NEW MATHEMATICAL PROPERTIES
OF THE LEAST SQUARE VALUE

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ABSTRACT

The Least Square Values for cooperative TU games (briefly, LS-values) represent a family of solutions of the following family of optimization problems associated with a cooperative TU game: minimize the sum of weighted squares of deviations of the excesses from their average, on the preimputation set; the weights were positive numbers associated to the coalitions, depending only on their size. For each system of weights a LS-value is obtained by the Lagrange multipliers method; the Shapley value belongs to the family for specified weights. L. Ruiz, F. Valenciano and J. Zarzuelo who introduced this class of values (see [9]), gave two axiomatic characterizations of the LS-values: one is based upon a set of axioms including linearity, efficiency, symmetry, inessential games additivity and weighted average marginal contribution monotonicity; the other is based upon a reduced game property relative to a Davis/Maschler type of reduced game (see [1]), and standardness for two person games.

In the present paper, we start by showing that for whatever system of weights we can express the corresponding LS-value by means of average per capita formulas similar to those proved earlier by the author for the Shapley value (see [4]). These formulas provide an algorithm for computing in parallel the LS-values with one operation more than in the computation of the Shapley value and very little memory capacity. Theoretically, the same formulas show that for each system of weights the corresponding LS-value is the Shapley value of a game easily obtained from the given game. This fact suggests that the LS-values have more properties analogue to the Shapley value than those considered in [9]. Therefore, we introduce a potential function for the LS-values and determine a potential basis for $G^N$, the space of TU games with the set of players $N$, relative to the LS-values. The potential basis is allowing us to solve for LS-values what we call the inverse problem: given a vector $\phi \in R^n$ and a system of weights $\gamma$, find out an explicit formula giving all games $v \in G^N$ for which the LS-value corresponding to $\gamma$ is $\phi$. In particular, for $\phi=0$, we find the null space of the LS-value; the nullity equals $2^n - n - 1$. The inverse problem for the Shapley value, the weighted Shapley value and the Banzhaf value have been solved earlier by the author (see [2], [3], and [5]).

Further, we introduce a new concept of consistency. Usually, a group of players in a game $v \in G^N$, who agreed upon a division rule, either leave the game with their payoffs or have a contract for getting later those payoffs. The set $T \subset N$ of remaining players would like to play a game (the reduced game) in which the same division rule would offer the same payoffs as in the initial game. Then, the division rule is said consistent relative to the reduced game. Now, we allow to the set of remaining players to use a new division rule if this rule gives them in the reduced game the same payoffs as in the initial game. The pair of rules will be called consistent relative to the reduced game. To develop such a consistency scheme, we introduce a new value, associated with a system of weights, called the Extended Least Square Value, and define a new reduced game of Hart/Mas-Colell type (see [6] and [7]), and prove that the pair (LS, ELS) is consistent relative to this...
reduced game. Then, we give an axiomatic characterization of ELS-values based upon the reduced game axiom and a weighted standardness axiom.

1. The average per capita formula for the Least Square Values.

A cooperative TU game is a pair \( G = (N, \nu) \), where \( N \) is a finite set, the set of players, and \( \nu \) is a function from the power set of \( N \) to the reals satisfying \( \nu(\emptyset) = 0 \). As \( N \) will be fixed, we denote by \( G^N \) the set of all TU games with the set of players \( N \) and a game in \( G^N \) will be determined by its function \( \nu \), so that the notation \( \nu \in G^N \) will be used with no confusion possible. It is well known that \( G^N \) can be identified with \( \mathbb{R}^{2^n-1} \), hence \( \dim(G^N) = 2^n - 1 \), by defining \( (\nu_1 + \nu_2)(S) = \nu_1(S) + \nu_2(S), \forall S \subseteq N \), and \( (a\nu)(S) = a \nu(S), \forall S \subseteq N \), for all \( a, \nu_1, \nu_2 \in G^N \) and \( a \in \mathbb{R} \). Beside the standard basis of \( \mathbb{R}^{2^n-1} \), given by \( e_S \in G^N \), \( \forall S \in P(N) - \{\emptyset\} \), with \( e_S(S) = 1 \) and \( e_S(T) = 0 \), for all \( T \neq S \), another popular basis is the unanimity basis, introduced by L.S. Shapley in [11], in order to derive the Shapley value formula; the unanimity basis is given by \( \omega_S \in G^N, \forall S \in P(N) - \{\emptyset\} \), with \( \omega_S(T) = 1 \), for all \( T \supseteq S \), and \( \omega_S(T) = 0 \), otherwise. In order to solve what will be called inverse problems, other bases for \( G^N \) have been introduced in [2], [3], and [5], as tools in approaching the inverse problems for the Shapley value, the weighted Shapley value and the Banzhaf value, respectively. In this paper, we shall introduce another family of bases for \( G^N \), as a tool in solving the inverse problem for the Least Square Values.

Ruiz, Valenciano and Zarzuelo in [9] have introduced the Least Square Values of cooperative TU games (briefly, LS-values), as follows. For any game \( \nu \in G^N \) and any \( x \in \mathbb{R}^n \), denote as usual \( e(S, x) = \nu(S) - x(S), \forall S \subseteq N \); assume that \( x \) is an efficient payoff, that is we have \( e(N, x) = 0 \). Then, denote

\[
(1.1) \quad \bar{e}(\nu) = (2^n - 1)^{-1} \sum_{S \subseteq N} e(S, x) = (2^n - 1)^{-1} \left( \sum_{S \subseteq N} \nu(S) - 2^{n-1} \nu(N) \right).
\]

which does not depend on \( x \); let \( m: P(N) - \{\emptyset\} \rightarrow \mathbb{R} \) be a function which is positive and symmetric, that is for all coalitions of size \( s \) the value of \( m \) is the same \( m(s) \).

Consider the optimization problem (P):

\[
(1.2) \quad \text{minimize } \sum_{S \subseteq N} m(s) \left( e(S, x) - \bar{e}(\nu) \right)^2 \quad \text{s.t. } x(N) = \nu(N).
\]

In [9], it has been proved that: if \( m \) is positive, then for any \( \nu \in G^N \) the problem (P) has a unique solution, namely

\[
(1.3) \quad LS_i(m, \nu) = n^{-1} \nu(N) + (an)^{-1} \left[ na_i(m, \nu) - \sum_{j \not\in N} a_j(m, \nu) \right], \forall i \in N,
\]

where

\[
(1.4) \quad \alpha = \sum_{s=1}^{n-1} m(s) \binom{n-2}{s-1}, \quad a_i(m, \nu) = \sum_{S : i \in S \subseteq N} m(s) \nu(S), \forall i \in N.
\]

Therefore, \( LS(m, \nu) \) has been called a Least Square Value. Many interesting properties of LS-values have been proved in [9], and among them it has been shown that in the
family of LS-values, for the weight function \( m(s) = (n - 1)^{-1} \left( \frac{n^2 - 1}{s - 1} \right)^{-1} \), \( s = 1, \ldots, n - 1 \), we get the Shapley value. This fact has been the starting point of our work, we have tried to see whether a LS-value could be represented by an average per capita formula similar to the Shapley value formula given by the author in [4]. Recall from [4] the notations

\[
(1.5) \quad v_s = \left( \frac{n}{s} \right)^{-1} \sum_{|S| = s} v(S), \quad v_s^i = \left( \frac{n-1}{s} \right)^{-1} \sum_{|S| = s, i \notin S} v(S), \forall i \in N,
\]

for \( s = 1, \ldots, n - 1 \). In words, \( v_s \) is the average worth of coalitions of size \( s \), while \( v_s^i \) is the average worth of coalitions of size \( s \) which do not contain the player \( i \). It has been shown in [4] that: for any \( v \in G^N \), the Shapley value \( Sh(N, v) \) can be represented by the formula

\[
(1.6) \quad Sh_i(N, v) = n^{-1}v(N) + \sum_{s=1}^{n-1} \frac{v_s - v_s^i}{s}, \forall i \in N,
\]

which has been called above the average per capita formula.

