A STUDY ON APPROXIMATIONS OF TOTALLY ACYCLIC COMPLEXES

by

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ABSTRACT

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Let R be a commutative local ring to which we associate the subcategory $\mathbf{K}_{\text{tac}}(R)$ of the homotopy category of R-complexes, consisting of totally acyclic complexes. Further suppose there exists a surjection of Gorenstein local rings $Q \xrightarrow{\varphi} R$ such that R can be viewed as a Q-module with finite projective dimension. Under these assumptions, Bergh, Jorgensen, and Moore define the notion of approximations of totally acyclic complexes. In this dissertation we make extensive use of these approximations and define several novel applications. In particular, we extend Auslander-Reiten theory from the category of R-modules over a Henselian Gorenstein ring and show that under the same assumptions, the triangulated category $\mathbf{K}_{\text{tac}}(R)$ has only finitely many distinct indecomposable totally acyclic complexes. We then present a classification scheme for this category based upon the decomposition into indecomposable complexes. Furthermore, we prove the existence of minimal approximations in the category. The authors above also apply the idea of right approximations to create resolutions of totally acyclic complexes. We provide further results with respect to these resolutions and introduce a minimality condition. Lastly,

we prove the uniqueness of such minimal resolutions and show several more properties which extend nicely from the module category.

TABLE OF CONTENTS

Introduction

The goal of this dissertation is to explore the theory of, and applications for, approximations of totally acyclic complexes. Given a commutative local ring R, the category of totally acyclic complexes, $\mathbf{K}_{\mathrm{tac}}(R)$, is a full triangulated subcategory of the (more well-known) homotopy category of chain complexes over R. Although first defined for totally acyclic complexes by Bergh, Jorgensen, and Moore in their 2019 paper *Totally Acyclic Approximations* (see [2]), the notion of approximations is hardly a novel concept as many categories exhibit objects with approximations associated to them. Right approximations, otherwise known as pre-covers and left approximations for pre-envelopes, are constructions developed as early as 1953 by Eckman and Schopf in [22]. They show that each module over any ring has an injective envelope or minimal left approximation. The dual analogue, a projective cover or minimal right approximation, was then given by Bass [24] in 1960 and each module over any perfect ring has a projective cover. In fact, given a pair of adjoint functors between any two categories, we may always obtain a pre-cover and pre-envelope via the counit and unit maps, respectively.

The modern, more categorical, notion of approximations was first defined by Auslander and Smalø in 1980 [17], with Enochs also defining them independently in 1981 [18]. The authors of [2] use this definition and utilize a pair of adjoint functors, between rings Q and R (where $Q \twoheadrightarrow R$ and R has finite projective dimension as a Q-module), to demonstrate existence of approximations for the category of totally acyclic complexes. In this dissertation, we take this one step further and show that under reasonable assumptions, *minimal* approximations also exist in this category.

To do this, we make use of categorical similarities between $\mathbf{K}_{\text{tac}}(R)$ and the stable category of totally reflexive R-modules denoted $\underline{\text{TR}}(R)$. Specifically, Buchweitz [21] proved in 1986 that the stable module category of maximal Cohen-Macaulay Q-modules, denoted $\underline{\text{MCM}}(Q)$, is a triangulated category and is equivalent to $\mathbf{K}_{\text{tac}}(Q)$ and the singularity category, denoted $\mathbf{D}^b_{sg}(Q)$. Furthermore, the same equivalences also apply to the categories $\underline{\text{TR}}(R)$ and $\mathbf{K}_{\text{tac}}(R)$. Later, in [25] Bergh, Jorgensen, and Oppermann show that the functor from $\mathbf{K}_{\text{tac}}(R)$ to $\mathbf{D}^b_{sg}(R)$ is fully faithful as depicted in the bottom right of the following diagram.

$$
\begin{array}{ccccc}
\underline{\text{MCM}}(Q) & \xleftarrow{\cong} & \mathbf{K}_{\text{tac}}(Q) & \xrightarrow{\cong} & \mathbf{D}^b_{\text{sg}}(Q) \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\
\underline{\text{TR}}(R) & \xleftarrow[\cong]{} & \mathbf{K}_{\text{tac}}(R) & \hookrightarrow & \mathbf{D}^b_{\text{sg}}(R)
\end{array}
$$

In essence, this communicates a deep connection between $\mathbf{K}_{\text{tac}}(R)$ and $\underline{\text{TR}}(R)$, as well as between $\mathbf{K}_{\text{tac}}(Q)$ and $\underline{\text{MCM}}(Q)$, thus giving reason to use structural properties of one to inspire what properties hold for the other. Therefore, we take the existence of covers in the stable category of totally reflexive modules as a "proof of concept" for the existence of covers in $\mathbf{K}_{\text{tac}}(R)$.

We begin this thesis by recalling some basic facts and definitions in Chapter One. Specifically, we give the full definition of a triangulated category and show the previously mentioned fact that $\mathbf{K}_{\text{tac}}(R)$ is a triangulated subcategory.

As stated by the authors of [2], the idea behind these approximations is to relate more complicated totally acyclic complexes with possibly infinite complexity to simpler, possibly periodic complexes. As such, we begin Chapter Two of this thesis by examining a class of rings for which there are only finitely many totally acyclic complexes to use as approximations. Namely, we extend the notion of Auslander-Reiten theory and finite Cohen-Macaulay (CM) type to the category of totally acyclic complexes. Furthermore, we provide a full account of all such finite quivers and show that they are completely analogous to those in the module category. All of which are discussed briefly, for the module case, at the end of Chapter One and to which the diligent reader may refer to [5] for a more thorough coverage.

In Chapter Three we aim to give a partial answer to a question posed in [2]: *Can one classify the objects of* $\mathbf{K}_{\text{tac}}(R)$ *with finite data, in terms of the objects of* $\boldsymbol{HMF}(P, x)$ *when* Q *has finite Cohen-Macaulay type?* Specifically, we provide a classification scheme of totally acyclic complexes by grouping them based on their decomposition into indecomposable components. That is, we count the number of indecomposables in the minimal approximation of each complex, and if the tuples are the same, we call them *Arnold equivalent*. However, before giving these definitions, we prove the existence of minimal right approximations in $\mathbf{K}_{\text{tac}}(R)$ via a slightly more general subcategory than that used by the authors of [2]. Originally intended to be defined for complexes whose approximations stem from rings of finite TAC type (rings with only finitely many distinct indecomposable totally acyclic complexes), we instead give the definition more generally in terms of a Henselian Gorenstein local ring. One clear advantage of the original setting is that there are only finitely many options for the summands, which in turn implies that the tuple is always finite. Nevertheless, in the more general scenario, the tuple will still always have only finitely many nonzero terms.

In the final chapter of this thesis, we turn towards the goal of developing the theory related to resolutions of complexes via approximations and mapping cones, first defined in §4.9 of [2]. The authors employed the categorical structure of $\mathbf{K}_{\text{tac}}(R)$ to build a resolution which describes a totally acyclic complex, similarly to how a free resolution describes a finitely generated R-module. Of course, just as in the latter construct, these new *triangle resolutions* are not unique unless we impose a condition of minimality. For this reason, we develop the notion of a minimal triangle resolution and prove their existence for any object in $\mathbf{K}_{\mathrm{tac}}(R)$. In the process of doing so, we employ the existence of minimal right approximations of totally acyclic complexes previously mentioned. Lastly, along with the development of minimal triangle resolutions, we provide properties that extend the classical constructions of free resolutions, such as an analogous notion to Betti numbers.

CHAPTER 1

Preliminaries

In this chapter we give some of the necessary background information upon which this thesis is built. We start by introducing the basic ideas and definitions of categories and, in particular, triangulated categories and chain complexes. We then discuss the homotopy category of chain complexes, $\mathrm{K}(R)$, with special attention to the subcategory comprised of totally acyclic complexes, $\mathbf{K}_{\mathrm{tac}}(R)$. Furthermore, we discuss many properties this subcategory possesses. Finally we give a brief overview of Auslander-Reiten theory, provide the quivers, and talk about the structures they exhibit.

1.1 Preliminaries on Rings and Modules

In order to provide context to this thesis, we begin by giving some preliminaries on ring and module theory. In particular, we recall definitions for specific types of rings and modules that will be used extensively in this, and later, chapters. We refer the interested reader to [29] and [30] for a more in-depth accounting of the definitions in this section.

Definition 1 (cf. [29])**.** A commutative *local* ring R is a commutative ring with a unique maximal ideal, say $\mathfrak{m}$. In this case we may unambiguously define the residue field $k = R/\mathfrak{m}$ and denote the whole affair by $(R, \mathfrak{m}, k)$.

Throughout this dissertation, we will assume that our rings are commutative Noetherian local rings and our modules are always finitely generated. Under these conditions, we recall a few more definitions. (The reader may note that these conditions may not be necessary for all definitions and results in this thesis, but shall be assumed anyway.)

Definition 2 (cf. [30])**.** Let R be a commutative ring. We call $x_1, ..., x_c$ a *regular sequence* if the sequence satisfies the following:

- $x_i \in R$ is a non-zero-divisor on $R/(x_1, ..., x_{i-1})$ for all $i = 1, ..., c$
- $(x_1, ..., x_c) \neq R$

Furthermore, for an R-module M, we call a sequence $x_1, x_2, ..., x_c$ in R an M-regular sequence if:

- x_i is a non-zero-divisor on $M/(x_1, ..., x_{i-1})M$ for all $i = 1, ..., c$
- $M/(x_1, x_2, ..., x_c)M \neq 0$

With this definition established, we recall what is meant by depth of a module and what it is to be a Cohen-Macaulay ring.

Definition 3 (cf. [30])**.** Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. For a finitely generated R-module M, all maximal M-regular sequences $x_1, x_2, ..., x_c$, where all $x_i \in \mathfrak{m}$, have the same length, n, equal to the $\mathfrak{m}$-*depth* of M, denoted $\text{depth}(M)$.

A finitely generated R-module M is called *Cohen-Macaulay* (CM) if $\text{depth}(M) = \dim(M)$, where $\dim(M)$ is the Krull dimension of M. Similarly, we call a ring R, *Cohen-Macaulay* if R is Cohen-Macaulay as an R-module over itself. Furthermore, we say a module M is *maximal Cohen-Macaulay* (MCM) if its depth is maximal and M is Cohen Macaulay; i.e. $\text{depth}(M) = \dim(M) = \dim(R)$. We now recall the definition of a Gorenstein local ring, as these rings will play an important role throughout this dissertation.

Definition 4 (cf. [30])**.** Let R be a commutative Noetherian local ring. We call such a ring *Gorenstein* if R has finite injective dimension as an R-module. If any ring is Gorenstein then it is also a CM ring. (see [30])

In order to discuss the Auslander-Reiten theory established later in this chapter, we further recall the following definitions. A local ring R is *Henselian* if any commutative R-algebra which is module-finite over R is a direct product of local R-algebras. Moreover, an R-module is called *indecomposable* if it has no nontrivial direct summands. In other words, an R-module M is indecomposable if whenever $M = M' \oplus M''$, either $M' = 0$ or $M'' = 0$. We now state Proposition 1.18 from [5] which connects these two ideas and introduces a notion of decomposition for arbitrary R-modules.

Proposition 5. [5, Proposition 1.18] Let R be a Henselian local ring and let M be an R-module. Then M is indecomposable if and only if the endomorphism ring $\mathrm{End}_R(M)$ is a local algebra; that is, sums of non-units in $\mathrm{End}_R(M)$ are non-units. This assures us that the category of finitely generated R-modules admits the Krull-Schmidt theorem. Namely, any R-module is uniquely a finite direct sum of indecomposable R-modules.

This proposition will be applied any time we discuss AR theory in the module case during this thesis. In fact, we later provide, and prove, an analogous proposition, 38, for totally acyclic complexes in a triangulated category.

1.2 Triangulated Categories

Much of the theory in this thesis involves properties that are derived from the categorical structure of the objects and morphisms we study. It would then behoove us to briefly discuss some basic category theory so that we may talk more in depth about the main type of category we work in – a triangulated category. Originally defined in Verdier's thesis [32] in the 1963, triangulated categories offer extra structure onto already existing additive categories. In particular, the triangulated structure comes about from a chosen suspension functor and a set of distinguished triangles which satisfy five axioms. As a matter of fact, a given additive category can have multiple triangulated structures. We again refer the reader [8] and [31] for more information on the topics in the following three sections.

Definition 6. [8, Definition 1.1] A category $\mathcal{A}$ is called an *additive category* if the following conditions hold:

i) For every pair of objects X, Y the set of morphisms $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group and the composition of morphisms

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

is bilinear over the integers.

ii) $\mathcal{A}$ contains a zero object, 0.

iii) For every pair of objects X, Y in $\mathcal{A}$ there exists a coproduct $X \oplus Y$ in $\mathcal{A}$.

Furthermore, an additive category $\mathcal{A}$ is called an *abelian category* if the following axioms are satisfied:

i) Every morphism in $\mathcal{A}$ has a kernel and cokernel.

ii) For every morphism $f : X \to Y$ in $\mathcal{A}$, the natural morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

A functor, Σ, between additive categories is called an *additive functor* if for every pair of objects X, Y the map $\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(\Sigma X, \Sigma Y)$ is a homomorphism of abelian groups. Now, let $\mathcal{T}$ be an additive category together with an invertible additive functor $\Sigma : \mathcal{T} \to \mathcal{T}$ called a *translation* or *suspension* functor. A *triangle* in $\mathcal{T}$ is a sequence of objects and morphisms of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

A *morphism of triangles* is a triple (f, g, h) of morphisms such that the following diagram commutes:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

We now have the necessary background information to define a triangulated category.

Definition 7. [8, Definition 3.1] A *triangulated category* is an additive category $\mathcal{T}$ together with an invertible endofunctor Σ, the translation or shift functor, and a collection of distinguished triangles satisfying the following axioms:

(TR0) Any triangle isomorphic to a distinguished triangle is again a distinguished triangle.

(TR1) For every object X in $\mathcal{T}$, the triangle $X \xrightarrow{\mathrm{Id}_X} X \to 0 \to \Sigma X$ is a distinguished triangle.

(TR2) For every morphism $f : X \to Y$ in $\mathcal{T}$ there is a distinguished triangle of the form $X \xrightarrow{f} Y \to Z \to \Sigma X$.

(TR3) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a distinguished triangle, then also $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is a distinguished triangle, and vice versa.

(TR4) Given distinguished triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$, then each commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

can be completed to a morphism of triangles (but not necessarily uniquely).

(TR5) Given distinguished triangles

$$X \xrightarrow{u} Y \to Z' \to \Sigma X,$$

$$Y \xrightarrow{v} Z \to X' \to \Sigma Y \text{ and}$$

$$X \xrightarrow{vu} Z \to Y' \to \Sigma X,$$

there exists a distinguished triangle $Z' \to Y' \to X' \to \Sigma Z'$ making the following diagram commute:

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma X \\
\downarrow{\scriptstyle Id_X} & & \downarrow{\scriptstyle v} & & \downarrow & & \downarrow{\scriptstyle Id_{\Sigma X}} \\
X & \xrightarrow{vu} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma X \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle Id_Z} & & \downarrow & & \downarrow{\scriptstyle \Sigma u} \\
Y & \xrightarrow{v} & Z & \longrightarrow & X' & \longrightarrow & \Sigma Y \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle Id'_X} & & \downarrow \\
Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & \Sigma Z'
\end{array}$$

A functor $\mathcal{F} : \mathcal{T} \to \mathcal{S}$ between triangulated categories is called a *triangle functor* if it is an additive functor together with natural isomorphisms for each $X \in \mathcal{T}$:

$$\Phi_X : \mathcal{F}\Sigma(X) \to \Sigma\mathcal{F}(X)$$

such that for any distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in $\mathcal{T}$, the triangle:

$$\mathcal{F}(X) \xrightarrow{\mathcal{F}(u)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(v)} \mathcal{F}(Z) \xrightarrow{\mathcal{F}(w)} \Sigma\mathcal{F}(X)$$

is distinguished in $\mathcal{S}$.

1.3 Chain Complexes

Since the notion of a triangulated category is quite abstract, we look to a more specific example to study. To do this we must first recall a few definitions.

Definition 8. [8, Section 1.1] A *complex* over an additive category $\mathcal{A}$ is a family $X = (X_n, \partial_n^X)_{n\in\mathbb{Z}}$ where $X_n \in \mathcal{A}$ and $\partial_n^X : X_n \to X_{n-1}$ are morphisms such that $\partial_n \circ \partial_{n-1} = 0$ for all n. A complex is usually written as follows:

$$\cdots \to X_{n+1} \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \to \cdots$$

Let $Y = (Y_n, \partial_n^Y)$ be another $\mathcal{A}$-complex, then a *morphism of complexes* $f : X \to Y$ is a family of morphisms $(f_n : X_n \to Y_n)_{n\in\mathbb{Z}}$ satisfying $\partial_n^Y \circ f_n = f_{n-1} \circ \partial_n^X$ for all n. In other words, we have the following commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_{n+1} & \xrightarrow{\partial_{n+1}^X} & X_n & \xrightarrow{\partial_n^X} & X_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \longrightarrow & Y_{n+1} & \xrightarrow[\partial_{n+1}^Y]{} & Y_n & \xrightarrow[\partial_n^Y]{} & Y_{n-1} & \longrightarrow & \cdots \end{array}$$

We must also recall a notion that will be instrumental in the subsequent chapters, namely, the mapping cone of a morphism of complexes. While the mapping cone can be defined for a morphism between any two complexes, it plays a particularly important role in the homotopy category of chain complexes, which will become apparent in the next section.

Definition 9. [8, Definition 6.3] Let f be a morphism between complexes $X = (X_n, \partial_n^X)$ and $Y = (Y_n, \partial_n^Y)$. The *mapping cone*, $\operatorname{cone}(f)$, is the complex defined by

$$\operatorname{cone}(f)_n = X_{n-1} \oplus Y_n \text{ and } \partial_n^{\operatorname{cone}(f)} := \begin{bmatrix} -\partial_{n-1}^X & 0 \\ f_{n-1} & \partial_n^Y \end{bmatrix}.$$

For a local ring $(R, \mathfrak{m}, k)$ we say that a complex C is *minimal* if $\operatorname{Im}(\partial_n^C) \subseteq \mathfrak{m}C_{n-1}$ for all $n \in \mathbb{Z}$. We also call C *contractible* if the identity morphism Id_C is null homotopic. (see Definition 12)

"Zooming out", if we take the collection of complexes over an abelian category $\mathcal{A}$, together with the morphisms between them, it forms an abelian category called the *category of complexes* over $\mathcal{A}$ and is denoted by $\mathbf{C}(\mathcal{A})$.

We now state a pair of lemmas, the proofs of which can be found in [6]:

Lemma 10. [6, Theorem B.54] Let C be a complex of projective R-modules such that R is a ring over which every finitely generated left module has a projective cover. Then we may write $C = M \oplus T$ where M is a minimal complex and T is contractible.

Lemma 11. [6, Theorem B.54(a)] The complex M from the previous lemma is unique in the following sense: If one also has $C = M' \oplus T'$, where M' is minimal and T' is contractible, then M' is isomorphic to M.

1.4 The Category of Totally Acyclic Complexes

Let $\mathcal{A}$ be an additive category and $f, g : X \to Y$ morphisms in $\mathbf{C}(\mathcal{A})$.

Definition 12. [8, Definition] The morphism $f : X \to Y$ is called *null homotopic*, denoted $f \sim 0$, if there exists a family of morphisms $(\sigma_n : X_n \to Y_{n+1})_{n\in\mathbb{Z}}$ such that

$$f_n = \partial^Y_{n+1}\,\sigma_n + \sigma_{n-1}\,\partial^X_n$$

for all n.

Furthermore, we say that the morphisms $f, g : X \to Y$ are homotopic if $f - g \sim 0$. In fact, it is well known that $\sim$ forms an equivalence relation.

Definition 13. [8, Definition 1.6] Let $\mathcal{A}$ be an additive category. The *homotopy category* $\mathbf{K}(\mathcal{A})$ has the same objects as the category, $\mathbf{C}(\mathcal{A})$, of complexes over $\mathcal{A}$. The morphisms in the homotopy category are the equivalence classes of morphisms in $\mathbf{C}(\mathcal{A})$ modulo homotopy; that is:

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X,Y) := \mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(X,Y)/\sim .$$

Proposition 14. [8, Theorem 6.7] Let $\mathcal{A}$ be an additive category. The homotopy category, $\mathbf{K}(\mathcal{A})$, with the suspension functor, Σ, defined by shifting one degree to the left. i.e.,

$$(\Sigma X)_n = X_{n-1} \text{ where } \partial^{\Sigma X}_n = -\,\partial^X_{n-1} \text{ and } \Sigma f_n = f_{n-1}$$

is a triangulated category.

Over an abelian category $\mathcal{A}$, for a complex $C \in \mathbf{C}(\mathcal{A})$ the requirement that $\partial_n^C \circ \partial_{n+1}^C = 0$ for all n is equivalent to saying $\operatorname{Im} \partial_{n+1}^C \subseteq \operatorname{Ker} \partial_n^C$ for all n. This means that we may consider the quotient module $\operatorname{Ker} \partial_n^C \, / \operatorname{Im} \partial_{n+1}^C$ which we call the n^{th} *homology*, denoted $\mathrm{H}_n(C)$.

Definition 15. [10, Section 2] Let R be a ring and C a complex of R-modules. We say that C is an *acyclic complex* if $\mathrm{H}_n(C) = 0$ for all n. Furthermore, if each C_n is a projective module and $\mathrm{H}_n(\operatorname{Hom}_R(C, R)) = 0 = \mathrm{H}_n(C)$, we say that C is *totally acyclic*. In other words, a complex of projective modules C is totally acyclic if both the complex and its dual $\operatorname{Hom}_R(C, R)$ are exact in each degree.

Since the ring R is local, each projective module in the complex C is free. We may then consider the ranks at each degree, which we call the *Betti number* and denote $\beta_i(C)$. In other words,

$$\beta_i^R(C) = \operatorname{rank}(C_i).$$

Furthermore, we can describe the growth of the Betti sequence via the notion of complexity. This is of particular importance if the sequence of Betti numbers is non-zero for infinitely many i.

