An Analytical Analysis of Fiber Waviness for Laminated Curved Beam Under Bending

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ABSTRACT

A closed-form analytical solution is developed for analyzing laminated composite curved beam with fiber waviness. Explicit expressions for evaluating equivalent axial and bending stiffness are formulated based upon modified lamination theory and taking into consideration the structural deformation characteristics of beam with narrow section. The maximum radial stress is computed for a laminated composite curved beam with fiber waviness under four-point bending at any given location. The maximum radial stress results were in a good agreement with numerical results obtained by ABAQUS. It is concluded that the present approach is an efficient method for analyzing laminated composite curved beam with fiber waviness.

INTRODUCTION

Fiber waviness is considered a common imperfection occurring in the manufacturing process of composite structures especially for thick composite laminates of compound curvature and in the region where the thickness is changing [1]. The imperfection is caused by non-uniform distribution of pressure and mismatch of thermal expansion (CTE) between tooling material, matrix, and fiber. This will cause longitudinal and transverse stresses in composite, including higher matrix contraction and fiber buckling as stated by Kantharaju [2]. Parameters on developing fiber waviness have been studied by Kugler and Moon [3]. They concluded that the influence of holding cure temperature is insignificant, but the cooling rate will affect the severity and the quantity of fiber waviness.

The concept of elastic moduli reduction for initial distortions of the reinforcing layers was first provided by Bolotin [4]. In his analysis, Kirchhoff hypothesis was used to describe the deformation of thin layers or slightly twisted plates with initial irregularities. In connection with the study of layered reinforced media with random initial irregularities, reduction on the modulus of elasticity in tension along the fibers of unidirectional glass-reinforced plastics (GRP) is proposed by Tarnopol'skii et al.
The shape of fiber irregularities is assumed to be a sinusoidal function. Bažant [6] advanced their approach by taking into account changes in wave amplitude due to radial forces. Three ideal cases of unidirectional fiber distributions were discussed. The first one is parallel, uniformly distributed fibers with sinusoidal curvature. The second one is not strictly parallel distributed fibers with sinusoidal curvature. The third one is that fiber waves are equal in amplitude but in opposite directions.

Extensive investigations of stiffness loss due to fiber waviness was conducted in [7-9]. Lo and Chim [10] predicted the compressive strength of unidirectional composite with fiber waviness. Adams and Hyer [11] experimentally investigated multi-directional composite laminates under static compression loading. They observed that severe waviness induced a static strength reduction of 36%, although the fiber waviness occurred in $0^\circ$ ply and accounted for only 20% of the load carrying capacity of the laminate. Rai et al. [12] numerically investigated lamina modulus as a function of fiber waviness, which is similar to [13-16]. They concluded that fiber waviness, which occurs in $0^\circ$ ply has significant influence on stiffness reduction. If fiber waviness occurs in $\pm 45^\circ$ ply, the influence in stiffness reduction is more pronounced in torsional cases than bending cases.

Fiber waviness can occur in either in-plane or out-of-plane for a laminated beam [17]. The effects of out-of-plane fiber waviness for a lamina were further investigated by Hsiao and Daniel [18-20]. Three types of fiber waviness are considered including uniform, graded and localized fiber waviness. They concluded that tensile and compressive elastic properties and nonlinear behavior in composite materials can be significantly influenced by fiber waviness. Several researchers applied numerical method for investigating effects of fiber waviness. Seon [21] studied tape composite with fiber waviness by linear and nonlinear Finite Element analysis (FE). The nonlinear interlaminar stress-strain relations can improve the delamination inset prediction. He observed that the failure load for a rectangular tape with small amplitude fiber waviness under tension is higher compared to fiber waviness with large amplitude. Nikishkov et al. [22] conducted a numerical model to investigate progressive fatigue damage in composites with fiber waviness. However, most of analytical researches are not focused on out-plane fiber waviness for a laminate composite beam. Therefore, the object of this research is to develop a feasible and efficient approach to analyze composite curved beam with out-of-plane fiber waviness.

