SCATTERING AND INVERSE SCATTERING ON THE LINE FOR A FIRST-ORDER SYSTEM WITH ENERGY-DEPENDENT POTENTIALS

by

RAMAZAN ERCAN

Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

May 2018

Acknowledgements

This dissertation is deeply indebted to Dr. Tuncay Aktosun for his invaluable guidance and support. Our almost daily meetings during my last year has taught me the impossibility of my advancement without his supervision. His supervision enabled me to acquire numerous skills and gain different points of view to attack the challenges experienced during research. He has inspired me all the time towards my degree.

I would like to thank everyone in the Mathematics Department at the University of Texas at Arlington, along with my committee members. I would like to thank the Department of Mathematics for support through an assistantship. I would also like to thank the Ministry of National Education of Turkey for their financial support to pursue my education.

I would like to thank my mother and father for their support over the years and understanding, even though I was not with them at times when they needed me around the most.

To Esra, I owe the pleasure of a life. I forever owe her a deep debt of gratitude for her self-sacrificing love and support. To Yusuf, his being and smile is always the first and biggest incentive to achieve any goal. This dissertation would be impossible without them.

April 10, 2018

Abstract

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Ramazan Ercan, Ph.D.

The University of Texas at Arlington, 2018

Supervising Professor: Tuncay Aktosun

A first-order system of two linear ordinary differential equations is analyzed. The linear system contains a spectral parameter, and it has two coefficients that are functions of the spatial variable x. Those two functions act as potentials in the linear system and they also linearly contain the spectral parameter λ, and hence they are referred to as energy-dependent potentials. Such a linear system arises in the solution to a pair of integrable nonlinear partial differential equations (known as the derivative nonlinear Schrödinger equations) via the so-called inverse scattering transform method.

The direct and inverse problems for the corresponding first-order linear system with energy-dependent potentials are investigated. In the direct problem, when the two potentials belong to the Schwartz class, the properties of the corresponding scattering coefficients and so-called bound-state data are derived. In the inverse problem, the two potentials are recovered from the scattering data set consisting of the scattering coefficients and bound-state data. The solutions to the direct and inverse problems are achieved by relating the scattering data and the potentials in

the energy-dependent system to those in a pair of first-order system with energy-independent potentials. An alternate solution to the inverse problem is given by formulating a linear integral equation (referred to as the alternate Marchenko integral equation), and the energy-dependent potentials are recovered with the help of the solution to the alternate Marchenko equation.

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Chapter 1

Introduction

In this thesis we consider the direct and inverse scattering problems for the first-order system with energy-dependent potentials, i.e.

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & \zeta q(x) \\ \zeta r(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad x \in \mathbb{R}, \tag{1.0.1}$$

where the prime denotes the x-derivative, ζ is spectral parameter, the scalar quantities α and β are the components of the column vector-valued wavefunction depending on both x and ζ, and $q(x)$ and $r(x)$ are complex-valued potentials. The parameter ζ^2 at times may be interpreted as energy. That is why we refer to (1.0.1) as the system with energy-dependent potentials to emphasize that $q(x)$ and $r(x)$ appearing in the coefficient matrix in (1.0.1) each contain the spectral parameter ζ as a coefficient. We assume that $q(x)$ and $r(x)$ belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. We recall the definition of the Schwartz class below.

Definition 1.1. *A function $f(x)$ is said to belong to the Schwartz class $\mathbb{S}(\mathbb{R})$ if that function belongs to $\mathbb{C}^\infty(\mathbb{R})$ and $x^m f^{(n)}(x) \to 0$ as $x \to \pm\infty$ for every pair of nonnegative integers m and n.*

Recall that $\mathbb{C}^\infty(\mathbb{R})$ denotes the class of functions $f(x)$ where all the derivatives $f^{(n)}(x)$ (including the zeroth derivative, i.e. the function itself) are continuous in $x \in \mathbb{R}$. The following are already known [17] about the Schwartz class. In the Schwartz class, $\left|f^{(n)}(x)\right|$ for each nonnegative integer n decays faster than any negative power of x as $x \to \pm\infty$. A function in the Schwartz class and all its derivatives belong to $L^p(\mathbb{R})$ for $1 \le p \le +\infty$. In other words, $\int_{-\infty}^{\infty} dx \, \left|f^{(n)}(x)\right|^p \le +\infty$ for every nonnegative integer n and every positive integer p. In the Schwartz class, $f^{(n)}(x)$ is uniformly bounded in $x \in \mathbb{R}$ for every nonnegative integer n.

One of the interesting features of (1.0.1) is its relation to the integrable system for partial differential equations, known as the derivative nonlinear Schrödinger (NLS) equation, given by

$$\begin{cases} iq_t + q_{xx} - i(qrq)_x = 0, \\ ir_t - r_{xx} - i(rqr)_x = 0, \end{cases} \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}^+. \tag{1.0.2}$$

The system in (1.0.2) is related to the scalar equation

$$iq_t + q_{xx} \pm i(q|q|^2)_x = 0, \tag{1.0.3}$$

which is obtained by setting $r(x) = \pm q(x)^*$, where the asterisk denotes complex conjugation. Kaup and Newell [1] studied (1.0.3) and show that the initial-value problem for (1.0.3) can be solved by using the method of the inverse scattering transform [2, 3, 4, 5, 6] related to (1.0.3). We elaborate on the application of (1.0.2) related to the inverse scattering transform in chapter 6.

The system in (1.0.1) is closely related to the standard system which is given by

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix}' = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \qquad x \in \mathbb{R}, \tag{1.0.4}$$

where λ is the spectral parameter, $u(x)$ and $v(x)$ are complex-valued potentials. We assume that $u(x)$ and $v(x)$ belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. The scalar quantities ξ and η are the components of the column vector-valued wavefunction depending on both x and λ. We refer to (1.0.4) as the standard system to emphasize that the potentials $u(x)$ and $v(x)$ appearing in the coefficient matrix in (1.0.4) do not contain the spectral parameter λ. The direct and inverse scattering problems for the standard system are well understood [3, 4, 5, 7]. Our goal is to exploit the relationship between (1.0.1) and (1.0.4) in order to investigate the direct and inverse scattering problem for (1.0.1).

Let us explain the meaning of the direct and inverse scattering problems for (1.0.1). Since the potentials in (1.0.1) vanish as $x \to \pm\infty$, for each real value of ζ the system (1.0.1) must have solutions of the form

$$\begin{bmatrix} a_\pm(\zeta)\, e^{-i\zeta^2 x} \\ b_\pm(\zeta)\, e^{i\zeta^2 x} \end{bmatrix}, \qquad x \to \pm\infty, \tag{1.0.5}$$

for some appropriate choices of $a_\pm(\zeta)$ and $b_\pm(\zeta)$. In quantum mechanics, $e^{i\zeta^2 x}$ can be interpreted as a plane wave moving in the positive x-direction and $e^{-i\zeta^2 x}$ can be interpreted as a plane wave moving in the negative x-direction. This tradition results from the fact that [8] the separation of the variable for the Schrödinger equation results in the time-dependent factor of the form $e^{-\omega t}$, where t is time variable and ω is the energy parameter.

With the choices of $a_+(\zeta) \equiv 1$ and $b_+(\zeta) \equiv 0$, we have the unit-amplitude plane wave $\begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix}$ that is sent from $x = +\infty$ onto the scatterers, i.e. the potentials $q(x)$ and $r(x)$. That plane wave is partly transmitted to $x = -\infty$ and this appears as $\begin{bmatrix} T_r(\zeta)\, e^{-i\zeta^2 x} \\ 0 \end{bmatrix}$ at $x = -\infty$. It is partly reflected to $x = +\infty$ and this appears as $\begin{bmatrix} 0 \\ R(\zeta)\, e^{i\zeta^2 x} \end{bmatrix}$ at $x = +\infty$. Thus, $T_r(\zeta)$ acts as the transmission coefficient from the right and $R(\zeta)$ acts as the reflection coefficient from the right. On the other hand, with the choices of $a_-(\zeta) \equiv 0$ and $b_-(\zeta) \equiv 1$, we have the unit-amplitude plane wave $\begin{bmatrix} 0 \\ e^{-i\zeta^2 x} \end{bmatrix}$ that is sent from $x = -\infty$ onto the scatterers, i.e. the potentials

$q(x)$ and $r(x)$. That plane wave is partly transmitted to $x = +\infty$ and this appears as $\begin{bmatrix} 0 \\ T_l(\zeta)\, e^{i\lambda x} \end{bmatrix}$ at $x = +\infty$. It is partly reflected to $x = -\infty$ and this appears as $\begin{bmatrix} \bar{L}(\zeta) e^{-i\zeta^2 x} \\ 0 \end{bmatrix}$ at $x = -\infty$. Hence, $\bar{T}_l(\zeta)$ acts as the transmission coefficient from the left and $\bar{L}(\zeta)$ acts as the reflection coefficient from the left.

The direct problem for (1.0.1) consists of determining the scattering caused by the pair of potentials $q(x)$ and $r(x)$. The relevant scattering is determined by evaluating the corresponding scattering coefficients, i.e. the transmission and reflection coefficients both from the left and from the right. In addition to the scattering solutions, the system (1.0.1) may also have certain solutions known as bound-state solutions, where the corresponding waves are trapped by the potentials. In the analysis of the direct problem for (1.0.1) one determines both the scattering solutions and the bound-state solutions, and hence one determines the scattering coefficients and the relevant information related to the bound-states.

The inverse problem for (1.0.1) consists of the determination of the potentials $q(x)$ and $r(x)$ in terms of their effect, i.e. in terms of the scattering and bound-states caused by those potentials. In other words, to solve the inverse problem for (1.0.1), one needs to determine the potentials $q(x)$ and $r(x)$ from an appropriate data set containing the scattering coefficients and the bound-state information.

In this thesis we actually use two standard problems, one of which is given by (1.0.4) and the other is given in (3.1.32). Even though (1.0.4) and (3.1.32) are both standard systems, the pair of potentials in (1.0.4) are $u(x)$ and $v(x)$ and the pair of potentials in (3.1.32) are $p(x)$ and $s(x)$, where $p(x)$ and $r(x)$ are different from $u(x)$ and $v(x)$. We analyze the direct and inverse scattering problems for (1.0.1) by utilizing the tools developed for the direct and inverse problems for the standard systems (1.0.4) and (3.1.32). By explicitly determining the relationships between the scattering coefficients for (1.0.1) and the scattering coefficients for (1.0.4) and (3.1.32), we establish the connections between the bound-states for (1.0.1) and the bound-states for (1.0.4) and (3.1.32). We establish the connection between the scattering solutions for the system (1.0.1) and the scattering solutions for the systems (1.0.4) and (3.1.32). We establish the connection between the bound-state data for (1.0.1) and the bound-state data for (1.0.4) and (3.1.32). We also establish the connection between the potentials $q(x)$ and $r(x)$ for (1.0.1) and the potentials $u(x)$ and $v(x)$ for (1.0.4), as well as the potentials $p(x)$ and $v(x)$ for (3.1.32).

This thesis is organized as follows. In Chapter 2 we review the basic results related to the direct and inverse problems for the standard system (1.0.4), i.e. for the first-order system with potentials $u(x)$ and $v(x)$ not depending on energy. We assume that the potentials $u(x)$ and $v(x)$ belong to the Schwartz class, and we introduce the four Jost solutions $\psi(\lambda, x)$, $\phi(\lambda, x)$, $\bar{\psi}(\lambda, x)$, $\bar{\phi}(\lambda, x)$ to (1.0.4) and explain their properties related to their dependence on the spatial coordinate x and the spectral parameter λ. Then, we introduce the scattering coefficients $T(\lambda)$, $R(\lambda)$, $L(\lambda)$, $\bar{T}(\lambda)$, $\bar{R}(\lambda)$, $\bar{L}(\lambda)$ for (1.0.4) and summarize the basic facts related to their properties. We introduce the bound states for (1.0.4) and the quantities relevant to the bound states, namely the bound-state energies, the bound-state dependency constants, the bound-state norming constants and the multiplicities of the bound states. After that, we describe the direct problem and explain how the potentials $u(x)$ and $v(x)$ uniquely determine the corresponding scattering data set for (1.0.4) given in (2.4.1). Next, we turn our attention to the inverse problem for (1.0.4), and we describe the Marchenko method of the recovery of the potentials $u(x)$ and $v(x)$ from the corresponding scattering data in (2.4.1). Although the results presented in this chapter are already known [3, 4, 7], we provide some details for the proofs. This helps establish our notation and helps obtain some further preliminary results needed later on for an energy-dependent system.

In Chapter 3 we develop a method to solve the direct and inverse problems for the first-order system (1.0.1) by using the theory for the direct and inverse problems that is presented for the standard system (1.0.4) in Chapter 2. This is done with the help of various transformations we establish between the energy-dependent system (1.0.1) and the two energy-independent systems (1.0.4) and (3.1.32). We describe the direct problem for (1.0.1), where the goal is to determine the scattering data set (3.5.1) from the potentials $q(x)$ and $r(x)$ appearing in (1.0.1). This is done by explicitly relating the scattering data set for (1.0.1) and scattering data sets for (1.0.4) and (3.1.32). We introduce the relationships among the corresponding Jost solutions for the first-order system (1.0.1) and the standard systems (1.0.4) and (3.1.32). Then, we introduce the relationships among the scattering coefficients for (1.0.1) and the scattering coefficients for the standard systems (1.0.4) and (3.1.32). After that, we relate the potentials $q(x)$ and $r(x)$ for (1.0.1) to the potentials $u(x)$, $v(x)$, $p(x)$, and $s(x)$ appearing in (1.0.4) and (3.1.32). Then, we introduce the bound states and the quantities relevant to the bound states, namely the bound-state energies, the bound-state dependency constants, the bound-state norming constants, and the multiplicities of the bound states. Next, we turn our attention to the inverse problem for (1.0.1), where the goal is to determine the potentials $q(x)$ and $r(x)$ from the scattering data set (3.5.1). We solve this inverse problem by exploiting the relationships we have established between the scattering data set for (1.0.1) and the scattering data sets for (1.0.4) and (3.1.32). The solutions to the inverse problems for the standard systems (1.0.4) and (3.1.32) are already known [3, 4, 5, 7]. We determine the potentials $q(x)$ and $r(x)$ in term of the solutions to the Marchenko equations relevant to (1.0.4) and (3.1.32). Since there are three distinct systems, namely (1.0.1), (1.0.4), and (3.1.32), introduced in our analysis, we carefully identify each relevant quantity by showing whether that quantity is related to (1.0.1), (1.0.4) or (3.1.32). In fact, in addition to the three distinct systems (1.0.1), (1.0.4), and (3.1.32) in our analysis, we use two additional systems given in (3.1.3) and (3.1.31). We carefully relate all the relevant quantities for these five distinct systems. This enables us to develop our method to solve our main inverse problem, i.e. the inverse problem for (1.0.1), by clarifying the relationships among all the five systems.

In Chapter 4 we analyze the inverse problem for (1.0.1) by using a different method. We determine the potentials $q(x)$ and $r(x)$ in (1.0.1) by formulating a Marchenko system, which we call the alternate Marchenko system. Our motivation comes from the work [9] by Tsuchida, where he formulated a Marchenko system to solve (1.0.1). Our alternate Marchenko system is not the same as the system formulated by Tsuchida [9]. Our Marchenko system has the appropriate symmetry properties and resembles the standard Marchenko system [3, 4, 5, 7] used to solve various other inverse problems. Furthermore, the derivation of our Marchenko system is clearly indicated, whereas the derivation of the Tsuchida's Marchenko formulation using certain gauge transformations is not as intuitive and not very clear to us. Nevertheless, we have greatly motivated by the important work of Tsuchida.

In Chapter 5 we analyze the scattering data set used by Kaup and Newell [1] and indicate how the Kaup-Newell data set is related to the scattering coefficients for (1.0.1). Since Kaup and Newell [1] considered only the special case $r(x) = \pm q(x)^*$, we extend the results in [1] by using the two potentials $q(x)$ and $r(x)$ without relating those two potentials to each other. In order to clearly indicate how the scattering data set of Kaup and Newell is related to the scattering theory for (1.0.1), we use the original notation of Kaup and Newell for the quantities used in [1], and we also use our own notation where we use a superscript on quantities to indicate the two potentials to identify the appropriate linear system. For example, we use $\psi^{(\zeta q,\zeta r)}$ for the Jost solution for (1.0.1), $T^{(\zeta q,\zeta r)}$ for the transmission coefficient, and $R^{(\zeta q,\zeta r)}$ for the right-reflection coefficient associated with (1.0.1). Kaup and Newell applied the inverse scattering transform to solve the initial value problem for (1.0.3). They associated (1.0.3) with the linear ordinary differential equation appearing in (1.0.1) in the special case $r(x) = \pm q(x)^*$. They used the Marchenko method, i.e. they set up a linear Marchenko integral equation and recovered $q(x)$ from the solution to the Marchenko equation. Since their goal was to develop the inverse scattering transform method for the derivative NLS equation given in (1.0.3), they were less interested in developing the scattering theory for (1.0.1). It is not quite clear how the scattering data set used by Kaup and Newell is related to the scattering coefficients for (1.0.1).

In Chapter 6 we introduce the AKNS method and list the AKNS pairs $\mathcal{X}$ and $\mathcal{T}$ corresponding to the integrable system (1.0.2). Then, we introduce the AKNS pair corresponding to the Chen-Lee-Liu system (6.0.11). This is done with the help of a transformation for the wave functions between the first-order system (1.0.1) and the first-order system (6.0.14) associated to Chen-Lee-Liu system (6.0.11). After that, we introduce the relationships among the corresponding Jost solutions for the first-order systems (1.0.1) and (6.0.14). Then, we introduce the relationships among the scattering coefficients for (1.0.1) and the scattering coefficients for (6.0.14). Then, we introduce the time evolution of scattering data sets for (1.0.1) and (6.0.14). Next, in the case of the bound states with multiplicities, we use the method of [10, 20, 21] and express the bound-state data in terms of matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$. Then, we introduce the Marchenko kernels $\Omega(y)$ and $\bar{\Omega}(y)$, defined in (2.5.10) and (2.5.11) respectively, in terms of matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$. Finally, we introduce the alternate Marchenko kernels $G^{(u,v)}(y)$, $\bar{G}^{(u,v)}(y)$, $G^{(p,s)}(y)$, and $\bar{G}^{(p,s)}(y)$, defined in (4.2.11) and (4.2.12) respectively, in terms of a matrix triplet (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$.

Chapter 2

Scattering and Inverse Scattering for a First-Order System

2.1 Scattering for the Standard System and Jost Solutions

In this chapter we review the direct and inverse problem for the standard system, i.e. for a first-order system with potentials not depending on the energy, which is given by

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix}' = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \qquad x \in \mathbb{R}, \tag{2.1.1}$$

where the prime denotes the x-derivative, λ is the spectral parameter, and $u(x)$ and $v(x)$ are complex-valued potentials. We assume that $u(x)$ and $v(x)$ belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. The scalar quantities ξ and η depend on both x and λ. Although the results presented in this section are already known [3, 4, 5, 7], we provide some details for the proofs. This helps to establish our notation and to obtain some further preliminary results needed later on for an energy-dependent system.