**Theorem 1:** For the Least Square Value \( LS(m, v) \) associated with a positive symmetric function \( m: P(N) - \{\phi, N\} \to R \), there exists a positive symmetric function \( \gamma: P(N) - \{\phi, N\} \to R \) with \( \sum_{s=1}^{n-1} \gamma(s) = n - 1 \), such that

\[
(1.7) \quad LS_i(\gamma, v) = n^{-1} \cdot v(N) + \sum_{s=1}^{n-1} \gamma(s) \frac{v_s - v_s^i}{s}, \forall i \in N,
\]

where \( LS(\gamma, v) \) denotes the number obtained when \( m \) has been expressed in terms of \( \gamma \) in \( LS(m, v) \).

**Proof:** A simple computation gives different expressions for the terms obtained after multiplying the bracket by the coefficient in (1.3). On the one hand we have for all \( i \in N \):

\[
(1.8) \quad \alpha^{-1}a_i(m, v) = \sum_{S: i \in S \subseteq N} \alpha^{-1}m(s)v(S) = \sum_{s=1}^{n-1} \alpha^{-1}m(s) \left( \sum_{|S| = s, i \notin S \subseteq N} v(S) \right);
\]

on the other hand, taking into account that a coalition \( S \) enters in \( s \) sums \( a_j(m, v) \), we can rewrite:

\[
(1.9) \quad (n\alpha)^{-1} \sum_{j \in N} a_j(m, v) = (n\alpha)^{-1} \sum_{j \in N} \left( \sum_{S: j \in S \subseteq N} m(s)v(S) \right) =
\]

\[
= n^{-1} \sum_{s=1}^{n-1} \left( \sum_{|S| = s, S \subseteq N} \alpha^{-1}m(s)v(S) \right) = \sum_{s=1}^{n-1} (n\alpha)^{-1} \sum_{|S| = s, S \subseteq N} m(s)v(S).
\]

The difference of the two terms (1.8) and (1.9) can be written as

\[
(1.10) \quad \sum_{s=1}^{n-1} (n\alpha)^{-1} (n - s)m(s) \left( \sum_{|S| = s, S \subseteq N} v(S) \right) - \sum_{s=1}^{n-1} \alpha^{-1}m(s) \left( \sum_{|S| = s, S \subseteq N, i \notin S} v(S) \right)
\]

or, by using (1.5), we get
\[
\sum_{s=1}^{n-1} \left( (n-1)^{-1} (n-s)m(s) \binom{n}{s} \right) v_s - \sum_{s=1}^{n-1} \frac{(n-1)^{-1} m(s) \left( \binom{n-1}{s} \right)}{s} v_s^i = \\
= \sum_{s=1}^{n-1} (n-1)m(s) \left( \binom{n-2}{s-1} \right) \frac{v_s - v_s^i}{s}.
\]

Here, it is enough to denote for \( s = 1, \ldots, n - 1 \):
\[
\gamma(s) = (n-1)m(s) \left( \binom{n-2}{s-1} \right), \quad \alpha = \sum_{t=1}^{n-1} m(t) \left( \binom{n-2}{t-1} \right),
\]
to notice that \( \gamma \) is a positive symmetric function satisfying \( \sum_{s=1}^{n-1} \gamma(s) = n - 1 \), and we got (1.7).

**Corollary 2:** The Least Square value \( LS(\gamma, v) \) associated with a positive symmetric function \( \gamma \) satisfying \( \sum_{s=1}^{n-1} \gamma(s) = n - 1 \) is the Shapley value, if and only if \( \gamma(s) = 1, \ s = 1, \ldots, n - 1 \).

**Proof:** The "if" part follows from (1.6) and (1.7). Conversely, the equality \( Sh(N, v) = LS(\gamma, v) = 0, \forall v \in G^N \), gives
\[
\sum_{s=1}^{n-1} \left( (\gamma(s) - 1) \frac{v_s - v_s^i}{s} \right) = 0, \forall i \in N,
\]
for all \( v \in G^N \). Let \( U \subset N \), \( U \neq \phi \), contains player \( i \), and let \( v \in G^N \) be the game \( v(U) = 1 \), and \( v(S) = 0, \forall S \neq U \). Then, \( v_s = 0 \) and \( v_s^i = 0, \forall i \in N \), whenever \( s \neq |U| \), we have \( v_u = \binom{n}{u}^{-1}, \ v_u^i = 0, \forall i \in N \), and we get \( \gamma(u) = 1 \). As there are coalitions of whatever size containing player \( i \), we get \( \gamma(s) = 1, \ s = 1, \ldots, n - 1 \).

As shown in [4], the terms in formula (1.7) can be computed in parallel and very little computer memory is needed to compute the n-vector \( (T_s^i) \), with \( T_s^i = \frac{v_s - v_s^i}{s} \), \( \forall i \in N \), for each \( s \), if the numbers \( v(S) \) with \( |S| = s \) are generated by the computer. The algorithm explained in [4] will work for a LS-value exactly as for the Shapley value except that each n-vector \( (T_s^i) \) must be multiplied by \( \gamma(s) \) before making the summation.

**Corollary 3:** The LS-value associated with a positive symmetric function \( \gamma : P(N) - \{ \phi \} \to R \) satisfying \( \sum_{s=1}^{n-1} \gamma(s) = n - 1 \) is the Shapley value of the game \( w \in G^N \) defined by:
\[
w(S) = \gamma(s)v(S), \forall S \subseteq N,
\]
where \( \gamma(n) = 1 \).
This follows from Theorem 1, noticing that (1.14) implies \( w_s = \gamma(s) v_s, \forall S \subseteq N \), and \( w^i_s = \gamma(s) v^i_s, \forall S \subseteq N, \forall i \in N \).