[5]
CONSTITUTIVE EQUATIONS OF FIBER WAVINESS

Fiber waviness is a misalignment of the fibers in a ply. The presence of the fiber waviness results in stiffness and strength loss and acts as a failure initiation in composite structures. This section describes the extension of analytical methodology for in-plane fiber waviness derived in [15] to application in composite curved beam with out-of-plane fiber waviness.

Geometry of Fiber Waviness

In-plane fiber waviness can be expressed by a sinusoidal wave function in 1-2 coordinate system as shown in Figure 1(a). The severity of fiber waviness can be expressed by a factor $R = A/L$, where $A$ is the amplitude of fiber waviness, and $L$ is the half sinusoidal length of fiber waviness. The waviness angle for any given locations can be introduced as

$$\phi = \tan^{-1}\left(\pi R \cos \frac{\pi x}{L}\right) \quad (1)$$

Fiber orientation is changed along fiber waviness direction. Therefore, the average compliance properties of a 0° lamina $[S']$ can be computed by integrating through fiber orientation where the direction is rotated based on sinusoidal function over the length of fiber waviness. Starts with compliance matrix for a 0° lamina without fiber waviness $[S]$, the transformation and averaging in-plane fiber waviness result for $[S']$ is obtained and can be found in [15]. The averaging stiffness matrix $[Q']$ can be computed, where $[Q'] = [S']^{-1}$.

On the other hand, the effect of out-of-plane fiber waviness can be obtained by rotating with respect to x-direction with an out-of-plane angle $\beta$ as shown in Figure 1(b). After rotation with respect to x-axis, the out-of-plane compliance $[S'']$ can be calculated by averaging out-of-plane fiber waviness property.

![Figure 1](image_url). (a) Geometry of in-plane fiber waviness. (b) Geometry of out-of-plane fiber waviness.
\[ [S'''] = [T_x(-\beta)]_x[S'][T_\sigma(\beta)]_x \]

where

\[
[T_\sigma(\beta)]_x = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & m^2 & n^2 & 2mn & 0 & 0 \\
0 & n^2 & m^2 & -2mn & 0 & 0 \\
0 & -mn & mn & m^2 - n^2 & 0 & 0 \\
0 & 0 & 0 & 0 & m & -n \\
0 & 0 & 0 & 0 & n & m
\end{bmatrix}
\]

\[(3)\]

\[
[T_x(\beta)]_x = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & m^2 & n^2 & mn & 0 & 0 \\
0 & m^2 & n^2 & -mn & 0 & 0 \\
0 & -2mn & 2mn & m^2 - n^2 & 0 & 0 \\
0 & 0 & 0 & 0 & m & -n \\
0 & 0 & 0 & 0 & n & m
\end{bmatrix}
\]

and \( m = \cos \beta, n = \sin \beta. \) The averaged out-of-plane stiffness matrix \([Q'''] = [S''']^{-1}.\]

Fiber waviness in thickness direction is shown in Figure 2. The out-of-plane compliance matrix can be computed by rotating in-plane properties with respect to \( x \)-axis for \( \beta = 90^\circ \). If different fiber orientations are considered based on the designed stacking sequence, the out-of-plane compliance matrix \([S_{out}]\) can be further computed by rotating \( \theta^\circ \) with respect to \( z \)-axis, where \( \theta \) is fiber orientation.

\[
[S_{out}] = [T_x(-\theta)]_z[T_x(-\beta)]_x[S'][T_\sigma(\beta)]_x[T_\sigma(\theta)]_z
\]

(4)

Figure 2. Out-of-plane fiber waviness in laminate stage with in-plane fiber orientation.

CONSTITUTIVE EQUATIONS OF COMPOSITE CURVED BEAM

The foundation of beam analysis is based upon the one-dimensional moment-curvature relationship along the longitudinal axis of the beam under bending. However, since composite is orthotropic material, which is inherent with two-dimensional property, an equivalent one-dimensional property of composite
beam is needed. The equivalent one-dimensional property is dependent of the structural response of the deformed beam and the structural response of the beam is dependent on the ratio of the width to height of the beam cross-section. The following section describes those behaviors.