There are two linearly independent column-vector solutions to (2.1.1), $\psi(\lambda, x)$ and $\phi(\lambda, x)$, known as the Jost solutions [3, 4, 5, 7, 13], which are uniquely determined by imposing the asymptotic conditions

$$\psi(\lambda, x) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad \bar{\psi}(\lambda, x) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{2.1.2}$$

$$\phi(\lambda, x) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad \bar{\phi}(\lambda, x) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{2.1.3}$$

where the overbar used does not indicate complex conjugation. In this thesis, we use the asterisk $*$ for complex conjugation. For convenience, let us define

$$m(\lambda,x) := \psi(\lambda,x)\, e^{-i\lambda x}, \quad \bar{m}(\lambda,x) := \bar{\psi}(\lambda,x)\, e^{i\lambda x}, \tag{2.1.4}$$

$$n(\lambda,x) := \phi(\lambda,x)\, e^{i\lambda x}, \quad \bar{n}(\lambda,x) := \bar{\phi}(\lambda,x)\, e^{-i\lambda x}. \tag{2.1.5}$$

Using (2.1.4) and (2.1.5) in (2.1.2) and (2.1.3) we get

$$m(\lambda,x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + o(1), \quad \bar{m}(\lambda,x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{2.1.6}$$

$$n(\lambda,x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + o(1), \quad \bar{n}(\lambda,x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + o(1), \qquad x \to -\infty. \tag{2.1.7}$$

In the next proposition, we show that $m(\lambda,x)$ and $\bar{m}(\lambda,x)$ defined in (2.1.4) satisfy certain integral relations, which are obtained by combining the differential equation for (2.1.4) and the asymptotic conditions of (2.1.6). Let us write $m(\lambda,x)$ and $\bar{m}(\lambda,x)$ in the component form as

$$m(\lambda,x) = \begin{bmatrix} m_1(\lambda,x) \\ m_2(\lambda,x) \end{bmatrix}, \quad \bar{m}(\lambda,x) = \begin{bmatrix} \bar{m}_1(\lambda,x) \\ \bar{m}_2(\lambda,x) \end{bmatrix}. \tag{2.1.8}$$

We can write (2.1.6) as

$$\begin{bmatrix} m_1(\lambda,x) \\ m_2(\lambda,x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + o(1), \quad \begin{bmatrix} \bar{m}_1(\lambda,x) \\ \bar{m}_2(\lambda,x) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty. \tag{2.1.9}$$

Proposition 2.1. *The components of vector-valued functions $m(\lambda,x)$ and $\bar{m}(\lambda,x)$ given in (2.1.8) satisfy the integral relations*

$$m_1(\lambda,x) = -\int_x^\infty dy\, u(y)\, m_2(\lambda,y)\, e^{2i\lambda(x-y)}, \tag{2.1.10}$$

$$m_2(\lambda, x) = 1 + \int_x^\infty dy \int_y^\infty dz\, v(y)\, u(z)\, m_2(\lambda, z)\, e^{2i\lambda(z-y)}, \tag{2.1.11}$$

$$\bar{m}_1(\lambda, x) = 1 + \int_x^\infty dy \int_y^\infty dz\, u(y)\, v(z)\, \bar{m}_1(\lambda, z)\, e^{2i\lambda(y-z)}, \tag{2.1.12}$$

$$\bar{m}_2(\lambda, x) = -\int_x^\infty dy\, u(y)\, \bar{m}_2(\lambda, y)\, e^{2i\lambda(y-x)}. \tag{2.1.13}$$

Proof. Since the Jost solution $\psi(\lambda, x)$ appearing in (2.1.2) is a solution to (2.1.1), we have

$$\psi'(\lambda, x) = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \psi(\lambda, x). \tag{2.1.14}$$

Using the first equality of (2.1.4) in (2.1.14) we obtain

$$[m(\lambda, x)\, e^{-i\lambda x}]' = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} m(\lambda, x)\, e^{i\lambda x},$$

or equivalently

$$i\lambda\, m(\lambda, x)\, e^{i\lambda x} + m'(\lambda, x)\, e^{i\lambda x} = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} m(\lambda, x)\, e^{i\lambda x}. \tag{2.1.15}$$

With the help of the first equality of (2.1.8), from (2.1.15) we get

$$\begin{bmatrix} m_1'(\lambda, x) \\ m_2'(\lambda, x) \end{bmatrix} + i\lambda \begin{bmatrix} m_1(\lambda, x) \\ m_2(\lambda, x) \end{bmatrix} = \begin{bmatrix} -i\lambda\, m_1(\lambda, x) + u(x)\, m_2(\lambda, x) \\ v(x)\, m_1(\lambda, x) + i\lambda\, m_2(\lambda, x \end{bmatrix},$$

which yields

$$\begin{cases} m_1'(\lambda, x) + 2i\lambda\, m_1(\lambda, x) = u(x)\, m_2(\lambda, x), \\ m_2'(\lambda, x) = v(x)\, m_1(\lambda, x). \end{cases} \tag{2.1.16}$$

Using an integrating factor in the first line of (2.1.16), we obtain

$$\begin{cases} [m_1(\lambda, x)\, e^{2i\lambda x}]' = u(x)\, m_2(\lambda, x)\, e^{2i\lambda x}, \\ m_2'(\lambda, x) = v(x)\, m_1(\lambda, x). \end{cases} \tag{2.1.17}$$

Integrating (2.1.17) with the asymptotic condition of the first equality in (2.1.9) we get

$$m_1(\lambda, x) = -\int_x^\infty dy\, u(y)\, m_2(\lambda, y)\, e^{2i\lambda(y-x)}, \tag{2.1.18}$$

$$m_2(\lambda, x) = 1 - \int_x^\infty dy\, v(y)\, m_1(\lambda, y). \tag{2.1.19}$$

Note that (2.1.18) agrees with (2.1.10). Using (2.1.18) in (2.1.19) we get

$$m_2(\lambda, x) = 1 + \int_x^\infty dy \int_y^\infty dz\, v(y)\, u(z)\, m_2(\lambda, z)\, e^{2i\lambda(z-y)},$$

which establishes (2.1.11). Next, we prove (2.1.12) and (2.1.13). Since the Jost solution $\bar{\psi}(\lambda, x)$ appearing in (2.1.2) is a solution to (2.1.1), we have

$$\bar{\psi}'(\lambda, x) = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \bar{\psi}(\lambda, x). \tag{2.1.20}$$

Using the second equality of (2.1.4) in (2.1.20) we get

$$[\bar{m}(\lambda, x)\, e^{-i\lambda x}]' = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \bar{m}(\lambda, x)\, e^{-i\lambda x},$$

which is equivalent to

$$-i\lambda\, \bar{m}(\lambda, x)\, e^{-i\lambda x} + \bar{m}'(\lambda, x)\, e^{-i\lambda x} = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \bar{m}(\lambda, x)\, e^{-i\lambda x}. \tag{2.1.21}$$

With the help of the second equality of (2.1.8), from (2.1.21) we obtain

$$\begin{bmatrix} \bar{m}_1'(\lambda, x) \\ \bar{m}_2'(\lambda, x) \end{bmatrix} - i\lambda \begin{bmatrix} \bar{m}_1(\lambda, x) \\ \bar{m}_2(\lambda, x) \end{bmatrix} = \begin{bmatrix} -i\lambda\, \bar{m}_1(\lambda, x) + u(x)\, \bar{m}_2(\lambda, x) \\ v(x)\, \bar{m}_1(\lambda, x) + i\lambda\, \bar{m}_2(\lambda, x \end{bmatrix},$$

which yields

$$\begin{cases} \bar{m}_1'(\lambda, x) = u(x)\, \bar{m}_2(\lambda, x), \\ \bar{m}_2'(\lambda, x) - 2i\lambda\, \bar{m}_2(\lambda, x) = v(x)\, \bar{m}_1(\lambda, x). \end{cases} \tag{2.1.22}$$

Using an integrating factor in the second line of (2.1.22), we have

$$\begin{cases} \bar{m}_1'(\lambda,x) = u(x)\,\bar{m}_2(\lambda,x), \\ [\bar{m}_2(\lambda,x)\,e^{-2i\lambda x}]' = v(x)\,\bar{m}_1(\lambda,x)\,e^{-2i\lambda x}. \end{cases} \tag{2.1.23}$$

Integrating (2.1.23) with the asymptotic conditions of the second equation in (2.1.9) we get

$$\bar{m}_1(\lambda,x) = 1 - \int_x^\infty dy\,u(y)\,\bar{m}_2(\lambda,y), \tag{2.1.24}$$

$$\bar{m}_2(\lambda,x) = -\int_x^\infty dy\,v(y)\,\bar{m}_1(\lambda,y)\,e^{2i\lambda(x-y)}. \tag{2.1.25}$$

Note that (2.1.25) coincides with (2.1.13). Substituting (2.1.25) in (2.1.24) we obtain

$$\bar{m}_1(\lambda,x) = 1 + \int_x^\infty dy \int_y^\infty dz\,u(y)\,v(z)\,\bar{m}_1(\lambda,z)\,e^{2i\lambda(y-z)},$$

which completes the proof of (2.1.12). □

The next proposition is the analog of Proposition 2.1, where we show that $n(\lambda,x)$ and $\bar{n}(\lambda,x)$ appearing in (2.1.5) satisfy certain integral relations. Let us write $n(\lambda,x)$ and $\bar{n}(\lambda,x)$ in terms of their components as

$$n(\lambda,x) = \begin{bmatrix} n_1(\lambda,x) \\ n_2(\lambda,x) \end{bmatrix}, \quad \bar{n}(\lambda,x) = \begin{bmatrix} \bar{n}_1(\lambda,x) \\ \bar{n}_2(\lambda,x) \end{bmatrix}. \tag{2.1.26}$$

We express the asymptotes in (2.1.7) as

$$\begin{bmatrix} n_1(\lambda,x) \\ n_2(\lambda,x) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + o(1), \quad \begin{bmatrix} \bar{n}_1(\lambda,x) \\ \bar{n}_2(\lambda,x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + o(1), \qquad x \to -\infty. \tag{2.1.27}$$

Proposition 2.2. *The components of vector-valued functions $n(\lambda,x)$ and $\bar{n}(\lambda,x)$ given in* (2.1.26) *satisfy the integral relations*

$$n_1(\lambda,x) = 1 + \int_{-\infty}^x dy \int_{-\infty}^y dz\,u(y)\,v(z)\,n_1(\lambda,z)\,e^{2i\lambda(y-z)}, \tag{2.1.28}$$

$$n_2(\lambda,x)=\int_{-\infty}^{x} dy\, v(y)\, n_1(\lambda,y)\, e^{2i\lambda(x-y)}, \tag{2.1.29}$$

$$\bar{n}_1(\lambda,x)=\int_{-\infty}^{x} dy\, u(y)\, \bar{n}_2(\lambda,y)\, e^{2i\lambda(x-y)}, \tag{2.1.30}$$

$$\bar{n}_2(\lambda,x)=1+\int_{-\infty}^{x} dy \int_{-\infty}^{y} dz\, v(y)\, u(z)\, \bar{n}_2(\lambda,z)\, e^{2i\lambda(z-y)}. \tag{2.1.31}$$

Proof. The Jost solution $\phi(\lambda,x)$ appearing in (2.1.3) satisfies (2.1.1), and hence we have

$$\phi'(\lambda,x)=\begin{bmatrix}-i\lambda & u(x)\\ v(x) & i\lambda\end{bmatrix}\phi(\lambda,x). \tag{2.1.32}$$

Using the first equality of (2.1.5) in (2.1.32) we get

$$[n(\lambda,x)\, e^{-i\lambda x}]'=\begin{bmatrix}-i\lambda & u(x)\\ v(x) & i\lambda\end{bmatrix} n(\lambda,x)\, e^{-i\lambda x},$$

or equivalently

$$-\,i\lambda\, n(\lambda,x)\, e^{-i\lambda x}+n'(\lambda,x)\, e^{-i\lambda x}=\begin{bmatrix}-i\lambda & u(x)\\ v(x) & i\lambda\end{bmatrix} n(\lambda,x)\, e^{-i\lambda x}. \tag{2.1.33}$$

With the help of the first equality of (2.1.26), from (2.1.33) we obtain

$$\begin{bmatrix}n_1'(\lambda,x)\\ n_2'(\lambda,x)\end{bmatrix}-i\lambda\begin{bmatrix}n_1(\lambda,x)\\ n_2(\lambda,x)\end{bmatrix}=\begin{bmatrix}-i\lambda\, n_1(\lambda,x)+u(x)\, n_2(\lambda,x)\\ v(x)\, n_1(\lambda,x)+i\lambda\, n_2(\lambda,x\end{bmatrix},$$

which yields

$$\begin{cases} n_1'(\lambda,x)=u(x)\, n_2(\lambda,x),\\ n_2'(\lambda,x)-2i\lambda\, n_2(\lambda,x)=v(x)\, n_1(\lambda,x).\end{cases} \tag{2.1.34}$$

Using an integrating factor in the second line of (2.1.34), we have

$$\begin{cases} n_1'(\lambda,x)=u(x)\, n_2(\lambda,x),\\ [n_2(\lambda,x)\, e^{-2i\lambda x}]'=v(x)\, n_1(\lambda,x)\, e^{-2i\lambda x}.\end{cases} \tag{2.1.35}$$

Integrating (2.1.35) with the help of the first asymptotic condition of (2.1.27), we get

$$n_1(\lambda,x) = 1 + \int_{-\infty}^{x} dy\, u(y)\, n_2(\lambda,y), \tag{2.1.36}$$

$$n_2(\lambda,x) = \int_{-\infty}^{x} dy\, v(y)\, n_1(\lambda,y)\, e^{2i\lambda(x-y)}. \tag{2.1.37}$$

Note that (2.1.37) coincide with (2.1.29). Substituting (2.1.37) into (2.1.36), we obtain

$$n_1(\lambda,x) = 1 + \int_{-\infty}^{x} dy \int_{-\infty}^{z} dz\, u(y)\, v(z)\, n_1(\lambda,z)\, e^{2i\lambda(y-z)},$$

which establishes (2.1.28). Next, we prove (2.1.30) and (2.1.31). The Jost solution $\bar{\phi}(\lambda,x)$ appearing in (2.1.3) satisfies (2.1.1), and thus we have

$$\bar{\phi}'(\lambda,x) = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \bar{\phi}(\lambda,x). \tag{2.1.38}$$

Using the second equality of (2.1.5) in (2.1.38) we get

$$[\bar{n}(\lambda,x)\, e^{i\lambda x}]' = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \bar{n}(\lambda,x)\, e^{i\lambda x},$$

which is equivalent to

$$i\lambda\, \bar{n}(\lambda,x)\, e^{i\lambda x} + \bar{n}'(\lambda,x)\, e^{i\lambda x} = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \bar{n}(\lambda,x)\, e^{i\lambda x}. \tag{2.1.39}$$

With the help of the second equality of (2.1.26), from (2.1.39) we obtain

$$\begin{bmatrix} \bar{n}_1'(\lambda,x) \\ \bar{n}_2'(\lambda,x) \end{bmatrix} + i\lambda \begin{bmatrix} \bar{n}_1(\lambda,x) \\ \bar{n}_2(\lambda,x) \end{bmatrix} = \begin{bmatrix} -i\lambda\, \bar{n}_1(\lambda,x) + u(x)\, \bar{n}_2(\lambda,x) \\ v(x)\, \bar{n}_1(\lambda,x) + i\lambda\, \bar{n}_2(\lambda,x \end{bmatrix},$$

or equivalently

$$\begin{cases} \bar{n}_1'(\lambda,x) + 2i\lambda\, \bar{n}_1(\lambda,x) = u(x)\, \bar{n}_2(\lambda,x), \\ n_2'(\lambda,x) = v(x)\, \bar{n}_1(\lambda,x). \end{cases} \tag{2.1.40}$$

Using an integrating factor in the first line of (2.1.40), we have

$$\begin{cases} [\bar{n}_1(\lambda,x)\, e^{2i\lambda x}]' = u(x)\, \bar{n}_2(\lambda,x)\, e^{2i\lambda x}, \\ \\ \bar{n}_2'(\lambda,x) = v(x)\, \bar{n}_1(\lambda,x). \end{cases} \tag{2.1.41}$$

Integrating (2.1.41) with the second asymptotic condition of (2.1.27) we get

$$\bar{n}_1(\lambda,x) = \int_{-\infty}^{x} dy\, u(y)\, \bar{n}_2(\lambda,y)\, e^{2i\lambda(y-x)}, \tag{2.1.42}$$

$$\bar{n}_2(\lambda,x) = 1 + \int_{-\infty}^{x} dy\, v(y)\, \bar{n}_1(\lambda,y). \tag{2.1.43}$$

Next, by using (2.1.42) in (2.1.43) we obtain

$$\bar{n}_1(\lambda,x) = \int_{-\infty}^{x} dy\, u(y)\, \bar{n}_2(\lambda,y)\, e^{2i\lambda(x-y)},$$
$$\bar{n}_2(\lambda,x) = 1 + \int_{-\infty}^{x} dy \int_{-\infty}^{y} dz\, v(y)\, u(z)\, \bar{n}_2(\lambda,z)\, e^{2i\lambda(z-y)},$$

which completes the proof of (2.1.30) and (2.1.31). □

The next result deals with the analyticity properties in λ for $m(\lambda,x)$, $n(\lambda,x)$, $\bar{m}(\lambda,x)$, and $\bar{n}(\lambda,x)$ appearing in (2.1.4) and (2.1.5).

Theorem 2.1. *Assume that the potentials $u(x)$ and $v(x)$ appearing in the system (2.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then:*

(a) *For each fixed $x \in \mathbb{R}$, the vector-valued functions $m(\lambda,x)$ and $n(\lambda,x)$ each have analytic extensions from $\lambda \in \mathbb{R}$ to $\lambda \in \mathbb{C}^+$. Furthermore, these two vector-valued functions are continuous in $\lambda \in \overline{\mathbb{C}^+}$ for each fixed $x \in \mathbb{R}$.*

(b) *For each fixed $x \in \mathbb{R}$, the vector-valued functions $\bar{m}(\lambda,x)$ and $\bar{n}(\lambda,x)$ each have analytic extensions from $\lambda \in \mathbb{R}$ to $\lambda \in \mathbb{C}^-$. Furthermore, these two vector-valued functions are continuous in $\lambda \in \overline{\mathbb{C}^-}$ for each fixed $x \in \mathbb{R}$.*

Proof. These analyticity properties are established by iterating the integral representation given in Proposition 2.1 and 2.2. The proof of (a) is obtained as follows. Iterating (2.1.11) we can express $m_2(\lambda, x)$ as an infinite summation as

$$m_2(\lambda, x) = \sum_{j=0}^{\infty} k_j(\lambda, x), \tag{2.1.44}$$

where we have

$$k_0(\lambda, x) := 1, \tag{2.1.45}$$

$$k_j(\lambda, x) := \int_x^\infty dy \int_y^\infty dz\, v(y)\, u(z)\, k_{j-1}(\lambda, z)\, e^{2i\lambda(z-y)}, \qquad j = 1, 2, 3, \dots. \tag{2.1.46}$$

Changing the order of integration in (2.1.46) we get

$$k_j(\lambda, x) = \int_x^\infty dz \int_x^z dy\, v(y)\, u(z)\, k_{j-1}(\lambda, z)\, e^{2i\lambda(z-y)}. \tag{2.1.47}$$

Let us define

$$M(\lambda, x, z) := \int_x^z dy\, v(y) u(z) e^{2i\lambda(z-y)}. \tag{2.1.48}$$

Using (2.1.48) in (2.1.47) we have

$$k_j(\lambda, x) = \int_x^\infty dz\, M(\lambda, x, z)\, k_{j-1}(\lambda, z). \tag{2.1.49}$$

With the help of $|e^{2i\lambda(z-y)}| \le 1$ when $y \le z$ and $\lambda \in \overline{C^+}$, from (2.1.48) we obtain

$$\int_x^\infty dz\, |M(\lambda, x, z)| \le \int_x^\infty dy\, |v(y)| \int_x^\infty dz\, |u(z)|. \tag{2.1.50}$$

Using (2.1.45) and (2.1.50) in (2.1.49) for $j = 1$, we get

$$|k_1(\lambda, x)| \le \int_x^\infty dy\, |v(y)| \int_x^\infty dz\, |u(z)|. \tag{2.1.51}$$

For $j = 2$, we have

$$k_2(\lambda, x) = \int_x^\infty dz\, M(\lambda, x, z)\, k_1(\lambda, z). \tag{2.1.52}$$

Substituting (2.1.50) and (2.1.51) in (2.1.52) we obtain

$$|k_2(\lambda,x)| \le \int_x^\infty dy\,|v(y)| \int_x^\infty dz\,|u(z)| \int_z^\infty dt\,|v(t)| \int_z^\infty ds\,|u(s)|,$$

or equivalently

$$|k_2(\lambda,x)| \le \left(\int_x^\infty dy\,|v(y)| \int_z^\infty dt\,|v(t)|\right)\left(\int_x^\infty dz\,|u(z)| \int_z^\infty ds\,|u(s)|\right). \quad (2.1.53)$$

Let us define

$$A(z) := \int_z^\infty dt\,|v(t)|, \qquad B(z) := \int_z^\infty ds\,|u(s)|, \quad (2.1.54)$$

which are uniformly decreasing in z and

$$A'(z) = -|v(z)|, \qquad B'(z) = -|u(z)|. \quad (2.1.55)$$

Using (2.1.54) and (2.1.55) in (2.1.53), we obtain

$$|k_2(\lambda,x)| \le \left(\int_x^\infty dy\,(-A'(y))\,A(y)\right)\left(\int_x^\infty dz\,(-B'(z)\,B(z)\right). \quad (2.1.56)$$

