# FREQUENCY RESPONSE APPROXIMATION METHODS <br> OF THE DISSIPATIVE MODEL OF FLUID TRANSMISSION LINES 

by<br>JOHN D. KING<br>Presented to the Faculty of the Graduate School of The University of Texas at Arlington in Partial Fulfillment of the Requirements for the Degree of<br>\title{ MASTER OF SCIENCE IN MECHANICAL ENGINEERING }

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To my wife Lilia
and my daughter Emily

## ACKNOWLEDGEMENTS

I want to thank Dr. Hullender for inspiring me in my decision to research fluid line dynamics while attending his course in Dynamic Systems Modeling. Dr. Hullender introduced the dissipative model of a fluid transmission and discussed the thesis by Tom Wongputorn, "Time Domain Simulation of Systems with Fluid Transmission Lines". This was the beginning for me to a much deeper understanding of modeling systems beyond the simple lump parameter models I have studied previously.

I also want to recognize Tom Wongputorn for his application of the Gauss Newton method and the standard he set for this research. It was personally very challenging for me to attempt to build on his work.

I am very grateful to Dr. Nomura who provided me with the mathematical tools to tackle this research and also sparked my interest in continuum mechanics and the Navier-Stokes equations. I really enjoyed attending his courses.

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March 20, 2006

# ABSTRACT <br> FREQUENCY RESPONSE APPROXIMATION METHODS OF THE DISSIPATIVE MODEL OF FLUID <br> TRANSMISSION LINES 

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This thesis introduces an accurate and efficient method of obtaining a linear approximation for a nonlinear model of a complete fluid transmission line system using Matlab ${ }^{\circledR}$ Signal Processing Toolbox and Symbolic Math Toolbox programs. This technique is then packaged in a Graphical User Interface program to streamline the process of analyzing a total system.

The nonlinear model applied in this thesis is called the dissipative model and is also referred to as the "exact" model, because it's derivation uses all of the NavierStokes equations as well as the equations of state. It has been studied and tested against real data and is recognized as the most accurate of all the known models.

Other modeling approaches are discussed in this work to illustrate the completeness of the dissipative model. The modal technique introduced in this thesis is inspired by the modal approximation method that is based on truncating the infinite series representation of the dissipative model. This modal approximation method is covered in depth in this document.

The method of approximating the frequency response of a fluid transmission line with a rational polynomial transfer function using the Matlab ${ }^{\circledR}$ 'invfreqs' least squares curve fitting algorithm has already been introduced. This work improves the technique by proving that an accurate result can be obtained by matching the mode with the resonant frequency, adding one additional order to the characteristic equation, and then normalizing the result by dividing the approximated transfer function by the steady state gain. It also improves the technique by applying the 'invfreqs' command to a total system rather than just the one line. The result is that the order of a linear transfer function for a total fluid transmission line system can be greatly reduced.

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## NOMENCLATURE

| $A$ | cross sectional area of fluid transmission line |
| :---: | :---: |
| $A(s)$ | denominator of linear rational polynomial transfer function |
| $a_{1}, \ldots, a_{2 m}$ | numerator polynomial coefficients for the propagation operator $\Gamma$ |
| $\bar{a}_{c i}, \bar{b}_{c i}$ | normalized numerator coefficients for the modal approximation of $\cosh \Gamma$ |
| $\bar{a}_{z i}, b_{z i}$ | normalized numerator coefficients for the modal approximation of $Z_{\mathrm{c}} \sinh \Gamma / \mathrm{Z}_{0} \cosh \Gamma$ |
| $\bar{a}_{s i}, \bar{b}_{s i}$ | normalized numerator coefficients for the modal approximation of $\mathrm{Z}_{0} \sinh \Gamma / \mathrm{Z}_{\mathrm{c}} \cosh \Gamma$ |
| $\alpha_{0, i}$ | $\mathrm{i}^{\text {th }}$ zero of the zero-order Bessel function |
| $\alpha_{1, i}$ | $i^{\text {th }}$ zero of the first-order Bessel function |
| $B(s)$ | numerator of linear rational polynomial transfer function |
| $b_{1}, \ldots, b_{2 m}$ | denominator polynomial coefficients for the propagation operator $\Gamma$ |
| $\beta$ | equivalent bulk modulus |
| $\beta$ | secant bulk modulus |
| $\beta_{r}$ | tangent bulk modulus |
| $B$ | Bessel function ration applied in the viscous line model |
| $B_{r}$ | Bessel function ratio applied in the dissipative line model |
| $B_{r \sigma}$ | Bessel function ratio with Prandtl number applied in the dissipative line model |
| $c_{2}, \ldots, c_{2 m+2}$ | numerator polynomial coefficients used in development of $\cosh \Gamma$ |
| $c_{p}$ | specific heat at constant pressure |
| $c_{v}$ | specific heat at constant volume |
| $C_{0}$ | $\text { speed of sound }(\mathrm{m} / \mathrm{s})=\sqrt{\beta_{e} / \rho}$ |
| $C_{1}$ | $\cosh \Gamma / Z_{c} \sinh \Gamma$ |
| $C_{2}$ | $1 / Z_{c} \sinh \Gamma$ |
| $C_{3}$ | $\sinh \Gamma / Z_{c} \cosh \Gamma$ |
| $C_{4}$ | $1 / \cosh \Gamma$ |
| $C_{5}$ | $Z_{c} \cosh \Gamma / \sinh \Gamma$ |
| $C_{6}$ | $\mathrm{Z}_{\delta} / \sinh \Gamma$ |
| $C_{7}$ | $Z_{c} \sinh \Gamma / \cosh \Gamma$ |
| $C_{p}$ | lumped capacitance of fluid line $=\mathrm{V} / \beta_{\mathrm{e}}$ |
| $d$ | fluid transmission line diameter |
| $D_{n}$ | dissipative number $=\omega_{\mathrm{v}} / \omega_{\mathrm{c}}=\nu \mathrm{L} / \mathrm{c}_{0} \mathrm{r}^{2}$ |


| $e(s)$ | error function |
| :---: | :---: |
| E | error function |
| $f_{1}, f_{2}, \ldots f_{n}$ | experimental data points used in the development of nonlinear least squares curve fitting technique |
| $f\left(x_{1}\right), f\left(x_{2}\right), \ldots f\left(x_{n}\right)$ | set of functions used in the development of nonlinear least squares curve fitting technique |
| $F$ | linear friction coefficient $=8 \omega_{\mathrm{v}}$ |
| $\gamma$ | specific heat ratio, $c_{p} / c_{v}$ |
| $\Gamma$ | propagation operator |
| $H(s)$ | linear rational polynomial transfer function |
| $i$ | second order mode number |
| $j$ | $\sqrt{-1}$ |
| $J_{0}$ | zeroth order Bessel function of the first kind |
| $J_{1}$ | first order Bessel function of the first kind |
| $J_{f}$ | Jacobian of a set of functions |
| $\bar{k}_{0}$ | normalized real term for the real pole of $\mathrm{Z}_{0} \sinh \Gamma / \mathrm{Z}_{\mathrm{c}} \cosh \Gamma$ |
| $\lambda$ | dimensionless root index for mode $i$, $=(\mathrm{i}-1 / 2) / \mathrm{D}_{\mathrm{n}}$ |
| $\lambda$ | dimensionless root index for mode $i,=i / D_{n}$ |
| $L$ | lumped inertance $=\rho \mathrm{l} / \mathrm{A}$ |
| $n$ | number of second order modes |
| $\theta$ | parameter vector |
| $\rho$ | fluid density |
| $\rho_{0}$ | fluid density at $\mathrm{t}=0$ |
| $p_{1}, p_{2}, \ldots p_{k}$ | parameter of functions $f\left(x_{1}\right), f\left(x_{2}\right), \ldots f\left(x_{n}\right)$ used in the development of nonlinear least squares curve fitting technique |
| $P_{0}$ | pressure at $\mathrm{t}=0$ |
| $\dot{P}$ | time derivative of pressure |
| $P_{i n}$ | input pressure |
| $P_{\text {out }}$ | output pressure |
| $P_{R}$ | pressure after input valve resistance |
| $\dot{Q}$ | time derivative of fluid flow rate |
| $Q_{\text {in }}$ | input fluid flow rate |
| $Q_{\text {out }}$ | output fluid flow rate |
| $Q(x, s)$ | fluid flow rate variable |
| $r$ | line radius |
| $R$ | lumped resistance $=128 \mathrm{v} \mathrm{\rho l} / \pi \mathrm{d}^{4}$ |
| $s$ | Laplace operator |
| $\bar{s}$ | normalized Laplace operator $=\mathrm{sr} / v=\mathrm{s} / \omega_{\mathrm{v}}$ |
| $\sigma$ | Prandtl number |
| $t$ | time |
| $T$ | temperature |


| $T_{o}$ | time average of temperature |
| :--- | :--- |
| $u$ | fluid velocity component in x-direction |
| $\bar{u}$ | average fluid velocity component in x-direction |
| $v$ | fluid velocity component in y-direction |
| $V$ | volume |
| $\nu$ | kinematic viscosity |
| $\omega$ | frequency $(\mathrm{rad} / \mathrm{sec})$ |
| $\omega_{c}$ | characteristic frequency $=\mathrm{c}_{0} / \mathrm{L}$ |
| $\omega_{v}$ | viscous frequency $=v / \mathrm{r}^{2}$ |
| $\bar{\omega}_{n i}$ | normalized natural frequency of $\mathrm{i}^{\text {th }}$ second order mode |
| $\zeta_{i}$ | damping ratio of $\mathrm{t}^{\mathrm{th}}$ second order mode |
| $\mathrm{Z}_{c}$ | characteristic impedance $=$ |
| $\mathrm{Z}_{0}$ | line impedance constant $=\rho_{0} \mathrm{c}_{0} / \pi \mathrm{r}^{2}$ |

## CHAPTER 1

## INTRODUCTION

### 1.1 System Modeling Background

Engineers want to predict the performance of a system before spending the time and capital to produce one that may not function as required. Models are produced to simulate the actual system and modifications are made to the design based on the performance of the model. The most convenient way to simulate a system is to produce a quantitative mathematical model. The performance of a system can often be very accurately predicted using a mathematical model that quantifies all the measurable dynamic behavior. Depending on the accuracy required these models can be very simple to extremely complex.

The simplest models are composed linear differential equations containing very generalized or "lumped" coefficients representing the system's physical parameters. Circuit theory is an example of this modeling approach and is quite adequate to obtain an accurate output for a small electric circuit operating at low frequencies, but not for a very long electrical transmission line or a circuit operating at very high frequencies. Fluid systems likewise, can be modeled with the "same" equations, but the accuracy limited. The model that an engineer uses has to represent all the dynamics that can have a measurable effect on the result [1/2].

### 1.2 Fluid Transmission Line Modeling History

Fluid and electrical transmission line dynamics have been studied extensively since their wide span application in the $20^{\text {th }}$ century. Heaviside is credited with the formulation of transmission line theory for electrical lines in 1887, and his equations are referred to as the telegrapher's equations. In contrast to circuit theory, Heaviside's equations are based on distributing the parameters of inductance and resistance along the length of the line. Electrical transmission line theory is derived from the Maxwell's equations which are nonlinear partial differential equations. He also proved that an electrical transmission line can be modeled using just two functions, the propagation operator $\Gamma$ and the characteristic impedance $Z_{0}[3]$.

A fluid transmission line system, like a long electrical transmission line, also needs to be modeled with distributed parameters to accurately quantify the dynamic behavior. The approach is very similar to electrical transmission line theory, but fluid transmission line theory is derived from the Navier-Stokes equations which are the foundational equations of fluid mechanics [4]. These equations obey the basic laws of conservation of momentum, mass, and energy:
1.) Momentum: The acceleration of fluid particles.
2.) Continuity: The conservation of mass.
3.) Energy: The dissipation of heat.

Since the Navier-Stokes equations do not cover the issue of compressibility whether it be a gas or a liquid, an additional equation needs to be included in the total solution:
4.) State: The influence of the compressibility of the fluid.

Iberall [5] was the first to produce a solution that included viscous friction and heat transfer effects. Gerlach [6] produced the first exact first order or classical model solution. From this work researchers have developed several distributed parameter models which is documented by Goodson and Leonard [7]. The dissipative model is considered to be the most accurate [8].

### 1.3 Rational Polynomial Approximations

The major obstacle to the distributed parameter model is that it is nonlinear and not in the form of a finite order rational polynomial familiar in classical modeling and control theory. This is a problem because the resulting transfer function cannot be transformed to the time domain using inverse Laplace transform techniques. To exactly represent a distributed parameter model in rational polynomial form would require an infinite order transfer function, because the frequency response of an exact solution oscillates to infinity. It is important to note that all real systems are distributed parameter and a finite order rational polynomial transfer function is simply an approximation. The goal in systems modeling is to have a transfer function approximation of the order that covers the required frequency range of operation [9].

Model Order Reduction is a branch of dynamic systems modeling research that seeks to simply or reduce the complexity of a system model without losing measurable output behavior [10]. The distributed parameter model is an infinite order representation of a system, and researchers have sought for methods to approximate it with a finite order model. Brown [11/12] was the first to approximate Iberall's solution in the Laplace domain to obtain a step and impulse time domain response. D'Souza and

Oldenburger [13] further developed Brown's approach to include the effects of line vibration. Hullender and Healey [14] developed a rational polynomial approximation by obtaining a Tailor's series expansion of the dissipative solution based on the mode number. Hullender and Hsue [15] applied the modal approximation approach to the seven unique solutions of the dissipative model. Hullender and Woods [16] applied the modal approximation method to the development of a minimum-order state-space model. Nursilo [17] introduced an approach to correct modal approximations at zero frequency. Wongputorn [18] introduced an approach by applying a least-squares curve fitting algorithm in Matlab ${ }^{\circledR}$ to the frequency response of the dissipative model.

## CHAPTER 2

## FLUID TRANSMISSION LINE MODELING

### 2.1 Modeling Overview

Each of the commonly accepted fluid transmission line models are developed and compared in this chapter in order to illustrate the importance of having a model that simulates all the dynamics and also the importance of minimizing the model to the frequency of operation.

### 2.2 Lumped Parameter Line Models

The simplest mathematical model of a fluid transmission line is the lumped parameter model. In this model the three physical parameters, resistance, inertance, and capacitance are assumed to be located in one or more discrete locations along the fluid transmission line. This model is constructed with a system of linear ordinary differential equations (ODEs). This model is useful since it can be integrated into a larger mechanical system of lumped parameter components to produce a rational polynomial transfer function. The inverse Laplace transform can then be applied to this result to obtain the time domain response of the system.


Figure 2.1 Basic lumped parameter model
The equations that define this model are as follows:

$$
\begin{gather*}
P_{\text {in }}-P_{R}=R Q_{i n}  \tag{2.1}\\
P_{R}-P_{o u t}=L \dot{Q}_{i n}  \tag{2.2}\\
Q_{\text {in }}-Q_{o u t}=C_{p} \dot{P}_{o u t} \tag{2.3}
\end{gather*}
$$

The Laplace transform of these equations when ignoring initial conditions are:

$$
\begin{gather*}
P_{\text {in }}-P_{R}=R Q_{\text {in }}  \tag{2.4}\\
P_{R}-P_{o u t}=L s Q_{\text {in }}  \tag{2.5}\\
Q_{\text {in }}-Q_{\text {out }}=C_{p} s P_{\text {out }} \tag{2.6}
\end{gather*}
$$

Combining these three equations results in the following matrix form:

$$
\left[\begin{array}{l}
P_{\text {out }}  \tag{2.7}\\
R Q_{\text {out }}
\end{array}\right]=\left[\begin{array}{cc}
1 & -\left(\frac{L}{R} s+1\right) \\
-R C_{p} s & \left(1+R C_{p} s+L C_{p} s^{2}\right)
\end{array}\right]\left[\begin{array}{l}
P_{\text {in }} \\
R Q_{\text {in }}
\end{array}\right]
$$

This is a second order model of the fluid line system. The problem with this model is that in reality are parameters are distributed along the line and not just located
at a discrete point as indicated by the diagram. The lumped parameter model can be modified to attempt to represent the distributed nature of the parameters as shown.


Figure 2.2 Two element lumped parameter model

In this example the line is split into two identical lumped parameter models where all the parameters are split into two lumped elements (lumping by length). This results in the following equation.

$$
\left[\begin{array}{l}
P_{\text {out }}  \tag{2.8}\\
R Q_{\text {out }}
\end{array}\right]=\left[\begin{array}{cc}
1 & -\left(\frac{L}{R} s+1\right) \\
-R C_{p} s & \left(1+R C_{p} s+L C_{p} s^{2}\right)
\end{array}\right]^{2}\left[\begin{array}{l}
P_{\text {in }} \\
R Q_{\text {in }}
\end{array}\right]
$$

If the line is divided n times then this equation would apply:

$$
\left[\begin{array}{l}
P_{\text {out }}  \tag{2.9}\\
R Q_{\text {out }}
\end{array}\right]=\left[\begin{array}{cc}
1 & -\left(\frac{L}{R} s+1\right) \\
-R C_{p} s & \left(1+R C_{p} s+L C_{p} s^{2}\right)
\end{array}\right]^{n}\left[\begin{array}{l}
P_{\text {in }} \\
R Q_{\text {in }}
\end{array}\right]
$$

The following figure shows the frequency response plots of a lumped parameter fluid transmission line.


Figure 2.3 Magnitude and phase frequency response plots of a transmission line modeled with single element lumped parameters


Figure 2.4 Magnitude and phase frequency response plots of a transmission line modeled with two element lumped parameters

This approach is really a form of finite element analysis. It is interesting to note that the frequency response quickly dies out after one peak in the case of the single element line and after two peaks with the two element line. The peaks are referred to as the "modes" of the frequency response. Each mode is equivalent to a second order rational polynomial transfer function. These lumped parameter models are derived from linear differential equations and produce a rational polynomial transfer function which can be transformed into a time domain function via the inverse Laplace transform.

In reality the frequency response of any system in nature has in infinite number of peaks as the magnitude dies out. An "exact" transfer function should then be a function of a cyclical function. The development of the distributed parameter model will show that the cyclical functions used to produce this model are hyperbolic sine and hyperbolic cosine functions. The only problem with distributed parameter models is that they are nonlinear and cannot be inverse Laplace transformed to produce a time domain response.

### 2.3 Distributed Parameter Line Models

The actual governing equations of a fluid transmission line are nonlinear partial differential equations (PDEs) that model the distributed nature of the three parameters of resistance, inertance, and capacitance. The governing equations used are the NavierStokes equations and the equation of state. The equation of state is used because the compressibility of both liquid and gas is taken into account. This results in a more accurate model of the line.


Figure 2.5 Distributed parameter model

The governing equations are as follows:
Momentum Equation
$\rho_{0}\left[\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial r}\right]=-\frac{\partial P}{\partial x}+\mu\left[\frac{4}{3} \frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial r^{2}}+\frac{1}{r} \frac{\partial u}{\partial r}+\frac{1}{3} \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)\right]$

Continuity Equation

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\rho_{0}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}\right]+v \frac{\partial \rho}{\partial r}+u \frac{\partial \rho}{\partial x}=0 \tag{2.11}
\end{equation*}
$$

Energy Equation

$$
\begin{equation*}
\frac{\partial T}{\partial t}+T_{0}(\gamma-1) \frac{\partial \rho}{\partial t}=\alpha_{0}\left[\frac{\partial^{2} T}{\partial^{2} r^{2}}+\frac{1}{r} \frac{\partial T}{\partial r}\right] \tag{2.12}
\end{equation*}
$$

State Equation for Liquids

$$
\begin{equation*}
\frac{d \rho}{\rho_{0}}=\frac{d P}{\beta} \tag{2.13}
\end{equation*}
$$

State Equation for Gases

$$
\begin{equation*}
\frac{d \rho}{\rho_{0}}=\frac{d P}{\gamma_{0}} \tag{2.14}
\end{equation*}
$$

The Laplace Transforms of the governing PDEs are as follows:

$$
\begin{align*}
& \frac{d P(x, \bar{s})}{d x}=-\frac{Z_{0} \Gamma^{2}(\bar{s})}{L \bar{s}} Q(x, \bar{s})  \tag{2.15}\\
& \frac{d Q(x, \bar{s})}{d x}=-\frac{\bar{s}}{L Z_{0}} P(x, \bar{s}) \tag{2.16}
\end{align*}
$$

where:

$$
\begin{gather*}
(x, \bar{s}) \in(0, L) \times \mathrm{C}  \tag{2.17}\\
Z_{0}=\frac{p_{0} c_{0}}{\pi r_{0}^{2}}  \tag{2.18}\\
\bar{s}=\omega s  \tag{2.19}\\
\omega=\frac{L}{c_{0}} \tag{2.20}
\end{gather*}
$$

Several models have been developed using these equations all of which have two functions in common:

Propagation Operator

$$
\Gamma(s)
$$

Characteristic Impedance $\quad \mathrm{Z}_{\mathrm{c}}(\mathrm{s})$
These functions are so named because of the following relationships:

$$
\begin{gather*}
\frac{P\left(x_{2}, s\right)}{P\left(x_{1}, s\right)}=e^{-\left[\Gamma(s) \frac{x_{2}-x_{2}}{\ell}\right]}  \tag{2.21}\\
\frac{P(x, s)}{Q(x, s)}=Z_{c}(s) \tag{2.22}
\end{gather*}
$$

The propagation operator governs the propagation of the input pressure through the line. The characteristic impedance governs the fluid flow. Note that the characteristic impedance is not a function of the length of the line. These two functions are sufficient to completely model a transmission line.

