SOME QUADRATIC REGULAR ALGEBRAS ON FOUR GENERATORS WITH A
1-DIMENSIONAL NONREDUCED LINE SCHEME

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SOME QUADRATIC REGULAR ALGEBRAS ON FOUR GENERATORS WITH A 1-DIMENSIONAL NONREDUCED LINE SCHEME

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In the 1980s, M. Artin, J. Tate, and M. Van den Bergh applied geometric techniques to noncommutative algebras. Their work introduced algebraic concepts called point modules and line modules and an associated geometric concept, which was later called the point scheme. In 2002, Shelton and Vancliff defined the concept of line scheme and developed a method for computing the line scheme of any quadratic algebra that satisfies certain conditions. Artin, Tate, and Van den Bergh were able to classify so-called quantum \( \mathbb{P}^2 \)s by their point scheme; quantum \( \mathbb{P}^3 \)s however are much more challenging and involve computing the line scheme.

In this thesis, we compute the point variety and line scheme of a certain quadratic algebra that is an iterated Ore extension of a polynomial ring on two variables. This algebra was provided in a paper authored by Stephenson and Vancliff as a counter example to a prior open problem. We find the closed points of the point scheme and compute the line scheme. Our work shows that the line scheme consists of three components: two of those components have increased multiplicity and direct us to a certain subalgebra of the algebra.

We also consider an algebra known as the Exotic Elliptic algebra. We investigate whether or not there is a relationship between central elements of the algebra and the line scheme of the algebra. In particular, we examine right ideals corresponding to (right) line modules for central elements.
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Dedications

I dedicate this thesis to my loving wife Amanda. Her toughness and compassion have seen me through some of my roughest times during graduate school. I am not sure I could have done this without her in my life.

I would also like to dedicate this thesis to all my family, friends and to those whom I consider my Texas family. Your love and support has been so greatly appreciated over the years.

Lastly, I want to dedicate this thesis to my mom Patricia. I know you have been watching over me all these years. While I know you are no longer with me in this world, your spirit still guides me and comforts me. I hope I have made you proud and that one day we find a way to prevent another son or daughter from losing their mother to breast cancer. Rest in peace mom, I love you.
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CHAPTER 1

Introduction

For centuries, geometry has been used to better understand and solve some of the toughest problems in mathematics. In the early 20th century, due to the work of people such as Weil, Zariski and others, a more formal and rigorous structure was created that merged ideas of geometry and commutative algebra. This subject is now known as Algebraic Geometry. In the mid-20th century, people such as Serre and Gothendieck added to these concepts through their work on sheaf theory and scheme theory. However, in the 1980s, advances in quantum theory had led to the discovery of some noncommutative algebras such as the Sklyanin algebra. Traditional techniques were not effective on these new algebras and produced limited results in studying these algebras, so a movement began to try to apply algebraic geometry to noncommutative algebras. One approach was to look at noncommutative algebras that, in some sense, behaved similarly to the polynomial ring. In the late 1980s, Artin and Schelter [2] proposed that regular algebras were a class of noncommutative algebras that “feel” close enough to the polynomial ring and thus a geometric structure could be applied. This work was then continued by Artin together with Tate and Van den Bergh [3] and a geometric framework was developed to study these regular algebras.

In [3], Artin, Tate and Van den Bergh introduced the concept of a point module and a line module. These modules could then be associated to points and lines in projective space. They then gave a method to compute what was later known as the point scheme. Once the point scheme is calculated, the point modules could then be determined from these points. An analogous method for line modules was developed later in 2002 by Shelton and Vancliff [20, 21] for algebras that are Artin-Schelter regular (also called AS-regular), of
global dimension four, that are quadratic domains, have four generators and six defining relations and the same Hilbert series as that of the polynomial ring on four variables. Artin-Schelter regular algebras of global dimension $n + 1$ are sometimes called quantum $\mathbb{P}^n$s and are considered to be noncommutative analogues of the polynomial ring on $n + 1$ variables.

Artin, Tate and Van den Bergh were able to classify all quantum $\mathbb{P}^2$s by their point scheme, but classifying quantum $\mathbb{P}^3$s is a much more challenging task. For this reason, since most of the quantum $\mathbb{P}^3$s are quadratic, we focus our attention on these algebras. For our purposes, a quadratic, noetherian, AS-regular algebra with Hilbert series $(1 - t)^{-4}$ will be called a quadratic quantum $\mathbb{P}^3$. We also try to determine what the geometry tells us about these algebras.

Not all quadratic algebras are quantum $\mathbb{P}^3$s. However, if we produce an Ore extension (subject to certain conditions) from an algebra that is AS-regular and noetherian, then the resulting algebra has the same properties. This thesis focuses on one such algebra introduced in [22] that is an iterated Ore extension and has the desired properties that we wish to study.

In Chapter 3, we introduce the algebra we wish to study and first compute the point variety. We not only verify the findings in [22] but also give the exact points of the point scheme. We then compute the line scheme by first computing the line variety and present its components in Theorem 3.4. In Theorem 3.7 we compute the multiplicity of each component and thereby identify the line scheme. In Corollary 3.8 we identify the intersection points of the components of the line scheme. We then conclude the chapter by giving our observations about what information the line scheme encodes about the algebra.

We also consider an algebra known as the Exotic Elliptic algebra [7, 8] in Chapter 4. We investigate whether or not there is a relationship between central elements of the algebra and the line scheme of the algebra. In particular, we examine right ideals corresponding to (right) line modules for central elements.

Throughout this work, our underlying goal is to seek information about an algebra that is encoded in geometric data (such as the line scheme) associated to the algebra.
CHAPTER 2

Definitions

This thesis assumes throughout that $k$ is an algebraically closed field and that char($k$) = 0. We assume that the reader is familiar with module theory and projective $n$-space, which is denoted $\mathbb{P}^n$. For a vector space $V$, we write $V^\times$ to represent the nonzero elements of $V$. For polynomials $f_1, \ldots, f_k$, we write $\mathcal{V}(f_1, \ldots, f_k)$ for the zero locus of $f_1, \ldots, f_k$.

Definition 2.1. (cf. [18]) Positively Graded, Connected $k$-Algebra

A $k$-algebra $A$ is positively graded if $A = \bigoplus_{j=0}^{\infty} A_j$, where $A_j$ is a subspace of $A$ for all $j$, and $A_iA_j \subset A_{i+j}$ for all $i$ and $j$. Such an algebra is sometimes called $\mathbb{N}$-graded. If $A_0 = k$, we say that $A$ is connected. The elements of $A_j^\times$ are called the homogeneous elements of degree $j$.

Definition 2.2. (cf. [19]) Quadratic Algebra

A $k$-algebra $A$ is called quadratic if:

- $A$ is generated by degree-one elements, and
- each one of the defining relations of $A$ is homogeneous of degree two.

Definition 2.3. (cf. [18]) Graded $A$-Module

If $M$ is a left (resp. right) $A$-module and $A = \bigoplus_{j=0}^{\infty} A_j$ is a positively graded $k$-algebra, we say $M$ is a graded module if $M = \bigoplus_{k=-\infty}^{\infty} M_k$, where $M_k$ is a subspace of $M$ for all $k$ and $A_jM_k \subset M_{j+k}$ (resp. $M_kA_j \subset M_{j+k}$) for all $j, k$.

Definition 2.4. ([3]) Point Module

Let $A = \bigoplus_{j=0}^{\infty} A_j$ denote an $\mathbb{N}$-graded, connected $k$-algebra generated by $A_1$ where dim$_k(A_1) =$
A graded right (resp. left) \( A \)-module \( M = \bigoplus_{i=0}^{\infty} M_i \) is called a right (resp. left) point module if

- \( M \) is cyclic with \( M = M_0A \) (resp. \( M = AM_0 \)), and
- \( \dim_k(M_i) = 1 \) for all \( i \).

**Example 2.5.** If \( A = k[x, y, z, w] \), the polynomial ring on \( x, y, z \) and \( w \), and if \( M = A/(x, y, z) \), then \( M \) is a point module with the associated point \((0 : 0 : 0 : 1) \in \mathbb{P}^3\).

**Definition 2.6.** ([3]) Line Module

Let \( A = \bigoplus_{j=0}^{\infty} A_j \) denote an \( \mathbb{N} \)-graded, connected \( k \)-algebra generated by \( A_1 \) where \( \dim_k(A_1) = n < \infty \). A graded right (resp. left) \( A \)-module \( M = \bigoplus_{i=0}^{\infty} M_i \) is called a right (resp. left) line module if:

- \( M \) is cyclic with \( M = M_0A \) (resp. \( M = AM_0 \)), and
- \( \dim_k(M_i) = i + 1 \) for all \( i \).

**Example 2.7.** If \( A = k[x, y, z, w] \), the polynomial ring on \( x, y, z \) and \( w \), and if \( M = A/(x, y) \), then \( M \) is a line module with the associated line \( \{(0 : 0 : \alpha : \beta) \in \mathbb{P}^3 \mid (\alpha : \beta) \in \mathbb{P}^1\} \).

**Definition 2.8.** (cf. [19]) Koszul Dual

Let \( V \) be a finite-dimensional \( k \)-vector space and \( T(V) \) the tensor algebra on \( V \). Let \( W \) be an ideal of \( T(V) \) generated by homogeneous quadratic elements. If \( A = T(V)/\langle W \rangle \), the Koszul dual, \( A^! \), of \( A \) is:

\[ A^! = T(V^*)/\langle W^\perp \rangle. \]

**Example 2.9.** Let

\[ A = k[x_1, x_2, x_3] \cong \frac{T(V)}{\langle W \rangle}, \]

where \( V = kx_1 \oplus kx_2 \oplus kx_3 \) and \( W \) is the span of

\[ \{x_1 \otimes x_2 - x_2 \otimes x_1, \ x_1 \otimes x_3 - x_3 \otimes x_1, \ x_2 \otimes x_3 - x_3 \otimes x_2\}. \]
Since \( \dim(V \otimes V) = 9 \) and \( \dim(W) = 3 \), \( \dim(W^\perp) = 9 - 3 = 6 \). If \( \{z_1, z_2, z_3\} \) is the dual basis to \( \{x_1, x_2, x_3\} \), we seek six linearly independent elements of \( T^2(k z_1 \oplus k z_2 \oplus k z_3) \) that vanish on \( W \). A computation shows that the six linearly independent elements
\[
z_1 \otimes z_1, \ z_2 \otimes z_2, \ z_3 \otimes z_3, \ z_1 \otimes z_2 + z_2 \otimes z_1, \ z_1 \otimes z_3 + z_3 \otimes z_1, \ z_2 \otimes z_3 + z_3 \otimes z_2
\]
vanish on \( W \). Therefore the Koszul dual of \( A \) is
\[
A^\dagger = \frac{T(k z_1 \oplus k z_2 \oplus k z_3)}{\langle z_1 \otimes z_1, \ z_2 \otimes z_2, \ z_3 \otimes z_3, \ z_1 \otimes z_2 + z_2 \otimes z_1, \ z_1 \otimes z_3 + z_3 \otimes z_1, \ z_2 \otimes z_3 + z_3 \otimes z_2 \rangle},
\]
which is isomorphic to
\[
\frac{k \langle z_1, z_2, z_3 \rangle}{\langle z_1^2, z_2^2, z_3^2, z_1 z_2 + z_2 z_1, z_1 z_3 + z_3 z_1, z_2 z_3 + z_3 z_2 \rangle}.
\]

**Definition 2.10.** (cf. [3]) Polynomial Growth

If \( A = \bigoplus_{j=0}^\infty A_j \) is a positively graded, connected \( k \)-algebra, we say that \( A \) has polynomial growth if there exist \( a, b \in \mathbb{R}_{>0} \) such that \( \dim_k(A_j) \leq a j^b \) for all \( j \).

**Definition 2.11.** (cf. [2]) Gorenstein Condition

A positively graded, connected \( k \)-algebra \( A \) satisfies the Gorenstein condition if:

- a minimal projective resolution of the left trivial module, \( _A k \), consists of finitely generated modules, and
- by dualizing this projective resolution of \( _A k \), we obtain a minimal projective resolution of \( k_A[e] \), shifted by some degree \( e \).

This is sometimes described by \( \text{Ext}^i_A(A_k, A) \cong \delta^i_d k_A[e] \), where \( \delta^i_d \) is the Kronecker-delta symbol.

**Definition 2.12.** ([2]) Artin-Schelter Regular

If \( A = \bigoplus_{j=0}^\infty A_j \) is a positively graded, connected \( k \)-algebra and is generated by \( A_1 \), we say that \( A \) is a regular (or AS-regular) algebra of global dimension \( d \) if:
- \( \text{gldim}(A) = d < \infty \), and
- \( A \) has polynomial growth, and
- \( A \) satisfies the Gorenstein condition.

**Definition 2.13.** Quantum \( \mathbb{P}^n \)

In this thesis, a quantum \( \mathbb{P}^n \) is an AS-regular algebra of global dimension \( n + 1 \).

**Definition 2.14.** (cf. [13]) Left \( \sigma \)-Derivation

Let \( R \) be a ring and \( \sigma : R \to R \) a ring homomorphism. A left \( \sigma \)-derivation \( \delta \) of \( R \) is a linear transformation \( \delta : R \to R \) such that:

\[
\delta(ab) = \sigma(a)\delta(b) + \delta(a)b, \text{ for all } a, b \in R.
\]

**Definition 2.15.** ([1, 13, 18]) Ore Extension

With notation as in Definition 2.14, the Ore extension \( R[y; \sigma, \delta] \) is the free left \( R \)-module with basis \( \{1, y, y^2, \ldots \} \) subject to the relation:

\[
yr = \sigma(r)y + \delta(r)
\]

for all \( r \in R \).

**Definition 2.16.** (cf. [14]) Gelfand-Kirillov Dimension

The Gelfand-Kirillov dimension (GK-dimension) of a \( \mathbb{k} \)-algebra \( A \) is:

\[
\text{GKdim}(A) = \sup \left\{ \limsup_{n \to \infty} \log_n (\dim_\mathbb{k} \sum_{i=0}^{n} V^i) : V \text{ is a finite-dimensional subspace of } A \right\},
\]

where \( V^0 = \mathbb{k} \) and \( V^i \) denotes the subspace spanned by all monomials \( v_1 \cdots v_i \) for all \( i \geq 1 \), where \( v_j \in V \) for all \( j \).