Definition 16. [26, Section 2] Let C be a complex of finitely generated free R-modules, then the complexity of C, denoted $\operatorname{cx}_R C$, is defined as

$$\operatorname{cx}_R C := \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists\ a \in \mathbb{R} \text{ s.t. } \beta_n(C) \leq a n^{t-1}\ \forall\ n \gg 0\}.$$

Let us now consider the idea of subcategories and specific types thereof that have useful properties. We say that a subcategory $\mathcal{C}$ of a category $\mathcal{T}$ is a *full subcategory* if

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) = \operatorname{Hom}_{\mathcal{T}}(X, Y)$$

whenever $X, Y \in \mathcal{C}$. If $\mathcal{T}$ is a triangulated category, we say that a full subcategory $\mathcal{C}$ is *triangulated* if it is closed under (de)suspensions and contains some mapping cone for each morphism between any two objects in the subcategory. We additionally say that a triangulated subcategory $\mathcal{C} \subseteq \mathcal{T}$ is *thick* if whenever $\mathcal{C}$ contains an object isomorphic to $X \oplus Y$, then it also contains both X and Y.

Moreover, Avramov and Iyengar give a construction in [9] of the smallest thick subcategory containing a collection of objects, Ω, in a triangulated category $\mathcal{T}$, denoted

$$\operatorname{thick}_{\mathcal{T}}(\Omega).$$

We do this in a series of steps.

For each $n \geq 0$ we define a full subcategory $\operatorname{thick}_{\mathcal{T}}^{n}(\Omega)$ called the n^{th} *thickening of* Ω as follows:

- $\operatorname{thick}_{\mathcal{T}}^{0}(\Omega) = \{0\}$.
- The objects of $\operatorname{thick}_{\mathcal{T}}^{1}(\Omega)$ are the retracts of finite direct sums of shifts of elements in Ω.
- For each $n \geq 2$, the objects of $\operatorname{thick}_{\mathcal{T}}^{n}(\Omega)$ are the retracts of those $C \in \mathcal{T}$ that appear in some exact triangle

  $$C' \to C \to C'' \to \Sigma C'$$

  with $C' \in \operatorname{thick}_{\mathcal{T}}^{n-1}(\Omega)$ and $C'' \in \operatorname{thick}_{\mathcal{T}}^{1}(\Omega)$.

Then

$$\operatorname{thick}_{\mathcal{T}}(\Omega) = \bigcup_{n \in \mathbb{N}} \operatorname{thick}_{\mathcal{T}}^{n}(\Omega)$$

Of particular importance to this thesis is the first thickening. While not thick itself, it is a full subcategory, closed under direct sums.

We now turn our attention to the object of focus in our studies throughout this thesis:

Definition 17. [2, Section 1] Let R be a ring and $\mathrm{K}(R)$ the homotopy category of R-complexes. Define the full subcategory $\mathbf{K}_{\mathrm{tac}}(R)$ of $\mathrm{K}(R)$ whose objects are the totally acyclic complexes and the morphisms are the the homotopy equivalence classes of R-complex chain maps.

For lack of a good reference, we now show that $\mathbf{K}_{\mathrm{tac}}(R)$ has all the useful properties set forth in the previous paragraphs.

Proposition 18. $\mathbf{K}_{\mathrm{tac}}(R)$ is a thick, triangulated subcategory of the homotopy category.

Proof. We begin by showing that $\mathbf{K}_{\mathrm{tac}}(R)$ is a triangulated subcategory of $\mathrm{K}(R)$.

Given a totally acyclic complex X it is clear that $\Sigma^i X$ is totally acyclic for all $i \in \mathbb{Z}$. Thus $\mathbf{K}_{\mathrm{tac}}(R)$ is closed under (de)suspensions, therefore it suffices to show that $\mathrm{cone}(f) \in \mathbf{K}_{\mathrm{tac}}(R)$ for any morphism $f \in \mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(R)}(X, Y)$. Since $\mathrm{cone}(f)$ is a complex we already have that

$$\mathrm{Im}(\partial_{n+1}^{\mathrm{cone}(f)}) \subseteq \mathrm{Ker}(\partial_n^{\mathrm{cone}(f)}).$$

To see the reverse containment, suppose $(x, y) \in \mathrm{Ker}(\partial_n^{\mathrm{cone}(f)})$. That is to say

$$\big(-\partial_{n-1}^X(x), f_{n-1}(x) + \partial_n^Y(y)\big) = (0, 0) \tag{1.1}$$

By (1.1) we have that $x \in \mathrm{Ker}(-\partial_{n-1}^X) = \mathrm{Im}(-\partial_n^X)$ and there exists $\alpha \in X_n$ such that $-\partial_n^X(\alpha) = x$. Furthermore, $f_{n-1}(x) = -\partial_n^Y(y)$. Now, as f is a chain map we have that $f_{n-1}\,\partial_n^X - \partial_n^Y\, f_n = 0$. In particular we have that

$$0 = f_{n-1}\,\partial_n^X(\alpha) - \partial_n^Y\, f_n(\alpha) =$$

$$-f_{n-1}(x) - \partial_n^Y\, f_n(\alpha) =$$

$$\partial_n^Y(y) - \partial_n^Y\, f_n(\alpha) = \partial_n^Y(y - f_n(\alpha)).$$

Thus, $y - f_n(\alpha) \in \mathrm{Ker}(\partial_n^Y) = \mathrm{Im}(\partial_{n+1}^Y)$ so there is a $\beta \in Y_{n+1}$ such that $\partial_{n+1}^Y(\beta) = y - f_n(\alpha)$.

Finally, consider the element $(\alpha, \beta) \in X_n \oplus Y_{n+1}$. Then

$$\begin{aligned}\partial_{n+1}^{\mathrm{cone}(f)}(\alpha, \beta) &= \big(- \partial_n^X(\alpha), f_n(\alpha) + \partial_{n+1}^Y(\beta)\big) \\ &= (x, f_n(\alpha) + y - f_n(\alpha)) = (x, y)\,.\end{aligned}$$

Thus,

$$\mathrm{Im}(\partial_{n+1}^{\mathrm{cone}(f)}) = \mathrm{Ker}(\partial_n^{\mathrm{cone}(f)})$$

and the mapping cone $\mathrm{cone}(f)$ is acyclic. It follows that $\mathrm{cone}(f)^* = \mathrm{Hom}_R(\mathrm{cone}(f), R)$ is acyclic from the fact that $\mathrm{Hom}_R(X, R)$ and $\mathrm{Hom}_R(Y, R)$ are exact by assumption and a similar argument to the previous one. We now turn our attention to proving that $\mathbf{K}_{\mathrm{tac}}(R)$ is a thick subcategory. To do this, assume that $X \oplus Y$ is a totally acyclic complex. Then, since homology respects finite coproducts we have that for each $n \in \mathbb{Z}$:

$$0 = \mathrm{H}_n(X \oplus Y) \cong \mathrm{H}_n(X) \oplus \mathrm{H}_n(Y).$$

Thus, X and Y are both acyclic. Furthermore, since $\mathrm{Hom}_R(-, R)$ respects finite coproducts as well, we have that:

$$0 = \mathrm{H}_n(\mathrm{Hom}_R(X \oplus Y, R)) \cong$$

$$\mathrm{H}_n(\mathrm{Hom}_R(X, R) \oplus \mathrm{Hom}_R(Y, R)) \cong$$

$$\mathrm{H}_n(\mathrm{Hom}_R(X, R)) \oplus \mathrm{H}_n(\mathrm{Hom}_R(Y, R))$$

for each $n \in \mathbb{Z}$. This proves the statement. □

Furthermore, $\mathbf{K}_{\mathrm{tac}}(R)$ has an interesting connection to the category of R-modules. Indeed, given a finitely generated module M over a Gorenstein ring, we may extend M to a totally acyclic complex $C \in \mathbf{K}_{\mathrm{tac}}(R)$ via a complete resolution.

Definition 19. [10, Section 3] A *complete resolution* of a finitely generated R-module M is a diagram

$$C \xrightarrow{\rho} P \xrightarrow{\pi} M$$

such that $C \in \mathbf{K}_{\mathrm{tac}}(R)$, P is a projective resolution of M, ρ is a morphism of R-complexes, and ρ_n is bijective for all $n \gg 0$. We will often abuse terminology and call C a complete resolution of M.

Though originally defined by Buchweitz in [21], the construction of complete resolutions is given by Avramov and Martsinkovsky in 3.6 and 3.7 in [10] and, for such a complete resolution it holds that $\rho_n = \mathrm{Id}_{P_n}$ for all $n \gg 0$. We also have the following lemma relating each totally acyclic complex with its minimal complex which follows easily using such complete resolutions.

Lemma 20. Let $C \in \mathbf{K}_{\mathrm{tac}}(R)$ such that $C = \overline{C} \oplus T$ where $\overline{C}$ is the minimal complex. Then C and $\overline{C}$ are homotopically equivalent.

Proof. Let $C \in \mathbf{K}_{\mathrm{tac}}(R)$. If C is minimal, we are done, therefore assume that C is non-minimal. We also know that C is the complete resolution of the R-module $X = \mathrm{Im}(\partial_0^C)$. However, we may also extend X to a complete resolution minimally by choosing the free resolution P' of X to be minimal. We then obtain the totally acyclic complex $\overline{C}$ and the complete resolution $\overline{C} \to P' \to X$. Then by Lemma 29 in §2.1, which shall be proven in time, we get the following commutative diagram

$$\begin{array}{ccccc} C & \longrightarrow & P & \longrightarrow & X \\ \simeq\downarrow & & \simeq\downarrow & & \| \\ \overline{C} & \longrightarrow & P' & \longrightarrow & X \end{array}$$

which shows that $C \simeq \overline{C}$. □

1.5 A Gentle Introduction to AR Theory

In the next chapter we provide an extension of Auslander-Reiten theory to the category of $\mathbf{K}_{\mathrm{tac}}(R)$. However, in order to fully contextualize this extension, we now give some preliminaries on AR theory in R-mod and introduce the notions of finite Cohen-Macaulay type and AR quivers. Throughout the rest of this section we have that $(R, \mathfrak{m}, k)$ is a Henselian hyper-surface defined by a nonzero divisor f in a regular local ring S:

$$R = S/(f).$$

In order to set the stage for the following definitions and descriptions we recall what is meant by a simple singularity from [5].

Definition 21. [5, Definition 8.1] Let S be a regular local ring. For a hypersurface $R = S/(f)$, consider the following ideals in S:

$$c(f) = \left\{ I \mid I \text{ is a proper ideal of } S \text{ with } f \in I^2 \right\}$$

We call such a ring R a *simple singularity* if the set $c(f)$ is finite.

We now state a classification, given in Theorem 8.8 in [5], for simple singularities which gives an explicit description for the possible forms of f. Let k be an algebraically closed field of characteristic 0 and $S = k[\![x, y, z_2, z_3..., z_d]\!]$. If $R = S/(f)$ is a simple singularity then f is equal to one of the following:

(A_n) $\quad x^2 + y^{n+1} + z_2^2 + z_3^2 + ... + z_d^2 \qquad (n \geq 1),$

(D_n) $\quad x^2 y + y^{n-1} + z_2^2 + z_3^2 + ... + z_d^2 \qquad (n \geq 4),$

(E_6) $\quad x^3 + y^4 + z_2^2 + z_3^2 + ... + z_d^2,$

(E_7) $\quad x^3 + xy^3 + z_2^2 + z_3^2 + ... + z_d^2,$

(E_8) $\quad x^3 + y^5 + z_2^2 + z_3^2 + ... + z_d^2.$

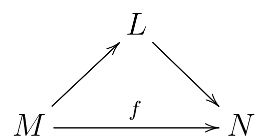
In order to give the definition of an AR quiver, we must first establish some notation. Let $\mathfrak{C}(R)$ be the full subcategory of R-mod consisting of MCM modules. Then for two indecomposable modules $M, N \in \mathfrak{C}(R)$ the *radical* of M and N is the submodule of $\operatorname{Hom}_R(M, N)$ consisting of all non-invertible morphisms $f : M \to N$ and is denoted $\operatorname{rad}(M, N)$. Furthermore,

$$\operatorname{rad}^2(M, N) = \sum \operatorname{rad}(L, N) \operatorname{rad}(M, L)$$

as L ranges over the subcategory $\mathfrak{C}(R)$. We may now define the submodule of irreducible morphisms as follows:

$$\operatorname{Irr}(M, N) = \frac{\operatorname{rad}(M, N)}{\operatorname{rad}^2(M, N)}.$$

We note that if $f \in \operatorname{Irr}(M, N)$ then for any $L \in \mathfrak{C}(R)$ and diagram of the form



f cannot be decomposed otherwise it would be factored out. Furthermore, we note that $\operatorname{Irr}(M, N)$ is a vector space over k and thus, has a k-dimension; which over the a fore mentioned rings is always finite and we define as:

$$\operatorname{irr}(M, N) = \dim_k(\operatorname{Irr}(M, N)).$$

Another important aspect of AR quivers is the encoded notions of AR sequences and AR translates. For an indecomposable CM module $M \in \mathfrak{C}(R)$, we define a set of short exact sequences $S(M)$ as follows:

$$S(M) =$$
$$\{s : 0 \to N_s \to E_s \to M \to 0 \mid s \text{ a nonsplit S.E.S. in } \mathfrak{C}(R) \text{ with } N_s \text{ indecomposable}\}.$$

By Lemma 2.2 of [5], $S(M)$ is nonempty if M is an indecomposable maximal Cohen Macaulay module. Furthermore, we may impose a partial ordering on this set of short exact sequences via the following definition.

Definition 22. [5, Definition 2.3] Let s and t be two elements of $S(M)$. We write $s > t$ if there is an $f \in \operatorname{Hom}_R(N_s, N_t)$ such that $\operatorname{Ext}^1_R(M, f)(s) = t$. In this case we say that s is bigger than t or t is smaller than s. This is equivalent to the existence of a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_s & \longrightarrow & E_s & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & N_t & \longrightarrow & E_t & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

We write $s \sim t$ if f is an isomorphism above. We often identify s with t when $s \sim t$.

We make use of this partially ordered set of short exact sequences to recall the following definition:

Definition 23 (Definition 2.8 , [5])**.** Let M be an indecomposable CM module over R. If a short exact sequence

$$s : 0 \to N_s \to E_s \to M \to 0$$

ending in M is the minimum element in $S(M)$ then, if it exists, is uniquely determined by M. Such a short exact sequence is called an *AR sequence*. In particular, the modules N_s and E_s are also unique up to an isomorphism. If s is the AR sequence ending in M, then we denote N_s by $\tau(M)$ and call it the *AR translation* of M.

We may now, finally, define an AR quiver.

Definition 24. [5, Definition 5.3] The AR quiver Γ of $\mathfrak{C}(R)$ for a simple singularity R is a directed graph where:

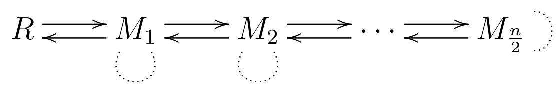
- each vertex corresponds to a non-isomorphic, indecomposable CM module,
- the number of arrows from vertex M to vertex N corresponds to the integer $\operatorname{irr}(M, N)$,
- also, to encode the information of the AR translation $\tau(M)$, we connect the vertex M to the vertex N with a dotted line if $N = \tau(M)$ such that there is an AR sequence $0 \to \tau(M) \to E \to M \to 0$ for some E.

1.6 AR Quivers Over Rings of Finite Type

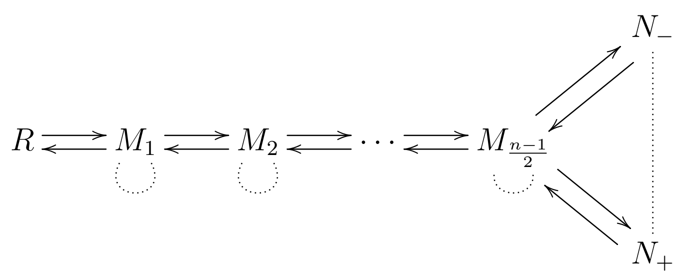
We now give a full accounting of the AR quivers when R is a 1-dimensional ring of finite CM type.

Let $R = k[\![x,y]\!]/(f)$ as a special case of the previous description.

- For A_n with n even we have that the AR quiver is:



- For A_n with n odd we have that the AR quiver is:

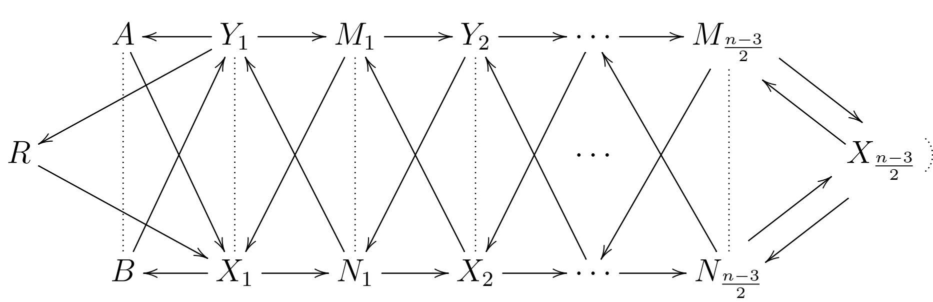


For both odd and even n, $M_0 \cong R$ and $M_j = \operatorname{Coker} \varphi_j$ where

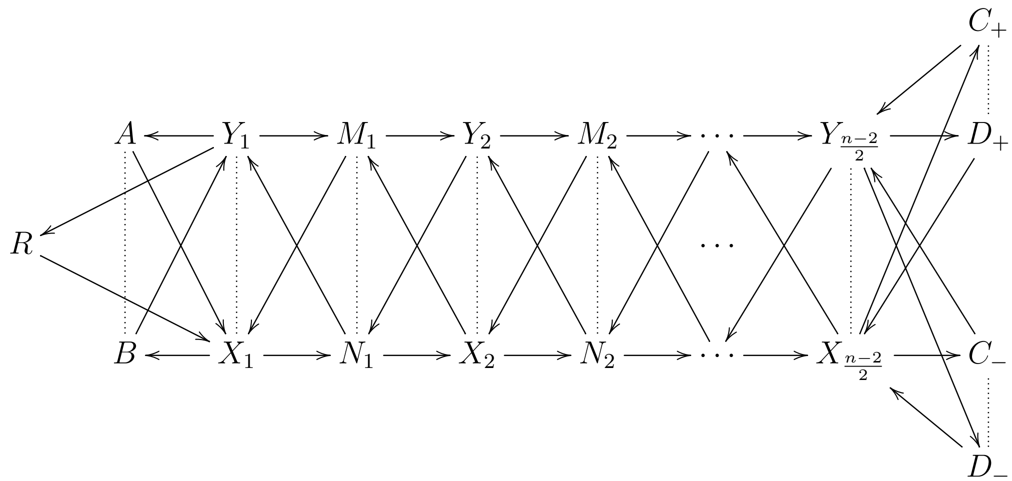
$$\varphi_j = \begin{bmatrix} x & y^j \\ y^{n+1-j} & -x \end{bmatrix}, \qquad 1 \le j < \frac{n+1}{2}.$$

In the case that n is odd we write $n = 2k-1$. In this case, $N_+ = R/(y^k + ix)$, $N_- = R/(y^k - ix)$ and $M_k = N_+ \oplus N_-$.