**Equivalent Stiffness for Composite Curved Beam**

In order to satisfy behavior of thick curved beam, shear deformation and rotary inertia are included in the derived equations. The force and moment resultants are the integrals of the stresses over beam thickness. The equivalent stiffness are the stiffness coefficients arising from the integration. More detail derivation can be found in [23].

\[
\bar{A}_c = R_m \sum_{k=1}^{n} [\bar{Q}_{x-y}]_{kth} \ln \frac{R_m + z_k}{R_m + z_{k-1}}
\]

\[
\bar{B}_c = R_m \sum_{k=1}^{n} [\bar{Q}_{x-y}]_{kth} \left( z_k - z_{k-1} \right) - R_m \ln \frac{R_m + z_k}{R_m + z_{k-1}}
\]

\[
\bar{D}_c = R_m \sum_{k=1}^{n} [\bar{Q}_{x-y}]_{kth} \left( \frac{1}{2} \left( z_k^2 - z_{k-1}^2 \right) \right) - R_m \left( z_k - z_{k-1} \right) + R_m^2 \ln \frac{R_m + z_k}{R_m + z_{k-1}}
\]

\[
GA = \frac{5}{4} \sum_{k=1}^{n} \left[ G_{13} \left( z_k - z_{k-1} \right) - \frac{4}{3z_k^3} \right) \right]
\]

where \([\bar{Q}_{x-y}]_{kth}\) is stiffness matrix which takes fiber orientation into account at k\(^{th}\) ply, n is total ply number, \(R_m\) is mean radius of the curved beam.

**Beam with Narrow Section under Bending**

If the width to height ratio of the cross-section is small (w/h < 6), the lateral curvature is induced due to the effect of Poisson’s ratio. Hence, the response of structure beam exhibits \(M_y = 0\) and \(K_y \neq 0\) for a narrow section beam.

With the above assumption, the constitutive equation for composite curved beam with narrow cross-section can be written as one dimensional relationship below:

\[
\begin{bmatrix}
N_x \\
M_x
\end{bmatrix} = 
\begin{bmatrix}
EA & EV \\
EV & EI
\end{bmatrix} \begin{bmatrix}
x_k^0 \\
k_x
\end{bmatrix} \text{ and } 
\begin{bmatrix}
EA & EV \\
EV & EI
\end{bmatrix}^{-1} = \begin{bmatrix}
a^* & b^* \\
b^* & d^*
\end{bmatrix}
\]

where

\[
a^* = a_{11} \frac{b_{16}^2}{d_{66}}, b^* = b_{16} \frac{d_{16}}{d_{66}} \text{ and } d^* = d_{11} \frac{d_{16}^2}{d_{66}}
\]
\[
\begin{bmatrix}
a & b \\
b & d
\end{bmatrix} = \begin{bmatrix}
\tilde{A}_c & \tilde{B}_c \\
\tilde{B}_c & \tilde{D}_c
\end{bmatrix}^{-1}
\]

(8)

\(\varepsilon_x^0\) and \(\kappa\) are the mid-plane strain and curvature along the longitudinal axis of the beam, \(EA\), \(EV\), and \(EI\) refer to the equivalent axial, coupling and bending stiffness and \(a^*, b^*\) and \(d^*\) are the compliance, coupling and flexibility components of beam laminate under bending, respectively. The matrices of \(a\), \(b\) and \(d\) are the conventional laminated plate properties.

CONSTITUTIVE EQUATION OF COMPOSITE CURVED BEAM WITH FIBER WAVINESS

Equivalent Stiffness Properties with In-Plane Fiber Waviness

Configuration of curved lamina with in-plane fiber waviness is shown in Figure 3(a). The average stiffness properties can be further obtained by replacing \([\bar{Q}_{x-y}]_{kth}\) to \([\bar{Q}'_{x-y}]_{kth}\) using Eqs. (5), where \([\bar{Q}'_{x-y}]\) is in-plane stiffness matrix with fiber waviness property.

