $C_{pk} \leq C_o$ (or H_o: $\sigma \geq \sigma_o$) can be determined using the following interval

$$\left(\frac{d - |\mu - m|}{3\sigma_U}, \frac{d - |\mu - m|}{3\sigma_L}\right)$$

where (σ_L^2, σ_U^2) is a $100(1- \alpha_1 - \alpha_2)\%$ conditional confidence interval of σ^2 following rejection of the preliminary test for testing H_o: $\sigma \geq \sigma_o$ vs. H_1: $\sigma < \sigma_o$. The value σ_U^2 is a $100(1- \alpha_1)\%$ conditional upper confidence limit of σ^2 which is determined by equation (2.4.1), and the value of σ_L^2 is a $100(1- \alpha_2)\%$ conditional lower confidence limit of σ^2 which is determined by equation (2.4.2).

The proof of the above result follows the result 2.4.2 and some simple calculations.

[Example 3.2.1] Conditional confidence interval of C_{pk} analysis for the data set in example 2.5.2 when μ is known and σ^2 is unknown

In example 2.5.2, suppose the process mean μ is known as 5.25, and the specified upper limit and lower limit are 5.45 and 4.85 respectively, so, d = (USL - LSL)/2 = 0.30, m = (LSL + USL)/2 = 5.15. From the sample, we obtain the sample mean and the sample standard deviation as $\overline{X}$ = 5.2110 and S = 0.0649, thus, the point estimate of C_{pk} is

$$\hat{C}_{pk} = \frac{d - |\mu - m|}{3S}$$

$$= \frac{0.30 - |5.25 - 5.15|}{3 * 0.0649}$$

$$= 1.03$$

Since in this example, the process mean is not equal to the midpoint of the specified interval, it's obvious that the value of the point estimate of C_{pk} is smaller than the value of the point estimate of C_p ($\hat{C}_p = d/3s = 1.54$).

Suppose the minimum required value of the process capability index C_{pk} for this process is 0.80, then we need to construct a test hypothesis as H_o: $C_{pk} \leq 0.80$ vs. H_1: $C_{pk} > 0.80$.

For the test hypothesis H_o: $C_{pk} \leq 0.80$ vs. H_1: $C_{pk} > 0.80$, It is equivalent to the hypothesis H_o: $\sigma \geq \sigma_o$ vs. H_1: $\sigma < \sigma_o$, where $\sigma_o = (d - |\mu - m|)/3C_o = (0.30 - |5.25 - 5.15|)/(3*0.80) = 0.0833$. We'll reject H_o at level $\alpha = 0.05$ if the test statistic $V = (n-1)\frac{s^2}{\sigma_o^2} \leq \chi^2_{59,0.05} = 42.3393$. Now the test statistic:

$$V = (n-1)\frac{s^2}{\sigma_o^2}$$

$$= 59*(0.0649/0.0833)^2$$

$$= 35.81$$

Therefore, we reject the null hypothesis H_o: $C_{pk} \leq 0.80$ (or H_o: $\sigma \geq 0.0833$) at level $\alpha = 0.05$.

According to result 2.4.2, a 97.5% conditional upper confidence limit of σ^2 following rejection of the null hypothesis H_o: $\sigma \geq 0.0833$ at level $\alpha = 0.05$ can be obtained by solving the equation

$$\frac{H(\frac{n-1}{\sigma_U^2} \cdot t)}{H(\frac{\sigma_o^2}{\sigma_U^2} \cdot \chi^2_{n-1;\alpha})} = \alpha_1 = 0.025$$

where $t = 0.0649^2$ is the observed value of S^2, $\sigma_o = 0.0833$, $n = 60$, and $\chi^2_{n-1,\alpha} = 42.34$. Use IMSL numerical library to solve the above equation, we obtain the solution

$$\sigma_U^2 = 0.01735$$

For the solution of a 97.5% conditional lower confidence limit of σ^2 following rejection of the null hypothesis H_o: $C_{pk} \leq 0.80$ (or H_o: $\sigma \geq 0.0833$) at level $\alpha = 0.05$, according to result 2.4.2, we can obtain the value of σ_L^2 by solving the equation

$$\frac{H(\frac{n-1}{\sigma_L^2} \cdot t)}{H(\frac{\sigma_o^2}{\sigma_L^2} \cdot \chi^2_{n-1;\alpha})} = 1 - \alpha_2 = 0.975$$

For the above equation, the IMSL numerical library is unable to reach a solution due to floating errors. So we use the result in Chapter 2, that is, section 2.6 equation (2.6.7), to find the ratio of the unconditional lower confidence limit of σ^2 over the conditional lower confidence limit of σ^2 first (in this case, α_2 = 0.025, n = 60, $\chi^2_{59,0.975} = 82.12$, and $\lambda = \chi^2_\alpha / V$ = 1.1824). The result shows that this ratio $\Lambda^2_U = \sigma^2_L / \sigma^{*2}_L$ = 0.9981^2 = 0.9962. Since the unconditional lower confidence limit of σ^2 is easy to obtain, and it is equal to 0.0030 from previous example, therefore, the conditional lower confidence limit of σ^2 is equal to

$$\sigma^2_L = 0.0030$$

Finally, a 95% conditional confidence interval of σ^2 following rejection of the null hypothesis H_o: $C_{pk} \le 0.80$ is given by:

$$(\sigma^2_L, \sigma^2_U) = (0.0030, 0.01735)$$

and consequently, a 95% conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $C_{pk} \le 0.80$ is given by

$$(C^L_p, C^U_p) = (\frac{d - |\mu - m|}{3\sigma_U}, \frac{d - |\mu - m|}{3\sigma_L})$$

$$= (0.51, 1.23)$$

3.3 CCIs of C_{pk} When μ Is Unknown and σ^2 Is Known

In some situations, if we have enough information about the variance of a process, i.e. the variance σ^2 of the process can be regarded as known. Then for the process capability index C_{pk} , there is only one unknown parameter, the process mean μ. If the measurement of a process follows a normal distribution, then a point estimator of μ is the sample mean $\overline{X}$. Therefore, a point estimator of the process capability index C_{pk} becomes

$$\hat{C}_{pk} = \frac{d - \left|\overline{X} - m\right|}{3\sigma}$$

$$= \begin{cases} \dfrac{d - (\overline{X} - m)}{3\sigma}, & if\ \overline{X} \geq m \\ \\ \dfrac{d + (\overline{X} - m)}{3\sigma}, & if\ \overline{X} < m \end{cases}$$

In this case, finding a conditional confidence interval of the process capability index C_{pk} is really a matter of finding a conditional confidence interval of the process mean μ.

For the process mean μ, if a random sample X_1, X_2, …… X_n drawn from the process follows a normal distribution $N(\mu, \sigma^2)$, then a 100(1-α)% unconditional confidence interval of μ is given by the interval

$$(\overline{X} - z_{1-\alpha/2}\sigma/\sqrt{n},\ \ \overline{X} + z_{1-\alpha/2}\sigma/\sqrt{n})$$

After we obtain an unconditional confidence interval of μ, then an unconditional confidence interval of C_{pk} can be easily determined by using the formula $C_{pk} = \dfrac{d - |\mu - m|}{3\sigma}$, since the only unknown parameter in this formula is μ.

The test hypothesis for the parameter C_{pk} for this case (μ is unknown, σ^2 is known) can be constructed as H_o: $C_{pk} = C_o$ vs. H_1: $C_{pk} \neq C_o$, or equivalent to the hypothesis H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$, where $\mu_o = m + d - 3\sigma C_o$ if $\mu_o \geq m$, and $\mu_o = m - d + 3\sigma C_o$ if $\mu_o < m$. For the same value of C_o, whether we choose the value μ_o by using the condition $\mu_o \geq m$ or $\mu_o < m$ depends on prior information. For example, if we allow more deviation from the lower side of the mean, then we need to use the condition $\mu_o \geq m$. That is, we choose $\mu_o = m + d - 3\sigma C_o$. Otherwise, we use the condition $\mu_o < m$ and choose $\mu_o = m - d + 3\sigma C_o$.

A common rule of how to use the above preliminary test is that, if the null hypothesis is not rejected, then we use μ_0 as an estimate of μ to give an estimate of C_{pk}, there is no need to construct a conditional confidence interval of C_{pk} in this case. But if the null hypothesis is rejected, we should use $\bar{x}$ as an estimate of μ to give the estimate of C_{pk} , and then we need to find a conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $C_{pk} = C_o$, or equivalently H_o: $\mu = \mu_o$.

As we already know, when the process variance σ^2 is known, the process capability index C_{pk} contains only one unknown parameter, the process mean μ. Therefore, in order to find a conditional confidence interval of C_{pk}, we only need to find a conditional confidence interval of the mean μ.

Arabatzis, Gregoire and Reynolds (1989) investigated the conditional confidence interval of the normal mean following rejection of a two-sided test when σ is known, although I don't quite agree with the main result they have for the conditional confidence interval of μ, but some partial results are still useful. Next, we'll follow the general method to find a conditional confidence interval of μ following rejection of the null hypothesis H_o: $\mu = \mu_o$.