With help of integration by parts, we can write (2.1.56) as

$$|k_2(\lambda,x)| \le \frac{[A(x)]^2}{2!}\ \frac{[B(x)]^2}{2!},$$

or equivalently

$$|k_2(\lambda,x)| \le \frac{1}{2!}\left(\int_x^\infty dy\,|v(y)|\right)^2 \frac{1}{2!}\left(\int_x^\infty dz\,|u(z)|\right)^2. \quad (2.1.57)$$

In a similar way for $j = n$ we obtain

$$|k_n(\lambda,x)| \le \frac{1}{n!}\left(\int_x^\infty dy\,|v(y)|\right)^n \frac{1}{n!}\left(\int_x^\infty dz\,|u(z)|\right)^n. \quad (2.1.58)$$

Since the potentials $u(x)$ and $v(x)$ belong to the Schwartz class $\mathbb{S}(\mathbb{R})$, each $k_j(\lambda,x)$ is bounded for each $\lambda \in \overline{\mathbb{C}^+}$. Thus, from $j = 1,2,3,...,n$ we obtain

$$\begin{aligned}\sum_{j=0}^{n} |k_j(\lambda,x)| \le 1 &+ \left(\int_x^\infty dy\,|v(y)|\right)\left(\int_x^\infty dz\,|u(z)|\right) \\ &+ \frac{1}{(2!)^2}\left(\int_x^\infty dy\,|v(y)|\right)^2\left(\int_x^\infty dz\,|u(z)|\right)^2 \\ &+ ... + \frac{1}{(n!)^2}\left(\int_x^\infty dy\,|v(y)|\right)^n\left(\int_x^\infty dz\,|u(z)|\right)^n,\end{aligned}$$

or equivalently

$$\sum_{j=0}^{n} |k_j(\lambda,x)| \le \sum_{j=0}^{n} \frac{\left[\left(\int_x^\infty dy\,|v(y)|\right)^{1/2}\left(\int_x^\infty dz\,|u(z)|\right)^{1/2}\right]^{2j}}{j!},$$

which can be written as

$$\begin{aligned}|m_2(\lambda,x)| &\le \sum_{j=0}^{\infty} \frac{\left[\left(\int_x^\infty dy\,|v(y)|\right)^{1/2}\left(\int_x^\infty dz\,|u(z)|\right)^{1/2}\right]^{2j}}{j!}\\ &= J_0\Big(2\Big(\int_x^\infty dy\,|v(y)|\Big)^{1/2}\Big(\int_x^\infty dz\,|u(z)|\Big)^{1/2}\Big),\end{aligned}$$

where J_0 is [6] the Bessel function of order zero. Thus, the series given in (2.1.44) is uniformly convergent. Each $k_j(\lambda,x)$ is analytic in λ for $\lambda \in \overline{\mathbb{C}^+}$ and continuous in λ for $\lambda \in \mathbb{C}^+$. Thus, by the Weierstrass theorem $m_2(\lambda,x)$ is the limit of a uniformly convergent sequence of analytic functions in every compact subset in $\mathbb{C}^+$. Consequently, $m_2(\lambda,x)$ is analytic in λ for $\lambda \in \mathbb{C}^+$ and continuous in λ for $\lambda \in \overline{\mathbb{C}^+}$. The analyticity and continuity of $n(\lambda,x)$, $\bar{m}(\lambda,x)$, $\bar{n}(\lambda,x)$ can be proved in the same way. □

In the next theorem, we present the large-λ asymptotics of the quantities $m(\lambda,x)$, $n(\lambda,x)$, $\bar{m}(\lambda,x)$, and $\bar{n}(\lambda,x)$ appearing in (2.1.4) and (2.1.5). Let us write $m(\lambda,x)$ and $\bar{m}(\lambda,x)$ in terms of their components as

$$m(\lambda,x) = \begin{bmatrix} m_1(\lambda,x) \\ m_2(\lambda,x) \end{bmatrix}, \quad \bar{m}(\lambda,x) = \begin{bmatrix} \bar{m}_1(\lambda,x) \\ \bar{m}_2(\lambda,x) \end{bmatrix}. \tag{2.1.59}$$

Theorem 2.2. *Assume that the potentials $u(x)$ and $v(x)$ appearing in the system (2.1.1) belong to the Schwarz class $\mathbb{S}(\mathbb{R})$. Then for the quantities appearing in (2.1.26) and (2.1.59) we have*

$$m_1(\lambda,x) = \frac{1}{2i\lambda}\,u(x) + o\left(\frac{1}{\lambda}\right), \qquad \lambda\to\infty \quad \text{in} \quad \overline{\mathbb{C}^+}, \tag{2.1.60}$$

$$m_2(\lambda,x) = 1 - \frac{1}{2i\lambda}\int_x^\infty dz\,u(z)\,v(z) + o\left(\frac{1}{\lambda}\right), \qquad \lambda\to\infty \quad \text{in} \quad \overline{\mathbb{C}^+}, \tag{2.1.61}$$

$$\bar{m}_1(\lambda,x) = 1 + \frac{1}{2i\lambda}\int_x^\infty dz\, u(z)\, v(z) + o\left(\frac{1}{\lambda}\right), \qquad \lambda\to\infty \quad \text{in} \quad \overline{\mathbb{C}^-}, \tag{2.1.62}$$

$$\bar{m}_2(\lambda,x) = -\frac{1}{2i\lambda}\, v(x) + o\left(\frac{1}{\lambda}\right), \qquad \lambda\to\infty \quad \text{in} \quad \overline{\mathbb{C}^-}, \tag{2.1.63}$$

$$n_1(\lambda,x) = 1 - \frac{1}{2i\lambda}\int_{-\infty}^x dz\, u(z)\, v(z) + o\left(\frac{1}{\lambda}\right), \qquad \lambda\to\infty \quad \text{in} \quad \overline{\mathbb{C}^+}, \tag{2.1.64}$$

$$n_2(\lambda,x) = \frac{1}{2i\lambda}\, v(x) + o\left(\frac{1}{\lambda}\right), \qquad \lambda\to\infty \quad \text{in} \quad \overline{\mathbb{C}^+}, \tag{2.1.65}$$

$$\bar{n}_1(\lambda,x) = -\frac{1}{2i\lambda}\, u(x) + o\left(\frac{1}{\lambda}\right), \qquad \lambda\to\infty \quad \text{in} \quad \overline{\mathbb{C}^-}, \tag{2.1.66}$$

$$\bar{n}_2(\lambda,x) = 1 + \frac{1}{2i\lambda}\int_{-\infty}^x dz\, u(z)\, v(z) + o\left(\frac{1}{\lambda}\right), \qquad \lambda\to\infty \quad \text{in} \quad \overline{\mathbb{C}^-}, \tag{2.1.67}$$

Proof. The proof for (2.1.61) is obtain as follows. Changing the order of integration in (2.1.11) we get

$$m_2(\lambda,x) = 1 + \int_x^\infty dz \int_x^z dy\, v(y)\, u(z)\, m_2(\lambda,z)\, e^{2i\lambda(y-z)},$$

which can be written as

$$m_2(\lambda,x) = 1 + \int_x^\infty dz\, u(z)\, m_2(\lambda,z) \int_x^z dy \left[\frac{d}{dy}\left(v(y)\,\frac{e^{2i\lambda(z-y)}}{-2i\lambda}\right) + \frac{1}{2i\lambda}\, v'(y)\, e^{2i\lambda(z-y)}\right],$$

or equivalently

$$\begin{aligned} m_2(\lambda,x) =& 1 - \frac{1}{2i\lambda}\int_x^\infty dz\, u(z)v(z)m_2(\lambda,z) \\ &+ \frac{1}{2i\lambda}\int_x^\infty dz\, v(x)\, m_2(\lambda,z)\, u(z)\, e^{2i\lambda(x-z)} \\ &+ \frac{1}{2i\lambda}\int_x^\infty dz \int_x^z dy\, u(z)\, m_2(\lambda,z)\, v'(y)\, e^{2i\lambda(y-z)}. \end{aligned} \tag{2.1.68}$$

Replacing $m_2(\lambda,x)$ by $1+[m_2(\lambda,x)-1]$ in the first integral in (2.1.68) we have

$$\begin{aligned} m_2(\lambda,x) =& 1 - \frac{1}{2i\lambda}\int_x^\infty dz\, u(z)\, v(z) - \frac{1}{2i\lambda}\int_x^\infty dz\, u(z)\, v(z)\Big[(m_2(\lambda,z)-1)\Big] \\ &+ \frac{1}{2i\lambda}\int_x^\infty dz\, v(x)\, m_2(\lambda,z)\, u(z)\, e^{2i\lambda(x-z)} \\ &+ \frac{1}{2i\lambda}\int_x^\infty dz \int_x^z dy\, u(z)\, m_2(\lambda,z)\, v'(y)\, e^{2i\lambda(y-z)}. \end{aligned} \tag{2.1.69}$$

Iterating (2.1.69) we get

$$m_2(\lambda,x) = 1 - \frac{1}{2i\lambda}\int_x^\infty u(z)v(z) + o\left(\frac{1}{\lambda}\right), \qquad \lambda\to\infty \quad \text{in} \quad \overline{\mathbb{C}^+}, \tag{2.1.70}$$

which completes the proof of (2.1.61). Let us now prove (2.1.60). Replacing $m_2(\lambda,x)$ by $1+[m_2(\lambda,x)-1]$ in (2.1.10) we get

$$m_1(\lambda,x) = -\int_x^\infty dy\, u(y)\, e^{2i\lambda(y-x)}\Big[1+(m_2(\lambda,y)-1)\Big],$$

or equivalently

$$m_1(\lambda,x) = -\int_x^\infty dy\, u(y)e^{2i\lambda(y-x)} - \int_x^\infty dy\, u(y)e^{2i\lambda(y-x)}\Big[m_2(\lambda,y)-1\Big]. \tag{2.1.71}$$

With the help of (2.1.70), from (2.1.71) we obtain

$$m_1(\lambda,x) = -\int_x^\infty dy\, u(y)e^{2i\lambda(y-x)} + o\left(\frac{1}{\lambda}\right), \qquad \lambda\to\infty \quad \text{in} \quad \overline{\mathbb{C}^+},$$

which can be written as

$$m_1(\lambda,x) = \int_x^\infty dy\left[-\frac{d}{dy}\left(u(y)\,\frac{e^{2i\lambda(y-x)}}{2i\lambda}\right) + u'(y)\,\frac{e^{2i\lambda(y-x)}}{2i\lambda}\right] + o\left(\frac{1}{\lambda}\right),$$

or equivalently

$$m_1(\lambda,x) = \frac{u(x)}{2i\lambda} + \frac{1}{2i\lambda}\int_x^\infty dy\, u'(y),\frac{e^{2i\lambda(y-x)}}{2i\lambda} + o\left(\frac{1}{\lambda}\right). \tag{2.1.72}$$

Since $u'(x)\in\mathbb{S}(\mathbb{R})$ and $|e^{2i\lambda(y-x)}|\le 1$ for $x\le y$ and $\lambda\in\overline{\mathbb{C}^+}$, the integral $\int_x^\infty$ in (2.1.72) is converges for all $x\in\mathbb{R}$. Hence, we obtain

$$m_1(\lambda,x) = \frac{1}{2i\lambda}\,u(x) + o\left(\frac{1}{\lambda}\right), \qquad \lambda\to\infty \quad \text{in} \quad \overline{\mathbb{C}^+},$$

which establishes (2.1.60). Similarly, the proof of (2.1.64) is obtained as follows. Changing the order of integration in (2.1.28) we get

$$n_1(\lambda,x) = 1 + \int_{-\infty}^x dz\int_z^x dy\, u(y)\, v(z)\, n_1(\lambda,z)\, e^{2i\lambda(y-z)},$$

or equivalently

$$n_1(\lambda,x) = 1 + \int_{-\infty}^{x} dz\, v(z)\, n_1(\lambda,z) \int_{z}^{x} dy \left[\frac{d}{dy}\Big(u(y)\, \frac{e^{2i\lambda(y-z)}}{2i\lambda} \Big) - \frac{1}{2i\lambda}\, u'(y)\, e^{2i\lambda(y-z)} \right],$$

which can be written as

$$\begin{aligned} n_1(\lambda,x) =& 1 - \frac{1}{2i\lambda} \int_{-\infty}^{x} dz\, u(z)\, v(z)\, n_1(\lambda,z) \\ &+ \frac{1}{2i\lambda} \int_{-\infty}^{x} dz\, v(z)\, n_1(\lambda,z)\, u(y)\, e^{2i\lambda(x-z)} \\ &- \frac{1}{2i\lambda} \int_{-\infty}^{x} dz \int_{z}^{x} dy\, v(z)\, n_1(\lambda,z)\, u'(y)\, e^{2i\lambda(y-z)}. \end{aligned} \tag{2.1.73}$$

Replacing $n_1(\lambda,z)$ by $1 + [n_1(\lambda,z) - 1]$ in the first integral in (2.1.73) we get

$$\begin{aligned} n_1(\lambda,x) =& 1 - \frac{1}{2i\lambda} \int_{-\infty}^{x} dz\, u(z)\, v(z) - \frac{1}{2i\lambda} \int_{-\infty}^{x} dz\, u(z)\, v(z) \Big[(n_1(\lambda,z) - 1) \Big] \\ &+ \frac{1}{2i\lambda} \int_{-\infty}^{x} dz\, v(z)\, n_1(\lambda,z)\, u(y)\, e^{2i\lambda(x-z)} \\ &- \frac{1}{2i\lambda} \int_{-\infty}^{x} dz \int_{z}^{x} dy\, v(z)\, n_1(\lambda,z)\, u'(y)\, e^{2i\lambda(y-z)}. \end{aligned} \tag{2.1.74}$$

By iterating (2.1.74) we obtain

$$n_1(\lambda,x) = 1 - \frac{1}{2i\lambda} \int_{-\infty}^{x} dz\, u(z)\, v(z) + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^+}, \tag{2.1.75}$$

which establishes (2.1.64). Now let us prove (2.1.65). Replacing $n_1(\lambda,y)$ by $1 + [n_1(\lambda,y) - 1]$ in (2.1.29) we get

$$n_2(\lambda,x) = \int_{-\infty}^{x} dy\, v(y) e^{2i\lambda(x-y)} + \int_{-\infty}^{x} dy\, v(y) e^{2i\lambda(x-y)} \Big[n_1(\lambda,y) - 1 \Big]. \tag{2.1.76}$$

Using (2.1.75) in (2.1.76) we have

$$n_2(\lambda,x) = \int_{-\infty}^{x} dy\, v(y) e^{2i\lambda(x-y)} + o\left(\frac{1}{\lambda}\right), \tag{2.1.77}$$

which can be written as

$$n_2(\lambda,x) = \int_{-\infty}^{x} dy \left[-\frac{d}{dy}\left(v(y)\, \frac{e^{2i\lambda(x-y)}}{2i\lambda} \right) + v'(y), \frac{e^{2i\lambda(x-y)}}{2i\lambda} \right] + o\left(\frac{1}{\lambda}\right),$$

or equivalently

$$n_2(\lambda, x) = -\frac{v(x)}{2i\lambda} + \frac{1}{2i\lambda}\int_{-\infty}^{x} dy\, v'(y)\frac{e^{2i\lambda(x-y)}}{2i\lambda} + o\left(\frac{1}{\lambda}\right). \tag{2.1.78}$$

Since $v(x)' \in \mathbb{S}(\mathbb{R})$ and $|e^{2i\lambda(x-y)}| \leq 1$ for $y \leq x$ and $\lambda \in \overline{\mathbf{C}^+}$, the integral $\int_{-\infty}^{x}$ in (2.1.78) is converges for all $x \in \mathbb{R}$. Hence, we obtain

$$n_2(\lambda, x) = -\frac{v(x)}{2i\lambda} + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^+}.$$

The proofs for (2.1.62), (2.1.63), (2.1.66), and (2.1.67) are given in a similar manner.

□

The next result deals with the continuity and the large x-asymptotics of $m(\lambda, x)$, $n(\lambda, x)$, $\bar{m}(\lambda, x)$, and $\bar{n}(\lambda, x)$ appearing in (2.1.4) and (2.1.5) for each fixed λ in their respective domain in the complex λ-plane.

Theorem 2.3. *Assume that the potentials $u(x)$ and $v(x)$ appearing in the system (2.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Let $m(\lambda, x)$ and $\bar{m}(\lambda, x)$ be the quantities defined in (2.1.4) and $n(\lambda, x)$ and $\bar{n}(\lambda, x)$ be the quantities defined in (2.1.5). Then:*

(a) *For each fixed $\lambda \in \overline{\mathbb{C}}^+$ we have*

$$m(\lambda, x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + o(1), \quad x \to +\infty; \quad n(\lambda, x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + o(1), \quad x \to -\infty, \tag{2.1.79}$$

and $m(\lambda, x)$ and $n(\lambda, x)$ are continuous and uniformly bounded in $x \in \mathbb{R}$.

(b) *For each fixed $\lambda \in \overline{\mathbb{C}}^-$ we have*

$$\bar{m}(\lambda, x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + o(1), \quad x \to +\infty; \quad \bar{n}(\lambda, x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + o(1), \quad x \to -\infty, \tag{2.1.80}$$

and $\bar{m}(\lambda, x)$ *and* $\bar{n}(\lambda, x)$ *are continuous and uniformly bounded in* $x \in \mathbb{R}$*.*

(c) *The Jost solutions* $\psi(\lambda, x)$ *and* $\phi(\lambda, x)$ *are continuous in* $x \in \mathbb{R}$ *for fixed* $\lambda \in \overline{\mathbb{C}^+}$ *and*

$$\psi(\lambda, x) = e^{i\lambda x} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + o(1) \right), \qquad x \to +\infty, \tag{2.1.81}$$

$$\phi(\lambda, x) = e^{-i\lambda x} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + o(1) \right), \qquad x \to -\infty. \tag{2.1.82}$$

(d) *The Jost solutions* $\bar{\psi}(\lambda, x)$ *and* $\bar{\phi}(\lambda, x)$ *are continuous in* $x \in \mathbb{R}$ *for fixed* $\lambda \in \overline{\mathbb{C}^-}$ *and*

$$\bar{\psi}(\lambda, x) = e^{-i\lambda x} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + o(1) \right), \qquad x \to +\infty, \tag{2.1.83}$$

$$\bar{\phi}(\lambda, x) = e^{i\lambda x} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + o(1) \right), \qquad x \to -\infty. \tag{2.1.84}$$

Proof. The proof of the first equality in (2.1.79) is obtained as follows. Let us copy the first equality in (2.1.8) for the convenience of the reader as

$$m(\lambda, x) = \begin{bmatrix} m_1(\lambda, x) \\ m_2(\lambda, x) \end{bmatrix}. \tag{2.1.85}$$

Iterating (2.1.11) we can express $m_2(\lambda, x)$ as an infinite summation as

$$m_2(\lambda, x) = \sum_{j=0}^{\infty} m_2^{(j)}(\lambda, x), \tag{2.1.86}$$

where we have

$$m_2^{(0)}(\lambda, x) := 1, \tag{2.1.87}$$

$$m_2^{(j)}(\lambda, x) := \int_x^\infty dy \int_y^\infty dz\, v(y)\, u(z)\, e^{2i\lambda(z-y)}\, m_2^{(j-1)}(\lambda, z), \quad j = 1, 2, 3, \dots. \tag{2.1.88}$$

Changing the order of integration in (2.1.88) we get

$$m_2^{(j)}(\lambda, x) = \int_x^\infty dz \int_x^z dy\, v(y)\, u(z)\, e^{2i\lambda(z-y)}\, m_2^{(j-1)}(\lambda, z). \tag{2.1.89}$$

Let us define

$$F(\lambda, x, z) := \int_x^z dy\, v(y) u(z) e^{2i\lambda(z-y)}. \tag{2.1.90}$$

Using (2.1.90) in (2.1.89) we have

$$m_2^{(j)}(\lambda, x) = \int_x^\infty dz\, F(\lambda, x, z)\, m_2^{(j-1)}(\lambda, z). \tag{2.1.91}$$

With the help of $|e^{2i\lambda(z-y)}| \leq 1$ when $y \leq z$ and $\lambda \in \overline{\mathbb{C}}^+$, from (2.1.90) we obtain

$$\int_x^\infty dz\, |F(\lambda, x, z)| \leq \int_x^\infty dy\, |v(y)| \int_x^\infty dz\, |u(z)|. \tag{2.1.92}$$

Using (2.1.87) and (2.1.92) in (2.1.91) for $j = 1$, we get

$$|m_2^{(1)}(\lambda, x)| \leq \int_x^\infty dy\, |v(y)| \int_x^\infty dz\, |u(z)|. \tag{2.1.93}$$