- For D_n, n odd, the AR quiver is:



- For D_n, n even, the AR quiver is:



For both even and odd n let

$$(\alpha, \beta) = (y, x^2 + y^{2l-2}),$$

$$(\gamma_+, \delta_+) = (y(x + iy^{l-1}), x - iy^{l-1}),$$

$$(\gamma_-, \delta_-) = (y(x - iy^{l-1}), x + iy^{l-1}).$$

Then

$$A = \operatorname{Coker} \alpha, \quad B = \operatorname{Coker} \beta,$$

$$C_+ = \operatorname{Coker} \gamma_+, \quad C_- = \operatorname{Coker} \gamma_-,$$

$$D_+ = \operatorname{Coker} \delta_+, \quad D_- = \operatorname{Coker} \delta_-.$$

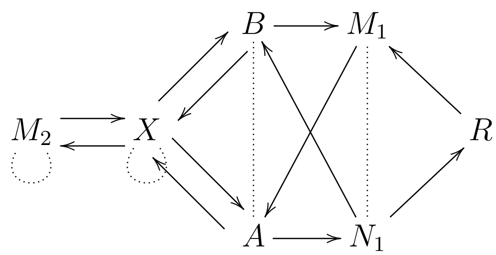
Furthermore, for the matrices:

$$\varphi_j = \begin{bmatrix} x & y^j \\ y^{n-j-2} & -x \end{bmatrix}, \ \psi_j = \begin{bmatrix} xy & y^{j+1} \\ y^{n-j-1} & -xy \end{bmatrix}, \ \xi_j = \begin{bmatrix} x & y^j \\ y^{n-j-1} & -xy \end{bmatrix}, \ \eta_j = \begin{bmatrix} xy & y^j \\ y^{n-j-1} & -x \end{bmatrix}$$

we have that:

$$M_j = \operatorname{Coker} \varphi_j, \quad N_j = \operatorname{Coker} \psi_j, \quad X_j = \operatorname{Coker} \xi_j \text{ and } \quad Y_j = \operatorname{Coker} \eta_j.$$

- For E_6 the AR quiver is:



Take the matrices:

$$\varphi_1 = \begin{bmatrix} x & y \\ y^3 & -x^2 \end{bmatrix}, \quad \psi_1 = \begin{bmatrix} x^2 & y \\ y^3 & -x \end{bmatrix},$$

$$\varphi_2 = \begin{bmatrix} x & y^2 \\ y^2 & -x^2 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} x^2 & y^2 \\ y^2 & -x \end{bmatrix},$$

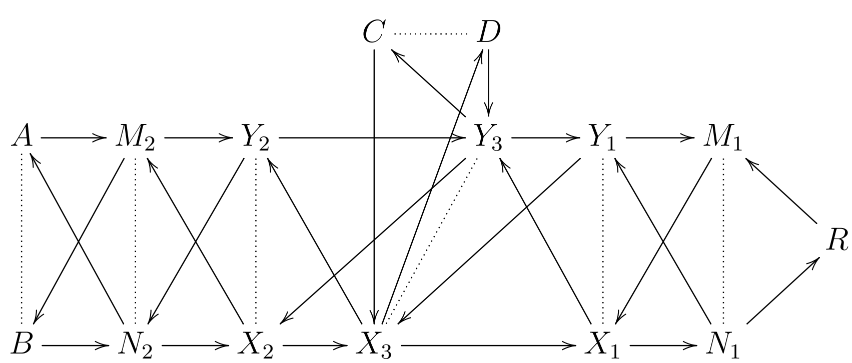
$$\alpha = \begin{bmatrix} y^3 & x^2 & xy^2 \\ xy & -y^2 & x^2 \\ x^2 & -xy & -y^3 \end{bmatrix}, \quad \beta = \begin{bmatrix} y & 0 & x \\ x & -y^2 & 0 \\ 0 & x & -y \end{bmatrix}$$

Then

$$M_i = \operatorname{Coker} \varphi_i, \quad N_i = \operatorname{Coker} \psi_i, \quad (i = 1, 2)$$

$$A = \operatorname{Coker} \alpha \text{ and } \quad B = \operatorname{Coker} \beta.$$

- For E_7 the AR quiver is:



Take the matrices:

$$\alpha = x, \qquad \beta = x^2 + y^3,$$

$$\gamma = \begin{bmatrix} x^2 & yx \\ xy^2 & -x^2 \end{bmatrix}, \qquad \delta = \begin{bmatrix} x & y \\ y^2 & -x \end{bmatrix},$$

$$\varphi_1 = \begin{bmatrix} x & y \\ xy^2 & -x^2 \end{bmatrix}, \qquad \psi_1 = \begin{bmatrix} x^2 & y \\ xy^2 & -x \end{bmatrix},$$

$$\varphi_2 = \begin{bmatrix} x & y^2 \\ xy & -x^2 \end{bmatrix}, \qquad \psi_2 = \begin{bmatrix} x^2 & y^2 \\ xy & -x \end{bmatrix},$$

$$\xi_1 = \begin{bmatrix} xy^2 & x^2 & -x^2y \\ xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{bmatrix}, \qquad \eta_1 = \begin{bmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{bmatrix},$$

$$\xi_2 = \begin{bmatrix} x^2 & -y^2 & -xy \\ xy & x & -y^2 \\ xy^2 & xy & x^2 \end{bmatrix}, \qquad \eta_2 = \begin{bmatrix} x & 0 & y \\ -xy & x^2 & 0 \\ 0 & -xy & x \end{bmatrix},$$

$$\xi_3 = \begin{bmatrix} \gamma & \epsilon \\ 0 & \delta \end{bmatrix}, \qquad \eta_3 = \begin{bmatrix} \delta & -\epsilon \\ 0 & \gamma \end{bmatrix}$$

where

$$\epsilon = \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}.$$

Then

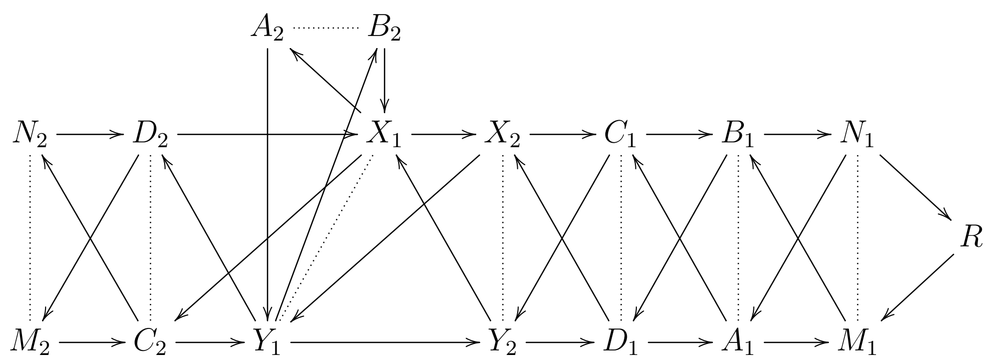
$$M_i = \operatorname{Coker} \varphi_i, \quad N_i = \operatorname{Coker} \psi_i, \quad (i = 1, 2)$$

$$X_i = \operatorname{Coker} \xi_i, \quad Y_i = \operatorname{Coker} \eta_i, \quad (i = 1, 2)$$

$$A = \operatorname{Coker} \alpha, \quad B = \operatorname{Coker} \beta$$

$$C = \operatorname{Coker} \gamma, \quad D = \operatorname{Coker} \delta.$$

- For E_8 the AR quiver is:



Take the matrices:

$$\varphi_1 = \begin{bmatrix} x & y \\ y^4 & -x^2 \end{bmatrix}, \qquad \psi_1 = \begin{bmatrix} x^2 & y \\ y^4 & -x \end{bmatrix},$$

$$\varphi_2 = \begin{bmatrix} x & y^2 \\ y^3 & -x^2 \end{bmatrix}, \qquad \psi_2 = \begin{bmatrix} x^2 & y^2 \\ y^3 & -x \end{bmatrix},$$

$$\alpha_1 = \begin{bmatrix} y & -x & 0 \\ 0 & y & -x \\ x & 0 & y^2 \end{bmatrix}, \qquad \beta_1 = \begin{bmatrix} y^4 & xy^3 & x^2 \\ -x^2 & y^4 & xy \\ -xy & -x^2 & y^2 \end{bmatrix},$$

$$\alpha_2 = \begin{bmatrix} y & -x & 0 \\ 0 & y^2 & -x \\ x & 0 & y^2 \end{bmatrix}, \qquad \beta_2 = \begin{bmatrix} y^4 & xy^3 & x^2 \\ -x^2 & y^4 & xy \\ -xy & -x^2 & y^2 \end{bmatrix},$$

$$\gamma_1 = \begin{bmatrix} y & -x & 0 & y^3 \\ x & 0 & -y^3 & 0 \\ -y^2 & 0 & -x^2 & 0 \\ 0 & -y^2 & -xy & -x^2 \end{bmatrix}, \qquad \delta_1 = \begin{bmatrix} 0 & x^2 & -y^3 & 0 \\ -x^2 & xy & 0 & -y^3 \\ 0 & -y^2 & -x & 0 \\ y^2 & 0 & y & -x \end{bmatrix},$$

$$\gamma_2 = \begin{bmatrix} x & y^2 & 0 & y \\ y^3 & -x^2 & -xy^2 & 0 \\ 0 & 0 & x^2 & y^2 \\ 0 & 0 & y^3 & -x \end{bmatrix}, \qquad \delta_2 = \begin{bmatrix} x^2 & y^2 & 0 & xy \\ y^3 & -x & -y^2 & 0 \\ 0 & 0 & x & y^2 \\ 0 & 0 & y^3 & -x^2 \end{bmatrix},$$

$$\xi_1 = \begin{bmatrix} y^4 & xy^2 & x^2 & 0 & 0 & xy \\ -x^2 & y^3 & xy & -x & 0 & 0 \\ -xy^2 & -x^2 & y^3 & 0 & -xy & 0 \\ 0 & 0 & 0 & y & -x & 0 \\ 0 & 0 & 0 & 0 & y^2 & -x \\ 0 & 0 & 0 & x & 0 & y^2 \end{bmatrix}, \quad \eta_1 = \begin{bmatrix} y & -x & 0 & 0 & 0 & -x \\ 0 & y^2 & -x & xy & 0 & 0 \\ x & 0 & y^2 & 0 & xy & 0 \\ 0 & 0 & 0 & y^4 & xy^2 & x^2 \\ 0 & 0 & 0 & -x^2 & y^3 & xy \\ 0 & 0 & 0 & -xy^2 & -x^2 & y^3 \end{bmatrix},$$

$$\xi_2 = \begin{bmatrix} y^4 & x^2 & 0 & -xy^2 & 0 \\ -x^2 & xy & 0 & -y^3 & 0 \\ 0 & -y^2 & -x & 0 & y^3 \\ -xy^2 & y^3 & 0 & x^2 & 0 \\ -y^3 & 0 & -y^2 & xy & -x^2 \end{bmatrix}, \quad \eta_2 = \begin{bmatrix} y & -x & 0 & 0 & 0 \\ x & 0 & 0 & y^2 & 0 \\ -y^2 & 0 & -x^2 & 0 & -y^3 \\ 0 & -y^2 & 0 & x & 0 \\ 0 & 0 & y^2 & y & -x \end{bmatrix}$$

Then

$$M_i = \operatorname{Coker} \varphi_i, \quad N_i = \operatorname{Coker} \psi_i,$$
$$A_i = \operatorname{Coker} \alpha_i, \quad B_i = \operatorname{Coker} \beta_i,$$
$$C_i = \operatorname{Coker} \gamma_i, \quad D_i = \operatorname{Coker} \delta_i,$$
$$X_i = \operatorname{Coker} \xi_i, \quad Y_i = \operatorname{Coker} \eta_i$$

for $i = 1, 2$.

CHAPTER 2

The Extension of AR Quivers

In this section we describe the extension of AR quivers from the category of Q-modules to the triangulated category $\mathbf{K}_{\text{tac}}(Q)$. In order to facilitate this extension we first define the notions of split and irreducible morphisms in $\mathbf{K}_{\text{tac}}(Q)$. We spend much of this chapter proving several lemmas in order to show that these notions extend nicely from the module category. In fact, we give a partial converse to Lemma 5.3 from [10], which we make extensive use of throughout this dissertation. We then describe how the vocabulary of AR theory in the module case extends to a triangulated category; namely, the ideas of AR triangles and AR translates. Furthermore, we not only show that these notions exist in $\mathbf{K}_{\text{tac}}(Q)$, but also provide concrete descriptions of both. Finally, we give a full accounting of the associated quivers and describe the corresponding vertices.

2.1 Split and Irreducible Morphisms

Let Q be a CM local ring with maximal ideal $\mathfrak{m}$ and residue field k. We first define what it means to be a split monomorphism and split epimorphism of complexes in the category $\mathbf{K}_{\text{tac}}(Q)$.

Definition 25. Let $f : C \to D$ be a morphism of totally acyclic complexes. We say that f is a *split monomorphism* if there exists a morphism of complexes $\gamma : D \to C$ such that $\gamma \circ f \sim \text{Id}_C$. Similarly, we say that f is a *split epimorphism* if there exists a morphism of complexes $\lambda : D \to C$ such that $f \circ \lambda \sim \text{Id}_D$.

When the distinction is irrelevant, we will often refer to a morphism which is either a split monomorphism or a split epimorphism as simply a split morphism. In particular, a morphism which does not split is a morphism which is neither a split epimorphism nor a split monomorphism.

Definition 26. [19, Section 4.3] Let C and D be totally acyclic complexes over R, and let $f : C \to D$ be a morphism of complexes. Then we say that f is an *irreducible chain morphism* if the following two conditions are satisfied:

i) f is not a split epimorphism nor a split monomorphism,

ii) if f can be decomposed as $f \sim h \circ g$ then either g is a split monomorphism or h is a split epimorphism.

One should note that this definition is simply a restriction of Happel's definition from [19] for an arbitrary category. We now give the definition of an irreducible morphism of totally acyclic complexes. The definitions of split and irreducible morphisms are analogous to those in the module category. The next few propositions will show that under certain conditions, an irreducible/split morphism in the category R-mod extends to one in $\mathbf{K}_{\text{tac}}(R)$ and vice-versa.

Lemma 27. Assume the chain maps $p, q : C \to D$ are homotopic with a homotopy morphism σ. Then p is a split epimorphism or a split monomorphism if and only if q is as well.

Proof. If $p \sim q$ then for each $n \in \mathbb{Z}$,

$$p_n - q_n = \sigma_{n-1}\partial_n^C + \partial_{n+1}^D \sigma_n. \tag{2.1}$$

Now assume that p is a split monomorphism, then there exists a chain map $\gamma : D \to C$ and homotopy τ such that

$$\mathrm{Id}_{C_n} - \gamma_n p_n = \tau_{n-1}\partial_n^C + \partial_{n+1}^C \tau_n \tag{2.2}$$

for each n. Then by composing equation (2.1) on the right by γ_n we obtain

$$\gamma_n p_n - \gamma_n q_n = \gamma_n \sigma_{n-1} \partial_n^C + \gamma_n \partial_{n+1}^D \sigma_n \tag{2.3}$$

and by substituting equation (2.2) into (2.3) we obtain

$$\mathrm{Id}_{C_n} - \tau_{n-1}\partial_n^C - \partial_{n+1}^C \tau_n - \gamma_n q_n = \gamma_n \sigma_{n-1} \partial_n^C + \gamma_n \partial_{n+1}^D \sigma_n.$$

Rearranging the previous equations and making the appropriate substitutions, we see that

$$\mathrm{Id}_{C_n} - \gamma_n q_n = \gamma_n \sigma_{n-1} \partial_n^C + \tau_{n-1} \partial_n^C + \partial_{n+1}^C \gamma_{n+1} \sigma_n + \partial_{n+1}^C \tau_n$$

which gives

$$\mathrm{Id}_{C_n} - \gamma_n q_n = \eta_{n-1} \partial_n^C + \partial_{n+1}^C \eta_n$$

where $\eta_n = \gamma_{n+1}\sigma_n + \tau_n$. Thus, $\mathrm{Id}_C \sim \gamma q$ and q is a split monomorphism.

Assume now that p is a split epimorphism. Then there exists a chain map $\lambda : D \to C$ and homotopy τ such that

$$\mathrm{Id}_{D_n} - p_n \lambda_n = \tau_{n-1} \partial_n^D + \partial_{n+1}^D \tau_n$$

for each n. Then, via an almost identical process, we obtain

$$\mathrm{Id}_{D_n} - q_n \lambda_n = \nu_{n-1} \partial_n^D + \partial_{n+1}^D \nu_n$$

where $\nu_n = \sigma_n \lambda_n + \tau_n$. Thus, $\mathrm{Id}_D \sim q\lambda$ and q is a split epimorphism. The reverse statement holds by switching p and q, which proves the statement. □

Proposition 28. Assume there exist chain maps $f, g : C \to D$ such that $f \sim g$. If f is an irreducible chain map, then so is g.

Proof. Let f be an irreducible morphism. Since f is not a split monomorphism nor a split epimorphism by Lemma 27, neither is g. Now suppose g decomposes so that

$g \sim k \circ h$ and note that by transitivity, $f \sim k \circ h$. Now since f is an irreducible morphism which decomposes, we must have that either k is a split monomorphism or h is a split epimorphism. This proves that g is an irreducible morphism. $\square$

We will now show that under particular assumptions, $f : M \to N$ is an irreducible morphism of R-modules if and only if the chain map $\hat{f} : C \to D$ is irreducible (where C and D are complete resolutions of M and N, respectively). We first state Lemma 5.3 from [10].

Lemma 29. [10, Lemma 5.3] Let $C \xrightarrow{\rho} P \xrightarrow{\pi} M$ and $D \xrightarrow{\rho'} P' \xrightarrow{\pi'} N$ be complete resolutions of finitely generated R-modules M and N, respectively. Further assume $f : M \to N$ is an R-module homomorphism, then there exists a unique (up to homotopy) morphism of R-complexes $\bar{f}$, making the right-hand square of the diagram

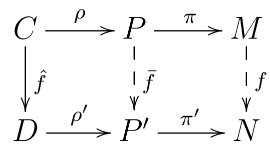
$$\begin{array}{ccccc} C & \xrightarrow{\rho} & P & \xrightarrow{\pi} & M \\ \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} \\ D & \xrightarrow{\rho'} & P' & \xrightarrow{\pi'} & N \end{array}$$

commute, and for each choice of $\bar{f}$ there exists a unique up to homotopy morphism $\hat{f}$ making the left-hand square commute up to homotopy. If two such $\bar{f}$ are homotopic, then so are the respective $\hat{f}$. If $f = 1_M$, then $\bar{f}$ and $\hat{f}$ are homotopy equivalences.

Effectively, this tells us that for each map $f : M \to N$ we have an induced morphism $\hat{f}$ of totally acyclic complexes. However, under certain conditions, we may also obtain $f : M \to N$ from the morphism of totally acyclic complexes $\hat{f}$. In effect, the following lemma is a converse to the previous one. This fact will play an important role in later applications.

Lemma 30. Let R be a Gorenstein local ring, and let M, N be maximal Cohen-Macaulay modules over R. Suppose further that C and D are the respective complete resolutions of M and N. Then for a morphism of totally acyclic complexes $\hat{f} : C \to D$

we obtain morphisms $\bar{f} : P \to P'$ and $f : M \to N$, such that the following diagram commutes.



Furthermore, these morphisms are unique in their respective categories.

Proof. Let M and N be maximal Cohen-Macaulay modules over a Gorenstein local ring R. Then by Theorem 3.1 in [10], M and N have complete resolutions

$$C \xrightarrow{\rho} P \xrightarrow{\pi} M$$

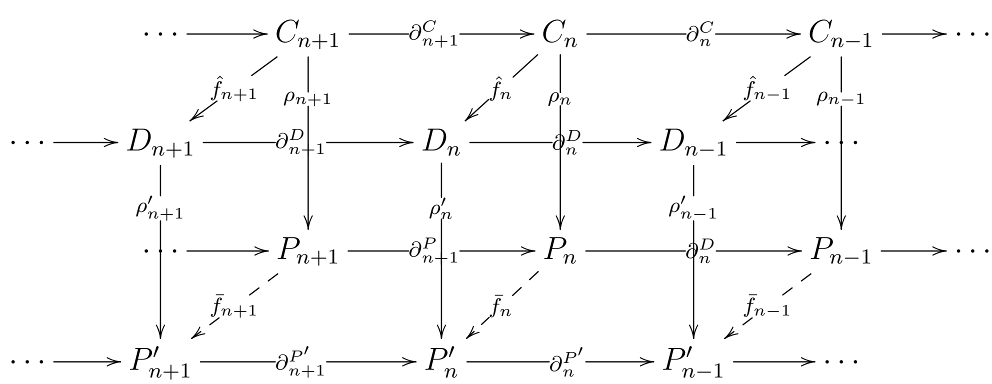
and

$$D \xrightarrow{\rho'} P' \xrightarrow{\pi'} N$$

(respectively), such that ρ_n and ρ'_n are bijective for all $n \geq 0$. Now we define $\bar{f} : P \to P'$ to be

$$\bar{f}_n = \rho'_n \hat{f}_n \rho_n^{-1}.$$

Our goal now is to show that $\bar{f}$ is a chain map. To do this we make use of the following diagram which we know commutes on all faces, save the bottom, which we will begin with proving.



Observe,

$$\bar{f}_n\,\partial^P_{n+1} = \rho'_n\hat{f}_n\rho_n^{-1}\,\partial^P_{n+1} =$$

$$\rho'_n\hat{f}_n\,\partial^C_{n+1}\,\rho^{-1}_{n+1} = \rho'_n\,\partial^D_{n+1}\,\hat{f}_{n+1}\rho^{-1}_{n+1} =$$

$$\partial^{P'}_{n+1}\,\rho'_{n+1}\hat{f}_{n+1}\rho^{-1}_{n+1} = \partial^{P'}_{n+1}\,\bar{f}_{n+1}$$

as needed. We finally show that if $\hat{f} \sim \hat{f}'$ then their induced maps, $\bar{f}$ and $\bar{f}'$ are homotopic as well. Let $\hat{f} \sim \hat{f}'$ so that there exists $\sigma : C \to \Sigma D$ such that for all n,

$$\hat{f}_n - \hat{f}'_n = \sigma_{n-1}\,\partial^C_n + \partial^D_{n+1}\,\sigma_n.$$

Now then,

$$\bar{f}_n - \bar{f}'_n = \rho'_n\hat{f}_n\rho_n^{-1} - \rho'_n\hat{f}'_n\rho_n^{-1} =$$

$$\rho'_n(\hat{f}_n - \hat{f}'_n)\rho_n^{-1} = \rho'_n(\sigma_{n-1}\,\partial^C_n + \partial^D_{n+1}\,\sigma_n)\rho_n^{-1} =$$

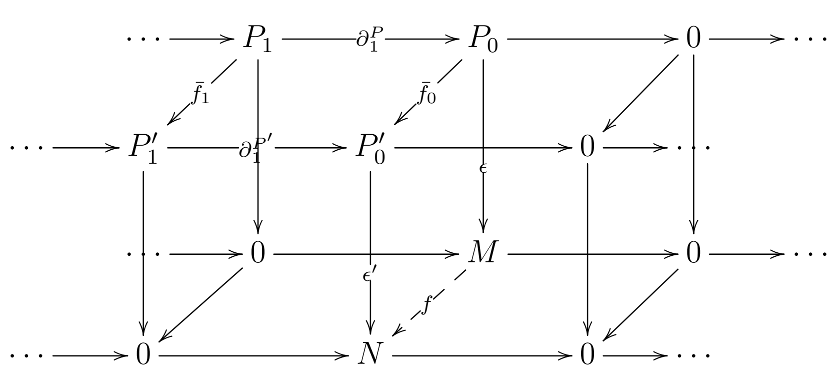
$$\rho'_n\sigma_{n-1}\,\partial^C_n\,\rho_n^{-1} + \rho'_n\,\partial^D_{n+1}\,\sigma_n\rho_n^{-1} =$$

$$\rho'_n\sigma_{n-1}\rho^{-1}_{n-1}\,\partial^P_n + \partial^{P'}_{n+1}\,\rho'_{n+1}\sigma_n\rho_n^{-1} =$$

$$\kappa_{n-1}\,\partial^P_n + \partial^{P'}_{n+1}\,\kappa_n$$

for $\kappa_n = \rho'_{n+1}\sigma_n\rho_n^{-1}$. Thus we have that $\bar{f} \sim \bar{f}'$.