### 2.3.1 Lossless Line Model

The lossless fluid transmission line model uses the momentum, continuity, and state equations but excludes the heat transfer governed by the energy equation and the dissipation effects.

## Momentum Equation (excluding dissipation terms)

$$
\begin{equation*}
\rho_{0} \frac{\partial \bar{u}}{\partial t}+\frac{\partial p}{\partial x}=0 \tag{2.23}
\end{equation*}
$$

Continuity Equation (excluding dissipation terms)

$$
\begin{equation*}
\rho_{0} \frac{\partial u}{\partial x}+u \frac{\partial \rho}{\partial x}+\frac{\partial \rho}{\partial t}=0 \tag{2.24}
\end{equation*}
$$

State Equation for Liquids

$$
\begin{equation*}
\frac{d \rho}{\rho_{0}}=\frac{d P}{\beta} \tag{2.25}
\end{equation*}
$$

State Equation for Gases

$$
\begin{equation*}
\frac{d \rho}{\rho_{0}}=\frac{d P}{\gamma_{0}} \tag{2.26}
\end{equation*}
$$

These form the following wave equations:

$$
\begin{align*}
& \frac{\partial^{2} P}{\partial x^{2}}=\frac{\rho_{0}}{\beta} \frac{\partial^{2} p}{\partial t^{2}}  \tag{2.27}\\
& \frac{\partial^{2} Q}{\partial x^{2}}=\frac{\rho_{0}}{\beta} \frac{\partial^{2} Q}{\partial t^{2}} \tag{2.28}
\end{align*}
$$

The solution to the wave equation in matrix form is as follows:

$$
\left[\begin{array}{l}
P_{\text {out }}  \tag{2.29}\\
Q_{\text {in }}
\end{array}\right]=\left[\begin{array}{cc}
\frac{1}{\cosh \Gamma} & -\frac{Z_{c} \sinh \Gamma}{\cosh \Gamma} \\
\frac{\sinh \Gamma}{Z_{c} \cosh \Gamma} & \frac{1}{\cosh \Gamma}
\end{array}\right]\left[\begin{array}{l}
P_{\text {in }} \\
Q_{\text {out }}
\end{array}\right]
$$

This form is consistent in all of the distributed parameter models. The only difference is in the calculation of the propagation operator and the characteristic impedance. The propagation operator and the characteristic impedance functions are defined in the lossless fluid transmission line model as:

$$
\begin{gather*}
\Gamma(s)=\frac{L s}{c_{0}}  \tag{2.30}\\
Z_{c}(s)=\frac{p_{0} c_{0}}{\pi r_{0}^{2}} \tag{2.31}
\end{gather*}
$$

### 2.3.2 Linear Friction Model

The linear friction transmission line model uses the following equations:

## Momentum Equation

(linear friction term resulting in pressure loss being proportional to average velocity)

$$
\begin{equation*}
\rho_{0}\left[\frac{\partial \bar{u}}{\partial t}+F \bar{u}\right]+\frac{\partial p}{\partial x}=0 \tag{2.32}
\end{equation*}
$$

The linear friction term is defined by the following equation.

$$
\begin{equation*}
\rho_{0} u F=-\frac{\Delta P}{\Delta x} \tag{2.33}
\end{equation*}
$$

The Hagen-Poiseuille theory for pressure drop in a pipe with laminar flow is given as:

$$
\begin{equation*}
\frac{\Delta P}{\Delta x}=-8 \frac{v_{0} \rho_{0} u}{r_{0}^{2}} \tag{2.34}
\end{equation*}
$$

This simplifies F to:

$$
\begin{equation*}
F=\frac{8 v_{0}}{r_{0}^{2}} \tag{2.35}
\end{equation*}
$$

The viscous frequency is defined as:

$$
\begin{equation*}
\omega_{v}=\frac{v_{0}}{r_{0}^{2}} \tag{2.36}
\end{equation*}
$$

Resulting in:

$$
\begin{equation*}
F=8 \omega_{v} \tag{2.37}
\end{equation*}
$$

Continuity Equation (excluding dissipation terms)

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\rho_{0} \frac{\partial \bar{u}}{\partial x}=0 \tag{2.38}
\end{equation*}
$$

State Equation for Liquids

$$
\begin{equation*}
\frac{d \rho}{\rho_{0}}=\frac{d P}{\beta} \tag{2.39}
\end{equation*}
$$

State Equation for Gases

$$
\begin{equation*}
\frac{d \rho}{\rho_{0}}=\frac{d P}{\gamma P_{0}} \tag{2.40}
\end{equation*}
$$

The propagation operator and characteristic impedance functions are defined by the linear friction transmission line model as:

$$
\begin{align*}
\Gamma(s) & =\frac{L s}{c_{0}} \sqrt{1+\frac{F}{s}}  \tag{2.41}\\
Z_{c}(s) & =\frac{p_{0} c_{0}}{\pi r_{0}^{2}} \sqrt{1+\frac{F}{s}} \tag{2.42}
\end{align*}
$$

### 2.3.3 Viscous Line Model

The viscous transmission line model applies the Navier-Stokes equations with the exception of heat transfer terms.

$$
\begin{array}{r}
\text { Momentum Equation } \\
\rho_{0} \frac{\partial u}{\partial t}=-\frac{\partial P}{\partial x}+\mu\left[\frac{\partial^{2} u}{\partial r^{2}}+\frac{1}{r} \frac{\partial u}{\partial r}\right] \\
\text { Continuity Equation } \\
\frac{\partial \rho}{\partial t}+\rho_{0}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}\right]=0 \tag{2.44}
\end{array}
$$

State Equation for Liquids

$$
\begin{equation*}
\frac{d \rho}{\rho_{0}}=\frac{d P}{\beta} \tag{2.45}
\end{equation*}
$$

State Equation for Gases

$$
\begin{equation*}
\frac{d \rho}{\rho_{0}}=\frac{d P}{\gamma_{0}} \tag{2.46}
\end{equation*}
$$

The propagation operator and characteristic impedance functions are defined by the viscous transmission line model as:

$$
\begin{gather*}
\Gamma(s)=\frac{B s}{\omega_{c}}  \tag{2.47}\\
B=\frac{1}{\sqrt{Z_{c}(s)=B Z_{0}}} \sqrt{1-\frac{J_{1}\left(j \sqrt{\frac{s}{\omega_{v}}}\right)}{j \sqrt{\frac{s}{\omega_{v}} J_{0}\left(j \sqrt{\frac{s}{\omega_{v}}}\right)}}} \tag{2.48}
\end{gather*}
$$

$\mathrm{J}_{0}$ and $\mathrm{J}_{1}$ are zero and first order Bessel functions of the first kind respectively. $\mathrm{Z}_{\mathrm{c}}$ and $\omega_{\mathrm{v}}$ are the characteristic and viscous frequency respectively.

### 2.3.4 Dissipative Transmission Line Model

The dissipative transmission line model is derived from all the original NavierStokes equations and the equation of state to include all the effects of viscosity, heat transfer, and compressibility. The characteristic impedance and the propagation operator are as follows:

$$
\begin{gather*}
\Gamma(s)=\frac{s}{\omega_{c}} \sqrt{\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}}  \tag{2.50}\\
Z_{c}(s)=\frac{Z_{0}}{\sqrt{\left(1-B_{r}\right)\left(1+(\gamma-1) B_{r \sigma}\right)}} \tag{2.51}
\end{gather*}
$$

where the two Bessel function ratios are as follows.

$$
\begin{align*}
B_{r}=\frac{2 J_{1}\left(j \sqrt{\frac{s}{\omega_{v}}}\right)}{\left.j \sqrt{\frac{s}{\omega_{v}} J_{0}\left(j \sqrt{\frac{s}{\omega_{v}}}\right.}\right)}  \tag{2.52}\\
B_{r \sigma}=\frac{2 J_{1}\left(j \sqrt{\frac{\sigma s}{\omega_{v}}}\right)}{\left.j \sqrt{\frac{\sigma s}{\omega_{v}} J_{0}\left(j \sqrt{\frac{\sigma s}{\omega_{v}}}\right.}\right)} \tag{2.53}
\end{align*}
$$

and the line impedance constant is defined as follows.

$$
\begin{equation*}
Z_{0}=\frac{\rho_{0} c_{0}}{\pi r^{2}} \tag{2.54}
\end{equation*}
$$

The specific heat ratio is given by the following equation.

$$
\begin{equation*}
\gamma=\frac{c_{p}}{c_{v}} \tag{2.55}
\end{equation*}
$$

### 2.4 Normalized Parameters

In practice, all of the distributed parameter models are normalized with respect to time for the purpose of frequency response comparison.

Normalized Laplace Operator

$$
\begin{equation*}
\bar{s}=\frac{s r^{2}}{v}=\frac{s}{\omega_{v}} \tag{2.56}
\end{equation*}
$$

$$
\begin{align*}
& \text { Viscous Frequency } \\
& \omega_{v}=\frac{v}{r^{2}} \tag{2.57}
\end{align*}
$$

Replacing the Laplace operator with the normalized operator produces the following.

$$
\begin{equation*}
s=\bar{s} \omega_{v} \tag{2.58}
\end{equation*}
$$

gives the normalized propagation operator as:

$$
\begin{equation*}
\Gamma(\bar{s})=\frac{\bar{s} \omega_{v}}{\omega_{c}} \sqrt{\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}} \tag{2.59}
\end{equation*}
$$

Defining the dissipation number as:

$$
\begin{equation*}
D_{n}=\frac{\omega_{v}}{\omega_{c}} \tag{2.60}
\end{equation*}
$$

and simplifying the normalized propagation operator gives the following equation.

$$
\begin{equation*}
\Gamma(\bar{s})=D_{n} \bar{s} \sqrt{\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}} \tag{2.61}
\end{equation*}
$$

The dissipation number is often a used as a reference point when comparing various frequency responses. The dissipation number can also be written as:

$$
\begin{equation*}
D_{n}=\frac{v L}{c_{0} r^{2}} \tag{2.62}
\end{equation*}
$$

The pressure waves in a fluid transmission line propagate at the speed of sound in the line, $c_{0}$, making the dissipative number a function of this value. The speed of sound is a function of the fluid density and the compressibility (inverse of bulk modulus) of the system.

The characteristic frequency is represented by the following relationship.

$$
\begin{equation*}
\omega_{c}=\frac{c_{0}}{L} \tag{2.63}
\end{equation*}
$$

The dissipative transmission line model approach is referred to as viscous theory and has been proven to be the most accurate model.


Figure 2.6 Frequency magnitude response of a blocked hydraulic line using the dissipative model


Figure 2.7 Frequency phase response of a blocked hydraulic line using the dissipative model

### 2.5 Transmission Line Model Comparison

This section applies each of the five models covered in the last section to a blocked fluid transmission line and compares the resulting frequency response plots.


Figure 2.8 Blocked transmission line illustration

## Fluid Properties

Density $=870 \mathrm{Kg} / \mathrm{m}^{3}$
Kinematic Viscosity $=4.6 \mathrm{e}-5 \mathrm{~m}^{2} / \mathrm{s}$
Prandtl Number $=1$
Specific Heat Ratio $=1$
Bulk Modulus $=1.21 \mathrm{e} 9 \mathrm{~N} / \mathrm{m}^{2}$

## Line Properties

Length $=2 \mathrm{~m}$
Diameter $=0.01 \mathrm{~m}$
Bulk Modulus $=1.73 \mathrm{e} 7 \mathrm{~N} / \mathrm{m}^{2}$

### 2.5.1 Lumped Parameter Model

The lumped parameter approach requires the computation of values for the line inertance and capacitance.

$$
\begin{gather*}
P_{\text {in }}-P_{\text {out }}=(R+L s) Q_{\text {in }}  \tag{2.64}\\
Q_{\text {in }}=C_{p} s P_{\text {out }}  \tag{2.65}\\
P_{\text {in }}-P_{\text {out }}=(R+L s) C_{p} s P_{\text {out }}  \tag{2.66}\\
P_{\text {in }}=\left(C_{p} L s^{2}+C_{p} R s+1\right) P_{o u t}  \tag{2.67}\\
P_{\text {out }}=\frac{1}{\left(C_{p} L s^{2}+C_{p} R s+1\right)} P_{\text {in }} \tag{2.68}
\end{gather*}
$$

$$
\begin{gather*}
R=\frac{128 v \rho \ell}{\pi d^{4}}  \tag{2.69}\\
R=\frac{128(4.6 e-5)(870)(2)}{\pi(0.01)^{4}}=3.261 \mathrm{e} 8 \quad \frac{\mathrm{~N}-\mathrm{s}}{\mathrm{~m}^{5}}  \tag{2.70}\\
L=\frac{\rho \ell}{A}  \tag{2.71}\\
L=\frac{(870)(2)}{\pi(0.01 / 2)^{2}}=2.215 \mathrm{e} 7 \frac{\mathrm{~N}-\mathrm{s}^{2}}{m^{5}}  \tag{2.72}\\
C_{p}=\frac{V}{\beta_{e}} \tag{2.73}
\end{gather*}
$$

The effective bulk modulus of the system takes into account the compressibility of both the hydraulic fluid and the hydraulic hose. The tangent bulk modulus is measured at a specific point and pressure whereas the secant bulk modulus is average change in pressure and volume. The secant bulk modulus is used in all computations.

$$
\begin{gather*}
\beta_{T}=-V_{0}\left[\frac{\partial P}{\partial V}\right]_{T, E}  \tag{2.74}\\
\beta_{s}=-V\left[\frac{\Delta P}{\Delta V}\right]_{T, E}  \tag{2.75}\\
\frac{1}{\beta_{s}}=-\frac{1}{V}\left[\frac{\Delta V}{\Delta P}\right]_{T, E}  \tag{2.76}\\
\frac{1}{\beta_{s}}=\frac{V_{\text {fluid }}}{V}\left[-\frac{\Delta V_{\text {fluid }}}{V_{\text {fluid }} \Delta P}\right]_{T, E}+\frac{V_{\text {hose }}}{V}\left[\frac{\Delta V_{\text {hose }}}{V_{\text {hose }} \Delta P}\right]_{T, E} \tag{2.77}
\end{gather*}
$$

In this system the liquid is compressing and the hose is expanding which explains the assignment of the positive and negative signs in the equation. The volume of the system is also the volume of the hose and the equation can be rewritten as follows.

$$
\begin{gather*}
\frac{1}{\beta_{s}}=\frac{V_{\text {fluid }}}{V}\left[-\frac{\Delta V_{\text {fluid }}}{V_{\text {fluid }} \Delta P}\right]_{T, E}+\left[\frac{\Delta V_{\text {hose }}}{V \Delta P}\right]_{T, E}  \tag{2.78}\\
\frac{1}{\beta_{s}}=\frac{V_{\text {fluid }}}{V}\left[\frac{1}{\beta_{\text {fluid }}}\right]_{T, E}+\left[\frac{1}{\beta_{\text {hose }}}\right]_{T, E} \tag{2.79}
\end{gather*}
$$

Assuming that the volume of the fluid is equal to the total volume gives

$$
\begin{gather*}
\frac{1}{\beta_{s}}=\left[\frac{1}{\beta_{\text {fluid }}}\right]_{T, E}+\left[\frac{1}{\beta_{\text {hose }}}\right]_{T, E}  \tag{2.80}\\
\beta_{e}=\beta_{s}=\frac{\beta_{\text {fluid }} \beta_{\text {hose }}}{\beta_{\text {fluid }}+\beta_{\text {hose }}}  \tag{2.81}\\
\beta_{e}=\frac{(1.21 \mathrm{e} 9)(1.73 \mathrm{e} 7)}{1.21 \mathrm{e} 9+1.73 \mathrm{e} 7}=1.706 \mathrm{e} 7 \frac{\mathrm{~N}}{\mathrm{~m}^{2}}  \tag{2.82}\\
C_{p}=\frac{V}{\beta_{e}}=\frac{\pi(0.01 / 2)^{2}(2)}{1.71 \mathrm{e} 7}=9.21 \mathrm{e}-12 \frac{\mathrm{~m}^{5}}{\mathrm{~N}} \tag{2.83}
\end{gather*}
$$

The transfer function for the lumped parameter model is:

$$
\begin{equation*}
P_{\text {out }}=\frac{1}{\left(0.20403 \mathrm{e}-3 s^{2}+0.3003 e-3 s+1\right)} P_{\text {in }} \tag{2.84}
\end{equation*}
$$

### 2.5.2 Distributed Parameter Models

The distributed parameter matrix transfer function for this causality reduces to

$$
\begin{equation*}
P_{\text {out }}=\frac{P_{i n}}{\cosh \Gamma} \tag{2.85}
\end{equation*}
$$

This transfer function will be applied to the next four models. The only difference is in the calculation of the propagation operator. The impedance constant is not required in this causality since the solution is the pressure output and not fluid flow.

### 2.5.2.1 Lossless Line Model

$$
\begin{gather*}
\Gamma(s)=\frac{L s}{c_{0}}  \tag{2.86}\\
c_{0}=\sqrt{\frac{\beta_{e}}{\rho}}  \tag{2.87}\\
c_{0}=\sqrt{\frac{1.71 \mathrm{e} 7}{870}}=140.2 \mathrm{~m} / \mathrm{s}  \tag{2.88}\\
\Gamma(s)=\frac{2 \mathrm{~s}}{140.2}=.01428 \mathrm{~s} \tag{2.89}
\end{gather*}
$$

2.5.2.2 Linear Friction Model

$$
\begin{gather*}
\Gamma(s)=\frac{L s}{c_{0}} \sqrt{1+\frac{8 \nu_{0}}{s r_{0}^{2}}}  \tag{2.90}\\
\Gamma(s)=\frac{2 s}{140.2} \sqrt{1+\frac{8(4.6 e-5)}{s(0.005)^{2}}}=0.0143 s \sqrt{1+\frac{14.72}{s}} \tag{2.91}
\end{gather*}
$$

### 2.5.2.3 Viscous Model

$$
\begin{equation*}
\Gamma(s)=\frac{s L}{c \sqrt{1-\frac{2 J_{1}\left(j \sqrt{\frac{r^{2} s}{v}}\right)}{j \sqrt{\frac{r^{2} s}{v} J_{0}\left(j \sqrt{\frac{r^{2} s}{v}}\right)}}}} \tag{2.92}
\end{equation*}
$$

### 2.5.2.4 Dissipative Model

$$
\begin{equation*}
\Gamma(s)=\frac{s}{\omega_{c}} \sqrt{\frac{1}{2 J_{1}\left(j \sqrt{\frac{s}{\omega_{v}}}\right)}} \tag{2.93}
\end{equation*}
$$

The viscous model and the dissipative model for the line have the same resulting propagation operator as well as the same transfer function due to the fact that the specific heat ratio of liquid is unity.


Figure 2.9 Frequency magnitude response of common fluid transmission line models

## CHAPTER 3

## MODAL APPROXIMATION APPROACH

### 3.1 Model Overview

As a result of being derived from partial differential equations the dissipative model transfer function is not in the rational polynomial form familiar in system modeling and control theory.

$$
\left[\begin{array}{c}
P_{\text {out }}  \tag{3.1}\\
Q_{\text {in }}
\end{array}\right]=\left[\begin{array}{cc}
\frac{1}{\cosh \Gamma} & -\frac{Z_{c} \sinh \Gamma}{\cosh \Gamma} \\
\frac{\sinh \Gamma}{Z_{c} \cosh \Gamma} & \frac{1}{\cosh \Gamma}
\end{array}\right]\left[\begin{array}{l}
P_{\text {in }} \\
Q_{\text {out }}
\end{array}\right]
$$

Fluid line systems actually contain several components in addition to the fluid line or lines.


Figure 3.1 Hydraulic log splitter

In order to integrate this model with other lumped parameters in a total system it is necessary to approximate the resulting transfer function as a rational polynomial transfer function. This effort has been the focus of much of the latest research in this field.