For $j = 2$, we have

$$m_2^{(2)}(\lambda, x) = \int_x^\infty dz\, F(\lambda, x, z)\, m_2^{(1)}(\lambda, z). \tag{2.1.94}$$

Substituting (2.1.92) and (2.1.93) in (2.1.94) we obtain

$$|m_2^{(2)}(\lambda, x)| \leq \int_x^\infty dy\, |v(y)| \int_x^\infty dz\, |u(z)| \int_z^\infty dt\, |v(t)| \int_z^\infty ds\, |u(s)|,$$

or equivalently

$$|m_2^{(2)}(\lambda, x)| \leq \left(\int_x^\infty dy\, |v(y)| \int_z^\infty dt\, |v(t)| \right) \left(\int_x^\infty dz\, |u(z)| \int_z^\infty ds\, |u(s)| \right). \tag{2.1.95}$$

Let us define

$$A(z) := \int_z^\infty dt\, |v(t)|, \qquad B(z) := \int_z^\infty ds\, |u(s)|, \tag{2.1.96}$$

which are uniformly decreasing in z and

$$A'(z) = -|v(z)|, \qquad B'(z) = -|u(z)|. \tag{2.1.97}$$

Using (2.1.96) and (2.1.97) in (2.1.95), we obtain

$$|m_2^{(2)}(\lambda, x)| \leq \left(\int_x^\infty dy \, (-A'(y)) \, A(y) \right) \left(\int_x^\infty dz \, (-B'(z) \, B(z) \right). \tag{2.1.98}$$

With help of integration by parts, we can write (2.1.98) as

$$|m_2^{(2)}(\lambda, x)| \leq \frac{[A(x)]^2}{2!} \, \frac{[B(x)]^2}{2!},$$

or equivalently

$$|m_2^{(2)}(\lambda, x)| \leq \frac{1}{2!} \left(\int_x^\infty dy \, |v(y)| \right)^2 \frac{1}{2!} \left(\int_x^\infty dz \, |u(z)| \right)^2. \tag{2.1.99}$$

In a similar way for $j = n$ we obtain

$$|m_2^{(n)}(\lambda, x)| \leq \frac{1}{n!} \left(\int_x^\infty dy \, |v(y)| \right)^n \frac{1}{n!} \left(\int_x^\infty dz \, |u(z)| \right)^n. \tag{2.1.100}$$

Since the potentials $u(x)$ and $v(x)$ belong to the Schwartz class $\mathbb{S}(\mathbb{R})$, each $m_2^{(j)}(\lambda, x)$ is bounded for fixed $\lambda \in \overline{\mathbb{C}}^+$. Thus, from $j = 1, 2, 3, ..., n$ we obtain

$$\begin{aligned} \sum_{j=0}^{n} |m_2^{(j)}(\lambda, x)| \leq 1 &+ \left(\int_x^\infty dy \, |v(y)| \right) \left(\int_x^\infty dz \, |u(z)| \right) \\ &+ \frac{1}{(2!)^2} \left(\int_x^\infty dy \, |v(y)| \right)^2 \left(\int_x^\infty dz \, |u(z)| \right)^2 \\ &+ ... + \frac{1}{(n!)^2} \left(\int_x^\infty dy \, |v(y)| \right)^n \left(\int_x^\infty dz \, |u(z)| \right)^n, \end{aligned}$$

or equivalently

$$\sum_{j=0}^{n} |m_2^{(j)}(\lambda, x)| \leq \sum_{j=0}^{n} \frac{\left[\left(\int_x^\infty dy \, |v(y)| \right)^{1/2} \left(\int_x^\infty dz \, |u(z)| \right)^{1/2} \right]^{2j}}{j!},$$

which can be written as

$$|m_2(\lambda,x)| \leq \sum_{j=0}^{\infty} \frac{\left[\left(\int_x^\infty dy\,|v(y)|\right)^{1/2}\left(\int_x^\infty dz\,|u(z)|\right)^{1/2}\right]^{2j}}{j!}$$
$$= J_0\Big(2\Big(\int_x^\infty dy\,|v(y)|\Big)^{1/2}\Big(\int_x^\infty dz\,|u(z)|\Big)^{1/2}\Big),$$

where J_0 is [6] the Bessel function of order zero. Thus, for each fixed $\lambda \in \overline{\mathbb{C}}^+$ the series given in (2.1.86) is uniformly convergent and uniformly bounded in $x \in \mathbb{R}$. Each $m_2^{(j)}(\lambda,x)$ is also continuous in $x \in \mathbb{R}$. Thus, by the Weierstrass theorem, $m_2(\lambda,x)$ is the limit of a uniformly convergent series of continuous terms. Hence, $m_2(\lambda,x)$ is continuous in $x \in \mathbb{R}$ for each fixed $\lambda \in \overline{\mathbb{C}}^+$. Now, let us also prove $m_1(\lambda,x)$. Let copy (2.1.10) for the convenience of the reader as

$$m_1(\lambda,x) = -\int_x^\infty dy\,u(y)\,m_2(\lambda,y)\,e^{2i\lambda(x-y)}. \tag{2.1.101}$$

By taking the absolute value of both side of (2.1.101) we get

$$|m_1(\lambda,x)| = \left|-\int_x^\infty dy\,u(y)\,m_2(\lambda,y)\,e^{2i\lambda(x-y)}\right|, \tag{2.1.102}$$

which yields

$$|m_1(\lambda,x)| \leq \int_x^\infty dy\;|u(y)|\;|m_2(\lambda,y)|\;\left|e^{2i\lambda(x-y)}\right|. \tag{2.1.103}$$

With the help of $\left|e^{2i\lambda(x-y)}\right| \leq 1$ when $y \leq x$ and $\lambda \in \mathbb{C}^+$, from (2.1.103) we obtain

$$|m_1(\lambda,x)| \leq \int_x^\infty dy\;|u(y)|\;|m_2(\lambda,y)|. \tag{2.1.104}$$

Since $u(x)$ belongs to the Schwartz class $\mathbb{S}(\mathbb{R})$ and $m_2(\lambda,y)$ is uniformly bounded in $y \in \mathbb{R}$ for each fixed $\lambda \in \overline{\mathbb{C}}^+$, from the expression in (2.1.101) we see that $m_1(\lambda,x)$ exists, uniformly bounded for each fixed $\lambda \in \overline{\mathbb{C}}^+$, and $m_1(\lambda,x) = o(1)$ as $x \to +\infty$. Hence, the first asymptotics in (2.1.79) is verified. From (2.1.101) we see that $m_1(\lambda,x)$ can be represented as an infinite summation by using the corresponding summation

(2.1.86) for $m_2(\lambda, y)$. We clearly know that the series given in (2.1.86) is uniformly convergent in $x \in \mathbb{R}$ and uniformly bounded in $x \in \mathbb{R}$ for each fixed $\lambda \in \overline{\mathbb{C}^+}$. Each $m_2^{(j)}(\lambda, x)$ is also continuous in $x \in \mathbb{R}$. Thus, by the Weierstrass theorem $m_1(\lambda, x)$ is represented as a uniformly convergent series of continuous terms. Hence, $m_1(\lambda, x)$ is continuous in $x \in \mathbb{R}$ for each fixed $\lambda \in \overline{\mathbb{C}}^+$. The continuity in $x \in \mathbb{R}$ and the large x-asymptotics of $n(\lambda, x)$ for each fixed $\lambda \in \overline{\mathbb{C}}^+$ are proved in a similar way. The continuity in $x \in \mathbb{R}$ and the large x-asymptotics of $\bar{m}(\lambda, x)$ and $\bar{n}(\lambda, x)$ for each fixed $\lambda \in \overline{\mathbb{C}}^-$ are established in the same manner.

We conclude (c) by using (a) and (2.1.4) and (2.1.5). Similarly, we obtain (d) by using (b) and (2.1.4) and (2.1.5). □

2.2 Scattering Coefficients

The scattering coefficients can be defined by using the x-asymptotics of the Jost solutions, or equivalently they can be obtained with the help of Wronskians of Jost solutions. Since the potentials $u(x)$ and $v(x)$ appearing in (2.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. we have

$$\phi(\lambda, x) = \begin{bmatrix} \frac{1}{T_{\rm r}(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T_{\rm r}(\lambda)} e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{2.2.1}$$

$$\bar{\phi}(\lambda, x) = \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}_{\rm r}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}_{\rm r}(\lambda)} e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{2.2.2}$$

$$\psi(\lambda, x) = \begin{bmatrix} \frac{L(\lambda)}{T_{\rm l}(\lambda)} e^{-i\lambda x} \\ \frac{1}{T_{\rm l}(\lambda)} e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{2.2.3}$$

$$\bar{\psi}(\lambda, x) = \begin{bmatrix} \frac{1}{\bar{T}_{\rm l}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}_{\rm l}(\lambda)} e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{2.2.4}$$

where $T_{\rm l}$ and $T_{\rm r}$ are called the transmission coefficients from left and from the right, respectively, and L and R are called the reflection coefficients from the left and from the right, respectively.

The analogy from quantum mechanics and wave propagation helps us with the scattering interpretation for physical solutions. When we multiply the Jost solution $\phi(\lambda, x)$ in (2.2.1) with $T_{\rm r}$ and we multiply the Jost solution $\bar{\psi}(\lambda, x)$ in (2.2.4) with $\bar{T}_{\rm l}$ we get the asymptotics

$$T_{\rm r}(\lambda)\phi(\lambda,x) = \begin{cases} \begin{bmatrix} e^{-i\lambda x} \\ R(\lambda)e^{i\lambda x} \end{bmatrix} + o(1), & x \to +\infty, \\ \\ \begin{bmatrix} T_{\rm r}(\lambda)e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), & x \to -\infty, \end{cases} \tag{2.2.5}$$

$$\bar{T}_{\rm l}(\lambda)\bar{\psi}(\lambda,x) = \begin{cases} \begin{bmatrix} \bar{L}(\lambda)e^{-i\lambda x} \\ e^{i\lambda x} \end{bmatrix} + o(1), & x \to -\infty, \\ \\ \begin{bmatrix} 0 \\ \bar{T}_{\rm l}(\lambda)e^{i\lambda x} \end{bmatrix} + o(1), & x \to +\infty. \end{cases} \tag{2.2.6}$$

We can interpret (2.2.5) as follows. The unit-amplitude plane wave $\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}$ is sent from $x = +\infty$ onto the potentials. It is partly transmitted and this appears as $\begin{bmatrix} T_{\rm r}(\lambda)\, e^{-i\lambda x} \\ 0 \end{bmatrix}$ at $x = -\infty$. It is partly reflected and this appears as $\begin{bmatrix} 0 \\ R(\lambda)\, e^{i\lambda x} \end{bmatrix}$ at $x = +\infty$. Thus, $T_{\rm r}$, the coefficient of $e^{-i\lambda x}$ at $x = -\infty$, is a transmission coefficient from the right and R, the coefficient of $e^{i\lambda x}$ at $x = +\infty$, is a reflection coefficient from the right. Using this analogy, we can interpret (2.2.6) as follows. The unit-amplitude plane wave $\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}$ is sent from $x = -\infty$ onto the potentials. It is partly transmitted and this appears as $\begin{bmatrix} 0 \\ \bar{T}_1(\lambda)\, e^{i\lambda x} \end{bmatrix}$ at $x = +\infty$. It is partly reflected and this appears as $\begin{bmatrix} \bar{L}(\lambda) e^{-i\lambda x} \\ 0 \end{bmatrix}$ at $x = -\infty$. Hence $\bar{T}_1$, the coefficient of $e^{i\lambda x}$ at $x = +\infty$, is a transmission coefficient from the left and $\bar{L}$, the coefficient of $e^{-i\lambda x}$ at $x = -\infty$, is a reflection coefficient from the left.

The scattering coefficients can also be obtained by using some Wronskian relations for (2.1.1). We define the Wronskian of two solutions $\begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix}$ and $\begin{bmatrix} \xi_2 \\ \eta_2 \end{bmatrix}$ to (2.1.1) as a determinant given by

$$\left[\begin{bmatrix} \xi_1 \\ \eta_2 \end{bmatrix} ; \begin{bmatrix} \xi_2 \\ \eta_2 \end{bmatrix} \right] := \det \begin{bmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{bmatrix} = \begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix}. \tag{2.2.7}$$

One can directly verify that the Wronskian of any two solutions to (2.1.1) is independent of x and may only depend on λ. To show that the Wronskian is independent of x, we can show that its x-derivative is zero, as the following argument shows. Taking the x-derivative of (2.2.7) we get

$$\begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix}' = \begin{vmatrix} \xi_1' & \xi_2 \\ \eta_1' & \eta_2 \end{vmatrix} + \begin{vmatrix} \xi_1 & \xi_2' \\ \eta_1 & \eta_2' \end{vmatrix}. \tag{2.2.8}$$

Using (2.1.1) in (2.2.8) we obtain

$$\begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix}' = \begin{vmatrix} -i\lambda\xi_1 + u\eta_1 & \xi_2 \\ i\lambda\eta_1 + v\xi_1 & \eta_2 \end{vmatrix} + \begin{vmatrix} \xi_1 & -i\lambda\xi_2 + u\eta_2 \\ \eta_1 & i\lambda\eta_2 + v\xi_2 \end{vmatrix}. \tag{2.2.9}$$

The determinants on the right-hand side of (2.2.9) can be evaluated to yield

$$\begin{vmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{vmatrix}' = (-i\lambda\xi_1 + u\eta_1)\eta_2 - (i\lambda\eta_1 + v\xi_1)\xi_2 + (i\lambda\eta_2 + v\xi_2)\xi_1 - (-i\lambda\xi_2 + u\eta_2)\eta_1$$
$$= 0.$$

Since the x-derivative of the Wronskian is zero, this shows that the Wronskian is independent of x. Thus, the Wronskian evaluated at $x = +\infty$ and the Wronskian evaluated at $x = -\infty$ yield the same value. We can use this to establish some properties of scattering coefficients. Evaluating the Wronskian $\left[\phi;\psi\right]$ at $x = +\infty$, we get

$$\left[\phi;\psi\right] = \begin{vmatrix} \frac{1}{T_{\rm r}(\lambda)}e^{-i\lambda x} & 0 \\ \frac{R(\lambda)}{T_{\rm r}(\lambda)}e^{i\lambda x} & e^{i\lambda x} \end{vmatrix} + o(1), \qquad x \to +\infty,$$

which yields

$$\left[\phi;\psi\right] = \frac{1}{T_{\rm r}(\lambda)}. \tag{2.2.10}$$

Similarly, evaluating the Wronskian $\left[\phi;\psi\right]$ at $x = -\infty$, we have

$$\left[\phi;\psi\right] = \begin{vmatrix} e^{-i\lambda x} & \frac{L(\lambda)}{T_{\rm l}(\lambda)}e^{-i\lambda x} \\ 0 & \frac{1}{T_{\rm l}(\lambda)}e^{i\lambda x} \end{vmatrix} + o(1), \qquad x \to -\infty,$$

or equivalently

$$\left[\phi;\psi\right] = \frac{1}{T_{\rm l}(\lambda)}. \tag{2.2.11}$$

Again, evaluating the Wronskian $\left[\bar{\psi};\bar{\phi}\right]$ at $x = +\infty$, we get

$$\left[\bar{\psi};\bar{\phi}\right] = \begin{vmatrix} e^{-i\lambda x} & \frac{\bar{R}(\lambda)}{\bar{T}_{\rm r}(\lambda)}e^{-i\lambda x} \\ 0 & \frac{1}{\bar{T}_{\rm r}(\lambda)}e^{i\lambda x} \end{vmatrix} + o(1), \qquad x \to +\infty,$$

from which we conclude that

$$\left[\bar{\psi};\bar{\phi}\right] = \frac{1}{\bar{T}_{\rm r}(\lambda)}. \tag{2.2.12}$$

Evaluating the Wronskian $\left[\bar{\psi};\bar{\phi}\right]$ at $x=-\infty$, we have

$$\left[\bar{\psi};\bar{\phi}\right]=\begin{vmatrix}\frac{1}{\bar{T}_{\mathrm{l}}(\lambda)}e^{-i\lambda x} & 0\\ \frac{\bar{L}(\lambda)}{\bar{T}_{\mathrm{l}}(\lambda)}e^{i\lambda x} & e^{-i\lambda x}\end{vmatrix}+o(1),\qquad x\to-\infty,$$

which yields

$$\left[\bar{\psi};\bar{\phi}\right]=\frac{1}{\bar{T}_{\mathrm{l}}(\lambda)}. \tag{2.2.13}$$

By Comparing (2.2.10) and (2.2.11) we obtain $T_{\mathrm{r}}(\lambda)$=$T_{\mathrm{l}}(\lambda)$ for any $\lambda\in\mathbb{R}$, and by comparing (2.2.12) and (2.2.13) we obtain $\bar{T}_{\mathrm{r}}(\lambda)$=$\bar{T}_{\mathrm{l}}(\lambda)$ for any $\lambda\in\mathbb{R}$. Since $T_{\mathrm{r}}(\lambda)$=$T_{\mathrm{l}}(\lambda)$ and $\bar{T}_{\mathrm{r}}(\lambda)$=$\bar{T}_{\mathrm{l}}(\lambda)$, we use $T(\lambda)$ to denote the common value of $T_{\mathrm{r}}(\lambda)$ and $T_{\mathrm{l}}(\lambda)$ and we use $\bar{T}(\lambda)$ to denote the common value of $\bar{T}_{\mathrm{r}}(\lambda)$ and $\bar{T}_{\mathrm{l}}(\lambda)$. Hence, we have

$$T_{\mathrm{r}}(\lambda)=T_{\mathrm{l}}(\lambda)=T(\lambda), \tag{2.2.14}$$

$$\bar{T}_{\mathrm{r}}(\lambda)=\bar{T}_{\mathrm{l}}(\lambda)=\bar{T}(\lambda). \tag{2.2.15}$$

Evaluating the Wronskian $\left[\bar{\psi};\phi\right]$ at $x=+\infty$ for any $\lambda\in\mathbb{R}$, and using (2.2.14) we have

$$\left[\bar{\psi};\phi\right]=\begin{vmatrix}e^{-i\lambda x} & \frac{1}{T(\lambda)}e^{-i\lambda x}\\ 0 & \frac{R(\lambda)}{T(\lambda)}e^{i\lambda x}\end{vmatrix}+o(1),\qquad x\to+\infty,$$

or equivalently

$$\left[\bar{\psi};\phi\right]=\frac{R(\lambda)}{T(\lambda)}. \tag{2.2.16}$$

Similarly, evaluating the Wronskian $\left[\bar{\psi};\phi\right]$ at $x=-\infty$ for any $\lambda\in\mathbb{R}$, and using (2.2.15) we obtain

$$\left[\bar{\psi};\phi\right]=\begin{vmatrix}\frac{1}{\bar{T}(\lambda)}e^{-i\lambda x} & e^{-i\lambda x}\\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}e^{i\lambda x} & 0\end{vmatrix}+o(1),\qquad x\to-\infty,$$

which yields

$$\left[\phi;\bar{\psi}\right]=-\frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}. \tag{2.2.17}$$

Thus, from(2.2.16) and (2.2.17) we have

$$\frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} = -\frac{R(\lambda)}{T(\lambda)}, \qquad \lambda \in \mathbb{R}. \tag{2.2.18}$$

In the same way, evaluating the Wronskian $\left[\phi; \bar{\phi}\right]$ at $x = +\infty$ for any $\lambda \in \mathbb{R}$ and using (2.2.14) and (2.2.15) we obtain

$$\left[\phi; \bar{\phi}\right] = \begin{vmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} & \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} & \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \end{vmatrix} + o(1), \qquad x \to +\infty,$$

which can be written as

$$\left[\phi; \bar{\phi}\right] = \frac{1}{T(\lambda)\,\bar{T}(\lambda)} - \frac{R(\lambda)\,\bar{R}(\lambda)}{T(\lambda)\bar{T}(\lambda)}. \tag{2.2.19}$$

The Wronskian of $\left[\phi; \bar{\phi}\right]$ at $x = -\infty$ for any $\lambda \in \mathbb{R}$ is written as

$$\left[\phi; \bar{\phi}\right] = \begin{vmatrix} e^{-i\lambda x} & 0 \\ 0 & e^{i\lambda x} \end{vmatrix} + o(1), \qquad x \to -\infty,$$

which yields

$$\left[\phi; \bar{\phi}\right] = 1. \tag{2.2.20}$$

Comparing (2.2.19) and (2.2.20) we get

$$\frac{1}{T(\lambda)\,\bar{T}(\lambda)} - \frac{R(\lambda)\,\bar{R}(\lambda)}{T(\lambda)\bar{T}(\lambda)} = 1, \qquad \lambda \in \mathbb{R},$$

or equivalently

$$T(\lambda)\,\bar{T}(\lambda) + R(\lambda)\,\bar{R}(\lambda) = 1, \qquad \lambda \in \mathbb{R}.$$