We now turn our attention to the existence of $f : M \to N$. Consider the diagram which we know commutes on all faces:



Since

$$\cdots \to P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\epsilon} M \to 0$$

is an exact sequence we have that ϵ is surjective. Thus, for each $x \in M$ there exists $p \in P_0$ such that $\epsilon(p) = x$; so define $f(x) = \epsilon' \bar{f}_0(p)$. To show that f is well defined take $p, p' \in P_0$ such that $\epsilon(p) = x = \epsilon(p')$. Then $p - p' \in \ker \epsilon = \operatorname{Im} \partial_1^P$ so there exists $y \in P_1$ such that $\partial_1^P(y) = p - p'$. So we have that

$$\begin{aligned} &\epsilon' \bar{f}_0(p) = \\ \epsilon' \bar{f}_0(\partial_1^P(y) + p') = \epsilon' \bar{f}_0\, \partial_1^P(y) + \epsilon' \bar{f}_0(p') &= \epsilon'\, \partial_1^{P'}\, \bar{f}_1(y) + \epsilon' \bar{f}_0(p') \\ &= \epsilon' \bar{f}_0(p') \end{aligned}$$

proving f is well defined. Furthermore, if $\bar{f} \sim \bar{f}'$ then their induced maps are equal. To see this, note that there exists $\sigma : P \to \Sigma P'$ such that $\bar{f}_0 - \bar{f}'_0 = \partial_1^{P'}\, \sigma_0 + 0$. Then

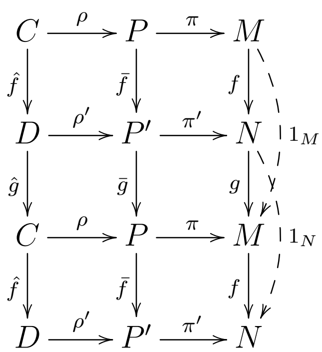
$$f - f' = \epsilon' \bar{f}_0 - \epsilon' \bar{f}'_0 = \epsilon'(\bar{f}_0 - \bar{f}'_0) = \epsilon'\, \partial_1^{P'}\, \sigma_0 = 0,$$

demonstrating that $f = f'$, and therefore f is unique. □

It should be noted that the maximal Cohen-Macaulay condition on modules in the previous lemma is necessary, as otherwise the bijectivity of ρ_0 is not guaranteed.

Lemma 31. Let R be a Gorenstein local ring, and let M, N be maximal Cohen-Macaulay modules over R. Then an R-module homomorphism $f : M \to N$ is a split monomorphism (resp. epimorphism) if and only if the induced morphism of totally acyclic complexes $\hat{f} : C \to D$ is a split monomorphism (resp. epimorphism).

Proof. Let $f : M \to N$ be a split R-module homomorphism, which implies there exists a $g : N \to M$ such that either $gf = 1_M$ or $fg = 1_N$. By Lemma 29 we have that there exist $\bar{f}, \bar{g}, \hat{f}$ and $\hat{g}$ such that the following diagram of complete resolutions commutes:



Also by Lemma 29, if $gf = \mathrm{Id}_M$ we must have that

$$\bar{g}\bar{f} \sim \mathrm{Id}_P \text{ and } \hat{g}\hat{f} \sim \mathrm{Id}_C\,.$$

Similarly, if $fg = \mathrm{Id}_N$ we have

$$\bar{f}\bar{g} \sim \mathrm{Id}_{P'} \text{ and } \hat{f}\hat{g} \sim \mathrm{Id}_D\,.$$

This shows that if f is a split monomorphism (resp. split epimorphism) then $\hat{f}$ is a split monomorphism (resp. split epimorphism).

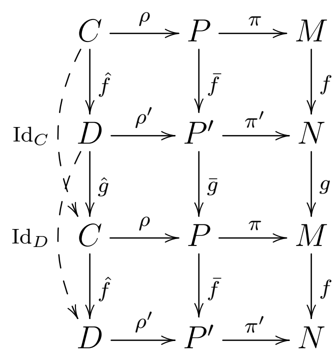
Now assume that $\hat{f} : C \to D$ is a split morphism of totally acyclic complexes. Then there exists a $\hat{g} : D \to C$ such that either, $\hat{g}\hat{f} \sim \mathrm{Id}_C$ or $\hat{f}\hat{g} \sim \mathrm{Id}_D$. In other words, there exists $\sigma : C \to \Sigma C$ such that

$$\hat{g}\hat{f} - \mathrm{Id}_C = \partial^C_{n+1}\,\sigma_n + \sigma_{n-1}\,\partial^C_n$$

or $\sigma' : D \to \Sigma D$ such that

$$\hat{f}\hat{g} - \mathrm{Id}_D = \partial^D_{n+1}\,\sigma'_n + \sigma'_{n-1}\,\partial^D_n\,.$$

By Lemma 30 we obtain the induced maps $\bar{g}, \bar{f}, g$ and f such that $\bar{f}_n = \rho'_n \hat{f}_n \rho_n^{-1}$ and similarly, $\bar{g}_n = \rho_n \hat{g}_n {\rho'}_n^{-1}$, such that the following diagram commutes:



Now

$$\bar{g}_n \bar{f}_n =$$

$$(\rho_n \hat{g}_n {\rho'}_n^{-1})(\rho'_n \hat{f}_n \rho_n^{-1}) = \rho_n \hat{g}_n {\rho'}_n^{-1} \rho'_n \hat{f}_n \rho_n^{-1} =$$

$$\rho_n \hat{g}_n \hat{f}_n \rho_n^{-1} = \rho_n (Id_{C_n} + \partial_{n+1}^C \, \sigma_n + \sigma_{n-1} \, \partial_n^C) \rho_n^{-1} =$$

$$\rho_n Id_{C_n} \rho_n^{-1} + \rho_n \, \partial_{n+1}^C \, \sigma_n \rho_n^{-1} + \rho_n \sigma_{n-1} \, \partial_n^C \, \rho_n^{-1} =$$

$$Id_{P_n} + \partial_{n+1}^P \, \rho_{n+1} \sigma_n \rho_n^{-1} + \rho_n \sigma_{n-1} \rho_{n-1}^{-1} \, \partial_n^P$$

$$= Id_{P_n} + \partial_{n+1}^P \, \tau_n + \tau_{n-1} \, \partial_n^P$$

where $\tau_n = \rho_{n+1} \sigma_n \rho_n^{-1}$. So we have that

$$\bar{g} \bar{f} \sim \mathrm{Id}_P \, .$$

We can similarly show that $\bar{f} \bar{g} \sim \mathrm{Id}_{P'}$.

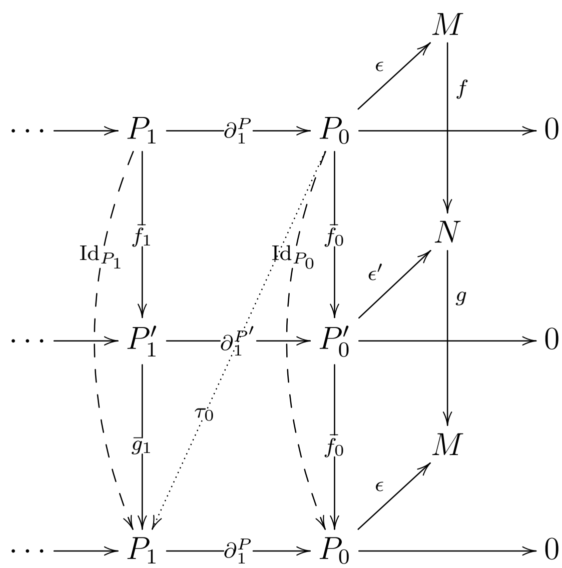
Thus far we have shown that if $\hat{f}$ splits then $\bar{f}$ splits as well. In other words we have that $\bar{f} \bar{g} \sim \mathrm{Id}_{P'}$ or $\bar{g} \bar{f} \sim \mathrm{Id}_P$. In particular, we have that

$$\bar{g}_0 \bar{f}_0 - \mathrm{Id}_{P_0} = \partial_1^P \, \tau_0 + 0$$

or

$$\bar{f}_0 \bar{g}_0 - \mathrm{Id}_{P'_0} = \partial_1^{P'} \, \tau'_0 + 0.$$

From this we obtain the following commutative diagram:



It should also be noted that $g : N \to M$ is given by $g(y) = \epsilon \bar{g}_0(p')$ where $\epsilon'(p') = y$ and is well defined. Now if $x \in M$ then $gf(x) = g(\epsilon' \bar{f}_0(p))$ where $\epsilon(p) = x$ and $p' = \bar{f}_0(p)$. It follows that

$$g(\epsilon' \bar{f}_0(p)) =$$

$$\epsilon \bar{g}_0 \bar{f}_0(p) = \epsilon(\mathrm{Id}_{P_0} + \partial_1^P \tau_0)(p) = \epsilon \, \mathrm{Id}_{P_0}(p) + \epsilon \, \partial_1^P \tau_0(p)$$

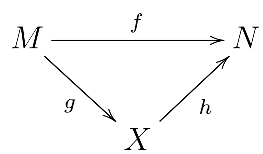
$$= \epsilon(p) = x$$

and therefore $gf = \mathrm{Id}_M$. We can similarly show that $fg = \mathrm{Id}_N$. Thus, we have shown that if $\hat{f}$ is a split monomorphism (resp. epimorphism) then f is a split monomorphism (resp. epimorphism). Thus, $\hat{f}$ splits if and only if f splits. $\square$

In the previous lemmas we have shown that the split and irreducible properties are preserved under homotopy and that split morphisms can be extended to and from the homotopy category of totally acyclic complexes. We now prove the last piece of the puzzle linking AR quivers over R-mod and $\mathbf{K}_{\mathrm{tac}}(R)$: that irreducible morphisms can be extended as well.

Proposition 32. Let R be a Gorenstein local ring, and let M, N be maximal Cohen-Macaulay modules over R. Then an R-module homomorphism $f : M \to N$ is irreducible if and only if the induced morphism of totally acyclic complexes $\hat{f} : C \to D$ is irreducible.

Proof. Let f be an irreducible R-module homomorphism. By definition $f : M \to N$ must not be a split morphism, so that by Lemma 31, the induced morphism $\hat{f} : C \to D$ is not split either. Now assume that f decomposes as in the following diagram:



In this case we must have that either $g : M \to X$ is a split monomorphism or $h : X \to N$ is a split epimorphism. Now by Lemma 29 and its uniqueness property, we have that there exist morphisms $\bar{f}, \bar{g}, \bar{h}$ and $\hat{f}, \hat{g}, \hat{h}$ such that each face of the following diagram commutes.

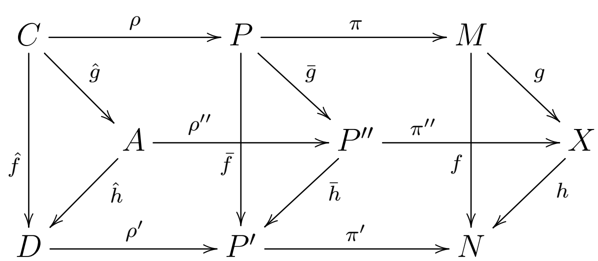


Fig. 1

So, we have that

$$\bar{f} \sim \bar{h}\bar{g} \text{ and } \hat{f} \sim \hat{h}\hat{g}.$$

Now by Lemma 31, if g is a split monomorphism then so are $\bar{g}$ and $\hat{g}$. Similarly, if h is a split epimorphism then so are $\bar{h}$ and $\hat{h}$. Thus, if f is irreducible, then so is $\hat{f}$.

Now assume that $\hat{f}$ is an irreducible morphism of totally acyclic complexes. By definition $\hat{f}$ is not a split morphism and, by Lemma 31, neither is f. Also, we have that if $\hat{f} \sim \hat{h}\hat{g}$, either $\hat{g}$ is a split monomorphism or $\hat{h}$ is a split epimorphism. Now, by Lemma 30, there exist morphisms $\bar{f}, \bar{g}, \bar{h}$ and f, g, h such that each face of Fig. 1 commutes. Therefore, we have that $\hat{f} \sim \hat{h}\hat{g}$ and $f = hg$. Furthermore, by Lemma 31 we have that if $\hat{g}$ is a split monomorphism or $\hat{h}$ is a split epimorphism, then so must be g or h, respectively. Thus, if $\hat{f}$ is irreducible then f is irreducible as well, proving the statement. □

2.2 Extension of AR Quivers to $\mathbf{K}_{\text{tac}}(Q)$

The extension of AR quivers to $\mathbf{K}_{\text{tac}}(Q)$ begins by recalling the definition of an AR triangle from [20]. See also [15].

Definition 33. [20, Section 3.1] A distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is called an *Auslander-Reiten triangle*, or AR triangle, if the following conditions are satisfied:

(AR1) X and Z are indecomposable objects,

(AR2) $w \neq 0$,

(AR3) If $f : W \to Z$ is not a retraction, then there exists $f' : W \to Y$ such that $vf' = f$.

We say that a triangulated category $\mathcal{T}$ has AR triangles if, for any indecomposable object Z of $\mathcal{T}$, there exists an AR-triangle ending at Z :

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

In this case, the AR-triangle is unique up to triangle isomorphism inducing the identity of Z. To show that $\mathbf{K}_{\text{tac}}(Q)$ has AR triangles, we must briefly discuss Serre functors. As defined in [15], a *Serre functor* of a category $\mathcal{T}$ is an auto-equivalence

$\nu : \mathcal{T} \to \mathcal{T}$ together with an isomorphism $D\operatorname{Hom}_{\mathcal{T}}(X, -) \simeq \operatorname{Hom}_{\mathcal{T}}(-, \nu X)$ for each $X \in \mathcal{T}$, where D is the duality, $\operatorname{Hom}_k(-, k)$. Applying Theorem 1.1.1 from [15] to $\mathbf{K}_{\text{tac}}(Q)$ we may deduce that $\mathbf{K}_{\text{tac}}(Q)$ has a Serre functor ν.

In fact, as shown by Auslander [27] in 1978, the stable category of maximal Cohen-Macaulay modules over a commutative isolated d-dimensional local Gorenstein singularity is $(d-1)$ Calabi-Yau. (See also [28]) Since the rings we are dealing with have dimension one, the Serre functor is given by $\nu = \text{Id}$. To prove that $\mathbf{K}_{\text{tac}}(Q)$ has AR triangles, consider the indecomposable totally acyclic complex Z. Then the AR triangle ending in Z is given by

$$\tau Z \to Y \to Z \to \nu Z$$

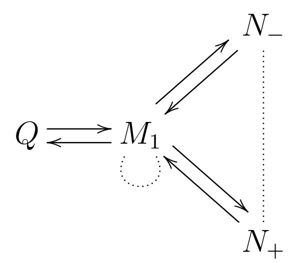
where τ, given by the composition $\tau = \Sigma^{-1}\nu$, is the AR translate. In other words we have the triangle:

$$\Sigma^{-1} Z \to Y \to Z \to \Sigma Z$$

Recall that in what follows the ring Q is a Henselian Gorenstein local ring of finite CM type. In order to construct AR quivers in $\mathbf{K}_{\text{tac}}(Q)$ we note that if a maximal Cohen-Macaulay module M is indecomposable then the totally acyclic complex C induced from the complete resolution $C \to P \to M$ is indecomposable as well. The proof of this fact is held until the next chapter (see Proposition 36), but has a few consequences we now mention. For one, we have that Q is of finite CM type if and only if Q is of finite TAC type; in other words, it has only finitely many indecomposable totally acyclic Q-complexes up to homotopy equivalence. Furthermore, by considering the vertices in the AR quivers of the Q-module category and extending those to totally acyclic complexes via their complete resolutions, we obtain the vertices of the AR quivers in $\mathbf{K}_{\text{tac}}(Q)$. For the edges, we consider the irreducible morphisms between the indecomposable modules. By applying Proposition 32 to each irreducible morphism in the AR quiver in Q-mod we obtain irreducible morphisms between the indecomposable totally acyclic complexes. Hence, we obtain the edges of the AR quiver in $\mathbf{K}_{\text{tac}}(Q)$.

Before presenting a full accounting of the AR quivers in the category $\mathbf{K}_{\text{tac}}(Q)$, it may be illuminating to see a specific example.

Example 34. Consider the case (A_3); that is, when $Q = \frac{k[\![x,y]\!]}{(x^2+y^4)}$. Then the AR quiver in the module category is given by:



where the vertices are given by:

$$M_1 = \text{Coker} \begin{bmatrix} x & y \\ y^3 & -x \end{bmatrix},$$

$$N_- = \text{Coker}\,(x - iy^2),$$

$$N_+ = \text{Coker}\,(x + iy^2).$$

Furthermore, the AR sequences ending in the modules M_1, N_+, N_- are:

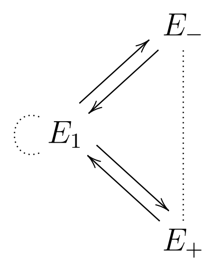
$$0 \to M_1 \to Q \oplus N_+ \oplus N_- \to M_1 \to 0,$$

$$0 \to N_- \to M_1 \to N_+ \to 0,$$

$$0 \to N_+ \to M_1 \to N_- \to 0$$

respectively.

Then by extending the modules and irreducible morphisms to their associated counterparts in $\mathbf{K}_{\text{tac}}(Q)$ we obtain the AR quiver:



where the totally acyclic complex associated to the module Q is the zero complex, since $\operatorname{pd}_Q(Q) = 0$. The other vertices are given by the totally acyclic complexes:

$$E_1 = \cdots \to \underset{1}{Q^2} \xrightarrow{\left[\begin{smallmatrix} x & y \\ y^3 & -x \end{smallmatrix}\right]} \underset{0}{Q^2} \xrightarrow{\left[\begin{smallmatrix} x & y \\ y^3 & -x \end{smallmatrix}\right]} \underset{-1}{Q^2} \to \cdots$$

$$E_- = \cdots \to \underset{1}{Q} \xrightarrow{(x-iy^2)} \underset{0}{Q} \xrightarrow{(x+iy^2)} \underset{-1}{Q} \to \cdots$$

$$E_+ = \cdots \to \underset{1}{Q} \xrightarrow{(x+iy^2)} \underset{0}{Q} \xrightarrow{(x-iy^2)} \underset{-1}{Q} \to \cdots$$

We may similarly extend the AR sequences to triangles given by:

$$E_1 \to E_+ \oplus E_- \to E_1 \to E_1,$$

$$E_- \to E_1 \to E_+ \to E_-,$$

$$E_+ \to E_1 \to E_- \to E_+$$

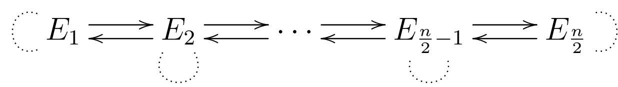
respectively.

2.3 AR Quivers in $\mathbf{K}_{\text{tac}}(Q)$

For completeness we now give the AR quivers for each case as we did previously. One should note that these are completely analogous to those in the module category. Let

$$Q = \frac{k[\![x, y]\!]}{(f)}$$

- When $f = x^2 + y^{n+1}$ we have two cases, for (A_n) with $n = 2l$ we have the quiver

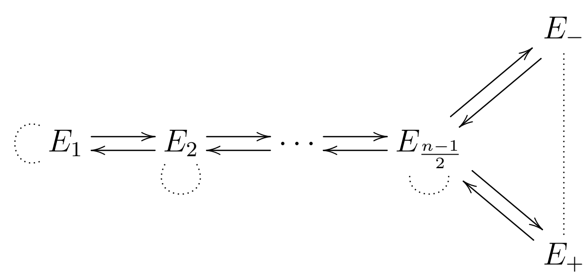


where:

$$E_j = \cdots \to Q^2 \xrightarrow{\begin{bmatrix} x & y^j \\ y^{n+1-j} & -x \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x & y^j \\ y^{n+1-j} & -x \end{bmatrix}} Q^2 \to \cdots$$

for $1 \leq j \leq \frac{n}{2}$.

For (A_n) with $n = 2l - 1$ we have the quiver



where, similar to the above example:

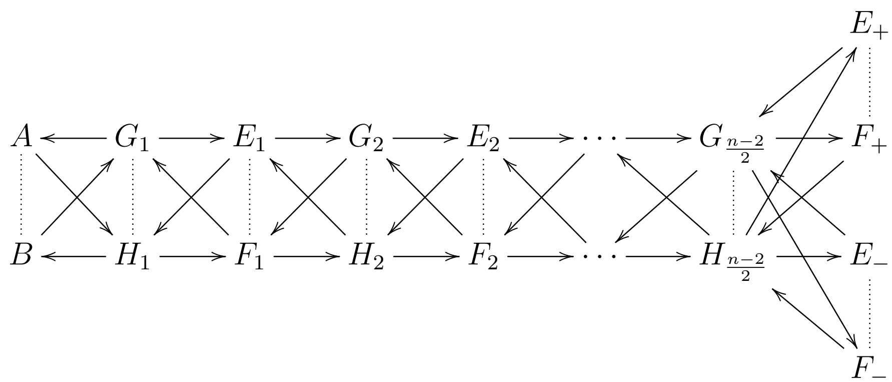
$$E_j = \cdots \to Q^2 \xrightarrow{\begin{bmatrix} x & y^j \\ y^{n+1-j} & -x \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x & y^j \\ y^{n+1-j} & -x \end{bmatrix}} Q^2 \to \cdots$$

$$E_- = \cdots \to Q \xrightarrow{(x-iy^l)} Q \xrightarrow{(x+iy^l)} Q \to \cdots$$

$$E_+ = \cdots \to Q \xrightarrow{(x+iy^l)} Q \xrightarrow{(x-iy^l)} Q \to \cdots$$

for $1 \leq j \leq \frac{n-1}{2}$.

- When $f = x^2y + y^{n-1}$ we again have two cases, for (D_n) with n even we have the quiver



where the vertices are given by the following totally acyclic complexes:

$$A = \cdots \to Q \xrightarrow{(y)} Q \xrightarrow{(x^2+y^{n-2})} Q \xrightarrow{(y)} Q \to \cdots$$

$$E_j = \cdots \to Q^2 \xrightarrow{\begin{bmatrix} x & y^j \\ y^{n-j-2} & -x \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} xy & y^{j+1} \\ y^{n-j-1} & -xy \end{bmatrix}} Q^2 \to \cdots$$

$$G_j = \cdots \to Q^2 \xrightarrow{\begin{bmatrix} x & y^j \\ y^{n-j-1} & -xy \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} xy & y^j \\ y^{n-j-1} & -x \end{bmatrix}} Q^2 \to \cdots$$

$$E_+ = \cdots \to Q \xrightarrow{y(x+iy^{\frac{n-2}{2}})} Q \xrightarrow{(x-iy^{\frac{n-2}{2}})} Q \xrightarrow{y(x+iy^{\frac{n-2}{2}})} Q \to \cdots$$

$$E_- = \cdots \to Q \xrightarrow{y(x-iy^{\frac{n-2}{2}})} Q \xrightarrow{(x+iy^{\frac{n-2}{2}})} Q \xrightarrow{y(x-iy^{\frac{n-2}{2}})} Q \to \cdots$$

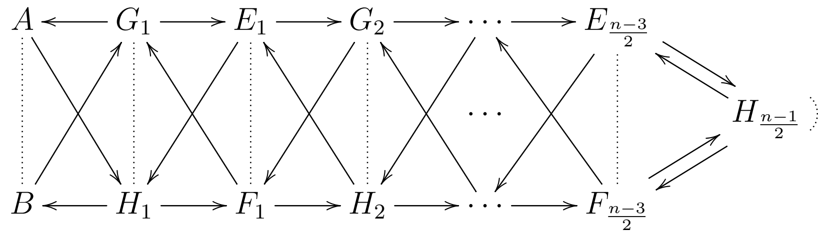
$$B = \Sigma^{-1}A, F_j = \Sigma^{-1}E_j,$$

$$F_+ = \Sigma^{-1}E_+, \quad F_- = \Sigma^{-1}E_-,$$

$$H_j = \Sigma^{-1}G_j$$

for $1 \leq j \leq n-3$.