2. The potential.

S. Hart and A. Mas-Colell have introduced in [6] and [7] a potential function for the Shapley value; Corollary 3 in the previous section suggests that a LS-value has also a potential. We define the potential for a game \( v \in G^N \) and a positive symmetric function \( \gamma: P(N) - \{ \phi \} \rightarrow R \) satisfying \( \sum_{s=1}^{n-1} \gamma(s) = n - 1 \) and \( \gamma(n) = 1 \), by the recursive formulas

\[
Q(\phi, v) = 0, \quad Q(\{i\}, v) = \gamma(1) v(\{i\}), \forall i \in N,
\]

and for all coalitions \( S \) with \(|S| > 1\)

\[
Q(S, v) = s^{-1} \left[ \gamma(s)v(S) + \sum_{j \in S} Q(S - \{j\}, v) \right].
\]

where \( s = |S| \).

Note that the potential depends on the weight function \( \gamma \), but to keep the notation simple we did not mention it; of course, the LS-value depends also on the weight function.

Note also that by the choice of the potential function a lot of results will follow from the Hart/Mas-Colell results on the Shapley value. The result which justifies the name of potential is given by:

**Theorem 4:** Let \( \gamma: P(N) - \{ \phi \} \rightarrow R \) be a positive symmetric function satisfying

\[
\sum_{s=1}^{n-1} \gamma(s) = n - 1 \quad \text{and} \quad \gamma(n) = 1,
\]

and let \( v \in G^N \) be any game. Let \( LS(\gamma, v) \) be the LS-value associated with \( \gamma \) and \( Q \) the function defined by the formulas (2.1) and (2.2). Then, we have

\[
Q(N, v) - Q(N - \{i\}, v) = LS_i(\gamma, v), \forall i \in N.
\]

**Proof:** By definition and an easy induction, we have for the potential of the Shapley value for the game \( w \) connected to \( v \) by (1.14), denoted \( P(S, w), \forall S \subseteq N \):

\[
P(\phi, w) = 0 = Q(\phi, v),
\]

\[
P(\{i\}, w) = w(\{i\}) = \gamma(1) v(\{i\}) = Q(\{i\}, v), \forall i \in N,
\]

\[
P(S, w) = s^{-1} \left[ w(S) + \sum_{j \in S} P(S - \{j\}, w) \right] =
\]

\[
= s^{-1} \left[ \gamma(s)v(S) + \sum_{j \in S} Q(S - \{j\}, v) \right] = Q(S, v), \forall S \subseteq N, |S| > 1.
\]

Hence for all coalitions \( S \subseteq N \) we have \( P(S, w) = Q(S, v) \). Then, from our Corollary 3 and the Hart/Mas-Colell Theorem on the potential of the Shapley value, we get
(2.5) \[ Q(N, v) - Q(N - \{i\}, v) = P(N, w) - P(N - \{i\}, w) = \]
\[ = Sh_i(N, w) = LS_i(\gamma, v), \forall i \in N. \]

Obviously, other results could follow from the Hart/Mas-Colell results, for example, it is easy to see that we have:

**Theorem 5:** If \( Q \) is the potential of the \( LS \)-value associated with the function \( \gamma \), then we have:

(2.6) \[ Q(T, v) = \sum_{S \subseteq T} \frac{(s-1)! (t-s)!}{t!} \gamma(s) v(S), \forall T \subseteq N. \]

**Proof:** We have from the proof of the previous theorem:

(2.7) \[ Q(T, v) = P(T, w) = \sum_{S \subseteq T} \frac{(s-1)! (t-s)!}{t!} w(S), \forall T \subseteq N, \]

where the last equality is given by the corresponding Hart/Mas-Colell result, and by using (1.14) the result (2.6) follows.

In connection with Theorem 4, note that while for whatever \( S \subseteq N \) we have \( P(S, w) - P(S - \{i\}, w) = Sh_i(S, w), \forall i \in N \), such a relation does not hold for the potential of the \( LS \)-value, because the last value depends on the weight function and the potential as well. To correct this fact, in order to get other results similar to the results for the Shapley value, we shall introduce in the fourth section what will be called an Extended Least Square value. Now, we continue this section with the determination of a potential basis for \( G^N \) relative to the \( LS \)-value.

Recall from [2], [3], [5], that for the Shapley value, the weighted Shapley value and the Banzhaf value, which have a potential, we have been able to determine a potential basis for \( G^N \) and by using this basis we have solved what we called the inverse problem for these values. Recall from [5], that a potential basis for \( G^N \) relative to a value \( \chi \), is a basis for \( G^N \) such that the coordinates of each \( v \in G^N \) relative to that basis are exactly the potentials of \( \chi \) for \( v \) and its subgames. Precisely, \( C = \{ c_S \in G^N : S \subseteq N, S \neq \phi \} \) is a potential basis for \( G^N \) relative to \( \chi \), if in the expansion \( v = \sum_{S \subseteq N} \alpha_S c_S \), we have \( \alpha_S = H(S, v), \forall S \subseteq N \), where \( H \) is the potential of \( \chi \) for \( v \) and its subgames. Recall also from [5], the result:

**Theorem 6:** If \( \chi \) is a value which has a potential \( H \), and there is a potential basis \( C \) for \( G^N \) relative to \( \chi \), then we must have for each coalition \( S \subseteq N \):

(2.8) \[ H(S, c_S) = 1, \quad H(T, c_S) = 0, \forall T \subseteq N, T \neq S. \]
Note that the definition (2.1) and (2.2) gives the potentials if the game $c_S \in G^N$ is known, but they give also the game $c_S$ for which the potentials of the value are known, when we write them as

$$c_S(\{i\}) = \left[\gamma(1)^{-1}\right] Q(\{i\}, c_S), \forall i \in N,$$

(2.9)  \[c_S(T) = \left[\gamma(t)^{-1}\sum_{j \in T} \left[Q(T, c_S) - Q(T - \{j\}, c_S)\right]\right], \forall T \subseteq N, |T| > 1.\]

We know the potentials of the Least Square value for the games $c_S$, given by Theorem 6, where $H$ is the potential $Q$, hence we can use (2.9).

**Theorem 7:** For each $S \subseteq N, S \neq \phi$, there is a unique game $c_S \in G^N$ such that $Q(S, c_S) = 1$ and $Q(T, c_S) = 0, \forall T \subseteq N, T \neq S$, where $Q$ is the potential of the LS-value associated with the weight function $\gamma$. Moreover, we have for each $S$

$$c_S(T) = \begin{cases} 
\left[\gamma(s)^{-1}\right] s & \text{if } T = S, \\
-\left[\gamma(s + 1)^{-1}\right] & \text{if } T = S \cup \{j\}, j \notin S, \\
0 & \text{otherwise.}
\end{cases}
$$

(2.10)

**Proof:** Fix $i \in N$ and consider $S = \{i\}$, then from (2.8) with $H = Q$ we get $c_{\{i\}}(\{i\}) = \left[\gamma(1)^{-1}\right] \text{ and } c_{\{i\}}(\{j\}) = 0, \forall j \in N, j \neq i$; then, if $T = \{i, j\}, j \neq i$, we get $c_{\{i\}}(\{i, j\}) = -\left[\gamma(2)^{-1}\right]$, otherwise $c_{\{i\}}(T) = 0$. We proved (2.10) for $S = \{i\}$. Consider any coalition $S$ with $|S| > 1$; obviously, from (2.8) and (2.9) we get $c_S(T) = 0$ whenever $|T| < |S|$ and $|T| = |S|$ but $T \neq S$, and $c_S(S) = \left[\gamma(s)^{-1}\right] s$; then, if $T = S \cup \{j\}, j \notin S$, we get $c_S(S \cup \{j\}) = -\left[\gamma(s + 1)^{-1}\right]$ otherwise $c_S(T) = 0$ if $|T| > |S|$. We proved (2.10) for any $S \subseteq N, |S| > 1$.