\[
\tilde{A}'_c = R_m \sum_{k=1}^{n} [\bar{Q}'_{x-y}]_{kth} \ln \frac{R_m + z_k}{R_m + z_{k-1}}
\]

\[
\tilde{B}'_c = R_m \sum_{k=1}^{n} [\bar{Q}'_{x-y}]_{kth} \left( (z_k - z_{k-1}) - r \ln \frac{R_m + z_k}{R_m + z_{k-1}} \right)
\]

\[
\tilde{D}'_c = R_m \sum_{k=1}^{n} [\bar{Q}'_{x-y}]_{kth} \left( \frac{1}{2} (z_k^2 - z_{k-1}^2) - R_m (z_k - z_{k-1}) + R_m^2 \ln \frac{R_m + z_k}{R_m + z_{k-1}} \right)
\]

(9)

Equivalent Stiffness Properties with Out-of-Plane Fiber Waviness

Out-of-plane fiber waviness will degrade the strength and fatigue performance of composite structure. In general, non-uniform fiber waviness is observed along the curved region instead of uniform fiber waviness. Therefore, a graded out-of-plane fiber waviness is assumed, and its configuration is shown in Figure 3(b), where \(L_{A_{end}}\) and \(L_{A_{bot}}\) are the location of plies where the zero amplitudes are observed on the configuration. \(L_{A_{max}}\) is the ply location where the maximum amplitude of fiber waviness is observed.

Fiber waviness length, \(L\), changes in the thickness direction where \(L_{kth}\) and \(r_{kth}\) are the length of fiber waviness and its corresponding radius for a lamina in \(kth\) layer.

\[
L_{kth} = 2\pi r_{kth} \left( \frac{\theta_{end} - \theta_{start}}{360} \right)
\]

(10)
Figure 3. (a) In-plane fiber waviness in a curved lamina. (b) Out-of-plane fiber waviness in a curved beam.

The amplitude of fiber waviness, $A_{\text{max}}$, is the maximum amplitude can be observed. The amplitude below and above the location $L_{\text{A} \text{max}}$, where contains the maximum amplitude are

\[
\begin{align*}
A_{\text{low}} &= \frac{A_{\text{max}}(\text{ply}^{\text{th}}_{\text{low}} - L_{\text{A} \text{bot}})}{L_{\text{A} \text{max}} - L_{\text{A} \text{bot}}} \\
A_{\text{upp}} &= \frac{A_{\text{max}}(\text{ply}^{\text{th}}_{\text{upp}} - L_{\text{A} \text{max}} + 1)}{L_{\text{A} \text{top}} - L_{\text{A} \text{max}}} \\
\end{align*}
\]

where $\text{ply}^{\text{th}}_{\text{low}} = L_{\text{A} \text{bot}} \sim L_{\text{A} \text{max}}$, and $\text{ply}^{\text{th}}_{\text{upp}} = L_{\text{A} \text{max}} \sim L_{\text{A} \text{top}} - 1$. Thus, the amplitude can be displaced as $\text{Amplitude} = [A_{\text{low}}, A_{\text{upp}}]$ from bottom ply to the top ply of entire curved beam.

The equivalent stiffness property can be computed by replacing $[\bar{Q}_{x-y}]_{kth}$ to $[Q_{\text{out}}]_{kth}$, where $[Q_{\text{out}}]_{kth} = [S_{\text{out}}]_{kth}^{-1}$.

\[
\begin{align*}
\bar{A}^{ij}_{c} &= R_{m} \sum_{k=1}^{n} [Q_{\text{out}}]_{kth} \ln \frac{R_{m} + z_{k}}{R_{m} + z_{k-1}} \\
\bar{B}^{ij}_{c} &= R_{m} \sum_{k=1}^{n} [Q_{\text{out}}]_{kth} \left( (z_{k} - z_{k-1}) - R_{m} \ln \frac{R_{m} + z_{k}}{R_{m} + z_{k-1}} \right) \\
\bar{D}^{ij}_{c} &= R_{m} \sum_{k=1}^{n} [Q_{\text{out}}]_{kth} \left( \frac{1}{2}(z_{k}^{2} - z_{k-1}^{2}) - R_{m}(z_{k} - z_{k-1}) + R_{m}^{2} \ln \frac{R_{m} + z_{k}}{R_{m} + z_{k-1}} \right)
\end{align*}
\]
**Maximum Radial Stress Prediction**