There are four possible causalities to a fluid line problem as follows:

$$
\begin{align*}
& {\left[\begin{array}{l}
Q_{\text {in }} \\
Q_{\text {out }}
\end{array}\right]=\left[\begin{array}{cc}
\frac{\cosh \Gamma}{Z_{c} \sinh \Gamma} & -\frac{1}{Z_{c} \sinh \Gamma} \\
\frac{1}{Z_{c} \sinh \Gamma} & -\frac{\cosh \Gamma}{Z_{c} \sinh \Gamma}
\end{array}\right]\left[\begin{array}{l}
P_{\text {in }} \\
P_{\text {out }}
\end{array}\right]}  \tag{3.2}\\
& {\left[\begin{array}{c}
P_{\text {in }} \\
P_{\text {out }}
\end{array}\right]=\left[\begin{array}{cc}
\frac{Z_{c} \cosh \Gamma}{\sinh \Gamma} & -\frac{Z_{c}}{\sinh \Gamma} \\
\frac{Z_{c}}{\sinh \Gamma} & -\frac{Z_{c} \cosh \Gamma}{\sinh \Gamma}
\end{array}\right]\left[\begin{array}{l}
Q_{\text {in }} \\
Q_{\text {out }}
\end{array}\right]}  \tag{3.3}\\
& {\left[\begin{array}{l}
P_{\text {in }} \\
Q_{\text {out }}
\end{array}\right]=\left[\begin{array}{cc}
\frac{1}{\cosh \Gamma} & \frac{Z_{c} \sinh \Gamma}{\sinh \Gamma} \\
-\frac{\sinh \Gamma}{Z_{c} \cosh \Gamma} & \frac{1}{\cosh \Gamma}
\end{array}\right]\left[\begin{array}{l}
P_{\text {out }} \\
Q_{\text {in }}
\end{array}\right]}  \tag{3.4}\\
& {\left[\begin{array}{l}
P_{\text {out }} \\
Q_{\text {in }}
\end{array}\right]=\left[\begin{array}{ll}
\frac{1}{\cosh \Gamma} & -\frac{Z_{c} \sinh \Gamma}{\cosh \Gamma} \\
\frac{\sinh \Gamma}{Z_{c} \cosh \Gamma} & \frac{1}{\cosh \Gamma}
\end{array}\right]\left[\begin{array}{l}
P_{\text {in }} \\
Q_{\text {out }}
\end{array}\right]} \tag{3.5}
\end{align*}
$$

In these four equations are seven unique transfer functions that will be defined as follows:

$$
\begin{align*}
& C_{1}=\frac{\cosh \Gamma}{Z_{c} \sinh \Gamma}  \tag{3.6}\\
& C_{2}=\frac{1}{Z_{c} \sinh \Gamma}  \tag{3.7}\\
& C_{3}=\frac{\sinh \Gamma}{Z_{c} \cosh \Gamma}  \tag{3.8}\\
& C_{4}=\frac{1}{\cosh \Gamma}  \tag{3.9}\\
& C_{5}=\frac{Z_{c} \cosh \Gamma}{\sinh \Gamma}  \tag{3.10}\\
& C_{6}=\frac{Z_{c}}{\sinh \Gamma}  \tag{3.11}\\
& C_{7}=\frac{Z_{c} \sinh \Gamma}{\cosh \Gamma} \tag{3.12}
\end{align*}
$$

$$
\begin{align*}
& \text { 3.2 Model Derivation } \\
& \cosh \Gamma=\prod_{i=1}^{\infty}\left\{1+\frac{\Gamma^{2}}{\pi^{2}\left(i-\frac{1}{2}\right)^{2}}\right\}  \tag{3.13}\\
& \sinh \Gamma=\Gamma\left[\prod_{i=1}^{\infty}\left\{1+\frac{\Gamma^{2}}{\pi^{2} \mathrm{i}^{2}}\right\}\right] \tag{3.14}
\end{align*}
$$

$$
\begin{align*}
& B_{r \sigma}=\prod_{i=1}^{\infty}\left\{\frac{1+\frac{\sigma \bar{s}}{\alpha_{1, i}^{2}}}{1+\frac{\sigma \bar{s}}{\alpha_{0, i}^{2}}}\right\}  \tag{3.15}\\
& B_{r}=\prod_{i=1}^{\infty}\left\{\frac{1+\frac{\bar{s}}{\alpha_{1, i}^{2}}}{1+\frac{\bar{s}}{\alpha_{0, i}^{2}}}\right\} \tag{3.16}
\end{align*}
$$

Where:
$\sigma$ is the Prandtl number
$\alpha_{0, \mathrm{i}}$ is the $\mathrm{i}^{\text {th }}$ zero of the zero-order Bessel function
$\alpha_{1, \mathrm{i}}$ is the $\mathrm{i}^{\text {th }}$ zero of the first-order Bessel function

The infinite product representation of the propagation operator is

$$
\begin{equation*}
\Gamma(\bar{s})=D_{n} \bar{s} \sqrt{\frac{1+(\gamma-1) \prod_{i=1}^{\infty}\left\{\frac{1+\frac{\sigma \bar{s}}{\alpha_{i, i}^{2}}}{1+\frac{\sigma \bar{s}}{\alpha_{0, i}^{2}}}\right\}}{1-\prod_{i=1}^{\infty}\left\{\frac{\left.1+\frac{\alpha_{i, i}^{2}}{1+\frac{\bar{s}}{\alpha_{0, i}^{2}}}\right\}}{}\right.} \sqrt{1}} \tag{3.17}
\end{equation*}
$$

Multiply both the numerator and denominator terms by $\prod_{i=1}^{\infty}\left[1+\frac{\bar{s}}{\alpha_{0, i}^{2}}\right]$ and by $\prod_{i=1}^{\infty}\left[1+\frac{\sigma \bar{s}}{\alpha_{0, i}^{2}}\right]$ appropriately and expanding to get the following equation:
$\Gamma(\bar{s})=D_{n} \bar{s} \sqrt{\left.\left[\begin{array}{l}\left.\left.\left(1+\frac{\sigma \bar{s}}{\alpha_{0,1}^{2}}\right)\left(1+\frac{\sigma \bar{s}}{\alpha_{0,2}^{2}}\right) \ldots+(\gamma-1)\left(1+\frac{\sigma \bar{s}}{\alpha_{1,1}^{2}}\right)\left(1+\frac{\sigma \bar{s}}{\alpha_{1,2}^{2}}\right) \ldots\right]\left(1+\frac{\bar{s}}{\alpha_{0,1}^{2}}\right)\left(1+\frac{\bar{s}}{\alpha_{0,2}^{2}}\right) \ldots\right] \\ {\left[\left(1+\frac{\bar{s}}{\alpha_{0,1}^{2}}\right)\right.}\end{array}\right)\left(1+\frac{\bar{s}}{\alpha_{0,2}^{2}}\right) \ldots-\left(1+\frac{\bar{s}}{\alpha_{1,1}^{2}}\right)\left(1+\frac{\bar{s}}{\alpha_{1,2}^{2}}\right) \cdots\right]\left[\left(1+\frac{\sigma \bar{s}}{\alpha_{0,1}^{2}}\right)\left(1+\frac{\sigma \bar{s}}{\alpha_{1,2}^{2}}\right) \ldots\right]}$

This can be reduced to:

$$
\begin{equation*}
\Gamma(\bar{s})=D_{n} \bar{s} \sqrt{\left[\frac{\gamma+a_{1} \bar{s}+a_{2} \bar{s}^{2}+\cdots+a_{2 m} \bar{s}^{2 m}}{b_{1} \bar{s}+b_{2} \bar{s}^{2}+\cdots+b_{2 m} \bar{s}^{2 m}}\right]} \tag{3.19}
\end{equation*}
$$

The polynomial coefficients $a_{i}$ and $b_{i}$ are functions of the Prandtl number and $m$ only.

### 3.2.1 Modal Approximation of $1 / \cosh \Gamma$

This result is applied to the series equation for $\cosh \Gamma$.

$$
\begin{equation*}
\cosh \Gamma=\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}}{\pi^{2}\left(i-\frac{1}{2}\right)^{2}}\left[\frac{\gamma+a_{1} \bar{s}+a_{2} \bar{s}^{2}+\cdots+a_{2 m} \bar{s}^{2 m}}{b_{1} \bar{s}+b_{2} \bar{s}^{2}+\cdots+b_{2 m} \bar{s}^{2 m}}\right]\right\} \tag{3.20}
\end{equation*}
$$

A new variable called the dimensionless root index for $\cosh \Gamma$ is introduced:

$$
\begin{equation*}
\lambda_{c}=\frac{1}{D_{n}}\left(i-\frac{1}{2}\right) \quad \mathrm{i}=1,2,3 \ldots \ldots \ldots \tag{3.21}
\end{equation*}
$$

$$
\begin{equation*}
\cosh \Gamma=\prod_{i=1}^{\infty}\left[1+\frac{\bar{s}^{2}}{\pi^{2} \lambda_{c}^{2}}\left[\frac{\gamma+a_{1} \bar{s}+a_{2} \bar{s}^{2}+\cdots+a_{2 m} \bar{s}^{2 m}}{b_{1} \bar{s}+b_{2} \bar{s}^{2}+\cdots+b_{2 m} \bar{s}^{2 m}}\right]\right] \tag{3.22}
\end{equation*}
$$

The goal is to simplify the equation as a rational polynomial first by combining the unity term and series term as follows.

$$
\begin{equation*}
\cosh \Gamma=\prod_{i=1}^{\infty}\left\{\frac{\left(b_{1} \bar{s}+b_{2} \bar{s}^{2}+\cdots+b_{2 m} \bar{s}^{2 m-1}\right)+\frac{\bar{s}^{2}}{\pi^{2} \lambda_{c}^{2}}\left(\gamma+a_{1} \bar{s}+a_{2} \bar{s}^{2}+\cdots+a_{2 m} \bar{s}^{2 m}\right)}{\left(b_{1} \bar{s}+b_{2} \bar{s}^{2}+\cdots+b_{2 m} \bar{s}^{2 m-1}\right)}\right\} \tag{3.23}
\end{equation*}
$$

Then divide both the numerator and denominator by $\bar{s}$ to get:

$$
\begin{equation*}
\cosh \Gamma=\prod_{i=1}^{\infty}\left\{\frac{\left(b_{1}+b_{2} \bar{s}+\cdots+b_{2 m} \bar{s}^{2 m-1}\right)+\frac{\bar{s}}{\pi^{2} \lambda_{c}^{2}}\left(\gamma+a_{1} \bar{s}+a_{2} \bar{s}^{2}+\cdots+a_{2 m} \bar{s}^{2 m}\right)}{\left(b_{1}+b_{2} \bar{s}+\cdots+b_{2 m} \bar{s}^{2 m-1}\right)}\right\} \tag{3.24}
\end{equation*}
$$

Combining terms in the numerator results in:

$$
\begin{equation*}
\cosh \Gamma=\prod_{i=1}^{\infty}\left[\frac{\left(b_{1}+c_{2} \bar{s}+c_{3} \bar{s}^{2}+\cdots+c_{2 m+2} \bar{s}^{2 m+1}\right)}{\left(b_{1} \bar{s}+b_{2} \bar{s}^{2}+\cdots+b_{2 m} \bar{s}^{2 m-1}\right)}\right] \tag{3.25}
\end{equation*}
$$

The polynomial coefficients $b_{i}$ are functions of the Prandtl number and $m$ only whereas the polynomial coefficients $c_{i}$ are functions of the Prandtl number, $m$, the specific heat ratio, and the dimensionless root index.

Factoring the numerator and denominator results in the following equation:

$$
\begin{equation*}
\cosh \Gamma=\prod_{i=1}^{\infty}\left[\frac{\left.\left(\bar{s}+z_{1}\right)\left(\bar{s}+z_{2}\right) \ldots . .\left(\bar{s}^{2}+2 \zeta \bar{\omega}_{i} \bar{s}+\bar{\omega}_{i j}\right)^{2}\right)}{\left(\bar{s}+\bar{p}_{1}\right)\left(\bar{s}+\bar{p}_{2}\right) \ldots . .\left(\bar{s}+\bar{p}_{2 m-1}\right)}\right] \tag{3.26}
\end{equation*}
$$

The real root terms in both the numerator and the denominator virtually cancel each other out for lines with low damping and cosh $\Gamma$ can be approximated as:

$$
\begin{equation*}
\cosh \Gamma=\prod_{i=1}^{\infty}\left[\frac{\bar{s}^{2}+2 \zeta_{i} \bar{\omega}_{n i} \bar{s}+\bar{\omega}_{n i}{ }^{2}}{\bar{a}_{c i} \bar{s}+\bar{b}_{c i}}\right] \tag{3.27}
\end{equation*}
$$

The transfer function $1 / \cosh \Gamma$ is:

$$
\begin{equation*}
\frac{1}{\cosh \Gamma}=\sum_{i=1}^{n} \frac{\bar{a}_{c i} \bar{s}+\bar{b}_{c i}}{\bar{S}^{2}+2 \varsigma_{i} \bar{\omega}_{n_{i}} \bar{s}+\bar{\omega}_{n_{i}}{ }^{2}} \tag{3.28}
\end{equation*}
$$

$n=$ number of second order modes

The values for the coefficients have been tabulated.

### 3.2.2 Modal Approximation of $Z_{c} \sinh \Gamma / \cosh \Gamma$

Substitute the infinite product series form for $\sinh \Gamma$ and $\cosh \Gamma$.

$$
\begin{equation*}
\frac{Z_{c} \sinh \Gamma}{\cosh \Gamma}=Z_{c} \frac{\Gamma\left[\prod_{i=1}^{\infty}\left\{1+\frac{\Gamma^{2}}{\pi^{2} i^{2}}\right\}\right]}{\prod_{i=1}^{\infty}\left\{1+\frac{\Gamma^{2}}{\pi^{2}\left(i-\frac{1}{2}\right)^{2}}\right\}} \tag{3.29}
\end{equation*}
$$

Substitute in the equation for the propagation operator.

$$
\begin{equation*}
\frac{Z_{c} \sinh \Gamma}{\cosh \Gamma}=Z_{c} D_{n} \bar{s} \sqrt{\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}} \frac{\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}}{\pi^{2} i^{2}}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]\right\}}{\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}}{\pi^{2}\left(i-\frac{1}{2}\right)^{2}}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]\right\}} \tag{3.30}
\end{equation*}
$$

Substitute the equation for the characteristic impedance.

$$
\begin{equation*}
\frac{Z_{c} \sinh \Gamma}{\cosh \Gamma}=\frac{Z_{0}}{\sqrt{\left(1-B_{r}\right)\left(1+(\gamma-1) B_{r \sigma}\right)}} D_{n} \bar{s} \sqrt{\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}} \frac{\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}}{\pi^{2} i^{2}}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]\right\}}{\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}}{\pi^{2}\left(i-\frac{1}{2}\right)^{2}}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]\right\}} \tag{3.31}
\end{equation*}
$$

The matrix transfer function when modal approximation method is applied is in the following form to remove the $\mathrm{Z}_{0}$ term in the development of the modal approximation model of the individual terms.

$$
\begin{gather*}
{\left[\begin{array}{c}
P_{\text {out }} \\
Z_{0} Q_{\text {in }}
\end{array}\right]=\left[\begin{array}{cc}
\frac{1}{\cosh \Gamma} & -\frac{Z_{c} \sinh \Gamma}{Z_{0} \cosh \Gamma} \\
\frac{\sinh \Gamma}{Z_{c} \cosh \Gamma} & \frac{1}{\cosh \Gamma}
\end{array}\right]\left[\begin{array}{l}
P_{\text {in }} \\
Z_{0} Q_{\text {out }}
\end{array}\right]}  \tag{3.32}\\
\frac{Z_{c} \sinh \Gamma}{Z_{0} \cosh \Gamma}=\frac{1}{\sqrt{\left(1-B_{r}\right)\left(1+(\gamma-1) B_{r \sigma}\right)}} D_{n} \bar{s} \sqrt{\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}} \frac{\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}}{\pi^{2} i^{2}}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]\right\}}{\left\{\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}}{\pi^{2}\left(i-\frac{1}{2}\right)^{2}}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]\right\}\right.} \tag{3.33}
\end{gather*}
$$

Reducing Bessel function terms gives

$$
\begin{equation*}
\frac{Z_{c} \sinh \Gamma}{Z_{0} \cosh \Gamma}=\frac{D_{n} \bar{s}}{\left(1-B_{r}\right)} \frac{\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}}{\pi^{2} i^{2}}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]\right\}}{\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}}{\pi^{2}\left(i-\frac{1}{2}\right)^{2}}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]\right\}} \tag{3.34}
\end{equation*}
$$

Substitute the infinite product series form of the Bessel functions produces:

Multiply both the numerator and denominator terms by $\prod_{i=1}^{\infty}\left[1+\frac{\bar{s}}{\alpha_{0, i}^{2}}\right]$ and by $\prod_{i=1}^{\infty}\left[1+\frac{\sigma \bar{s}}{\alpha_{0, i}^{2}}\right]$ appropriately to get the following equation:


Another dimensionless root index for the sinh $\Gamma$ polynomial is introduced.

$$
\begin{equation*}
\lambda_{s}=\frac{i}{D_{n}} \tag{3.37}
\end{equation*}
$$

Substitute the dimensionless root indices of both $\cosh \Gamma$ and $\sinh \Gamma$ into the equation.


This equation can be simplified into the following form.

$$
\begin{equation*}
\frac{Z_{c} \sinh \Gamma}{Z_{0} \cosh \Gamma}=\left[\frac{\left(\bar{s}+\bar{u}_{1}\right)\left(\bar{s}+\bar{u}_{2}\right) \ldots .\left(\bar{s}+\bar{u}_{m}\right)}{\left.\left(\bar{s}_{1}\right) \bar{l}_{1}\right)\left(\bar{s}+\bar{l}_{2}\right) \ldots .\left(\bar{s}+\bar{l}_{m}\right)} \frac{\prod_{i=1}^{\infty}\left[\left(\bar{s}+\bar{r}_{1}\right)\left(\bar{s}+\bar{r}_{2}\right) \ldots .\left(\bar{s}+\bar{r}_{2 m-1}\right)\left(\bar{s}^{2}+2 \zeta_{s i} \omega_{v s i} \bar{s}+\omega_{v s i}^{2}\right)\right]}{\prod_{i=1}^{\infty}\left[\left(\bar{s}+\bar{p}_{1}\right)\left(\bar{s}+\bar{p}_{2}\right) \ldots . .\left(\bar{s}+\bar{p}_{2 m-1}\right)\left(\bar{s}^{2}+2 \zeta_{i} \omega_{v i} \bar{s}+\omega_{v i}^{2}\right)\right]}\right. \tag{3.39}
\end{equation*}
$$

The real root terms in both the numerator and the denominator do not necessarily cancel each other out for lines with low damping as in the case of $\cosh \Gamma$.

If $\frac{Z_{c} \sinh \Gamma}{Z_{0} \cosh \Gamma}$ is divided by the first real root $\left(\bar{s}+\bar{u}_{1}\right)$, the residues of the resulting real roots approach zero.

$$
\begin{equation*}
\frac{Z_{c} \sinh \Gamma}{Z_{0} \cosh \Gamma}=(\bar{s}+\bar{u}) \sum_{1}^{n} \frac{\bar{a}_{z i} \bar{s}+\bar{b}_{z i}}{\bar{s}^{2}+2 \zeta_{i} \bar{\omega}_{v i} \bar{s}+\omega_{v i}^{2}} \tag{3.40}
\end{equation*}
$$

$n=$ number of second order modes
The values for the coefficients have been tabulated.