For (D_n) with n odd we have the quiver

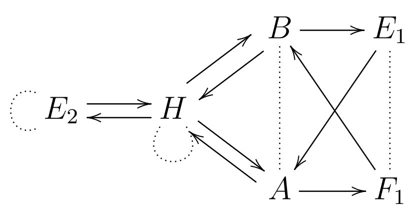


where the vertices are the same as those in the even case, with the caveat that

$$H_{\frac{n-1}{2}} \simeq G_{\frac{n-1}{2}}$$

and as such, is its own AR translate.

- When $f = x^3 + y^4$ we have the case (E_6) and the is quiver given by



where the vertices are given by the following totally acyclic complexes:

$$E_1 = \cdots \to Q^2 \xrightarrow{\begin{bmatrix} x & y \\ y^3 & -x^2 \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x^2 & y \\ y^3 & -x \end{bmatrix}} Q^2 \to \cdots$$

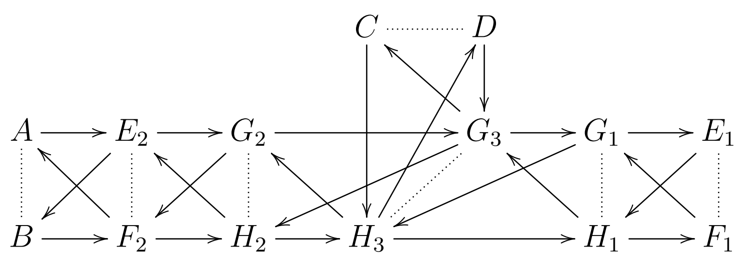
$$E_2 \simeq F_2 = \cdots \to Q^2 \xrightarrow{\begin{bmatrix} x & y^2 \\ y^2 & -x^2 \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x^2 & y^2 \\ y^2 & -x \end{bmatrix}} Q^2 \to \cdots$$

$$A = \cdots \to Q^3 \xrightarrow{\begin{bmatrix} y^3 & x^2 & xy^2 \\ xy & -y^2 & x^2 \\ x^2 & -xy & -y^3 \end{bmatrix}} Q^3 \xrightarrow{\begin{bmatrix} y & 0 & x \\ x & -y^2 & 0 \\ 0 & x & -y \end{bmatrix}} Q^3 \to \cdots$$

$$H = \cdots \to Q^4 \xrightarrow{\begin{bmatrix} x & y^2 & 0 & y \\ y^2 & -x^2 & -xy & 0 \\ 0 & 0 & x^2 & y \\ 0 & 0 & y^3 & -x \end{bmatrix}} Q^4 \xrightarrow{\begin{bmatrix} x^2 & y & 0 & xy \\ y^3 & -x & y & 0 \\ 0 & 0 & x & y^2 \\ 0 & 0 & y^2 & -x^2 \end{bmatrix}} Q^4 \to \cdots$$

$$F_1 = \Sigma^{-1} E_1, B = \Sigma^{-1} A.$$

- When $f = x^3 + xy^3$ we have the case (E_7) and the quiver is given by



where the vertices are given by the following totally acyclic complexes:

$$A = \cdots \to Q \xrightarrow{(x)} Q \xrightarrow{(x^2+y^3)} Q \to \cdots$$

$$C = \cdots \to Q^2 \xrightarrow{\begin{bmatrix} x^2 & yx \\ xy^2 & -x^2 \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x & y \\ y^2 & -x \end{bmatrix}} Q^2 \to \cdots$$

$$E_1 = \cdots \to Q^2 \xrightarrow{\begin{bmatrix} x & y \\ xy^2 & -x^2 \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x^2 & y \\ xy^2 & -x \end{bmatrix}} Q^2 \to \cdots$$

$$E_2 = \cdots \to Q^2 \xrightarrow{\begin{bmatrix} x & y^2 \\ xy & -x^2 \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x^2 & y^2 \\ xy & -x \end{bmatrix}} Q^2 \to \cdots$$

$$G_1 = \cdots \to Q^3 \xrightarrow{\begin{bmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{bmatrix}} Q^3 \xrightarrow{\begin{bmatrix} xy^2 & -x^2 & -x^2y \\ xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{bmatrix}} Q^3 \to \cdots$$

$$G_2 = \cdots \to Q^3 \xrightarrow{\begin{bmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{bmatrix}} Q^3 \xrightarrow{\begin{bmatrix} xy^2 & -x^2 & -x^2y \\ xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{bmatrix}} Q^3 \to \cdots$$

$$G_3 = \cdots \to Q^4 \xrightarrow{\begin{bmatrix} x & y & -y & 0 \\ y^2 & -x & 0 & -y \\ 0 & 0 & x^2 & xy \\ 0 & 0 & xy^2 & -x^2 \end{bmatrix}} Q^4 \xrightarrow{\begin{bmatrix} x^2 & xy & y & 0 \\ xy^2 & -x^2 & 0 & y \\ 0 & 0 & x & y \\ 0 & 0 & y^2 & -x \end{bmatrix}} Q^4 \to \cdots$$

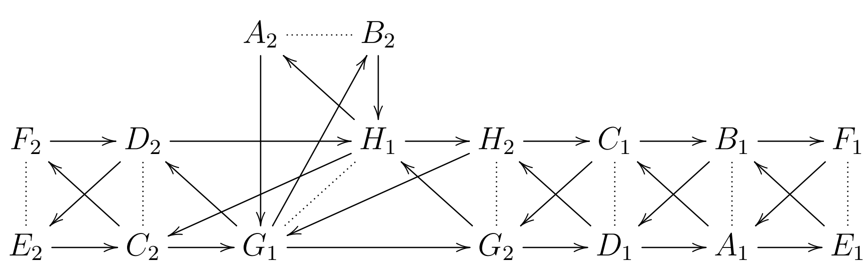
$$B = \Sigma^{-1}A, D = \Sigma^{-1}C$$

and

$$F_i = \Sigma^{-1}E_i \ , \ H_j = \Sigma^{-1}G_j$$

for $1 \leq i \leq 2$ and $1 \leq j \leq 3$.

- Finally, when $f = x^3 + y^5$ we have the case (E_8) and the quiver is given by



where the vertices are given by the following totally acyclic complexes:

$$A_1 = \cdots \to Q^3 \xrightarrow{\begin{bmatrix} y & -x & 0 \\ 0 & y & -x \\ x & 0 & y^2 \end{bmatrix}} Q^3 \xrightarrow{\begin{bmatrix} y^4 & xy^3 & x^2 \\ -x^2 & y^4 & xy \\ -xy & -x^2 & y^2 \end{bmatrix}} Q^3 \to \cdots$$

$$A_2 = \cdots \to Q^3 \xrightarrow{\begin{bmatrix} y & -x & 0 \\ 0 & y^2 & -x \\ x & 0 & y^2 \end{bmatrix}} Q^3 \xrightarrow{\begin{bmatrix} y^4 & xy^3 & x^2 \\ -x^2 & y^4 & xy \\ -xy & -x^2 & y^2 \end{bmatrix}} Q^3 \to \cdots$$

$$E_1 = \cdots \to Q^2 \xrightarrow{\begin{bmatrix} x & y \\ y^4 & -x^2 \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x^2 & y \\ y^4 & -x \end{bmatrix}} Q^2 \to \cdots$$

$$E_2 = \cdots \to Q^2 \xrightarrow{\begin{bmatrix} x & y^2 \\ y^3 & -x^2 \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x^2 & y^2 \\ y^3 & -x \end{bmatrix}} Q^2 \to \cdots$$

$$C_1 = \cdots \to Q \xrightarrow{\begin{bmatrix} y & -x & 0 & y^3 \\ x & 0 & -y^3 & 0 \\ -y^2 & 0 & -x^2 & 0 \\ 0 & -y^2 & -xy & -x^2 \end{bmatrix}} Q \xrightarrow{\begin{bmatrix} 0 & x^2 & -y^3 & 0 \\ -x^2 & xy & 0 & -y^3 \\ 0 & -y^2 & -x & 0 \\ y^2 & 0 & y & -x \end{bmatrix}} Q \to \cdots$$

$$C_2 = \cdots \to Q \xrightarrow{\begin{bmatrix} x & y^2 & 0 & y \\ y^3 & -x^2 & -xy^2 & 0 \\ 0 & 0 & x^2 & y^2 \\ 0 & 0 & y^3 & -x \end{bmatrix}} Q \xrightarrow{\begin{bmatrix} x^2 & y^2 & 0 & xy \\ y^3 & -x & -y^2 & 0 \\ 0 & 0 & x & y^2 \\ 0 & 0 & y^3 & -x^2 \end{bmatrix}} Q \to \cdots$$

$$G_1 = \cdots \to Q \xrightarrow{\begin{bmatrix} y & -x & 0 & 0 & 0 & -x \\ 0 & y^2 & -x & xy & 0 & 0 \\ x & 0 & y^2 & 0 & xy & 0 \\ 0 & 0 & 0 & y^4 & xy^2 & x^2 \\ 0 & 0 & 0 & -x^2 & y^3 & xy \\ 0 & 0 & 0 & -xy^2 & -x^2 & y^3 \end{bmatrix}} Q \xrightarrow{\begin{bmatrix} y^4 & xy^2 & x^2 & 0 & 0 & xy \\ -x^2 & y^3 & xy & -x & 0 & 0 \\ -xy^2 & -x^2 & y^3 & 0 & -xy & 0 \\ 0 & 0 & 0 & y & -x & 0 \\ 0 & 0 & 0 & 0 & y^2 & -x \\ 0 & 0 & 0 & x & 0 & y^2 \end{bmatrix}} Q \to \cdots$$

$$G_2 = \cdots \to Q \xrightarrow{\begin{bmatrix} y & -x & 0 & 0 & 0 \\ x & 0 & 0 & y^2 & 0 \\ -y^2 & 0 & -x^2 & 0 & -y^3 \\ 0 & -y^2 & 0 & x & 0 \\ 0 & 0 & y^2 & y & -x \end{bmatrix}} Q \xrightarrow{\begin{bmatrix} y^4 & x^2 & 0 & -xy^2 & 0 \\ -x^2 & xy & 0 & -y^3 & 0 \\ 0 & -y^2 & -x & 0 & y^3 \\ -xy^2 & y^3 & 0 & x^2 & 0 \\ -y^3 & 0 & -y^2 & xy & -x^2 \end{bmatrix}} Q \to \cdots$$

and

$$B_j = \Sigma^{-1} A_j, \quad D_j = \Sigma^{-1} C_j,$$
$$F_j = \Sigma^{-1} E_j, \quad H_j = \Sigma^{-1} G_j,$$

for $1 \leq j \leq 2$.

CHAPTER 3

Classification of $\mathbf{K}_{\text{tac}}(R)$

In this section we define a classification scheme for totally acyclic complexes over a ring R. Let Q be a Henselian Gorenstein local ring and $Q \overset{\varphi}{\twoheadrightarrow} R$ be a surjective ring homomorphism such that R has finite projective dimension as a Q-module. We start by showing that over such a ring, $\mathbf{K}_{\text{tac}}(Q)$ is a Krull-Schmidt category. Afterward, we make use of a pair of functors which induce approximations (discussed at length in [2]), along with some basic properties of totally acyclic complexes to define an Arnold-Tuple. As necessitated by this definition, we prove the existence of minimal right approximations in the category of $\mathbf{K}_{\text{tac}}(R)$ as an inevitable extension of the existence of a non-minimal approximation. We then show that this definition is well defined up to homotopy, although a more "coarse" description than that of homotopy equivalence. Lastly, by building upon the work of Bergh, Jorgensen, and Moore in [3], we discuss an extension of their result which gives a concrete description of approximations, to the relative codimension $c \geq 2$ setting.

3.1 $\mathbf{K}_{\text{tac}}(Q)$ is a Krull-Schmidt category

To begin, we recall a definition by Claus Michael Ringel in [13]. We note that in the first chapter we stated Proposition 5, which relates Henselian rings to the Krull-Schmidt-Remak theorem. It is now that we build the previously advertised extension to $\mathbf{K}_{\text{tac}}(R)$.

Definition 35. [13, Section 2.2] A k-additive category $\mathcal{T}$ is called a *Krull-Schmidt category* if the endomorphism ring $\operatorname{End}(X)$ of any indecomposable object C of $\mathcal{T}$ is a local ring.

Let Q be a Henselian Gorenstein local ring, it is in this section we show that $\mathbf{K}_{\text{tac}}(Q)$ is a Krull-Schmidt category. However, we first state two facts which hold in this category.

Proposition 36. Let Q be a Henselian Gorenstein local ring and M be a maximal Cohen Macaulay Q-module. Furthermore, let

$$C \xrightarrow{\rho} P \xrightarrow{\pi} M$$

be the complete resolution of M with $C \in \mathbf{K}_{\text{tac}}(Q)$. Then we have that M is indecomposable if and only if C is as well.

Proof. Assume for contraposition that M is decomposable, so that we may write $M = A \oplus B$ with neither A nor B contractible. We can then consider the complete resolution of $A \oplus B$,

$$C_A \oplus C_B \xrightarrow{\rho'} P_A \oplus P_B \xrightarrow{\pi'} A \oplus B,$$

which gives us the following diagram:

$$\begin{array}{ccccc} C & \xrightarrow{\rho} & P & \xrightarrow{\pi} & M \\ \downarrow{\scriptstyle \hat{\mu}} & & \downarrow{\scriptstyle \bar{\mu}} & & \downarrow{\scriptstyle Id} \\ C_A \oplus C_B & \xrightarrow{\rho'} & P_A \oplus P_B & \xrightarrow{\pi'} & A \oplus B \end{array}$$

Then by Lemma 29 we have that $\hat{\mu}$ is a homotopy equivalence, and therefore C is a decomposable complex in $\mathbf{K}_{\text{tac}}(Q)$. Similarly, if C is a decomposable complex in $\mathbf{K}_{\text{tac}}(Q)$ we have that $C \simeq C_A \oplus C_B$ with neither C_A nor C_B contractible. Now let $A = \operatorname{Im} \partial_0^{C_A}$ and $B = \operatorname{Im} \partial_0^{C_B}$ and note that by Lemma 30 in conjunction with the previous diagram, we have that $C \simeq C_A \oplus C_B$ implies $M = A \oplus B$. Thus, M is decomposable as well. $\square$

Proposition 37. Let Q be a Henselian Gorenstein local ring and M be an MCM Q-module. Furthermore, let

$$C \xrightarrow{\rho} P \xrightarrow{\pi} M$$

be the complete resolution of M with $C \in \mathbf{K}_{\mathrm{tac}}(Q)$. Then $\mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(Q)}(C) \cong \mathrm{End}_Q(M)$ as rings.

Proof. We begin by defining a map $\phi : \mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(Q)}(C) \to \mathrm{End}_Q(M)$. Let $\hat{f} \in \mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(Q)}(C)$ and consider the following diagram:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_1 & \xrightarrow{\partial_1^C} & C_0 & \xrightarrow{\partial_0^C} & \mathrm{Im}\,\partial_0^C \\
& \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\
\cdots \longrightarrow & C_1 & \xrightarrow{\partial_1^C} & C_0 & \xrightarrow{\partial_0^C} & \mathrm{Im}\,\partial_0^C
\end{array}$$

Where $\mathrm{Im}\,\partial_0^C = M$ and define ϕ as follows:

$$\phi(\hat{f}) = f \text{ where } f(x) = \partial_0^C\, f_0(a) \text{ such that } \partial_0^C(a) = x.$$

To see that this map makes sense we first show that f is independent of the choice in preimage of x. Suppose that $\partial_0^C(a) = x = \partial_0^C(b)$ so that $\partial_0^C(a-b) = 0$ implying $a - b \in \mathrm{Ker}(\partial_0^C) = \mathrm{Im}(\partial_1^C)$ by exactness of C. Thus, there exists $y \in C_1$ such that $\partial_1^C(y) = a - b$, from which it follows that

$$\partial_1^C\, f_1(y) = f_0\, \partial_1^C(y) = f_0(a-b). \tag{3.1}$$

By composing (3.1) on the left with ∂_0^C we obtain

$$\partial_0^C\, \partial_1^C\, f_1(y) = \partial_0^C\, f_0(a-b)$$

where the left-hand term is clearly zero. Thus, $\partial_0^C\, f_0(a) = \partial_0^C\, f_0(b)$ as needed. Now Lemma 30 states that two homotopic maps, $\hat{f}, \hat{g} \in \mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(Q)}(C)$, give the same morphism in $\mathrm{End}_Q(M)$. Thus, ϕ is well defined.

Next we define a map $\psi : \mathrm{End}_Q(M) \to \mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(Q)}(C)$ by $\psi(f) = \hat{f}$ where $\hat{f}$ is the morphism of complete resolutions described by Lemma 29.

Now we will show that

$$\psi\phi = \mathrm{Id}_{\mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(Q)}(C)} \text{ and } \phi\psi = \mathrm{Id}_{\mathrm{End}_Q(M)} .$$

Let $\hat{f} \in \mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(Q)}(C)$ so that $\psi\phi(\hat{f}) = \psi f = \hat{f}'$ and by the uniqueness property of Lemma 29, $\hat{f} \sim \hat{f}'$. Thus, $\psi\phi = \mathrm{Id}_{\mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(Q)}(C)}$. Now, if instead $f \in \mathrm{End}_Q(M)$ then $\phi\psi(f) = \phi(\hat{f}) = f'$ and again by the uniqueness property of Lemma 30, $f = f'$. Thus, $\phi\psi = \mathrm{Id}_{\mathrm{End}_Q(M)}$.

This proves the ring isomorphism $\mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(Q)}(C) \cong \mathrm{End}_Q(M)$. □

Theorem 38. Let Q be a Henselian Gorenstein local ring. The category $\mathbf{K}_{\mathrm{tac}}(Q)$ is a Krull-Schmidt category.

Proof. Let Q be a Henselian Gorenstein local ring. We aim to show that for any indecomposable totally acyclic complex $C \in \mathbf{K}_{\mathrm{tac}}(Q)$, its endomorphism ring, $\mathrm{End}\,\mathbf{K}_{\mathrm{tac}}(Q)(C)$, is local.

We first note that for two complexes C and D, $\mathrm{Hom}_{\mathbf{K}_{\mathrm{tac}}(Q)}(C, D)$ is a Q-module and composition of morphisms is Q-bilinear (cf. §1, [4]). Therefore, $\mathbf{K}_{\mathrm{tac}}(Q)$ is a Q-additive category.

Let $C \in \mathbf{K}_{\mathrm{tac}}(Q)$ be an indecomposable totally acyclic complex. By Proposition 36, $M = \mathrm{Im}(\partial_0^C)$ must also be an indecomposable Q-module. Since M is indecomposable, Proposition 5 implies that $\mathrm{End}_Q(M)$ is a local ring, which by Proposition 37 implies that $\mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(Q)}(C)$ is a local ring as well. Thus $\mathbf{K}_{\mathrm{tac}}(Q)$ is a Krull-Schmidt category, as stated. □

3.2 Approximations of $\mathbf{K}_{\text{tac}}(R)$

We now work towards building the classification scheme previously mentioned, although in order to fully contextualize what is to come, we must first recall a few notions from [2]. We begin by establishing a connection between the categories $\mathbf{K}_{\text{tac}}(Q)$ and $\mathbf{K}_{\text{tac}}(R)$. Recall the assumption that there exists a surjective ring homomorphism $Q \overset{\varphi}{\twoheadrightarrow} R$ such that R has finite injective dimension as a Q-module. Under these conditions Bergh, Jorgensen and Moore define a pair of adjoint functors:

$$\mathbf{K}_{\text{tac}}(Q) \underset{T_\varphi}{\overset{S_\varphi}{\rightleftarrows}} \mathbf{K}_{\text{tac}}(R).$$

As it turns out, the descension functor, defined in §3 of [2], is relatively simple; $S = S_\varphi : \mathbf{K}_{\text{tac}}(Q) \to \mathbf{K}_{\text{tac}}(R)$ is the change of rings functor defined by:

$$SC = R \otimes_Q C \text{ and } Sf = R \otimes_Q f.$$

However, the ascension functor, defined in §2 of [2], is much more interesting; $T = T_\phi : \mathbf{K}_{\text{tac}}(R) \to \mathbf{K}_{\text{tac}}(Q)$ is defined as follows: On objects, say $C \in \mathbf{K}_{\text{tac}}(R)$, TC is the complete resolution of $\text{Im}(\partial_0^C)$ as a Q-module. For a chain map $f \in \text{Hom}_{\mathbf{K}_{\text{tac}}(R)}(C, C')$ we have $Tf = \hat{\mu}$, the homotopy equivalence class of the lifting of the induced map from Lemma 29.

Furthermore, the authors show that both functors are triangulated (See §2, §3) and form an adjoint pair (See §3); specifically, they induce a unit and counit:

$$\eta : \text{Id}_{\mathbf{K}_{\text{tac}}(Q)} \to TS$$

and

$$\epsilon : ST \to \text{Id}_{\mathbf{K}_{\text{tac}}(R)}$$

respectively.

We now provide the definition of an approximation (given by Auslander and Smalø, and independently by Enochs), the origins of which seem far more elusive than the definition itself.

Definition 39. [2, Section 4] Given a full subcategory $\mathcal{X}$ of a category $\mathcal{C}$. We define a *right* $\mathcal{X}$*-approximation* (*pre-cover*) of $C \in \mathcal{C}$ as a map $\phi : X \to C$ such that for all objects $Y \in \mathcal{X}$ and any map $f : Y \to C$, there exists a map $g : Y \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & & \\ {\scriptstyle g}\downarrow & \searrow^{f} & \\ X & \xrightarrow[\phi]{} & C \end{array}$$

We define *left* $\mathcal{X}$*-approximations* (*pre-envelopes*) dually.