If a random sample $X_1, X_2, \ldots, X_n$ is taken from a normal distribution $N(\mu, \sigma^2)$, where μ is unknown and σ^2 is known. Then a level α test for testing H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ has the critical region

$$K = \left\{ \overline{X} : \left| \overline{X} - \mu_o \right| > z_{1-\alpha/2} (\sigma / \sqrt{n}) \right\}$$

where $z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution. The null hypothesis is rejected if $\overline{X} \in K$, and a conditional confidence interval of μ is computed only after we rejected the null hypothesis.

The conditional pdf of $\overline{X}$ can be expressed as

$$f_c(\bar{x}) = \begin{cases} f(\bar{x})/D, & if\ |\bar{x}-\mu_o| > z_{1-\alpha/2}(\sigma/\sqrt{n}) \\ 0, & otherwise \end{cases}$$

where $f(\bar{x})$ is the unconditional pdf of $\overline{X}$, and D is the power of the test which is given by

$$D = P(|\bar{x}-\mu_o| > z_{1-\alpha/2}(\sigma/\sqrt{n})|\mu)$$

$$= 1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}$$

$$= 1-\Phi\left\{z_{1-\alpha/2}-\gamma\right\}+\Phi\left\{-z_{1-\alpha/2}-\gamma\right\}$$

where $\gamma = \sqrt{n}(\mu-\mu_o)/\sigma$, and $\Phi(\cdot)$ is the CDF of the standard normal distribution. Under H_o, D = α. When $\gamma \to \infty$, D approaches 1.

The conditional CDF of $\overline{X}$ can be expressed as

$$F_c(\bar{x}) = \begin{cases} \dfrac{\Phi\left\{\sqrt{n}(\bar{x}-\mu)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\gamma\right\}+\Phi\left\{-z_{1-\alpha/2}-\gamma\right\}}, & if\ \bar{x} < \mu_o - z_{1-\alpha/2}\sigma/\sqrt{n} \\ \dfrac{\Phi\left\{\sqrt{n}(\bar{x}-\mu)/\sigma\right\}-\Phi\left\{z_{1-\alpha/2}-\gamma\right\}+\Phi\left\{-z_{1-\alpha/2}-\gamma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\gamma\right\}+\Phi\left\{-z_{1-\alpha/2}-\gamma\right\}}, & if\ \bar{x} > \mu_o + z_{1-\alpha/2}\sigma/\sqrt{n} \end{cases}$$

The above formula implies that if the null hypothesis is rejected by a small observation of $\bar{X}$, i.e., if $\bar{x} < \mu_o - z_{1-\alpha/2}\sigma/\sqrt{n}$, then the conditional CDF of $\bar{X}$ can be expressed as

$$F_c(\bar{x}) = \frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\gamma\right\}+\Phi\left\{-z_{1-\alpha/2}-\gamma\right\}}$$

$$= \frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}} \quad (3.3.1)$$

If the null hypothesis is rejected by a large observation of $\bar{X}$, i.e., if $\bar{x} > \mu_o + z_{1-\alpha/2}\sigma/\sqrt{n}$, then the conditional CDF of $\bar{X}$ can be expressed as

$$F_c(\bar{x}) = \frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu)/\sigma\right\}-\Phi\left\{z_{1-\alpha/2}-\gamma\right\}+\Phi\left\{-z_{1-\alpha/2}-\gamma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\gamma\right\}+\Phi\left\{-z_{1-\alpha/2}-\gamma\right\}}$$

$$= \frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu)/\sigma\right\}-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu-\mu_o)/\sigma\right\}} \quad (3.3.2)$$

It's quite obvious from equations (3.3.1) and (3.3.2) that the conditional CDF of $\bar{X}$ depends only on the parameter μ, but not on any other nuisance parameters, so we can use the general method mentioned before in Chapter 2 to find a conditional confidence interval of μ. And in this case, it still can be verified numerically that the two functions $h_1(\mu)$ and $h_2(\mu)$ constructed by the following equations

$$F_c(h_1(\mu);\mu) = \alpha_1$$

and

$$F_c(h_2(\mu);\mu) = 1-\alpha_2$$

are increasing functions. Therefore, apply the general method for finding a confidence interval of an unknown parameter, we get the following result.

[Result 3.3.1] Suppose the random sample X_1, X_2, …… X_n is taken from a normal distribution $N(\mu, \sigma^2)$, where μ is unknown and σ^2 is known. Let $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ with $\alpha_1 + \alpha_2 < 1$, and $\bar{x}$ be an observed value of $\bar{X}$. Let $\Phi(\cdot)$ denote the CDF of the standard normal distribution. If the observed value $\bar{x}$ results in rejecting the null hypothesis H_o: $\mu = \mu_o$ at level α by the condition $\bar{x} < \mu_o - z_{1-\alpha/2}\sigma/\sqrt{n}$, then the solutions of

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_u^c)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}} = \alpha_1 \tag{3.3.3}$$

and

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_l^c)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}} = 1-\alpha_2 \tag{3.3.4}$$

construct a $100(1-\alpha_1-\alpha_2)$% conditional confidence interval (μ_l^c, μ_u^c) of μ. Otherwise, if the observed value $\bar{x}$ results in rejecting the null hypothesis H_o: $\mu = \mu_o$ at level α by the condition $\bar{x} > \mu_o + z_{1-\alpha/2}\sigma/\sqrt{n}$, then the solutions of

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_u^c)/\sigma\right\}-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}} = \alpha_1 \tag{3.3.5}$$

and

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_l^c)/\sigma\right\}-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}}=1-\alpha_2 \quad (3.3.6)$$

construct a 100(1- α_1 - α_2)% conditional confidence interval (μ_l^c, μ_u^c) of μ.

The above equations look like complicated, but if we use IMSL numerical library, we still can solve these equations for the conditional lower and upper confidence limits of μ.

Once we obtain the conditional confidence interval of μ as (μ_l^c, μ_u^c), to obtain a conditional confidence interval of C_{pk} just follows some simple calculations. We'll use an example to illustrate the above procedure for finding a conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $\mu = \mu_o$.

[Example 3.3.1] Conditional confidence interval of C_{pk} analysis when μ is unknown and σ^2 is known

Consider the same data set as in example 2.5.2. Suppose that the specified target value of the process is 5.25, the specified upper limit and lower limit are 5.45 and 4.85 respectively. Therefore, d = (USL - LSL)/2 = 0.60*0.5 = 0.30, and m = (USL + LSL)/2 = (5.45 + 4.85)*0.5 = 5.15. Suppose the standard deviation of this process is known as σ = 0.06 (the process mean μ is unknown). After doing some basic analysis to the data set, we obtain the sample mean $\bar{X}$ = 5.2110 and the sample standard deviation S = 0.0649.

Since in this case $\overline{X}$ = 5.2110 > m = 5.15, so, a point estimate of C_{pk} for this process is

$$\hat{C}_{pk} = \frac{d - |\bar{x} - m|}{3\sigma}$$

$$= \frac{d - (\bar{x} - m)}{3\sigma}$$

$$= \frac{0.30 - (5.211 - 5.15)}{3 * 0.06}$$

= 1.33

A 95% unconditional confidence interval (μ_l , μ_u) of μ can be determined by

$$(\bar{x} - z_{1-\alpha/2}\sigma / \sqrt{n}, \quad \bar{x} + z_{1-\alpha/2}\sigma / \sqrt{n})$$

$$= (5.211 - 1.96 * 0.06 / \sqrt{60}, \; 5.211 + 1.96 * 0.06 / \sqrt{60})$$

= (5.1958, 5.2262)

Since $C_{pk} = \frac{d - |\mu - m|}{3\sigma}$, use the above information of the unconditional confidence interval of μ, we can determine a 95% unconditional confidence interval (C^L_{pk}, C^U_{pk}) of C_{pk}. Since in this case the value of m is located on the left side of the above interval (μ_l , μ_u), therefore, a 97.5% upper confidence limit of C_{pk} happens at $\mu = \mu_l = 5.1958$, and a 97.5% lower confidence limit of C_{pk} happens at $\mu = \mu_u = 5.2262$, therefore

$$(C_{pk}^L, C_{pk}^U) = \left(\frac{d - |\mu_u - m|}{3\sigma}, \frac{d - |\mu_l - m|}{3\sigma} \right)$$

$$= \left(\frac{0.30 - |5.2262 - 5.15|}{3 * 0.06}, \frac{0.30 - |5.1958 - 5.15|}{3 * 0.06} \right)$$

$$= (1.24, 1.41)$$

Note, if m happened to be inside of the interval (μ_l, μ_u), then the upper confidence limit of C_{pk} equals to the value of C_p (= d/3σ), and the lower confidence limit of C_{pk} can be determined by one of the two values μ_l and μ_u which has the longer distance from the midpoint point m of the specified interval.

So far, all the above results based on the situation that there is no preliminary test for the process mean has been performed. If for any reason a preliminary test for testing H_o: μ = μ_o vs. H_1: μ ≠ μ_o has been done (the value of μ_0 depends on the prior information of the process mean μ, or of the process capability index C_{pk}, which can be obtained from previous experiences), then the above result of the unconditional confidence interval of C_{pk} is no longer valid. So we need the following procedure to find a conditional confidence interval of C_{pk}.