### 3.2.3 Modal Approximation of $\sinh \Gamma / Z_{c} \cosh \Gamma$

As stated earlier, the form of the transfer function when the Modal Approximation method is applied is in the following form to remove the $\mathrm{Z}_{0}$ term in the development of the modal approximation model of the individual terms.

$$
\begin{equation*}
\frac{Z_{0} \sinh \Gamma}{Z_{c} \cosh \Gamma} \tag{3.41}
\end{equation*}
$$

Substitute the infinite product series form for $\sinh \Gamma$ and $\cosh \Gamma$.

$$
\begin{equation*}
\frac{Z_{0} \sinh \Gamma}{Z_{c} \cosh \Gamma}=\frac{Z_{0} \Gamma\left[\prod_{i=1}^{\infty}\left\{1+\frac{\Gamma^{2}}{\pi^{2} \mathrm{i}^{2}}\right\}\right]}{Z_{c} \prod_{i=1}^{\infty}\left\{1+\frac{\Gamma^{2}}{\pi^{2}\left(i-\frac{1}{2}\right)^{2}}\right\}} \tag{3.42}
\end{equation*}
$$

Substitute in the equation for the propagation operator.

$$
\begin{equation*}
\frac{Z_{0} \sinh \Gamma}{Z_{c} \cosh \Gamma}=\frac{Z_{0} D_{n} \bar{s} \sqrt{\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}}\left[\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]}{\pi^{2} \mathrm{i}^{2}}\right\}\right]}{Z_{c} \prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]}{\pi^{2}\left(i-\frac{1}{2}\right)^{2}}\right\}} \tag{3.43}
\end{equation*}
$$

Substitute in the equation for the characteristic impedance.

$$
\begin{equation*}
\frac{Z_{0} \sinh \Gamma}{Z_{c} \cosh \Gamma}=\frac{Z_{0} D_{n} \bar{s} \sqrt{\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}}\left[\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]}{\pi^{2} \mathrm{i}^{2}}\right\}\right]}{\frac{Z_{0}}{\left.\sqrt{\left(1-B_{r}\right)\left(1+(\gamma-1) B_{r \sigma}\right.}\right)} \prod_{=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]}{\pi^{2}\left(i-\frac{1}{2}\right)^{2}}\right\}} \tag{3.44}
\end{equation*}
$$

Reducing Bessel function terms and canceling out the impedance constant gives

$$
\begin{equation*}
\frac{Z_{0} \sinh \Gamma}{Z_{c} \cosh \Gamma}=\left[D_{n} \bar{s}\left(1+(\gamma-1) B_{r \sigma}\right)\right] \frac{\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]}{\pi^{2} \mathrm{i}^{2}}\right\}}{\prod_{i=1}^{\infty}\left\{1+\frac{D_{n}^{2} \bar{s}^{2}\left[\frac{1+(\gamma-1) B_{r \sigma}}{1-B_{r}}\right]}{\pi^{2}\left(i-\frac{1}{2}\right)^{2}}\right\}} \tag{3.45}
\end{equation*}
$$

Substitute the infinite product series form of the Bessel functions produces


Combine the following term as one quotient.

$$
\begin{equation*}
\left[1+(\gamma-1) \prod_{i=1}^{\infty}\left\{\frac{1+\frac{\sigma \bar{s}}{\alpha_{1, i}^{2}}}{1+\frac{\sigma \bar{s}}{\alpha_{0, i}^{2}}}\right\}\right] \tag{3.47}
\end{equation*}
$$

To get the following:

$$
\begin{equation*}
\left[\frac{\prod_{i=1}^{\infty}\left\{1+\frac{\sigma \bar{s}}{\alpha_{0, i}^{2}}\right\}+(\gamma-1) \prod_{i=1}^{\infty}\left\{1+\frac{\sigma \bar{s}}{\alpha_{1, i}^{2}}\right\}}{\prod_{i=1}^{\infty}\left\{1+\frac{\sigma \bar{s}}{\alpha_{0, i}^{2}}\right\}}\right] \tag{3.48}
\end{equation*}
$$

Substitute into the equation to get


Combine the following term as one quotient

$$
\begin{equation*}
1-\prod_{i=1}^{\infty}\left\{\frac{1+\frac{\bar{s}}{\alpha_{1, i}^{2}}}{1+\frac{\bar{s}}{\alpha_{0, i}^{2}}}\right\} \tag{3.50}
\end{equation*}
$$

To get the following

$$
\begin{equation*}
\left[\frac{\prod_{i=1}^{\infty}\left\{1+\frac{\bar{s}}{\alpha_{0, i}^{2}}\right\}+\prod_{i=1}^{\infty}\left\{1+\frac{\bar{s}}{\alpha_{1, i}^{2}}\right\}}{\prod_{i=1}^{\infty}\left\{1+\frac{\bar{s}}{\alpha_{0, i}^{2}}\right\}}\right] \tag{3.51}
\end{equation*}
$$

Substitute into the equation to get


Factor out the following term.

$$
\begin{equation*}
\frac{\prod_{i=1}^{\infty}\left\{1+\frac{\bar{s}}{\alpha_{0, i}^{2}}\right\}}{\prod_{i=1}^{\infty}\left\{1+\frac{\sigma \bar{s}}{\alpha_{0, i}^{2}}\right\}} \tag{3.53}
\end{equation*}
$$

The result is the following equation.

Introduce the dimensionless root indices to get the following equation.

This equation can be simplified into the following form.

$$
\begin{equation*}
\frac{Z_{c} \sinh \Gamma}{Z_{0} \cosh \Gamma}=D_{n} \bar{s}\left[\frac{\left(\bar{s}+\bar{a}_{1}\right)\left(\bar{s}+\bar{a}_{2}\right) \ldots .\left(\bar{s}+\bar{a}_{m}\right)}{\left(\bar{s}+\bar{b}_{1}\right)\left(\bar{s}+\bar{b}_{2}\right) \ldots .\left(\bar{s}+\bar{b}_{m}\right)}\right] \frac{\prod_{i=1}^{\infty}\left[\left(\bar{s}+\bar{r}_{1}\right)\left(\bar{s}+\bar{r}_{2}\right) \ldots . .\left(\bar{s}+\bar{r}_{2 m-1}\right)\left(\bar{s}^{2}+2 \zeta_{s i} \omega_{v s i} \bar{s}+\omega_{v s i}^{2}\right)\right]}{\left.\left.\prod_{i=1}+\bar{p}_{1}\right)\left(\bar{s}+\bar{p}_{2}\right) \ldots .\left(\bar{s}+\bar{p}_{2 m-1}\right)\left(\bar{s}^{2}+2 \zeta_{i} \omega_{v i} \bar{s}+\omega_{v i}^{2}\right)\right]} \tag{3.56}
\end{equation*}
$$

The real root terms in both the numerator and the denominator do not necessarily cancel each other out for lines with low damping as in the case of $\cosh \Gamma$. If $\frac{Z_{c} \sinh \Gamma}{Z_{0} \cosh \Gamma}$ is divided by the Laplace operator, the residues of the resulting real roots approach zero.

$$
\begin{equation*}
\frac{Z_{c} \sinh \Gamma}{\bar{s} Z_{0} \cosh \Gamma}=\sum_{i=1}^{n}\left[\frac{\bar{a}_{s i} \bar{s}+\bar{b}_{s i}}{\bar{s}^{2}+2 \zeta_{i} \bar{\omega}_{v i} \bar{s}+\omega_{v i}^{2}}+\frac{\bar{k}_{1}}{\left(\bar{s}+\bar{p}_{1}\right)}+\frac{\bar{k}_{2}}{\left(\bar{s}+\bar{p}_{2}\right)}+\ldots\right] \tag{3.57}
\end{equation*}
$$

The inclusion of one real pole gives a very accurate approximation.

$$
\begin{equation*}
\frac{Z_{c} \sinh \Gamma}{\bar{s} Z_{0} \cosh \Gamma}=\frac{\bar{k}_{0}}{\left(\bar{s}+\bar{p}_{0}\right)}+\sum_{i=1}^{n}\left[\frac{\bar{a}_{s i} \bar{s}+\bar{b}_{s i}}{\bar{s}^{2}+2 \zeta_{i} \bar{\omega}_{v i} \bar{s}+\omega_{v i}^{2}}\right] \tag{3.58}
\end{equation*}
$$

### 3.3 Modal Approximation Residue Coefficient Tables

Table 3.1 Residue Coefficients for Air (Pneumatic Transmission Lines)

| $\begin{array}{\|cc\|} \hline \lambda_{\mathrm{c}} \\ & \\ & \\ & \lambda_{\mathrm{s}} \\ \hline \end{array}$ | $\omega_{\mathrm{n}}$ | $\zeta$ | $1 / \cosh \Gamma$ |  | $\mathrm{Z}_{\mathrm{d}} / \sinh \Gamma$ |  | $1 / \mathrm{Z}_{\mathrm{c}} \sinh \Gamma$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  | (-1) ${ }^{1} \mathrm{D}_{\mathrm{n}} \mathrm{Z}_{i} / Z_{0}$ | (-1)' $\mathrm{D}_{\mathrm{nb}} \mathrm{b}_{\mathrm{i}} / \mathrm{Z}_{0}$ | $(-1)^{1} \mathrm{Z}_{0} \mathrm{D}_{\mathrm{n}} \mathrm{a}_{\mathrm{i}} \mid$ | $(-1)^{\prime} \mathrm{Z}_{0} \mathrm{D}_{\mathrm{n}} \mathrm{b}_{\mathrm{i}}$ |
|  |  |  | $(-1)^{i+1}(1-2 i) a_{i}$ | $(-1)^{i+1}(1-2 i) b_{i}$ | $\mathrm{Z}_{\mathrm{c}} \cosh \Gamma / \sinh \Gamma$ <br> $Z_{\mathrm{c}} \sinh \Gamma / \cosh \Gamma$ |  | $\cosh \Gamma / Z_{c} \sinh \Gamma$ <br> $\sinh \Gamma / Z_{c} \cosh \Gamma$ |  |
|  |  |  |  |  | $\mathrm{D}_{\mathrm{n}} \mathrm{a}_{\mathrm{i}} / \mathrm{Z}_{0}$ | $\mathrm{D}_{\mathrm{n}} \mathrm{b}_{\mathrm{i}} / \mathrm{Z}_{0}$ | $\mathrm{Z}_{0} \mathrm{D}_{\mathrm{n}} \mathrm{a}_{\mathrm{i}}$ | $\mathrm{Z}_{0} \mathrm{D}_{\mathrm{n}} \mathrm{b}_{\mathrm{i}}$ |
| 0.02 | 0.0451 | 64.163 | -0.000159 | -0.002586 | 1.4286 | 8.2617 | 1.3797 | -2.14E-5 |
| 0.04 | 0.0901 | 32.083 | -0.000635 | -0.010345 | 1.4287 | 8.2618 | 1.3798 | -8.54E-5 |
| 0.06 | 0.1352 | 21.390 | -0.001428 | -0.023279 | 1.4288 | 8.2618 | 1.3800 | -1.92E-4 |
| 0.08 | 0.1803 | 16.045 | -0.002539 | -0.041390 | 1.4289 | 8.2619 | 1.3802 | -3.40E-4 |
| 0.10 | 0.2254 | 12.838 | -0.003966 | -0.064685 | 1.4291 | 8.2621 | 1.3805 | -5.31E-4 |
| 0.20 | 0.4509 | 6.4266 | -0.015830 | -0.25915 | 1.4306 | 8.2631 | 1.3829 | -0.00208 |
| 0.30 | 0.6768 | 4.2930 | -0.035494 | -0.58462 | 1.4332 | 8.2650 | 1.3869 | -0.00451 |
| 0.40 | 0.9032 | 3.2288 | -0.062787 | -1.0431 | 1.4369 | 8.2677 | 1.3925 | -0.00759 |
| 0.50 | 1.1304 | 2.5922 | -0.097464 | -1.6375 | 1.4416 | 8.2714 | 1.3996 | -0.01101 |
| 0.60 | 1.3584 | 2.1695 | -0.13919 | -2.3713 | 1.4473 | 8.2762 | 1.4083 | -0.01436 |
| 0.70 | 1.5875 | 1.8688 | -0.18755 | -3.2489 | 1.4542 | 8.2822 | 1.4185 | -0.01713 |
| 0.80 | 1.8177 | 1.6444 | -0.24200 | -4.2751 | 1.4620 | 8.2898 | 1.4301 | -0.01873 |
| 0.90 | 2.0493 | 1.4708 | -0.30192 | -5.4556 | 1.4710 | 8.2991 | 1.4431 | -0.01848 |
| 1.00 | 2.2824 | 1.3327 | -0.36653 | -6.7962 | 1.4809 | 8.3105 | 1.4573 | -0.01563 |
| 1.20 | 2.7535 | 1.1272 | -0.50624 | -9.9828 | 1.5038 | 8.3406 | 1.4887 | 0.001035 |

Table 3.1 - Continued

| 1.30 | 2.9917 | 1.0488 | -0.57924 | -11.841 | 1.5166 | 8.3600 | 1.5055 | 0.01646 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1.35 | 3.1115 | 1.0142 | -0.61602 | -12.839 | 1.5233 | 8.3709 | 1.5140 | 0.02626 |
| 1.38 | 3.1836 | 0.99459 | -0.6381 | -13.460 | 1.5274 | 8.3779 | 1.5192 | 0.03286 |
| 1.40 | 3.2317 | 0.98202 | -0.6528 | -13.884 | 1.5302 | 8.3827 | 1.5227 | 0.03757 |
| 1.50 | 3.4737 | 0.92437 | -0.7257 | -16.116 | 1.5444 | 8.4091 | 1.5400 | 0.06496 |
| 2.00 | 4.7114 | 0.72401 | -1.0434 | -30.217 | 1.6199 | 8.5995 | 1.6170 | 0.30086 |
| 3.00 | 7.2868 | 0.52103 | -1.3385 | -72.731 | 1.7364 | 9.1569 | 1.6597 | 0.95333 |
| 4.00 | 9.9163 | 0.41536 | -1.5376 | -133.45 | 1.7917 | 9.6372 | 1.6268 | 1.2416 |
| 5.00 | 12.571 | 0.35102 | -1.8441 | -213.52 | 1.8236 | 10.007 | 1.6052 | 1.2040 |
| 10.00 | 26.286 | 0.22229 | -3.4749 | -940.55 | 1.9113 | 12.073 | 1.6641 | 0.57002 |
| 15.00 | 40.539 | 0.17476 | -4.2906 | -2230.3 | 1.9395 | 14.396 | 1.7273 | 0.83770 |
| 20.00 | 55.034 | 0.14745 | -4.8375 | -4084.1 | 1.9494 | 16.347 | 1.7617 | 1.2725 |
| 25.00 | 69.661 | 0.12923 | -5.2970 | -6508.5 | 1.9547 | 18.016 | 1.7838 | 1.6704 |
| 30.00 | 84.376 | 0.11602 | -5.6900 | -9508.2 | 1.9581 | 19.506 | 1.7997 | 2.0472 |
| 35.00 | 99.157 | 0.10588 | -6.0263 | -13086 | 1.9605 | 20.868 | 1.8119 | 2.4182 |
| 40.00 | 113.99 | 0.09778 | -6.3149 | -17243 | 1.9622 | 22.131 | 1.8215 | 2.7879 |
| 45.00 | 128.86 | 0.09112 | -6.5633 | -21982 | 1.9635 | 23.331 | 1.8294 | 3.1576 |
| 50.00 | 143.76 | 0.08551 | -6.7775 | -27302 | 1.9644 | 24.427 | 1.8358 | 3.5271 |
| 55.00 | 158.69 | 0.08071 | -6.9623 | -33204 | 1.9652 | 25.482 | 1.8413 | 3.8964 |
| 60.00 | 173.64 | 0.07653 | -7.1213 | -39690 | 1.9657 | 26.486 | 1.8459 | 4.2654 |
| 70.00 | 203.59 | 0.06958 | -7.3734 | -54410 | 1.9665 | 28.364 | 1.8534 | 5.0024 |
| 80.00 | 233.60 | 0.06400 | -7.5525 | -71464 | 1.9669 | 30.095 | 1.8591 | 5.7372 |
| 90.00 | 263.66 | 0.05937 | -7.6720 | -90851 | 1.9671 | 31.703 | 1.8536 | 6.4684 |
| 10.00 | 293.74 | 0.05547 | -7.7425 | -112572 | 1.9672 | 33.205 | 1.8774 | 7.1946 |
| 150.00 | 444.46 | 0.04220 | -7.6011 | -256130 | 1.9663 | 39.516 | 1.8814 | 10.698 |
| 200.00 | 595.39 | 0.03429 | -7.0233 | -457800 | 1.9647 | 44.366 | 1.8714 | 13.883 |
| 300.00 | 897.35 | 0.02500 | -5.5689 | -1034823 | 1.9617 | 51.192 | 1.8836 | 19.052 |
| 400.00 | 1199.2 | 0.01964 | -4.2872 | -1842813 | 1.9595 | 55.539 | 1.8835 | 22.734 |
| 500.00 | 1500.8 | 0.01614 | -3.3103 | -2881478 | 1.9579 | 58.387 | 1.8830 | 25.300 |
| 600.00 | 1802.3 | 0.01367 | -2.5920 | -4150748 | 1.9568 | 60.313 | 1.8825 | 27.099 |
| 700.00 | 2103.8 | 0.01185 | -2.0646 | -5650623 | 1.9560 | 61.655 | 1.8820 | 28.384 |
| 800.00 | 2405.1 | 0.01044 | -1.6728 | -7381119 | 1.9554 | 62.619 | 1.8816 | 29.321 |
| 900.00 | 2706.4 | 0.00933 | -1.3771 | -9342256 | 1.9550 | 63.330 | 1.8814 | 30.019 |
| 1000.00 | 3007.5 | 0.00843 | -1.1501 | -11534000 | 1.9547 | 63.866 | 1.8811 | 30.550 |

Table 3.2 Residue Coefficients for Liquid (Hydraulic Transmission Lines)