Now, it takes only a repetition of the proof given in [5] for the similar result, to prove:

**Theorem 8:** The set of games $C = \{c_S \in G^N : S \subseteq N, S \neq \phi\}$, given by (2.10) in Theorem 7, forms a basis for $G^N$, and this is a potential basis relative to the LS-value.

**Example 1:** If $|N| = 3$, then the games $c_S$ given by (2.10) are

$$c_1 = \left(\frac{1}{\gamma(1)}, 0, 0, -\frac{1}{\gamma(2)}, -\frac{1}{\gamma(2)}, 0, 0\right),$$

$$c_2 = \left(0, \frac{1}{\gamma(1)}, 0, -\frac{1}{\gamma(2)}, 0, -\frac{1}{\gamma(2)}, 0\right),$$

$$c_3 = \left(0, 0, \frac{1}{\gamma(1)}, 0, -\frac{1}{\gamma(2)}, -\frac{1}{\gamma(2)}, 0\right),$$

$$c_{12} = \left(0, 0, 0, \frac{2}{\gamma(2)}, 0, 0, -\frac{1}{\gamma(3)}\right),$$
where the components of each vector are corresponding to the coalitions taken in the order \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, and \(\gamma(3) = 1\).

Note that these games form a basis for \(\mathbb{R}^7\) because they are linearly independent. Note also that for \(\gamma(1) = \gamma(2) = 1\), this is the potential basis for the Shapley value (see [2]).

We can check that the above basis is a potential basis for \(G^N\) relative to the LS-value, by using the expansion \(v = \sum_{s \subseteq N} \alpha_S c_S\), to write the coordinates of an arbitrary \(v \in \mathbb{R}^7\) relative to this basis:

\[
\begin{align*}
v(1) &= [\gamma(1)]^{-1} \alpha_1, & v(2) &= [\gamma(1)]^{-1} \alpha_2, & v(3) &= [\gamma(1)]^{-1} \alpha_3, \\
v(1, 2) &= [\gamma(2)]^{-1}(2\alpha_{12} - \alpha_1 - \alpha_2), & v(1, 3) &= [\gamma(2)]^{-1}(2\alpha_{13} - \alpha_1 - \alpha_3), \\
v(2, 3) &= [\gamma(2)]^{-1}(2\alpha_{23} - \alpha_2 - \alpha_3), & v(1, 2, 3) &= [\gamma(3)]^{-1}(3\alpha_{123} - \alpha_{12} - \alpha_{13} - \alpha_{23}),
\end{align*}
\]

and using (2.1) and (2.2) to compute potentials. For example

\[
Q(\{1, 2\}, v) = \frac{1}{2} [\gamma(2)v(1, 2) + Q(\{1\}, v) + Q(\{2\}, v)] = \alpha_{12},
\]

3. The inverse problem.

Consider the following problem: let \(L \in \mathbb{R}^n\) be given: determine the set of all games \(v \in G^N\) for which \(LS(\gamma, v) = L\); this will be called the inverse problem for the LS-values. To prove the result needed in solving the inverse problem we must first give:

**Theorem 9:** If \(c_S \in G^N\), for all \(S \subseteq N\), \(S \neq \emptyset\), are the games forming the potential basis for \(G^N\) relative to the LS-value, then their LS-values are:

\[
LS(\gamma, c_S) = 0, \forall S \subseteq N, |S| \leq n - 2,
\]

(3.1)

\[
LS(\gamma, c_{N-\{j\}}) = -e_j, \forall j \in N, \quad LS(\gamma, c_N) = e,
\]

where \(e_j, \forall j \in N\), are the vectors of the standard basis for \(\mathbb{R}^n\), and \(e \in \mathbb{R}^n\) is the vector with all components equal to one.

**Proof:** To compute \(LS(\gamma, c_S)\) we can use Theorem 4, because the potentials of the games \(c_S\) satisfy (2.8). Thus, if \(|S| \leq n - 2\), then both \(Q(N, c_S) = 0\) and \(Q(N - \{i\}, c_S) = 0, \forall i \in N\), hence \(LS(\gamma, c_S) = 0\); if \(S = N - \{j\}\), then
\( Q(N, c_{N-\{j\}}) = 0 \) but \( Q(N - \{i\}, c_{N-\{j\}}) = \delta_{ij} \), where \( \delta_{ii} = 1 \) and \( \delta_{ij} = 0, i \neq j \), therefore we have \( LS_i(\gamma, c_{N-\{j\}}) = -\delta_{ij}, \forall i \in N \), so that \( LS(\gamma, c_{N-\{j\}}) = -e_j \).

Finally, we have \( Q(N, c_N) = 1 \) and \( Q(N - \{i\}, c_N) = 0, \forall i \in N \), hence \( LS_i(\gamma, c_N) = 1, \forall i \in N \), so that \( LS(\gamma, c_N) = e \).

Now, we can prove:

**Theorem 10**: If \( L \in R^N \) is given, then any game \( v \in G^N \) for which \( LS(\gamma, v) = L \), is obtained for some numerical values of the parameters \( a_N \) and \( a_S, \forall S \subset N, \) \( |S| \leq n - 2 \), from the formula

\[
(3.2) \quad v = \sum_{|S| \leq n-2} a_S c_S + a_N (c_N + \sum_{j \in N} c_{N-\{j\}}) - \sum_{j \in N} L_j c_{N-\{j\}},
\]

where \( C = \{ c_S : S \subseteq N, S \neq \phi \} \) is the potential basis for \( G^N \) relative to the LS-value.

**Proof**: In the expansion of \( v \) in the potential basis we take \( Q(N, v) = a_N \), \( Q(S, v) = a_S \) for all \( S \subset N \) with \( |S| \leq n - 2 \), and \( Q(N - \{j\}, v) = Q(N, v) - L_j \), for all \( j \in N \).

Conversely, if \( v \in G^N \) is given by (3.2), then by linearity we obtain

\[
LS_i(\gamma, v) = \sum_{|S| \leq n-2} a_S LS_i(\gamma, c_S) + a_N LS_i(\gamma, c_N + \sum_{j \in N} c_{N-\{j\}}) - \sum_{j \in N} L_j LS_i(\gamma, c_{N-\{j\}}), \forall i \in N;
\]

By Theorem 9, the first sum equals zero, it easy to check that the second term equals zero, and we get

\[
LS_i(\gamma, v) = -\sum_{j \in N} L_j (-\delta_{ij}) = L_i, \forall i \in N.
\]

Note that for \( L = 0 \), we obtain the following:

**Corollary 11**: The null space of the linear operator \( LS : G^N \to R^N \) is spanned by the set of vectors \( \{ c_S \in G^N : S \subseteq N, |S| \leq n - 2 \} \cup \{ c_N + \sum_{j \in N} c_{N-\{j\}} \} \), hence dim \( Nu(LS) = 2^n - n - 1 \).