Lekhnitskii [24] provided closed-form methods for obtaining interlaminar stress (through thickness stress) of a laminated curved beam under bending and shearing load. Chung and Harold [25] provided a closed-form solution for a laminated curved beam under axial load. By equating $\frac{\partial \sigma_r}{\partial r}$ to zero, the maximum delamination stress along thickness direction $\sigma_{r,max}$ is obtained [26, 27]. Regarding Classical Lamination Theory (CLT), no through thickness stress can be calculated because of plane stress assumption. Therefore, González [28] developed a method which takes interlaminar stresses, $\sigma_r$ and $\tau_{r\theta}$ into account based on CLT and equilibrium equations of the elasticity in polar coordinates. The maximum radial stress for composite curved beam in this section, is modified from [28, 29] and will be extended to apply for out-of-plane fiber waviness in composite curved beam under bending.

$$
\sigma_{r}^i(r, \theta) = \frac{N_i R_m (EA)_i}{wtr} \left( \frac{N(\theta)}{EA} - \frac{M(\theta)}{EV} + (r - R) \left( \frac{M(\theta)}{EI} - \frac{N(\theta)}{EV} \right) \right)
$$

where $N_i$ is total ply number, $R_m$ is mean radius of the curved beam, $EA$, $EV$, and $EI$ are axial, coupling, and bending stiffness, respectively. $N(\theta)$ and $M(\theta)$ are axial force and moment, respectively. $w$ is the width of the beam and $t$ is the thickness of the complete beam. $(EA)_i$ is the equivalent axial stiffness to EA but in a ply only. Once the circumferential stress has been obtained, the other stresses are obtained due to equilibrium equations of the elasticity in polar coordinates.

$$
\frac{\partial \sigma_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial (r^2 \tau_{r\theta})}{\partial r} = 0 \quad , \quad \frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{r\theta}}{\partial \theta} = \sigma_\theta
$$

Substituting Eqs. (13) to Eqs. (14), the radial stress $\sigma_r$ due to applied moment is given by

$$
\sigma_{r,M}^i(r, \theta) = \sigma_{r,M}^{i-1}(r_{oi}, \theta) \frac{r_{oi}}{r} - \frac{N_i R_m (EA)_i M(\theta)}{wtr EI} \left[ r_{oi} - r - \left( R_m + \frac{EI}{EV} \right) \log \frac{r_o}{r_i} \right]
$$

where $\sigma_{r,M}^0 = 0$. However, Eqs. (15) is very sensitive with the given total ply number. If the ply number is greater than 15 plies, the $\sigma_r$ distribution is inaccurate compared to closed-form solution provided by Lekhnitskii [24]. Therefore, a modified equation is proposed to satisfy the specimen with large ply number.

$$
\sigma_{r,M}^i(r, \theta) = \sigma_{r,M}^{i-1}(r_{oi}, \theta) \frac{r_{oi}}{r} + \frac{R_m Q(1,1) M(\theta)}{r EI} \left[ r_o - r_i - \left( R_m + \frac{EI}{EV} \right) \log \frac{r_o}{r_i} \right]
$$

where $r_i$ is the inner radius of the ply, $r_o$ is the outer radius of the ply, $Q(1,1)$ is the stiffness component along $\theta$ direction for a given ply.
In order to satisfy equivalent stiffness in composite curved beam with fiber waviness, in Eqs. (17), \( Q(1,1) \) can be replaced to \([Q(1,1)_{out}]_{kth} \). It should be noted that constant ply thickness is assumed in Eqs. (16). However, if fiber waviness is presented, ply thickness is going to be functional of the amplitude of fiber waviness in each ply. Therefore, Eqs (16) can be modified as

\[
\sigma_{r,M}(r,\theta) = \sigma_{r,M}\left(\frac{r_0}{r}\right) + \frac{R_m[Q(1,1)_{out}]_{kth}M(\theta)}{rE_I} \left[ r_0 - r_i - \left( R_m + \frac{EI}{EV} \right) \log \frac{r_{oo}}{r_{ii}} \right] \quad (17)
\]

where \( r_{ii} = r_i + \text{Amplitude}_i \) and \( r_{oo} = r_o + \text{Amplitude}_i \). \( \text{Amplitude}_i \) is the amplitude of fiber waviness in \( i \)th ply.