In this example, the target value of process is specified as μ = 5.25, if the standard deviation of the process is known as σ = 0.06, then we expect the value of C_{pk} for this process as $C_{pk} = \frac{d - |\mu - m|}{3\sigma}$ = (0.30 - (5.25 - 5.15))/3*0.06 = 1.11. So normally, if we have any prior information which shows that the process mean will be around the value of 5.25, then we will construct a test hypothesis H_o: $C_{pk} = 1.11$ vs. H_1: $C_{pk} \neq 1.11$, or equivalently to H_o: $\mu = 5.25$ vs. H_1: $\mu \neq 5.25$. If the null hypothesis is not rejected, then we accept the test value μ = 5.25 as known. That is, we accept for this process, the process mean μ is equal to 5.25, and consequently the process capability index C_{pk} is equal to 1.11, no conditional confidence interval is needed. If the null hypothesis is rejected, then we need to use $\hat{C}_{pk} = \frac{d - |\overline{X} - m|}{3\sigma}$ to give a point estimate of C_{pk}, and then construct a conditional confidence interval of C_{pk} following rejection of the test.

For the testing hypothesis H_o: $C_{pk} = 1.11$ vs. H_1: $C_{pk} \neq 1.11$ (or equivalently H_o: $\mu = 5.25$ vs. H_1: $\mu \neq 5.25$) in this example, we'll reject H_o at level α = 0.05 if the value of the test statistic $\overline{X}$ falls into the following reject region

$$K = \left\{\bar{x} : |\bar{x} - \mu_o| > z_{1-\alpha/2}(\sigma/\sqrt{n})\right\}$$

Now the test statistic $\bar{x}$ = 5.2110, and the value

$$\mu_o - z_{1-\alpha/2}\sigma/\sqrt{n} = 5.25 - 1.96 * 0.06/\sqrt{60}$$

$$= 5.2348$$

Therefore, we have $\bar{x} < \mu_o - Z_{1-\alpha/2}\sigma/\sqrt{n}$, and we reject the null hypothesis H_o: $\mu = 5.25$ at level α = 0.05.

Next, we will construct a 95% conditional confidence interval of the process mean μ following rejection of the null hypothesis H_o: $\mu = 5.25$.

Since in this example the rejection of the null hypothesis is caused by a small value of $\overline{X}$, i.e., the rejection is due to $\bar{x} < \mu_o - Z_{1-\alpha/2}\sigma/\sqrt{n}$. According to result 3.3.1, a 97.5% conditional upper confidence limit of μ following rejection of the null hypothesis H_o: $\mu = 5.25$ at level α = 0.05 can be determined by solving the equation

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_u^c)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}} = \alpha_1 = 0.025$$

where $\bar{x}$ = 5.2110 is the observed value of $\bar{X}$, μ_o = 5.25, σ = 0.06, n = 60, and $z_{1-\alpha/2} = 1.96$. Use IMSL numerical library to solve the above equation for μ_u^c, we get the solution

$$\mu_u^c = 5.227$$

To obtain a 97.5% conditional lower confidence limit of μ following rejection of the null hypothesis H_o: $\mu = 5.25$ at level α = 0.05, according to result 3.3.1, we can find μ_l^c by solving the equation

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_l^c)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}+\Phi\left\{-z_{1-\alpha/2}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}} = 1 - \alpha_2 = 0.975$$

Again, use the IMSL numerical library, we obtain the solution of the above equation as

$$\mu_l^c = 4.954$$

Therefore, a 95% conditional confidence interval of μ following rejection of the null hypothesis H_o: $\mu = 5.25$ is given by:

$$(\mu_l^c, \mu_u^c) = (4.954, 5.227)$$

After we obtain a conditional confidence interval of the process mean μ, now we can determine a conditional confidence interval of the process capability index C_{pk} following rejection of the null hypothesis H_o: $C_{pk} = 1.11$ (or equivalently H_o: $\mu = 5.25$). Since in this case, the value of m (= 5.15) is inside of the conditional confidence interval (μ_l^c, μ_u^c), therefore, a 97.5% conditional upper confidence limit of C_{pk} happens at the value of $\mu = m = 5.15$, and a 97.5% conditional lower confidence limit of C_{pk} happens at the value of $\mu = \mu_l^c = 4.954$. Thus, a 95% conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $C_{pk} = 1.11$ can be determined as

$$(C_{pk}^L, C_{pk}^U) = \left(\frac{d - \left|\mu_l^c - m\right|}{3\sigma}, \frac{d - \left|m - m\right|}{3\sigma} \right)$$

$$= \left(\frac{0.30 - \left|4.954 - 5.15\right|}{3 * 0.06}, \frac{0.30 - \left|5.15 - 5.15\right|}{3 * 0.06} \right)$$

$$= (0.58, 1.67)$$

The relationship between the conditional confidence interval of C_{pk} and the unconditional confidence interval of C_{pk} for the case that μ is unknown and σ is known can also be analyzed by using a similar method discussed in section 2.6. Except in this case, we need first to find the relationship between the conditional confidence limits of μ and the unconditional confidence limits of μ. Since this procedure is so complicated, we will not discuss in detail at this time.

In some special cases, we still need to test a one-sided hypothesis for the process capability $C_{pk,}$ this include the following two different situations, H_o: $C_{pk} \le C_o$ vs. H_1: $C_{pk} > C_o$ or H_o: $C_{pk} \ge C_o$ vs. H_1: $C_{pk} < C_o$. To find a conditional confidence interval of C_{pk} following rejection of the above null hypotheses follows a similar procedure discussed in this section. First, we need to find a conditional confidence interval of the process mean μ following rejection of the tests, and then we use the relationship $C_{pk} = \dfrac{d - |\mu - m|}{3\sigma}$ to obtain a conditional confidence interval of C_{pk}. Appendix B gives some brief results for finding conditional confidence intervals of the process mean μ following rejection of one-sided tests.

3.4 CCIs of C_{pk} When both μ and σ^2 Are Unknown

Previously, we discussed the conditional confidence intervals of the process capability index C_{pk} for the two difference cases: either μ is known and σ^2 is unknown or μ is unknown and σ^2 is known. But in most situations, both the true values of the two parameters μ and σ^2 are unknown. So next, we'll discuss the conditional confidence intervals of C_{pk} when both μ and σ^2 are unknown.

The testing hypotheses we need to consider for this situation depends on how much prior information we have. If we have prior information for both parameters μ and σ^2, then we need to construct testing hypotheses for the two parameters μ and σ^2. But in some cases, we only have information for one of the two parameters. If this is the case, then we only need to construct one testing hypothesis. Next, we will discuss these two different cases.

3.4.1 Testing for both Parameters

As we mentioned at the beginning of Chapter 3, if both the mean μ and the variance σ^2 of a process are unknown parameters, and we have uncertain prior information for both μ and σ^2, then we will test the parameters μ and σ^2 separately using two sequential tests. The conditional confidence interval of C_{pk} will be considered following rejection of any of the two tests. The procedure is given as the following. First, test the hypothesis H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$, if the null hypothesis is not rejected, we regard σ as given ($\sigma = \sigma_o$), and then test H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ for the parameter μ, this test is a normal test since σ is given. If the null hypothesis H_o: $\mu = \mu_o$ is also not rejected, then we use μ_o and σ_o as two estimates of μ and σ to give the estimate of C_{pk}, no conditional confidence interval of C_{pk} is needed. But if the null hypothesis H_o: $\mu = \mu_o$ is rejected, we use $\bar{x}$ and σ_o as two estimates to give the estimate of C_{pk}. And then we will find a conditional confidence interval of μ following rejection of the null hypothesis H_o: $\mu = \mu_o$ of the two sequential tests. Finally, we use the above conditional confidence interval of μ together with the value of σ_o (since $\sigma = \sigma_o$ is regarded as known in this case) to obtain a conditional confidence interval of C_{pk}. The procedure for finding this conditional confidence interval of C_{pk} is almost the same as the one we discussed in the last section (section 3.3).

If the null hypothesis of the first test for testing H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$ has been rejected, in this case, we need to use the sample standard deviation s as an estimate of σ, and then regard σ as unknown to construct the second hypothesis H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ for testing the process mean μ. This time the test is a t-test since σ is unknown. If the null hypothesis of the second test is not rejected, we need to use μ_o and s as two estimates of μ and σ to give the point estimate of C_{pk}, and then try to find a conditional confidence interval of σ following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ (method refers to Appendix C, conditional confidence intervals of σ following rejection of a two-sided test). The conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ of the two sequential tests can be obtained as following. We regard μ as known ($\mu = \mu_o$) and σ as unknown and use the conditional confidence interval of σ together with the known value of μ ($\mu = \mu_o$) to construct a conditional confidence interval of C_{pk} (this procedure is similar to the one we discussed in section 3.2, except the conditional confidence interval of σ in this case follows rejection of a two-sided test).

[Example 3.4.1] In example 2.5.2, suppose the process mean μ and variance σ^2 are all unknown. The specified upper limit and lower limit are 5.45 and 4.85 respectively, so, d = (USL - LSL)/2 = 0.30, m = (LSL + USL)/2 = 5.15. From the sample, we get the sample mean and sample standard deviation as $\overline{X}$ = 5.2110 and S = 0.0649.

Suppose from prior information, we know that the process standard deviation might be around 0.05, and the process mean μ might be around 5.20. So we use two sequential tests to test H_o: σ = 0.05 vs. H_1: σ ≠ 0.05 and H_o: μ = 5.20 vs. H_1: μ ≠ 5.20 for the process standard deviation and mean separately.