**Example 2**: If \( |N| = 3 \) and \( L \in R^3 \) is given, then the solution of the inverse problem for the LS-value is obtained from

\[
(3.3) \quad v = a_1 c_1 + a_2 c_2 + a_3 c_3 + a_{123} (c_{123} + c_{12} + c_{13} + c_{23}) - L_1 c_{23} - L_2 c_{13} - L_3 c_{12}
\]

where the vectors \( c_S \) are those shown in the Example 1. Hence, the set of games having \( LS(\gamma, v) = L \) is obtained for some numerical values of parameters \( a_1, a_2, a_3, \) and \( a_{123} \) from

\[
(3.4) \quad v(1) = \gamma(1)^{-1} a_1, v(2) = \gamma(1)^{-1} a_2, v(3) = \gamma(1)^{-1} a_3,
\]

\[
v(1, 2) = \gamma(2)^{-1} (2a_{123} - a_1 - a_2 - 2L_3),
\]

\[
v(1, 3) = \gamma(2)^{-1} (2a_{123} - a_1 - a_3 - 2L_2),
\]

\[
v(2, 3) = \gamma(2)^{-1} (2a_{123} - a_2 - a_3 - 2L_1),
\]

9
\( v(1, 2, 3) = L_1 + L_2 + L_3, \)

derived from (3.3).


In [9], two axiomatic characterizations for the LS-values are given. One is based upon a group of axioms including the linearity, efficiency, symmetry, additivity for inessential games and weighted average marginal contribution monotonicity. The other is based upon consistency relative to a new Davis/Maschler type of reduced game and standardness for two person games. Recall that Hart/Mas-Colell have shown that the Shapley value, which is a LS-value, can be characterized by consistency relative to another type of reduced game and standardness for two person games (see [6], [7]). This fact suggests that the LS-values may also be characterized by some kind of consistency relative to a new concept of consistency, which uses also a new concept of value to be called Extended Least Square Value.

The LS-values depend on the weight functions \( m: P(N) - \{\phi, N\} \rightarrow R \) as it becomes clear by looking at (1.3) and (1.4). Moreover, one can say that these formulas associate a unique LS-value to an entire set of weight functions determined up to a multiplicative factor. In the set there is a unique normalized weight function, that is one satisfying the condition

\[
(4.1) \quad \alpha(n) = \sum_{s=1}^{n-1} m(s) \binom{n-2}{s-1} = 1,
\]

so that we may say that there is an one-to-one correspondence between the normalized weight functions and the LS-values. In the first section, we have introduced a positive symmetric function \( \gamma: P(N) - \{\phi, N\} \rightarrow R, \) by

\[
(4.2) \quad \gamma(s) = \frac{n-1}{\alpha(n)} \binom{n-2}{s-1} m(s), s = 1, 2, ..., n - 1.
\]

Note that \( \gamma \) satisfies

\[
(4.3) \quad \sum_{s=1}^{n-1} \gamma(s) = n - 1,
\]

for all weight functions \( m. \) If \( \alpha(n) = 1, \) then we uniquely get

\[
(4.4) \quad m(s) = (n - 1)^{-1} \binom{n-2}{s-1}^{-1} \gamma(s), s = 1, 2, ..., n - 1,
\]

so that there is also an one-to-one correspondence between the weight functions \( \gamma \) satisfying (4.3) and the LS-values. We preferred to work with these last weight functions because of the simple representation of the LS-values given in terms of \( \gamma \) by formula (1.7). We shall see that this is also the case in consistency problems.

In a game \( v \in G^N, \) a group of players \( T \subset N \) would play a new game in \( G^T, \) after the payoffs of the other players in \( N - T, \) obtained by some division rule \( \Psi \) have
been put aside. It is highly desirable that in the new game $v_T^\Psi$ the payoffs to be obtained by using $\Psi$ are consistent with the payoffs obtainable by the same players in $v$. Clearly, in passing from the game $v \in G^N$ to the game $v_T^\Psi \in G^T$ we have to know how the division rule $\Psi$ applies to $v_T^\Psi$. Coming to our case, where $\Psi$ is the LS-value, which depends on $m$ (or $\gamma$), beside the usual problems of defining appropriately the reduced game $v_T^\Psi$ and proving the consistency, we have a new question: how do we take the LS-value of a game in $G^T$, while the LS-value of $v \in G^N$ has been determined by an already selected weight function $m$?

A very natural way would be to take the LS-value corresponding to the restriction $m^t$ of $m$ to $P(T) - \{\phi, T\}$, even though this restriction is not yet normalized. Alternatively, we may use the corresponding function $\gamma^t$, which would be worse; indeed, if $\gamma^n$ is the weight corresponding to $m$ by (4.2), then in the simplest case when $|T| = n - 1$ the weight $\gamma^{n-1}$ will be given by the long expression

$$
\gamma^{n-1}(s) = \frac{(n-2)(n-s-1)\gamma^n(s)}{(n-2)\gamma^n(1)+(n-3)\gamma^n(2)+...+(n-2)\gamma^n(n-2)}, \ s = 1, 2, ..., n - 2.
$$

Therefore, we shall not use this approach in defining a reduced game such that the LS-value is consistent relative to this game.

In [9], the authors have taken another approach. Starting from a LS-value for games in $G^N$ corresponding to a normalized weight function $m^n$, the LS-value for games in $G^T$ with $|T| = n - 1$ corresponds to a weight function $m^{n-1}$ defined by

$$
m^{n-1}(s) = m^n(s) + m^n(s + 1), \ s = 1, 2, ..., n - 2.
$$

It is easy to see that the weight $m^{n-1}$ is also normalized, because $\alpha(n - 1) = \alpha(n) = 1$, and more importantly a general rule for obtaining $m^t$ can also be derived from (4.6). In this way, an appropriately chosen reduced game helped the authors in proving the consistency property relative to a new Davis/Maschler type of reduced game. Alternatively, we may try to use the weight $\gamma^t$ corresponding to $m^t$, because the recursion formula similar to (4.6) is simple

$$
\gamma^{n-1}(s) = \frac{n-s-1}{n-1} \gamma^n(s), \ s = 1, 2, ..., n - 2,
$$

and the general rule for obtaining $\gamma^t$ is obvious. Perhaps the work done in [9] may be written in terms of function $\gamma$, but we shall not do that, we shall use a third approach which does not make any assumption on the weight functions. However, the philosophy of the consistency will be new, so that we must describe first this new way of understanding the consistency.

As above, in a game $v \in G^N$ a group of players $T \subset N$ would play a new game in $G^T$ after the payoffs of the other players in $N - T$ obtained by some division rule $\Psi$ have been put aside. Now, the players are more liberal than before so that they will accept in the new game a new division rule $\chi$, which may be different of $\Psi$, but again it is highly desirable that in the new game $v_T^\Psi \chi$ the payoffs given by $\chi$ are consistent with the payoffs
obtainable by the same players in \( v \) where the rule \( \Psi \) was used. Coming to our case, where \( \Psi \) is the LS-value, we shall use as \( \chi \) a new value to be introduced, an ELS-value. So, we shall define a new reduced game of Hart/Mas-Colell type, in which the ELS-value will give the same payoffs to players in \( T \) as the payoffs given by the LS-value to these players in \( v \). We formalize this concept of consistency as follows.