If \( A \) is a \( \mathbb{k} \)-algebra, and \( M \) is a right \( A \)-module, the Gelfand-Kirillov dimension (GK-dimension) of \( M \) is:

\[
\text{GKdim}(M) = \sup_{V,F} \left\{ \limsup_{n \to \infty} \log_n (\dim_\mathbb{k} (FV^n)) \right\},
\]

where the supremum is taken over all finite-dimensional subspaces \( V \) of \( A \) containing 1 and all finite-dimensional subspaces \( F \) of \( M \).
Definition 2.17. (cf. [15]) Auslander Condition
A \( \mathbb{k} \)-algebra \( A \) satisfies the Auslander condition if, for every graded, finitely generated \( A \)-module \( M \) and for all \( n \geq 0 \),
\[
\inf\{i : \text{Ext}^i_A(N, A) \neq \{0\}\} \geq n
\]
for all graded \( A \)-submodules \( N \) of \( \text{Ext}^n_A(M, A) \).

Definition 2.18. (cf. [15]) Auslander Regular
A \( \mathbb{k} \)-algebra \( A \) is Auslander regular of global dimension \( d \) if:

- \( A \) is noetherian, and
- \( \text{gldim}(A) = d < \infty \), and
- \( A \) satisfies the Auslander condition.

Definition 2.19. (cf. [17]) Cohen-Macaulay Property
A \( \mathbb{k} \)-algebra \( A \) satisfies the Cohen-Macaulay property if there exists \( n \geq 0 \) such that
\[
\text{GKdim}(M) + \inf\{i : \text{Ext}^i_A(M, A) \neq 0\} = n,
\]
for all nonzero finitely generated \( A \)-modules \( M \); when satisfied, \( n = \text{GKdim}(A) \).

Definition 2.20. (cf. [10]) Plücker Coordinates
For any line in \( \mathbb{P}^3 \) through distinct points \( a, b \) that have homogeneous coordinates \((a_1 : a_2 : a_3 : a_4)\) and \((b_1 : b_2 : b_3 : b_4)\), respectively, the Plücker coordinate \( M_{ij} \) for \( 1 \leq i < j \leq 4 \) is defined as:
\[
M_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} = a_i b_j - b_i a_j.
\]
CHAPTER 3

The Point Variety and Line Scheme of Some Algebras in a Family of Iterated Ore Extensions

In this chapter we compute the point variety and line scheme of a family of algebras first introduced by Stephenson and Vancliff in 2002 [22]. In Section 3.1, we define this family of algebras (denoted $A(a, b, c, d, \lambda)$), which, for simplicity, we call $A$. We describe the method to compute the point variety and then compute the closed points of the point variety in Section 3.2. We verify that there are at most seven closed points in the point scheme. Our algebra depends on parameters $a, b, c, d$, and if they are all zero and if the coefficient field is algebraically closed, then the point scheme has exactly seven closed points. The set of closed points of the point scheme is given in Proposition 3.3. In Section 3.3, we focus our attention on the line scheme. Section 3.3.1 describes the method to compute the line scheme using the algebra $A$. However, for our purposes, we found that it was convenient to focus on a certain algebra $B$ within this family of algebras. We compute the line scheme of $B$ in Section 3.3.

In Theorem 3.4, we identify the components of the line scheme as a subscheme of $\mathbb{P}^5$. We find the line variety is the union of three curves: namely, a line, a conic and a third curve. In Theorem 3.7, we find the multiplicity of each of the components of our line scheme and find that the line has multiplicity four as a subscheme of the line scheme and that the conic has multiplicity four. This means that, since the conic has degree 2, as a subscheme of the line scheme, the conic has degree 8. The third curve has multiplicity one and degree 8. We describe the intersection points of the components of the line scheme in Corollary 3.8. We remind the reader that part of our goal is to find what the geometry tells us about
the algebra; therefore, we conclude this chapter with comments concerning the relationship between algebraic properties of the algebra and geometric properties of the line scheme.

3.1. The Algebra $A(a, b, c, d, \lambda)$.

We now introduce the family of algebras that we will be analyzing in the rest of the chapter. Unless otherwise stated, let $\mathbb{k}$ be an algebraically closed field such that $\text{char}(\mathbb{k}) = 0$.

These algebras were first introduced in [22] as a means to support the main theorem of that paper and provide additional candidates for a generic quantum $\mathbb{P}^3$. M. Artin, J. Tate and M. Van den Bergh proved in the late 1980s that a quantum $\mathbb{P}^2$ is a finite module over its center if and only if the automorphism of the point scheme has finite order [4, Theorem II]. Prior to [22], it had been unknown if the analogous result holds for a quantum $\mathbb{P}^3$. The algebra, which we define below, is one of two algebras presented in [22] that show the existence of algebras that are infinite modules over their centers even though the corresponding automorphism of the point scheme has finite order.

Definition 3.1. [22] Let $a, b, c, d, \lambda \in \mathbb{k}$ and $A(a, b, c, d, \lambda)$ denote the $\mathbb{k}$-algebra with generators $x_1, \ldots, x_4$ subject to the defining relations:

$x_2 x_1 = -x_1 x_2 + 2x_3^2, \quad x_4 x_1 = -x_1 x_4 + x_2^2 + ax_2 x_3 + bx_3^2 + \lambda x_1 x_3,$

$x_3 x_1 = x_1 x_3, \quad x_4 x_2 = -x_2 x_4 + x_1^2 + cx_1 x_3 + dx_3^2 + (\lambda - 2)x_2 x_3,$

$x_3 x_2 = x_2 x_3, \quad x_4 x_3 = x_3 x_4 + x_1 x_2 - x_3^2.$

Each algebra in this family was formed as an iterated Ore extension of the polynomial ring on two variables. Ore extensions can preserve several algebraic properties. In particular, if $R$ is a noetherian, Artin-Schelter regular (and Auslander-regular) domain that satisfies the Cohen-Macaulay property, and if:

1. $\sigma \in \text{Aut}(R)$ and $\delta$ is a left $\sigma$-derivation, and
2. $R = \bigoplus_{i=0}^{\infty} R_i$ is a connected, graded $\mathbb{k}$-algebra where $\sigma(R_i) \subseteq R_i$ for every $i \geq 0$,

then the Ore extension $R[w; \sigma, \delta]$ is also a noetherian, Artin-Schelter regular (and Auslander-regular) domain that satisfies the Cohen-Macaulay property [16]. For simplicity, we will refer to the algebra $A(a, b, c, d, \lambda)$ as simply $A$ unless otherwise noted.
According to [22, Proposition 2.2(a)], $A \cong R[x_4; \sigma, \delta]$, where $\sigma$ is an automorphism of $R$ as in (2) above and $\delta$ is a left $\sigma$-derivation of $R$. The proof of this was left to the reader; for completeness we restate and prove that result.

**Proposition 3.2.** [22] The algebra $A$ is a noetherian, Artin-Schelter regular domain of global dimension four.

**Proof.** Let $S$ be the polynomial ring generated by $x_1$ and $x_3$. The maps

$$
\sigma_s(x_1) = -x_1, \quad \sigma_s(x_3) = x_3,
$$

extend to an automorphism of $S$. We will show that the maps

$$
\delta_s(x_1) = 2x_3^2, \quad \delta_s(x_3) = 0,
$$

extend to a left $\sigma_s$-derivation of $S$. To prove this, it suffices to show that $\delta_s$ is well defined on $S$ as follows:

$$
\delta_s(x_3x_1 - x_1x_3) = \sigma_s(x_3)\delta_s(x_1) + \delta_s(x_3)x_1 - \sigma_s(x_1)\delta_s(x_3) - \delta_s(x_1)x_3
\begin{align*}
&= 2x_3^3 + 0 - 0 - 2x_3^3 \\
&= 0.
\end{align*}
$$

Since $S$ is $k[x_1, x_3]$, $S$ is a noetherian, Artin-Schelter regular (and Auslander-regular) domain that satisfies the Cohen-Macaulay property. By the lemma in [16], $S[x_2; \sigma_s, \delta_s]$ is as well.

Now let $R = S[x_2; \sigma_s, \delta_s]$. The following maps:

$$
\sigma(x_1) = -x_1, \quad \sigma(x_2) = -x_2, \quad \sigma(x_3) = x_3,
$$

extend to an automorphism of $R$. We will show that the maps

$$
\begin{align*}
\delta(x_1) &= x_2^2 + ax_2x_3 + bx_3^2 + \lambda x_1x_3, \\
\delta(x_2) &= x_1^2 + cx_1x_3 + dx_3^2 + (\lambda - 2)x_2x_3, \\
\delta(x_3) &= x_1x_2 - x_3^2,
\end{align*}
$$

continue the action of $\sigma_s$ and $\delta_s$. The proof of these claims will be given in the next section.
extend to a left $\sigma$-derivation of $R$. Using the same method from earlier:
\[
\delta(x_3x_1 - x_1x_3) = \sigma(x_3)\delta(x_1) + \delta(x_3)x_1 - \sigma(x_1)\delta(x_3) - \delta(x_1)x_3 \\
= x_3(x_2^2 + ax_2x_3 + bx_3^2 + \lambda x_1x_3) + (x_1x_2 - x_3^2)x_1 \\
- (-x_1)(x_1x_2 - x_3^2) - (x_2^2 + ax_2x_3 + bx_3^2 + \lambda x_1x_3)x_3 \\
= 0,
\]
\[
\delta(x_3x_2 - x_2x_3) = \sigma(x_3)\delta(x_2) + \delta(x_3)x_2 - \sigma(x_2)\delta(x_3) - \delta(x_2)x_3 \\
= x_3(x_1^2 + cx_1x_3 + dx_3^2 + (\lambda - 2)x_2x_3) + (x_1x_2 - x_3^2)x_2 \\
- (-x_2)(x_1x_2 - x_3^2) - (x_1^2 + cx_1x_3 + dx_3^2 + (\lambda - 2)x_2x_3)x_3 \\
= 0,
\]
\[
\delta(x_2x_1 + x_1x_2 - 2x_3x_3) = \sigma(x_2)\delta(x_1) + \delta(x_2)x_1 + \sigma(x_1)\delta(x_2) + \delta(x_1)x_2 - 2\sigma(x_3)\delta(x_3) - 2\delta(x_3)x_3 \\
= -x_2(x_2^2 + ax_2x_3 + bx_3^2 + \lambda x_1x_3) + (x_1^2 + cx_1x_3 + dx_3^2 + (\lambda - 2)x_2x_3)x_1 \\
- x_1(x_1^2 + cx_1x_3 + dx_3^2 + (\lambda - 2)x_2x_3) + (x_2^2 + ax_2x_3 + bx_3^2 + \lambda x_1x_3)x_2 \\
- 2x_3(x_1x_2 - x_3^2) - 2(x_1x_2 - x_3^2)x_3 \\
= 0.
\]
Therefore, $A \cong R[w; \sigma, \delta]$ and the result follows from the lemma in [16].

3.2. The Point Variety.

In [22], the point scheme of $A$ was found to have at most seven closed points for arbitrary $a, b, c, d, \lambda \in k$. However, if $k$ is algebraically closed and $a = b = c = d = 0$, then the point scheme has exactly seven closed points. In this section we will compute the closed points of the point scheme; that is, the point variety of $A$.

3.2.1. Method for Computing the Point Variety.

We compute the point variety of $A$ by using the method outlined in [3] as follows. Let $V = \sum_{i=1}^{4} |kx_i|$. We write the defining relations of $A$ by using a matrix equation, $M\hat{x}$, where
\( \hat{x} = [x_1, x_2, x_3, x_4]^T \). For \( A \), our matrix \( M \) is the following \( 6 \times 4 \) matrix:

\[
M = \begin{bmatrix}
  x_2 & x_1 & -2x_3 & 0 \\
  x_3 & 0 & -x_1 & 0 \\
  0 & x_3 & -x_2 & 0 \\
  x_4 & -x_2 & -ax_2 - bx_3 - \lambda x_1 & x_1 \\
 -x_1 & x_4 & -cx_1 - dx_3 - x_2(\lambda - 2) & x_2 \\
  0 & -x_1 & x_3 + x_4 & -x_3
\end{bmatrix}.
\]

Let \( \Gamma_A \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \) be the scheme that is the zero locus of the defining relations of \( A \). Let \( p_A \) denote the image of \( \Gamma_A \) under the projection map \( \pi: \mathbb{P}(V^*) \times \mathbb{P}(V^*) \to \mathbb{P}(V^*) \) onto the first component of \( \mathbb{P}(V^*) \times \mathbb{P}(V^*) \). The point scheme can be identified with \( p_A \) and can be computed using the \( 4 \times 4 \) minors of \( M \).

### 3.2.2. The Point Variety of \( A \).

The polynomials in the Appendix, in Section 5.1, are the fifteen \( 4 \times 4 \) minors of \( M \). The polynomial 5.1.3 implies that either \( x_3 = 0 \) or \( x_1x_2 - x_3^2 = 0 \). Using Mathematica to find a Gröbner Basis and setting \( x_3 \) to be zero gives us that \( x_1 = x_2 = 0 \) and that \( x_4 \) is free. We may then take \( x_4 = 1 \) thereby giving us the point \( e_4 = (0, 0, 0, 1) \) as one of our closed points in \( p_A \). Now setting \( x_3 = 1 \) and \( x_1x_2 - x_3^2 = 0 \) and computing a Gröbner Basis results in the following polynomials:

\[
\begin{align*}
(1) \ & x_1^6 + cx_1^5 + dx_1^4 - 2x_1^3 - bx_1^2 - ax_1 - 1 = 0, \\
(2) \ & 2x_4 - x_1^3 - cx_1^2 - dx_1 + 2 - \lambda = 0, \\
(3) \ & x_2 - x_1^5 - cx_1^4 - dx_1^3 + 2x_1^2 + bx_1 + a = 0.
\end{align*}
\]

Here, we notice that \( x_2 \) and \( x_4 \) can be solved in terms of \( x_1 \) and since the polynomial in (1) is a degree-6 polynomial solely in terms of \( x_1 \), we have at most 6 solutions to this system of polynomial equations. This verifies [22, Proposition 2.2(c)]; namely, \( p_A \) has at most seven closed points.