Again, evaluating the Wronskian $\left[\bar{\phi}; \psi\right]$ at $x = +\infty$ for any $\lambda \in \mathbb{R}$ and using (2.2.15) we obtain

$$\left[\bar{\phi}; \psi\right] = \begin{vmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} & 0 \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} & e^{i\lambda x} \end{vmatrix} + o(1), \qquad x \to +\infty,$$

which yields

$$\left[\bar{\phi};\psi\right]=\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}. \tag{2.2.21}$$

Similarly, evaluating the Wronskian $\left[\bar{\phi};\psi\right]$ at $x=-\infty$ for any $\lambda\in\mathbb{R}$ and using (2.2.14) we have

$$\left[\bar{\phi};\psi\right]=\begin{vmatrix}0 & \frac{L(\lambda)}{T(\lambda)}e^{-i\lambda x}\\ e^{i\lambda x} & \frac{1}{T(\lambda)}e^{i\lambda x}\end{vmatrix}+o(1),\qquad x\to-\infty,$$

or equivalently

$$\left[\psi;\bar{\phi}\right]=-\frac{L(\lambda)}{T(\lambda)}. \tag{2.2.22}$$

Thus, from (2.2.21) and (2.2.22) we obtain

$$\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}=-\frac{L(\lambda)}{T(\lambda)},\qquad \lambda\in\mathbb{R}. \tag{2.2.23}$$

Finally, evaluating the Wronskian $\left[\bar{\psi};\psi\right]$ at $x=+\infty$ for any $\lambda\in\mathbb{R}$ we have

$$\left[\bar{\psi};\psi\right]=\begin{vmatrix}e^{-i\lambda x} & 0\\ 0 & e^{i\lambda x}\end{vmatrix}+o(1),\qquad x\to+\infty,$$

which can be written as

$$\left[\bar{\psi};\psi\right]=1. \tag{2.2.24}$$

Evaluating the Wronskian $\left[\bar{\psi};\psi\right]$ at $x=-\infty$ for any $\lambda\in\mathbb{R}$ and using (2.2.14) and (2.2.15) we have

$$\left[\bar{\psi};\psi\right]=\begin{vmatrix}\frac{1}{\bar{T}(\lambda)}e^{-i\lambda x} & \frac{L(\lambda)}{T(\lambda)}e^{-i\lambda x}\\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}e^{i\lambda x} & \frac{1}{T(\lambda)}e^{i\lambda x}\end{vmatrix}+o(1),\qquad x\to-\infty,$$

or equivalently

$$\left[\bar{\psi};\psi\right]=\frac{1}{T(\lambda)\bar{T}(\lambda)}-\frac{L(\lambda)\,\bar{L}(\lambda)}{T(\lambda)\bar{T}(\lambda)}. \tag{2.2.25}$$

Therefore, from (2.2.24) and (2.2.25) we obtain

$$T(\lambda)\,\bar{T}(\lambda) + L(\lambda)\,\bar{L}(\lambda) = 1, \qquad \lambda \in \mathbb{R}.$$

The Jost solutions $\psi(\lambda, x)$ and $\bar{\psi}(\lambda, x)$ are linearly independent vector solutions to (2.1.1). Thus, $\phi(\lambda, x)$ and $\bar{\phi}(\lambda, x)$ can be expressed as a linear combination of $\psi(\lambda, x)$ and $\bar{\psi}(\lambda, x)$. We have the following relationships among the Jost solutions for $\lambda \in \mathbb{R}$:

$$\begin{cases} \phi(\lambda, x) = C_1\,\bar{\psi}(\lambda, x) + C_2\psi(\lambda, x), \\ \bar{\phi}(\lambda, x) = C_3\,\psi(\lambda, x) + C_4\,\bar{\psi}(\lambda, x), \end{cases} \tag{2.2.26}$$

where C_1, C_2, C_3, C_4 are functions of λ alone. The coefficients in (2.2.26) are related to the scattering coefficients. With the help of Wronskians from the first line of (2.2.26) we get

$$\left[\bar{\psi}; \phi\right] = C_1 \left[\bar{\psi}; \bar{\psi}\right] + C_2 \left[\bar{\psi}; \psi\right]. \tag{2.2.27}$$

It is clear that first Wronskian on the right-hand side of (2.2.27) vanishes. Using (2.2.16) and (2.2.24) we evaluate C_2 appearing in (2.2.27) and get $C_2 = \frac{R(\lambda)}{T(\lambda)}$. Similarly, from the first line of (2.2.26) we obtain

$$\left[\phi; \psi\right] = C_1 \left[\bar{\psi}; \psi\right] + C_2 \left[\psi; \psi\right]. \tag{2.2.28}$$

The second Wronskian on the right-hand side of (2.2.28) vanishes. Using (2.2.10), (2.2.14) and (2.2.24) we evaluate C_1 appearing in (2.2.28) and obtain $C_1 = \frac{1}{T(\lambda)}$. With the help of the Wronskians from the second line of (2.2.26) we get

$$\left[\bar{\phi}; \psi\right] = C_3 \left[\bar{\psi}; \psi\right] + C_4 \left[\psi; \psi\right]. \tag{2.2.29}$$

The second Wronskian on the right-hand side of (2.2.29) vanishes. Using (2.2.21) and (2.2.24) we evaluated C_1 appearing in (2.2.29) and get $C_3 = \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}$. Similarly, from the second line of (2.2.26) we obtain

$$\left[\bar{\psi}; \bar{\phi}\right] = C_3 \left[\bar{\psi}; \bar{\psi}\right] + C_4 \left[\bar{\psi}; \psi\right]. \tag{2.2.30}$$

The first Wronskian on the right-hand side of (2.2.30) vanishes. Using (2.2.12), (2.2.15), and (2.2.24) we evaluate C_4 appearing in (2.2.30) and get $C_4 = \frac{1}{T(\lambda)}$. We can now put C_1, C_2, C_3, C_4 together and write (2.2.26) as

$$\begin{cases} T(\lambda)\,\phi(\lambda,x) = \bar{\psi}(\lambda,x) + R(\lambda)\,\psi(\lambda,x), \\ \bar{T}(\lambda)\,\bar{\phi}(\lambda,x) = \psi(\lambda,x) + \bar{R}(\lambda)\,\bar{\psi}(\lambda,x), \end{cases} \qquad \lambda \in \mathbb{R}, \tag{2.2.31}$$

which yields a Riemann-Hilbert problem. In this Riemann-Hilbert problem, given the coefficients $T(\lambda), R(\lambda), \bar{T}(\lambda), \bar{R}(\lambda)$ for $\in \mathbb{R}$, the goal is to obtain the Jost solutions $\phi(\lambda,x), \psi(\lambda,x), \bar{\phi}(\lambda,x), \bar{\psi}(\lambda,x)$ in such a way that these Jost solutions have the appropriate analyticity and asymptotic properties in λ for each fixed $x \in \mathbb{R}$.

The analyticity of transmission coefficients are established in the following theorem.

Theorem 2.4. *Assume the potentials $u(x)$ and $v(x)$ appearing in the system (2.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the quantity $\frac{1}{T(\lambda)}$ is analytic in λ for $\lambda \in \mathbb{C}^+$, and the quantity $\frac{1}{\bar{T}(\lambda)}$ is analytic in λ for $\lambda \in \mathbb{C}^-$.*

Proof. Using (2.2.14) in (2.2.11) and (2.2.15) in (2.2.12) we have

$$\begin{cases} \Big[\phi;\psi\Big] = \frac{1}{T(\lambda)}, \\ \Big[\bar{\psi};\bar{\phi}\Big] = \frac{1}{\bar{T}(\lambda)}. \end{cases} \tag{2.2.32}$$

From (2.1.4), (2.1.5), and theorem 2.1(a), we know that ϕ, ψ are analytic in $\lambda \in \mathbb{C}^+$ for each fixed $x \in \mathbb{R}$. Similarly, from (2.1.4), (2.1.5), and theorem 2.1(b), we know that $\bar{\phi}, \bar{\psi}$ are analytic in $\lambda \in \mathbb{C}^-$ for each fixed $x \in \mathbb{R}$. From the first equality in (2.2.32) we conclude that the Wronskian $\Big[\phi;\psi\Big]$ is also analytic in $\lambda \in \mathbb{C}^+$ which indicated that the quantity $\frac{1}{T(\lambda)}$ is analytic in $\lambda \in \mathbf{C}^+$. Similarly, from the second equality in (2.2.32) we conclude that the Wronskian $\Big[\bar{\psi};\bar{\phi}\Big]$ is also analytic in $\lambda \in \mathbb{C}^-$ which indicated that the quantity $\frac{1}{\bar{T}(\lambda)}$ is analytic in $\lambda \in \mathbf{C}^-$. □

In the next proposition, we show that $\frac{1}{T(\lambda)}$, $\frac{1}{\bar{T}(\lambda)}$, $\frac{R(\lambda)}{T(\lambda)}$, and $\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}$ appearing in (2.2.1)-(2.2.4) satisfy certain integral relations, which are obtained by comparing the integral representation for Jost solutions and the x-asymptotics of Jost solutions.

Proposition 2.3. *Assume that the potentials $u(x)$ $v(x)$ belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the quantities $\frac{1}{T(\lambda)}$, $\frac{R(\lambda)}{T(\lambda)}$, $\frac{1}{\bar{T}(\lambda)}$, and $\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}$ can be expressed as*

$$\frac{1}{T(\lambda)} = 1 + \int_{-\infty}^{\infty} dy\, u(y)\, n_2(\lambda, y), \tag{2.2.33}$$

$$\frac{R(\lambda)}{T(\lambda)} = \int_{-\infty}^{\infty} dy\, v(y)\, n_1(\lambda, y)\, e^{-2i\lambda y}, \tag{2.2.34}$$

$$\frac{1}{\bar{T}(\lambda)} = 1 + \int_{-\infty}^{\infty} dy\, v(y)\, \bar{n}_1(\lambda, y), \tag{2.2.35}$$

$$\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} = \int_{-\infty}^{\infty} dy\, u(y)\, \bar{n}_2(\lambda, y)\, e^{2i\lambda y}. \tag{2.2.36}$$

Proof. With the help of first equality of (2.1.26), from (2.1.36) and (2.1.37) we obtain

$$\begin{bmatrix} n_1(\lambda, x) \\ n_2(\lambda, x) \end{bmatrix} = \begin{bmatrix} 1 + \int_{-\infty}^{x} dy\, u(y)\, n_2(\lambda, y) \\ \int_{-\infty}^{x} dy\, v(y)\, n_1(\lambda, y)\, e^{2i\lambda(x-y)} \end{bmatrix}. \tag{2.2.37}$$

Now, let us write $\phi(\lambda, x)$ and $\bar{\phi}(\lambda, x)$ in the component form as

$$\phi(\lambda, x) = \begin{bmatrix} \phi_1(\lambda, x) \\ \phi_2(\lambda, x) \end{bmatrix}, \qquad \bar{\phi}(\lambda, x) = \begin{bmatrix} \bar{\phi}_1(\lambda, x) \\ \bar{\phi}_2(\lambda, x) \end{bmatrix}. \tag{2.2.38}$$

Using (2.2.1) and (2.2.14) in the first equality of (2.2.38) we get

$$\begin{bmatrix} \phi_1(\lambda, x) \\ \phi_2(\lambda, x) \end{bmatrix} = \begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty,$$

or equivalently

$$\begin{bmatrix} \phi_1(\lambda, x)\, e^{i\lambda x} \\ \phi_2(\lambda, x)\, e^{i\lambda x} \end{bmatrix} = \begin{bmatrix} \frac{1}{T(\lambda)} \\ \frac{R(\lambda)}{T(\lambda)} e^{2i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{2.2.39}$$

with the help of first equalities in (2.1.5) and (2.1.26), from (2.2.39) we have

$$\begin{bmatrix} n_1(\lambda,x) \\ n_2(\lambda,x) \end{bmatrix} = \begin{bmatrix} \frac{1}{T(\lambda)} \\ \frac{R(\lambda)}{T(\lambda)} e^{2i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty. \tag{2.2.40}$$

Comparing (2.2.37) and (2.2.40) we obtain

$$\begin{bmatrix} \frac{1}{T(\lambda)} \\ \frac{R(\lambda)}{T(\lambda)} e^{2i\lambda x} \end{bmatrix} = \begin{bmatrix} 1 + \int_{-\infty}^{\infty} dy\, u(y)\, n_2(\lambda,y) \\ \int_{-\infty}^{\infty} dy\, v(y)\, n_1(\lambda,y)\, e^{2i\lambda(x-y)} \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} \frac{1}{T(\lambda)} \\ \frac{R(\lambda)}{T(\lambda)} \end{bmatrix} = \begin{bmatrix} 1 + \int_{-\infty}^{\infty} dy\, u(y)\, n_2(\lambda,y) \\ \int_{-\infty}^{\infty} dy\, v(y)\, n_1(\lambda,y)\, e^{-2i\lambda y} \end{bmatrix},$$

which establishes (2.2.33) and (2.2.34). Similarly, with the help of second equality in (2.1.26), from (2.1.42) and (2.1.43) we get

$$\begin{bmatrix} \bar{n}_1(\lambda,x) \\ \bar{n}_2(\lambda,x) \end{bmatrix} = \begin{bmatrix} \int_{-\infty}^{x} dy\, u(y)\, \bar{n}_2(\lambda,y)\, e^{2i\lambda(x-y)} \\ 1 + \int_{-\infty}^{x} dy\, v(y)\, \bar{n}_1(\lambda,y) \end{bmatrix}. \tag{2.2.41}$$

Using (2.2.2) and (2.2.15) in the second equality of (2.2.38) we have

$$\begin{bmatrix} \bar{\phi}_1(\lambda,x) \\ \bar{\phi}_2(\lambda,x) \end{bmatrix} = \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty,$$

or equivalently

$$\begin{bmatrix} \bar{\phi}_1(\lambda,x)\, e^{-i\lambda x} \\ \bar{\phi}_2(\lambda,x)\, e^{-i\lambda x} \end{bmatrix} = \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-2i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{2.2.42}$$

with the help of second equalities in (2.1.5) and (2.1.26), from (2.2.42) we obtain

$$\begin{bmatrix} \bar{n}_1(\lambda,x) \\ \bar{n}_2(\lambda,x) \end{bmatrix} = \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-2i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} \end{bmatrix} + o(1), \qquad x \to +\infty. \tag{2.2.43}$$

Comparing (2.2.41) and (2.2.43) we obtain

$$\begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-2i\lambda x} \\ \\ \frac{1}{\bar{T}(\lambda)} \end{bmatrix} = \begin{bmatrix} \int_{-\infty}^{\infty} dy\, u(y)\, \bar{n}_2(\lambda, y)\, e^{2i\lambda(y-x)} \\ 1 + \int_{-\infty}^{\infty} dy\, v(y)\, \bar{n}_1(\lambda, y) \end{bmatrix},$$

which completes the proof of (2.2.35) and (2.2.36). □

In the next theorem, we present the large-λ asymptotics of $\frac{1}{T(\lambda)}$ and $\frac{1}{\bar{T}(\lambda)}$.

Theorem 2.5. *Assume that the potentials $u(x)$ and $v(x)$ appearing in the system (2.1.1) belong to Schwarz class $\mathbb{S}(\mathbb{R})$. Then, the asymptotics expansions of scattering coefficients $\frac{1}{T(\lambda)}$ and $\frac{1}{\bar{T}(\lambda)}$ for large λ are given by*

$$\frac{1}{T(\lambda)} = 1 - \frac{1}{2i\lambda} \int_{-\infty}^{\infty} dy\, u(y)\, v(y) + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^+}, \tag{2.2.44}$$

$$\frac{1}{\bar{T}(\lambda)} = 1 + \frac{1}{2i\lambda} \int_{-\infty}^{\infty} dy\, u(y)\, v(y) + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^-}. \tag{2.2.45}$$

Proof. By comparing (2.1.64) and the first component of (2.2.40) we obtain

$$\frac{1}{T(\lambda)} = 1 - \frac{1}{2i\lambda} \int_{-\infty}^{\infty} dy\, u(y)\, v(y) + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^+},$$

which establishes (2.2.44). Similarly, comparing (2.1.67) and the second component of (2.2.43) we get

$$\frac{1}{\bar{T}(\lambda)} = 1 + \frac{1}{2i\lambda} \int_{-\infty}^{\infty} dy\, u(y)\, v(y) + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^-},$$

which completes the proof. □

2.3 Bound-State Solutions to the Standard System

A bound-state solution to (2.1.1) is a square-integrable column-vector solution in $x \in \mathbb{R}$. As we show in this section, such a solution occurs at a λ-value at which the transmission coefficient $T(\lambda)$ has a pole somewhere in the upper-half complex

λ-plane as well as at a λ-value at which the transmission coefficient $\bar{T}(\lambda)$ has a pole somewhere in the lower-half complex λ-plane. We denote the poles of $T(\lambda)$ in $\mathbb{C}^+$ by λ_j and assume that there are N such poles. Similarly, we denote the poles of $\bar{T}(\lambda)$ in $\mathbb{C}^-$ by $\bar{\lambda}_j$ and assume that there are $\bar{N}$ such poles. It may be possible that $N = 0$ or $\bar{N} = 0$. It may also be possible that each bound state is not a simple one, i.e. the corresponding pole has a multiplicity greater than one. We assume that the multiplicity of the pole λ_j of $T(\lambda)$ in $\mathbb{C}^+$ is equal to the positive integer m_j and assume that the multiplicity of the pole $\bar{\lambda}_j$ of $\bar{T}(\lambda)$ in $\mathbb{C}^-$ is equal to the positive integer $\bar{m}_j$.

Let us first consider the case of simple bound states, i.e. when the poles of $T(\lambda)$ in $\mathbb{C}^+$ and the poles of $\bar{T}(\lambda)$ in $\mathbb{C}^-$ are all simple. From (2.2.10) we know

$$\left[\phi;\psi\right] = \frac{1}{T}. \tag{2.3.1}$$

Hence, if T has a pole at $\lambda_j \in \mathbf{C}^+$, the Jost solutions $\phi(\lambda, x)$ and $\psi(\lambda, x)$ are linearly dependent at that λ_j-value. Thus, there exist scalar constant γ_j-values such that

$$\phi(\lambda_j, x) = \gamma_j\,\psi(\lambda_j, x), \qquad j = 1, ..., N. \tag{2.3.2}$$

Similarly, from (2.2.12) we know that

$$\left[\bar{\psi};\bar{\phi}\right] = \frac{1}{\bar{T}}. \tag{2.3.3}$$

Therefore, if $\bar{T}(\lambda)$ has a pole at $\bar{\lambda}_j$, the Jost solutions $\bar{\phi}(\lambda, x)$ and $\bar{\psi}(\lambda, x)$ are linearly dependent at that $\bar{\lambda}_j$-value. Thus, there exist scalar constant $\bar{\gamma}_j$-values such that

$$\bar{\phi}(\bar{\lambda}_j, x) = \bar{\gamma}_j\,\bar{\psi}(\bar{\lambda}_j, x), \qquad j = 1, ..., \bar{N}. \tag{2.3.4}$$

The constants γ_j and $\bar{\gamma}_j$ are usually called the dependency constants because (2.3.2) and (2.3.4) indicate the linear dependence of the corresponding Jost solutions. Let us now explain why the bound-states occur at the poles of $T(\lambda)$ in $\mathbb{C}^+$ or at the poles of $\bar{T}(\lambda)$ in $\mathbb{C}^-$. With the help of (2.1.81) and (2.1.82) we get

$$\psi(\lambda_j, x) = e^{i\lambda_j x}\left(\begin{bmatrix}0\\1\end{bmatrix} + o(1)\right), \qquad x \to +\infty, \tag{2.3.5}$$

$$\phi(\lambda_j, x) = e^{-i\lambda_j x}\left(\begin{bmatrix}1\\0\end{bmatrix} + o(1)\right), \qquad x \to -\infty, \tag{2.3.6}$$

where $\lambda_j \in \mathbb{C}^+$. From (2.3.5) we observe that $\psi(\lambda_j, x)$ decays exponentially as $x \to +\infty$, and from (2.3.6) we see that $\phi(\lambda_j, x)$ decays exponentially as $x \to -\infty$. Similarly, with the help of (2.1.83) and (2.1.84) we have

$$\bar{\psi}(\bar{\lambda}_j, x) = e^{-i\bar{\lambda}_j x}\left(\begin{bmatrix}1\\0\end{bmatrix} + o(1)\right), \qquad x \to +\infty, \tag{2.3.7}$$

$$\bar{\phi}(\bar{\lambda}_j, x) = e^{i\bar{\lambda}_j x}\left(\begin{bmatrix}0\\1\end{bmatrix} + o(1)\right), \qquad x \to -\infty, \tag{2.3.8}$$

where $\bar{\lambda}_j \in \mathbb{C}^-$. From (2.3.7) we see that $\bar{\psi}(\bar{\lambda}_j, x)$ decays exponentially as $x \to +\infty$, and from (2.3.8) we observe that $\bar{\phi}(\bar{\lambda}_j, x)$ decays exponentially as $x \to -\infty$.