For the hypothesis H_o: $\sigma = 0.05$ vs. H_1: $\sigma \neq 0.05$, we will reject the null hypothesis at level α = 0.05 if the test statistic V = (n-1)$\frac{s^2}{\sigma_o^2}$ ≤ $\chi^2_{n-1;\alpha/2}$, or V = (n-1)$\frac{s^2}{\sigma_o^2}$ ≥ $\chi^2_{n-1;1-\alpha/2}$. Now in this example, the observed test statistic

$$V = (n-1)\frac{s^2}{\sigma_o^2}$$

$$= (60-1)*0.0649^2/0.05^2$$

$$= 99.40$$

and $\chi^2_{n-1;1-\alpha/2}$ = $\chi^2_{59;0.975}$ = 82.12, so we reject the null hypothesis H_o: $\sigma = 0.05$ at level α = 0.05.

After we tested for the process variance, now we need to test for the process mean μ. Since the null hypothesis in the first test has been rejected, so the process variance is now unknown.

For testing hypothesis H_o: $\mu = 5.20$ vs. H_1: $\mu \neq 5.20$, since the true value of σ is unknown, we use a two-sided t-test. We'll reject the null hypothesis H_o: $\mu = 5.20$ at level α = 0.05 if the test statistic ($\bar{X}$, S) falls in the following rejection region

$$K = \left\{(\bar{x}, s) : \left|\bar{x} - \mu_o\right| > t_{1-\alpha/2}(s/\sqrt{n})\right\}$$

Now from the sample: $\bar{x}$ = 5.2110, s = 0.0649, so

$$\left|\bar{x} - \mu_o\right| = \left|5.211 - 5.20\right| = 0.011$$

and $$t_{1-\alpha/2}(s/\sqrt{n}) = 2.00 * 0.0649/\sqrt{60}$$

$$= 1.01$$

Thus, we have $\left|\bar{x} - \mu_o\right| < t_{1-\alpha/2}(s/\sqrt{n})$, and we do not reject the null hypothesis H_o: $\mu = 5.20$ at level α = 0.05. Therefore, we regard $\mu = 5.20$ as known for this process.

For the conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ of the two sequential tests, we use the conditional confidence interval of σ together with the known value of μ (μ = 5.20) to construct it. So next, we need to find a conditional confidence interval of σ following rejection of the null hypothesis H_o: σ = 0.05.

Since the rejection is due to a small observed value of *s*, i.e., $\frac{(n-1)s^2}{\sigma_o^2} < \chi^2_{n-1;\alpha/2}$ (this should be always the case in analysis of this type of conditional confidence intervals of C_{pk}, because if the rejection is due to a large observed value of *s*, then the process is obviously not capable, there is no needs to construct a conditional confidence interval before we improved the current process). According to result C.1 in appendix C, a 97.5% conditional upper confidence limit of σ^2 following rejection of the null hypothesis H_o: σ = 0.05 at level α = 0.05 can be obtained by solving the equation

$$\frac{H(\frac{n-1}{\sigma_U^2}\cdot s^2)}{1-H(\frac{\sigma_o^2}{\sigma_U^2}\cdot \chi^2_{n-1;1-\alpha/2})+H(\frac{\sigma_o^2}{\sigma_U^2}\cdot \chi^2_{n-1;\alpha/2})} = \alpha_1 = 0.025$$

where t = 0.0649^2 is the observed value of S^2, σ_o = 0.05, n = 60, $\chi^2_{n-1,\alpha/2} = 39.66$ and $\chi^2_{n-1,1-\alpha/2} = 82.12$. Use IMSL numerical library to solve the above equation, we get the solution as

$$\sigma_U^2 = 0.006267$$

To obtain a 97.5% conditional lower confidence limit of σ^2 following rejection of the null hypothesis H_o: σ = 0.05 at level α = 0.05, according to result C.1, we can find the conditional lower limit by solving the equation

$$\frac{H(\frac{n-1}{\sigma_U^2}\cdot s^2)}{1-H(\frac{\sigma_o^2}{\sigma_U^2}\cdot \chi^2_{n-1;1-\alpha/2})+H(\frac{\sigma_o^2}{\sigma_U^2}\cdot \chi^2_{n-1;\alpha/2})} = 1-\alpha_2 = 0.975$$

The solution of the conditional lower confidence limit of σ^2 is

$$\sigma_L^2 = 0.003896$$

Therefore, a 95% conditional confidence interval of σ following rejection of the null hypothesis H_o: σ = 0.05 at level α = 0.05 is given by:

$$(\sigma_l^c, \sigma_u^c) = (0.0624, 0.0792)$$

Consequently, a 95% conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ of the two sequential tests can be determined as

$$(C_{pk}^L, C_{pk}^U) = \left(\frac{d - |\mu_o - m|}{3\sigma_u^c}, \frac{d - |\mu_o - m|}{3\sigma_l^c} \right)$$

$$= \left(\frac{0.30 - |5.20 - 5.15|}{3 * 0.0792}, \frac{0.30 - |5.20 - 5.15|}{3 * 0.0624} \right)$$

$$= (1.05, 1.34)$$

If the null hypothesis of the second test for testing H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ is also rejected, In this case, both μ and σ need to be considered as unknown now, and the conditional confidence interval of C_{pk} should be considered following rejection of the two preliminary tests. This situation is more complicated than all the cases we discussed before. In order to find a conditional confidence interval of C_{pk}, we should first consider to find a joint confidence region of μ and σ. Next, we'll give some basic analyses for how to find a conditional confidence interval of C_{pk} in this situation.

Following the general method, in order to find a conditional joint confidence region of μ and σ, first we need to find the conditional joint CDF of $\bar{X}$ and S, so we start with finding the unconditional joint pdf of $\bar{X}$ and S.

If a random sample $X_1, X_2, \ldots, X_n$ is taken from a normal distribution N(μ, σ^2), then $\bar{X}$ ~ N(μ, σ^2/n), $S^2 \sim \frac{\sigma^2}{n-1}\chi^2_{n-1}$, and also $\bar{X}$ and S are independent. Follows Arabatzis, Gregoire and Reynolds (1989), the unconditional joint pdf of $\bar{X}$ and S can be expressed as

$$f(\bar{x}, s) = g(\bar{x})q(s)$$

$$= \frac{2s^{n-2}\sqrt{n}((n-1)/2)^{((n-1)/2}}{\sqrt{2\pi}\sigma^n\Gamma((n-1)/2)} e^{(-[n(\bar{x}-\mu)^2+(n-1)s^2]/2\sigma^2}$$

for $-\infty < \bar{x} < \infty, 0 < s < \infty$, where $\Gamma(\cdot)$ is the Gamma function.

The conditional joint pdf of $\bar{X}$ and S following rejection of the two tests for testing H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$ and H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ can be expressed as

$$f_c(\bar{x}, s) = \begin{cases} f(\bar{x}, s)/D, & if\ (\bar{x}, s) \in K \\ 0, & otherwise \end{cases}$$

where K is the critical region of the two tests determined by the intersection of $|\bar{x} - \mu_o| > t_{1-\alpha/2}(s/\sqrt{n})$ and $\frac{(n-1)s^2}{\sigma_o^2} < \chi^2_{n-1;\alpha/2}$ or $\frac{(n-1)s^2}{\sigma_o^2} > \chi^2_{n-1;1-\alpha/2}$, which is also the total shaded open regions of I, II, III and IV shown in figure 3.1; D is the total unconditional probability of $(\bar{x}, s)$ falling into the above critical region, which is determined by the following double integral.

$$D = \iint_K f(\bar{x}, s) d\bar{x} ds$$

The conditional joint CDF of $\bar{X}$ and S following rejection of the two tests for testing H$_o$: σ = σ$_o$ vs. H$_1$: σ ≠ σ$_o$ and H$_o$: μ = μ$_o$ vs. H$_1$: μ ≠ μ$_o$ can be expressed as

$$F_c(\bar{x}, s) = \frac{\iint f(\bar{x}, s) d\bar{x} ds}{D}, \quad for\ (\bar{x}, s) \in K \tag{3.4.1}$$

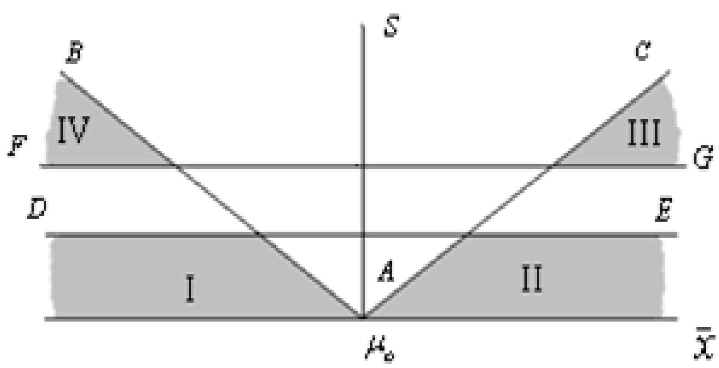

Figure 3.1 The joint domain of $(\bar{x}, s)$ for the conditional joint CDF of $\bar{X}$ and S following rejection of two tests. The lines AB and AC are determined by $|\bar{x} - \mu_o| = t_{1-\alpha/2}(s/\sqrt{n})$. The lines DE and FG are determined by $\frac{(n-1)s^2}{\sigma_o^2} = \chi^2_{n-1;\alpha/2}$ *and* $\frac{(n-1)s^2}{\sigma_o^2} = \chi^2_{n-1;1-\alpha/2}$

It should be noticed that the calculations of the double integral $\iint f(\bar{x}, s) d\bar{x} ds$ in equation (3.4.1) are quite different when the pair of observations $(\bar{x}, s)$ falls into different regions of I, II, III or IV shown in figure 3.4.1. From equation (3.4.1), it's quite obvious that the conditional joint CDF of $\bar{X}$ and S only depends on the two unknown parameters μ and σ. Although the expression of the conditional joint CDF of $\bar{X}$ and S is somewhat complicated, but with powerful computer programs, it's still possible to calculate the cumulated probability for any observed value of $(\bar{x}, s)$, if the two parameters μ and σ^2 are given.