If in fact \( a = b = c = d = 0 \) and \( k \) is algebraically closed, we now have the following system of polynomials:
\( (1) \ x_1^6 - 2x_1^3 - 1 = 0, \)
\( (2) \ 2x_4 - x_4^3 + 2 - \lambda = 0, \)
\( (3) \ x_2 - x_1^5 + 2x_1^2 = 0. \)

From the polynomial in (1) and its derivative, computing a Gröbner Basis tells us that (1) and its derivative have no solutions. Therefore, (1) has no repeated zeros. This shows that we have six distinct solutions to the above system of equations. These six solutions, along with \( e_4, \) agrees with the other part of [22, Proposition 2.2(c)]; namely that \( p_A \) has exactly seven closed points.

From the discussion above, we have the following result.

**Proposition 3.3.** Let \( A \) and \( p_A \) be as above with \( a = b = c = d = 0. \) The following points are the set of closed points of \( p_A: \)

i. \( e_4 = (0,0,0,1) \)

ii. \( (\alpha, \ alpha^5 - 2alpha^2, 1, \frac{1}{2}(alpha^3 - 2 + \lambda)) \) where \( alpha^6 - 2alpha^3 - 1 = 0. \)

**Proof.** The proof follows from the preceding discussion.

**3.3. The Line Scheme.**

In this section, we compute the line scheme \( \mathcal{L} \) of the algebra \( A(0, 0, 0, 0, 2) \) which is in the family of algebras \( A. \) We will denote \( A(0, 0, 0, 0, 2) \) by \( B \) for simplicity. We summarize the method in Section 3.3.1 using the algebra \( A, \) and in Section 3.3.2 we compute the line scheme of \( B. \)

**3.3.1. Method to Compute the Line Scheme.**

We can compute the line scheme of any quadratic quantum \( \mathbb{P}^3 \) since the line scheme can be identified with a subscheme of \( \mathbb{P}^5. \) The method was introduced in a paper by Shelton and Vancliff [21] and we summarize that method using our algebra \( A. \)

In order to describe the line scheme, we must first construct the Koszul dual of \( A \) which we denote as \( A^! \). Since \( A \) has four generators and six defining relations, \( A^! \) is a quadratic algebra on four generators with ten defining relations. We take \( \{z_1, z_2, z_3, z_4\} \) as our dual
basis for $V^*$ of $V = \sum_{i=1}^4 k x_i$. It is straightforward to check that the defining relations of $A^1$ are:

$$
\begin{align*}
    z_4^2 &= 0, & z_2^2 + z_1 z_4 &= 0, \\
    z_4 z_3 + z_3 z_4 &= 0, & z_3 z_1 + z_1 z_3 - \lambda z_2^2 - cz_1^2 &= 0, \\
    z_4 z_2 - z_2 z_4 &= 0, & z_3 z_2 + z_2 z_3 - (\lambda - 2) z_1^2 - a z_2^2 &= 0, \\
    z_4 z_1 - z_1 z_4 &= 0, & z_2 z_1 - z_1 z_2 + z_3 z_4 &= 0, \\
    z_1^2 + z_2 z_4 &= 0, & z_3^2 + 2z_1 z_2 - z_3 z_4 - b z_2^2 - d z_1^2 &= 0.
\end{align*}
$$

Following our work with the point scheme, we write these defining relations in the form of a matrix equation, $M\hat{z}$, where $\hat{z} = [z_1, z_2, z_3, z_4]^T$ and $M$ is as follows:

$$
M = \begin{bmatrix}
0 & 0 & 0 & z_4 \\
0 & 0 & z_4 & z_3 \\
0 & z_4 & 0 & -z_2 \\
z_4 & 0 & 0 & -z_1 \\
z_1 & 0 & 0 & z_2 \\
0 & z_2 & 0 & z_1 \\
z_3 - c z_1 & -\lambda z_2 & z_1 & 0 \\
-(\lambda - 2) z_1 & z_3 - a z_2 & z_2 & 0 \\
z_2 & -z_1 & 0 & z_3 \\
-d z_1 & 2z_1 - b z_2 & z_3 & -z_3
\end{bmatrix}.
$$

We now produce a $10 \times 8$ matrix by concatenating two $10 \times 4$ matrices coming from $M$ as follows. We first replace each $z_i$ in $M$ with $u_i \in k$ and then do this again by replacing each $z_i$ with $v_i \in k$, where $k \sum_{i=1}^4 u_i x_i \in \mathbb{P}(V)$ and $k \sum_{i=1}^4 v_i x_i \in \mathbb{P}(V)$. The two matrices thus obtained are then concatenated and give us the following $10 \times 8$ matrix:
There are forty-five $8 \times 8$ minors of this matrix where each is a bi-homogeneous polynomial of bi-degree $(4, 4)$ in $u_i$ and $v_i$. Each minor is a linear combination of products of polynomials of the form $N_{ij} = u_i v_j - u_j v_i$ where $1 \leq i < j \leq 4$. We now use the following maps to convert to Plücker coordinates, $M_{ij}$ for $1 \leq i < j \leq 4$:

$N_{12} \mapsto M_{34}$, $N_{13} \mapsto -M_{24}$, $N_{14} \mapsto M_{23}$,

$N_{23} \mapsto M_{14}$, $N_{24} \mapsto -M_{13}$, $N_{34} \mapsto M_{12}$.

The line scheme of $A$ is determined from the zeros of the 45 polynomials in $M_{ij}$ and the Plücker polynomial,

$$\mathcal{P} = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}.$$  

The map $\{\text{lines in } \mathbb{P}^3\} \to \mathcal{V}(\mathcal{P})$ which sends a line in $\mathbb{P}^3$ to a point in $\mathcal{V}(\mathcal{P})$ represented by its Plücker coordinates is a bijection, hence we may associate lines in $\mathbb{P}^3$ to points on the quadric $\mathcal{V}(\mathcal{P}) \subset \mathbb{P}^5$. For our algebra $A$, the forty-five polynomials and $\mathcal{P}$ are provided in Section 5.2.

### 3.3.2. The Closed Points of the Line Scheme.

Using Mathematica, we compute a Gröbner basis of the forty-six polynomials in Section 5.2. Our initial computations yielded polynomials that were not user friendly. For this reason, we specialize to one particular algebra in the family, namely $B = A(0, 0, 0, 0, 2)$. 


Polynomial 5.2.2 in the Appendix has a factor of $M_{12}$. Setting $M_{12} = 0$ and computing a new Gröbner basis yields that $M_{23}^4$ and $M_{13}^4$ are elements in the ideal. It follows that $M_{23} = M_{13} = 0$. With this data we compute another Gröbner basis which contains the three polynomials:

\[ M_{34}^2(M_{14}M_{24} - M_{34}^2), \]
\[ M_{24}M_{34}(M_{14}M_{24} - M_{34}^2), \]
\[ M_{14}M_{34}(M_{14}M_{24} - M_{34}^2). \]

For all three of these equations to vanish, either $M_{34} = 0$ or $M_{14}M_{24} - M_{34}^2 = 0$. Therefore, the line variety of $B$ contains the following components:

\[ \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{34}), \]
\[ \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{14}M_{24} - M_{34}^2). \]

We now assume $M_{12} \neq 0$ by setting $M_{12} = 1$. We obtain the polynomials listed in Section 5.3 of the Appendix after we compute a new Gröbner basis (using degree reverse lexicographical ordering) and re-homogenize. It follows that the final component of the line variety is the zero locus of the polynomials in Section 5.3.

We now summarize our conclusions from this section in the following theorem.

**Theorem 3.4.** The line variety of $B$ is the union of the following varieties:

(I) $\mathcal{L}_1 = \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{34})$, which is a line.

(II) $\mathcal{L}_2 = \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{14}M_{24} - M_{34}^2)$, which is a conic.

(III) $\mathcal{L}_3$ which is the zero locus of the polynomials listed in Section 5.3 of the Appendix.

For simplicity, we present an affine open subset of this component by setting $M_{12} = 1$ which gives us the variety:

\[ \mathcal{V}(M_{12} - 1, \ 2M_{13}M_{23} + 1, \ 1 + 2M_{13}^3 + 2M_{23}^3 + 2M_{34}, \]
\[ 2M_{13}^4 + 2M_{14} - M_{23}^2, \ M_{13}^2 - 2M_{23} - 2M_{23}^4 + 2M_{24}). \]

**Proof.** The result follows from the above discussion. \[\blacksquare\]
3.3.3. Converting the Plücker coordinates to lines in $\mathbb{P}^3$.

We now describe the lines in $\mathbb{P}^3$ that are parametrized by $\mathfrak{L}$. If $l$ is any line in $\mathbb{P}^3$ and if $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$ are two distinct points that lie on $l$, we may represent $l$ as the following $2 \times 4$ matrix of rank two:

$$ l = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}. $$

Every point on $l$ may be represented as a linear combination of the rows of the above matrix in homogeneous coordinates. The Plücker coordinates, $M_{ij}$, are the $2 \times 2$ minors $a_i b_j - a_j b_i$ of the matrix $l$.

**Note 3.5.** The Plücker polynomial, $\mathcal{P} = M_{12} M_{34} - M_{13} M_{24} + M_{14} M_{23}$, vanishes on $l$ thus, as stated just prior to Section 3.3.2., we may associate lines in $\mathbb{P}^3$ to points on the quadric $\mathcal{V}(\mathcal{P}) \subset \mathbb{P}^5$ and $\mathcal{V}(\mathcal{P})$ parametrizes all lines in $\mathbb{P}^3$.

**Theorem 3.6.** The lines in $\mathbb{P}(V^*)$ that are parametrized by the line scheme are as follows.

1. From Theorem 3.4, the first component of our line variety is the line $\mathfrak{L}_1 = \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{34})$ in $\mathbb{P}^5$. The $2 \times 4$ rank-two matrices that correspond to points on this line are given by:

$$ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

where $(a_1, a_2) \in \mathbb{P}^1$. Hence, $\mathfrak{L}_1$ parametrizes all the lines in $\mathbb{P}^3$ that contain $e_4$ and lie on the plane $\mathcal{V}(x_3)$.

2. Our next component is $\mathfrak{L}_2 = \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{14} M_{24} - M_{34}^2)$ which is a conic in $\mathbb{P}^5$. The $2 \times 4$ rank-two matrices that correspond to points on this conic are given by:

$$ \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

where $(a_1, a_2, a_3) \in \mathbb{P}^2$ satisfies $a_1 a_2 - a_3^2 = 0$. It follows that $\mathfrak{L}_2$ parametrizes all the lines in $\mathbb{P}^3$ that contain $e_4$ and lie on the singular quadric $\mathcal{V}(x_1 x_2 - x_3^2)$. 

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(3) Our last component is \( \mathfrak{L}_3 \). The \( 2 \times 4 \) rank-two matrices that correspond to points in the affine open subset (where \( M_{12} \neq 0 \)) of this component are given by:

\[
\begin{bmatrix}
1 & 0 & a_3 & a_4 \\
0 & 1 & b_3 & b_4
\end{bmatrix},
\]

where \( a_i, b_i \in k \) and satisfy:

\[
\begin{align*}
2a_3b_3 - 1 &= 0, \\
2b_4 - a_3^2 + 2b_3^2 &= 0, \\
2a_4 + 2a_3^2 - 2a_3 - b_3^2 &= 0.
\end{align*}
\]

**Proof.** The proof of (1) and (2) follows directly from Theorem 3.4(I) and (II), so is omitted. The proof of (3) is similar to that of (1) and (2), but entails noting that the cubic polynomial in Theorem 3.4(III) is contained in the ideal generated by the other polynomials in Theorem 3.4(III) and the image of \( \mathcal{P} \).

3.3.4. Multiplicity of the Components of the Line Scheme.

Some quadratic quantum \( \mathbb{P}^3 \)s were considered in [5, 6, 8, 23, 24]. In those articles, the line variety and the line scheme were the same; in other words the scheme was a reduced scheme. Our algebra does not have a reduced line scheme. Therefore, to identify the line scheme, we must now determine the multiplicity of each component. In order to compute these multiplicities, we intersect each component of \( \mathfrak{L} \) with a generic hyperplane.

Let \( I(\mathcal{V}) \) be the ideal that is generated by the forty-six polynomials in Section 5.2 where \( a = b = c = d = 0 \) and \( \lambda = 2 \). Let \( K[\mathcal{V}] = k[M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34}] / I(\mathcal{V}) \) denote the coordinate ring of \( \mathfrak{L} \). By [20, Corollary 2.6] there are no embedded points in the line scheme, and so the multiplicity of each component of \( \mathfrak{L} \) equals the multiplicity of those points \( p \) on that component that do not lie on any other component. It follows that the multiplicity of the component is given by the dimension of the localized ring associated to such a point \( p \) that is the intersection of the component with a generic hyperplane. In particular, the point \( p \) should not be an intersection point of the components of \( \mathfrak{L} \). We choose \( g \in K[\mathcal{V}] \) where \( g \) is homogeneous of degree 1 with arbitrary coefficients such that
Using Mathematica we compute a Gröbner basis for the ideal generated by \( g \) and the forty-six polynomials in Section 5.2. Once we localize to \( p \), we then use the Affine package in Maxima to find the dimension of the localized ring. The Affine package, which was written by William Schelter, uses Bergman’s Diamond Lemma. We use this method in the proof of the next result.

**Theorem 3.7.** The multiplicity of each component of \( \mathfrak{L} \) is as follows.

i. The line, \( \mathfrak{L}_1 \), has multiplicity 4.

ii. The conic, \( \mathfrak{L}_2 \), has multiplicity 4.

iii. The component \( \mathfrak{L}_3 \) has multiplicity 1 and is an octic curve.