By theorem 2.3 (c) we know that the right-hand of (2.3.2) decays exponentially as $x \to +\infty$ and the left-hand side of (2.3.2) decays exponentially as $x \to -\infty$. Thus, from (2.3.2) we conclude that both $\phi(\lambda_j, x)$ and $\psi(\lambda_j, x)$ decay exponentially as $x \to \pm\infty$. Similarly, by theorem 2.3 (d) we know that the right-hand of (2.3.4) decays exponentially as $x \to +\infty$ and the left-hand side of (2.3.4) decays exponentially as $x \to -\infty$. Thus, from (2.3.4) we conclude that both $\bar{\phi}(\bar{\lambda}_j, x)$ and $\bar{\psi}(\bar{\lambda}_j, x)$ decay exponentially as $x \to \pm\infty$.

By theorem 2.3(c) we also know that $\phi(\lambda_j, x)$ and $\psi(\lambda_j, x)$ are continuous in $x \in \mathbb{R}$. Thus, each of $\phi(\lambda_j, x)$ and $\psi(\lambda_j, x)$ is square integrable in $x \in \mathbb{R}$, and hence each can be used as a bound-state solution at $\lambda = \lambda_j$. Similarly, By theorem 2.3(d) we know that $\bar{\phi}(\bar{\lambda}_j, x)$ and $\bar{\psi}(\bar{\lambda}_j, x)$ are continuous in $x \in \mathbb{R}$. Thus, each of $\bar{\phi}(\bar{\lambda}_j, x)$ and $\bar{\psi}(\bar{\lambda}_j, x)$ is square integrable in $x \in \mathbb{R}$, and hence each can be used as a bound-state solution at $\bar{\lambda} = \bar{\lambda}_j$.

In the next theorem, we present the relationships between the residues of transmission coefficients $T(\lambda)$ and $\bar{T}(\lambda)$ at the bound-state poles and the dependency constants γ_j, $\bar{\gamma}_j$.

Theorem 2.6. (a) *The residue of $T(\lambda)$ at $\lambda_j \in \mathbb{C}^+$ and the dependency constant γ_j are related as*

$$\frac{1}{\operatorname{Res}(T, \lambda_j)} = -2i\gamma_j \int_{-\infty}^{\infty} ds\, \psi_1(\lambda_j, s)\, \psi_2(\lambda_j, s). \tag{2.3.9}$$

(b) *The residue of $\bar{T}(\lambda)$ at $\bar{\lambda}_j \in \mathbb{C}^-$ and the dependency constant $\bar{\gamma}_j$ are related as*

$$\frac{1}{\operatorname{Res}(\bar{T}, \bar{\lambda}_j)} = 2i\bar{\gamma}_j \int_{-\infty}^{\infty} ds\, \bar{\psi}_1(\lambda_j, s)\, \bar{\psi}_2(\lambda_j, s). \tag{2.3.10}$$

Proof. If $T(\lambda)$ has a simple pole at λ_j, we have the expansion of $T(\lambda)$ about λ_j as

$$T(\lambda) = \frac{z_j}{(\lambda - \lambda_j)} + O(1), \qquad \lambda \to \lambda_j,$$

or equivalently

$$\frac{1}{T(\lambda)} = \frac{1}{z_j}(\lambda - \lambda_j) + O\left((\lambda - \lambda_j)^2\right), \qquad \lambda \to \lambda_j. \tag{2.3.11}$$

By taking the λ-derivative of both side of (2.3.11) at $\lambda = \lambda_j$ we get

$$\frac{1}{z_j} = \frac{d}{d\lambda}\left(\frac{1}{T(\lambda)}\right)\Big|_{\lambda=\lambda_j},$$

or equivalently

$$\frac{1}{\operatorname{Res}(T, \lambda_j)} = \frac{d}{d\lambda}\left(\frac{1}{T(\lambda)}\right)\Big|_{\lambda=\lambda_j}. \tag{2.3.12}$$

Using (2.2.10) in (2.3.12) we have

$$\frac{1}{\operatorname{Res}(T, \lambda_j)} = \frac{d}{d\lambda}\left[\phi, \psi\right]\Big|_{\lambda=\lambda_j},$$

or equivalently

$$\frac{1}{\text{Res}(T,\lambda_j)} = \begin{vmatrix} \dot\phi_1 & \psi_1 \\ \dot\phi_2 & \psi_2 \end{vmatrix}_{\lambda=\lambda_j} + \begin{vmatrix} \phi_1 & \dot\psi_1 \\ \phi_2 & \dot\psi_2 \end{vmatrix}_{\lambda=\lambda_j}, \tag{2.3.13}$$

where an overdot indicates the λ-derivative. Let us define

$$M(\lambda,x) := \frac{d}{dx}\begin{vmatrix} \dot\phi_1 & \psi_1 \\ \dot\phi_2 & \psi_2 \end{vmatrix}, \quad N(\lambda,x) := \frac{d}{dx}\begin{vmatrix} \phi_1 & \dot\psi_1 \\ \phi_2 & \dot\psi_2 \end{vmatrix},$$

or equivalently

$$M(\lambda,x) := \begin{vmatrix} \dot\phi_1' & \psi_1 \\ \dot\phi_2' & \psi_2 \end{vmatrix} + \begin{vmatrix} \dot\phi_1 & \psi_1' \\ \dot\phi_2 & \psi_2' \end{vmatrix}, \tag{2.3.14}$$

$$N(\lambda,x) := \begin{vmatrix} \phi_1' & \dot\psi_1 \\ \phi_2' & \dot\psi_2 \end{vmatrix} + \begin{vmatrix} \phi_1 & \dot\psi_1' \\ \phi_2 & \dot\psi_2' \end{vmatrix}. \tag{2.3.15}$$

By taking the λ-derivative (2.1.1) we obtain

$$\begin{bmatrix} \dot\xi' \\ \dot\eta' \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix}\begin{bmatrix} \dot\xi \\ \dot\eta \end{bmatrix}, \qquad x\in\mathbb{R}. \tag{2.3.16}$$

Using (2.3.16) in (2.3.14) and (2.3.15) we obtain

$$\begin{aligned} M(\lambda,x) &= \begin{vmatrix} -i\,\phi_1 - i\lambda\,\dot\phi_1 + u(x)\,\dot\phi_2 & \psi_1 \\ i\,\phi_2 + v(x)\,\dot\phi_1 + i\lambda\,\dot\phi_2 & \psi_2 \end{vmatrix} + \begin{vmatrix} \dot\phi_1 & -i\lambda\,\psi_1 + u(x)\,\psi_2 \\ \dot\phi_2 & v(x)\,\psi_1 + i\lambda\,\psi_2 \end{vmatrix} \\ &= -i(\phi_1\,\psi_2 + \phi_2\,\psi_1). \end{aligned} \tag{2.3.17}$$

$$\begin{aligned} N(\lambda,x) &= \begin{vmatrix} -i\lambda\,\phi_1 + u(x)\,\phi_2 & \dot\psi_1 \\ v(x)\,\phi_1 + i\lambda\,\phi_2 & \dot\psi_2 \end{vmatrix} + \begin{vmatrix} \phi_1 & -i\,\psi_1 - i\lambda\,\dot\psi_1 + u(x)\,\dot\psi_2 \\ \phi_2 & i\,\psi_2 + v(x)\,\dot\psi_1 + i\lambda\,\dot\psi_2 \end{vmatrix} \\ &= i(\phi_1\,\psi_2 + \phi_2\,\psi_1). \end{aligned} \tag{2.3.18}$$

Comparing (2.3.17) and (2.3.18) we have

$$M(\lambda,x) = -N(\lambda,x) \tag{2.3.19}$$

With the help of (2.3.2), from (2.3.19) we obtain

$$M(\lambda,x)=-N(\lambda,x)=-i[\gamma_j\,\psi_1(\lambda_j,x)\,\psi_2(\lambda_j,x)+\gamma_j\,\psi_2(\lambda_j,x)\,\psi_1(\lambda_j,x)]$$
$$=-2i\gamma_j\,\psi_1(\lambda_j,x)\,\psi_2(\lambda_j,x).$$

Since $\psi_1(\lambda_j,x)$, $M(\lambda_j,x)$, and $N(\lambda_j,x)$ each vanish as $x\to+\infty$, we have

$$\begin{vmatrix}\dot{\phi}_1 & \psi_1\\ \dot{\phi}_2 & \psi_2\end{vmatrix}=\int_{\infty}^{x}ds\,M(\lambda_j,s)=-2i\gamma_j\int_{\infty}^{x}ds\,\psi_1(\lambda_j,s)\,\psi_2(\lambda_j,s),\tag{2.3.20}$$

$$\begin{vmatrix}\phi_1 & \dot{\psi}_1\\ \phi_2 & \dot{\psi}_2\end{vmatrix}=\int_{-\infty}^{x}ds\,N(\lambda_j,s)=2i\gamma_j\int_{-\infty}^{x}ds\,\psi_1(\lambda_j,s)\,\psi_2(\lambda_j,s).\tag{2.3.21}$$

From (2.3.13), (2.3.20), and (2.3.21) we obtain

$$\frac{1}{\operatorname{Res}(T,\lambda_j)}=-2i\gamma_j\int_{-\infty}^{\infty}ds\,\psi_1(\lambda_j,s)\,\psi_2(\lambda_j,s).$$

The proof for (b) is given in a similar manner, and we get

$$\frac{1}{\operatorname{Res}(\bar{T},\bar{\lambda}_j)}=2i\bar{\gamma}_j\int_{-\infty}^{\infty}ds\,\bar{\psi}_1(\lambda_j,s)\,\bar{\psi}_2(\lambda_j,s),$$

which completes the proof. □

We use c_j and $\bar{c}_j$ to denote the bound-state norming constants at λ_j and $\bar{\lambda}_j$, respectively, when the bound states are all simple. The bound-state norming constants c_j and $\bar{c}_j$ [11, 12] are related to the residues of $T(\lambda)$ and $\bar{T}(\lambda)$ at the poles λ_j and $\bar{\lambda}_j$, respectively, as

$$c_j=i\gamma_j\operatorname{Res}(T,\lambda_j)=\frac{-1}{2\int_{-\infty}^{\infty}ds\,\psi_1(\lambda_j,s)\,\psi_2(\lambda_j,s)},\tag{2.3.22}$$

$$\bar{c}_j=i\bar{\gamma}_j\operatorname{Res}(\bar{T},\bar{\lambda}_j)=\frac{1}{2\int_{-\infty}^{\infty}ds\,\bar{\psi}_1(\lambda_j,s)\,\bar{\psi}_2(\lambda_j,s)}.\tag{2.3.23}$$

2.4 Direct Problem for the Standard System

The direct problem consists of determining the scattering data set when the pair of potentials $u(x)$ and $v(x)$ are given. The appropriate scattering data set can be constructed with help of the scattering coefficients and bound-state data.

We evaluate the scattering coefficients by solving the first-order system (2.1.1) with $u(x)$ and $v(x)$. In section 2.1, we determine the Jost solutions $\phi(\lambda, x)$, $\bar{\phi}(\lambda, x)$, $\psi(\lambda, x)$, and $\bar{\psi}(\lambda, x)$ uniquely by imposing the appropriate asymptotic conditions. Then in section 2.2, we obtain the scattering coefficients $T(\lambda), R(\lambda), L(\lambda), \bar{T}(\lambda), \bar{R}(\lambda)$, and $\bar{R}(\lambda)$ from x-asymptotics of the Jost solutions, or equivalently from the Wronskian relations among the Jost solutions. From (2.2.18) we know that $\bar{L}(\lambda)$ can be expressed in terms of $\bar{R}(\lambda), T(\lambda)$, and $\bar{T}(\lambda)$. Similarly, from (2.2.23) we know that $L(\lambda)$ can be expressed in terms of $\bar{R}(\lambda), T(\lambda)$, and $\bar{T}(\lambda)$. Hence, instead of using the six scattering coefficients $T(\lambda), R(\lambda), L(\lambda), \bar{T}(\lambda), \bar{R}(\lambda)$, and $\bar{L}(\lambda)$ in the scattering data set, it is enough to use the four scattering coefficients $T(\lambda), R(\lambda), \bar{T}(\lambda)$, and $\bar{R}(\lambda)$.

A bound-state solution to (2.1.1) is a column-vector solution where a square integrable in $x \in \mathbb{R}$, and such a solution occurs at a λ-value at which $T(\lambda)$ has a pole in the upper-half complex λ-plane or at λ-value at which $\bar{T}(\lambda)$ has a pole in the lower- half complex λ-plane. We denote the poles of $T(\lambda)$ in $\mathbb{C}^+$ by λ_j and assume that there are N such poles. Similarly, we denote the poles of $\bar{T}(\lambda)$ in $\mathbb{C}^-$ by $\bar{\lambda}_j$ and assume that there are $\bar{N}$ such poles. It is possible that $N = 0$ or $\bar{N} = 0$. We use c_j to denote the corresponding bound-state norming constant if λ_j is a simple pole. Similarly, we use $\bar{c}_j$ to denote the corresponding bound-state norming constant if $\bar{\lambda}_j$ is a simple pole. From (2.3.22) and (2.3.23), we know that the bound-state norming constant c_j and $\bar{c}_j$ are related to the residues of $T(\lambda)$ and $\bar{T}(\lambda)$ at poles λ_j and $\bar{\lambda}_j$, respectively.

It may be possible that each bound state is not a simple one, i.e. the corresponding pole has a multiplicity greater than one. We assume that the multiplicity of the pole λ_j of $T(\lambda)$ in $\mathbb{C}^+$ is equal to the positive integer m_j and assume that the multiplicity of the pole $\bar{\lambda}_j$ of $\bar{T}(\lambda)$ in $\mathbb{C}^-$ is equal to the positive integer $\bar{m}_j$. Hence, for each λ_j, there are m_j norming [10, 13] constants c_{jk} for $k = 0, 1, ..., m_j - 1$. Similarly, for each $\bar{\lambda}_j$ there are $\bar{m}_j$ norming constants $\bar{c}_{jk}$ for $k = 0, 1, ..., \bar{m}_j - 1$. Let us write scattering data set as

$$\boldsymbol{S} := \left\{ R, \bar{R}, \left\{ \lambda_j, \left\{ c_{jk} \right\}_{k=0}^{m_j-1} \right\}_{j=1}^{N}, \left\{ \bar{\lambda}_j, \left\{ \bar{c}_{jk} \right\}_{k=0}^{\bar{m}_j-1} \right\}_{j=1}^{\bar{N}} \right\} \tag{2.4.1}$$

The scattering data set $\boldsymbol{S}$ is uniquely determined by the pair of potentials $u(x)$ and $v(x)$. We can visualize the direct problem as the mapping

$$\{u(x), v(x)\} \mapsto \boldsymbol{S}.$$

2.5 Inverse Problem for the Standard System

The inverse problem for (2.1.1) consists of the determination of the potentials $u(x)$ and $v(x)$ from the scattering data. The appropriate scattering data set to use consists of the scattering coefficients and the bound-state information. From (2.2.31) we have

$$\begin{cases} T(\lambda)\ \phi(\lambda, x) = \bar{\psi}(\lambda, x) + R(\lambda)\ \psi(\lambda, x), \\ \bar{T}(\lambda)\ \bar{\phi}(\lambda, x) = \psi(\lambda, x) + \bar{R}(\lambda)\ \bar{\psi}(\lambda, x), \end{cases} \qquad \lambda \in \mathbb{R}, \tag{2.5.1}$$

which yields a Riemann-Hilbert problem. One can solve the relevant inverse problem by solving the corresponding Riemann-Hilbert problem. In this Riemann-Hilbert problem, given the coefficients $T(\lambda), R(\lambda), \bar{T}(\lambda), \bar{R}(\lambda)$ for $\in \mathbb{R}$, the goal is to obtain the Jost solutions $\phi(\lambda, x), \psi(\lambda, x), \bar{\phi}(\lambda, x), \bar{\psi}(\lambda, x)$. Once the Jost solutions are known, the potentials $u(x)$ and $v(x)$ can be recovered from (1.0.4).

An alternative procedure is to solve the inverse problem by the Marchenko method [4, 6, 13, 10]. In this section, we will use the Marchenko integral equation to solve the inverse problem. Now let us derive the Marchenko integral equation from the Riemann-Hilbert problem when $T(\lambda)$ has simple poles at λ_j-values in $\mathbb{C}^+$ and $\bar{T}(\lambda)$ has simple poles at $\bar{\lambda}_j$-values in $\mathbb{C}^-$. From (2.1.4) and theorem 2.1(a), we know that $\psi(\lambda, x)$ is analytic in $\lambda \in \mathbb{C}^+$ for each fixed $x \in \mathbb{R}$ and continuous in $\lambda \in \overline{\mathbb{C}^+}$, and behaves like

$$\psi(\lambda, x) - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} = O\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^+}. \tag{2.5.2}$$

By taking the Fourier transform of (2.5.2) via $\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, e^{-i\lambda y}$ we get

$$K(x, y) := \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \left(\psi(\lambda, x) - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} \right) e^{-i\lambda y}, \tag{2.5.3}$$

Similarly, from (2.1.4) and theorem 2.1(b), we know that $\bar{\psi}(\lambda, x)$ is analytic in $\lambda \in \mathbb{C}^-$ for each fixed $x \in \mathbb{R}$ and continuous in $\lambda \in \overline{\mathbb{C}^-}$, and behaves like

$$\bar{\psi}(\lambda, x) - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} = O\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^-}. \tag{2.5.4}$$

By taking the Fourier transform of (2.5.4) via $\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, e^{i\lambda y}$ we obtain

$$\bar{K}(x, y) := \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \left(\bar{\psi}(\lambda, x) - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} \right) e^{i\lambda y}. \tag{2.5.5}$$

Taking the inverse Fourier transform of (2.5.3) and (2.5.5) we get

$$\psi(\lambda, x) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + \int_x^{\infty} dz\, K(x, z) e^{i\lambda z}, \tag{2.5.6}$$

$$\bar{\psi}(\lambda, x) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + \int_x^{\infty} dz\, \bar{K}(x, z) e^{-i\lambda z}. \tag{2.5.7}$$

Theorem 2.7. *The Marchenko integral equations associated with* (2.1.1) *are given by*

$$\bar{K}(x,y)+\Omega(x+y)\begin{bmatrix}0\\1\end{bmatrix}+\int_x^\infty dz\,K(x,z)\,\Omega(z+y)=0,\qquad y>x, \tag{2.5.8}$$

$$K(x,y)+\bar{\Omega}(x+y)\begin{bmatrix}1\\0\end{bmatrix}+\int_x^\infty dz\,\bar{K}(x,z)\,\bar{\Omega}(z+y)=0,\qquad y>x, \tag{2.5.9}$$

where

$$\Omega(y):=\hat{R}(y)+\sum_{j=1}^{N}c_j\,e^{i\lambda_j y}, \tag{2.5.10}$$

$$\bar{\Omega}(y):=\hat{\bar{R}}(y)+\sum_{j=1}^{\bar{N}}\bar{c}_j\,e^{-i\bar{\lambda}_j y}, \tag{2.5.11}$$

with

$$\hat{R}(y):=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda R(\lambda)\,e^{i\lambda y}, \tag{2.5.12}$$

$$\hat{\bar{R}}(y):=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\bar{R}(\lambda)\,e^{-i\lambda y}. \tag{2.5.13}$$