Now we have all the information we need for finding a conditional joint confidence region of μ and σ, namely, the conditional joint CDF of $\bar{X}$ and S which only depends on the two unknown parameters but not on any other unknown nuisance parameters. For any observed value of $(\bar{x}, s)$, if a conditional joint confidence region of μ and σ exists, it could be found by using the above information. Next, we'll try to extend the general method of finding a confidence interval for an unknown parameter to the two parameters case.

Suppose $\Omega \in K$ is one relatively small region of $\bar{X}$ and S such that $P[(\bar{X}, S) \in \Omega] = 1 - \alpha$, if we regard $(\bar{x}, s)$ as random statistics and let (μ, σ) change jointly, then the statement $(\bar{x}, s) \in \Omega$ is equivalent to the statement $\alpha_1 < F_c(\bar{x}, s) < 1 - \alpha_2$ for some α_1 and α_2 such that $\alpha_1 + \alpha_2 = \alpha$. Therefore, if the inequality $\alpha_1 < F_c(\bar{x}, s) < 1 - \alpha_2$ has a solution for the region of (μ, σ), then this solution should construct a 100(1-α)% joint confidence region of μ and σ. In other words, if we plug any pair of (μ, σ) values into the above inequality and make the inequality a true statement for a pair of observed statistics $\bar{x}$ and *s*. then this pair of (μ, σ) value should be in a 100(1- α_1 - α_2)% conditional joint confidence region of μ and σ which is related to this observed pair of statistics $\bar{x}$ and *s*. In this way, we can extend the general method to the two parameters case, and obtain the following result.

[Result 3.4.1] Suppose a random sample X_1, X_2, …… X_n is taken from a normal distribution $N(\mu, \sigma^2)$, where μ and σ^2 are both unknown. Let $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ such that $0 < 1 - \alpha_1 - \alpha_2 < 1$. Let $\bar{x}$ and s be the observed values of $\bar{X}$ and S, and let $F_c(\bar{x}, s)$ denote the conditional joint CDF of $\bar{X}$ and S (see equation (3.4.1)). If the observed values of $\bar{x}$ and s result in rejecting the two null hypotheses H_o: $\mu = \mu_o$ and H_o: $\sigma = \sigma_o$ at level α, then the solution of

$$\alpha_1 < F_c(\bar{x}, s) < 1 - \alpha_2 \tag{3.4.2}$$

for all pairs of (μ, σ) construct a $100(1- \alpha_1 - \alpha_2)\%$ conditional joint confidence region of μ and σ.

The resulting joint confidence region of the solution of equation (3.4.2) is not easy to figure out, but we may think in the following way to get a rough picture. In equation (3.4.2), if we fix one of the two unknown parameters, say σ, at one value σ_1, then the problem becomes to finding a conditional confidence interval of one single unknown parameter. By the general method we discussed in Chapter 2, the solution should be a finite interval if the value of σ_1 is within the joint confidence region. If we change σ to another fixed value σ_2, then the solution of μ is another finite interval if σ_2 is still in the joint confidence region. Same situation happens when we fix μ at one value and try to find the solution of σ. So, we may conclude that the solution of equation (3.4.2) is just one connected region of μ and σ, and this region should contain the pair of observed value of $(\bar{x}, s)$.

In order to verify this, we may take an example using the same data set as in example 2.5.2. Suppose the process mean and variance are all unknown, and we are interested in the values of $\sigma = \sigma_o = 0.1$ and $\mu = \mu_o = 5.25$, so we take two sequential tests for testing H_o: σ = 0.1 vs. H_1: σ ≠ 0.1 and H_o: μ = 5.25 vs. H_1: μ ≠ 5.25. Since the observed statistics are s = 0.0649 and $\bar{x}$ = 5.211, the two null hypothesis are all rejected at level α = 0.05 by the above two observed statistics. And in this case, the pair of observed statistics $(\bar{x}, s)$ falls into the reject region I as shown in figure 3.4.1.

The total critical region is determined by the lines $s = \sqrt{\frac{\chi^2_{\alpha/2}}{n-1}} \cdot \sigma_o = 0.082$, $s = \sqrt{\frac{\chi^2_{1-\alpha/2}}{n-1}} \cdot \sigma_o = 0.118$ and $\bar{x} = \mu_o - t_{1-\alpha/2}(s/\sqrt{n}) = 5.25 - 0.26s$, $\bar{x} = \mu_o + t_{1-\alpha/2}(s/\sqrt{n})$ $= 5.25 + 0.26s$. Since in this case, the observed pair of statistics $(\bar{x}, s)$ falls in region I, therefore, the double integral in the numerator of equation (3.4.1) can be written as

$$\iint f(\bar{x}, s) d\bar{x} ds = \int_0^s \int_{-\infty}^{\bar{x}} f(x, y) dx dy$$

and the power D can be expressed as

D =

$$\int_0^{0.082} \int_{-\infty}^{5,25-0.26s} f(\bar{x}, s) d\bar{x} ds + \int_0^{0.082} \int_{5.25+0.26s}^{+\infty} f(\bar{x}, s) d\bar{x} ds + \int_{0.118}^{+\infty} \int_{5.25+0.26s}^{+\infty} f(\bar{x}, s) d\bar{x} ds + \int_{0.118}^{+\infty} \int_{-\infty}^{5.25-0.26s} f(\bar{x}, s) d\bar{x} ds$$

Thus, the conditional joint CDF of $\bar{X}$ and S following rejection of the two sequential tests can be expressed as

$$F_c(\bar{x}, s) = \frac{\iint f(\bar{x}, s) d\bar{x} ds}{D}$$

$$= \frac{\int_0^s \int_{-\infty}^{\bar{x}} f(x,y)\,dx\,dy}{\int_0^{0.082} \int_{-\infty}^{5,25-0.26s} f(\bar{x},s)\,d\bar{x}\,ds + \int_0^{0.082} \int_{5.25+0.26s}^{+\infty} f(\bar{x},s)\,d\bar{x}\,ds + \int_{0.118}^{+\infty} \int_{5.25+0.26s}^{+\infty} f(\bar{x},s)\,d\bar{x}\,ds + \int_{0.118}^{+\infty} \int_{-\infty}^{5.25-0.26s} f(\bar{x},s)\,d\bar{x}\,ds}$$

If we plug the unconditional joint pdf $f(\bar{x},s)$ of $\bar{X}$ and S into the above CDF and simplify, we obtain the conditional CDF of $\bar{X}$ and S as

$F_c(\bar{x},s) =$

$$\frac{\int_0^s \int_{-\infty}^{\bar{x}} \frac{y^{58}}{\sigma^{60}} \cdot e^{-\frac{60(x-\mu)^2+59y^2}{2\sigma^2}}\,dx\,dy}{\int_0^{0.082} \int_{-\infty}^{5,25-0.26s} \frac{s^{58}}{\sigma^{60}} \cdot e^{-\frac{60(\bar{x}-\mu)^2+59s^2}{2\sigma^2}}\,d\bar{x}\,ds + \int_0^{0.082} \int_{5.25+0.26s}^{+\infty} \varphi\,d\bar{x}\,ds + \int_{0.118}^{+\infty} \int_{5.25+0.26s}^{+\infty} \varphi\,d\bar{x}\,ds + \int_{0.118}^{+\infty} \int_{-\infty}^{5.25-0.26s} \varphi\,d\bar{x}\,ds}$$

where $\varphi = \frac{s^{58}}{\sigma^{60}} \cdot e^{-\frac{60(\bar{x}-\mu)^2+59s^2}{2\sigma^2}}$ is used only for simplification of the expression.

For the observed pair of statistics $(\bar{x},s)$ = (5.211, 0.0649), the inequality for a 95% conditional joint confidence region of μ and σ (equation 3.4.2) can be written as

0.025 ≤

$$\frac{\int_{0}^{0.0649}\int_{-\infty}^{5.211}\frac{s^{58}}{\sigma^{60}}\cdot e^{-\frac{60(\bar{x}-\mu)^2+59s^2}{2\sigma^2}}d\bar{x}ds}{\int_{0}^{0.082}\int_{-\infty}^{5,25-0.26s}\frac{s^{58}}{\sigma^{60}}\cdot e^{-\frac{60(\bar{x}-\mu)^2+59s^2}{2\sigma^2}}d\bar{x}ds+\int_{0}^{0.082}\int_{5.25+0.26s}^{+\infty}\varphi\, d\bar{x}ds+\int_{0.118}^{+\infty}\int_{5.25+0.26s}^{+\infty}\varphi\, d\bar{x}ds+\int_{0.118}^{+\infty}\int_{-\infty}^{5.25-0.26s}\varphi\, d\bar{x}ds}$$

$$\leq 0.975$$

Now if we fix the value of σ at 0.065, then the above inequality only contains one unknown parameter μ, this situation is similar to the one for finding the conditional confidence interval of a single parameter. Using powerful computer program, we can obtain the solution of μ as an interval. If we change the value of σ to another number 0.06, we can obtain another solution of interval if σ = 0.06 is still within the joint confidence region of μ and σ.