**Proof.** We illustrate the proof using the conic, \( \mathfrak{L}_2 \), as the proof for the other components is similar. Let \( \beta \in k^\times \). The point \( p_2 = (0, 0, 1, 0, \beta^2, \beta) \) lies on the conic and does not lie on any of the other components. Let \( h_2 \in K[V] \) where
\[
h_2 = \beta M_{14} - M_{24} + B_4 (\beta M_{14} - M_{34}) + B_1 M_{12} + B_2 M_{13} + B_3 M_{23},
\]
with \( B_i \in \kappa \) for all \( i \). We compute a Gröbner basis for the ideal generated by \( h_2 \) and the forty-six polynomials in Section 5.2. Next, we localize repeatedly until the remaining list of polynomials vanishes on \( p_2 \) and no other points on the line scheme. This process yields the list of polynomials in Section 5.4 of the Appendix. We obtain a localized ring isomorphic to a polynomial ring on three variables \( x, y, z \) modulo the relations in Section 5.4 after substituting \( M_{12} = x, M_{13} = y, \) and \( M_{14} = z \). Use of the Affine package in Maxima to compute the dimension of this localized ring yields that it is four. It follows that \( \mathfrak{L}_2 \) has multiplicity four. Since the conic has degree 2, the degree of \( \mathfrak{L}_2 \) as a subscheme of \( \mathfrak{L} \) is 8.

The hyperplane used for the line, \( \mathfrak{L}_1 \), is given by
\[
h_1 = \beta M_{14} - M_{24} + B_1 M_{12} + B_2 M_{13} + B_3 M_{23} + B_4 M_{34}
\]
and for \( \mathfrak{L}_3 \) is given by
\[
h_3 = M_{23} + M_{12} + B_1 (M_{13} - \frac{1}{2} M_{12}) + B_2 (M_{14} - \frac{7}{16} M_{12}) + B_3 (M_{24} + \frac{1}{8} M_{12}) + B_4 (M_{34} - \frac{3}{8} M_{12})
\]
with $B_i \in \mathbb{k}$ for all $i$ and $\beta \in \mathbb{k}^\times$. Repeating the above process for the third component yields a localized ring of dimension one, but since the hyperplane given by $h_3$ intersects $\mathfrak{L}_3$ at 8 distinct points, the degree of $\mathfrak{L}_3$ is 8. Thus, the third component is an octic curve in $\mathbb{P}^5$. The sum of the degrees of the components of $\mathfrak{L}$ is $4 + 8 + 8 = 20$, as expected by [9, Theorem 2.1].

3.3.5. The Intersection Points of the Line Scheme.

Now that we have computed the line variety, we consider points of intersection between the components of $\mathfrak{L}$. We use the following notation:

$$E_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}, \delta_{i4}, \delta_{i5}, \delta_{i6}),$$

where $\delta_{ij}$ is the Kronecker-delta function.

**Corollary 3.8.** The components of $\mathfrak{L}$ intersect at two distinct points; namely, $\mathfrak{L}_1 \cap \mathfrak{L}_2 = \mathfrak{L}_1 \cap \mathfrak{L}_3 = \mathfrak{L}_2 \cap \mathfrak{L}_3 = \{E_3, E_5\}$.

**Proof.** This follows from Theorem 3.4 as a direct computation using the polynomials given in the theorem.

We conclude this subsection with an observation about the potential number of intersection points of a line scheme of a generic quantum $\mathbb{P}^3$.

**Conjecture 3.9.** For a generic quantum $\mathbb{P}^3$, the number of intersection points of the components of the line scheme is even.

We base this conjecture on the results in this thesis and on some results in [5, 6, 8, 23, 24].

3.3.6. Conclusions on the Line Scheme of $B$.

In [5, 6, 23, 24], some algebras were considered for which it was found that the intersection points of the components of the respective line schemes of the algebras correspond to certain right ideals that have a larger than usual intersection with a certain normalizing sequence of length four of homogeneous degree-2 elements in the algebra. However, for our algebra,
B, we found only one normalizing sequence, \( \{ x_1, x_2 \} \), and it has length two. On the other hand, intersection points of the components of any scheme are simply points of increased multiplicity and the line scheme of \( B \) has many points of increased multiplicity, namely all the points on the line and the conic. Indeed, the line \( \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{34}) \subset \mathbb{P}^5 \) has a multiplicity of four and the conic \( \mathcal{V}(M_{12}, M_{13}, M_{23}, M_{14}M_{24} - M_{23}^2) \subset \mathbb{P}^5 \) has a multiplicity of four. These schemes parametrize the lines in \( \mathbb{P}^3 \) on the plane \( \mathcal{V}(x_3) \) that pass through \( e_4 \) and the lines in \( \mathbb{P}^3 \) on the singular quadric \( \mathcal{V}(x_1x_2 - x_3^2) \) that pass through \( e_4 \).

Let \( \mathcal{V} = \mathcal{V}(x_1x_2 - x_3^2) \cup \mathcal{V}(x_3) \subset \mathbb{P}^3 \). Notice that \( \mathcal{V}(x_1x_2 - x_3^2) \cap \mathcal{V}(x_3) = \mathcal{V}(x_1, x_3) \cup \mathcal{V}(x_2, x_3) \) (see Figure 1 below). These two lines correspond to the intersection points \( E_3 \) and \( E_5 \) of the components of the line scheme of \( B \). The subalgebra \( C \) of \( B \) that is generated by \( x_1, x_2, x_3 \) is the Ore extension \( C = S[x_2; \sigma_s, \delta_s] \), where \( \sigma_s \) and \( \delta_s \) are defined in the proof of Proposition 3.2. The point scheme of \( C \) is \( \mathcal{V}(x_1x_2 - x_3^2) \cup \mathcal{V}(x_3) \) in \( \mathbb{P}^2 \). So there exist infinitely many slices of \( \mathcal{V} \) where each slice is isomorphic to the point scheme of \( C \) (such as the slice given by \( \mathcal{V} \cap \mathcal{V}(x_4) \)). Therefore, it appears that the line scheme of \( B \) is identifying the subalgebra \( C \) of \( B \). In other words, we can view the subalgebra \( C \) of \( B \) as being information about the algebra \( B \) that is encoded in geometric data associated to \( B \), thereby fitting our underlying goal.
Figure 1. Depiction of the union of the plane $\mathcal{V}(x_3)$ and the singular quadric $\mathcal{V}(x_1 x_2 - x_3^2)$ in $\mathbb{P}^3$.
Central Elements in the Exotic Elliptic Algebra

Continuing our theme of finding what information the geometry tells us about the algebra, we examine the degree-two central elements in the exotic elliptic algebra. The algebra was first defined in [7, 11, 12], its point scheme was described in [8] and the line scheme was computed in [8, 23]. The algebra is defined as follows.

**Definition 4.1.** (cf. [7, 11, 12]) The Exotic Elliptic Algebra

Let \( A_{EE} \) denote the \( k \)-algebra on degree-one generators \( x_1, \ldots, x_4 \) with the following defining relations:

\[
\begin{align*}
    x_1x_2 - x_2x_1 &= \alpha(x_3x_4 - x_4x_3), & x_1x_2 + x_2x_1 &= x_3x_4 + x_4x_3, \\
    x_1x_3 - x_3x_1 &= \beta(x_4x_2 - x_2x_4), & x_1x_3 + x_3x_1 &= x_4x_2 + x_2x_4, \\
    x_1x_4 - x_4x_1 &= \gamma(x_2x_3 - x_3x_2), & x_1x_4 + x_4x_1 &= x_2x_3 + x_3x_2,
\end{align*}
\]

where \( \alpha, \beta, \gamma \in k^\times \) satisfy

\[
\{\alpha, \beta, \gamma\} \cap \{\pm 1\} = \emptyset \text{ and } \alpha + \beta + \gamma + \alpha\beta\gamma = 0.
\]

This algebra was called the exotic elliptic algebra in [7]. In [23, Theorem 4.2], it was shown that the line scheme of the exotic elliptic algebra has seven components: three quartic elliptic curves \( (E_1, E_2, E_3) \) and four nonsingular conics \( (\Lambda_1, \ldots, \Lambda_4) \). It was found in [23] that the components of the line scheme intersect at 24 distinct points and that \( E_i \cap E_j \) is empty for all \( i \neq j \) and \( \Lambda_i \cap \Lambda_j \) is empty for all \( i \neq j \). However, the reader should note that the statement in Corollary 4.4 in [23] that lists the intersection points contains typographical errors.

Our analysis concentrates on the algebra given by \( \alpha = -2, \beta = 3, \) and \( \gamma = 1/5 \) and we take \( k = \mathbb{C} \). Using the Affine package in Maxima, we find that \( A_{EE} \) contains a 2-dimensional
subspace of central homogeneous elements of degree two spanned by the central elements
\[ x_1^2 - x_2^2 - 4x_4^2 \text{ and } 2x_2^2 + x_3^2 + 5x_4^2. \]
We now examine a right ideal that corresponds to an intersection point and determine whether a central element belongs to the span of the degree-two elements of the right ideal. In the following result, we continue to order the Plücker coordinates: \( M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34} \). We also use the notation \( I_2 = (A_{EE})_2 \cap I \) for any right ideal \( I \) of \( A_{EE} \).

**Lemma 4.2.**

(a) For \( p \in E_i \cap \Lambda_j \), the right ideal of \( A_{EE} \) that determines the right line module corresponding to \( p \) contains a central homogeneous element of degree two.

(b) For \( p = (0, 0, d, 0, 0, 1) \in E_1 \setminus (\Lambda_1 \cup \ldots \cup \Lambda_4) \), where \( d^2 = 8/5 \), the right ideal of \( A_{EE} \) that determines the right line module corresponding to \( p \) contains a central homogeneous element of degree two.

(c) For \( p = (2i, -i, -1, i, -i) \in \Lambda_3 \setminus (E_1 \cup E_2 \cup E_3) \), where \( i^2 = -1 \), the right ideal of \( A_{EE} \) that determines the right line module corresponding to \( p \) contains a central homogeneous element of degree two.

**Proof.**

(a) By symmetry, it suffices to consider \( p = (0, c, \gamma, 1, c, 0) \in E_3 \cap \Lambda_1 \), where \( c^2 = \gamma = \frac{1}{5} \).

The \( 2 \times 4 \) matrix corresponding to such a point is
\[
\begin{bmatrix}
  c & 1 & 0 & 0 \\
  0 & 0 & 1 & c
\end{bmatrix},
\]
so the right ideal that corresponds to this point is \( I = (x_1 - cx_2)A_{EE} + (x_4 - cx_3)A_{EE} \) and \( I_2 \) is the span of the following degree-two elements:
\[
\begin{align*}
  &x_2x_1 - \sqrt{5}x_1^2, \quad x_2x_3, \quad x_2^2 - \sqrt{5}x_1x_2, \\
  &x_2x_4 - \sqrt{5}x_1x_4, \quad x_1x_3, \quad 2x_2^2 - 3x_3^2 + 5x_4^2, \\
  &5x_1^2 + 3x_2^2 - 4x_3^2.
\end{align*}
\]
Since
\[5x_1^2 - x_2^2 + 2x_3^2 - 10x_4^2 = (5x_1^2 + 3x_2^2 - 4x_3^2) - 2(2x_2^2 - 3x_3^2 + 5x_4^2) = 5(x_1^2 - x_2^2 - 4x_3^2) + 2(2x_2^2 + x_3^2 + 5x_4^2),\]
we see that \(I_2\) contains a central element.

(b) Next, we consider the point \(p = (0, 0, d, 0, 0, 1) \in E_1 \setminus (\Lambda_1 \cup \ldots \cup \Lambda_4)\), where \(d^2 = \frac{8}{5}\).

The \(2 \times 4\) matrix corresponding to such a point is
\[
\begin{bmatrix}
d & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

The right ideal that corresponds to this point is \(J = (x_1 - dx_3)A_{EE} + x_2A_{EE}\) and \(J_2\) is the span of the following degree-two elements:
\[
\begin{align*}
x_2x_1, & \quad x_2^2, \quad 2x_1x_2 - 5dx_1x_4, \quad x_1x_3 - dx_3^2, \\
x_2x_4, & \quad x_2x_3, \quad 2x_1^2 + dx_1x_3.
\end{align*}
\]

Since \(x_2^2, 2x_1^2 + dx_1x_3, x_1x_3 - dx_3^2\) are elements of \(J_2\), the element
\[
\frac{5}{2}(\frac{6}{5}x_2^2 + 2x_1^2 + dx_1x_3 - d(x_1x_3 - dx_3^2)) = \frac{5}{2}(2x_1^2 + \frac{6}{5}x_2^2 + \frac{8}{5}x_3^2)
\]
\[
= 5x_1^2 + 3x_2^2 + 4x_3^2
\]
\[
= 5(x_1^2 - x_2^2 - 4x_3^2) + 4(2x_2^2 + x_3^2 + 5x_4^2),
\]
of \(J_2\) is a central element.

(c) We now consider the point \(p = (2i, -i, -1, 1, i, -i) \in \Lambda_3 \setminus (E_1 \cup E_2 \cup E_3)\). The \(2 \times 4\) matrix corresponding to such a point is
\[
\begin{bmatrix}
1 & -i & i & 0 \\
1 & i & 0 & -1
\end{bmatrix}.
\]

It follows that the right ideal corresponding to this point is \(K = (ix_1 - x_2 - 2x_3)A_{EE} + (ix_1 + x_2 + 2ix_4)A_{EE}\) and \(K_2\) is the span of the following degree-two elements:
\[ix_1^2 + x_1x_3 - x_2x_1 - 3x_2x_4, \quad x_2^2 - 3x_2x_3 + 5x_1x_4 - ix_1x_2, \quad 2ix_4^2 + x_2x_4 + ix_1x_4,\]
\[x_2^2 + ix_2x_4 + ix_1x_3 + ix_1x_2, \quad 2x_2x_4 + 3x_2x_1 - 2ix_1x_4 + x_1x_2, \quad 2x_3^2 + x_2x_3 - ix_1x_3,\]
\[ix_1^2 + 3ix_1x_4 + x_2x_1 - ix_2x_3.\]

Transferring this data into a matrix and performing row operations using Mathematica, we find that \(x_1^2 - 2ix_2x_1, 3x_2^2 + 2ix_2x_1 - 2x_2x_3 + 6ix_2x_4, 2x_3^2 - 3ix_2x_1 + x_2x_3 - 3ix_2x_4\) and \(6x_4^2 - ix_2x_1 + x_2x_3 - 3ix_2x_4\) belong to \(K_2\), and thus \(x_1^2 - 2x_3^2 + 6x_4^2\) and \(x_2^2 + 4x_4^2\) are elements in \(K_2\). It follows that

\[
x_1^2 - 2x_3^2 + 6x_4^2 - 5(x_2^2 + 4x_4^2) = x_1^2 - 5x_2^2 - 2x_3^2 - 14x_4^2
\]
\[
= \frac{1}{5}(5x_1^2 + 3x_2^2 + 4x_4^2) - \frac{14}{5}(2x_2^2 + x_3^2 + 5x_4^2)
\]
\[
= (x_1^2 - x_2^2 - 4x_4^2) - 2(2x_2^2 + x_3^2 + 5x_4^2),
\]

which is, therefore, an element of \(K_2\) and also a central element.