|  | $\omega_{\mathrm{n}}$ | $\zeta$ | 1/cosh $\Gamma$ |  | $\mathrm{Z}_{\mathrm{c}} / \sinh \Gamma$ |  | $1 / \mathrm{Z}_{\mathrm{c}} \sinh \Gamma$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  | (-1) ${ }^{\text {i }} \mathrm{D}_{\mathrm{n}} \mathrm{a}_{i} / \mathrm{Z}_{0}$ | $(-1)^{i} \mathrm{D}_{\mathrm{nb}} \mathrm{b}_{i} / Z_{0}$ | (-1) ${ }^{\text {i }} \mathrm{Z}_{0} \mathrm{D}_{\mathrm{n}} \mathrm{a}_{\mathrm{i}}$ | $(-1)^{i} \mathrm{Z}_{0} \mathrm{D}_{\mathrm{n}} \mathrm{b}_{\mathrm{i}}$ |
|  |  |  | $(-1)^{i+1}(1-2 i) \mathrm{a}_{\mathrm{i}}$ | $(-1)^{i+1}(1-2 i) b_{i}$ | $Z_{\mathrm{c}} \cosh \Gamma / \sinh \Gamma$ <br> $\mathrm{Z}_{\mathrm{c}} \sinh \Gamma / \cosh \Gamma$ |  | $\cosh \Gamma / Z_{\mathrm{c}} \sinh \Gamma$ $\sinh \Gamma / Z_{\mathrm{c}} \cosh \Gamma$ |  |
|  |  |  |  |  | $\mathrm{D}_{\mathrm{n}} \mathrm{a}_{\mathrm{i}} \mathrm{Z}_{0}$ | $\mathrm{D}_{\mathrm{n}} \mathrm{b}_{\mathrm{i}} / \mathrm{Z}_{0}$ | $\mathrm{Z}_{0} \mathrm{D}_{\mathrm{n}} \mathrm{a}_{\mathrm{i}}$ | $\mathrm{Z}_{0} \mathrm{D}_{\mathrm{n}} \mathrm{b}_{\mathrm{i}}$ |
| 0.01 | 0.0267 | 108.31 | -6.7412E-6 | -9.0755E-4 | 2.0000 | 11.566 | 1.3824 | -7.6476E-6 |
| 0.02 | 0.0534 | 54.154 | -2.6965E-5 | -3.6302E-3 | 2.0000 | 11.566 | 1.3824 | -3.0590E-5 |
| 0.03 | 0.0801 | 36.103 | -6.0670E-5 | -8.1680E-3 | 2.0000 | 11.566 | 1.3824 | -6.8828E-5 |
| 0.04 | 0.1068 | 27.077 | -1.0786E-4 | -1.4521E-2 | 2.0000 | 11.566 | 1.3824 | -1.2236E-4 |
| 0.05 | 0.1335 | 21.662 | -1.6852E-4 | -2.2689E-2 | 2.0000 | 11.566 | 1.3824 | -1.9118E-4 |
| 0.06 | 0.1602 | 18.052 | -2.4267E-4 | -3.2672E-2 | 2.0000 | 11.566 | 1.3824 | -2.7530E-4 |
| 0.07 | 0.1869 | 15.473 | -3.3030E-4 | -4.4471E-2 | 2.0000 | 11.566 | 1.3824 | -3.7471E-4 |
| 0.08 | 0.2136 | 13.339 | -4.3141E-4 | -5.8084E-2 | 2.0000 | 11.566 | 1.3824 | -4.8941E-4 |
| 0.09 | 0.2403 | 12.035 | -5.4599E-4 | -7.3513E-2 | 2.0000 | 11.567 | 1.3824 | -6.1940E-4 |
| 0.10 | 0.2670 | 10.832 | -6.7404E-4 | -9.0758E-2 | 2.0000 | 11.567 | 1.3824 | -7.6467E-4 |
| 0.20 | 0.5340 | 5.4172 | -2.6952E-3 | -3.6306E-1 | 2.0001 | 11.567 | 1.3825 | -3.0577E-3 |
| 0.30 | 0.8010 | 3.6130 | -6.0607E-3 | -8.702E-1 | 2.0003 | 11.568 | 1.3828 | -6.8764E-3 |
| 0.40 | 1.0681 | 2.7113 | -1.0765E-2 | -1.4528 | 2.0005 | 11.569 | 1.3831 | -1.2216E-2 |
| 0.50 | 1.3352 | 2.1707 | -1.6803E-2 | -2.2706 | 2.0007 | 11.571 | 1.3835 | -1.9069E-2 |
| 0.60 | 1.6023 | 1.8106 | -2.4165E-2 | -3.2707 | 2.0011 | 11.573 | 1.3840 | $-2.7428 \mathrm{E}-2$ |
| 0.70 | 1.8696 | 1.5536 | -3.2840E-2 | -4.4535 | 2.0015 | 11.575 | 1.3846 | -3.7283E-2 |
| 0.80 | 2.1369 | 1.3611 | -4.2817E-2 | -5.8194 | 2.0019 | 11.578 | 1.3852 | -4.8619E-2 |
| 0.90 | 2.4043 | 1.2116 | -5.4081E-2 | -7.3689 | 2.0024 | 11.581 | 1.3860 | -6.1425E-2 |
| 1.00 | 2.6718 | 1.0921 | -6.6615E-2 | -9.1026 | 2.0030 | 11.584 | 1.3868 | -7.5681E-2 |
| 1.05 | 2.8056 | 1.0410 | -7.3354E-2 | -10.039 | 2.0033 | 11.586 | 1.3873 | -8.3351E-2 |
| 1.10 | 2.9394 | 0.9945 | -8.0403E-2 | -11.021 | 2.0036 | 11.588 | 1.3878 | -9.1376E-2 |
| 1.20 | 3.2072 | 0.9134 | -9.5425E-2 | -13.125 | 2.0043 | 11.593 | 1.3888 | $-1.0849 \mathrm{E}-1$ |
| 1.30 | 3.4751 | 0.8448 | -1.1166E-1 | -15.414 | 2.0050 | 11.597 | 1.3899 | -1.2698E-1 |
| 1.40 | 3.7432 | 0.7862 | -1.2908E-1 | -17.891 | 2.0058 | 11.603 | 1.3911 | -1.4686E-1 |
| 1.60 | 4.2796 | 0.6912 | -1.6739E-1 | -23.408 | 2.0075 | 11.615 | 1.3937 | -1.9062E-1 |
| 1.80 | 4.8171 | 0.6177 | -2.1015E-1 | -29.684 | 2.0094 | 11.629 | 1.3967 | $-2.3954 \mathrm{E}-1$ |
| 2.00 | 5.3553 | 0.5593 | -2.5710E-1 | -36.725 | 2.0114 | 11.645 | 1.4000 | -2.9338E-1 |
| 3.00 | 8.0609 | 0.3868 | -5.4497E-1 | -83.734 | 2.0242 | 11.765 | 1.4206 | -6.2620E-1 |
| 4.00 | 10.796 | 0.3036 | -8.9161E-1 | -151.34 | 2.0394 | 11.967 | 1.4472 | -1.0338 |
| 5.00 | 13.566 | 0.2554 | -1.2545 | -240.78 | 2.0550 | 12.267 | 1.4775 | -1.4699 |
| 10.0 | 27.867 | 0.1613 | -2.4968 | -1037.6 | 2.0957 | 14.935 | 1.6156 | -3.0903 |
| 15.0 | 42.640 | 0.1262 | -3.0243 | -2419.6 | 2.0904 | 17.737 | 1.6885 | -3.8995 |
| 20.0 | 57.558 | 0.1061 | -3.4562 | -4388.6 | 2.0808 | 19.996 | 1.7294 | -4.5559 |
| 25.0 | 72.570 | 0.0930 | -3.8498 | -6952.0 | 2.0736 | 21.941 | 1.7571 | -5.1628 |
| 30.0 | 87.647 | 0.0836 | -4.2000 | -10113 | 2.0680 | 23.698 | 1.7776 | -5.7056 |
| 35.0 | 102.77 | 0.0764 | -4.5144 | -13875 | 2.0634 | 25.313 | 1.7934 | -6.1952 |
| 40.0 | 117.94 | 0.0707 | -4.8011 | -18238 | 2.0596 | 26.815 | 1.8062 | -6.6428 |
| 45.0 | 133.14 | 0.0661 | -5.0651 | -23204 | 2.0564 | 28.223 | 1.8167 | -7.0562 |
| 50.0 | 148.36 | 0.0622 | -5.3101 | -28774 | 2.0536 | 29.552 | 1.8256 | -7.4404 |
| 55.0 | 163.60 | 0.0589 | -5.5387 | -34950 | 2.0511 | 30.813 | 1.8332 | -7.7995 |
| 60.0 | 178.86 | 0.0560 | -5.7528 | -41731 | 2.0490 | 32.017 | 1.8398 | -8.1363 |

Table 3.2-Continued

| 70.0 | 209.43 | 0.0513 | -6.1438 | -57113 | 2.0454 | 34.275 | 1.8508 | -8.7528 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 80.0 | 240.05 | 0.0475 | -6.4932 | -74925 | 2.0424 | 36.369 | 1.8596 | -9.3049 |
| 90.0 | 270.71 | 0.0444 | -6.8079 | -95171 | 2.0399 | 38.327 | 1.8668 | -9.8032 |
| 100 | 301.41 | 0.0418 | -7.0929 | -1.1785 E 5 | 2.0377 | 40.172 | 1.8728 | -1.0255 E 1 |
| 150 | 455.23 | 0.0330 | -8.1857 | -2.6786 E 5 | 2.0303 | 48.150 | 1.8930 | -1.1999 E 1 |
| 200 | 609.32 | 0.0278 | -8.8940 | -4.7895 E 5 | 2.0257 | 54.724 | 1.9045 | -1.3143 E 1 |
| 250 | 763.82 | 0.0243 | -9.3460 | -7.5117 E 5 | 2.0224 | 60.376 | 1.9120 | -1.3885 E 1 |
| 300 | 918.36 | 0.0217 | -9.6132 | -1.0846 E 6 | 2.0200 | 65.354 | 1.9172 | -1.4337 E 1 |
| 350 | 1073.0 | 0.0197 | -9.7418 | -1.4791 E 6 | 2.0180 | 69.807 | 1.9211 | -1.4572 E 1 |
| 400 | 1227.7 | 0.0180 | -9.7642 | -1.9348 E 6 | 2.0164 | 73.831 | 1.9241 | -1.4639 E 1 |
| 450 | 1382.4 | 0.0166 | -9.7046 | -2.4515 E 6 | 2.0150 | 77.496 | 1.9264 | -1.4576 E 1 |
| 500 | 1537.2 | 0.0155 | -9.5817 | -3.0294 E 6 | 2.0138 | 80.850 | 1.9283 | -1.4414 E 1 |
| 550 | 1692.0 | 0.0145 | -9.4101 | -3.6683 E 6 | 2.0128 | 83.933 | 1.9298 | -1.4174 E 1 |
| 600 | 1846.8 | 0.0136 | -9.2018 | -4.3683 E 6 | 2.0119 | 86.776 | 1.9310 | -1.3875 E 1 |
| 650 | 2001.6 | 0.0129 | -8.966 | -5.1293 E 6 | 2.0111 | 89.402 | 1.9320 | -1.3532 E 1 |
| 700 | 2156.4 | 0.0122 | -8.7111 | -5.9513 E 6 | 2.0103 | 91.834 | 1.9329 | -1.3157 E 1 |
| 800 | 2466.1 | 0.0110 | -8.1668 | -7.7783 E 6 | 2.0090 | 96.182 | 1.9342 | -1.2351 E 1 |
| 900 | 2775.7 | 0.0100 | -7.6061 | -9.8490 E 6 | 2.0080 | 99.937 | 1.9351 | -1.1515 E 1 |
| 1000 | 3085.4 | 0.0092 | -70529 | -1.2164 E 7 | 2.0071 | 103.19 | 1.9358 | -1.0686 E 1 |

Table 3.3 Residue Coefficients for the Real Poles of $Z_{0} / \sinh \Gamma$ and $Z_{0} \cosh \Gamma / \sinh \Gamma$

| $j$ | Pole | Residue Coefficients for $Z_{0} / \sinh \Gamma$ and <br> $Z_{0} \cosh \Gamma / \sinh \Gamma$ <br> $K_{j} D_{n} / Z_{0}$ |
| :---: | :---: | :---: |
| 1 | -10.1400 | 0.169386 |
| 2 | -45.1979 | 0.046074 |
| 3 | -107.923 | 0.019571 |
| 4 | -198.487 | 0.010568 |
| 5 | -316.905 | 0.006128 |
| 6 | -462.886 | 0.004161 |
| 7 | -638.069 | 0.003323 |
| 8 | -837.815 | 0.000873 |
| 9 | -1069.95 | 0.001907 |
| 10 | -1326.08 | 0.000560 |

Table 3.4 Residue Coefficients for the Real Poles of $1 / Z_{0} \sinh \Gamma$ and $\cosh \Gamma / Z_{0} \sinh \Gamma$

| $j$ | Pole | Residue Coefficients for $1 / Z_{0} \sinh \Gamma$ and <br> $\cosh \Gamma / Z_{0} \sinh \Gamma$ <br> $K_{j} D_{n} / Z_{0}$ |
| :---: | :---: | :---: |
| 0 | -5.78319 | 0.689837 |
| 1 | -30.4713 | 0.129436 |
| 2 | -74.8862 | 0.51558 |
| 3 | -139.043 | 0.26872 |
| 4 | -222.939 | 0.016075 |
| 5 | -326.461 | 0.009828 |
| 6 | -450.314 | 0.007266 |
| 7 | -592.386 | 0.004908 |
| 8 | -756.437 | 0.003949 |
| 9 | -938.303 | 0.001649 |

### 3.4 Modal Approximation of a Blocked Hydraulic Line

The tabulated values for residue coefficients are a function of the dimensionless root Indices $\lambda_{c}$ and $\lambda_{s}$ which are functions of the dissipative number, $D_{n}$.

$$
\begin{equation*}
D_{n}=\frac{v L}{c r^{2}}=\frac{(4.6 \mathrm{e}-5)(2)}{(140.0)(0.01 / 2)^{2}}=0.02628 \tag{3.59}
\end{equation*}
$$

For a four mode approximation ( $\mathrm{i}=1,2,3,4$ ):

$$
\begin{gather*}
\lambda_{C}=\frac{1}{0.02628}\left(i-\frac{1}{2}\right)  \tag{3.60}\\
\lambda_{C}=\left[\begin{array}{llll}
19.02 & 57.07 & 95.12 & 133.17
\end{array}\right] \tag{3.61}
\end{gather*}
$$



Figure 3.2 Log-log plot of natural frequency of a hydraulic line as a function of the dimensionless root index

Interpolation gives the following values for the natural frequencies for each mode.

$$
\omega_{C}=\left[\begin{array}{llll}
54.65 & 169.92 & 286.43 & 403.45 \tag{3.62}
\end{array}\right]
$$



Figure 3.3 Log-log plot of natural frequency of a hydraulic line as a function of the dimensionless root index

Interpolation gives the following values for the damping ratios for each mode.

$$
\varsigma=\left[\begin{array}{llll}
0.1100 & 0.0577 & 0.0431 & 0.0360 \tag{3.63}
\end{array}\right]
$$



Figure 3.4 Log-log plot of residue coefficient, (-1) $\mathrm{i}+1(1-2 \mathrm{i}) \mathrm{a}_{\mathrm{i}}$ of a hydraulic line as a function of the dimensionless root index

Interpolation gives the following values for the residue coefficient, $a_{i}$ for each mode.

$$
\begin{gather*}
(-1)^{i+1}(1-2 i) a_{i}=\left[\begin{array}{lllll}
-3.3719 & -5.6274 & -6.9538 & -7.8178
\end{array}\right]  \tag{3.64}\\
a_{i}=\left[\begin{array}{llll}
3.3719 & -1.8758 & 1.3908 & -1.1168
\end{array}\right] \tag{3.65}
\end{gather*}
$$



Figure 3.5 Log-log plot of residue coefficient, (-1) $\mathrm{i}+1(1-2 \mathrm{i}) \mathrm{b}_{\mathrm{i}}$ of a hydraulic line as a function of the dimensionless root index

Interpolation gives the following values for the residue coefficient, $b_{i}$ for each mode.

$$
\begin{gather*}
(-1)^{i+1}(1-2 i) b_{i}=\left[\begin{array}{llll}
-4004 & -37,760 & -1.0678 \mathrm{e} 5 & -2.1736 \mathrm{e} 5
\end{array}\right]  \tag{3.66}\\
b_{i}=\left[\begin{array}{llll}
4004 & -12,587 & 2,1357 & -3,1052
\end{array}\right] \tag{3.67}
\end{gather*}
$$



Figure 3.6 Frequency response of modal approximations

## CHAPTER 4

## FREQUENCY RESPONSE CURVE FITTING

### 4.1 Least Squares Method for Linear Curve Fitting

The modal method of Hullender and Healey [14] approximates each modal response by appropriately truncating the Taylor series form of the exact solution. Another approach is to apply a curve fit algorithm to the frequency response of the exact solution. Linear regression is the use of algorithms to model data points with a linear equation. The least squares method developed by Gauss and Legendre [19] is the most common technique used to model linear data in the form of a line or a polynomial.

The following are experimental data points.

$$
\begin{equation*}
f\left(x_{1}\right)=f_{1}, f\left(x_{2}\right)=f_{2}, \cdots f\left(x_{n}\right)=f_{n} \tag{4.1}
\end{equation*}
$$

The goal is to obtain the best approximation of this data by a linear equation representation.

$$
\begin{equation*}
f(x) \approx a x+b \tag{4.2}
\end{equation*}
$$

The least squares method obtains an equation in which the total squared error is minimized.

$$
\begin{equation*}
E^{2}(a, b)=\sum_{j=1}^{n}\left[F\left(x_{j}\right)-f_{j}\right]^{2} \tag{4.3}
\end{equation*}
$$

$$
\begin{gather*}
a=\frac{N \sum x_{j} f_{j}-\left(\sum f_{j}\right)\left(\sum x_{j}\right)}{N \sum x_{j}^{2}-\left(\sum x_{j}\right)^{2}}  \tag{4.4}\\
b=\frac{\left(\sum x_{j}^{2}\right)\left(\sum f_{j}\right)-\left(\sum x_{j}\right)\left(\sum x_{j} f_{j}\right)}{N \sum x_{j}^{2}-\left(\sum x_{j}\right)^{2}} \tag{4.5}
\end{gather*}
$$

### 4.2 Nonlinear Least Squares Overview

System models derived from partial differential equations are nonlinear and cannot be approximated with the simple least squares method described in the previous section.

### 4.2.1 Newton's Iterative Method

Newton's method [20] is an iterative process to approximate a real zero of a differentiable function. As in all iterate processes, a first approximation of the zero $r$ is made. The first approximation is chosen as the $x$-intercept of a tangent line 1 .


Figure 4.1 First approximation using tangent line x -intercept

The equation of the tangent line can be written as:

$$
\begin{equation*}
y-f\left(x_{1}\right)=f^{\prime}\left(x_{1}\right)\left(x-x_{1}\right) \tag{4.6}
\end{equation*}
$$

Solve for ( $\mathrm{x}_{2}$ ):

$$
\begin{equation*}
x_{2}=x_{1}-\frac{f\left(x_{1}\right)}{f^{\prime}\left(x_{1}\right)} \tag{4.7}
\end{equation*}
$$

In general the process is repeated until the desired convergence is reached.

$$
\begin{align*}
& x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}  \tag{4.8}\\
& f^{\prime}\left(x_{n}\right) \neq 0
\end{align*}
$$

It is important that the first approximation is sufficiently close to $r$ that successive approximations converge.

### 4.2.2 Newton's Method in Optimization

Newton's method for determining the real roots of a function can be modified to approximate local the maxima and minima of a function. Local maxima and minima points are stationary points and the slope of the derivative at these points is zero.

$$
\begin{align*}
& x_{n+1}=x_{n}-\frac{f^{\prime}\left(x_{n}\right)}{f^{\prime \prime}\left(x_{n}\right)}  \tag{4.9}\\
& f^{\prime \prime}\left(x_{n}\right) \neq 0
\end{align*}
$$

### 4.2.3 The Gauss-Newton Method

The Gauss-Newton Method [21] is an iterative approach to solve nonlinear least squares problems. Since nonlinear functions are multivariable, the Jacobian is used in the same fashion (with some modification) as the derivative in the single variable

Newton's method of Optimization. The Jacobian is the matrix of all first order partial derivatives and is analogous to the derivative of a multivariable function.

$$
\begin{align*}
p^{k+1}=p^{k}- & \left(J_{f}\left(p^{k}\right) J_{f}\left(p^{k}\right)^{T}\right)^{-1} J_{f}\left(p^{k}\right) f\left(p^{k}\right)  \tag{4.10}\\
J_{f} & =\left|\begin{array}{cccc}
\frac{\partial f_{1}}{\partial p_{1}} & \frac{\partial f_{1}}{\partial p_{2}} & \cdots & \frac{\partial f_{1}}{\partial p_{n}} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial f_{n}}{\partial p_{1}} & \frac{\partial f_{n}}{\partial p_{2}} & \cdots & \frac{\partial f_{n}}{\partial p_{n}}
\end{array}\right| \tag{4.11}
\end{align*}
$$

In the iterative process an initial guess $\mathrm{p}^{0}$ is made and subsequent approximations for p are made using equation 4.10. The inverse matrix of equation 4.10 is not computed directly. Instead the following equation is solved

$$
\begin{equation*}
p^{k+1}=p^{k}+\delta^{k} \tag{4.12}
\end{equation*}
$$

Where $\delta^{\mathrm{k}}$ is solved by the following linear equation

$$
\begin{equation*}
J_{f}\left(p^{k}\right) J_{f}\left(p^{k}\right)^{T} \delta^{k}=-J_{f}\left(p^{k}\right) f\left(p^{k}\right) \tag{4.13}
\end{equation*}
$$

### 4.3 Approximation of the Dissipative Model Using Gauss-Newton Method

The following rational polynomial transfer function is to be obtained by applying the Gauss-Newton method to the frequency response data points of the exact solution of a fluid line system.

$$
\begin{equation*}
H(s)=\frac{B(s)}{A(s)}=\frac{b_{1} s^{n-1}+b_{2} s^{n-2}+\cdots+b_{n}}{s^{n}+a_{1} s^{n-1}+a_{2} s^{n-2}+\cdots+a_{n}} \tag{4.14}
\end{equation*}
$$

Levi [22] was the first to document the application of the linear least-squares technique to approximate a transfer function with a rational polynomial. Levi's
technique was limited to systems with no poles on the imaginary axis. Sanathanan and Koerner [23] applied an iterative search method to solve for any system (SK iteration).

The Matlab ${ }^{\circledR}$ command 'invfreqs' program applies two algorithms to obtain the solution in the form of equation 4.13. The non-iterative algorithm is based on Levi's work and the iterative algorithm is based on the damped Gauss-Newton method.