After we obtained the conditional joint confidence region of μ and σ, the conditional confidence interval of C_{pk} following rejection of the two tests for testing H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$ and H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$ can also be determined, but the computation is still very complicated, we need to use powerful computer programs to calculate it.

3.4.2 Testing for One of the Two Parameters

In some situations, we may have uncertain prior information on one of the two unknown parameters. If this is the case, then we can only construct one preliminary test. First, consider the case that we have some prior information about the process mean μ, and we test the hypothesis H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$. If the null hypothesis is not rejected, then we regard $\mu = \mu_o$ as known, no conditional confidence interval of C_{pk} is needed (the conditional confidence interval of C_{pk} in this case follows not rejecting a preliminary test, which is not in our topic). If the null hypothesis H_o: μ = μ_o is rejected, then we need to use $\bar{x}$ and s as two estimates of μ and σ to give a point estimate of C_{pk}. The conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: μ = μ_o can be obtained by using a similar procedure discussed in section 3.4.1.

The conditional joint pdf of $\bar{X}$ and S following rejection of the null hypothesis H_o: μ = μ_o can be expressed as

$$f_c(\bar{x}, s) = \begin{cases} f(\bar{x}, s)/D, & if\ (\bar{x}, s) \in K \\ 0, & otherwise \end{cases}$$

where $f(\bar{x}, s)$ is the unconditional joint pdf of $\bar{X}$ and S; K is the critical region of the test which is determined by $|\bar{x} - \mu_o| > t_{1-\alpha/2}(s/\sqrt{n})$, i.e., the regions I and II shown in figure 3.2; D is the total unconditional probability of $(\bar{x}, s)$ falling into the above critical region, which is determined by the following double integral.

$$D = \iint_K f(\bar{x}, s) d\bar{x} ds$$

In this situation, D is also the power of the test for testing H_o: μ = μ_o vs. H_1: μ ≠ μ_o , which can be calculated by using the non-central t-distribution, that is

$$D = P(|\bar{X} - \mu_o| > t_{1-\alpha/2}(S/\sqrt{n}) | \mu)$$

$$= P\left(\left| \frac{\bar{X} - \mu + \mu - \mu_o}{S/\sqrt{n}} \right| > t_{1-\alpha/2} \middle| \mu \right)$$

$$= 1 - H(t_{1-\alpha/2}) + H(-t_{1-\alpha/2})$$

where $H(\cdot)$ is the CDF of the non-central t-distribution with (n-1) degrees of freedom and with non-centrality parameter $\delta = \frac{\mu - \mu_0}{\sigma / \sqrt{n}}$. It's quite obvious that D involves the two unknown parameters μ and σ.

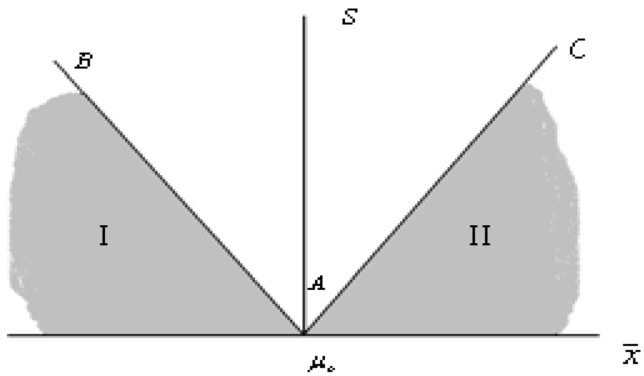

Figure 3.2 The joint domain of $(\bar{x}, s)$ for the conditional joint CDF of $\bar{X}$ and S following rejection of one test for the mean H_o: $\mu = \mu_o$ vs. H_1: $\mu \neq \mu_o$. The lines AB and AC are determined by $|\bar{x} - \mu_o| = t_{1-\alpha/2}(s/\sqrt{n})$.

The conditional joint CDF of $\bar{X}$ and S following rejection of the null hypothesis H_o: $\mu = \mu_o$ can be expressed as

$$F_c(\bar{x}, s) = \frac{\iint f(\bar{x}, s) d\bar{x} ds}{D}, \quad for\ (\bar{x}, s) \in K \tag{3.4.2}$$

The calculation of the double integral $\iint f(\bar{x}, s) d\bar{x} ds$ in equation (3.4.2) is still quite different when the pair of observation $(\bar{x}, s)$ falls into different regions of I and II shown in figure 3.4.2. This conditional joint CDF of $\bar{X}$ and S depends only on the two unknown parameters μ and σ but not on any other nuisance parameters, so we can follow the same procedure discussed in section 3.4.1 to find a conditional joint confidence region of μ and σ following rejection of the null hypothesis H_o: $\mu = \mu_o$. After we obtained the joint confidence region of μ and σ, we can use it to obtain a conditional confidence interval of C_{pk}.

In case we only have uncertain prior information about the process variance σ^2, then we need to test the hypothesis H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$. If the null hypothesis is not rejected, then we regard $\sigma = \sigma_o$ as known. The confidence interval of C_{pk} in this case will not be discussed at this time, since there is no test hypothesis has been rejected. Thus, no conditional confidence interval of C_{pk} is needed. If the null hypothesis H_o: $\sigma = \sigma_o$ is rejected, then we need to use $\bar{x}$ and s as two estimates of μ and σ to give a point estimate of C_{pk}. The conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ can be constructed similarly to the previous case, except the rejection region is different.

The conditional joint pdf of $\bar{X}$ and S following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ can be expressed as

$$f_c(\bar{x}, s) = \begin{cases} f(\bar{x}, s)/D, & if \ (\bar{x}, s) \in K \\ \\ 0, & otherwise \end{cases}$$

where $f(\bar{x}, s)$ is the unconditional joint pdf of $\bar{X}$ and S. K is the critical region of the test which is determined by $\frac{(n-1)s^2}{\sigma_o^2} < \chi^2_{n-1;\alpha/2}$ and $\frac{(n-1)s^2}{\sigma_o^2} > \chi^2_{n-1;1-\alpha/2}$, that is, the regions I and II shown in figure 3.3. D is the total unconditional probability of $(\bar{x}, s)$ falling into the above critical region, which is also determined by the double integral.

$$D = \iint_K f(\bar{x}, s) d\bar{x} ds$$

The conditional joint CDF of $\overline{X}$ and S following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ can be expressed as

$$F_c(\bar{x}, s) = \frac{\iint f(\bar{x}, s) d\bar{x} ds}{D}, \quad for\ (\bar{x}, s) \in K \tag{3.4.3}$$

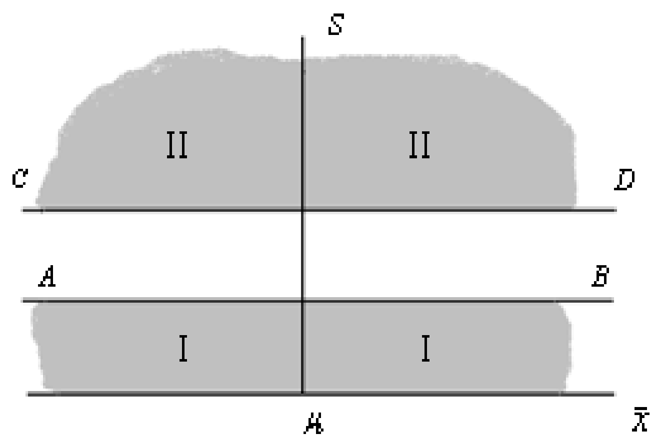

Figure 3.3 The joint domain of $(\bar{x}, s)$ for the conditional joint CDF of $\overline{X}$ and S following rejection of one test for σ H_o: $\sigma = \sigma_o$ vs. H_1: $\sigma \neq \sigma_o$.
The lines AB and CD are determined by $\frac{(n-1)s^2}{\sigma_o^2} = \chi^2_{n-1;\alpha/2}$ and $\frac{(n-1)s^2}{\sigma_o^2} = \chi^2_{n-1;1-\alpha/2}$

The same situation as in the previous two cases, the calculations of the double integral $\iint f(\bar{x}, s) d\bar{x} ds$ in equation (3.4.3) are different when the pair of observations $(\bar{x}, s)$ falls into different regions of I and II shown in figure 3.4.3. As we can check, this conditional joint CDF of $\overline{X}$ and S only depends on the two unknown parameters μ and σ. Therefore, we can use result 3.4.1 to find a conditional joint confidence region of μ and σ following rejection of the null hypothesis H_o: $\sigma = \sigma_o$. Once the conditional joint confidence region of μ and σ is determined, a conditional confidence interval of C_{pk} following rejection of the null hypothesis H_o: $\sigma = \sigma_o$ can also be determined.