**FINITE ELEMENT ANALYSIS**

**Mesh, Element Used and Model Validation**

FE analysis was conducted for validating the analytical results using ABAQUS [30]. 2D plane strain linear element without fully integrated was employed. Convergence study was performed before finalizing the mesh density. The model was first validated by using the stress distribution provided in [24] when the amplitude of fiber waviness is zero as shown in Figure 4 and Table I.

![Figure 4. \( \sigma_r \) and \( \sigma_\theta \) in ABAQUS under bending.](image-url)
Table I. \( \sigma_\theta \) AND \( \sigma_r \) COMPARISON BETWEEN ABAQUS AND CLOSED-FORM SOLUTION IN [24].

<table>
<thead>
<tr>
<th></th>
<th>( \sigma_\theta ) (Pa)</th>
<th>( \sigma_r ) (Pa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>min</td>
<td>max</td>
</tr>
<tr>
<td>Lekhnitskii</td>
<td>(-2.20e^8)</td>
<td>(3.048e^8)</td>
</tr>
<tr>
<td>ABAQUS</td>
<td>(-2.06e^8)</td>
<td>(3.20e^8)</td>
</tr>
<tr>
<td>Error %</td>
<td>6.8</td>
<td>4.75</td>
</tr>
</tbody>
</table>

**Loads and Boundary Conditions**

A bending moment of 20 N-m is applied at one end of the curved beam. A node with coupling constrains connected with the end surface was implemented to present constant moment applied at the end surface of the curved beam. Layers are perfectly bonded with upper and lower adjacent layers. The boundary condition considered was a curved cantilever boundary condition as shown in Figure 5.

![Figure 5. (a) Perfect bonded layers (b) Boundary conditions and applied moment.](image)

**NUMERICAL RESULTS**

In this study, the inner radius of the composite curved beam is 6.4 mm and the outer radius of the composite curved beam is 12.988 mm. The width of the beam is 12.7 mm. Therefore, the mean radius \( R_m \) is 9.694 mm and the total thickness of the beam is 6.588 mm, which means it contains 36 plies and the ply thickness is 0.183 mm for IM7/8552 material. The material properties for IM7/8552 [31] are:

\[
\begin{align*}
E_1 &= 157 \text{ GPa} & E_2 &= 8.96 \text{ GPa} & E_3 &= 8.96 \text{ GPa} \\
G_{12} &= 5.08 \text{ GPa} & G_{23} &= 2.99 \text{ GPa} & G_{13} &= 5.08 \text{ GPa} \\
\nu_{12} &= 0.32 & \nu_{23} &= 0.5 & \nu_{13} &= 0.32
\end{align*}
\]

where \( E_1, E_2, \) and \( E_3 \) are the Young’s moduli of the composite lamina along the material coordinates. \( G_{12}, G_{23}, G_{13} \) and are the Shear moduli and \( \nu_{12}, \nu_{23}, \) and \( \nu_{13} \) are Poisson’s ratio with respect to the 1-2, 2-3 and 1-3 planes, respectively. The curved beam has unidirectional stacking sequence in 0° combined with out-of-plane fiber waviness. The location of fiber waviness can be located at any hoop and radial location in this analysis.
Composite Curved Beam Stiffness Validation

Stiffness was validated by investigating stress distribution using present method and comparing with FE analysis results. In Figure 6(a), $\sigma_\theta$ distributions are compared with the closed-form solution provided by Lekhnitskii [24], Classical Lamination Theory (CLT), Eqs. (13) provided from González et al. [29], and FE results using ABAQUS. A pure bending loading condition is considered, which results in $N(\theta) = 0$ and $M(\theta) = M$ in Eqs. (13). Since $\sigma_r$ cannot be obtained from CLT method, comparison between the closed-form solution provided by Lekhnitskii [24], Eqs. (16) provided from [29], and numerical results using ABAQUS are shown in Figure 6(b). Both $\sigma_\theta$ and $\sigma_r$ distributions have excellent agreement with the results from ABAQUS. Since Eqs. (13) and (16) contains $EI$, $EV$, and $EA$, and both CLT and JMG methods are in agreement with numerical solution, the stiffnesses of composite curved beam are validated.