**Definition 11:** Let \( \gamma : P(N) - \{\phi\} \rightarrow R \) be a positive symmetric function; an Extended Least Square Value is an operator \( ELS: G^N \rightarrow R^n \) defined by

\[
(4.8) \quad ELS_i(\gamma, v) = \gamma(n) \frac{v(N)}{n} + \sum_{s=1}^{n-1} \gamma(s) \frac{v_s - v'_s}{s}, \quad i \in N, \quad v \in G^N.
\]

where the numbers \( v_s \) and \( v'_s \), \( i \in N \), are given for \( s = 1, 2, \ldots, n - 1 \) by (1.5).

Note that if \( \gamma(n) = 1 \) and \( \sum_{s=1}^{n-1} \gamma(s) = n - 1 \), then the ELS-value equals the LS-value with the same weights, as shown by Theorem 1; this explains the name, even though for other weights which do not satisfy the above conditions the ELS-value is not an optimal solution of problem (1.2). Note also that while for subgames of \( v \in G^N \) to obtain a LS-value we have to change the weights, for an ELS-value with weight function \( \gamma \) satisfying the above conditions, the weight function should stay the same, except that some weights will not be used. Therefore, while we have \( ELS(\gamma, v) = LS(\gamma, v), \forall v \in G^N \), for subgames of \( v \) this equality does not hold anymore. Finally, notice that if \( \gamma(s) = 1 \) for \( s = 1, 2, \ldots, n - 1 \), then the ELS-value is the Shapley value, but a more interesting relationship is offered by the following

**Lemma 12:** Let \( \gamma : P(N) - \{\phi\} \rightarrow R \) be a positive symmetric function and \( ELS: G^N \rightarrow R^n \) be the operator (4.8) associated with \( \gamma \); then, for any subgame \( v_T \in G^T \) of a game \( v \in G^N \) we have

\[
(4.9) \quad ELS(\gamma, v_T) = Sh(T, \gamma v_T), \quad \forall T \subseteq N.
\]

**Proof:** Follows from Definition 11, which is given for whatever set of players, and the result (1.6) obtained in [4].

**Lemma 13:** If ELS is the operator (4.8) corresponding to a positive symmetric function \( \gamma : P(N) - \{\phi\} \rightarrow R \), then we have for any subgame \( v_T \in G^T \) of a game \( v \in G^N \) the equality

\[
(4.10) \quad \sum_{i \in T} ELS_i(\gamma, v_T) = \gamma(t)v(T), \quad \forall T \subseteq N,
\]

**Proof:** This follows from Lemma 12 and the efficiency of the Shapley value.

Now, return to the purpose of introducing the new value ELS. In our procedure of defining a new concept of consistency, we shall take \( \Psi \) to be \( LS(\gamma, v) \) and \( \chi \) to be \( ELS(\gamma, v) \), both associated with the same weight function, and for both add \( \sum_{s=1}^{n-1} \gamma(s) = n - 1 \), and \( \gamma(n) = 1 \). In other words, we shall take ELS such that
ELS(γ, v) = LS(γ, v) holds for all \( v \in G^N \) but for the subgames ELS can be obtained for the same weights while LS can not be obtained with these weights. The next step in the procedure is that of introducing the concept of reduced game:

**Definition 14:** Let \( \gamma : P(N) - \{ \emptyset \} \to R \) be a positive symmetric function satisfying

\[
\gamma(n) = 1, \quad \sum_{s=1}^{n-1} \gamma(s) = n - 1,
\]

and consider the associated values \( \Psi(\gamma, v) = LS(\gamma, v), v \in G^N, \) and \( \chi(\gamma, v_T) = ELS(\gamma, v_T), \) \( \forall v_T \in G^T, \forall T \subseteq N; \) the reduced game of \( v \) relative to the pair \( (\Psi, \chi) \) and any \( T \subset N \) is

\[
\Psi^\chi_T(S) = [\gamma(s)]^{-1} \cdot \left[ [\gamma(n - t + s)v(S \cup T^c) - \sum_{j \in T^c} \chi_j(\gamma, v_{SUT^c})], S \subseteq T, \right.
\]

where \( T^c = N - T. \)

Taking into account that \( ELS(\gamma, v) = LS(\gamma, v) \) and Lemma 13 we can rewrite (4.12) as

\[
\Psi^\chi_T(S) = \left\{ \begin{array}{ll}
\gamma(t)]^{-1} \sum_{j \in T} LS_j(\gamma, v) & \text{if } S = T, \\
[\gamma(s)]^{-1} \sum_{j \in S} ELS_j(\gamma, v_{SUT^c}) & \text{if } S \subset T.
\end{array} \right.
\]

Note that when we write \( ELS(\gamma, v_T) \) like in (4.9), or \( ELS_j(\gamma, v_{SUT^c}) \) like in (4.12), the subscript of the game shows that we consider the restriction of \( \gamma \) to the power set of that subscript. Note also that if \( \gamma(s) = 1, s = 1, 2, ..., n - 1, \) then (4.12) is the Hart/Mas-Colell reduced game for the Shapley value. Let us illustrate Definition 14:

**Example 3:** For \( |N| = 3, T = \{1, 2\} \) and \( \gamma = (\gamma_1, \gamma_2, 1) \) with \( \gamma_1 + \gamma_2 = 2, \) and the LS-value given by Theorem 1 and the ELS-value given by Definition 11, we shall compute the reduced game defined in (4.12) or (4.13); we get

\[
\gamma_1 \Psi_{T^1}^\chi(1) = ELS_1(\gamma, v_{\{1,3\}}) = \frac{1}{2} \gamma_2 v(1, 3) + \gamma_1 \frac{v(1) - v(3)}{2},
\]

\[
\gamma_1 \Psi_{T^2}^\chi(2) = ELS_2(\gamma, v_{\{2,3\}}) = \frac{1}{2} \gamma_2 v(2, 3) + \gamma_1 \frac{v(2) - v(3)}{2},
\]

\[
\gamma_2 \Psi_{T^1}^\chi(1, 2) = LS_1(\gamma, v) + LS_2(\gamma, v) = \frac{2}{3} v(1, 2, 3) + \frac{1}{6} \gamma_2 [2v(1, 2) - v(1, 3) - v(2, 3)] + \frac{1}{6} \gamma_1 [v(1) + v(2) - 2v(3)].
\]