We conjecture that the symmetry of \(A_{EE}\) allows the results of (b) (respectively, (c)) of Lemma 4.2 to apply to all points in \(E_i \setminus (\Lambda_1 \cup \ldots \cup \Lambda_4)\) for all \(i\) (respectively, all points in \(\Lambda_i \setminus (E_1 \cup E_2 \cup E_3)\) for all \(i\)). If this holds, then it follows that the annihilator of every line module of \(A_{EE}\) contains a homogeneous central element of degree two, in which case, it appears that the geometry is directing us to central elements in the algebra \(A_{EE}\). In other words, it seems as though the geometric data of the line scheme is encoding algebraic data of the algebra, thereby again fitting our underlying goal.
CHAPTER 5

Appendix
5.1. Point Scheme Polynomials for \( A \).
5.1.1 \( 2x_1x_3(x_1x_2 - x_3^2) \)
5.1.2 \( 2x_2x_3(x_1x_2 - x_3^2) \)
5.1.3 \( -2x_3^2(x_1x_2 - x_3^2) \)
5.1.4 \( -x_1^4 - x_1^2x_3 - cx_1^2x_3 + 2x_1^2x_2x_3 + ax_1x_2^2x_3 - dx_1^2x_3^2 + bx_1x_2x_3^2 + 2x_2^2x_3 - 2x_1^2x_2x_4 + 2x_1x_2^2x_4 \)
5.1.5 \( x_3^4x_2 + x_1^2x_2x_3 - x_1^2x_3^2 - ax_1x_2x_3^2 - bx_1x_3^3 - 2x_2x_3^3 + 2x_1^2x_3x_4 - \lambda x_1^2x_3^2 \)
5.1.6 \( x_3^4x_2 - x_1^2x_3 - cx_1^2x_3^2 + x_1x_2x_3^2 - dx_1x_3^3 + 2x_3^3x_4 - \lambda x_1x_2x_3^2 \)
5.1.7 \( -x_1^4x_2 - x_1^2x_3 - cx_1^2x_2x_3 - 2x_1^2x_2x_3 - ax_1x_2^2x_3 - dx_1^2x_2x_3 + bx_1x_2x_3^2 - 2x_2x_3^3 + 2x_1x_3x_4 + 2x_2x_3x_4 \)
5.1.8 \( x_3^4x_2 + x_3^4x_3 - x_1x_2x_3^2 + ax_2x_3^3 - bx_2x_3^3 - 2x_3^3x_4 + \lambda x_1x_2x_3^3 \)
5.1.9 \( x_3^4x_2 - x_1x_2x_3^2 + cx_1x_2x_3^3 - 3x_2x_3^3 + 2x_1x_3^3 + dx_2x_3^3 - 2x_2x_3x_4 + \lambda x_2x_3^2 \)
5.1.10 \( -cx_1^2x_2 + 2x_1^2x_3^2 + ax_1x_2x_3^3 - x_1x_2x_3 - ax_1x_2x_3^2 - dx_1^2x_2x_3 + bx_1x_2^2x_3 - cx_1x_2x_3^3 + 3x_2x_3 - bx_1^2x_3^2 - 2x_1x_2x_3^2 - dx_2x_3^3 + x_1x_2x_3^4 + x_3^3x_4 - cx_1x_2x_3^3 + 2x_1x_2x_3x_4 - ax_2x_3x_4 - dx_3^3x_4 - 2x_2x_3x_4^2 + 2x_3^3x_4^2 - \lambda x_1^3x_3 - \lambda x_2^3x_3 - 2x_1x_2x_3x_4 \)
5.1.11 \( x_3(x_1^3 - x_2^3 + cx_1^2x_3 - 2x_1x_2x_3 - ax_2^2x_3 + dx_1x_3^2 - bx_2x_3^2) \)
5.1.12 \( x_3(x_1^3 - x_2^3 + ax_2x_3^2 + bx_3^3 - 2x_1x_2x_3 + \lambda x_1x_3^2) \)
5.1.13 \( x_3(x_1^2 + x_2^2 + x_3^2 - 3x_2x_3^2 + dx_3^3 - 2x_2x_3x_4 + \lambda x_2x_3^3) \)
5.1.14 \( x_3(x_1^3 + x_2^3 - dx_2x_3^3 + x_1x_3^2 + ax_2x_3^2 - bx_2^3 + 2x_1x_2x_3^2 - cx_1x_2x_3^3 + 3x_2x_3^2 - dx_2x_3^3 - x_1^2x_2x_4 + x_2^3x_4 + x_1x_3^4 - ax_2x_3^4 - bx_2^2x_4 + 2x_1x_3x_4 - \lambda x_2^2x_3^2 - \lambda x_1x_3^2x_4 \)
5.1.15 \( -x_3^4x_2 - x_1^2x_3^3 + x_2^3x_3 - ax_1x_2x_3^2 - bx_1^3 - x_1x_2x_4 + x_1^2x_3x_4 - cx_1x_3^3x_4 + 3x_2x_3^2x_4 - dx_3^3x_4 - 2x_2x_3x_4^2 - \lambda x_1^3x_3^2 - 2x_2x_3x_4^2) \)
5.2. Line Scheme Polynomials for \( A \).
5.2.1 \( P = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23} \)
5.2.2 \( -\frac{1}{2}M_{12}(-M_{13}^3 + M_{12}M_{13}M_2 - 2M_{23}^3 + 2M_{13}M_{23}M_{34}) \)
5.2.3 \( -\frac{1}{2}M_{13}(M_{12}M_{13}^2 - dM_{13}^3 + 2M_{12}M_{13}M_{14} + M_{12}M_{23} + aM_{12}M_{13}M_{23} + bM_{13}M_{23} + 2M_{13}M_{23}^2 - 2M_{13}^2M_{34} - \lambda M_{12}M_{13}^2) \)
5.2.4 \( \frac{1}{2}M_{23}(M_{12}M_{13}^2 - dM_{13}^3 + 2M_{12}M_{13}M_{14} + M_{12}M_{23} + aM_{12}M_{13}M_{23} + bM_{13}M_{23} + 2M_{13}M_{23}^2 - 2M_{13}^2M_{34} - \lambda M_{12}M_{13}^2) \)
5.2.5 $\frac{1}{2}(-M_{13}^2 + M_{13}^2 M_{14} + 2M_{12} M_{13}^2 M_{23} - dM_{13}^2 M_{23} + M_{12} M_{13} M_{14} M_{23} + M_{12}^2 M_{23}^2 + aM_{12} M_{13} M_{23}^2 + bM_{12} M_{13} M_{23}^2 + M_{13} M_{23}^2 + M_{14} M_{23}^2 - 2M_{13} M_{14} M_{23} M_{34} - \lambda M_{12} M_{13} M_{23})$  
5.2.6 $\frac{1}{2}(M_{12} M_{13}^2 - M_{12}^2 M_{14}^2 - 3M_{12} M_{13} M_{23} - aM_{12} M_{13} M_{14} M_{23} + M_{12}^2 M_{13}^2 M_{23} - bM_{12} M_{13}^2 M_{23} + bM_{12} M_{13} M_{23}^2 + 2M_{12} M_{13} M_{23} M_{34} + dM_{12} M_{13}^2 M_{34} + aM_{12} M_{13} M_{34} + 2M_{12} M_{23} M_{34} + \lambda M_{12} M_{13} M_{23} - \lambda M_{12} M_{13} M_{34} + \lambda M_{12} M_{13} M_{23} M_{34} )$  
5.2.7 $\frac{1}{2}(M_{12} M_{13}^2 - cM_{12} M_{13}^2 + M_{12} M_{13}^2 M_{23} + dM_{13}^3 M_{23} - M_{12} M_{13} M_{14} M_{23} - M_{12}^2 M_{13}^2 M_{23} - bM_{13} M_{23}^2 - 2M_{13} M_{23} M_{24} + M_{12} M_{13} M_{23} M_{24} + M_{12} M_{13}^2 M_{23})$  
5.2.8 $-\frac{1}{2}M_{12}(M_{12} M_{13} - cM_{12} M_{13} M_{23} + 2M_{12} M_{23} - M_{12} M_{23}^2 + dM_{13} M_{23}^2 - bM_{23} - 2M_{12} M_{23} M_{24} - 2M_{23} M_{34} + \lambda M_{12} M_{23}^2 )$  
5.2.9 $\frac{1}{2}(-M_{12} M_{13}^2 + cM_{12} M_{13}^2 M_{23} - M_{12} M_{13}^2 M_{23} - dM_{12} M_{13}^2 M_{23} + bM_{13} M_{23}^2 + M_{13} M_{23}^2 + M_{13} M_{23}^2 + M_{12} M_{13} M_{23} + M_{23} M_{24} - 2M_{13} M_{23} M_{24} M_{34} - \lambda M_{12} M_{13} M_{23})$  
5.2.10 $\frac{1}{2}M_{13}(M_{12} M_{13} - cM_{12} M_{13} M_{23} + 2M_{12} M_{23} - M_{12} M_{23}^2 + dM_{13} M_{23}^2 - bM_{23} - 2M_{12} M_{23} M_{24} - 2M_{23} M_{34} + \lambda M_{12} M_{23}^2 )$  
5.2.11 $\frac{1}{2}(M_{12} M_{13}^2 - cM_{12} M_{13}^2 M_{23} + M_{12} M_{13} M_{23}^2 + dM_{13} M_{23}^2 - bM_{13} M_{23}^2 - 2M_{23} - 2M_{12} M_{13} M_{23} M_{24} + 2M_{13} M_{23} M_{34} + M_{12} M_{13} M_{23})$  
5.2.12 $M_{12} M_{13}^2 - cM_{13} M_{23} + 2M_{13} M_{23}^2 + M_{13} M_{14} M_{23} + M_{12} M_{23}^2 + aM_{13} M_{23}^2 - M_{13} M_{23} M_{24} - M_{12} M_{13} M_{23} M_{34}$  
5.2.13 $\frac{1}{2}M_{23}(M_{12} M_{13} - cM_{12} M_{13} M_{23} + 2M_{12} M_{23} - M_{12} M_{23}^2 + bM_{23} + M_{13} M_{23} M_{34} + \lambda M_{12} M_{23}^2 )$  
5.2.14 $\frac{1}{2}(-M_{12} M_{13} + M_{12} M_{13}^2 M_{23} - 2M_{12} M_{13} M_{23} - M_{12} M_{13} M_{23}^2 + 2cM_{12} M_{23}^2 + bM_{12} M_{23}^2 - 4M_{13} M_{23}^2 - 2aM_{13}^2 + 2M_{12} M_{23} M_{24} + 2M_{12} M_{23} M_{34} - \lambda M_{12} M_{13} M_{23})$  
5.2.15 $\frac{1}{2}(M_{12} M_{13}^2 M_{14} + M_{12} M_{13} M_{23} + bM_{13} M_{23}^2 - cM_{12} M_{13} M_{23}^2 + 2M_{12} M_{23}^2 - M_{12} M_{23} M_{24} - M_{12} M_{23} M_{34} + M_{12} M_{13} M_{23} M_{34} + dM_{23} M_{34} - bM_{13} M_{23} M_{34} - 2M_{13} M_{23} M_{34}^2 - \lambda M_{12} M_{13} M_{23} + \lambda M_{12} M_{13} M_{23} M_{34} )$
5.2.16 \( \frac{1}{2} (M_{12}^3 M_{13}^4 - cM_{12}^2 M_{13} M_{32} + M_{12} M_{13}^2 M_{23} - M_{12} M_{13} M_{14} M_{23} - M_{12} M_{13}^2 M_{24} + dM_{12} M_{13} M_{24} - bM_{12}^2 M_{23} + cM_{12} M_{23} - 2M_{13} M_{23} - dM_{13}^2 - M_{12} M_{13}^2 M_{24} - cM_{12} M_{13} M_{23} M_{34} + M_{12} M_{23}^2 M_{34} + dM_{13} M_{23} M_{34} + bM_{23}^3 + 2M_{12} M_{23} M_{24} M_{34} + 2M_{23}^2 M_{34} + \lambda M_{12} M_{23}^2 M_{34} - \lambda M_{12} M_{23} M_{34} + bM_{23}^3 + 2M_{12} M_{23} M_{24} M_{34} + 2M_{23}^2 M_{34} + \lambda M_{12} M_{23}^2 M_{34} - \lambda M_{12} M_{23} M_{34}) \)

5.2.17 \( \frac{1}{2} (M_{12}^3 M_{13} M_{14} - cM_{12} M_{13} M_{14} M_{23} + dM_{13} M_{14} M_{23} - M_{12} M_{13} M_{24} + bM_{12} M_{13} M_{24} - M_{12} M_{13}^2 M_{24} - M_{12}^2 M_{24}^2 + M_{12} M_{13} M_{24} + M_{13} M_{14} M_{24} + bM_{13} M_{24} - cM_{12} M_{24}^3 + 2M_{13} M_{24}^3 + M_{14} M_{24}^3 + dM_{24}^3 - M_{13} M_{24} M_{34} + M_{14} M_{24} M_{34} + M_{13} M_{24} M_{34} + cM_{12} M_{13} M_{24} - dM_{13} M_{24} M_{34} - bM_{24}^3 M_{34} - M_{12} M_{24} M_{34} - 2M_{24} M_{34} + \lambda M_{12} M_{13} M_{24} - \lambda M_{12} M_{23} M_{34} + \lambda M_{12} M_{23} M_{34}) \)