![Figure 6. (a) $\sigma_\theta$ comparison. (b) $\sigma_r$ comparison.](image)

The difference between beam theories with conventional, wide and narrow cross-section was discussed in [32] and is shown in Figure 7. As seen in Figure 7, bending stiffness along longitudinal direction decreases when the mean radius of the curved beam increases. The stiffness under narrow beam assumption has higher bending stiffness compared with stiffness using wide and general assumptions.

Equivalent Stiffness for Composite Curved Beam with Fiber Waviness

A single out-of-plane fiber waviness in the unidirectional composite curved beam is investigated in this section. The inner radius ($R_i$) is 6.4 mm and the outer radius ($R_o$) is 12.7 mm. The maximum amplitude of fiber waviness is selected at 15$^{th}$ ply. The top and bottom plies with zero amplitude are selected at 5$^{th}$ and 33$^{rd}$ plies. Fiber waviness initiates at $\theta = 20^\circ$ and ends at $\theta = 70^\circ$, and $A_{\max} = 0.2(R_o - R_i)$ as shown in Figure 8. Comparison for axial and bending stiffness for composite beam with and without curvature and fiber waviness is shown in Table II.
Figure 7. Bending stiffness difference between general, wide, and narrow cross-section beam with and without initial curvature.

Table II. COMPARISON FOR AXIAL AND BENDING STIFFNESS FOR COMPOSITE BEAM WITH AND WITHOUT CURVATURE AND WAVY, RESPECTIVELY.

<table>
<thead>
<tr>
<th></th>
<th>A_x (N/m)</th>
<th>D_x (N-m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Perfect</td>
<td>1.034E9</td>
<td>3.741E3</td>
</tr>
<tr>
<td>Wavy</td>
<td>7.111E8</td>
<td>3.386E3</td>
</tr>
<tr>
<td>Curved</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Perfect</td>
<td>1.077E9</td>
<td>4.024E3</td>
</tr>
<tr>
<td>Wavy</td>
<td>7.402E8</td>
<td>3.637E3</td>
</tr>
</tbody>
</table>

Figure 8. Single out-of-plane fiber waviness for composite curved beam.

For a given fiber waviness amplitude approximately equals to 10 % of total thickness of composite curved beam with 10 wavy plies, the effect of fiber waviness in different thickness location can be investigated as shown in Table III.
According to Table III, bending stiffness reaches the maximum when the location of fiber waviness is approximately located in the middle axis along longitudinal direction. Effect of fiber waviness amplitude is investigated and shown in Table IV. LA$^\text{bot}_{\text{end}}$ is selected to be 5$^{\text{th}}$ ply and LA$^\text{top}_{\text{end}}$ is selected to be 33$^{\text{rd}}$ ply. The location where contains the maximum fiber waviness amplitude is chosen to be 15$^{\text{th}}$ ply. The maximum amplitude varies from 0 % to 30 % out of total thickness of composite curved beam are investigated. According to Table IV, as amplitude increases, both axial and bending stiffness decrease. The axial and bending stiffness comparison between perfect composite curved beam and curved beam with fiber waviness is also presented. It is more pronounced for axial stiffness since significant axial stiffness reduction is observed when the amplitude of fiber waviness increases.