Now, for the new game \( (\Psi_{T^1}^\chi(1), \Psi_{T^2}^\chi(2), \Psi_{T^1}^\chi(1, 2)) \) we can compute \( ELS(\gamma, \Psi_{T^1}^\chi), \) to get
\[ ELS_1(\gamma, v_T^{\Psi, x}) = \frac{1}{2} \gamma_2 v_T^{\Psi, x}(1, 2) + \gamma_1 \frac{v_T^{\Psi, x}(1) - v_T^{\Psi, x}(2)}{2} = \]
\[ = \frac{1}{3} v(1, 2, 3) + \frac{1}{2} \gamma_2 \frac{v(1, 2) + v(1, 3) - 2v(2, 3)}{3} + \gamma_1 \frac{2v(1) - v(2) - v(3)}{6}, \]
\[ ELS_2(\gamma, v_T^{\Psi, x}) = \frac{1}{2} \gamma_2 v_T^{\Psi, x}(1, 2) + \gamma_1 \frac{-v_T^{\Psi, x}(1) + v_T^{\Psi, x}(2)}{2} = \]
\[ = \frac{1}{3} v(1, 2, 3) + \frac{1}{2} \gamma_2 \frac{v(1, 2) - 2v(1, 3) + v(2, 3)}{3} + \gamma_1 \frac{-v(1) + 2v(2) - v(3)}{6}. \]

It is easy to see that we have

\[ ELS_i(\gamma, v_T^{\Psi, x}) = LS_i(\gamma, v), \quad i = 1, 2, \]

hence in the reduced game the remaining players will get the same payoffs as in the original game, even though the division rule has been changed. This example illustrates the following:

**Definition 15:** A pair of values \((\Psi, \chi)\) is consistent relative to a reduced game concept \(v_T^{\Psi, x}\), if for all \(v \in G^N\) and all \(T \subseteq N\), we have

\[ (4.14) \quad \chi_i(v_T^{\Psi, x}) = \Psi_i(v), \quad \forall i \in T. \]

If \((4.14)\) holds for \(\Psi = \chi\), then \(\Psi\) is simply consistent.

Our Example 3 above shows that for 3-person games the pair \((LS, ELS)\) associated with a weight function \(\gamma\) satisfying \((4.11)\) is consistent relative to the reduced game concept \((4.12)\); indeed, we proved that \((4.14)\) holds for \(|T| = 2\) and it is easy to check it for \(|T| = 1\). We intend to prove that for any set of players \(N\) and positive symmetric weight function \(\gamma\) satisfying \((4.11)\) the pair \((LS, ELS)\) is consistent relative to the reduced game concept \((4.12)\). A potential approach will be indirectly used together with the reduced game property of the Shapley value proved by Hart/Mas-Colell (see [6], [7]) in the last step of our procedure.

**Theorem 16:** Let \(\gamma : P(N) - \{\emptyset\} \to R\) be a positive symmetric weight function satisfying \((4.11)\); the associated pair of values \((\Psi, \chi)\) defined by

\[ (4.15) \quad \Psi(\gamma, v) = LS(\gamma, v), \quad v \in G^N, \]
\[ \chi(\gamma, v_T) = ELS(\gamma, v_T), \quad v_T \in G^T, \quad T \subseteq N, \]

is consistent relative to the reduced game concept \((4.12)\), or \((4.13)\).

**Proof:** As the Shapley value has the reduced game property relative to Hart/Mas-Colell reduced game

\[ (4.16) \quad w_T^{Sh}(S) = w(S \cup T^c) - \sum_{j \in T^c} S_{h_j}(S \cup T^c, w) = \sum_{j \in S} S_{h_j}(S \cup T^c, w), \quad S \subseteq T, \]

14
we have for all \( w \in G^N \) and all \( T \subset N \):

\[
(4.17) \quad Sh_i(T, w_T^{SH}) = Sh_i(N, w), \forall i \in N.
\]

Consider the game \( w \) obtained from \( v \) by \( w(S) = \gamma(s)v(S), S \subseteq N \). By Corollary 3, we have

\[
(4.18) \quad Sh_i(N, w) = LS_i(\gamma, v),
\]

so that we must prove that the left hand side in (4.17) is the ELS for our reduced game. Let us denote by \( w_T^{\psi,x} \in G^T \) the game \( w_T^{\psi,x}(S) = \gamma(s)u_T^{\psi,x}(S), \forall S \subseteq T \); then we get

\[
w_T^{\psi,x}(S) = w(S \cup T^c) - \sum_{j \in T^c} \chi_j(\gamma, v_{S \cup T^c}) =
\]

\[
= w(S \cup T^c) - \sum_{j \in T^c} ELS_j(\gamma, v_{S \cup T^c}) = w(S \cup T^c) - \sum_{j \in T^c} Sh_j(S \cup T^c, w),
\]

where the first equality sign follows from (4.12), the second equality sign is the result of the choice (4.15) and the last follows from Lemma 12. Hence, by comparing with (4.16) we have \( w_T^{SH} = w_T^{\psi,x} \); therefore the left hand side in (4.17) is

\[
(4.19) \quad Sh_i(T, w_T^{SH}) = Sh_i(T, w_T^{\psi,x}) = ELS_i(T, u_T^{\psi,x}), \forall i \in N,
\]

where Lemma 12 has been used again. From (4.17), (4.18) and (4.19) we have \( ELS(T, u_T^{\psi,x}) = LS(\gamma, v), \forall T \subset N \), which proves the theorem.

Theorem 16 will be a half of an axiomatic characterization of ELS-values to be given in the next section.

5. Axiomatic Characterization of ELS-values.

Recall from [6] and [7], that a value \( \chi \) is standard for two person games (\( \{i, j\}, v \)) if we have

\[
(5.1) \quad \chi_i(\{i, j\}, v) = v(i) + \frac{1}{2} [v(i, j) - v(i) - v(j)],
\]

\[
\chi_j(\{i, j\}, v) = v(j) + \frac{1}{2} [v(i, j) - v(i) - v(j)].
\]

A similar concept will be introduced now to fit the case of the ELS-values.

Definition 17: Let \( \gamma: P(N) = \{\emptyset\} \rightarrow R \) be a positive symmetric weight function satisfying (4.11), where \( |N| > 2 \). A value \( \chi \) associated with \( \gamma \) is standard weighted for two person games if we have for \( |T| = 2, T \subset N \), and the game \( (N, v) \).

\[
(5.2) \quad \chi_i(\gamma, v_T) = \frac{1}{2} \gamma(2)v(T) + \frac{1}{2} \gamma(1)[v(i) - v(j)], i, j = 1, 2, i \neq j.
\]

Note that any value which is standard for two person games, like the Shapley value, the Banzhaf value, etc., is also standard weighted with all weights equal to one.
Lemma 18: Any ELS-value associated with a weight function $\gamma$ is standard weighted.

Proof: From Definition 11 applied to a two person game we get immediately (5.2).

From now on a weight function $\gamma$ satisfying (4.11) and the LS-value $\Psi(\gamma, v) = LS(\gamma, v)$ associated with this weight are fixed, while a value $\chi$ associated with the same weight will be arbitrary. We shall give a necessary and sufficient condition for $\chi$ to be the ELS-value.

Theorem 19: A value $\chi$ associated with a weight function $\gamma$ is
(i) forming a consistent pair $(\Psi, \chi)$ with the LS-value associated with the same weights,
(ii) standard weighted for two person games,
if and only if $\chi$ is the ELS-value associated with $\gamma$.