5.2.18 \( \frac{1}{2} (M_{12}^3 M_{13} - cM_{12}^2 M_{13} M_{23} + 2M_{12} M_{13} M_{23} + M_{12} M_{13} M_{14} M_{23} + dM_{12} M_{13} M_{24} - cM_{13}^2 M_{23} - cM_{13} M_{14} M_{23} - cM_{13}^2 M_{23}^2 + bM_{12} M_{33}^2 + 2M_{13} M_{23} + aM_{23}^4 - M_{12} M_{13}^2 M_{24} - M_{12} M_{23}^2 M_{24} - M_{12} M_{23} M_{24} + aM_{13} M_{23} M_{24} - 2M_{12}^2 M_{23} M_{34} - M_{12}^2 M_{23} M_{34} - dM_{13} M_{23} M_{34} + bM_{23} M_{24} + M_{12} M_{24} M_{34} + 2M_{23} M_{24} + \lambda M_{12} M_{23} - \lambda M_{13} M_{23} M_{34}) \)

5.2.19 \( \frac{1}{2} (M_{12}^3 M_{13} + M_{12} M_{13} M_{14} - cM_{13}^2 M_{23} - cM_{13}^2 M_{23}^2 + M_{12} M_{13}^2 M_{23} + aM_{13} M_{23} M_{24} + M_{12} M_{23}^2 M_{24} + aM_{13} M_{23} M_{24} - 2M_{13} M_{23}^2 - dM_{13} M_{23} M_{34} + bM_{13} M_{23} M_{34} + 2M_{13} M_{23} M_{34}^2 + \lambda M_{13} M_{14} M_{23} - \lambda M_{13} M_{23} M_{34}) \)

5.2.20 \( \frac{1}{2} (-M_{12}^3 M_{13} + M_{12} M_{13} M_{14} + cM_{13}^2 M_{23} - cM_{13}^2 M_{23}^2 - M_{12} M_{13}^2 M_{23} - aM_{13} M_{23}^2 + 3M_{13} M_{23} M_{24} + M_{12} M_{23}^2 M_{24} + aM_{13} M_{23} M_{24} - dM_{13} M_{23} M_{34} + bM_{13} M_{23} M_{34} + 2M_{13} M_{23} M_{34} - bM_{13} M_{23} M_{34} + \lambda M_{13} M_{14} M_{23} - \lambda M_{13} M_{23} M_{34}) \)

5.2.21 \( -(\frac{1}{2} M_{12} M_{13}^2 - dM_{13}^2 + 2M_{12} M_{13} M_{14} + M_{12} M_{23} + aM_{12} M_{13} M_{23} + bM_{13} M_{23} + 2M_{13} M_{23}^2 - 2M_{13}^2 M_{34} - \lambda M_{12} M_{13}^2) \)

5.2.22 \( \frac{1}{2} (-3M_{12} M_{13}^2 + dM_{12} M_{13}^3 + 2cM_{13}^4 - 2M_{12}^2 M_{13} M_{14} - M_{12}^3 M_{23} - aM_{12} M_{13} M_{23} - bM_{12} M_{13}^2 M_{23} - 4M_{12} M_{13} M_{23} - 2aM_{12} M_{13} M_{23} + 2M_{12} M_{13} M_{23} + \lambda M_{12} M_{13} M_{23} \)

5.2.23 \( \frac{1}{2} (-2M_{12} M_{13}^3 + dM_{12} M_{13}^3 + cM_{13}^4 - M_{12} M_{13} M_{14} - M_{12} M_{13}^2 M_{23} - bM_{12} M_{13} M_{23} - 2M_{13} M_{23} - M_{13} M_{23} M_{24} - 2M_{12} M_{13} M_{23} - aM_{12} M_{13} M_{23} + 2M_{13} M_{24} + M_{13} M_{14} M_{24} + M_{12} M_{13} M_{23} M_{24} + aM_{12} M_{13} M_{23} + M_{12} M_{13} M_{23} - dM_{13} M_{24} + M_{12} M_{13} M_{24} + bM_{13} M_{23} M_{34} + 2M_{13} M_{23} M_{34} - 2M_{13} M_{23} M_{34} + \lambda M_{12} M_{13}^2 + \lambda M_{12} M_{13}^2 + \lambda M_{13} M_{14} M_{23} - \lambda M_{13} M_{23} M_{24} \)

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\[5.2.24 \quad \frac{1}{2}(2M_{13}^4 - 3M_{12}M_{13}^2M_{23} + dM_{13}^3M_{23} - 2M_{12}M_{13}M_{14}M_{23} - M_{12}^2M_{23}^2 - aM_{12}M_{13}M_{23}^2 - bM_{12}^2M_{23}^2 - 2M_{12}^2M_{23}M_{34} + \lambda M_{12}M_{13}M_{23})
\]
\[5.2.25 \quad \frac{1}{2}(-M_{12}^2M_{13}^2 + M_{12}^3M_{23} - 2M_{12}^3M_{23} - 3M_{12}M_{13}M_{23}^2 - dM_{13}^2M_{23}^2 + M_{12}M_{14}M_{23}^2 + aM_{12}M_{23}^3 + bM_{12}M_{23}^3 + M_{12}M_{13}M_{23}^2 + M_{23}^2M_{24} + M_{13}^2M_{23}M_{34} + 2M_{13}^2M_{23}M_{34} - \lambda M_{12}M_{13}M_{23})
\]
\[5.2.26 \quad \frac{1}{2}(-M_{12}^2M_{13}^2 - aM_{12}M_{12}^2M_{13}M_{23}^2 + bM_{12}^3M_{23} + aM_{12}M_{13}M_{23}^2 - 2M_{12}^3M_{23} + 3M_{12}M_{13}M_{23}^2 - dM_{13}^2M_{23}^2 + M_{12}M_{14}M_{23}^2 + M_{12}M_{13}M_{23}M_{24} - M_{12}^2M_{13}M_{34} + dM_{12}^2M_{13}M_{34} - 2M_{12}M_{13}M_{14}M_{34} + aM_{12}M_{13}M_{23}M_{34} + bM_{13}^2M_{23}M_{34} + 2M_{13}^2M_{34} - \lambda M_{12}M_{13}M_{23}^2 + \lambda M_{12}M_{13}M_{34})
\]
\[5.2.27 \quad \frac{1}{2}(-M_{12}^2M_{12}M_{13}^2 - 4M_{12}^2M_{13}M_{23} + 2cM_{13}^3M_{23} - aM_{12}M_{13}M_{23}^2 - 4M_{13}^2M_{23}^2 + 2M_{12}M_{14}M_{23}^2 - 2aM_{12}M_{13}M_{23}^2 - 2M_{12}M_{23}M_{24} + 2M_{12}M_{13}M_{23}M_{34})
\]
\[5.2.28 \quad \frac{1}{2}(M_{13}^3M_{14} - M_{13}^4M_{23}^2 - 2M_{12}M_{13}M_{14}M_{23} + dM_{12}M_{14}M_{23} - M_{12}^2M_{14}M_{23} - M_{12}^2M_{13}M_{24} + aM_{12}M_{13}M_{23}M_{24} - bM_{13}M_{23}M_{24} - M_{13}^2M_{23}M_{24} - M_{14}^2M_{23}M_{24} + M_{13}^2M_{14}M_{23}M_{24} + aM_{12}M_{13}M_{23}M_{34} + bM_{13}M_{23}M_{34} + 2M_{13}^2M_{23}M_{34} + M_{13}M_{14}M_{23}M_{34} + M_{14}^2M_{23}M_{34} - \lambda M_{12}M_{13}M_{14}M_{23}^2 + \lambda M_{12}M_{13}M_{23}M_{34} - \lambda M_{12}M_{13}M_{13}M_{34}^2)
\]
\[5.2.29 \quad \frac{1}{2}(-M_{12}^2M_{13}^3 - aM_{12}^2M_{13}^3 + 2M_{13}^3M_{14} + cM_{12}^2M_{13}M_{23} - 2M_{12}M_{13}^2M_{23} + acM_{12}M_{13}M_{23} - 2aM_{12}M_{13}M_{23} + bcM_{13}M_{23} + M_{12}^2M_{23}^2 + aM_{12}M_{13}M_{23}^2 - 2bM_{13}M_{23}^2 - adM_{13}^2M_{23}^2 - bM_{13}M_{14}M_{23}^2 - 3M_{13}^2M_{14}M_{23}^2 + dM_{13}^2M_{14}M_{23}^2 - 2M_{12}M_{13}M_{14}M_{24} + M_{12}M_{23}M_{24} + aM_{12}M_{13}M_{23}M_{24} + 2cM_{13}^2M_{34} + bM_{12}M_{13}M_{23}M_{34} - 4M_{13}^2M_{23}M_{34} - 2M_{12}M_{14}M_{23}M_{34} + 2M_{12}M_{13}M_{23}M_{34}^2 - 2\lambda M_{13}^2M_{23}M_{34} - 2\lambda M_{12}M_{13}M_{23} + 3\lambda M_{12}M_{13}^2M_{23}^2 + 3\lambda M_{12}M_{13}M_{23}^2 + d\lambda M_{13}^2M_{23}^2 + 2\lambda M_{12}M_{13}M_{14}M_{23} + b\lambda M_{12}^2M_{23}^2 + a\lambda M_{12}M_{13}M_{23}^2 + 2\lambda M_{13}^2M_{23}M_{34} - \lambda M_{12}M_{13}^2M_{23})
5.2.30 \( \frac{1}{2} (2M_{12}^3 M_{13} - dM_{12}^2 M_{23}^2 - cM_{12} M_{13}^3 + M_{12}^3 M_{14} + cM_{12} M_{13}^2 M_{14} + cM_{12} M_{13} M_{23} +
2M_{12} M_{13}^2 M_{23} + ac M_{12} M_{13}^2 M_{23} + bc M_{13}^3 M_{23} - M_{12} M_{13} M_{14} M_{23} + dM_{13}^3 M_{14} M_{23} -
M_{12} M_{13}^4 M_{23} - 2M_{12}^2 M_{23}^2 - 2b M_{13}^2 M_{23}^2 + 2c M_{13}^2 M_{23}^2 - ad M_{13} M_{14} M_{23} -
4M_{13}^3 M_{23}^2 - 2M_{12} M_{13}^2 M_{24} - M_{12} M_{13} M_{14} M_{24} - M_{12}^2 M_{23} M_{24} - 2M_{13} M_{14} M_{23} M_{24} - M_{12}^2 M_{13} M_{34} +
dM_{12} M_{13}^2 M_{34} - M_{12}^2 M_{13} M_{34} - aM_{12} M_{13} M_{23} M_{34} - bM_{13} M_{13} M_{23} M_{34} + 2M_{13} M_{14} M_{23} M_{34} -
2aM_{13} M_{23} M_{34} + 2\lambda M_{12} M_{13} M_{14}^2 M_{23} - M_{12}^3 M_{13}^2 M_{23} + \lambda M_{12} M_{13}^2 M_{14} M_{23} - \lambda^2 M_{12} M_{13}^2 M_{23})

5.2.31 \( \frac{1}{2} M_{23} (M_{12}^2 + a M_{12} M_{13} + b M_{12} M_{13}^2 - M_{13}^2 M_{14} + M_{13} M_{14}^2 - cM_{12} M_{23} + 2M_{12} M_{13} M_{23} -
ac M_{12} M_{13} M_{23} + a M_{12} M_{13}^2 M_{23} + b M_{12} M_{13} M_{14} M_{23} + dM_{12} M_{13}^2 M_{23} + 2b M_{13} M_{23} - cM_{13} M_{23}^2 +
ad M_{13} M_{23}^2 + b M_{14} M_{23}^2 - c M_{14} M_{23}^2 + 2M_{13}^2 M_{23} + M_{12} M_{13} M_{24} - aM_{12} M_{14} M_{24} - dM_{13} M_{24} +
M_{12} M_{14} M_{24} + 3M_{23} M_{24} + M_{23} M_{14} - b M_{13} M_{34} - c M_{13} M_{34} + c M_{13} M_{14} M_{34} + M_{14} M_{23} M_{34} +
aM_{23} M_{34} - M_{23} M_{34} - 3M_{14} M_{24} M_{34} - 2M_{14} M_{24} M_{34} + aM_{23} M_{24} M_{34} + dM_{13} M_{34}^2 +
bM_{23} M_{34} + 2\lambda M_{14}^2 M_{13} M_{14} - \lambda^2 M_{12} M_{13} M_{23} + dM_{13}^2 M_{14} M_{23} - \lambda M_{12} M_{13} M_{14} M_{23} -
b M_{13} M_{12} M_{13} M_{23} - \lambda M_{12} M_{13} M_{23} M_{34} + M_{14} M_{23} M_{34} + \lambda M_{13} M_{24} M_{34} + \lambda^2 M_{12} M_{13} M_{23})