Proof. For simplicity, we may suppress the arguments and write ϕ for $\phi(\lambda,z)$, ψ for $\phi(\lambda,z)$, $\bar{\phi}$ for $\bar{\phi}(\lambda,z)$, R for $R(\lambda)$, T for $T(\lambda)$, $\bar{R}$ for $\bar{R}(\lambda)$, and $\bar{T}$ for $\bar{T}(\lambda)$. Rewriting (2.5.1) as

$$\begin{cases}(T-1)\,\phi+\phi-\begin{bmatrix}e^{-i\lambda x}\\0\end{bmatrix}=\bar{\psi}-\begin{bmatrix}e^{-i\lambda x}\\0\end{bmatrix}+R\left(\psi-\begin{bmatrix}0\\e^{i\lambda x}\end{bmatrix}+\begin{bmatrix}0\\e^{i\lambda x}\end{bmatrix}\right),\\[2ex](\bar{T}-1)\,\bar{\phi}+\bar{\phi}-\begin{bmatrix}0\\e^{i\lambda x}\end{bmatrix}=\psi-\begin{bmatrix}0\\e^{i\lambda x}\end{bmatrix}+\bar{R}\left(\bar{\psi}-\begin{bmatrix}e^{-i\lambda x}\\0\end{bmatrix}+\begin{bmatrix}e^{-i\lambda x}\\0\end{bmatrix}\right).\end{cases} \tag{2.5.14}$$

Applying $\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{i\lambda y}$ on both sides of the first equation in (2.5.14) we get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} (T-1)\, \phi\, e^{i\lambda y} + \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \left(\phi - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} \right) e^{i\lambda y} \\ = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \left(\bar{\psi} - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} \right) e^{i\lambda y} \\ + \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} R \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} e^{i\lambda y} + \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} R \left(\psi - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} \right) e^{i\lambda y}. \end{aligned} \tag{2.5.15}$$

Now let us consider each term in (2.5.15) separately. For the first term on the left-hand side of (2.5.15), since $T(\lambda)$ has a pole at λ_j for $j = 1, ..., N$ in the upper-half complex λ-plane the integral term can be written as

$$\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} (T-1)\, \phi\, e^{i\lambda y} = i \sum_{j=1}^{N} \operatorname{Res}(T, \lambda_j)\, \phi(\lambda_j, x)\, e^{i\lambda_j y}. \tag{2.5.16}$$

Using (2.3.2) in (2.5.16) we have

$$\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} (T-1)\, \phi\, e^{i\lambda y} = i \sum_{j=1}^{N} \operatorname{Res}(T, \lambda_j)\, \gamma_j\, \psi(\lambda_j, x)\, e^{i\lambda_j y}. \tag{2.5.17}$$

Substituting (2.5.6) in (2.5.17), we get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} (T-1)\, \phi\, e^{i\lambda y} \frac{d\lambda}{2\pi} \\ = i \sum_{j=1}^{N} \operatorname{Res}(T, \lambda_j)\, \gamma_j \left(\begin{bmatrix} 0 \\ e^{i\lambda_j x} \end{bmatrix} + \int_{x}^{\infty} dz\, K(x,z) e^{i\lambda_j z} \right) e^{i\lambda_j y} \\ = i \sum_{j=1}^{N} \operatorname{Res}(T, \lambda_j)\, \gamma_j \begin{bmatrix} 0 \\ e^{i\lambda_j (x+y)} \end{bmatrix} \\ + i \sum_{j=1}^{N} \operatorname{Res}(T, \lambda_j)\, \gamma_j \int_{x}^{\infty} dz\, K(x,z)\, e^{i\lambda_j (z+y)}. \end{aligned} \tag{2.5.18}$$

From (2.1.5) and theorem 2.1(a), we know that $\left(\phi - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right)$ is analytic in λ for $\lambda \in \mathbb{C}^+$ and continuous in λ for $\lambda \in \overline{\mathbb{C}^+}$. Hence, for the second term on the left-hand side of (2.5.15) we have

$$\int_{-\infty}^{\infty} \left(\phi - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right) e^{i\lambda y} \frac{d\lambda}{2\pi} = 0, \qquad y > x. \tag{2.5.19}$$

For the first term on the right-hand side of (2.5.15), from (2.5.5) we have

$$\int_{-\infty}^{\infty} \left(\bar{\psi} - \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}\right) e^{i\lambda y} \frac{d\lambda}{2\pi} = \bar{K}(x,y). \tag{2.5.20}$$

From (2.5.12), the second term on the right-hand side of (2.5.15) is related $\hat{R}(x+y)$ as

$$\int_{-\infty}^{\infty} R \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} e^{i\lambda y} \frac{d\lambda}{2\pi} = \hat{R}(x+y) \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.5.21}$$

By substituting (2.5.6) in the last term on the right-hand side of (2.5.15), we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} R(\lambda) \left(\psi - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}\right) e^{i\lambda y} \frac{d\lambda}{2\pi} &= \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \int_{x}^{\infty} dz\, R(\lambda) K(x,z)\, e^{i\lambda(z+y)} \\ &= \int_{x}^{\infty} dz\, K(x,z) \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} R(\lambda)\, e^{i\lambda(z+y)}. \end{aligned} \tag{2.5.22}$$

Using (2.5.19) in (2.5.22) we get

$$\int_{-\infty}^{\infty} R(\lambda) \left(\psi - \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}\right) e^{i\lambda y} \frac{d\lambda}{2\pi} = \int_{x}^{\infty} dz\, K(x,z)\, \hat{R}(z+y). \tag{2.5.23}$$

We can now put all of these together and write (2.5.15) as

$$i\sum_{j=1}^{N}\operatorname{Res}(T,\lambda_j)\,\gamma_j\begin{bmatrix}0\\ e^{i\lambda_j(x+y)}\end{bmatrix}+i\sum_{j=1}^{N}\operatorname{Res}(T,\lambda_j)\,\gamma_j\int_x^\infty dz\,K(x,z)\,e^{i\lambda_j(z+y)}$$
$$=\bar{K}(x,y)+\hat{R}(x+y)\begin{bmatrix}0\\1\end{bmatrix}+\int_x^\infty dz\,K(x,z)\,\hat{R}(z+y),$$

or equivalently

$$\begin{aligned}\bar{K}(x,y)&+\left(\hat{R}(x+y)-i\sum_{j=1}^{N}\operatorname{Res}(T,\lambda_j)\,\gamma_j\,e^{i\lambda_j(x+y)}\right)\begin{bmatrix}0\\1\end{bmatrix}\\&+\int_x^\infty dz\,K(x,z)\left(\hat{R}(z+y)-i\sum_{j=1}^{N}\operatorname{Res}(T,\lambda_j)\,\gamma_j\,e^{i\lambda_j(z+y)}\right)=0.\end{aligned}\tag{2.5.24}$$

By substituting (2.3.22) in (2.5.24), we have

$$\begin{aligned}\bar{K}(x,y)&+\left(\hat{R}(x+y)+i\sum_{j=1}^{N}c_j\,e^{i\lambda_j(x+y)}\right)\begin{bmatrix}0\\1\end{bmatrix}\\&+\int_x^\infty dz\,K(x,z)\left(\hat{R}(z+y)+i\sum_{j=1}^{N}c_j\,e^{i\lambda_j(z+y)}\right)=0.\end{aligned}\tag{2.5.25}$$

Using (2.5.10) in (2.5.25) we obtain

$$\bar{K}(x,y)+\Omega(x+y)\begin{bmatrix}0\\1\end{bmatrix}+\int_x^\infty dz\,K(x,z)\,\Omega(z+y)=0,\qquad y>x,$$

which completes the proof of (2.5.8). We can prove (2.5.9) in the same way by applying $\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\,e^{-i\lambda y}$ on both sides of the second equation in (2.5.14) we get

$$K(x,y)+\bar{\Omega}(x+y)\begin{bmatrix}1\\0\end{bmatrix}+\int_x^\infty dz\,\bar{K}(x,z)\,\bar{\Omega}(z+y)=0,\qquad y>x,$$

which completes the proof. □

In case the poles of $T(\lambda)$ in $\mathbb{C}^+$ have multiplicities, in the definition of $\Omega(y)$ given in (2.5.10), the summation term needs to be modified to include the multiplicities. From the scattering data $\left\{R, \lambda_j, \{c_{jk}\}_{k=0}^{m_j-1}\right\}_{j=1}^{N}$, the Marchenko kernel [10, 11] $\Omega(y)$ appearing in (2.5.8) can be written as

$$\Omega(y) := \hat{R}(y) + \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} c_{jk} \frac{y^k}{k!} e^{i\lambda_j y}. \tag{2.5.26}$$

Similarly, in case the poles of $\bar{T}(\lambda)$ in $\mathbb{C}^-$ have multiplicities, in the definition of $\bar{\Omega}(y)$ given in (2.5.11), the summation term needs to be modified to include the multiplicities. From the scattering data $\left\{\bar{R}, \bar{\lambda}_j, \{\bar{c}_{jk}\}_{k=0}^{\bar{m}_j-1}\right\}_{j=1}^{\bar{N}}$, the Marchenko kernel [10, 11] $\bar{\Omega}(y)$ appearing in (2.5.9) can be written as

$$\bar{\Omega}(y) := \hat{\bar{R}}(y) + \sum_{j=1}^{\bar{N}} \sum_{k=0}^{\bar{m}_j-1} \bar{c}_{jk} \frac{y^k}{k!} e^{-i\bar{\lambda}_j y}. \tag{2.5.27}$$

Let us write $K(x,y)$ and $\bar{K}(x,y)$ in term of their components as

$$\bar{K}(x,y) = \begin{bmatrix} \bar{K}_1(x,y) \\ \bar{K}_2(x,y) \end{bmatrix}, \qquad K(x,y) = \begin{bmatrix} K_1(x,y) \\ K_2(x,y) \end{bmatrix}. \tag{2.5.28}$$

With the help of (2.5.28) we can write the Marchenko equations given in (2.5.8) and (2.5.9) in the component form as four scalar integral equations. We have

$$\bar{K}_1(x,y) + \int_x^\infty dz\, K_1(x,z)\, \Omega(z+y) = 0, \qquad y > x, \tag{2.5.29}$$

$$\bar{K}_2(x,y) + \Omega(x+y) + \int_x^\infty dz\, K_2(x,z)\, \Omega(z+y) = 0, \qquad y > x, \tag{2.5.30}$$

$$K_1(x,y) + \bar{\Omega}(x+y) + \int_x^\infty dz\, \bar{K}_1(x,z)\, \bar{\Omega}(z+y) = 0, \qquad y > x, \tag{2.5.31}$$

$$K_2(x,y) + \int_x^\infty dz\, \bar{K}_2(x,z)\, \bar{\Omega}(z+y) = 0, \qquad y > x. \tag{2.5.32}$$

From (2.5.29) and (2.5.31), by eliminating $\bar{K}_1(x,y)$ we get

$$K_1(x,y) + \bar{\Omega}(x+y) - \int_x^\infty dz \int_x^\infty dt\, K_1(x,t)\ \Omega(t+z)\, \bar{\Omega}(z+y) = 0, \quad y > x. \tag{2.5.33}$$

Using (2.5.30) and (2.5.32), by eliminating $K_2(x,y)$ we obtain

$$\bar{K}_2(x,y)+\Omega(x+y)+\int_x^\infty dz\int_x^\infty dt\,\bar{K}_2(x,t)\,\bar{\Omega}(t+z)\,\Omega(z+y)=0,\quad y>x. \tag{2.5.34}$$

If Ω and $\bar{\Omega}$ are given, we can solve (2.5.33) and obtain $K_1(x,y)$ and then use in (2.5.29) to get $\bar{K}_1(x,y)$. Similarly, if Ω and $\bar{\Omega}$ are given we can solve (2.5.34) and obtain $\bar{K}_2(x,y)$ and then use in (2.5.32) to get $K_2(x,y)$.

In the next proposition, we present the large-λ asymptotics of the quantities $m(\lambda,x)$ and $\bar{m}(\lambda,x)$ appearing in (2.1.4) in terms of the solutions of Marchenko integral equation $K(x,y)$ and $\bar{K}(x,y)$.

Proposition 2.4. *Assume that the potentials $u(x)$ and $v(x)$ appearing in the system (2.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, for large λ we have*

$$m(\lambda,x)=\begin{bmatrix}0\\1\end{bmatrix}-\frac{1}{i\lambda}K(x,x)+o\left(\frac{1}{\lambda}\right),\qquad \lambda\to\infty\quad\text{in}\quad\lambda\in\overline{\mathbb{C}^+}, \tag{2.5.35}$$

$$\bar{m}(\lambda,x)=\begin{bmatrix}1\\0\end{bmatrix}-\frac{1}{i\lambda}\bar{K}(x,x)+o\left(\frac{1}{\lambda}\right),\qquad \lambda\to\infty\quad\text{in}\quad\lambda\in\overline{\mathbb{C}^-}, \tag{2.5.36}$$

where $K(x,x):=K(x,x^+)$, $\bar{K}(x,x):=\bar{K}(x,x^+)$, and the functions $m(\lambda,x)$ and $\bar{m}(\lambda,x)$ are defined in (2.1.4).

Proof. With the help of the first equality in (2.1.4), from (2.5.6) we have

$$m(\lambda,x)\,e^{i\lambda x}=\begin{bmatrix}0\\e^{i\lambda x}\end{bmatrix}+\int_x^\infty dy\,K(x,y)\,e^{i\lambda y}, \tag{2.5.37}$$

or equivalently

$$m(\lambda,x)=\begin{bmatrix}0\\1\end{bmatrix}+\int_x^\infty dy\,K(x,y)\,e^{i\lambda(y-x)}. \tag{2.5.38}$$

The integral term in (2.5.38) can be written as

$$\int_x^\infty dy\,K(x,y)\,e^{i\lambda(y-x)}=\int_x^\infty dy\left[K(x,y)\,\frac{d}{dy}\,\frac{e^{i\lambda(y-x)}}{i\lambda}\right],$$

or equivalently

$$\int_x^\infty dyle, K(x,y)\, e^{i\lambda(y-x)} = \int_x^\infty dy \left[\frac{d}{dy} \left(K(x,y)\, \frac{e^{i\lambda(y-x)}}{i\lambda} \right) - K_y(x,y)\, \frac{e^{i\lambda(y-x)}}{i\lambda} \right]. \tag{2.5.39}$$

By evaluating the right-hand side of (2.5.39) we obtain

$$\int_x^\infty dy\, K(x,y)\, e^{i\lambda(y-x)} = K(x,y)\, \frac{e^{i\lambda(y-x)}}{i\lambda} \Bigg|_{y=x}^{y=\infty} - \int_x^\infty dy\, K_y(x,y)\, \frac{e^{i\lambda(y-x)}}{i\lambda},$$

or equivalently

$$\int_x^\infty dy\, K(x,y)\, e^{i\lambda(y-x)} = \frac{-K(x,x)}{i\lambda} - \int_x^\infty dy\, K_y(x,y)\, \frac{e^{i\lambda(y-x)}}{i\lambda}. \tag{2.5.40}$$

Since $K_y(x,y) \in \mathbb{S}(\mathbb{R})$ and $|e^{2i\lambda(y-x)}| \le 1$ for $x \le y$ and $\lambda \in \overline{\mathbb{C}^+}$, the integral $\int_x^\infty$ in (2.5.40) is convergent for all $x \in \mathbb{R}$. Thus, we obtain

$$m(\lambda,x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{i\lambda}\, K(x,x) + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \lambda \in \overline{\mathbb{C}^+},$$

which establishes (2.5.35). Similarly, with the help of the second equality in (2.1.4), from (2.5.7) we have

$$\bar{m}(\lambda,x)\, e^{-i\lambda x} = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + \int_x^\infty dy\, \bar{K}(x,y)\, e^{-i\lambda y}. \tag{2.5.41}$$

We can write (2.5.41) as

$$\bar{m}(\lambda,x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_x^\infty dy\, \bar{K}(x,y)\, e^{i\lambda(x-y)}. \tag{2.5.42}$$

The integral term in (2.5.42) can be written as

$$\int_x^\infty dy\, \bar{K}(x,y)\, e^{i\lambda(x-y)} = \int_x^\infty dy \left[\bar{K}(x,y)\, \frac{d}{dy}\, \frac{e^{i\lambda(x-y)}}{-i\lambda} \right],$$

which can be written as

$$\int_x^\infty dy\,\bar K(x,y)\,e^{i\lambda(x-y)} = \int_x^\infty dy\left[\frac{d}{dy}\left(\bar K(x,y)\,\frac{e^{i\lambda(x-y)}}{-i\lambda}\right) + \bar K_y(x,y)\,\frac{e^{i\lambda(x-y)}}{i\lambda}\right]. \tag{2.5.43}$$

By evaluating the right-hand side of (2.5.43) we obtain

$$\int_x^\infty dy\,\bar K(x,y)\,e^{i\lambda(x-y)} = \bar K(x,y)\,\frac{e^{i\lambda(x-y)}}{-i\lambda}\Bigg|_{y=x}^{y=\infty} + \frac{1}{i\lambda}\int_x^\infty dy\,\bar K_y(x,y)\,e^{i\lambda(x-y)},$$

or equivalently

$$\int_x^\infty dy\,\bar K(x,y)\,e^{i\lambda(x-y)} = \frac{\bar K(x,x)}{i\lambda} + \frac{1}{i\lambda}\int_x^\infty dy\,\bar K_y(x,y)\,e^{i\lambda(x-y)}. \tag{2.5.44}$$

Since $\bar K_y(x,y) \in \mathbb{S}(\mathbb{R})$ and $|e^{2i\lambda(y-x)}| \le 1$ for $x \le y$ and $\lambda \in \overline{\mathbb{C}^+}$, the integral $\int_x^\infty$ in (2.5.44) is convergent for all $x \in \mathbb{R}$. Thus, we obtain

$$\int_x^\infty dy\,\bar K(x,x)\,e^{i\lambda(x-y)} = \frac{\bar K(x,x)}{i\lambda} + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \lambda \in \overline{\mathbb{C}^+}. \tag{2.5.45}$$

Using (2.5.45) in (2.5.42) we get

$$\bar m(\lambda,x) = \begin{bmatrix}1\\0\end{bmatrix} + \bar K(x,y)\,\frac{e^{i\lambda(x-y)}}{i\lambda} + o\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \lambda \in \overline{\mathbb{C}^+},$$

which completes the proof. □

In the next theorem, we show how to recover the potentials $u(x)$ and $v(x)$ from the solutions $K(x,y)$ and $\bar K(x,y)$ to the Marchenko integral equation .

Theorem 2.8. *Assume that the potentials $u(x)$ and $v(x)$ appearing in the system (2.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, we have*

$$u(x) = -2\,K_1(x,x), \tag{2.5.46}$$

$$v(x) = -2\,\bar K_2(x,x), \tag{2.5.47}$$

$$\int_x^\infty dz\,u(z)\,v(z) = 2\,K_2(x,x) = 2\,\bar K_1(x,x), \tag{2.5.48}$$

where $K_1(x,y)$, $\bar K_1(x,y)$, $K_2(x,y)$ and $\bar K_2(x,y)$ are the quantities appearing in (2.5.28).

Proof. With the help of (2.1.60) and (2.1.61), from the first equality of (2.1.8) we obtain

$$m(\lambda,x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{2i\lambda}\, u(x) + o\left(\dfrac{1}{\lambda}\right) \\ -\dfrac{1}{2i\lambda}\displaystyle\int_x^\infty dz\, u(z)\, v(z) + o\left(\dfrac{1}{\lambda}\right) \end{bmatrix}, \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^+}. \tag{2.5.49}$$

Similarly, using (2.1.62) and (2.1.63) in the second equality of (2.1.8) we get

$$\bar{m}(\lambda,x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{2i\lambda}\displaystyle\int_x^\infty dz\, u(z)\, v(z) + o\left(\dfrac{1}{\lambda}\right) \\ -\dfrac{1}{2i\lambda}\, v(x) + o\left(\dfrac{1}{\lambda}\right) \end{bmatrix}, \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^-}. \tag{2.5.50}$$

Hence, by comparing (2.5.35) and (2.5.49) we have

$$K(x,x) = \begin{bmatrix} -\dfrac{1}{2} u(x) \\ \dfrac{1}{2}\displaystyle\int_x^\infty dz\, u(z)\, v(z) \end{bmatrix}. \tag{2.5.51}$$

With the help of the second equality in (2.5.28), from (2.5.51) we obtain

$$K_1(x,x) = -\frac{1}{2}\, u(x),$$

$$K_2(x,x) = \frac{1}{2}\int_x^\infty dz\, u(z)\, v(z),$$

or equivalently

$$u(x) = -2\, K_1(x,x),$$

$$\int_x^\infty dz\, u(z)\, v(z) = 2\, K_2(x,x).$$

Similarly, by comparing (2.5.36) and (2.5.50) we have

$$\bar{K}(x,x) = \begin{bmatrix} \dfrac{1}{2}\displaystyle\int_x^\infty dz\, u(z)\, v(z) \\ -\dfrac{1}{2} v(x) \end{bmatrix}. \tag{2.5.52}$$

Using the first equality of (2.5.28) in (2.5.52) we get

$$\begin{aligned}\bar{K}_1(x,x) &= \frac{1}{2}\int_x^\infty dz\, u(z)\, v(z),\\ \bar{K}_2(x,x) &= -\frac{1}{2}\, v(x),\end{aligned}$$

or equivalently

$$\begin{aligned}v(x) &= -2\,\bar{K}_2(x,x),\\ \int_x^\infty dz\, u(z)\, v(z) &= 2\,\bar{K}_1(x,x),\end{aligned}$$

which completes the proof. □

Chapter 3

Scattering and Inverse Scattering for a First-Order System with Energy-Dependent Potentials

3.1 Scattering with Energy-Dependent Potentials

In this chapter, we develop a method to analyze the direct and inverse problems for the first-order system with energy-dependent potentials. Such a system appears in (1.0.1) and we quote it here again for the convenience of the reader as

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & \zeta q(x) \\ \zeta r(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad x \in \mathbb{R}, \tag{3.1.1}$$

where we recall that the prime denotes the x-derivative, ζ is the spectral parameter, the scalar quantities α and β depend on both x and ζ, and $q(x)$ and $r(x)$ are complex-valued potentials. We assume that $q(x)$ and $r(x)$ belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Recall that the spectral parameter ζ can sometimes be interpreted as energy. Compared to the system (2.1.1), we see that $q(x)$ and $r(x)$ appearing in (3.1.1) contain ζ as the coefficient. For this reason, we refer to the system (3.1.1) as the system with energy-dependent potentials.