Taking this idea one step further we may define a minimal approximation, which provides some notion of uniqueness.

Definition 40. [2, Section 4] We define a *minimal* right $\mathcal{X}$-approximation (*cover*) to be a right $\mathcal{X}$-approximation $\phi : X \to C$ for $C \in \mathcal{C}$ such that for any diagram of the form:

$$\begin{array}{ccc} X & & \\ {\scriptstyle g}\downarrow & \searrow^{\phi} & \\ X & \xrightarrow[\phi]{} & C \end{array}$$

we must have that g is an automorphism of X. A *minimal* left $\mathcal{X}$-approximation (*envelope*) is defined dually.

Indeed, any such minimal approximation is unique up to isomorphism.

Remark. $\mathrm{Im}(S) \subset \mathbf{K}_{\mathrm{tac}}(R)$ is *contravariantly finite* or, in other words, right $\mathrm{Im}(S)$-approximations exist in $\mathbf{K}_{\mathrm{tac}}(R)$, and are given by the counit of the previously mentioned adjoint functors.

The existence of left approximations also holds, and is shown in Theorem 4.1 of [2]. The fact that both right and left approximations exists follows directly from the fact that S and T are adjoint; however, we now provide a more in-depth proof. We claim that the morphism $\epsilon_C : STC \to C$ is the right approximation of C in $\mathbf{K}_{\text{tac}}(R)$. To see this, let $Y \in \mathbf{K}_{\text{tac}}(Q)$ and $f : SY \to C$ be any morphism. We must now find a morphism such that the following diagram commutes:

$$\begin{array}{ccc} SY & & \\ \downarrow & \searrow^{f} & \\ STC & \xrightarrow[\epsilon_C]{} & C \end{array}$$

Now, by the naturality of $\epsilon : ST \to \mathrm{Id}_{\mathbf{K}_{\text{tac}}(R)}$, we have the commutative diagram

$$\begin{array}{ccc} STSY & \xrightarrow{ST(f)} & STC \\ {\scriptstyle \epsilon_{SY}}\downarrow & & \downarrow{\scriptstyle \epsilon_C} \\ SY & \xrightarrow[f]{} & C \end{array}$$

and therefore, we obtain the relation:

$$\epsilon_C \circ ST(f) \sim f \circ \epsilon_{SY}. \tag{3.2}$$

We then note that $\epsilon_{SY} \circ S_{\eta_Y} \sim \mathrm{Id}_{SY}$. By composing the relation in (3.2) on the right with S_{η_Y}, we see that

$$\epsilon_C \circ ST(f) \circ S_{\eta_Y} \sim f \circ \epsilon_{SY} \circ S_{\eta_Y} \sim f$$

and $ST(f) \circ S_{\eta_Y}$ is the morphism we seek. This proves that ϵ_C is the right approximation of the totally acyclic complex C.

If we wish to ensure that the following definitions make sense, we need to show that minimal approximations exist in the category $\mathbf{K}_{\text{tac}}(R)$. In order to show this, we look to Proposition 2.5 from [16], and provide the following extension of the previous remark. In particular, we make use of the fact that (possibly non-minimal) approximations exist, and show that they can be "reduced" in some way to a minimal approximation.

Theorem 41. Let $\mathcal{X}$ be a full subcategory of $\mathbf{K}_{\mathrm{tac}}(R)$ closed under direct summands and suppose that R is Henselian.

i) Let $N \xrightarrow{\psi} X \xrightarrow{\phi} M \to \Sigma N$ be a distinguished triangle in $\mathbf{K}_{\mathrm{tac}}(R)$ where ϕ is a precover of M. Then the following are equivalent:

   (a) ϕ is not an $\mathcal{X}$-cover.

   (b) There exists a sub-complex L of N such that $\psi(L) \neq 0$ and is a direct summand of X.

ii) The following are equivalent for a totally acyclic complex M:

   (a) M has an $\mathcal{X}$-precover.

   (b) M has an $\mathcal{X}$-cover.

Proof. *i*) We begin by showing that (b) implies (a). To this end, assume there exists a sub-complex L of N such that $\psi(L) \neq 0$ and is also a direct summand of X. As $\psi(L)$ is a non-zero summand of X we may take X' to be its compliment. Let $\theta : X' \to X$ be the natural inclusion and $\pi : X \to X'$ the natural projection. Then set $f = \theta\pi$ and note that, as $\psi(L)$ would map to zero under ϕ, we have that $\phi = \phi f$. Now, suppose for contradiction that ϕ is an $\mathcal{X}$-cover. It follows that f is an isomorphism and hence, also θ and π. Therefore, we must have that $\psi(L) = 0$, contradicting our assumption. Thus, ϕ is not an $\mathcal{X}$-cover.

To show that (a) implies (b), assume that ϕ is not an $\mathcal{X}$-cover. As such, there exists a non-isomorphism $f \in \mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(R)}(X)$ such that $\phi = \phi f$. Now, let $S = R[f]$ be the subalgebra of $\mathrm{End}_{\mathbf{K}_{\mathrm{tac}}(R)}(X)$ generated by f over R and note that S is a commutative ring.

Assuming S is a local ring, we will show that the approximation $\phi = 0$. Since S is local, set $\mathfrak{n}$ as the unique maximal ideal. Then, as S is a finitely generated R-module, the factor ring $S/\mathfrak{m}S$ is an artinian local ring with maximal ideal $\mathfrak{n}/\mathfrak{m}S$. Hence, $\mathfrak{n}^r \subseteq \mathfrak{m}S$ for some integer r. Since $f \in \mathfrak{n}$ we have that $f^r = a_0 + a_1 f + a_2 f^2 + ... + a_s f^s$ with $a_i \in \mathfrak{m}$ for all $i = 0, ..., s$. Since $\phi = \phi f$, and in particular, since $\phi = \phi f^l$ for all $0 \leq l \leq \infty$, we observe that

$$\begin{aligned} \phi &= \phi f^r \\ &= \phi(a_0 + a_1 f + a_2 f^2 + ... + a_s f^s) \\ &= a_0 + a_1 \phi f + a_2 \phi f^2 + ... + a_s \phi f^s \\ &= (a_0 + a_1 + a_2 + ... + a_s)\phi \in \mathfrak{m}\phi. \end{aligned}$$

It follows from Nakayama's Lemma that $\phi = 0$, and so the triangle $N \xrightarrow{\psi} X \xrightarrow{0} M \to \cdot$ splits. That is, we get the following commutative diagram:

$$\begin{array}{ccccccc} N & \xrightarrow{\psi} & X & \xrightarrow{0} & M & \longrightarrow & \Sigma N \\ \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \\ \Sigma^{-1}M \oplus X & \longrightarrow & X & \longrightarrow & M & \longrightarrow & M \oplus \Sigma X \end{array}$$

Thus, by the triangulated five lemma $N \cong \Sigma^{-1}M \oplus X$ and $L := X$ satisfies condition (b) in the statement.

We must now consider the case that S is not a local ring. Since R is Henselian, the finite R-algebra S is a product of local rings, and hence there exists a non-trivial idempotent,

$$e = b_0 + b_1 f + ... + b_t f^t \in S$$

where $b_i \in R$ for all $i = 0, ..., t$. By taking $1 - e$ instead of e if $b_0 + b_1 + ... + b_t \in \mathfrak{m}$, we may assume that it is not. In other words, $b_0 + b_1 + ... + b_t$ is a unit of R.

Given the decomposition

$$X = \mathrm{Im}(e) \oplus \mathrm{Im}(1 - e)$$

and since e is not an isomorphism, we can see that $\mathrm{Im}(e) \neq 0 \neq \mathrm{Im}(1-e)$. We now claim that $\mathrm{Im}(1-e) \subseteq \mathrm{Im}(\psi)$. To see this, note that the triangle $N \xrightarrow{\psi} X \xrightarrow{\phi} M \to \Sigma N$ induces a long exact sequence of abelian groups:

$$\cdots \to \mathrm{Hom}(X, N) \xrightarrow{\psi_*} \mathrm{Hom}(X, X) \xrightarrow{\phi_*} \mathrm{Hom}(X, M) \to \cdots$$

Since $\phi e = \phi(b_0 + b_1 f + ... + b_t f^t) = (b_0 + b_1 + ... + b_t)\phi$ and $b_0 + b_1 + ... + b_t$ is a unit, we have that $\phi = (b_0 + b_1 + ... + b_t)^{-1}\phi e$. Now take $1 - e \in \mathrm{Hom}(X, X)$, and note that

$$\begin{aligned} \phi_*(1-e) &= \phi(1-e) \\ &= (b_0 + b_1 + ... + b_t)^{-1}\phi e(1-e) \\ &= 0 \end{aligned}$$

showing that $1 - e \in \mathrm{Ker}\,\phi_* = \mathrm{Im}(\psi_*)$. Thus there exists $\alpha \in \mathrm{Hom}(X, N)$ such that $\psi_*(\alpha) = 1 - e$. Then for $x \in \mathrm{Im}(1-e)$ there exists $y \in X$ such that $x = (1-e)(y)$. Moreover, observe that,

$$\begin{aligned} x &= (1-e)(y) \\ &= \psi_*(\alpha)(y) \\ &= \psi(\alpha(y)). \end{aligned}$$

Therefore, we have shown that $\mathrm{Im}(1-e) \subseteq \mathrm{Im}(\psi)$. Thus, $L := \psi^{-1}(\mathrm{Im}(1-e))$ satisfies condition (b).

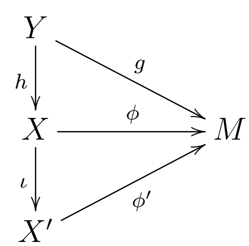
ii) It is trivial that (b) implies (a). Assume now that $\phi : X \to M$ is an $\mathcal{X}$-precover. We will show that there exists an $\mathcal{X}$-cover. Letting $N = \Sigma^{-1}\,\mathrm{cone}(\phi)$, we have the following distinguished triangle:

$$\Sigma^{-1}\,\mathrm{cone}(\phi) \xrightarrow{\psi} X \xrightarrow{\phi} M \to \mathrm{cone}(\phi')$$

Suppose that ϕ is not an $\mathcal{X}$-cover. Then from $i)$ there exists a sub-complex L of $\Sigma^{-1}\operatorname{cone}(\phi)$ such that $\psi(L) \neq 0$ and is a direct summand of X. Let X' be the complement of $\psi(L)$ and $\phi' : X' \to M$ the induced map, noting that we also get the following distinguished triangle:

$$\Sigma^{-1}\operatorname{cone}(\phi') \xrightarrow{\psi'} X' \xrightarrow{\phi'} M \to \operatorname{cone}(\phi')$$

Then as $\mathcal{X}$ is a thick subcategory, $X' \in \mathcal{X}$. To show that X' is a $\mathcal{X}$-precover, let $Y \in \mathcal{X}$ and $g \in \operatorname{Hom}_{\mathbf{K}_{\text{tac}}(R)}(Y, M)$. Since we know that X is an $\mathcal{X}$-precover we obtain the following diagram:



where $g = \phi'\iota h$. Thus, X' is an $\mathcal{X}$-precover. Since X' is strictly smaller than X, we may repeat this process and eventually arrive at an $\mathcal{X}$-cover. Indeed, since R is Henselian, X is a finite sum of irreducible complexes, so this process must eventually terminate. This proves the statement. □

In other words, for a Henselian Gorenstein ring if the complex C belongs to $\operatorname{thick}^1_{\mathbf{K}_{\text{tac}}(R)}\operatorname{Im}(S)$, then the approximation is the identity map; otherwise we may take any approximation and cut out the redundant summands. We leave the reader with the following open question.

Question: For a totally acyclic complex C, is the approximation STC minimal whenever $C \notin \operatorname{thick}^1_{\mathbf{K}_{\text{tac}}(R)}\operatorname{Im}(S)$?

3.3 Arnold-Tuples

Unless otherwise stated, we assume that Q is a Henselian Gorenstein local ring and that there exists a surjective ring homomorphism $\varphi : Q \to R$. We may now state the definition of an Arnold-tuple.

Definition 42 (Arnold-Tuple)**.** Let Λ be the set of distinct, indecomposable totally acyclic complexes in $\text{thick}^1_{\mathbf{K}_{\text{tac}}(R)} \operatorname{Im}(S)$. Then for any $C \in \mathbf{K}_{\text{tac}}(R)$ we may write

$$\overline{STC} = (\bigoplus_{E \in \Lambda} E^{k_E}) \oplus T$$

where $\overline{STC}$ is the minimal approximation, T is contractible and all but finitely many k_E are zero. Then the *Arnold-tuple* of C over Q is defined to be:

$$A_Q(C) = (k_E)_{E \in \Lambda}.$$

One should note the condition that "all but finitely many k_E are zero" in the previous definition is ensured, as $\mathbf{K}_{\text{tac}}(R)$ is a Krull-Schmidt category. However, we should check that if C and D are homotopic, then $A_Q(C) = A_Q(D)$. The following proposition confirms this.

Proposition 43. If C and D are totally acyclic complexes over R and $C \simeq D$, then $A_Q(C) = A_Q(D)$.

Proof. Let C and D be totally acyclic complexes in $\mathbf{K}_{\text{tac}}(R)$ such that $C \simeq D$. Then for $\overline{STC}, \overline{STD} \in \mathbf{K}_{\text{tac}}(R)$ each has a decomposition into finite direct sums of indecomposable complexes. Suppose that $\overline{STC} = (\bigoplus_{E \in \Lambda} E^{k_E}) \oplus T$ and $\overline{STD} = (\bigoplus_{E \in \Lambda} E^{l_E}) \oplus T'$. Then by Lemmas 11 and 20,

$$\bigoplus_{E \in \Lambda} E^{k_E} \cong \bigoplus_{E \in \Lambda} E^{l_E}.$$

Now because $\mathbf{K}_{\text{tac}}(R)$ is a Krull-Schmidt category, we have unique decomposition, and as such, for each $E \in \Lambda$ that appears in the previous sums it holds that $E^{k_E} \cong E^{l_E}$, thus implying that $k = l$. Consequently, if $C \simeq D$ then $A_Q(C) = A_Q(D)$, and so Arnold-tuples are well defined up to homotopy. $\square$

We may now state our classification scheme, which groups totally acyclic complexes by their Arnold-Tuples. In other words, two totally acyclic complexes are in the same class if they have the same Arnold-tuple, in which case we call two such complexes Arnold equivalent.

Definition 44 (Arnold Equivalence)**.** Let $C, D \in \mathbf{K}_{\text{tac}}\, R$, we say that C is *Arnold equivalent* to D if $A_Q(C) = A_Q(D)$. We denote this by

$$C \simeq_Q D.$$

As Arnold equivalence is an equality of tuples, it should be clear that it forms an equivalence relation. We also note that Arnold equivalence is a less restrictive notion than that of homotopy. To illustrate this we provide an example exhibiting two complexes which are Arnold equivalent but not homotopically equivalent.

Example 45. Let $Q = k[[x, y]]/(x^2)$ and $R = Q/(y^2)$, with:

$$M = \operatorname{Coker} \left(\begin{smallmatrix} -x & 0 \\ y & x \end{smallmatrix}\right) \text{ and,}$$

$$N = \operatorname{Coker} \left(\begin{smallmatrix} x & y & 0 & 0 \\ 0 & 0 & xy & y^2 \end{smallmatrix}\right).$$

Then the complete resolution of $M \oplus M$ is:

$$C = \cdots \to R^4 \xrightarrow{\left(\begin{smallmatrix} -x & 0 & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & y & x \end{smallmatrix}\right)} R^4 \xrightarrow{\left(\begin{smallmatrix} -x & 0 & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & y & x \end{smallmatrix}\right)} R^4 \xrightarrow{\left(\begin{smallmatrix} -x & 0 & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & y & x \end{smallmatrix}\right)} R^4 \to \cdots$$

and the complete resolution of N is:

$$D = \cdots \to R^7 \xrightarrow{\partial_3} R^5 \xrightarrow{\partial_2} R^3 \xrightarrow{\partial_1} R^2 \xrightarrow{\partial_0} R^3 \xrightarrow{\partial_{-1}} R^5 \xrightarrow{\partial_{-2}} R^7 \to \cdots$$

Hence, we note that C has complexity 1 and D has complexity 2; therefore, C and D are clearly not homotopically equivalent. However, when we look at the minimal approximations over the ring Q, we have the following complex for both C and D:

$$\cdots \to R^4 \to R^4 \xrightarrow{\begin{pmatrix} -x & 0 & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & y & x \end{pmatrix}} R^4 \to R^4 \to \cdots$$

Thus, while $C \not\simeq D$, we have that $A_Q(C) = A_Q(D)$ and so $C \simeq_Q D$. Therefore, we have that Arnold equivalence is a more coarse notion than that of homotopy.

3.4 Approximations and Mapping Cones

We now turn our attention to the approximation and classification of mapping cones. For a simple example, consider the zero morphism, so that the mapping cone is given by a direct sum of shifts of the source and target complexes. Thus, the Arnold-tuple can easily be deduced. This is illustrated by the following example:

Example 46. Let $Q = k[[x, y]]/(x^2 + y^2)$, type (A_1), and $R = Q/(x^2)$. Then take the zero map from $E_+ \to E_-$:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & R & \xrightarrow{x+iy} & R & \xrightarrow{x-iy} & R & \longrightarrow & \cdots \\ & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \\ \cdots & \longrightarrow & R & \xrightarrow[x-iy]{} & R & \xrightarrow[x+iy]{} & R & \longrightarrow & \cdots \end{array}$$

The mapping cone is then given by $\text{cone}(f) = \Sigma E^+ \oplus E^-$ with differential

$$\partial^{\text{cone}(f)} = \begin{pmatrix} \partial^{\Sigma E_+} & 0 \\ 0 & \partial^{E_-} \end{pmatrix}.$$

However, as we have that $\Sigma E_+ = E_-$, the approximation of the mapping cone is in fact the mapping cone itself, $E_- \oplus E_-$, and therefore has an Arnold-tuple of (0,0,2).

In [2] the authors go further to describe the approximation of the mapping cone in the case that R has relative codimension one.

Proposition 47. [2, Corollary 6.4] Let f be a non-zero-divisor contained in the maximal ideal of Q, $R = Q/(f)$, and $C \in \mathbf{K}_{\mathrm{tac}}(R)$. If $[\epsilon_C] : STC \to C$ is the right approximation of C, then cone($[\epsilon_C]$) is isomorphic to $\Sigma^2 C$ in $\mathbf{K}_{\mathrm{tac}}(R)$, and we have the distinguished triangle

$$STC \xrightarrow{[\epsilon_C]} C \xrightarrow{[t]} \Sigma^2 C \to \Sigma STC.$$

We present a generalization of this theorem to codimension c as well as provide a concrete description of approximations. Let Q be a Gorenstein local ring and $R = Q/(\bar{f})$ where $\bar{f} = f_1, ..., f_c$ is a Q-regular sequence of length c. Then for any $C \in \mathbf{K}_{\mathrm{tac}}(R)$, its approximation, STC, has the following form:

$$STC : \cdots \to \bigoplus_{i=0}^{c} (C_{n+1-i})^{\binom{c}{i}} \xrightarrow{\partial_{n+1}} \bigoplus_{i=0}^{c} (C_{n-i})^{\binom{c}{i}} \xrightarrow{\partial_n} \bigoplus_{i=0}^{c} (C_{n-1-i})^{\binom{c}{i}} \to \cdots \qquad (3.3)$$

In an effort to understand the construction of the above form, we now state Theorem 5.1 from [3]

Theorem 48. [3, Theorem 5.1] Assume that $R = Q/(f_1, ..., f_c)$, where $f_1, ..., f_c$ is a Q-regular sequence, and let $C \in \mathbf{K}_{\mathrm{tac}}(R)$. Letting $\overline{F} = C$, the complex $F \otimes_Q K$ defined above is TC. That is, $F \otimes_Q K$ is a complete resolution of $\operatorname{Im} \partial_0^C$. It follows that STC is $(F \otimes_Q K) \otimes_Q R$, and the morphism $\epsilon_C : STC \to C$ is the map that projects the copy $\overline{F} \otimes_Q K_0$ of $\overline{F}$ in $(F \otimes_Q K) \otimes_Q R$ onto $\overline{F} = C$.

In this theorem K denotes the Koszul complex on $f_1, ..., f_c$, which has the form:

$$0 \to \bigwedge^{c}(Q^c) \to \cdots \to \bigwedge^{2}(Q^c) \to Q^c \to Q \to R \to 0.$$

We note that $\bigwedge^i (Q^c) \cong Q^{\binom{c}{i}}$ as Q^c is a free module of rank c. Thus, K is equivalent to the complex:

$$0 \to Q^{\binom{c}{c}} \to Q^{\binom{c}{c-1}} \to \cdots \to Q^{\binom{c}{1}} \to Q^{\binom{c}{0}} \to 0.$$

To see that the approximation has the advertised form, let $C \in \mathbf{K}_{\text{tac}}(R)$, then lift C to a graded module over Q, say $\widetilde{C}$. Then the complex described in the theorem above, $\widetilde{C} \otimes_Q K$, would be TC. Here this complex has the form:

$$(\widetilde{C} \otimes_Q K)_n = \begin{array}{ccccc} (\widetilde{C}_n \otimes_Q K_0) & = & (\widetilde{C}_n \otimes_Q Q^{\binom{c}{0}}) & = & \widetilde{C}_n^{\binom{c}{0}} \\ \bigoplus & & \bigoplus & & \bigoplus \\ (\widetilde{C}_{n-1} \otimes_Q K_1) & = & (\widetilde{C}_{n-1} \otimes_Q Q^{\binom{c}{1}}) & = & \widetilde{C}_{n-1}^{\binom{c}{1}} \\ \bigoplus & & \bigoplus & & \bigoplus \\ \vdots & & \vdots & & \vdots \\ \bigoplus & & \bigoplus & & \bigoplus \\ (\widetilde{C}_{n-c+1} \otimes_Q K_{c-1}) & = & (\widetilde{C}_{n-c+1} \otimes_Q Q^{\binom{c}{c-1}}) & = & \widetilde{C}_{n-c+1}^{\binom{c}{c-1}} \\ \bigoplus & & \bigoplus & & \bigoplus \\ (\widetilde{C}_{n-c} \otimes_Q K_c) & = & (\widetilde{C}_{n-c} \otimes_Q Q^{\binom{c}{c}}) & = & \widetilde{C}_{n-c}^{\binom{c}{c}} \end{array}$$

Alternatively presented in a more condensed format as,

$$(\widetilde{C} \otimes_Q K)_n = \bigoplus_{i=0}^{c} (\widetilde{C}_{n-i})^{\binom{c}{i}}.$$

Then, again by Theorem 5.1 in [3], $STC = (\widetilde{C} \otimes_Q K) \otimes_Q R$ where:

$$((\widetilde{C} \otimes_Q K) \otimes_Q R)_n = \bigoplus_{i=0}^{c} (C_{n-i})^{\binom{c}{i}}.$$

We now discuss the differentials on this approximation, although we refer the interested reader to [3] for a more complete accounting. To begin, let $\mathcal{B}$ be a basis of the Koszul complex together with 0:

$$\mathcal{B} = \left\{ e_{i_1} \wedge \cdots \wedge e_{i_j} | i_1 < \cdots < i_j, 1 \leq j \leq c \right\} \cup \{0, 1\}.$$

In order to obtain the desired differential we perturb the differential on $\widetilde{C} \otimes_Q K$, which is $\partial^{\widetilde{C}} \otimes K + \widetilde{C} \otimes \partial^K$, to

$$\partial = \sum_{\alpha \in \mathcal{B}} t^\alpha \otimes s_\alpha$$

where $t^0 = \mathrm{Id}_{\widetilde{C}}, t^1 = \partial^{\widetilde{C}}, s_0 = \partial^K, t^\alpha$ are as defined by Lemma 2.2 in [3] for $\alpha \neq 0$, and s_α is multiplication by α for $\alpha \neq 0$.