### Table III. PARAMETER STUDY FOR LOCATION OF FIBER WAVINESS FOR COMPOSITE CURVED BEAM.

<table>
<thead>
<tr>
<th>LA$^\text{bot}_{\text{end}}$ (ply)</th>
<th>LA$^\text{max}$ (ply)</th>
<th>LA$^\text{top}_{\text{end}}$ (ply)</th>
<th>A$_{x}$ ($N/m$)</th>
<th>D$_{x}$ ($N - m$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>10</td>
<td>9.849E8</td>
<td>3.463E3</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>15</td>
<td>9.967E8</td>
<td>3.812E3</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>20</td>
<td>10.14E8</td>
<td>3.988E3</td>
</tr>
<tr>
<td>15</td>
<td>20</td>
<td>25</td>
<td>10.27E8</td>
<td>4.015E3</td>
</tr>
<tr>
<td>20</td>
<td>25</td>
<td>30</td>
<td>10.36E8</td>
<td>3.964E3</td>
</tr>
<tr>
<td>25</td>
<td>30</td>
<td>35</td>
<td>10.44E8</td>
<td>3.875E3</td>
</tr>
</tbody>
</table>

### Table IV. COMPARISON BETWEEN EFFECT OF FIBER WAVINESS AMPLITUDE ON AXIAL AND BENDING STIFFNESS.

<table>
<thead>
<tr>
<th>A$_{\text{max}}$</th>
<th>A$_{x}$ ($N/m$)</th>
<th>D$_{x}$ ($N - m$)</th>
<th>A$_{x}$ reduction %</th>
<th>D$_{x}$ reduction %</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 %</td>
<td>1.077E9</td>
<td>4.024E3</td>
<td>100%</td>
<td>100%</td>
</tr>
<tr>
<td>5 %</td>
<td>1.022E9</td>
<td>3.975E3</td>
<td>95%</td>
<td>99%</td>
</tr>
<tr>
<td>10 %</td>
<td>9.146E8</td>
<td>3.866E3</td>
<td>85%</td>
<td>96%</td>
</tr>
<tr>
<td>15 %</td>
<td>8.166E8</td>
<td>3.747E3</td>
<td>76%</td>
<td>93%</td>
</tr>
<tr>
<td>20 %</td>
<td>7.402E8</td>
<td>3.637E3</td>
<td>69%</td>
<td>90%</td>
</tr>
<tr>
<td>25 %</td>
<td>6.819E8</td>
<td>3.539E3</td>
<td>63%</td>
<td>88%</td>
</tr>
<tr>
<td>30 %</td>
<td>6.369E8</td>
<td>3.455E3</td>
<td>59%</td>
<td>86%</td>
</tr>
</tbody>
</table>

**Maximum Radial Stress**

Maximum radial stress can be predicted well using closed-form solution provided by [24] for a perfect curved beam without fiber waviness. However,
maximum radial stress will be relocated and varied if fiber waviness is introduced. The $\sigma_r$ comparison between present method using Eqs. (17) and FE results with and without fiber waviness is shown in Figure 9. The $\sigma_r$ distribution has excellent agreement with FE results. The maximum $\sigma_r$ predicted from present method is 48.00 MPa and maximum $\sigma_r$ obtained from FE analysis is 49.85 MPa. The error percentage is less than 4% between the result from present method and FE analysis. Moreover, the location which has maximum $\sigma_r$ using present method is $r = 10.27$ mm, and the location which has maximum $\sigma_r$ using FE analysis is $r = 10.26$ mm. The error percentage is less than 1% between the result from present method and FE analysis.

![Figure 9. $\sigma_r$ Comparison between present and FE results with and without fiber waviness.](image)

**CONCLUDING REMARKS**

A closed-form analytical solution is developed for analyzing laminated composite curved beam with presence fiber waviness. The explicit expressions for evaluating equivalent axial and bending stiffness are formulated based upon modified lamination theory and taking into consideration the structural deformation characteristics of beam with narrow section. The maximum radial stress is computed for a laminated composite curved beam with fiber waviness under four-point bending at any random location. The present stiffness and stress results are in good agreement with numerical results from ABAQUS. It is found that the geometry of fiber waviness such as location and amplitude have great impact on
both equivalent axial and bending stiffness. However, fiber waviness has less impact on the bending stiffness if the plies affected by fiber waviness are near the middle axis of composite curved beam. It is concluded that the present approach is an efficient method for analyzing laminated composite curved beam with fiber waviness.

REFERENCES