APPENDIX A

IMSL PROGRAM CODE

A.1 Main Program Code for the Conditional Upper Confidence Limit of σ^2.

```
ROGRAM CCI_UpperLimit
!
!    Purpose:
!          To calculate the conditional upper confidence limits of the variance
!     following rejecting the null hypothesis that the variance is greater than
!     or equal to a certain number
!
!          This program uses the IMSL library subroutine ZREAL (Find the
!     real zeros of a real function using Müller's method) together with the
!     IMSL function CHIDF(CHSQ, DF)
!
!     Record of revisions:
!     Date              Programmer          Description of change
!     ====              ==========          ================
!     02/25/2009        Jianchun Zhang      Original code
!
!
USE MSIMSL      ! Invoke the IMSL library
!
IMPLICIT NONE     ! All the variables used in the program should be defined
!
!    Declare variables
!
    INTEGER        ITMAX, NROOT
    REAL           EPS, ERRABS, ERRREL, ETA
    PARAMETER  (NROOT=1)
!
```

```
     INTEGER    INFO(NROOT)
     REAL           F, X(NROOT), XGUESS(NROOT)
!
!    Declare the input function for finding the upper limit
!
     EXTERNAL   F
!
!    Set values of initial guess:      XGUESS = 0.01
!
     DATA XGUESS/0.01/
!
!    Stop criteria
!
     EPS    = 1.0E-5
     ERRABS = 1.0E-5
     ERRREL = 1.0E-5
     ETA    = 1.0E-2
!
     ITMAX  = 100
!
!    Calculate the upper limit and output the result
!
     CALL ZREAL (F, ERRABS, ERRREL, EPS, ETA, NROOT, ITMAX,
XGUESS , X, INFO)
     CALL WRRRN ('The upper limit is', 1, NROOT, X, 1, 0)
!
     END PROGRAM CCI_UpperLimit   ! Main program end here
!
!    Input the external function F    (This function F comes from the equation
```

```
!     of the upper limit. Here X represents the upper limit)
!
      REAL  FUNCTION F (X)
      REAL                      X
!
      F = CHIDF(47.0*0.0828**2/X,47.0)/CHIDF(32.27*0.1253**2/X,47.0)-0.025
      RETURN
      END
```

A.2 Some Other Functions

```
F = ANORDF(sqrt(60.0)*(5.211-X)/0.06)/(1-ANORDF(1.96-sqrt(60.0)*(X-
        5.25)/0.06)+ANORDF(-1.96-sqrt(60.0)*(X-5.25)/0.06))-0.025

F = TDF(sqrt(60.0)*(5.211-X)/0.0649,59.0)/(1-TNDF(2.0,59,sqrt(60.0)*(X-
        5.25)/0.0649)+TNDF(-2.0,59,sqrt(60.0)*(X-5.25)/0.0649))-0.025
```

APPENDIX B

CONDITIONAL CONFIDENCE INTERVALS OF THE MEAN μ FOLLOWING REJECTION OF A ONE-SIDED TEST

If a process has a known variance σ^2 and unknown mean μ, then the conditional confidence interval of the process capability C_{pk} only depends on the corresponding conditional confidence interval of the process mean μ.

In Chapter 3, we investigated how to find a conditional confidence interval of the process mean μ following rejection of a two-sided preliminary test. Although most of the tests for testing the process mean in process capability analysis are two-sided tests, but in some special cases, we still need a one-sided test for testing the process mean μ. In this appendix, we'll briefly state out the results of the conditional confidence interval of the process mean μ following rejection of a one-sided test.

Meeks & D'Agostino (1983) investigated the conditional confidence interval of the normal mean μ following rejection of a one-sided test for testing H_o: $\mu \le \mu_o$ vs. H_1: $\mu > \mu_o$. Instead of solving for the conditional confidence limits directly, they tried to find out the relationship between the conditional confidence limits and the unconditional confidence limits, and they provided two formulas for the differences of the conditional confidence limits (upper and lower) and the corresponding unconditional confidence limits. Now, we'll state the result in a different way by solving for the conditional confidence limits directly.

[Result B.1] Suppose the random sample X_1, X_2, …… X_n is taken from a normal distribution $N(\mu, \sigma^2)$, where μ is unknown and σ^2 is known. Let $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ with $\alpha_1 + \alpha_2 < 1$, and $\bar{x}$ be an observed value of $\bar{X}$, let $\Phi(\cdot)$ denote the CDF of standard normal distribution. If the observed value $\bar{x}$ results in rejecting the null hypothesis H_o: $\mu \le \mu_o$ at level α, then the solutions of

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_u^c)/\sigma\right\}-\Phi\left\{z_{1-\alpha}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}} = \alpha_1 \tag{B.1}$$

and

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_l^c)/\sigma\right\}-\Phi\left\{z_{1-\alpha}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}}{1-\Phi\left\{z_{1-\alpha}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}} = 1 - \alpha_2 \qquad \text{(B.2)}$$

construct a 100(1- α_1 - α_2)% conditional confidence interval (μ_l^c, μ_u^c) of μ.

Use the example they provided in their paper, If μ_o = 10, σ = 5, n = 25, $\alpha = 0.05$, and $\bar{x} = 14.245$. Since we have the test statistic

$$z = \frac{\sqrt{n}(\bar{x}-\mu_o)}{\sigma} = \frac{\sqrt{25}(14.245-10)}{5} = 4.245 > z_{1-\alpha}(=1.645)$$

so we reject the null hypothesis at level $\alpha = 0.05$. According to result B.1, a 95% conditional upper confidence limit of μ can be obtained by solving the equation (B.1), and a 95% conditional lower confidence limit of μ can be obtained by solving the equation (B.2). Again, using the IMSL numerical library, we can find these two confidence limits directly. After we run the IMSL FORTRAN program, we get the solution for the equation (B.1) as μ_u^c =15.89, and the solution for the equation (B.2) as μ_l^c= 12.50. Therefore, a 90% conditional confidence interval of μ is determined by

$$(\mu_l^c,\ \mu_u^c) = (12.50,\ 15.89)$$

If the test hypothesis for the mean μ happened to be another type of one-sided test, i.e., H_o: μ ≥ μ_o vs. H_1: μ < μ_o, following the procedure provided in Chapter 3 (section 3.3), we can derive the following result

[Result B.2] Suppose the random sample X_1, X_2, …… X_n is taken from a normal distribution $N(\mu, \sigma^2)$, where μ is unknown and σ^2 is known. Let $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ with $\alpha_1 + \alpha_2 < 1$, and $\bar{x}$ be an observed value of $\bar{X}$, let $\Phi(\cdot)$ denote the CDF of standard normal distribution. If the observed value $\bar{x}$ results in rejecting the null hypothesis H_o: $\mu \geq \mu_o$ at level α, then the solutions of

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_u^c)/\sigma\right\}}{\Phi\left\{-z_{1-\alpha}-\sqrt{n}(\mu_u^c-\mu_o)/\sigma\right\}} = \alpha_1 \tag{B.3}$$

and

$$\frac{\Phi\left\{\sqrt{n}(\bar{x}-\mu_l^c)/\sigma\right\}}{\Phi\left\{-z_{1-\alpha}-\sqrt{n}(\mu_l^c-\mu_o)/\sigma\right\}} = 1 - \alpha_2 \tag{B.4}$$

construct a $100(1-\alpha_1-\alpha_2)\%$ conditional confidence interval (μ_l^c, μ_u^c) of μ.

The above two formulas (B.3) and (B.4) can also be obtained from equations (3.3.3) and (3.3.4) by letting $z_{1-\alpha/2} \to \infty$ in the second term in the denominator. In the situation of the two-sided test, if we let the right side critical value goes to $+\infty$, then the test becomes the above one-sided test.

In the last example, if the observed value of $\bar{x}$ changes to $\bar{x} = 5.755$, and all the other values keep unchanged, then for the hypothesis H_o: $\mu \geq \mu_o$ vs. H_1: $\mu < \mu_o$, the test statistic becomes

$$z = \frac{\sqrt{n}(\bar{x}-\mu_o)}{\sigma} = \frac{\sqrt{25}(5.755-10)}{5} = -4.245 < -z_{1-\alpha} (= -1.645)$$

So we still reject the null hypothesis at level $\alpha = 0.05$. According to result B.2, a 95% conditional upper confidence limit of μ can be obtained by solving equation (B.3), and a 95% conditional lower confidence limit of μ can be obtained by solving equation (B.4). Using the IMSL numerical library, we can find these two conditional confidence limits as the following. The solution for the equation (B.3) is μ_u^c = 7.504, and the solution for the equation (B.4) is μ_l^c = 4.110. Therefore, a 90% conditional confidence interval of μ following rejection of the null hypothesis H_o: $\mu \geq \mu_o$ is determined by

$$(\mu_l^c,\ \mu_u^c) = (4.110,\ 7.504)$$

APPENDIX C

CONDITIONAL CONFIDENCE INTERVALS OF σ^2 FOLLOWING REJECTION OF A TWO-SIDED TEST

Suppose a random sample $X_1, X_2, \ldots, X_n$ is taken from $N(\mu, \sigma^2)$, $\overline{X}$ and S are the sample mean and sample standard deviation, let $\bar{x}$ and s are the observed values of $\overline{X}$ and S. A level α test for testing $H_o: \sigma = \sigma_o$ vs. $H_o: \sigma \neq \sigma_o$ has the critical region

$$K = \{ S^2 : (n-1) S^2/\sigma_o^2 < \chi^2_{n-1;\alpha/2} \text{ or } (n-1) S^2/\sigma_o^2 > \chi^2_{n-1;1-\alpha/2} \}$$