5.2.32 \( \frac{1}{2} (-M_{12}^2 M_{13} M_{14} + a M_{12} M_{13} M_{23} - 2M_{12} M_{13} M_{23} + b M_{12} M_{23}^2 - a M_{12} M_{13} M_{23} +
2a M_{13} M_{23}^2 - b M_{13}^2 M_{23}^2 - a M_{12} M_{13}^2 M_{23} + 2b M_{13} M_{23}^2 + a M_{12} M_{13}^2 M_{24} +
3 M_{12} M_{13} M_{23} M_{24} - d M_{12}^2 M_{23} M_{24} + M_{12} M_{13} M_{23} M_{24} - a M_{12} M_{23} M_{24} + b M_{13} M_{23}^2 M_{24} +
M_{12} M_{13} M_{24}^2 - M_{12} M_{13} M_{34} + M_{12} M_{23} M_{34} + d M_{12} M_{13} M_{34}^2 - 2 M_{13} M_{23} M_{34} -
b M_{12} M_{23}^2 M_{34} + 4 M_{13} M_{23} M_{34}^2 - M_{12} M_{24} M_{34} + 2 M_{13} M_{23} M_{24} M_{34} - 2 M_{12} M_{23} M_{34} +
M_{12}^2 M_{13}^2 M_{23} - \lambda M_{12}^2 M_{13}^2 M_{23} + 2 M_{13} M_{23}^2 M_{23} - 3 M_{12} M_{13} M_{23}^2 + d M_{13} M_{13}^2 M_{23} - \lambda M_{12} M_{14} M_{23}^2 -
b M_{13} M_{12} M_{13} M_{23} - 2 M_{12} M_{13} M_{23} M_{34} - 2 M_{13} M_{23} M_{34}^2 + \lambda^2 M_{12} M_{13} M_{23})

\[\begin{array}{c}
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\end{array}\]
5.2.33 \[ \frac{1}{2} (\lambda M_{14} - M_{12} M_{23} - 2M_{12} M_{13} M_{23} - aM_{12} M_{23}^2 - dM_{12} M_{23}^2 + cM_{12} M_{13} M_{23}^2 + cM_{12} M_{14} M_{23}^2 + 2M_{13} M_{14} M_{23}^2 - 2M_{12} M_{23}^3 + acM_{12} M_{23} - 2aM_{13} M_{23}^3 + bcM_{13} M_{23}^2 - 2bM_{23}^4 - adM_{23}^4 + M_{12} M_{13} M_{24} - 5M_{12} M_{23} M_{24} + dM_{13} M_{23} M_{24} - bM_{23} M_{24} - 2M_{12} M_{23} M_{24}^{2} + cM_{12} M_{23} M_{34} + 2M_{12} M_{13} M_{23} M_{34} + 2M_{13} M_{23} M_{34} - 4M_{13} M_{23} M_{34} - 2M_{23} M_{24} M_{34} - \lambda M_{12} M_{13} M_{23} - 2 \lambda M_{12} M_{23}^3 + 3 \lambda M_{12} M_{23}^3 - d \lambda M_{13} M_{23}^3 + b \lambda M_{13}^2 M_{23}^3 + 3 \lambda M_{12} M_{23}^2 M_{24} + 2 \lambda M_{23} M_{34} - \lambda^2 M_{12} M_{23}^2) \]

5.2.34 \[-(\frac{1}{2}) M_{24} (M_{12}^3 + aM_{12} M_{13} + bM_{12} M_{13}^2 - M_{12} M_{13} M_{14} + M_{13} M_{14}^2 - cM_{12} M_{23} + 2M_{12} M_{13} M_{23} - acM_{12} M_{13} M_{23} + aM_{12} M_{13} M_{23} - bcM_{12} M_{23} + M_{12} M_{14} M_{23} + dM_{12} M_{23}^2 + bM_{13} M_{23} - cM_{13} M_{23}^2 + bM_{14} M_{23}^2 + dM_{23} M_{24} + M_{12} M_{12} M_{24} + 3M_{23} M_{24} + M_{23} M_{24} - bM_{13} M_{23} - cM_{13} M_{23} + cM_{14} M_{23} - 2M_{12} M_{23} M_{34} + aM_{13} M_{14} M_{34} + M_{12} M_{23} M_{34} + aM_{23} M_{24} M_{34} + dM_{13} M_{34} + bM_{23} M_{34} + 2M_{34} + \lambda M_{13} - \lambda M_{13} M_{14} - 2 \lambda M_{12} M_{13} M_{23} + d \lambda M_{13} M_{23} - \lambda M_{12} M_{13} M_{23} - bM_{12} M_{23} - \lambda M_{23} M_{24} + \lambda M_{12} M_{24} + \lambda M_{14} M_{23} M_{34} + \lambda M_{13} M_{24} M_{34} + \lambda^2 M_{12} M_{13} M_{23} \]

5.2.35 \[\frac{1}{2} (M_{12} M_{13} - M_{12} M_{13} - cM_{12} M_{14} M_{23} + cM_{13} M_{14} M_{23} + 2M_{13} M_{14} M_{23}^2 + M_{14} M_{23}^2 - 2M_{13} M_{14} M_{23} M_{24} - M_{14} M_{23} M_{24} + aM_{13} M_{23} M_{24} - M_{13} M_{23} M_{24} - aM_{13} M_{23} M_{24} M_{34} + M_{13} M_{23} M_{34} + bM_{12} M_{23} M_{34} - cM_{12} M_{23} M_{34} + \lambda M_{12} M_{13} M_{23} M_{34} + dM_{12} M_{23} M_{34} + bM_{12} M_{23} M_{34} + 2M_{12} M_{23} M_{34} + 2M_{12} M_{23} M_{34} - \lambda M_{13} M_{13} M_{23} + \lambda M_{12} M_{13} M_{23} - \lambda M_{12} M_{13} M_{23} - bM_{12} M_{23} M_{24} M_{34} + dM_{13} M_{13} M_{23} M_{34} - \lambda M_{12} M_{13} M_{23} M_{34} + 2M_{14} M_{23} M_{34} + 2M_{14} M_{23} M_{34} + aM_{13} M_{23} M_{34} + bM_{12} M_{23} M_{34} + dM_{13} M_{13} M_{23} M_{34} - \lambda M_{12} M_{13} M_{23} M_{34} + 2 \lambda M_{13} M_{23} M_{34} + \lambda^2 M_{12} M_{13} M_{23} M_{34} \]
5.2.36 \( \frac{1}{2} M_{34}(M_{12}^2 + aM_{12}^2 M_{13} + bM_{12} M_{13}^2 - M_{13} M_{14} + M_{13} M_{14}^2 - cM_{12}^2 M_{23} + 2M_{12} M_{13} M_{23} - \\
acM_{12} M_{13} M_{23} + aM_{12}^2 M_{13} - bcM_{13}^2 M_{23} + dM_{12} M_{13} M_{23} + bM_{13} M_{23} - cM_{13} M_{23}^2 + \\
adM_{13} M_{23}^2 + bM_{14} M_{23}^2 - cM_{14} M_{23}^2 + 2M_{23}^2 + M_{12} M_{13} M_{24} - aM_{13}^2 M_{24} - dM_{13}^2 M_{24} + \\
M_{12} M_{14} M_{23} + 3M_{23}^2 M_{24} - M_{23}^2 M_{34} + cM_{13} M_{14} M_{34} + M_{14} M_{23} M_{34} + \\
aM_{23}^2 M_{34} - dM_{13}^2 M_{34} - 3M_{13} M_{24} M_{34} - 2M_{14} M_{24} M_{34} + aM_{23} M_{24} M_{34} + dM_{13} M_{34}^2 + \\
bM_{23} M_{34}^2 + 2\lambda M_{34}^3 + M_{13}^2 - \lambda M_{13} M_{14} - 2\lambda M_{12} M_{13} M_{23} + d\lambda M_{13}^2 M_{23} - \lambda M_{12} M_{14} M_{23} - \\
\lambda M_{13}^2 M_{23} - \lambda M_{12} M_{13} M_{24} - \lambda M_{13}^2 M_{24} + \lambda M_{14} M_{23} M_{34} + \lambda M_{13} M_{24} M_{34} + \\
\lambda^2 M_{12} M_{13} M_{23}) \\
5.2.37 \( \frac{1}{2}(-M_{12} M_{13} M_{14} - M_{12} M_{13} M_{14} - cM_{13} M_{14} M_{23} + 2M_{13} M_{14} M_{23}^2 - M_{14} M_{23}^2 - \\
2M_{13} M_{14} M_{23} M_{24} - M_{14}^2 M_{23} M_{24} - M_{12} M_{13}^2 M_{24} - aM_{13} M_{14}^2 M_{24} + M_{13} M_{24} M_{24} - \\
M_{12} M_{13} M_{24}^2 - aM_{13} M_{23} M_{24} + M_{13} M_{24} + M_{14} M_{23} M_{24} - \\
M_{12} M_{13} M_{23} M_{34} - aM_{12} M_{13} M_{23} M_{34} + 2aM_{13} M_{23} M_{34} - bM_{13} M_{23} M_{34} + \\
dM_{12} M_{13} M_{23} M_{34} + dM_{12} M_{23} M_{34}^2 + 2M_{13} M_{23} M_{34} + 2M_{14} M_{23} M_{34} + \\
2M_{12} M_{13} M_{24} M_{34} - dM_{13} M_{24} M_{34} - M_{12} M_{14} M_{24} M_{34} - bM_{13} M_{23} M_{24} M_{34} - 2cM_{13}^2 M_{34} + \\
4M_{13} M_{23} M_{34}^2 + 2M_{12} M_{23} M_{34} - 2M_{13} M_{24} M_{34} + \lambda M_{13} M_{14} M_{23} - \lambda M_{13} M_{23} M_{24} + \\
\lambda M_{13} M_{23} M_{34} - 2\lambda M_{13} M_{13} M_{23} M_{34} + d\lambda M_{13} M_{23} M_{34} - \lambda M_{12} M_{14} M_{23} M_{34} - \lambda M_{13} M_{23} M_{34} - \\
\lambda M_{12} M_{13} M_{24} M_{34} - \lambda M_{13} M_{24} M_{34} + \lambda^2 M_{12} M_{13} M_{23} M_{34}) \\
5.2.38 \( \frac{1}{2}(-M_{12}^2 - aM_{13}^2 M_{13}^2 - bM_{12} M_{13}^2 + cM_{12}^2 M_{13}^2 + acM_{12} M_{13}^3 + bcM_{13}^4 - M_{12} M_{13}^2 M_{14} + \\
dM_{12} M_{14} - 2M_{12} M_{13} M_{14} - 4M_{12} M_{13} M_{23} - aM_{12} M_{13}^2 M_{23} - 2bM_{13} M_{23} + 2M_{13}^2 M_{23} - \\
adM_{13} M_{23} - M_{12} M_{14} M_{23} - bM_{13} M_{14} M_{23} - 4M_{13}^2 M_{23} + aM_{12} M_{13} M_{24} - 2M_{13}^2 M_{24} - \\
aM_{12}^2 M_{13} M_{34} + 2M_{12}^2 M_{14} M_{34} + 2M_{12} M_{13} M_{23} M_{34} - 2aM_{13}^2 M_{23} M_{34} + cM_{12} M_{13} - dM_{13} M_{14} + \\
3\lambda M_{12} M_{13} M_{14} + \lambda M_{12} M_{13} M_{23} + bM_{13} M_{23} + 2M_{13}^2 M_{23} - 2M_{13} M_{13} M_{34} - \lambda^2 M_{12} M_{13}) \\
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5.2.39 \( -(\frac{1}{3})M_{12}(M_{13}^3 + aM_{12}^2M_{13} + bM_{12}M_{13}^2 - M_{13}M_{14} + M_{13}M_{14}^2 - cM_{12}M_{23} + 2M_{12}M_{13}M_{23} - acM_{12}M_{13}M_{23} + aM_{12}M_{13}^2 - bcM_{13}^2M_{23} + M_{12}M_{14}M_{23} + dM_{12}M_{23}^2 + 2bM_{13}M_{23}^2 - cM_{13}M_{23}^2 + adM_{13}M_{23}^2 + bM_{14}M_{23}^2 - cM_{14}M_{23}^2 + 2M_{23}^2 + M_{12}M_{13}M_{24} - aM_{13}^2M_{24} - dM_{13}M_{24} + M_{12}M_{14}M_{24} + 3M_{23}M_{24} + M_{23}M_{24}^2 - cM_{13}M_{34} + cM_{13}M_{14}M_{34} + M_{14}M_{23}M_{34} + \)
\( aM_{23}M_{34} - dM_{23}^2M_{34} - 3M_{13}M_{24}M_{34} - 2M_{14}M_{24}M_{34} + aM_{23}M_{24}M_{34} + dM_{13}M_{34}^2 + bM_{23}M_{34}^2 + 2\lambda M_{34}^3 + M_{13}^3 - \lambda M_{13}^2M_{14} - 2\lambda M_{12}M_{13}M_{23} + d\lambda M_{13}^2M_{23} - \lambda M_{12}M_{14}M_{23} - b\lambda M_{13}M_{23}^2 - \lambda M_{13}^2M_{14} - \lambda M_{12}M_{13}M_{24} + \lambda M_{14}M_{23}M_{34} + M_{13}M_{14}M_{34} + \lambda M_{12}M_{13}M_{23}) \)

5.2.40 \( \frac{1}{2}M_{13}(M_{12}^2 + aM_{12}^2M_{13} + bM_{12}M_{13}^2 - M_{13}M_{14} + M_{13}M_{14}^2 - cM_{12}M_{23} + 2M_{12}M_{13}M_{23} - acM_{12}M_{13}M_{23} + aM_{12}M_{13}^2 - bcM_{13}^2M_{23} + M_{12}M_{14}M_{23} + dM_{12}M_{23}^2 + 2bM_{13}M_{23}^2 - cM_{13}M_{23}^2 + adM_{13}M_{23}^2 + bM_{14}M_{23}^2 - cM_{14}M_{23}^2 + 2M_{23}^2 + M_{12}M_{13}M_{24} - aM_{13}^2M_{24} - dM_{13}M_{24} + M_{12}M_{14}M_{24} + 3M_{23}M_{24} + M_{23}M_{24}^2 - cM_{13}M_{34} + cM_{13}M_{14}M_{34} + M_{14}M_{23}M_{34} + \)
\( aM_{23}M_{34} - dM_{23}^2M_{34} - 3M_{13}M_{24}M_{34} - 2M_{14}M_{24}M_{34} + aM_{23}M_{24}M_{34} + dM_{13}M_{34}^2 + bM_{23}M_{34}^2 + 2\lambda M_{34}^3 + M_{13}^3 - \lambda M_{13}^2M_{14} - 2\lambda M_{12}M_{13}M_{23} + d\lambda M_{13}^2M_{23} - \lambda M_{12}M_{14}M_{23} - b\lambda M_{13}M_{23}^2 - \lambda M_{13}^2M_{14} - \lambda M_{12}M_{13}M_{24} + \lambda M_{14}M_{23}M_{34} + M_{13}M_{14}M_{34} + \lambda M_{12}M_{13}M_{23}) \)