### 4.3.1 Levi's Algorithm

The exact transfer function provided by the dissipative model is sampled for $\omega_{\mathrm{k}}$ frequencies $(k=1,2, \ldots \mathrm{~m})$. The transfer function $H(s)$ can be represented in the following form.

$$
\begin{gather*}
\sum_{k=1}^{m} H\left(j \omega_{k}\right)=\sum_{k=1}^{m}\left(h_{k}\right)  \tag{4.15}\\
\sum_{k=1}^{m} H\left(j \omega_{k}\right)=\sum_{k=1}^{m} \frac{B\left(j \omega_{k}\right)}{A\left(j \omega_{k}\right)}=\sum_{k=1}^{m} \frac{b_{1}\left(j \omega_{k}\right)^{n-1}+b_{2}\left(j \omega_{k}\right)^{n-2}+\cdots+b_{n}}{\left(j \omega_{k}\right)^{n}+a_{1}\left(j \omega_{k}\right)^{n-1}+a_{2}\left(j \omega_{k}\right)^{n-2}+\cdots+a_{n}} \tag{4.16}
\end{gather*}
$$

The error in this algorithm is

$$
\begin{equation*}
e(s)=H(s)-\frac{B(s)}{A(s)} \tag{4.17}
\end{equation*}
$$

This equation is multiplied on both sides to remove the denominator term.

$$
\begin{equation*}
A(s) e(s)=A(s) H(s)-B(s) \tag{4.18}
\end{equation*}
$$

$\mathrm{A}(\mathrm{s}) e(\mathrm{~s})$ is defined as the new error function E .

$$
\begin{gather*}
E=A(s) H(s)-B(s)  \tag{4.19}\\
E=\sum_{k=1}^{m} A\left(j \omega_{k}\right) H\left(j \omega_{k}\right)-B\left(j \omega_{k}\right) \tag{4.20}
\end{gather*}
$$

The goal is to minimize the Frobenius or Euclean norm of E. This is the square root of the sum of the absolute squares of its elements.

$$
\begin{equation*}
\min _{a, b} \sum_{k=1}^{m}\|E\|_{2} \tag{4.21}
\end{equation*}
$$

The Matlab ${ }^{\circledR}$ 'invfreqs' program is based on Levi’s work, but does not use the method directly. The program uses the following equations to define the numerator $\mathrm{A}(\mathrm{s})$ and the denominator $\mathrm{B}(\mathrm{s})$ of the transfer function $\mathrm{H}(\mathrm{s})$.

$$
\begin{gather*}
A\left(j \omega_{k}\right)=\left[\widetilde{\omega}_{k}\right][\widetilde{a}]+\left(j \omega_{k}\right)^{n}  \tag{4.22}\\
B\left(j \omega_{k}\right)=\left[\widetilde{\omega}_{k}\right][\widetilde{b}]  \tag{4.23}\\
{\left[\widetilde{\omega}_{k}\right]_{1 \times n}=\left[\begin{array}{llll}
\left(j \omega_{k}\right)^{n-1} & \left(j \omega_{k}\right)^{n-2} & \cdots & 1
\end{array}\right]}  \tag{4.24}\\
{[\widetilde{a}]_{n \times 1}=\left[\begin{array}{llll}
a_{1} & a_{2} & \cdots & a_{n}
\end{array}\right]^{T}}  \tag{4.25}\\
{[\widetilde{b}]_{n \times 1}=\left[\begin{array}{llll}
b_{1} & b_{2} & \cdots & b_{n}
\end{array}\right]^{T}} \tag{4.26}
\end{gather*}
$$

The error function E , is then represented in the following form.

$$
\begin{gather*}
E=\sum_{k=1}^{m}\left(\left[\widetilde{\omega}_{k}\right]\left[\widetilde{a}_{k}\right]+\left(j \omega_{k}\right)\right) H\left(j \omega_{k}\right)-\left[\widetilde{\omega}_{k}\right]\left[\widetilde{b}_{k}\right]  \tag{4.27}\\
E=\sum_{k=1}^{m}\left(\left[\widetilde{\omega}_{k}\right]\left[\widetilde{a}_{k}\right]+\left(j \omega_{k}\right)\right) h_{k}\left(j \omega_{k}\right)-\left[\widetilde{\omega}_{k}\right]\left[\widetilde{b}_{k}\right]  \tag{4.28}\\
E=\sum_{k=1}^{m} h_{k}\left(j \omega_{k}\right)\left[\widetilde{\omega}_{k}\right]\left[\widetilde{a}_{k}\right]+h_{k}\left(j \omega_{k}\right)\left(j \omega_{k}\right)-\left[\widetilde{\omega}_{k}\right]\left[\widetilde{b}_{k}\right]  \tag{4.29}\\
\min _{a, b} \sum_{k=1}^{m}\left\|h_{k}\left(j \omega_{k}\right)\left[\widetilde{\omega}_{k}\right]\left[\widetilde{a}_{k}\right]+h_{k}\left(j \omega_{k}\right)\left(j \omega_{k}\right)-\left[\widetilde{\omega}_{k}\right]\left[\widetilde{b}_{k}\right]\right\|_{2} \tag{4.30}
\end{gather*}
$$

This is then represented as follows.

$$
\left.\begin{array}{c}
\min _{a, b} \sum_{k=1}^{m}\|[\widetilde{A}] \boldsymbol{\theta}-[\widetilde{c}]\|_{2} \\
{[\widetilde{A}]_{n \nless 2 n}=\left[\begin{array}{ccccccc}
h_{1}\left(j \omega_{1}\right)^{n-1} & h_{1}\left(j \omega_{1}\right)^{n-2} & \cdots & h_{1} & -\left(j \omega_{1}\right)^{n-1} & -\left(j \omega_{1}\right)^{n-2} & \cdots \\
h_{2}\left(j \omega_{2}\right)^{n-1} & h_{2}\left(j \omega_{2}\right)^{n-2} & \cdots & h_{2} & -\left(j \omega_{2}\right)^{n-1} & -\left(j \omega_{2}\right)^{n-2} & \cdots \\
h_{3}\left(j \omega_{3}\right)^{n-1} & h_{3}\left(j \omega_{3}\right)^{n-2} & \cdots & h_{3} & -\left(j \omega_{3}\right)^{n-1} & -\left(j \omega_{3}\right)^{n-2} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\vdots & h_{m}\left(j \omega_{m}\right)^{n-2} & \cdots & h_{m} & -\left(j \omega_{m}\right)^{n-1} & -\left(j \omega_{m}\right)^{n-2} & \cdots
\end{array}\right]-1} \tag{4.32}
\end{array}\right] .
$$

The following is the vector containing the polynomial coefficients that are to be approximated using this method.

$$
\begin{align*}
& \theta_{2 n \times 1}=\left[\begin{array}{llllllll}
a_{1} & a_{2} & \cdots & a_{n} & b_{1} & b_{2} & \cdots & b_{n}
\end{array}\right]^{T} \tag{4.33}
\end{align*}
$$

$$
\begin{align*}
& {[\tilde{A} \mid \widetilde{A}]^{T} \theta=[\tilde{A}]^{T}[\tilde{c}]} \tag{4.35}
\end{align*}
$$

The parameter vector $\theta$ is solved using orthogonal matrix triangularization.

### 4.3.2 Iterative Algorithm to Estimate $\theta$

As in section 4.2.3 the Jacobian is used as a multivariable derivative.

$$
\begin{equation*}
J(\theta)=\frac{\partial R(\theta)}{\partial \theta} \tag{4.36}
\end{equation*}
$$

$$
\begin{gather*}
J(\theta)=\frac{R\left(\theta^{+}\right)-R(\theta)}{\theta^{+}-\theta}  \tag{4.37}\\
R\left(\theta^{+}\right)=R(\theta)+J(\theta)\left(\theta^{+}-\theta\right)  \tag{4.38}\\
\min _{\theta^{+}}\left\|R(\theta)+J(\theta)\left(\theta^{+}-\theta\right)\right\|  \tag{4.39}\\
J(\theta)^{T} J(\theta)\left(\theta^{+}-\theta\right)=-J(\theta)^{T} R(\theta)  \tag{4.40}\\
\theta^{+}=\theta-\left(J(\theta)^{T} J(\theta)\right)^{-1} J(\theta)^{T} R(\theta) \tag{4.41}
\end{gather*}
$$

The following equation is equation 4.41 with the damping term $\lambda$ added. This is the Damped Gauss-Newton method.

$$
\begin{gather*}
\theta^{+}=\theta-\lambda\left(J(\theta)^{T} J(\theta)\right)^{-1} J(\theta)^{T} R(\theta)  \tag{4.44}\\
R\left(\theta^{+}\right)^{T} R\left(\theta^{+}\right) \leq R(\theta)^{T} R(\theta)  \tag{4.45}\\
\left\|\left(J(\theta)^{T} J(\theta)\right)^{-1} J(\theta)^{T} R(\theta) \leq 0.01\right\| \tag{4.46}
\end{gather*}
$$

## CHAPTER 5

## APPLICATION OF MATLAB ${ }^{\circledR}$ FUNCTIONS TO OBTAIN A FINITE ORDER TRANSFER FUNCTION OF A TOTAL FLUID TRANSMISSION LINE SYSTEM

### 5.1 Total Fluid Transmission Line System

The hyperbolic transfer function matrix defines the input/output relationship of a single fluid line. Typical hydraulic and pneumatic systems contain a number of fluid lines and other resistive, inductive, and capacitive components.


Figure 5.1 Hydraulic brake valve schematic (Mico)

### 5.2 Application of Matlab ${ }^{\circledR}$ Symbolic Toolbox Commands to Model a Fluid Transmission Line

The following section provides example applications of Matlab ${ }^{\circledR}$ commands to model a fluid transmission line system. Both a lumped and a distrubuted parameter
model are presented to illustrate the need to approximate the distributed parameter model after the frequency domain transfer function is obtained.

### 5.2.1 Frequency and Time Response of Lumped Model

The Matlab ${ }^{\circledR}$ Symbolic Toolbox provides the 'solve.m' program that will solve an ' $n$ ' number of symbolic equations for an ' $n$ ' number of unknowns. For example, the simple lumped parameter model for the blocked hydraulic transmission line discussed earlier can be solved using the following command:

$$
\begin{equation*}
\text { Sol }=\text { solve }(\text { Pin }- \text { Pout }=(\mathrm{RL}+\mathrm{LI} * \mathrm{~s}) * \text { Qin','Qin }=\mathrm{CP} * \text { Pout*s','Pout,Qin' }) \tag{5.1}
\end{equation*}
$$

Matlab ${ }^{\circledR}$ responds with the following output.

$$
\begin{align*}
& \text { Sol= } \\
& \text { Pout:[1x1sym] }  \tag{5.2}\\
& \text { Qin:[1x1sym] }
\end{align*}
$$

To view the individual solution of $\mathrm{P}_{\text {out }}$ type:

$$
\begin{equation*}
\text { Pout }=\text { Sol.Pout } \tag{5.3}
\end{equation*}
$$

Matlab ${ }^{\circledR}$ responds with the following output:
Pout $=$

$$
\begin{equation*}
\operatorname{Pin} /\left(1+\mathrm{CP} * \mathrm{~s} * \mathrm{RL}+\mathrm{CP} * \mathrm{~s}^{\wedge} 2 * \mathrm{LI}\right) \tag{5.4}
\end{equation*}
$$

The input pressure value, $\mathrm{P}_{\mathrm{in}}$, needs to expressed as unity in order to obtain the transfer function of the line rather than the particular solution. This can be accomplished using the 'subs' command to replace a symbolic term with a real value.

$$
\begin{equation*}
\mathrm{TF}=\operatorname{subs}\left(\text { Pout, }{ }^{\prime} \operatorname{Pin} ', 1\right) \tag{5.5}
\end{equation*}
$$

$$
\begin{align*}
& \mathrm{TF}= \\
& 1 /\left(1+\mathrm{CP} * \mathrm{~s}^{*} \mathrm{RL}+\mathrm{CP} * \mathrm{~s}^{\wedge} 2 * \mathrm{LI}\right) \tag{5.6}
\end{align*}
$$

An optional command can be used to make the solution in a typeset form.
pretty(Sol.TF)

1

$$
\begin{equation*}
1+\mathrm{CP} \text { s RL }+\mathrm{CP} \mathrm{~s}^{2} \mathrm{LI} \tag{5.8}
\end{equation*}
$$

The present form of the solution is symbolic and needs to be converted in a form that Matlab ${ }^{\circledR}$ can process. The 'subs' command is used to replace all the symbolic elements with actual values. Before the 'subs' command can be run the values for each symbolic parameter must already be in the Matlab ${ }^{\circledR}$ workspace.

$$
\begin{align*}
& \mathrm{p}=870 \\
& \mathrm{Vis}=4.6 \mathrm{e}-5 ; \\
& \mathrm{Bf}=1.21 \mathrm{e} 9 \\
& \mathrm{Bl}=1.73 \mathrm{e} 7 \\
& \mathrm{Be}=(\mathrm{Bf} * \mathrm{Bl}) /(\mathrm{Bf}+\mathrm{Bl}) ; \\
& \mathrm{L}=2 ; \\
& \mathrm{d}=0.01  \tag{5.9}\\
& \mathrm{r}=\mathrm{d} / 2 \\
& \mathrm{~V}=\mathrm{L}^{*} \mathrm{pi}^{*} \mathrm{r}^{\wedge} 2 ; \\
& \mathrm{LI}=(\mathrm{p} * \mathrm{~L}) /\left(\mathrm{pi}^{*} \mathrm{r}^{\wedge} 2\right) ; \\
& \mathrm{CP}=\mathrm{V} / \mathrm{Be} ; \\
& \mathrm{RL}=\left(128^{*} \mathrm{Vis}^{*} \mathrm{p}^{*} \mathrm{~L}\right) /\left(\mathrm{pi}^{*} \mathrm{~d}^{\wedge} 4\right) ; \\
& \mathrm{Pin}=1 ;
\end{align*}
$$

Note: all parameters are derived from metric units ( $\mathrm{m}, \mathrm{Kg}$, and s ).

$$
\begin{aligned}
& \mathrm{TF}=\operatorname{subs}\left(\mathrm{TF}, \mathrm{LI}^{\prime}, \mathrm{LI}\right) \\
& \mathrm{TF}=\operatorname{subs}\left(\mathrm{TF}, \mathrm{CP}^{\prime}, \mathrm{CP}\right) \\
& \mathrm{TF}=\operatorname{subs}\left(\mathrm{TF}, \mathrm{RL}^{\prime}, \mathrm{RL}\right)
\end{aligned}
$$

Matlab ${ }^{\circledR}$ will return the solution in whole number form.

$$
\left.\begin{array}{l}
\mathrm{TF}=1 /(1+311885959964924424757420014269 \quad 27 / \ldots \\
10384593717069655257060992658440192 * \mathrm{~s}+\ldots \\
3390064782  \tag{5.11}\\
22743915169174379291 \\
1661534994 \\
7311448411
\end{array} 2975882535043072 * \mathrm{~s}^{\wedge} 2\right) \text { 2) }
$$

This can be cleaned up using the 'vpa' command to convert the rational whole number expressions to decimal form and the symbolic propagation operator can be replaced with this result using the 'subs' command.

$$
\begin{align*}
& \mathrm{TF}=\operatorname{vpa}(\mathrm{TF}, 5)  \tag{5.12}\\
& \mathrm{TF}= \\
& 1 /\left(1 .+.30034 \mathrm{e}-2 * \mathrm{~s}+.20403 \mathrm{e}-3 * \mathrm{~s}^{\wedge} 2\right) \tag{5.13}
\end{align*}
$$

This result cannot be processed by Matlab ${ }^{\circledR}$ commands designed for transfer functions such as the 'step' and 'bode' commands. One reason for this is that the Laplace operator in this result is symbolic. This can be converted to a Matlab ${ }^{\circledR}$ transfer function by performing the following operations.

$$
\begin{align*}
& \text { Num }=[1] \\
& \text { Den }=\left[\begin{array}{lll}
0.30034 e-2 & 0.20403 e-3 & 1
\end{array}\right]  \tag{5.14}\\
& \mathrm{TF}=\operatorname{tf}(\text { Num, Den })
\end{align*}
$$

This produces the following output in the Matlab ${ }^{\circledR}$ workspace command line.

## Transferfunction:

1
$0.000204 \mathrm{~s}^{\wedge} 2+0.003003 \mathrm{~s}+1$
This result can be processed by the commands 'bode' and 'step' to produce the frequency response and time domain response respectively.

$$
\begin{align*}
& \text { bode(TF) }  \tag{5.16}\\
& \text { step(TF) } \tag{5.17}
\end{align*}
$$

The natural frequency can be obtained from the characteristic equation of the second order RLC model as follows:

$$
\begin{gather*}
s^{2}+2 \varsigma_{n} \omega_{n} s+\omega_{n}^{2}  \tag{5.18}\\
s^{2}+\frac{R}{L} s+\frac{1}{L C}  \tag{5.19}\\
\omega_{n}=\sqrt{\frac{1}{L C}}=70 \tag{5.20}
\end{gather*}
$$



Figure 5.2 Frequency response plot of lumped parameter model


Figure 5.3 Time domain step input response plot of lumped parameter model

### 5.2.2 Frequency and Time Response of the Dissipative Model

This approach can also be applied to the more complex dissipative model. One major advantage of this approach is that the causality does not need to be considered as in the modal approximation method. The following symbolic equations can always be used to define the line dynamics.

Distributed Parameter Line Equations

$$
\begin{align*}
& \text { Qin }=\mathrm{C} 11 * \text { Pin }-\mathrm{C} 21 * \text { Pout }  \tag{5.21}\\
& \text { Qout }=\mathrm{C} 21 * \text { Pin }-\mathrm{C} 11 * \text { Pout } \tag{5.22}
\end{align*}
$$

Distributed Parameter Transfer Functions, $\mathrm{C}_{1}$ and $\mathrm{C}_{2}$

$$
\begin{align*}
& \mathrm{C} 1=\cosh (\mathrm{G}) /(\mathrm{Z} * \sinh (\mathrm{G}))  \tag{5.23}\\
& \mathrm{C} 2=1 /(\mathrm{Z} * \sinh (\mathrm{G})) \tag{5.24}
\end{align*}
$$

Impedance Constant, $\mathrm{Z}_{0}$

$$
\begin{equation*}
\mathrm{Z}=\mathrm{Z} 0 /(\operatorname{sqrt}(1-\mathrm{B}) * \operatorname{sqrt}(1+(\mathrm{v}-1) * \operatorname{sigma})) \tag{5.25}
\end{equation*}
$$

Normalized Propagation Operator, $\Gamma$

$$
\begin{equation*}
\mathrm{G}=\mathrm{Dn} * \mathrm{~s} * \operatorname{sqrt}((1+(\mathrm{v}-1) * \operatorname{Bigma}) /(1-\mathrm{B})) \tag{5.26}
\end{equation*}
$$

Unnormalized Propagation Operator, $\Gamma$

$$
\begin{equation*}
\mathrm{G}=\mathrm{Dn} *\left(\mathrm{~s}^{*}\left(\mathrm{r}^{\wedge} 2\right) / \mathrm{Vis}\right) * \operatorname{sqrt}((1+(\mathrm{v}-1) * \mathrm{Bsigma}) /(1-\mathrm{B})) \tag{5.27}
\end{equation*}
$$

Normalized Bessel Function Ratio with Prandtl Number, $B_{r}$ o

$$
\begin{align*}
\text { Bsigma }= & 2 * \operatorname{besselj}(1, \mathrm{j} * \operatorname{sqrt}(\operatorname{sigma} * \mathrm{~s})) / \ldots \\
& (\mathrm{j} * \operatorname{sqrt}(\operatorname{sigma} * \mathrm{~s}) * \operatorname{besselj}(0, \mathrm{j} * \operatorname{sqrt}(\operatorname{sigma} * \mathrm{~s}))) \tag{5.28}
\end{align*}
$$

Unnormalized Bessel Function Ratio with Prandtl Number, $\mathrm{B}_{\mathrm{r}}$

$$
\begin{align*}
\operatorname{Bsigma}= & 2 * \operatorname{besselj}\left(1, \mathrm{j}^{*} \operatorname{sqrt}\left(\operatorname{sigma} *\left(\mathrm{~s}^{*}\left(\mathrm{r}^{\wedge} 2\right) / \mathrm{Vis}\right)\right)\right) / \ldots \\
& \left(\mathrm{j}^{*} \operatorname{sqrt}\left(\operatorname{sigma} *\left(\mathrm{~s}^{*}\left(\mathrm{r}^{\wedge} 2\right) / \mathrm{Vis}\right)\right) * \ldots\right.  \tag{5.29}\\
& \left.\operatorname{besselj}\left(0, \mathrm{j}^{*} \operatorname{sqrt}\left(\operatorname{sigma} *\left(\mathrm{~s}^{*}\left(\mathrm{r}^{\wedge} 2\right) / \mathrm{Vis}\right)\right)\right)\right)
\end{align*}
$$

Normalized and Unnormalized Bessel Function Ratio, $\mathrm{B}_{\mathrm{r}}$

$$
\begin{align*}
& B=2 * \operatorname{besselj}\left(1, j^{*} \operatorname{sqrt}(\mathrm{~s})\right) /\left(\mathrm{j}^{*} \operatorname{sqrt}(\mathrm{~s}) * \operatorname{besselj}\left(0, \mathrm{j}^{*} \operatorname{sqrt}(\mathrm{~s})\right)\right)  \tag{5.30}\\
& B=2 * \operatorname{besselj}\left(1, \mathrm{j}^{*} \operatorname{sqrt}\left(\left(\mathrm{~s}^{*}\left(\mathrm{r}^{\wedge} 2\right) / \operatorname{Vis}\right)\right)\right) / \ldots \\
& \left(\mathrm{j}^{*} \operatorname{sqrt}\left(\left(\mathrm{~s}^{*}\left(\mathrm{r}^{\wedge} 2\right) / \operatorname{Vis}\right)\right) * \operatorname{besselj}\left(0, \mathrm{j}^{*} \operatorname{sqrt}\left(\left(\mathrm{~s}^{*}\left(\mathrm{r}^{\wedge} 2\right) / \operatorname{Vis}\right)\right)\right)\right) \tag{5.31}
\end{align*}
$$

The two dissipative line equations can be solved as follows using the 'solve' command to obtain a symbolic solution for the transfer function.

$$
\begin{equation*}
\text { Sol }=\text { solve }(' \text { Qin }=\mathrm{C} 11-\mathrm{C} 21 * \text { Pout', } ' 0=\mathrm{C} 21-\mathrm{C} 11 * \text { Pout', 'Qin, Pout' }) \tag{5.32}
\end{equation*}
$$

As in the lumped parameter model, the input pressure value, $\mathrm{P}_{\mathrm{in}}$, is expressed as unity in order to obtain the transfer function of the line rather than the particular solution.