The null hypothesis is rejected if $S^2 \in K$, and a conditional confidence interval of σ^2 is computed only if the null hypothesis has been rejected. The conditional pdf of S^2 can be expressed in the following way

$$f_c(s^2) = \begin{cases} f(s^2)/D, & if \ \frac{(n-1)s^2}{\sigma_o^2} < \chi^2_{n-1;\alpha/2} \ \ or \ \ \frac{(n-1)s^2}{\sigma_o^2} > \chi^2_{n-1;1-\alpha/2} \\ 0, & otherwise \end{cases}$$

where $f(s^2)$ is the unconditional pdf which is determined by $S^2 \sim \frac{\sigma^2}{n-1}\chi^2_{n-1}$ and D is the power of the test given by

$$D = P((n-1)\frac{s^2}{\sigma_o^2} < \chi^2_{n-1;\alpha/2} \ \ or \ \ (n-1)\frac{s^2}{\sigma_o^2} > \chi^2_{n-1;1-\alpha/2} | \sigma)$$

$$= 1 - P((n-1)\frac{s^2}{\sigma^2} < \frac{\sigma_o^2}{\sigma^2} \cdot \chi^2_{n-1;1-\alpha/2}) + P((n-1)\frac{s^2}{\sigma^2} < \frac{\sigma_o^2}{\sigma^2} \cdot \chi^2_{n-1;\alpha/2})$$

$$= 1 - H(\psi \cdot \chi^2_{n-1;1-\alpha/2}) + H(\psi \cdot \chi^2_{n-1;\alpha/2})$$

where $\psi = \frac{\sigma_o^2}{\sigma^2}$ and H(·) is the CDF of chi-square distribution with (n-1) degrees of freedom. Under H_o, D = α . When $\sigma \to 0$ or $\sigma \to \infty$, the power D approaches 1.

The unconditional CDF of S^2 is given by

$$F(s^2) = P(S^2 \le s^2)$$

$$= P[\frac{(n-1)S^2}{\sigma^2} \le \frac{n-1}{\sigma^2} \cdot s^2]$$

$$= H(\frac{n-1}{\sigma^2} \cdot s^2), \qquad s^2 > 0$$

The conditional CDF of S^2 following rejecting the null hypothesis H_o: $\sigma = \sigma_o$ is given by

$$F_c(s^2) = \begin{cases} \dfrac{F(s^2)}{D}, & if \quad s^2 < \dfrac{\sigma_o^2}{n-1}\chi^2_{n-1;\alpha/2} \\ \\ \dfrac{F(s^2) - (1-D)}{D}, & if \quad s^2 > \dfrac{\sigma_o^2}{n-1}\chi^2_{n-1;1-\alpha/2} \end{cases}$$

$$= \begin{cases} \dfrac{H(\frac{n-1}{\sigma^2} \cdot s^2)}{1 - H(\psi \cdot \chi^2_{n-1;1-\alpha/2}) + H(\psi \cdot \chi^2_{n-1;\alpha/2})}, & if \quad s^2 < \dfrac{\sigma_o^2}{n-1}\chi^2_{n-1;\alpha/2} \\ \\ \dfrac{H(\frac{n-1}{\sigma^2} \cdot s^2) - H(\psi \cdot \chi^2_{n-1;1-\alpha/2}) + H(\psi \cdot \chi^2_{n-1;\alpha/2})}{1 - H(\psi \cdot \chi^2_{n-1;1-\alpha/2}) + H(\psi \cdot \chi^2_{n-1;\alpha/2})}, & if \quad s^2 > \dfrac{\sigma_o^2}{n-1}\chi^2_{n-1;1-\alpha/2} \end{cases}$$

The above formula tells us that if the observed value s^2 of the statistic S^2 is small, i.e., if $s^2 < \frac{\sigma_o^2}{n-1}\chi^2_{n-1;\alpha/2}$, then the conditional CDF of S^2 can be expressed as

$$F_c(s^2) = \frac{H(\frac{n-1}{\sigma^2}\cdot s^2)}{1 - H(\psi \cdot \chi^2_{n-1;1-\alpha/2}) + H(\psi \cdot \chi^2_{n-1;\alpha/2})}$$

(C.1)

On the other hand, if the value of s^2 is large, i.e., if $s^2 > \frac{\sigma_o^2}{n-1}\chi^2_{n-1;1-\alpha/2}$, then the conditional CDF of S^2 is

$$F_c(s^2) = \frac{H(\frac{n-1}{\sigma^2}\cdot s^2) - H(\psi \cdot \chi^2_{n-1;1-\alpha/2}) + H(\psi \cdot \chi^2_{n-1;\alpha/2})}{1 - H(\psi \cdot \chi^2_{n-1;1-\alpha/2}) + H(\psi \cdot \chi^2_{n-1;\alpha/2})}$$

(C.2)

The above two expressions of the conditional CDF of S^2 only contains only one unknown parameter σ, so we can apply the general method mentioned in Chapter 2 to find a conditional confidence interval of σ^2. And in this case, we still can verify numerically that the two functions $h_1(s^2)$ and $h_2(s^2)$ which are constructed by the following two equations

$$F_c(h_1(s^2); s^2) = \alpha_1$$

and

$$F_c(h_2(s^2); s^2) = 1 - \alpha_2$$

are increasing functions. So apply the general method, we can obtain a conditional confidence interval of σ^2 following rejection of the null hypothesis H_o: $\sigma = \sigma_o$. We summarize the above derivations in the following result.

[Result C.1] Suppose a random sample X_1, X_2, …… X_n is taken from a normal distribution $N(\mu, \sigma^2)$. Let $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ with $\alpha_1 + \alpha_2 < 1$, and s^2 be an observed value of S^2. Let $H(\cdot)$ denote the CDF of chi-square distribution with (n-1) degrees of freedom. If the observed value s^2 results in rejecting the null hypothesis H_o: $\sigma = \sigma_o$ at level α by the condition $s^2 < \frac{\sigma_o^2}{n-1}\chi^2_{n-1;\alpha/2}$, then the solutions of

$$\frac{H(\frac{n-1}{\sigma_U^2}\cdot s^2)}{1 - H(\frac{\sigma_o^2}{\sigma_U^2}\cdot \chi^2_{n-1;1-\alpha/2}) + H(\frac{\sigma_o^2}{\sigma_U^2}\cdot \chi^2_{n-1;\alpha/2})} = \alpha_1 \qquad \text{(C.3)}$$

and

$$\frac{H(\frac{n-1}{\sigma_L^2}\cdot s^2)}{1 - H(\frac{\sigma_o^2}{\sigma_L^2}\cdot \chi^2_{n-1;1-\alpha/2}) + H(\frac{\sigma_o^2}{\sigma_L^2}\cdot \chi^2_{n-1;\alpha/2})} = 1 - \alpha_2 \qquad \text{(C.4)}$$

construct a $100(1-\alpha_1-\alpha_2)\%$ conditional confidence interval (σ_L^2, σ_U^2) of σ^2.

If the observed value s^2 results in rejecting the null hypothesis H_o: $\sigma = \sigma_o$ at level α by the condition $s^2 > \frac{\sigma_o^2}{n-1}\chi^2_{n-1;1-\alpha/2}$, then the solutions of

$$\frac{H(\frac{n-1}{\sigma_U^2}\cdot s^2) - H(\frac{\sigma_o^2}{\sigma_U^2}\cdot \chi^2_{n-1;1-\alpha/2}) + H(\frac{\sigma_o^2}{\sigma_U^2}\cdot \chi^2_{n-1;\alpha/2})}{1 - H(\frac{\sigma_o^2}{\sigma_U^2}\cdot \chi^2_{n-1;1-\alpha/2}) + H(\frac{\sigma_o^2}{\sigma_U^2}\cdot \chi^2_{n-1;\alpha/2})} = \alpha_1 \tag{C.5}$$

and

$$\frac{H(\frac{n-1}{\sigma_L^2}\cdot s^2) - H(\frac{\sigma_o^2}{\sigma_L^2}\cdot \chi^2_{n-1;1-\alpha/2}) + H(\frac{\sigma_o^2}{\sigma_L^2}\cdot \chi^2_{n-1;\alpha/2})}{1 - H(\frac{\sigma_o^2}{\sigma_L^2}\cdot \chi^2_{n-1;1-\alpha/2}) + H(\frac{\sigma_o^2}{\sigma_L^2}\cdot \chi^2_{n-1;\alpha/2})} = 1 - \alpha_2 \tag{C.6}$$

construct a 100(1- α_1 - α_2)% conditional confidence interval (σ_L^2, σ_U^2) of σ^2.

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BIOGRAPHICAL STATEMENT

Jianchun Zhang received his Bachelor degree in Mechanical Engineering from TongJi university, China, in 1989. After his graduation from TongJi university, he was employed as a (senior) engineer by the construction company in China - Shanghai Construction Group. He worked there for more than 10 years until he went into USA.

In the year 2005, he joined the University of Texas at Arlington (UTA) to continue his graduate study in statistics. He received his Master degree in statistics in 2006, and completed his Ph.D degree study in statistics in 2010. His main work as a Graduate Teaching Assistant in the Math Department of UTA was teaching undergraduate courses, including College Algebra and Pre-Calculus.

His research interests in statistics are statistical inferences, biostatistics, experimental design, and SAS programming.