Proof: As stated in Lemma 18, the ELS-value is standard weighted for two person games; in Theorem 16 it has been proved that the ELS-value is forming a consistent pair with the LS-value. Hence, the "if" part has already been proved; it remains to be shown that if a value $\chi$ satisfies (i) and (ii), like the ELS-value, then we should have $\chi(\gamma, v) = ELS(\gamma, v), \forall v \in G^N$. The proof will be carried out similarly to the corresponding proofs in [5], [6], and [7], that is we shall prove in two stages, proving first that such a value $\chi$ satisfying (i) and (ii) must satisfy

\[ (5.3) \sum_{j \in S} \chi_j(\gamma, v_S) = \gamma(s)v(S), \forall S \subseteq N, |S| \geq 2, \forall v \in G^N, \]

then, we shall pass to the proof of the uniqueness. The equality (5.3) will be proved by induction, starting with coalitions $S$ with $|S| = 2$: as $\chi(\gamma, v_S)$ and $ELS(\gamma, v_S)$ satisfy both (ii) we get more than (5.3), namely we obtain $\chi_j(\gamma, v_S) = ELS_j(\gamma, v_S), \forall j \in S$, and by summation (5.3) follows. Assume that (5.3) holds for all coalitions $T \subset N$ with $|T| < s$ and prove it for $S$ with $|S| = s$. Let $U \subset S$ has $|U| = 2$; we get

\[ \sum_{j \in S} \chi_j(\gamma, v_S) = \sum_{j \in U} \chi_j(\gamma, v_S) + \sum_{j \in U^c} \chi_j(\gamma, v_S) = \]

\[ = \sum_{j \in U} \chi_j(\gamma, v_U^\chi) + \sum_{j \in U^c} \chi_j(\gamma, v_S) = \gamma(2)v_U^\chi(U) + \sum_{j \in U^c} \chi_j(\gamma, v_S) = \gamma(s)v(S), \]

where the second equality follows from (i), the third follows from (5.3) proved for two person games and the last equality is implied by the Definition (4.12) of the reduced game. Hence (5.3) holds.

As noticed above, for coalitions $S$ with $|S| = 2$ we proved

\[ (5.4) \chi_j(\gamma, v_S) = ELS_j(\gamma, v_S), \forall j \in S. \]

Now we intend to prove (5.4) for all $S \subseteq N$, by induction over the size of $S$; this will be the second stage of the proof, the uniqueness of ELS. We assume that (5.4) holds for all coalitions $L$ with $2 \leq |L| < s$ and we prove it for $S$. Consider any $i$ and $j$ in $S$, and
$U = \{i, j\}$; let $v_{U}^{\Psi, X}$ and $v_{U}^{\Psi, ELS}$ be the reduced games defined by (4.12) for the pairs $(\Psi, \chi)$ and $(\Psi, ELS)$. From (4.12) and (5.3) already proved we have

$$v_{U}^{\Psi, X}(i) = [\gamma(1)]^{-1}[\gamma(s - 1)v(S - \{j\}) - \sum_{k \in U^c} \chi_k(\gamma, v_{S - \{j\}})] = [\gamma(1)]^{-1}\chi_i(\gamma, v_{S - \{j\}})$$

$$v_{U}^{\Psi, ELS}(i) = [\gamma(1)]^{-1}[\gamma(s - 1)v(S - \{j\}) - \sum_{k \in U^c} ELS_k(\gamma, v_{S - \{j\}})]$$

$$= [\gamma(1)]^{-1}ELS_i(\gamma, v_{S - \{j\}})$$

where in the second we have used Lemma 13. By the induction assumption $\chi_i(\gamma, v_{S - \{j\}}) = ELS_i(\gamma, v_{S - \{j\}})$ holds, hence the last side on each row is the same and we proved the equality $v_{U}^{\Psi, X}(i) = v_{U}^{\Psi, ELS}(i)$. Similarly, we prove $v_{U}^{\Psi, X}(j) = v_{U}^{\Psi, ELS}(j)$. These results show that $\chi_i(\gamma, v_{U}^{\Psi, X}) \geq \chi_i(\gamma, v_{U}^{\Psi, ELS})$ if and only if $v_{U}^{\Psi, X}(U) \geq v_{U}^{\Psi, ELS}(U)$ and also if and only if $\chi_j(\gamma, v_{U}^{\Psi, X}) \geq \chi_j(\gamma, v_{U}^{\Psi, ELS})$. We proved

(5.5) \[ \chi_i(\gamma, v_{U}^{\Psi, X}) \geq \chi_i(\gamma, v_{U}^{\Psi, ELS}) \iff \chi_j(\gamma, v_{U}^{\Psi, X}) \geq \chi_j(\gamma, v_{U}^{\Psi, ELS}), \]

for all coalitions $U \subseteq S$ with $|U| = 2$. Consider now again $U = \{i, j\}$; we get

$$\chi_i(\gamma, v_{S}) = \chi_i(\gamma, v_{U}^{\Psi, X}) \geq \chi_i(\gamma, v_{U}^{\Psi, ELS}) = ELS_i(\gamma, v_{U}^{\Psi, ELS}) = ELS_i(\gamma, v_{S}),$$

if and only if

$$\chi_j(\gamma, v_{S}) = \chi_j(\gamma, v_{U}^{\Psi, X}) \geq \chi_j(\gamma, v_{U}^{\Psi, ELS}) = ELS_j(\gamma, v_{U}^{\Psi, ELS}) = ELS_j(\gamma, v_{S}),$$

where the first equality on each row follows from (i), the second equality follows from the fact that $\chi = ELS$ for two person games, the last equality follows again from (i), and the double implication is given by (5.5). For each pair $i, j \in S$ we have

(5.6) \[ \chi_i(\gamma, v_{S}) \geq ELS_i(\gamma, v_{S}) \iff \chi_j(\gamma, v_{S}) \geq ELS_j(\gamma, v_{S}); \]

from (5.3) and Lemma 13 we have also

(5.7) \[ \sum_{k \in S} \chi_k(\gamma, v_{S}) = \sum_{k \in S} ELS_k(\gamma, v_{S}), \]

hence (5.6) can hold only if (5.4) holds, so that (5.4) is proved.

Note that the proof has the same type of steps as the proof of the similar result in [6], but there are differences due to the new definition of the reduced game and the new concept of consistency.

Remarks: a) W. Thomson in [12], gives a comprehensive study of various types of consistency; this monograph has a rich section on cooperative TU games. b) The new type of consistency discussed in our paper lead to an axiomatic characterization of the
ELS-values because of the axiomatic characterization of the Shapley value due to Hart/Mas-Colell and the relationship between the ELS-values and the Shapley value. The situation was similar in the case of the Banzhaf value (see Dragan, [5]).

c) The LS - values corresponding to the weights \( m(s) = 1, s = 1, 2, ..., n - 1 \), were studied in [10] and called the least square prenucleolus.

d) The unique solution (1.3) of the optimization problem (1.2) has been obtained by Keane (1969) in [8].

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