The solve command provides the following symbolic solution for the transfer function.

## $\mathrm{TF}=$

C21/C11
The next step is to replace the symbolic terms by actual numeric values using the 'subs' command. This requires that the numeric values for C 11 and C 21 already exist in the Matlab ${ }^{\circledR}$ workspace.

$$
\begin{align*}
& \mathrm{TF}=\operatorname{subs}\left(\mathrm{TF}, \mathrm{C}^{\prime} 21 ', \mathrm{C} 21\right)  \tag{5.34}\\
& \mathrm{TF}=\operatorname{subs}\left(\mathrm{TF},{ }^{\prime} \mathrm{C} 111^{\prime}, \mathrm{C} 11\right) \tag{5.35}
\end{align*}
$$

If the propagation operator, $\Gamma$, is treated as a symbol, the result is the familiar distributed parameter transfer function for a blocked line, $1 / \cosh \Gamma$. The computation of the propagation operator produces a rather complex result.

$$
\begin{aligned}
& \mathrm{G}= \\
& 60603375026040832 / 4242751136953196875 * \mathrm{~s} . . \\
& *(1 /(1-25 / 1073741824 * \text { besseli }(1,1073741824 / \ldots \\
& 169710045478127875 * 13576803638250230^{\wedge}(1 / 2) \ldots \\
& \left.* \mathrm{~s}^{\wedge}(1 / 2)\right)^{*} 13576803638250230^{\wedge}(1 / 2) / \mathrm{s}^{\wedge}(1 / 2) / \ldots \\
& \operatorname{besseli}(0,1073741824 / 169710045478127875 * \ldots \\
& \left.\left.\left.13576803638250230^{\wedge}(1 / 2) * \mathrm{~s}^{\wedge}(1 / 2)\right)\right)\right)^{\wedge}(1 / 2)
\end{aligned}
$$

This can be cleaned up using the 'vpa' command and the symbolic propagation operator can be replaced with this result using the 'subs' command.

$$
\begin{align*}
& \mathrm{G}= \\
& .14284 \mathrm{e}-1 * \mathrm{~s}^{*}\left(1 /\left(1 .-2.7129 * \operatorname{besseli}\left(1, .73721^{*} \mathrm{~s}^{\wedge}(1 / 2)\right) / \mathrm{s}^{\wedge}(1 / 2) \ldots\right.\right.  \tag{5.37}\\
& \left.\left./ \operatorname{besseli}\left(0, .73721^{*} \mathrm{~s}^{\wedge}(1 / 2)\right)\right)\right)^{\wedge}(1 / 2) \\
& \mathrm{TF}= \\
& 1 / \cosh \left(.14284 \mathrm{e}-1 * \mathrm{~s}^{*}\left(1 /\left(1 .-2.7129 * \operatorname{besseli}\left(1 ., .73721 * \mathrm{~s}^{\wedge}(1 / 2)\right) / \mathrm{s}^{\wedge}(1 / 2) / \ldots\right.\right.\right.  \tag{5.38}\\
& \left.\left.\left.\operatorname{besseli}\left(0, .73721 * \mathrm{~s}^{\wedge}(1 / 2)\right)\right)\right)^{\wedge}(1 / 2)\right)
\end{align*}
$$

It is important to mention that not only is this result not a convenient rational polynomial form but also that this form is not recognized as a transfer function by Matlab ${ }^{\circledR}$. This result cannot be processed by Matlab ${ }^{\circledR}$ commands designed for transfer functions such as the 'step' and 'bode' commands. The Laplace operator in this result is symbolic and needs to be replaced by $\mathrm{j} \omega$ which must be an array of a specified frequency range. A range of $1: 10,000$ is usually sufficient to cover all significant modes of the response.
$\mathrm{w}=[1: 10000]$
TFfreqs $=\operatorname{subs}\left(\mathrm{TF}, \mathrm{s}^{\prime}, \mathrm{j}^{*} \mathrm{w}\right)$

The result is 10,000 data points that can now be graphed using the following commands.

TFmag $=20 * \log 10($ abs(TFfreqs $))$
TFphase $=$ angle $($ TFfreqs $) * 180 /$ pi
$\mathrm{g}=$ figure('Name', 'Magnitude Plot'); semilogx(w, TFmag) title('Magnitude Plot');
xlabel('Frequency rad/sec');
ylabel('Decibels');
h = figure('Name',' Phase Plot');
semilogx(w, TFphase)
title('Phase Plot');
xlabel('Frequencyrad/sec'); ylabel('Degrees');


Figure 5.4 Frequency magnitude response of dissipative model


Figure 5.5 Frequency response phase plot of dissipative model
The complex transfer function needs to be processed by the 'invfreqs' command to produce a rational polynomial transfer function. By inspection of the frequency response graph, the appropriate order can be determined for the desired frequency range.


Figure 5.6 Determination of required number of modes
In this example the frequency range to be analyzed is $1000 \mathrm{rad} / \mathrm{sec}$. This requires the rational polynomial approximation to contain five $2^{\text {nd }}$ order modes.

$$
\begin{equation*}
\mathrm{wa}=[1: 1000] \tag{5.45}
\end{equation*}
$$

Here the exact solution needs to be scaled back from $10,000 \mathrm{rad} / \mathrm{sec}$ to $1,000 \mathrm{rad} / \mathrm{sec}$ in order to be processed by the 'invfreqs' command.

TFfreqswa $=\operatorname{subs}\left(T F, s^{\prime}, j^{*}\right.$ wa $)$

Since there are five $2^{\text {nd }}$ order modes, a $10^{\text {th }}$ order characteristic equation needs to used. The numerator should be one order less than the denominator to produce a stable transfer function.

$$
\begin{align*}
& \text { [Num, Den] = invfreqs(TFfreqswa, wa,9,10) }  \tag{5.47}\\
& \mathrm{TFA}=\operatorname{tf}(\mathrm{Num}, \mathrm{Den})  \tag{5.48}\\
& \text { Transfer function : } \\
& 160.3 \mathrm{~s}^{\wedge} 9+1.354 \mathrm{e} 005 \mathrm{~s}^{\wedge} 8-2.618 \mathrm{e} 008 \mathrm{~s}^{\wedge} 7+1.592 \mathrm{e} 011 \mathrm{~s}^{\wedge} 6-1.437 \mathrm{e} 014 \mathrm{~s}^{\wedge} 5+\ldots \\
& 6.554 \mathrm{e} 016 \mathrm{~s}^{\wedge} 4-2.792 \mathrm{e} 019 \mathrm{~s}^{\wedge} 3+6.89 \mathrm{e} 021 \mathrm{~s}^{\wedge} 2-1.346 \mathrm{e} 024 \mathrm{~s}+2.952 \mathrm{e} 026  \tag{5.49}\\
& \mathrm{~s}^{\wedge} 10+241.2 \mathrm{~s}^{\wedge} 9+1.891 \mathrm{e} 006 \mathrm{~s}^{\wedge} 8+3.435 \mathrm{e} 008 \mathrm{~s}^{\wedge} 7+1.141 \mathrm{e} 012 \mathrm{~s}^{\wedge} 6+1.444 \mathrm{e} 014 \\
& 2.511 \mathrm{e} 017 \mathrm{~s}^{\wedge} 4+1.904 \mathrm{e} 019 \mathrm{~s}^{\wedge} 3+1.696 \mathrm{e} 022 \mathrm{~s}^{\wedge} 2+7.076 \mathrm{e} 023 \mathrm{~s}+1.814 \mathrm{e} 026
\end{align*}
$$

The following commands are used to obtain the frequency response of the approximate rational polynomial transfer function and plot a comparison graph with the frequency response of the exact solution.

TFAfreqs $=$ freqs(Num,Den, w)
TFAmag $=20 * \log 10(\operatorname{abs}($ TFAfreqs $))$
TFmag $=20 * \log 10($ abs(TFfreqs $))$
$\mathrm{g}=$ figure('Name','Magnitude Plot');
semilogx(w, TFmag,'b', w, TFAmag,'r');
title('Magnitude Plot');
xlabel('Frequency rad/sec');
ylabel('Decibels');


Figure 5.7 Frequency magnitude response of approximation without fit-error weighting

This result accurately models the exact response at the high frequencies but fails at the low end of the frequency range. The gain of the approximation is also very inaccurate. The 'invfreqs' command can be modified to allow fit-errors to be weighted verses frequency.
[Num, Den] = invfreqs(TFfreqs, wa, $9,10, \mathrm{wt}, 100)$;


Figure 5.8 Frequency magnitude response of approximation with fit-error weighting

This result almost exactly models the exact solution out to $1000 \mathrm{rad} / \mathrm{sec}$. The steady state gain of the approximation should be unity. The frequency magnitude response curve shows a small error. The gain can be calculated with the following command.

$$
\begin{align*}
& \text { Gain }=\text { dcgain(TFA) }  \tag{5.55}\\
& \text { Gain }= \\
& 1.04894105026723 \tag{5.56}
\end{align*}
$$

The approximated transfer function can be divided by this amount to achieve a steady state gain of unity.


Figure 5.9 Frequency magnitude response of approximation with normalized gain

Setting the steady state gain to unity resulted in a small increase in error of the response at higher frequencies. Figure 5.10 shows the comparison magnitude and phase plots of modal approximations using 'invfreqs' where the maximum frequency for each mode is the resonant frequency of the respective mode.


Figure 5.10 Frequency response of approximation with normalized gain for the first five modes

These approximations more accurately model the exact solution than the modal approached covered in Chapter 3. A more accurate approximation can be achieved by increasing the order of each mode by a factor of one. The drawback is that the transfer function is of higher order, which is contrary to the objective of order reduction.


Figure 5.11 Frequency magnitude response of approximation using additional order for each mode


Figure 5.12 Frequency phase response of approximation using additional order for each mode


Figure 5.13 Step input response of modal approximations using 'invfreqs'

As expected, a step input to each model does not result in much variation in the time domain response, but an impulse input results in significant variation in the. This illustrates the importance of having a model that is accurate in the frequency range that the system operates.


Figure 5.14 Impulse input response of modal approximations using 'invfreqs'

The goal of obtaining a finite order rational polynomial transfer function that accurately models that exact solution has been accomplished. Now the result can be implemented with other block elements to model a total system using classical control theory methods. An even better approach is to use the 'solve' command and 'invfreqs' to obtain an approximated transfer function for the total system.

### 5.3 Application of Matlab ${ }^{\circledR}$ Symbolic Toolbox Commands to Model a Total Fluid Transmission Line System

The 'solve' command can provide symbolic solutions for problems with any number of unknowns. In the following hydraulic brake system, there are seven transmission lines, four capacitive elements, and one resistive element.


Figure 5.15 Multiple line system with capacitive and resistive elements Each line is modeled with two equations for a total of 14 dissipative equations. There is three summation equations where lines are joined. Resistive and Capacitive Elements are modeled as lumped elements in the total system. This system is defined by a total of 22 equations. The 'solve' command is as follows:

$$
\begin{align*}
& \text { Sol }=\text { solve('1-P11 }=\text { Rin } * \text { Q11' }, \ldots \\
& \text { 'Q11 = C } 11 * \mathrm{P} 11-\mathrm{C} 21 * \mathrm{Ps} 1 ', \mathrm{Q} 21=\mathrm{C} 21 * \mathrm{P} 11-\mathrm{C} 11 * \mathrm{Ps} 1 ', \ldots \\
& ' \mathrm{Q} 12=\mathrm{C} 12 * \mathrm{Ps} 1-\mathrm{C} 22 * \mathrm{Ps} 2 ', \mathrm{Q} 22=\mathrm{C} 22 * \mathrm{Ps} 1-\mathrm{C} 12 * \mathrm{Ps} 2{ }^{\prime}, \ldots \\
& \text { 'Q13 }=\mathrm{C} 13 * \mathrm{Ps} 1-\mathrm{C} 23 * \operatorname{Ps} 3 ', ' \mathrm{Q} 23=\mathrm{C} 23 * \operatorname{Ps} 1-\mathrm{C} 13 * \text { Ps3', } \ldots \\
& \text { 'Q14 = C14*Ps2-C24*P24','Q24 = C24*Ps2-C14*P24', .. } \\
& \text { 'Q15 = C15*Ps2-C25*P25','Q25 = C25*Ps2-C15*P25', } \ldots \\
& \text { 'Q16 = C16*Ps3-C26*P26','Q26 = C26*Ps3-C16*P26',... }  \tag{5.57}\\
& \text { 'Q17 = C17 * Ps3-C27*P27','Q27 = C27 * Ps3-C17*P27', } \ldots \\
& \text { 'Q24 }=\mathrm{Cp} 4 * \mathrm{P} 24 * \mathrm{~s}^{\prime}, \mathrm{Q} 25=\mathrm{Cp} 5 * \mathrm{P} 25 * \mathrm{~s} \text { ', } \ldots \\
& \text { 'Q26 }=\mathrm{Cp} 6 * \mathrm{P} 26 * \mathrm{~s}^{\prime}, \mathrm{Q} 27=\mathrm{Cp} 7 * \mathrm{P} 27 * \mathrm{~s}^{\prime}, \ldots \\
& \text { 'Q21 = Q12 + Q13', 'Q22 = Q14 + Q15','Q23 = Q16 + Q17', } \ldots \\
& \text { 'Q11, Q21,Q12,Q22,Q13,Q23,Q14, Q24,Q15,Q25,Q16,Q26,Q17,Q27... } \\
& \text { P11,Ps1,Ps2, Ps3, P24, P25, P26,P27') }
\end{align*}
$$

The individual solutions (symbolic and numeric) contain too many terms to attempt to document. The following figures are the magnitude and phase plots for the pressure output of line number 7 .


Figure 5.16 Frequency response plots for seven line brake system


Figure 5.17 Frequency response modal approximation plots for seven line brake system


Figure 5.18 Step response modal approximation plot for seven line brake system


Figure 5.19 Impulse response modal approximation plot for seven line brake system

The advantage of combining the total system in 'solve' is that the order of the system transfer function for a given frequency range will be reduced. If this is not possible then the resulting transfer function must be combined in the classical method with transfer functions of the remaining system components.

### 5.4 Application of Matlab ${ }^{\circledR}$ Matrix Commands to Model a Total Fluid Transmission Line System

Another method of solving the system equations is to use a matrix. In this approach, the only symbolic term is the Laplace operator.

$$
\begin{equation*}
B=[A][X] \tag{5.58}
\end{equation*}
$$



The following Matlab ${ }^{\circledR}$ command solves this equation using Gaussian Elimination.

$$
\begin{equation*}
X=A \backslash B \tag{5.60}
\end{equation*}
$$

### 5.5 Application of Matlab ${ }^{\circledR}$ Graphical User Interface to Produce the Fluid Transmission System Analyzer

The Matlab ${ }^{\circledR}$ tools previously covered have been incorporated into a Graphical User Interface program that simplifies the process of obtaining a transfer function of a fluid transmission line system. This program is prompted by inputting 'FLRAR1' in the Matlab ${ }^{\circledR}$ command line. The Graphical User Interface initially appears as follows:


Figure 5.20 GUI opening screen shot
The first thing required is to input the material constants of the fluid and the line in the upper left corner as in the following example.

Note: all parameters are derived from metric units ( $\mathrm{m}, \mathrm{kg}$, and s ).


Figure 5.21 GUI material constants section


Figure 5.22 GUI model properties section
After completing the Model Properties section, the Model Diagram section will automatically illustrate the model with a line diagram as shown here.


Figure 5.23 GUI model diagram section
The individual line lengths are specified in the Line Lengths section.


Figure 5.24 GUI line lengths section
The program will place the word "OUTPUT" above the lines that are output lines. The input valve resistance, the output valve resistance and output capacitance values are then placed in the following fields.


Figure 5.25 GUI line lengths section

Rout and CPout is restricted to the same values for all outputs. The program is now ready to calculate the frequency response of the system using the Dissipative line model. Click on the Frequency Response Analysis button to calculate the frequency response.


Figure 5.26 GUI system solution section before analysis
The program uses the following algorithm to determine the modal frequencies.

```
a=0
for m=1
for k=1:9990
if TFmag(k+1) >= TFmag(k)
if TFmag(k+1) >= TFmag(k+2)
MF(m)= w(k+1)
m=m+1
end
end
a=a+1
end
end
```



Figure 5.27 GUI system solution section after analysis
Occasionally the algorithm fails to match mode to frequency, and a manual override is necessary. The algorithm fails when modes do not have a local maximum point. The manual frequency input can be determined by inspecting the frequency magnitude response plot. The Upper Frequency value automatically shows after selecting the number of modes. Finally, the Run Configuration button is pushed to calculate the modal approximation and plot the approximated response against the exact response.


Figure 5.28 GUI frequency response comparison output
The transfer function is displayed in the Matlab ${ }^{\circledR}$ command line.

Transfer function:

```
7.122 s^8-1347 s^7 + 4.346e005 s^6-2.016e006 s^5 + 1.803e010 s^4-4.622e012 s^3
        +1.981e013 s^2 + 1.167e017 s+3.724e018
    s}\mp@subsup{s}{}{\wedge}9+208.9\mp@subsup{s}{}{\wedge}8+2.1e005\mp@subsup{s}{}{\wedge}7+3.197e007 \mp@subsup{s}{}{\wedge}6+1.217e010 s^5 + 1.21le012 s^
+1.923e014 s^3 + 1.033e016 s^2 + 2.607e017 s + 3.724e018
```

The file can also be saved and loaded as needed. This command only saves the solution parameters. The Frequency Response Analysis button needs to be pushed after opening the file.


Figure 5.29 GUI complete screen shot


Figure 5.30 GUI layout editor screen shot


Figure 5.31 GUI files

### 5.6 Fluid Transmission System Analyzer Frequency Selection Application

A frequency response may be obtained that does not have modal frequencies defined by a local maxima. The following figure is the result of a five mode approximation of the same seven line system covered in the previous section. The program fails to produce a perfect curve fit because the incorrect modal frequency is selected by the program.


Figure 5.32 Failure of frequency determination algorithm
The frequency selected for the fifth mode is $566 \mathrm{rad} / \mathrm{sec}$. By inspection of the plot this is the frequency of the sixth mode. The problem in this example is the fourth mode does not have a local maxima. In cases like this the Frequency Override feature
must be used. A frequency of $425 \mathrm{rad} / \mathrm{sec}$ is determined by visual inspection and entered into the Frequency Override field and the configuration is rerun to produce the following approximation.


Figure 5.33 Accurate approximation using frequency override

## CHAPTER 6

## SUMMARY OF RESEARCH AND RECOMMENDATION FOR FUTURE STUDY

### 6.1 Summary of Research

This research presents a method of solving for a total fluid transmission line system using the dissipative model and then applying the Matlab ${ }^{\circledR}$ 'invfreqs' algorithm to the frequency response of the total system rather than individual lines. This can be accomplished using the symbolic 'solve' command or by matrix operations using a symbolic Laplace operator. The advantage of this is apparent when there are several lines in the system. Previously each line would be approximated in the Laplace domain and the linear transfer functions lumped together. In this research, one approximation is made on the total system rather than combining several approximations eliminating combined error.

In addition to efficiently combining multiple fluid lines all the elements can be combined before performing a modal approximation. Lumped resistive and capacitive components as well as higher order elements can be combined with the lines and solved for the frequency response of a particular output. Any element in the system can be nonlinear. The only requirement is that the element is represented in the Laplace domain.

An algorithm for matching mode to frequency is also introduced and all the tools presented in this work are combined into a single user friendly program using the

Matlab ${ }^{\circledR}$ Graphical User Interface. This program provides an efficient method to analyze a fluid transmission line system. To obtain a copy of the program send an email request to: johnduaneking@hotmail.com.

### 6.2 Recommendation for Future Study

The modifications made to the modal solution provided by 'invfreqs' was to normalize the transfer function to produce a steady state gain of unity and to increase the order of the total number of modes by one. Improving the accuracy of the approximation process so that the additional order can be reduced would be a major improvement. The algorithm to determine modal frequency is based on the local maxima of the frequency magnitude response. Occasionally, the modal frequency is not defined by a local maxima and a manual determination of the frequency needs to be made. Perhaps and algorithm based on inflection points would perform better. The phase plot appears to have distinct inflection points.

The Matlab ${ }^{\circledR}$ features applied in this work are only a few of the tools available to compute and analyze system models. One suggestion would be to produce a graphical user interface that permits the inclusion of all system components. The program introduced in this work is limited to multiple connected lines, an input resistive element and output resistive and capacitive elements. A more comprehensive program should be more flexible to allow for the input of all components of the system. Another approach would be to find a method to automatically input a transfer function into Matlab ${ }^{\circledR}$ Simulink and integrate this feature into an analysis program.