Remark. The t^α can be thought of as a set of Eisenbud operators when $|\alpha| = 1$ and "higher order" Eisenbud operators when $|\alpha| \geq 2$.

In general, providing a concrete description of this differential presents some technical challenges. However, it will be informative to explore this concept in a specific case, for instance, when the relative codimension is 3.

Example 49. Let Q and R be as before and let $c = 3$. In other words, $R = Q/(f_1, f_2, f_3)$ where f_1, f_2, f_3 is a Q-regular sequence. Let $C \in \mathbf{K}_{\mathrm{tac}}(R)$ so that

$$C = \cdots \to C_{n+1} \xrightarrow{\partial^C_{n+1}} C_n \xrightarrow{\partial^C_n} C_{n-1} \to \cdots$$

and let $\widetilde{C}$ be the lifting to a graded module over Q. Now, from the previous construction we obtain the fact that TC has the form:

$$\cdots \to \partial^{\widetilde{C}}_{n+1} \oplus (\partial^{\widetilde{C}}_n)^3 \oplus (\partial^{\widetilde{C}}_{n-1})^3 \oplus \partial^{\widetilde{C}}_{n-2} \xrightarrow{\widetilde{\partial}_{n+1}} \partial^{\widetilde{C}}_n \oplus (\partial^{\widetilde{C}}_{n-1})^3 \oplus (\partial^{\widetilde{C}}_{n-2})^3 \oplus \partial^{\widetilde{C}}_{n-3} \to \cdots$$

and for each n the differential $\widetilde{\partial}$ is given by the matrix:

$$\widetilde{\partial}_n = \begin{bmatrix} \partial^{\widetilde{C}}_{n-3} & (-1)^n t^{e_1} & (-1)^{n-1} t^{e_2} & (-1)^n t^{e_3} & t^{e_1 \wedge e_2} & -t^{e_1 \wedge e_3} & t^{e_2 \wedge e_3} & (-1)^{3n} t^{e_1 \wedge e_2 \wedge e_3} \\ (-1)^n f_1 & \partial^{\widetilde{C}}_{n-2} & 0 & 0 & (-1)^n t^{e_2} & (-1)^{n-1} t^{e_3} & 0 & t^{e_2 \wedge e_3} \\ (-1)^{n-1} f_2 & 0 & \partial^{\widetilde{C}}_{n-2} & 0 & (-1)^n t^{e_1} & 0 & (-1)^{n-1} t^{e_3} & t^{e_1 \wedge e_3} \\ (-1)^n f_3 & 0 & 0 & \partial^{\widetilde{C}}_{n-2} & 0 & (-1)^n t^{e_1} & (-1)^{n-1} t^{e_2} & t^{e_1 \wedge e_2} \\ 0 & (-1)^{n-1} f_2 & (-1)^{n-1} f_1 & 0 & \partial^{\widetilde{C}}_{n-1} & 0 & 0 & (-1)^n t^{e_3} \\ 0 & (-1)^{n-1} f_3 & 0 & (-1)^n f_1 & 0 & \partial^{\widetilde{C}}_{n-1} & 0 & (-1)^n t^{e_2} \\ 0 & 0 & (-1)^{n-1} f_3 & (-1)^{n-1} f_2 & 0 & 0 & \partial^{\widetilde{C}}_{n-1} & (-1)^n t^{e_1} \\ 0 & 0 & 0 & 0 & (-1)^n f_3 & (-1)^n f_2 & (-1)^n f_1 & \partial^{\widetilde{C}}_n \end{bmatrix}$$

Then by applying the functor $S = - \otimes_Q R$, we see that STC is given by:

$$\cdots \to \partial^C_{n+1} \oplus (\partial^C_n)^3 \oplus (\partial^C_{n-1})^3 \oplus \partial^C_{n-2} \xrightarrow{\partial_{n+1}} \partial^C_n \oplus (\partial^C_{n-1})^3 \oplus (\partial^C_{n-2})^3 \oplus \partial^C_{n-3} \to \cdots$$

wherein each n^{th} differential is given by:

$$\partial_n = \begin{bmatrix} \partial^C_{n-3} & (-1)^n t^{e_1} & (-1)^{n-1} t^{e_2} & (-1)^n t^{e_3} & t^{e_1\wedge e_2} & -t^{e_1\wedge e_3} & t^{e_2\wedge e_3} & (-1)^{3n} t^{e_1\wedge e_2\wedge e_3} \\ 0 & \partial^C_{n-2} & 0 & 0 & (-1)^n t^{e_2} & (-1)^{n-1} t^{e_3} & 0 & t^{e_2\wedge e_3} \\ 0 & 0 & \partial^C_{n-2} & 0 & (-1)^n t^{e_1} & 0 & (-1)^{n-1} t^{e_3} & t^{e_1\wedge e_3} \\ 0 & 0 & 0 & \partial^C_{n-2} & 0 & (-1)^n t^{e_1} & (-1)^{n-1} t^{e_2} & t^{e_1\wedge e_2} \\ 0 & 0 & 0 & 0 & \partial^C_{n-1} & 0 & 0 & (-1)^n t^{e_3} \\ 0 & 0 & 0 & 0 & 0 & \partial^C_{n-1} & 0 & (-1)^n t^{e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial^C_{n-1} & (-1)^n t^{e_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial^C_n \end{bmatrix}$$

Furthermore, if we assume that the lifting of C to $\widetilde{C}$ is actually a complex over Q, we have that each t^α is zero. In this case, the complex stays the same, but the differentials are now given by:

$$\partial_n = \begin{bmatrix} \partial^C_{n-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial^C_{n-2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial^C_{n-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial^C_{n-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial^C_{n-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial^C_{n-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial^C_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial^C_n \end{bmatrix}$$

and it is clear that the approximation complex is then

$$STC = \Sigma^3 C \oplus (\Sigma^2 C)^3 \oplus (\Sigma C)^3 \oplus C.$$

Now, from Theorem 5.1, we note that the approximation of C is

$$\epsilon_C : \Sigma^3 C \oplus (\Sigma^2 C)^3 \oplus (\Sigma C)^3 \oplus C \twoheadrightarrow C$$

which projects the copy of C in STC onto C. Moreover, the minimal subcomplex of the mapping cone of the approximation can be given by

$$\text{cone}(\epsilon_C) = \Sigma^4 C \oplus (\Sigma^3 C)^3 \oplus (\Sigma^2 C)^3.$$

In light of the previous example we make the following proposition.

Proposition 50. Assume Q is a Gorenstein local ring and $R = Q/(\bar{f})$ where $\bar{f} = f_1, ..., f_c$ is a Q-regular sequence of length c. Further, let $C \in \mathbf{K}_{\text{tac}}(R)$ be a minimal complex such that its lifting to a graded Q-module is a Q-complex. Then the approximation of C is given by:

$$\epsilon_C : \bigoplus_{i=0}^{c} (\Sigma^i C)^{\binom{c}{i}} \twoheadrightarrow C$$

which projects the copy of C in STC to C. Furthermore, the minimal subcomplex of the mapping cone, of the approximation of C is given by

$$\operatorname{cone}(\epsilon_C) \simeq \bigoplus_{i=1}^{c} (\Sigma^{i+1} C)^{\binom{c}{i}}. \tag{3.4}$$

Proof. Let $C \in \mathbf{K}_{\mathrm{tac}}(R)$ such that its lifting as a graded Q-module is a Q-complex. We begin with reiterating that under these assumptions, each t^α previously described is zero. As such, and following from Theorem 5.1 in [3], it is easily seen that the approximation of C is given by $\epsilon_C : \bigoplus_{i=0}^{c} (\Sigma^i C)^{\binom{c}{i}} \twoheadrightarrow C$. Specifically, the approximation is shown in the following diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \bigoplus_{i=0}^{c} (\Sigma^i C_{n+1})^{\binom{c}{i}} & \xrightarrow{\partial_{n+1}} & \bigoplus_{i=0}^{c} (\Sigma^i C_n)^{\binom{c}{i}} & \xrightarrow{\partial_n} & \bigoplus_{i=0}^{c} (\Sigma^i C_{n-1})^{\binom{c}{i}} & \longrightarrow \cdots \\
& \downarrow {\scriptstyle [1\ 0\ \cdots\ 0]} & & \downarrow {\scriptstyle [1\ 0\ \cdots\ 0]} & & \downarrow {\scriptstyle [1\ 0\ \cdots\ 0]} & \\
\cdots \longrightarrow & C_{n+1} & \xrightarrow{\partial^C_{n+1}} & C_n & \xrightarrow{\partial^C_n} & C_{n-1} & \longrightarrow \cdots
\end{array}$$

To find the mapping cone of the approximation, we simply apply the definition to obtain

$$\operatorname{cone}(\epsilon_C) = \bigoplus_{i=0}^{c} (\Sigma^{i+1} C)^{\binom{c}{i}} \oplus C$$

wherein the differential at each degree n is given by:

$$\left[\begin{array}{ccccc|c}
(-1)^{c+1}\,\partial^C_{n-(c+1)} & 0 & \cdots & 0 & 0 & \\
0 & E_c & \cdots & 0 & 0 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \mathbf{0} \\
0 & 0 & \cdots & E_1 & 0 & \\
0 & 0 & \cdots & 0 & -\,\partial^C_{n-1} & \\
\hline
 & 0\ 0 & \cdots & 0\ 1 & & \partial^C_n
\end{array}\right]$$

Here, E_j is a $\binom{c}{j} \times \binom{c}{j}$ matrix with $(-1)^j \, \partial^C_{n-1-j}$ on the diagonal for all $1 \leq j \leq c$. We now wish to show that the minimal subcomplex is as previously stated; to do this, we will exhibit a homotopy equivalence between the mapping cone and (3.4). Consider the following diagram:

$$\begin{array}{ccccccc} \cdots \longrightarrow & \bigoplus_{i=0}^{c} (\Sigma^{i+1} C_n)^{\binom{c}{i}} \oplus C_{n+1} & \xrightarrow{\partial^{\mathrm{cone}(\epsilon_C)}_{n+1}} & \bigoplus_{i=0}^{c} (\Sigma^{i+1} C_{n-1})^{\binom{c}{i}} \oplus C_n & \longrightarrow \cdots \\ & \iota_{n+1} \uparrow \downarrow \pi_{n+1} & & \iota_n \uparrow \downarrow \pi_n & \\ \cdots \longrightarrow & \bigoplus_{i=1}^{c} (\Sigma^{i+1} C_n)^{\binom{c}{i}} & \xrightarrow{\partial^C_{n-i}} & \bigoplus_{i=1}^{c} (\Sigma^{i+1} C_{n-1})^{\binom{c}{i}} & \longrightarrow \cdots \end{array}$$

where π and ι are the obvious projection and injection, respectively. It should be clear that $\pi_n \iota_n$ is the identity on the subcomplex for all n. However, to show that $\iota_n \pi_n$ is homotopic to the identity on the mapping cone, we must look to homotopy maps. Using the following $(2^c + 1) \times (2^c + 1)$ square matrices as homotopy maps:

$$\sigma_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

for all n, we can see that

$$\mathrm{cone}(\epsilon_C) \simeq \bigoplus_{i=1}^{c} (\Sigma^{i+1} C)^{\binom{c}{i}}$$

as needed, thus proving the statement. $\square$

CHAPTER 4

Triangle Resolutions

In this chapter we build on the notion of a resolution of totally acyclic complexes outlined in Section 4.9 of [2]. We first describe the construction of such triangle resolutions, after which we give the definition of a minimal resolution. In order to show the existence of such minimal resolutions we call on the main theorem of the previous chapter, Theorem 41, wherein we proved the existence of minimal approximations in $\mathbf{K}_{\mathrm{tac}}(R)$. Then, given such minimal resolutions, we exhibit the extension of a few properties from the category of R-modules. Finally, we describe the notion a triangulated Betti sequence and posit a conjecture.

4.1 Building a Triangle Resolution

In many respects, the construction of a triangle resolution is much like that of a classical free resolution; the main exception being the lack of kernels with which to do so. To get around this, the mapping cone of a morphism plays the analogous role in a triangulated category. In the following construction we keep this in mind and look to Definition 8.1.2 in [11]. We also discuss some basic properties of the resulting triangle resolutions, many of which are completely analogous to those in the classical case.

In Construction 4.9 of [2], the authors provide a construction of triangle resolutions in $\mathbf{K}_{\mathrm{tac}}(R)$ with approximating class $\mathrm{Im}(S)$. However, the following construction is given in an arbitrary triangulated category $\mathcal{T}$ with a full subcategory $\mathcal{X}$.

For an object $C \in \mathcal{T}$ we can complete its $\mathcal{X}$-approximation, $\phi_C : B_0 \to C$, to a triangle in $\mathcal{T}$, and rotate it to obtain:

$$\Sigma^{-1}\operatorname{cone}(\phi_C) \longrightarrow B_0 \xrightarrow{\phi_C} C \longrightarrow \operatorname{cone}(\phi_C)$$

Having done so, we may now use the same method to approximate the complex $\Sigma^{-1}\operatorname{cone}(\phi_C)$. We can again complete and rotate the $\mathcal{X}$-approximation $\phi_{\Sigma^{-1}\operatorname{cone}(\phi_C)} : B_1 \to \Sigma^{-1}\operatorname{cone}(\phi_C)$ to a triangle thusly:

$$\Sigma^{-1}\operatorname{cone}(\phi_{\Sigma^{-1}\operatorname{cone}(\phi_C)}) \to B_1 \to \Sigma^{-1}\operatorname{cone}(\phi_C) \to \operatorname{cone}(\phi_{\Sigma^{-1}\operatorname{cone}(\phi_C)})$$

Then, similar to the classical case in the construction of a free resolution, we compose the maps as seen below:

$$\begin{array}{ccccc} & B_1 & \dashrightarrow & B_0 \to C & \\ \nearrow & & \searrow \quad \nearrow & & \\ \Sigma^{-1}\operatorname{cone}(\phi_{\Sigma^{-1}\operatorname{cone}(\phi_C)}) & & \Sigma^{-1}\operatorname{cone}(\phi_C) & & \end{array}$$

Repeating this process we arrive at a resolution of an object in $\mathbf{K}_{\text{tac}}(R)$:

$$\mathbf{B} : \cdots \longrightarrow B_3 \longrightarrow B_2 \longrightarrow B_1 \longrightarrow B_0 \longrightarrow C$$

where each $B_i \in \mathcal{X}$ and B_0 is an approximation of C, B_1 is an approximation of $\Sigma^{-1}\operatorname{cone}(B_0 \to C)$, etc.

In general, a triangle resolution is not unique. To see this we first note that if $D \in \mathbf{K}_{\text{tac}}(Q)$ then $STSD$ is not a minimal approximation by Proposition 4.6 in [2]. However, the morphism Id_{SD} is a minimal $\operatorname{Im}(S)$-approximation. We illustrate the non-uniqueness of triangle resolutions with an example.

Example 51. Let $Q = \frac{k[[x,y]]}{(x^2)}$ and $R = Q/(y^2)$, then

$$D : \cdots \to Q^2 \xrightarrow{\left[\begin{smallmatrix} -x & 0 \\ y & x \end{smallmatrix}\right]} Q^2 \xrightarrow{\left[\begin{smallmatrix} -x & 0 \\ y & x \end{smallmatrix}\right]} Q^2 \to \cdots$$

is a totally acyclic complex in $\mathbf{K}_{\text{tac}}(Q)$ and note $SD \in \text{Im}(S)$. Its approximation, $STSD$ is given by:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & R^4 & \xrightarrow{\begin{bmatrix} -x & 0 & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & y & x \end{bmatrix}} & R^4 & \xrightarrow{\begin{bmatrix} -x & 0 & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & y & x \end{bmatrix}} & R^4 & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} & & \downarrow{\scriptstyle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} & & \downarrow{\scriptstyle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}} & \\
\cdots \longrightarrow & R^2 & \xrightarrow[\begin{bmatrix} -x & 0 \\ y & x \end{bmatrix}]{} & R^2 & \xrightarrow[\begin{bmatrix} -x & 0 \\ y & x \end{bmatrix}]{} & R^2 & \longrightarrow \cdots
\end{array}$$

Then the mapping cone of the approximation is given by the complex:

$$\cdots \to R^6 \xrightarrow{\begin{bmatrix} x & 0 & 0 & 0 & 0 & 0 \\ -y & -x & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 & 0 \\ 0 & 0 & y & x & 0 & 0 \\ 1 & 0 & 0 & 0 & -x & 0 \\ 0 & 1 & 0 & 0 & y & x \end{bmatrix}} R^6 \xrightarrow{\begin{bmatrix} x & 0 & 0 & 0 & 0 & 0 \\ -y & -x & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 & 0 \\ 0 & 0 & y & x & 0 & 0 \\ 1 & 0 & 0 & 0 & -x & 0 \\ 0 & 1 & 0 & 0 & y & x \end{bmatrix}} R^6 \to \cdots$$

Of course, this complex is homotopically equivalent to SD, via the morphism: $\left[\begin{smallmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{smallmatrix}\right]$ and, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, we have that the approximation of $\text{cone}(\left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right]) \sim STSD$ and so, the triangle resolution can be given by:

$$\cdots \to STSD \to STSD \to STSD \to SD.$$

However, since the minimal approximation of SD is given by $\epsilon : SD \xrightarrow{\text{Id}_{SD}} SD$, we see that $\text{cone}(\epsilon) \sim 0$. Thus, the triangle resolution can also be given by:

$$0 \to STSD \to SD.$$

This leads us to the necessity for defining the notion of a *minimal* triangle resolution, which, in similar fashion to the classical case, encodes a notion of uniqueness.

4.2 Minimal Triangle Resolutions and Properties

Definition 52. Let $\mathcal{X}$ be a full subcategory of a triangulated category $\mathcal{C}$, which is closed under direct summands, such that for each $C \in \mathcal{C}$ there exists a minimal right

$\mathcal{X}$-approximation. We then have that the $\mathcal{X}$-triangle resolution can be constructed such that each $\mathcal{X}$-approximation is a minimal right approximation. In this case we say that the sequence of morphisms:

$$\mathbf{B} : \cdots \to B_2 \to B_1 \to B_0 \to C$$

is a *minimal $\mathcal{X}$-triangle resolution*.

For the previous definition to be of any consequence in this thesis, we should first establish that $\mathbf{K}_{\mathrm{tac}}(R)$ does, in fact, have minimal triangle resolutions. This fact wholly relies on the existence of minimal approximations. The following corollary follows directly from the definition of the first thickening and Theorem 41.

Corollary 53. Minimal $(\mathrm{thick}^1_{\mathbf{K}_{\mathrm{tac}}(R)} \operatorname{Im}(S))$-triangle resolutions exist in $\mathbf{K}_{\mathrm{tac}}(R)$.

We now look towards stating a few basic properties, but shall first establish some notation to simplify proofs and allow for ease of understanding.

Given an $\mathcal{X}$-triangle resolution of C, say

$$\mathbf{B} : \cdots \to B_2 \xrightarrow{\Delta_2} B_1 \xrightarrow{\Delta_1} B_0 \xrightarrow{\Delta_0} C \to 0$$

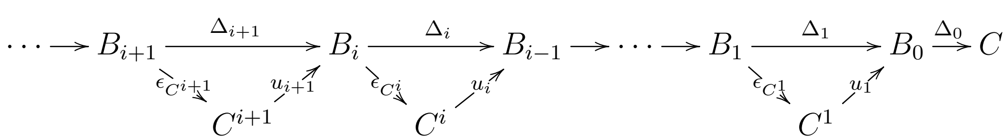
set

$$C^0 = C$$

$$C^1 = \Sigma^{-1} \operatorname{cone}([\epsilon_C])$$

$$C^n = \Sigma^{-1} \operatorname{cone}([\epsilon_{C^{n-1}}]) \text{ for } n \geq 2.$$

Note that each C^n fits into the following commutative diagram:



where $\Delta_i = u_i \epsilon_{C^i}$ and ϵ_{C^i} is the approximation of C^i. We also note that by construction

$$C^{i+1} \xrightarrow{u_{i+1}} B_i \xrightarrow{\epsilon_{C^i}} C_i \to \cdots$$

We begin by discussing the uniqueness property of minimal triangle resolutions.