5.2.41 \( \frac{1}{2}(M_{13}^2M_{13} + aM_{12}^2M_{13} + bM_{12}M_{13}^2 - cM_{12}M_{13}M_{23} + 2M_{12}M_{13}M_{23} - acM_{12}M_{13}M_{23} - bcM_{13}M_{23} + M_{12}M_{23}^2 + aM_{12}M_{13}M_{23} + dM_{12}M_{13}M_{23} + 2bM_{13}M_{23}^2 - cM_{13}M_{23}^2 + adM_{13}M_{23}^2 + bM_{14}M_{23}^2 - cM_{14}M_{23}^2 + 4M_{13}M_{23}^2 + M_{12}M_{23}^2 - 4M_{13}^2M_{23} + 2M_{12}M_{13}M_{14}M_{24} - M_{12}M_{23}M_{24} - aM_{12}M_{13}M_{23}M_{24} + 2M_{13}^2M_{23}M_{24} - bM_{12}M_{13}M_{23}M_{34} - 2M_{12}M_{23}^2M_{34} + 2aM_{13}M_{23}^2M_{34} - 2M_{13}M_{24}M_{34} - \lambda M_{12}^2M_{13}M_{23} + d\lambda M_{13}^2M_{23} - 2\lambda M_{12}M_{13}M_{14}M_{23} - b\lambda M_{13}M_{23}^2 - \lambda M_{12}M_{13}^2M_{23} + 2\lambda M_{13}M_{23}M_{34} + \lambda^2M_{12}M_{13}M_{23} \)

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5.2.42 $-\left(\frac{1}{2}\right)M_{14}(M_{12}^3 + aM_{12}^2M_{13} + bM_{12}M_{13}^2 - M_{12}^3M_{14} + M_{13}M_{14}^2 - cM_{12}^2M_{23} + 2M_{12}M_{13}M_{23} - acM_{12}M_{13}M_{23} + aM_{12}^3M_{23} - bcM_{13}^3M_{23} + M_{12}M_{14}M_{23} + dM_{12}M_{23}^2 + 2bM_{13}M_{23}^2 - cM_{13}M_{23}^2 + adM_{13}M_{23}^2 + bM_{14}M_{23}^2 - cM_{14}M_{23}^2 + 2M_{23}^2 + M_{12}M_{13}M_{24} - aM_{13}M_{24} - dM_{13}^2M_{24} + M_{12}M_{14}M_{24} + 3M_{23}^2M_{24} + M_{23}^2M_{24} - bM_{13}^2M_{24} - cM_{14}^2M_{24} - cM_{13}M_{24}^2 + M_{12}M_{24}M_{34} + aM_{23}^2M_{34} + dM_{13}M_{34}^2 + bM_{23}M_{34}^2 + 2\lambda M_{23}^3 + M_{13}^3 + \lambda M_{12}^3M_{14} - 2\lambda M_{12}M_{13}M_{23} + d\lambda M_{13}^2M_{23} - \lambda M_{12}M_{14}M_{23} - \lambda M_{13}^2M_{23} - \lambda M_{12}M_{13}M_{24} = \lambda M_{23}M_{34} + \lambda M_{14}M_{23}M_{34} + \lambda M_{13}M_{24}M_{34} + \lambda^2 M_{12}M_{13}M_{23})$

5.2.43 $\frac{1}{2}(-M_{12}^3M_{13}^2 - M_{12}^2M_{13}M_{14} - M_{12}^3M_{23} - aM_{12}M_{13}^2M_{23} + bM_{12}^2M_{14}M_{23} + cM_{12}M_{13}M_{14}M_{23} + 2M_{12}M_{14}M_{23} + cM_{12}^2M_{23}^2 - 4M_{12}M_{13}M_{23}^2 + acM_{12}M_{13}M_{23} - 2aM_{13}M_{23}^2 + bcM_{13}^2M_{23}^2 - M_{12}M_{14}M_{23}^2 - dM_{12}M_{23}^3 - 2bM_{13}M_{23}^2 - adM_{13}M_{23}^2 - bM_{14}M_{23}^2 - 4M_{12}M_{13}M_{23}M_{24} + 2M_{12}M_{14}M_{23}M_{24} + 2M_{12}M_{13}M_{24}M_{34} + dM_{12}M_{13}M_{23}M_{34} + 2cM_{12}^2M_{23}M_{34} - 2M_{12}M_{23}M_{34} - 2M_{12}M_{23}^2M_{34} - 2\lambda M_{12}^3M_{23}M_{34} + 3\lambda M_{12}M_{13}M_{23}^2 + d\lambda M_{12}^2M_{23} + \lambda M_{12}M_{14}M_{23}^2 + b\lambda M_{12}M_{13}^3M_{23} + 2\lambda M_{12}M_{13}M_{23}M_{24} + 2\lambda M_{12}^2M_{23}^2M_{34} - \lambda^2 M_{12}M_{13}M_{23}^2)$

5.2.44 $\frac{1}{2}(M_{12}^2M_{13}^3 - M_{12}^2M_{13}M_{14} - M_{12}^3M_{23} - aM_{12}M_{13}M_{23} - bM_{12}M_{13}^2M_{23} - cM_{12}M_{13}M_{23} + cM_{12}M_{13}M_{14}M_{23} + cM_{12}^2M_{23} + acM_{12}M_{13}M_{23}^2 + bcM_{13}^2M_{23}^2 + M_{12}M_{14}M_{23}^2 + dM_{12}M_{13}M_{23}^2 - 2bM_{13}M_{23}^2 - adM_{13}M_{23}^2 - bM_{14}M_{23}^2 - 4M_{12}M_{13}M_{23}M_{24} + dM_{12}M_{13}M_{23}M_{34} + dM_{12}M_{13}M_{23}M_{24} - 2M_{12}M_{14}M_{23}M_{24} + dM_{12}M_{13}M_{23}M_{34} - 2aM_{13}M_{23}M_{34} + 2M_{13}M_{23}M_{34} + 2M_{12}M_{23}M_{34} + \lambda M_{12}M_{13}M_{23}^2 - d\lambda M_{13}^2M_{23} + \lambda M_{12}M_{14}M_{23}^2 + \lambda M_{12}M_{13}^2M_{23} + \lambda M_{12}M_{13}^2M_{23}^2 + 2\lambda M_{23} + 2\lambda M_{12}M_{13}M_{23}M_{24} - \lambda M_{13}M_{23}M_{34} - \lambda^2 M_{12}M_{13}M_{23}^2)$
5.2.45 \( \frac{1}{2}(2M_{12}^2M_{13}M_{14} - dM_{12}M_{13}^2M_{24} - cM_{13}^3M_{14} + M_{12}^2M_{14}^2 + cM_{13}^2M_{14}^2 + \\
2M_{12}^2M_{14}M_{23} + M_{13}M_{14}^2M_{23} + M_{12}^3M_{24} + aM_{12}^2M_{13}M_{24} + bM_{12}M_{13}^2M_{24} - 3M_{12}^2M_{14}M_{24} + \\
2M_{12}M_{13}M_{23}M_{24} + aM_{12}M_{13}M_{23}M_{24} + M_{12}M_{14}M_{23}M_{24} - M_{12}M_{13}M_{24}^2 - aM_{12}^2M_{24}^2 - \\
cM_{12}M_{13}M_{34} - acM_{12}M_{13}^2M_{34} - bcM_{13}^3M_{34} + M_{12}M_{13}M_{14}M_{34} + 2M_{12}^2M_{23}M_{34} + \\
2bM_{13}^3M_{23}M_{34} + 2cM_{13}^2M_{23}M_{34} + adM_{13}^2M_{23}M_{34} + bM_{13}M_{14}M_{23}M_{34} + 4M_{13}M_{23}M_{34} + \\
M_{12}^2M_{24}M_{34} - bM_{13}M_{24}M_{34} + aM_{12}^2M_{24}M_{34} + bM_{12}M_{13}M_{24}^2 + 2aM_{13}M_{23}M_{34}^2 - \lambda M_{12}^3M_{13}M_{14} - \\
\lambda M_{13}^2M_{14}M_{23} + \lambda M_{13}^3M_{24} - 2\lambda M_{12}M_{13}M_{34} + d\lambda M_{13}M_{34} - 2\lambda M_{12}M_{13}M_{14}M_{34} - \\
\lambda M_{12}^3M_{23}M_{34} - b\lambda M_{13}^3M_{23}M_{34} - 2\lambda M_{13}^2M_{23}M_{34} + 2\lambda M_{13}^3M_{34} + \lambda^2 M_{12}M_{13}^2M_{34}) \\
5.2.46 \frac{1}{2}(-M_{12}^3M_{14} + cM_{12}M_{13}M_{23} - 2M_{12}M_{13}M_{14}M_{23} - M_{12}M_{14}M_{23} - dM_{12}M_{14}^2M_{23} + \\
cM_{13}M_{14}M_{23}^2 + cM_{14}M_{23}^2 - 2M_{14}M_{23}^3 + M_{12}M_{13}M_{14}M_{24} + bM_{12}M_{23}M_{24} - 3M_{14}M_{23}M_{24} - \\
aM_{23}M_{24} + M_{12}M_{24}^2 + M_{13}M_{23}M_{24}^2 - aM_{23}^2M_{24}^2 - M_{12}^2M_{14}M_{34} + aM_{12}^3M_{23}M_{34} - \\
acM_{12}M_{23}^2M_{34} + 2aM_{13}M_{23}^2M_{34} - bcM_{13}M_{23}^2M_{34} + dM_{14}M_{23}^2M_{34} + 2bM_{23}^2M_{34} + \\
adM_{23}^3M_{34} + 3M_{12}M_{23}M_{24}M_{34} - dM_{13}M_{23}M_{24}M_{34} - cM_{12}M_{34}^2 + dM_{12}M_{23}M_{34}^2 - \\
2cM_{13}M_{23}^2M_{34} + 4M_{23}M_{34}^2 + \lambda M_{14}M_{23}M_{24} - \lambda M_{12}M_{23}M_{24} - \lambda M_{13}M_{23}M_{24} - \lambda M_{12}^3M_{13}M_{34} + \\
2\lambda M_{13}^2M_{23}M_{34} - 2\lambda M_{12}M_{23}M_{34} + d\lambda M_{13}M_{23}M_{34} - b\lambda M_{23}^3M_{34} - 2\lambda M_{12}M_{23}M_{24}M_{34} - \\
2\lambda M_{23}^2M_{34}^2 + \lambda^2 M_{12}M_{23}M_{34})
5.3. Polynomials for $\mathcal{L}_3$.

5.3.1 $M_{12}^4 + 2M_{13}M_{23}$
5.3.2 $-2M_{13}^4 - 2M_{12}^3M_{14} + M_{12}^2M_{23}^2$
5.3.3 $-M_{12}^2 - 4M_{14}M_{23} - 2M_{12}M_{34}$
5.3.4 $-M_{12}^2 - 4M_{13}M_{24} + 2M_{12}M_{34}$
5.3.5 $M_{12}^2 M_{13} - 2M_{12}M_{14} + 2M_{13}M_{34}$
5.3.6 $M_{12}M_{13} - 2M_{12}M_{24} + 2M_{13}M_{34}$
5.3.7 $-M_{13}^3M_{14} - 2M_{13}^3M_{14} + M_{12}^2M_{23}^2 - M_{12}^2M_{23}M_{24} - 2M_{12}^2M_{14}M_{34}$
5.3.8 $M_{12}M_{23} - 2M_{12}M_{24} - 2M_{23}M_{34}$
5.3.9 $M_{12}^2 + 8M_{14}M_{24} - 4M_{34}^2$
5.3.10 $-M_{12}^2 - 4M_{13}^3 + 4M_{12}^2M_{14} - 4M_{12}M_{24} + 4M_{12}M_{34}$
5.3.11 $-M_{12}^2M_{14} - 4M_{13}^2M_{14} + 2M_{12}^2M_{23}^2 - 4M_{12}^2M_{23}M_{24} + 2M_{12}^2M_{24}^2 - 4M_{12}^2M_{14}M_{34} - 4M_{12}^2M_{14}M_{34}^2$
5.3.12 $-M_{12}^2 - 8M_{13}^3 + 16M_{13}M_{14} - 8M_{13}M_{24}^2 - 8M_{34}M_{24} + 2M_{12}M_{34} + 4M_{12}M_{34}^2 - 8M_{34}^3$

5.4. Polynomials for Multiplicity of the Conic.

5.4.1 $-1 + 3M_{14} - 3M_{14}^2 + M_{14}^3$
5.4.2 $M_{13} - 2M_{13}M_{14} + M_{13}M_{23}^2$
5.4.3 $-M_{13}^2 + M_{13}M_{14}^2$
5.4.4 $M_{13}^3$
5.4.5 $M_{12} - 2M_{12}M_{14} + M_{12}M_{23}^2$
5.4.6 $M_{12}M_{13} - M_{13}M_{34}^2$
5.4.7 $M_{13}^2 - 2M_{13}M_{34}^2$
5.4.8 $3M_{13}^2 + 2B_3M_{13}^2 - 2M_{13}M_{14} + 4B_2M_{13}M_{34} + 8M_{12}M_{34}^2 - 8B_3M_{12}M_{34}^2 - 4M_{12}M_{14}M_{34}^2 + 4B_3M_{12}M_{14}M_{34}^2 - 12B_3M_{13}M_{34} + 12B_3M_{13}M_{34}^2 + M_{13}M_{34}^2 + 16M_{14}M_{34}^2 + 8M_{13}M_{14}M_{34}^2 - 8B_3M_{13}M_{14}M_{34}^3 - 4M_{14}M_{34}^3 - 16M_{12}M_{34}^2 - 8M_{12}M_{14}M_{34}^2 - 12M_{13}M_{34}^6 - 5M_{13}M_{34}^6 + 10M_{13}M_{14}M_{34}^6 - 2M_{13}M_{34}^6 - 4\alpha M_{34}^6 + 4\alpha M_{14}M_{34}^6$
5.4.9 $-4M_{12}M_{34} + 4M_{12}M_{14}M_{34} + 4M_{13}M_{34}^2 + 2M_{13}M_{34}^2 - 2B_3M_{13}M_{34}^2 - 4M_{13}M_{14}M_{34}^2 + M_{13}M_{34}^2 - 2\alpha M_{12} + 2\alpha M_{12}M_{14} + 2\alpha M_{13}M_{34} - 2\alpha M_{13}M_{14}M_{34}$
References