## APPENDIX A

MODEL COMPARISON PROGRAM

The following Matlab ${ }^{\circledR}$ program plots a comparative frequency response of each of the five models.

```
function system = MODELCOMP
syms s
%%%%%%%%%%%%%%%%%Model Properties%%%%%%%%%%%%%%%%%%%%%%%
p=870;
Vis=4.6e-5;
sigma=1;
v=1;
Bf=1.21e9;
Bl=1.73e7;
L=2;
d=0.01;
r=d/2;
V=L*pi*r^2;
w=[1:1000];
Pin=5e6;
Be}=(\textrm{Bf}*\textrm{Bl})/(\textrm{Bf}+\textrm{Bl}
c=sqrt(Be/p)
Dn=(Vis*L/(c*r`^2))
wc=c/L;
wv=Vis/r^2;
%%%%%%%%%Model 1: Lumped Parameter Model Calculation%%%%%%%%%%%%%
LI=(p*L)/(pi*r^^2)
CP=V/Be
RL=(128*Vis*p*L)/(pi*d^4)
Sol=solve('Pin-Pout=(RL+LI*s)*Qin','Qin=CP*Pout*s','Pout,Qin')
Pout=Sol.Pout
TFM1=Pout
TFM1=subs(TFM1,'LI',LI);
TFM1=subs(TFM1,'CP',CP);
TFM1=subs(TFM1,'RL',RL);
TFM1=subs(TFM1,'wv',wv);
TFM1=subs(TFM1,'Pin',Pin);
TFM1=subs(TFM1,'s',j*w);
TFM1MAG=20*}\operatorname{log}10(abs(TFM1))
```

$\% \% \% \% \% \% \% \% \% \% \% \% \%$ Model 2 : Lossless Line Model Calculation\%\%\%\%\%\%\%\%\%\%\%
$\mathrm{G} 2=\mathrm{L} * \mathrm{~s} / \mathrm{c}$;
TFM2 $=$ Pin $/ \cosh (\mathrm{G} 2)$;
TFM2=subs(TFM2,'s',j*w);
TFM2MAG $=20 * \log 10(\operatorname{abs}($ TFM2 $)$ );
TFM2PHASE=angle(TFM2)*180/pi;
\%\%\%\%\%\%\%\%\%\%\%\%Model 3: Linear Friction Model Calculation\%\%\%\%\%\%\%\%\%\%\%
$\mathrm{G} 3=((\mathrm{s} / \mathrm{wc})){ }^{*} \operatorname{sqrt}\left(1+\left(8^{*} \mathrm{wv} /(\mathrm{s})\right)\right)$;
TFM3 $=$ Pin $/ \cosh (\mathrm{G} 3)$;
TFM3=subs(TFM3,'s',j*w);
TFM3MAG $=20 * \log 10(\operatorname{abs}(\mathrm{TFM} 3))$;
TFM3PHASE=angle(TFM3)*180/pi;
\%\%\%\%\%\%\%\%\%\%Model 4: Viscous Model Calculation\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%
$\mathrm{B}=1 / \operatorname{sqrt}(1-(2 * \operatorname{besselj}(1, \mathrm{j} * \operatorname{sqrt}((\mathrm{~s} / \mathrm{wv}))) /(\mathrm{j} * \operatorname{sqrt}((\mathrm{~s} / \mathrm{wv})) * \operatorname{besselj}(0, \mathrm{j} * \operatorname{sqrt}((\mathrm{~s} / \mathrm{wv}))))))$;
$\mathrm{G} 4=\mathrm{B}^{*} \mathrm{~s} / \mathrm{wc}$;
TFM4 $=$ Pin $/ \cosh (\mathrm{G} 4)$;
TFM4=subs(TFM4,'s'; ${ }^{*}$ *w);
TFM4MAG $=20 * \log 10(\mathrm{abs}(\mathrm{TFM} 4))$;
TFM4PHASE=angle(TFM4)*180/pi;
$\% \% \% \% \% \% \% \% \% \% \%$ Model 5 : Dissipative Model Calculation $\% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \%$
Bsigma=2*besselj(1,j*sqrt(sigma*(s/wv)))/(j*sqrt(sigma*(s/wv))*besselj(0,j*sqrt(sigma*(s/wv))));
$\mathrm{B}=2 * \operatorname{besselj}(1, \mathrm{j} * \operatorname{sqrt}((\mathrm{~s} / \mathrm{wv}))) /\left(\mathrm{j}^{*} \operatorname{sqrt}((\mathrm{~s} / \mathrm{wv})) * \operatorname{besselj}(0, \mathrm{j} * \operatorname{sqrt}((\mathrm{~s} / \mathrm{wv})))\right)$;
$\mathrm{G} 5=\mathrm{Dn} *(\mathrm{~s} / \mathrm{wv}) * \operatorname{sqrt}((1+(\mathrm{v}-1) *$ Bsigma $) /(1-\mathrm{B})) ;$
TFM5 $=$ Pin/ $\cosh (G 5)$;
TFM5=subs(TFM5,'s',j*w);
TFM5MAG=20* $\log 10(\operatorname{abs}(\mathrm{TFM} 5))$;
TFM5PHASE=angle(TFM5)*180/pi;
\%\%\%\%\%\%\%\%\%\%\%Comparison Plot\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%
$\mathrm{g}=$ figure('Name','Magnitude Plot');
semilogx(w,TFM1MAG,w,TFM2MAG,w,TFM3MAG,w,TFM4MAG,w,TFM5MAG)
title('Magnitude Plot');
xlabel('Frequency rad/sec');
ylabel('Decibels');
end

## APPENDIX B

TRUNCATED PRODUCT SERIES MODAL APPROXIMATION PROGRAM

The following Matlab ${ }^{\circledR}$ program calculates the modal approximation transfer function for $1 / \cosh \Gamma$ and compares the resulting frequency response with $1 / \cosh \Gamma$. Note that you must have the tables imported.

```
function system = MODAL(sigma,v,p,Vis,Bf,Bl,L,d,i)
%%%%%%%%%%%%%%%%LOAD TABLES%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
syms H P;
form = input('Hydraulic or Pneumatic (H/P):')
if form = H
data = xlsread('data-liquid.xls');
else
data = xlsread('data-air.xls');
end
%%%%%%%%%%%%CALCULATE DISSIPATION NUMBER%%%%%%%%%%%%%%%%%%%
r=d/2;
Be=(Bf*Bl)/(Bf+Bl)
c=sqrt(Be/p)
Dn=(Vis*L/(c*r^2))
%%%%%%%%%%%%CALCULATE COSH ROOT INDICES%%%%%%%%%%%%%%%%%%%%%
if i==1
i=[1];
else if i==2
i=[1 2];
else if i==3
i=[lllll
else if i==4
i=[lllllll
else if i==5
i=[lllllll
end
end
end
end
end
Lambda_c=(1/Dn)*(i-1/2)
```

```
%%%%%%%%%%%%CALCULATE RESIDUE COEFFICIENTS%%%%%%%%%%%%%%%%%%%
DI = data(:,1);
NF = data(:,2);
DR = data(:,3);
CA1 = data(:,4);
CB1 = data(:,5);
w = interpl(DI,NF,Lambda_c)
d = interp1(DI,DR,Lambda_c)
cosh_residue_a = interp1(DI,CA1,Lambda_c)
cosh_residue_b = interp1(DI,CB1,Lambda_c)
a = cosh_residue_a./(((-1).^(i+1)).*(1-2.*i))
b}=\mathrm{ cosh_residue_b./(((-1).^(i+1)).*(1-2.*i))
%%%%%%%%%%% CALCULATE RATIONAL POLYNOMIAL APPROXIMATION %%%%%%%%
Li=length(i);
for }\textrm{k}=1:\textrm{Li
num=[a(k)b(k)]
den=[1 2*d(k)*w(k) w(k)^2]
APPROXTF(k)=tf(num,den)
k=k+1;
end
%%%%%%%%%%%% EXACT MODEL CALCULATION %%%%%%%%%%%%%%%%%%%%%
w}=[0.01:1000]
syms s
Z0=(p*c)/(pi*r^2);
Bsigma=2*besselj(1,j*sqrt(sigma*(s)))/(j*sqrt(sigma*(s))*besselj(0,j*sqrt(sigma*(s))));
B=2*\operatorname{besselj(1,j*sqrt((s)))/(j*sqrt((s))*besselj(0,j*sqrt((s))));}
Z=Z0/(sqrt(1-B)*sqrt(1+(v-1)*Bsigma));
G=Dn*(s)*sqrt((1+(v-1)*Bsigma)/(1-B));
EXACT=1/cosh(G);
EXACT=vpa(EXACT,5)
EXACT=subs(EXACT,'s',j*w);
EXACTMAG=20*}\operatorname{log}10(abs(EXACT))
EXACTPHASE=angle(EXACT)*180/pi;
%%%%%%%%%%%%% MODAL COMPARISON TO EXACT %%%%%%%%%%%%%%%%%%%%
if length(i)==1
Gainsum1=dcgain(APPROXTF(1));
TFsum1=APPROXTF(1)/Gainsum1
[n1,d1] = tfdata(TFsum1,'v');
APPROX1=freqs(n1,d1,w);
APPROXMAG1=20*log10(abs(APPROX1));
APPROXPHASE1=angle(APPROX1)*180/pi;
g = figure('Name','Magnitude Plot');
```

```
semilogx(w,EXACTMAG,w,APPROXMAG1)
title('Magnitude Plot');
xlabel('Normalized Frequency rad/sec');
ylabel('Decibels');
h = figure('Name','Phase Plot');
semilogx(w,EXACTPHASE,w,APPROXPHASE1)
title('Phase Comparison Plots');
xlabel('Normalized Frequency rad/sec');
ylabel('Degrees');
elseif length(i)=2
Gainsum1=dcgain(APPROXTF(1));
TFsum1=APPROXTF(1)/Gainsum1;
TFsum2=(APPROXTF(1)+APPROXTF(2));
Gainsum2=dcgain(TFsum2);
TFsum2=TFsum2/Gainsum2;
[n1,d1] = tfdata(TFsum1,'v');
[n2,d2] = tfdata(TFsum2,'v');
APPROX1=freqs(n1,d1,w);
APPROX2=freqs(n2,d2,w);
APPROXMAG1=20*\operatorname{log}10(abs(APPROX1));
APPROXMAG2=20*log10(abs(APPROX2));
APPROXPHASE1=angle(APPROX1)*180/pi;
APPROXPHASE2=angle(APPROX2)*180/pi;
g = figure('Name','Magnitude Plot');
semilogx(w,EXACTMAG,w,APPROXMAG1,w,APPROXMAG2)
title('Magnitude Plot');
xlabel('Normalized Frequency rad/sec');
ylabel('Decibels');
h = figure('Name','Phase Plot');
semilogx(w,EXACTPHASE,w,APPROXPHASE1,w,APPROXPHASE2)
title('Phase Comparison Plots');
xlabel('Normalized Frequency rad/sec');
ylabel('Degrees');
elseif length(i)==3
Gainsum1=dcgain(APPROXTF(1));
TFsum1=APPROXTF(1)/Gainsum1
TFsum2=(APPROXTF(1)+APPROXTF(2))
Gainsum2=dcgain(TFsum2);
TFsum2=TFsum2/Gainsum2
TFsum3=(APPROXTF(1)+APPROXTF(2)+APPROXTF(3))
Gainsum3=dcgain(TFsum3);
TFsum3=TFsum3/Gainsum3
[n1,d1] = tfdata(TFsum1,'v');
```

```
[n2,d2] = tfdata(TFsum2,'v');
[n3,d3] = tfdata(TFsum3,'v');
APPROX1=freqs(n1,d1,w);
APPROX2=freqs(n2,d2,w);
APPROX3=freqs(n3,d3,w);
APPROXMAG1=20*log10(abs(APPROX1));
APPROXMAG2=20*log10(abs(APPROX2));
APPROXMAG3=20*log10(abs(APPROX3));
APPROXPHASE1=angle(APPROX1)*180/pi;
APPROXPHASE2=angle(APPROX2)*180/pi;
APPROXPHASE3=angle(APPROX3)*180/pi;
g = figure('Name','Magnitude Plot');
semilogx(w,EXACTMAG,w,APPROXMAG1,w,APPROXMAG2,w,APPROXMAG3)
title('Magnitude Plot');
xlabel('Normalized Frequency rad/sec');
ylabel('Decibels');
h = figure('Name','Phase Plot');
semilogx(w,EXACTPHASE,w,APPROXPHASE1,w,APPROXPHASE2)
title('Phase Comparison Plots');
xlabel('Normalized Frequency rad/sec');
ylabel('Degrees');
elseif length(i)==4
Gainsum1=dcgain(APPROXTF(1));
TFsum1=APPROXTF(1)/Gainsum1
TFsum2=(APPROXTF(1)+APPROXTF(2));
Gainsum2=dcgain(TFsum2);
TFsum2=TFsum2/Gainsum2
TFsum3=(APPROXTF(1)+APPROXTF(2)+APPROXTF(3));
Gainsum3=dcgain(TFsum3);
TFsum3=TFsum3/Gainsum3
TFsum4=(APPROXTF(1)+APPROXTF(2)+APPROXTF(3)+APPROXTF(4));
Gainsum4=dcgain(TFsum4);
TFsum4=TFsum4/Gainsum4
[n1,d1] = tfdata(TFsum1,'v');
[n2,d2] = tfdata(TFsum2,'v');
[n3,d3] = tfdata(TFsum3,'v');
[n4,d4] = tfdata(TFsum4,'v');
APPROX1=freqs(n1,d1,w);
APPROX2=freqs(n2,d2,w);
APPROX3=freqs(n3,d3,w);
APPROX4=freqs(n4,d4,w);
APPROXMAG1=20*log10(abs(APPROX1));
APPROXMAG2=20*log10(abs(APPROX2));
APPROXMAG3=20*log10(abs(APPROX3));
APPROXMAG4=20*log10(abs(APPROX4));
```

```
APPROXPHASE1=angle(APPROX1)*180/pi;
APPROXPHASE2=angle(APPROX2)*180/pi;
APPROXPHASE3=angle(APPROX3)*180/pi;
APPROXPHASE4=angle(APPROX4)*180/pi;
g = figure('Name','Magnitude Plot');
semilogx(w,EXACTMAG,w,APPROXMAG1,w,APPROXMAG2,w,APPROXMAG3,w,APPROXMAG
4)
title('Magnitude Plot');
xlabel('Normalized Frequency rad/sec');
ylabel('Decibels');
h = figure('Name','Phase Plot');
semilogx(w,EXACTPHASE,w,APPROXPHASE1,w,APPROXPHASE2,w,APPROXPHASE3,w,APPRO
XPHASE4)
title('Phase Comparison Plots');
xlabel('Normalized Frequency rad/sec');
ylabel('Degrees');
end
```


## APPENDIX C

'INVFREQS' MODAL APPROXIMATION COMPARISON PROGRAM

The following Matlab ${ }^{\circledR}$ program plots the frequency response of several modal approximations of a given fluid line using the 'invfreqs' command.

```
function system = MODALDISS
warning off
syms s
%%%%%%%%%%%%%%%%%%%%%%% Model Properties %%%%%%%%%%%%%%%%%%%%%%
p=870;
Vis=4.6e-5;
Bf=1.21e9;
Bl=1.73e7;
L=2;
d=0.01;
r=d/2;
V=L*pi*r^2;
Be=(Bf * Bl)/(Bf+Bl);
c=sqrt(Be/p);
Dn=(Vis*L/(c*r^2));
sigma=1;
v=1;
\(\% \% \% \% \% \% \% \% \% \%\) Number of Modal Approximations to Plot \(\% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \% \%\) ModeNum=5
\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\% Exact \%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%
\(B=2 * \operatorname{besselj}\left(1, j^{*} \operatorname{sqrt}\left(\left(s^{*}\left(r^{\wedge} 2\right) / V i s\right)\right)\right) /\left(j^{*} \operatorname{sqrt}\left(\left(s^{*}\left(r^{\wedge} 2\right) / V i s\right)\right)^{*} \operatorname{besselj}\left(0, j^{*} \operatorname{sqrt}\left(\left(s^{*}\left(r^{\wedge} 2\right) / V i s\right)\right)\right)\right)\);
Bsigma \(=2 * \operatorname{besselj}\left(1, j * \operatorname{sqrt}\left(\operatorname{sigma} *\left(s^{*}\left(r^{\wedge} 2\right) / V i s\right)\right)\right) /\left(j^{*} \operatorname{sqrt}\left(\operatorname{sigma} *\left(s^{*}\left(r^{\wedge} 2\right) / V i s\right)\right)^{*} \operatorname{besselj}\left(0, j *\right.\right.\) sqrt(sigma*\(\left(s^{*}\right.\) ( \(\left.\left.\mathrm{r}^{\wedge} 2\right) / \mathrm{Vis}\right)\) )));
\(\mathrm{Z} 0=(\mathrm{p} * \mathrm{c}) /\left(\mathrm{pi}^{*} \mathrm{r}^{\wedge} 2\right)\);
\(\mathrm{Z}=\mathrm{Z} 0 /\left(\operatorname{sqrt}(1-\mathrm{B}) * \operatorname{sqrt}\left(1+(\mathrm{v}-1)^{*}\right.\right.\) Bsigma \(\left.)\right)\);
\(\mathrm{G}=\mathrm{Dn} *\left(\mathrm{~s}^{*}\left(\mathrm{r}^{\wedge} 2\right) / \mathrm{Vis}\right) * \operatorname{sqrt}((1+(\mathrm{v}-1) *\) Bsigma \() /(1-\mathrm{B}))\);
\(\mathrm{w}=[1: 10000]\);
```

```
C11=cosh(G)/(Z*}\operatorname{sinh}(\textrm{G}))
C21=1/(Z*}\operatorname{sinh}(\textrm{G}))
Sol=solve('QIN=C11-C21*POUT','0=C21-C11*POUT','QIN,POUT');
TF=Sol.POUT;
TF=subs(TF,'C11',C11);
TF=subs(TF,'C21',C21);
TFfreqs=subs(TF,'s',j*w);
TFmag=20*log10(abs(TFfreqs));
TFphase=angle(TFfreqs)*180/pi;
```

```
%%%%%%%%%%%%%%%%%%%%%%Modal Frequency Range%%%%%%%%%%%%%%%%%%%%%%%
a=0;
for m=1;
for k=1:9990;
if TFmag(k+1) >= TFmag(k);
if TFmag(k+1)>= TFmag(k+2);
Modefreqs(m)= w(k+1);
m=m+1;
end;
end;
a=a+1;
end
end
```

\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\% Approximation Plots \%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%
$\mathrm{g}=$ figure('Name','Magnitude Plot');
for $\mathrm{m}=1$ :ModeNum;
Modefreqs(m);
wa $=[1: \operatorname{Modefreqs}(\mathrm{m})]$;
TFfreqs=subs(TF,'s',j*wa);
$\mathrm{wt}=$ ones(size(wa));
[Num,Den]=invfreqs(TFfreqs,wa,(2*m-1),(2*m),wt,100);
TFA=tf(Num,Den);
Gain=dcgain(TFA);
TFA=TFA/Gain;
[Num,Den] = tfdata(TFA,'v');
TFAfreqs=freqs(Num,Den,w);
TFAmag $=20 * \log 10$ (abs(TFAfreqs));
semilogx(w,TFmag,'b',w,TFAmag,'r');
hold on
title('Magnitude Plot');
xlabel('Frequency rad/sec');
ylabel('Decibels');
end
hold off
h = figure('Name','Phase Plot');
for $\mathrm{m}=1: 5$;
Modefreqs(m);
wa $=[1: \operatorname{Modefreqs}(\mathrm{m})]$;

TFfreqs=subs(TF,'s',j*wa);
wt=ones(size(wa));
[Num,Den]=invfreqs(TFfreqs,wa,(2*m-1),(2*m),wt,100);
TFA=tf(Num,Den);
Gain=dcgain(TFA);
TFA=TFA/Gain;
[Num,Den] = tfdata(TFA,'v');
TFAfreqs=freqs(Num,Den,w);
TFAphase=angle(TFAfreqs)*180/pi;
semilogx(w,TFphase,'b',w,TFAphase,'r');
hold on
title('Phase Plot');
xlabel('Frequency rad/sec');
ylabel('Degrees');
end

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## BIOGRAPHICAL STATEMENT

John King was born in Janesville, Wisconsin in 1962. After high school he joined the U.S. Air Force where he worked as a Target Intelligence Specialist at RAF Upper Heyford, England. He received a Bachelor of Science in Mechanical Engineering from the University of Texas at Arlington in 1993. He has worked in the telecommunications industry since 1994 and has been Sales Engineer at Fujitsu Network Communications, Inc. since 1998. He received his Master of Science in Mechanical Engineering from the University of Texas at Arlington in May 2006. His research interests are dynamic systems modeling and transmission line theory.