Proposition 54 (Uniqueness)**.** A minimal triangle resolution, if it exists, is unique in the sense that there exists a family of homotopy equivalences γ_i such that the following diagram commutes.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & B_2 & \xrightarrow{\Delta_2} & B_1 & \xrightarrow{\Delta_1} & B_0 & \xrightarrow{\Delta_0} & C \\
 & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_0} & & \Vert \\
\cdots \longrightarrow & B_2' & \xrightarrow{\Delta_2'} & B_1' & \xrightarrow{\Delta_1'} & B_0' & \xrightarrow{\Delta_0'} & C
\end{array}$$

Proof. Let $\mathbf{B}'$ be another minimal triangle resolution of C, say

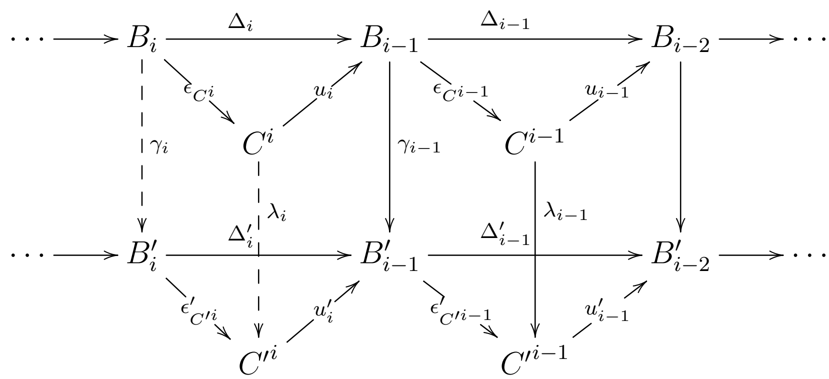
$$\mathbf{B}' : \cdots \to B_2' \to B_1' \to B_0' \to C \to 0.$$

We will use induction on the degree of B_i. When $i = 0$, there exist homotopy equivalences $C^0 \to C^0$ (in this case the identity) and $B_0 \simeq B_0'$. The latter statement follows from the fact that both are minimal approximations of C in conjunction with the diagram below:

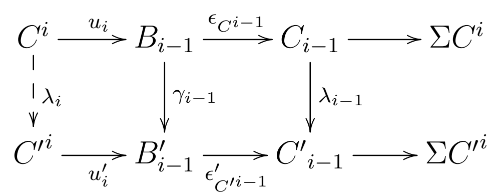
$$\begin{array}{ccc}
B_0 & & \\
\downarrow{\scriptstyle \gamma_0} & \searrow & \\
B_0' & \longrightarrow & C \\
\downarrow{\scriptstyle \gamma_0'} & \nearrow & \\
B_0 & &
\end{array}$$

By definition of minimal approximation, $\gamma_0\gamma_0' \sim Id_{B_0}$. Reversing the role of B_0 and B_0' gives the other condition for homotopy equivalence. Furthermore, by the definition of approximations, we have that $\Delta_0'\gamma_0 \sim \mathrm{Id}_C\Delta_0$.

To apply the inductive step, assume that there exist homotopy equivalences $\lambda_{i-1} : C^{i-1} \xrightarrow{\simeq} C'^{i-1}$ and $\gamma_{i-1} : B_{i-1} \xrightarrow{\simeq} B'_{i-1}$. Then we have the following diagram:



To show that $C^i \simeq C'^i$, we construct the diagram of triangles



where, by TR3, a dashed arrow exists such that $u'_i\lambda_i \sim \gamma_{i-1}u_i$ and furthermore, by the triangulated five lemma, it is a homotopy equivalence. Now, since B_i, B'_i are both minimal approximations of C^{i-1}, C'^{i-1} (respectively) and $C^{i-1} \simeq C'^{i-1}$, we have that there exists $\gamma_i : B_i \simeq B'_i$ such that $\epsilon'_{C'^i}\gamma_i \sim \lambda_i\epsilon_{C^i}$. Now we will show that $\Delta'_i\gamma_i \sim \gamma_{i-1}\Delta_i$.

$$\Delta'_i\gamma_i \sim$$

$$u'_i\epsilon'_{C'_i}\gamma_i \sim u'_i\lambda_i\epsilon_{C^i} \sim \gamma_{i-1}u_i\epsilon_{C^i}$$

$$\sim \gamma_{i-1}\Delta_i$$

thus proving our statement. $\square$

In a triangulated category $\mathcal{T}$, we unfortunately do not have the notion of exactness. However, as with distinguished triangles, we can say something about the composition of consecutive morphisms in a resolution.

Proposition 55 (Composition of consecutive maps is 0)**.** Let **B** be an $\mathcal{X}$-triangle resolution. Then $\Delta_i \Delta_{i+1} = 0$.

Proof. Let

$$\cdots \to B_i \xrightarrow{\Delta_i} B_{i-1} \to \cdots \to B_1 \xrightarrow{\Delta_1} B_0 \xrightarrow{\Delta_0} C$$

be a triangle resolution of the complex C. Then the composition $\Delta_i \Delta_{i+1}$ is given by:

$$B_{i+1} \xrightarrow{\epsilon_{C^{i+1}}} C^{i+1} \xrightarrow{u_{i+1}} B_i \xrightarrow{\epsilon_{C^i}} C^i \xrightarrow{u_i} B_{i-1}.$$

Since

$$C^{i+1} \xrightarrow{u_{i+1}} B_i \xrightarrow{\epsilon_{C^i}} C^i \to \Sigma C^{i+1}$$

is a distinguished triangle and the composition of two maps in a triangle is zero, the statement is proved. □

Continuing with the similarities to resolutions in R-mod, we state the following proposition.

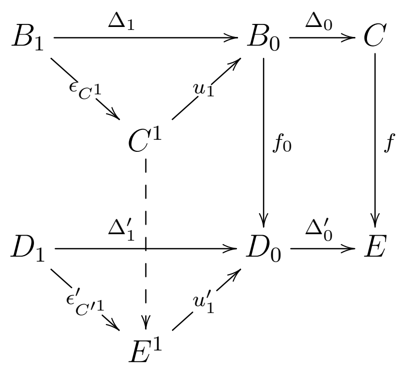
Proposition 56 (Comparison Theorem)**.** Let $\mathbf{B} : \cdots \to B_2 \to B_1 \to B_0 \to C$ and $\mathbf{D} : \cdots \to D_2 \to D_1 \to D_0 \to E$ be $\mathcal{X}$-Triangle Resolutions of C and E respectively and let $f : C \to E$ be a morphism of objects in the category $\mathcal{T}$. Then there exists a family of morphisms $(f_i) : B_i \to D_i$ making the following diagram commute:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & B_2 & \longrightarrow & B_1 & \longrightarrow & B_0 & \longrightarrow & C \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow f \\
\cdots & \longrightarrow & D_2 & \longrightarrow & D_1 & \longrightarrow & D_0 & \longrightarrow & E
\end{array}$$

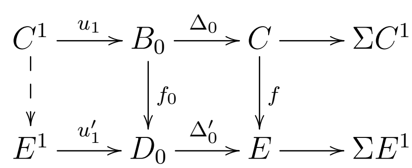
Proof. We will proceed by induction on the degree. For the base case $i = 0$, consider the diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & B_2 & \xrightarrow{\Delta_2} & B_1 & \xrightarrow{\Delta_1} & B_0 & \xrightarrow{\Delta_0} & C \\
 & & & & & & \downarrow & & \downarrow f \\
\cdots & \longrightarrow & D_2 & \xrightarrow{\Delta_2'} & D_1 & \xrightarrow{\Delta_1'} & D_0 & \xrightarrow{\Delta_0'} & E
\end{array}$$

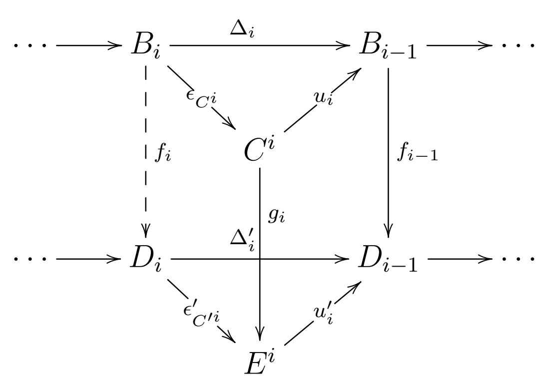
Since D_0 is an $\mathcal{X}$-approximation of E, $B_0 \in \mathcal{X}$ and $f\Delta_0$ is a morphism from $B_0 \to E$, there exists a morphism $f_0 : B_0 \to D_0$ such that $\Delta'_0 f_0 \sim f\Delta_0$. For the purpose of induction, we must also show that there exists a morphism, say $g_1 : C^1 \to E^1$, making the following diagram commute:



In order to do so, we apply TR3 to the following diagram of triangles:



Thus we obtain the morphism $g_1 : C^1 \to E^1$ such that $u'_1 g_1 \sim f_0 u_1$. Now, for the inductive step, assume that there exist morphisms $f_{i-1} : B_{i-1} \to D_{i-1}$ and $g_i : C^i \to E^i$ such that $u'_i g_i \sim f_{i-1} u_i$. Consider again the following commutative diagram of complexes:



Then $g_i\epsilon_{C^i}$ is a morphism $B_i \to E^i$ where $B_i \in \mathcal{X}$ and since D_i is an $\mathcal{X}$-approximation we have that there exists a morphism $f_i : B_i \to D_i$ such that $g_i\epsilon_{C^i} \sim \epsilon'_{C'^i} f_i$. Now note that

$$f_{i-1}\Delta_i \sim$$
$$f_{i-1}u_i\epsilon_{C^i} \sim u'_i g_i \epsilon_{C^i} \sim u'_i \epsilon'_{C'^i} f_i$$
$$\sim \Delta'_i f_i$$

which follows from the fact that homotopy is preserved under composition. Thus, the statement is proved. □

4.3 Triangle Betti Numbers

Seeing as we now have the notion of a minimal resolution in $\mathbf{K}_{\text{tac}}(R)$ we would also like to establish some way of discussing the size of the complexes which comprise it. This motivates a triangulated analogue of Betti numbers.

In a Krull-Schmidt category, objects have a unique decomposition into a finite direct sum of indecomposable objects. This leads to a natural notion for the “rank” of the complexes in the resolution; however, as opposed to copies of the ring, we use the number of indecomposable components. Set Λ to be the collection of all such indecomposable objects.

Definition 57. Assume $\mathcal{T}$ is a triangulated Krull-Schmidt category with a full subcategory, $\mathcal{X}$, closed under direct summands. Furthermore, let

$$\mathbf{B} : \cdots \to B_2 \to B_1 \to B_0 \to C$$

be a minimal $\mathcal{X}$-triangle resolution of an object $C \in \mathcal{T}$, and note for each $i \geq 0$ we can write $B_i = \bigoplus_{E\in\Lambda} E^{k_E}$. Then we say that the i^{th} *triangle Betti number* is defined to be:

$$\beta_i^{\mathcal{T}}(C) = \sum_{E\in\Lambda} k_E.$$

<u>Remark</u>: We note that the i^{th} triangle Betti number is given by the sum of the entries in the Arnold-tuple, $A_Q(B_i)$.

As a direct corollary to Theorem 6.1 in [1], any totally acyclic complex over a hypersurface is periodic of period at most two, moreover, as previously mentioned, any codimension-one approximation can be given by a shift of the original complex. The more interesting examples arise when we consider instead a pair of rings with codimension two or more; of course, this also comes with an added increase in difficulty of providing interesting examples. However, of particular interest in the forthcoming examples is the complexities of C, the approximations in the sequence, and that of the triangle resolution itself. For context, we first define complexity of the triangle resolution.

Definition 58. Let

$$\mathbf{B} : \cdots \to B_2 \to B_1 \to B_0 \to C$$

be a minimal $\mathcal{X}$-triangle resolution of an object $C \in \mathcal{T}$. Then the complexity of the triangle resolution is given by

$$\operatorname{cx}_{\mathcal{T}} C := \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists\ a \in \mathbb{R} \text{ s.t. } \beta_n^{\mathcal{T}}(C) \leq an^{t-1}\ \forall\ n \gg 0\}$$

We now give a few examples:

Example 59. Let $Q = \frac{k[[x,y]]}{(x^2)}$ and $R = Q/(y^2)$, and

$$D : \cdots \to Q^2 \xrightarrow{\begin{bmatrix} -x & 0 \\ y & x \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} -x & 0 \\ y & x \end{bmatrix}} Q^2 \to \cdots$$

a totally acyclic complex in $\mathbf{K}_{\text{tac}}(Q)$. Consider the triangle resolution of SD given in Example 51:

$$0 \to SD \to SD.$$

Here, the triangle Betti numbers are

$$\beta_i^{\mathbf{K}_{\text{tac}}(R)}(SD) = \begin{cases} 1 & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

therefore, $\operatorname{cx}_{\mathbf{K}_{\text{tac}}(R)} SD = 0$.

In the case of relative codimension one, we have that triangle resolutions will always have complexity of at most one. This can be seen as an analogue of Eisenbud's Theorem 6.1 in [1], that complexes are periodic in this situation.

Example 60. Let $Q = k[x,y]/(x^2)$, $R = Q/(y^2)$, and C be the totally acyclic R-complex with $\operatorname{Im}(\partial_0^C) = Rxy \cong k$:

$$\cdots \to R^3 \xrightarrow{\left[\begin{smallmatrix} x & 0 & -y \\ 0 & y & x \end{smallmatrix}\right]} R^2 \xrightarrow{[x\ y]} R \xrightarrow{xy} R \xrightarrow{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]} R^2 \to \cdots$$

Then the approximation of C with respect to $\mathbf{K}_{\text{tac}}(Q)$, is given by the map $STC \xrightarrow{\epsilon_C} C$, depicted in the diagram below:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & R^2 & \xrightarrow{\left[\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right]} & R^2 & \xrightarrow{\left[\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right]} & R^2 & \xrightarrow{\left[\begin{smallmatrix} x & -y \\ 0 & x \end{smallmatrix}\right]} & R^2 & \xrightarrow{\left[\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right]} & R^2 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]} & & \downarrow{\scriptstyle [1\ 0]} & & \downarrow{\scriptstyle [y\ 0]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} y & 0 \\ 0 & y \end{smallmatrix}\right]} & & \\
\cdots & \longrightarrow & R^3 & \xrightarrow[\left[\begin{smallmatrix} x & 0 & -y \\ 0 & y & x \end{smallmatrix}\right]]{} & R^2 & \xrightarrow[{[x\ y]}]{} & R & \xrightarrow[xy]{} & R & \xrightarrow[\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]]{} & R^2 & \longrightarrow & \cdots
\end{array}$$

with mapping cone:

$$\operatorname{cone}(\epsilon_C) : \cdots \to R^5 \xrightarrow{\left[\begin{smallmatrix} -x & -y & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & 0 \\ 1 & 0 & x & 0 & -y \\ 0 & 1 & 0 & y & x \end{smallmatrix}\right]} R^4 \xrightarrow{\left[\begin{smallmatrix} -x & y & 0 & 0 \\ 0 & -x & 0 & 0 \\ 1 & 0 & x & y \end{smallmatrix}\right]} R^3 \xrightarrow{\left[\begin{smallmatrix} -x & -y & 0 \\ 0 & -x & 0 \\ y & 0 & xy \end{smallmatrix}\right]} R^3 \xrightarrow{\left[\begin{smallmatrix} -x & y & 0 \\ 0 & -x & 0 \\ y & 0 & x \\ 0 & y & y \end{smallmatrix}\right]} R^4 \to \cdots$$

Furthermore, Proposition 6.5 in [2] tells us that the approximation of the mapping cone is given by the complex ΣSTC. By repeating this process, we see that the mapping cone of the approximation of this mapping cone is given by $\Sigma^2 STC = STC$ (as STC is periodic of period two). Therefore, the triangle resolution is given by

$$\cdots \to \Sigma STC \to STC \to \Sigma STC \to STC \to C.$$

We note that $\beta_i^{\mathbf{K}_{\text{tac}}(R)}(C) = 1$ for each $i \in \mathbb{Z}$, thus $\operatorname{cx}_{\mathbf{K}_{\text{tac}}(R)} C = 1$.

Interestingly, we note that in each of the examples above, the complexity of the original complex is equal to the sum of the complexity of the triangle resolution plus the largest complexity of the complexes in the resolution. With this in mind, we leave the reader with a final conjecture:

Conjecture: *Let* $C \in \mathbf{K}_{\text{tac}}(R)$ *with minimal triangle resolution* $\boldsymbol{B} = \cdots \to B_1 \to B_0 \to C$. *Furthermore, set* $\upsilon = \sup\{t \mid \operatorname{cx}_R B_i = t,\ \forall\ i \in \mathbb{N}\}$. *We then have the following equality:*

$$\operatorname{cx}_R C = \operatorname{cx}_{\mathbf{K}_{\text{tac}}(R)} \boldsymbol{B} + \upsilon.$$

REFERENCES

[1] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc., 260, (1980), no. 1, 35–64.

[2] P.A. Bergh, D.A. Jorgensen, W.F. Moore, *Totally acyclic approximations*, Appl Categor Struct (2021). https://doi.org/10.1007/s10485-021-09633-1.

[3] P.A. Bergh, D.A. Jorgensen, W.F. Moore *A converse to a construction of Eisenbud–Shamash*, J. Commut. Algebra, 12 (4), (2020), 467–477.

[4] L.W. Christensen, D.A. Jorgensen, *Tate (co)homology via pinched complexes*, Transactions of the American Mathematical Society, (2011), 366. 10.1090/S0002-9947-2013-05746-7.

[5] Y. Yoshino, *Maximal Cohen-Macaulay Modules over Cohen-Macaulay Rings*, London Mathematical Society Lecture Note Series, Cambridge: Cambridge University Press, doi:10.1017/CBO9780511600685.

[6] L.W. Christensen, H.B. Foxby, H. Holm, *Derived Category Methods in Commutative Algebra*, Unpublished Manuscript, https://www.math.ttu.edu/ lchriste/download/dcmca.pdf.

[7] G.J. Leuschke, R. Wiegand, *Cohen-Macaulay Representations*, American Mathematical Soc., (2012), ISBN 0821875817, 9780821875810.

[8] T. Holm, P. Jørgensen, *Triangulated categories: Definitions, properties, and examples* In T. Holm, P. Jørgensen, & R. Rouquier (Eds.), Triangulated Categories (London Mathematical Society Lecture Note Series, pp. 1-51), Cambridge: Cambridge University Press, doi:10.1017/CBO9781139107075.002.

[9] L.L. Avramov, S. Iyengar, *Cohomology over complete intersections via exterior algebras*, In T. Holm, P. Jørgensen, & R. Rouquier (Eds.), Triangulated Categories (London Mathematical Society Lecture Note Series, pp. 52-75), Cambridge: Cambridge University Press, (2010), doi:10.1017/CBO9781139107075.003.

[10] L.L. Avramov, A. Martsinkovsky, *Absolute, Relative and Tate Cohomology of Modules of Finite Gorenstein Dimension*, Proceedings of the London Mathematical Society, Volume 85, Issue 2, (2002), Pages 393–440, https://doi.org/10.1112/S0024611502013527.

[11] E.E. Enochs, O. Jenda, *Relative Homological Algebra*, Berlin, Boston: De Gruyter, (2011), https://doi.org/10.1515/9783110215212.

[12] H. Krause, *Krull-Schmidt categories and projective covers*, Expositiones Mathematicae, (2014), 33. 10.1016/j.exmath.2015.10.001.

[13] C.M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics, 1099. Springer-Verlag, Berlin, (1984), xiii+376 pp. ISBN: 3-540-13905-2 16A64 (11E81 16A46).

[14] S. Liu, *Auslander-Reiten theory in a Krull-Schmidt category*, The São Paulo Journal of Mathematical Sciences, (2010), 4, 10.11606/issn.2316-9028.v4i3p425-472.

[15] C. Amiot, *On the structure of triangulated categories with finitely many indecomposables*, 2007, (hal-00118624v2).

[16] R. Takahashi, *On the category of modules of Gorenstein dimension zero*, Math. Z., 251, (2005), 249–256, https://doi.org/10.1007/s00209-005-0795-8.

[17] M. Auslander, S.O. Smalø, *Preprojective modules over Artin algebras*, J. Algebra 66 (1980), no. 1, 61-122.

[18] E.E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J. Math. 39 (1981), no. 3, 189-209.

[19] D. Happel, *Triangulated Categories in the Representation of Finite Dimensional Algebras*, London Mathematical Society Lecture Note Series. Cambridge: Cambridge University (1988), Press. doi:10.1017/CBO9780511629228.

[20] D. Happel, *On the derived category of a finite-dimensional algebra*, Commentarii mathematici Helvetici 62 (1987): 339-389, http://eudml.org/doc/140089.

[21] R.O. Buchweitz, *Maximal Cohen–Macaulay Modules and Tate Cohomology over Gorenstein Rings*, 1987

[22] B. Eckmann, A. Schopf, *Über injektive Moduln*, Archiv der Mathematik, (1953), 4 (2): 75–78, doi:10.1007/BF01899665, ISSN 0003-9268, MR 0055978.

[23] E. Matlis, *Injective modules over Noetherian rings*, Pacific Journal of Mathematics, (1958), 8: 511–528, doi:10.2140/pjm.1958.8.511, ISSN 0030-8730, MR 0099360.

[24] H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc, 95 (1960), 466-488

[25] P.A. Bergh, D.A. Jorgensen, S. Oppermann, *The Gorenstein defect category*, Quarterly Journal Math, 66 (2015), no. 2, 459-471.

[26] P.A. Bergh, D.A. Jorgensen, *The depth formula for modules with reducible complexity.*, Illinois Journal of Mathematics, (2009), 55, 10.1215/ijm/1359762397.

[27] M. Auslander, *Functors and morphisms determined by objects*, Representation theory of algebras, (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976), pp. 1–244. Lecture Notes in Pure Appl. Math., Vol. 37, Dekker, New York, 1978.

[28] C. Amiot, O. Iyama, I. Reiten, *Stable categories of Cohen-Macaulay modules and cluster categories*, American Journal of Mathematics, (2015), 137(3), 813-857. doi:10.1353/ajm.2015.0019.

[29] D. Eisenbud *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Math., vol. 150, Springer-Verlag, Berlin and New York, 1995, xvi+785 pp., ISBN 0-387-94268-8.

[30] W. Bruns, H. Herzog, *Cohen-Macaulay Rings*, (2nd ed., Cambridge Studies in Advanced Mathematics), Cambridge: Cambridge University Press, (1998) doi:10.1017/CBO9780511608681.

[31] A. Neeman, *Triangulated Categories*, (AM-148), Princeton University Press, 2001, JSTOR, www.jstor.org/stable/j.ctt7zvcqc.

[32] J.L. Verdier, *Des Catégories Dérivées des Catégories Abéliennes*, Astérisque (Paris: Société Mathématique de France), 239.

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