Our main motivation in this chapter is to solve the inverse problem for the first-order system (3.1.1) by using the theory developed for the standard system (2.1.1). This is done with the help of various transformations we establish between the linear system (3.1.1) and linear system (2.1.1). Through these transformations, all the relevant quantities for (3.1.1), such as the Jost solutions ϕ, $\bar{\phi}$, ψ, $\bar{\psi}$; the scattering coefficients T, R, L, $\bar{T}$, $\bar{R}$, $\bar{L}$; the bound state dependency constants γ_j and $\bar{\gamma}_j$, the bound state norming constants c_j and $\bar{c}_j$, and the potentials $q(x)$ and $r(x)$ can all be expressed in terms of corresponding quantities for the standard system (2.1.1).

The properties of the Jost solutions for (3.1.1) are different from those for the standard system (2.1.1); For example, the asymptotics of the Jost solutions for the large-ζ are more complicated compared to those for (2.1.1). Essentially, the only viable way to analyze the direct and inverse scattering problem for (3.1.1) is to relate the relevant quantities for (3.1.1) to the corresponding quantities for (2.1.1). In this chapter we establish all the relevant relationships between (3.1.1) and (2.1.1).

Let us rewrite (3.1.1) in a more convenient form by multiplying the second row of (3.1.1) with ζ to get

$$\begin{bmatrix} \alpha \\ \zeta\,\beta \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & q(x) \\ \zeta^2\,r(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \zeta\,\beta \end{bmatrix}, \qquad x \in \mathbb{R}. \tag{3.1.2}$$

Note that the coefficient matrix in (3.1.2) contains ζ as ζ^2. As a result α and $\zeta\beta$ can be viewed as functions of x and λ, where $\lambda := \zeta^2$. Letting $\theta := \alpha$ and $\omega := \zeta\,\beta$, we can write (3.1.2) as

$$\begin{bmatrix} \theta \\ \omega \end{bmatrix}' = \begin{bmatrix} -i\lambda & q(x) \\ \lambda\,r(x) & i\lambda \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}, \qquad x \in \mathbb{R}. \tag{3.1.3}$$

Next, we relate (3.1.3) to (2.1.1) by using another transformation. In the next theorem, we present the transformation for the wave functions between the first-order system (3.1.1) and the standard system (2.1.1) through the intermediate system (3.1.3). The relevant steps are outlined in the following diagram:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & \zeta q(x) \\ \zeta r(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \theta \\ \omega \end{bmatrix}' = \begin{bmatrix} -i\lambda & q(x) \\ \lambda r(x) & i\lambda \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix}' = \begin{bmatrix} -i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

As the diagram indicates, we first transform (3.1.1) into (3.1.3) and then transform (3.1.3) into the standard system (2.1.1).

Theorem 3.1. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the first order systems (2.1.1), (3.1.1), and (3.1.3) are related to each other as*

$$\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \tag{3.1.4}$$

$$\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \delta \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{3.1.5}$$

where ϵ and δ are some scalar quantities depending on λ but not on x, $\zeta = \sqrt{\lambda}$, and

$$E(x) := \exp\left(\frac{i}{2}\int_{-\infty}^{x} dt\, q(t)\, r(t)\right), \tag{3.1.6}$$

$$u(x) = q(x)\, E^{-2}, \tag{3.1.7}$$

$$v(x) = \left(-\frac{i\,r'(x)}{2} + \frac{q(x)\,r(x)^2}{4}\right) E^2. \tag{3.1.8}$$

Proof. Note that even though E depends on x, we suppress its x-dependence and write mostly E instead of $E(x)$. The proof of (3.1.4) is obtained as follows. Using (3.1.4) in (3.1.1) we get

$$\frac{1}{\epsilon}\begin{bmatrix}1 & 0\\ 0 & 1/\zeta\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix}' = \begin{bmatrix}-i\zeta^2 & \zeta\, q(x)\\ \zeta\, r(x) & i\zeta^2\end{bmatrix}\frac{1}{\epsilon}\begin{bmatrix}1 & 0\\ 0 & 1/\zeta\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix},$$

or equivalently

$$\begin{bmatrix}1 & 0\\ 0 & 1/\zeta\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix}' = \begin{bmatrix}-i\zeta^2 & \zeta\, q(x)\\ \zeta\, r(x) & i\zeta^2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/\zeta\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix},$$

which can be written as

$$\begin{bmatrix}\theta\\ \omega\end{bmatrix}' = \begin{bmatrix}1 & 0\\ 0 & \zeta\end{bmatrix}\begin{bmatrix}-i\zeta^2 & \zeta\, q(x)\\ \zeta\, r(x) & i\zeta^2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/\zeta\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix}. \tag{3.1.9}$$

The product of the first three matrices on the right-hand side of (3.1.9) yields

$$\begin{bmatrix}1 & 0\\ 0 & \zeta\end{bmatrix}\begin{bmatrix}-i\zeta^2 & \zeta\, q(x)\\ \zeta\, r(x) & i\zeta^2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1/\zeta\end{bmatrix} = \begin{bmatrix}-i\zeta^2 & q(x)\\ \zeta^2\, r(x) & i\zeta^2\end{bmatrix}. \tag{3.1.10}$$

Using $\lambda = \zeta^2$ in (3.1.10), from (3.1.9) we obtain (3.1.3). This establishes that (3.1.1) is equivalent (3.1.3) via (3.1.4). Next, we will prove (3.1.5). Let us look for a transformation between (3.1.3) and (2.1.1) in the form

$$\begin{bmatrix}\theta\\ \omega\end{bmatrix} := \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix}, \tag{3.1.11}$$

where A, B, C, D are scalar quantities not depending on λ. Since $\begin{bmatrix}\theta\\ \omega\end{bmatrix}$ appearing in (3.1.11) is a solution to (3.1.3), it needs to satisfy (3.1.3). Hence, using (3.1.11) in (3.1.3) we get

$$\begin{bmatrix}A & B\\ C & D\end{bmatrix}'\begin{bmatrix}\xi\\ \eta\end{bmatrix} + \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix}' = \begin{bmatrix}-i\lambda & q(x)\\ \lambda\, r(x) & i\lambda\end{bmatrix}\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix},$$

which can be written as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}' = \left(- \begin{bmatrix} A & B \\ C & D \end{bmatrix}' + \begin{bmatrix} -i\lambda & q(x) \\ \lambda\, r(x) & i\lambda \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \begin{bmatrix} \xi \\ \eta \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix}' = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \left(- \begin{bmatrix} A & B \\ C & D \end{bmatrix}' + \begin{bmatrix} -i\lambda & q(x) \\ \lambda\, r(x) & i\lambda \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{3.1.12}$$

Comparing (3.1.12) with (2.1.1) we obtain

$$\begin{bmatrix} i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \left(- \begin{bmatrix} A & B \\ C & D \end{bmatrix}' + \begin{bmatrix} -i\lambda & q(x) \\ \lambda\, r(x) & i\lambda \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right),$$

or equivalently

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} i\lambda & u(x) \\ v(x) & i\lambda \end{bmatrix} = \left(- \begin{bmatrix} A & B \\ C & D \end{bmatrix}' + \begin{bmatrix} -i\lambda & q(x) \\ \lambda\, r(x) & i\lambda \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right). \tag{3.1.13}$$

Since A, B, C, D are assumed not to depend on λ, each side of (3.1.13) contains λ at most linearly. By equating the terms that do not contain λ and also equating the coefficients of λ on each side of (3.1.13) we get

$$\begin{cases} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & u(x) \\ v(x) & 0 \end{bmatrix} = - \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} + \begin{bmatrix} 0 & q(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \\ \\ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{bmatrix} = \begin{bmatrix} -i\lambda & 0 \\ \lambda\, r(x) & i\lambda \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \end{cases}$$

or equivalently

$$\begin{cases} \begin{bmatrix} B\,v(x) & A\,u(x) \\ D\,v(x) & C\,u(x) \end{bmatrix} = \begin{bmatrix} -A' + C\,q(x) & -B' + q(x)\,D \\ -C' & -D' \end{bmatrix}, \\ \\ \begin{bmatrix} -i\,A & i\,B \\ -i\,C & i\,D \end{bmatrix} = \begin{bmatrix} -i\,A & -i\,B \\ r(x)\,A + i\,C & r(x)\,B + i\,D \end{bmatrix}. \end{cases} \tag{3.1.14}$$

From the first line of the second equality in (3.1.14) we obtain $B = 0$. Hence, we can rewrite (3.1.14) as

$$\begin{cases} \begin{bmatrix} 0 & A\,u(x) \\ D\,v(x) & C\,u(x) \end{bmatrix} = \begin{bmatrix} -A' + C\,q(x) & q(x)\,D \\ -C' & -D' \end{bmatrix}, \\ \\ \begin{bmatrix} 0 & 0 \\ -i\,C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ r(x)\,A + i\,C & 0 \end{bmatrix}. \end{cases} \tag{3.1.15}$$

From (3.1.15), by comparing the corresponding matrix entries, we obtain

$$A' = q(x)\,C, \tag{3.1.16}$$

$$A\,u(x) = q(x)\,D, \tag{3.1.17}$$

$$D\,v(x) = -C', \tag{3.1.18}$$

$$C\,u(x) = -D', \tag{3.1.19}$$

$$r(x)\,A + 2i\,C = 0. \tag{3.1.20}$$

Using (3.1.16) and (3.1.20) we have

$$\begin{cases} C = \dfrac{i\, r(x)}{2}\, A, \\ A' = \dfrac{i\, q(x)\, r(x)}{2}\, A. \end{cases} \tag{3.1.21}$$

Note that the second equality in (3.1.21) is a first-order linear homogeneous ordinary differential equation. The solution to (3.1.21) is given by

$$\begin{cases} C = \dfrac{i\, r(x)}{2}\, \delta_1 \, \exp\left(\dfrac{i}{2} \displaystyle\int_{-\infty}^{x} q\, r\right), \\ A = \delta_1 \, \exp\left(\dfrac{i}{2} \displaystyle\int_{-\infty}^{x} q\, r\right), \end{cases} \tag{3.1.22}$$

where δ_1 is a constant and for simplicity we have suppressed the dummy integration variables. Using (3.1.6) we can write (3.1.22) as

$$\begin{cases} C = \dfrac{i\, r(x)}{2}\, \delta_1\, E, \\ A = \delta_1\, E, \end{cases} \tag{3.1.23}$$

where we have used the definition of E given in (3.1.6). Similarly, from (3.1.17), (3.1.18), and (3.1.19), respectively we obtain

$$\begin{cases} u(x) = \dfrac{q(x)\, D}{A}, \\ v(x) = -\dfrac{C'}{D}, \\ D' = -\dfrac{q(x)\, C\, D}{A}. \end{cases} \tag{3.1.24}$$

Using (3.1.22) in the third line of (3.1.24) we get

$$D' = -\frac{i\, q(x)\, r(x)}{2}\, D, \tag{3.1.25}$$

which is a first-order linear homogeneous ordinary differential equation. The solution to (3.1.25) is given by

$$D = \delta_2 \, \exp\left(-\frac{i}{2} \int_{-\infty}^{x} q\, r\right), \tag{3.1.26}$$

where δ_2 is a constant. Using the definition of E given in (3.1.6), we write (3.1.26) as

$$D = \delta_2 \, E^{-1}. \tag{3.1.27}$$

Using (3.1.23) and (3.1.27) in (3.1.24) we obtain

$$\begin{cases} u(x) = \dfrac{\delta_2}{\delta_1} \, q(x) \, E^{-2}, \\ v(x) = \dfrac{\delta_1}{\delta_2} \left(\dfrac{i \, r'(x)}{2} - \dfrac{r(x)^2 \, q(x)}{4} \right) E^2. \end{cases} \tag{3.1.28}$$

Let us choose $\delta_1 = \delta_2$ and use δ to denote their common value. Then, using (3.1.23), (3.1.27), and (3.1.28) we get

$$\begin{cases} A = \delta \, E, \\ B = 0, \\ C = \dfrac{i \, r(x)}{2} \, \delta \, E, \\ D = \delta \, E^{-1}, \\ u(x) = q(x) \, E^{-2}, \\ v(x) = \left(\dfrac{i \, r'(x)}{2} - \dfrac{r(x)^2 \, q(x)}{4} \right) E^2. \end{cases} \tag{3.1.29}$$

With the help of (3.1.29), we can write (3.1.11) as

$$\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \delta \begin{bmatrix} E & 0 \\ \frac{i \, r(x)}{2} \, E & E^{-1} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

which completes the proof of (3.1.5). □

In establishing the link between (3.1.1) an (2.1.1) we have multiplied the second line of (3.1.1) by the spectral parameter ζ. Next we establish a link between (3.1.1) and another standard system similar to (2.1.1). For this, we multiply the first line of (3.1.1) by the spectral parameter ζ we get

$$\begin{bmatrix} \zeta \, \alpha \\ \beta \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & \zeta^2 \, q(x) \\ r(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} \zeta \, \alpha \\ \beta \end{bmatrix}, \qquad x \in \mathbb{R}. \tag{3.1.30}$$

Note that the coefficient matrix in (3.1.30) contains ζ as ζ^2. As a result, the quantities $\alpha\zeta$ and β can be viewed as functions of x and λ, where we recall that $\lambda = \zeta^2$. Letting $\tilde{\theta} := \zeta\,\alpha$ and $\tilde{\omega} := \beta$, we can write (3.1.30) as

$$\begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix}' = \begin{bmatrix} -i\lambda & \lambda\, q(x) \\ r(x) & i\lambda \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix}, \qquad x \in \mathbb{R}. \tag{3.1.31}$$

In the next theorem, we present another set of transformations relating the first-order system (3.1.1) and another form of the standard system but with different potentials, and that standard system is given by

$$\begin{bmatrix} \tilde{\xi} \\ \tilde{\eta} \end{bmatrix}' = \begin{bmatrix} -i\lambda & p(x) \\ s(x) & i\lambda \end{bmatrix} \begin{bmatrix} \tilde{\xi} \\ \tilde{\eta} \end{bmatrix}. \tag{3.1.32}$$

The relevant steps are outlined in the following diagram:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & \zeta q(x) \\ \zeta r(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix}' = \begin{bmatrix} -i\lambda & \lambda\, q(x) \\ r(x) & i\lambda \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \tilde{\xi} \\ \tilde{\eta} \end{bmatrix}' = \begin{bmatrix} -i\lambda & p(x) \\ s(x) & i\lambda \end{bmatrix} \begin{bmatrix} \tilde{\xi} \\ \tilde{\eta} \end{bmatrix}.$$

Note that the system given in (3.1.32) is similar to that given in (2.1.1) but the potentials $p(x)$ and $s(x)$ in (3.1.32) are not the same as the potentials $u(x)$ and $v(x)$ appearing in (2.1.1). As the diagram indicates, we first transform (3.1.1) into (3.1.31) and then transform (3.1.31) into the standard system (3.1.32).

Theorem 3.2. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the first order system (3.1.1), (3.1.31), and (3.1.32) are related to each other as*

$$\begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix} = \tilde{\epsilon} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \tag{3.1.33}$$

$$\begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix} = \tilde{\delta} \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\xi} \\ \tilde{\eta} \end{bmatrix}, \tag{3.1.34}$$

where $\tilde{\epsilon}$ and $\tilde{\delta}$ are some scalar quantities depending only on λ but not on x, $\zeta = \sqrt{\lambda}$, E is the quantity defined in (3.1.6), and

$$p(x) = \left(-\frac{i\,q'(x)}{2} + \frac{q(x)^2\,r(x)}{4}\right) E^{-2}, \tag{3.1.35}$$

$$s(x) = r(x)\,E^2. \tag{3.1.36}$$

Proof. The proof of (3.1.33) is obtained as follows. In (3.1.1), by replacing $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ with $1/\tilde{\epsilon} \begin{bmatrix} 1/\zeta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix}$ we get

$$1/\tilde{\epsilon} \begin{bmatrix} 1/\zeta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & \zeta\,q(x) \\ \zeta\,r(x) & i\zeta^2 \end{bmatrix} 1/\tilde{\epsilon} \begin{bmatrix} 1/\zeta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} 1/\zeta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & \zeta\,q(x) \\ \zeta\,r(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} 1/\zeta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix},$$

which can be written as

$$\begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix}' = \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -i\zeta^2 & \zeta\,q(x) \\ \zeta\,r(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} 1/\zeta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}. \tag{3.1.37}$$

The product of the first three matrices on the right-hand side of (3.1.37) yields

$$\begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -i\zeta^2 & \zeta\, q(x) \\ \zeta\, r(x) & i\zeta^2 \end{bmatrix} \begin{bmatrix} 1/\zeta & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -i\zeta^2 & \zeta^2\, q(x) \\ r(x) & i\zeta^2 \end{bmatrix}. \tag{3.1.38}$$

Using $\lambda = \zeta^2$ in (3.1.38), from (3.1.37) we obtain (3.1.31). This establishes that (3.1.1) is equivalent to (3.1.31) via (3.1.33). Next, we will prove (3.1.34). Let us look for a transformation between (3.1.31) and (3.1.32) in the form

$$\begin{bmatrix} \tilde\theta \\ \tilde\omega \end{bmatrix} := \begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix} \begin{bmatrix} \tilde\xi \\ \tilde\eta \end{bmatrix}, \tag{3.1.39}$$

where $\tilde A, \tilde B, \tilde C, \tilde D$ are scalar quantities not depending on λ. Since $\begin{bmatrix} \tilde\theta \\ \tilde\omega \end{bmatrix}$ appearing in (3.1.39) is a solution to (3.1.31), it needs to satisfy (3.1.31). Hence, using (3.1.39) in (3.1.31) we get

$$\begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix}' \begin{bmatrix} \tilde\xi \\ \tilde\eta \end{bmatrix} + \begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix} \begin{bmatrix} \tilde\xi \\ \tilde\eta \end{bmatrix}' = \begin{bmatrix} -i\lambda & \lambda q(x) \\ r(x) & i\lambda \end{bmatrix} \begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix} \begin{bmatrix} \tilde\xi \\ \tilde\eta \end{bmatrix},$$

which can be written as

$$\begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix} \begin{bmatrix} \tilde\xi \\ \tilde\eta \end{bmatrix}' = \left(- \begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix}' + \begin{bmatrix} -i\lambda & \lambda\, q(x) \\ r(x) & i\lambda \end{bmatrix} \begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix} \right) \begin{bmatrix} \tilde\xi \\ \tilde\eta \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} \tilde\xi \\ \tilde\eta \end{bmatrix}' = \begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix}^{-1} \left(- \begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix}' + \begin{bmatrix} -i\lambda & \lambda\, q(x) \\ r(x) & i\lambda \end{bmatrix} \begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix} \right) \begin{bmatrix} \tilde\xi \\ \tilde\eta \end{bmatrix}. \tag{3.1.40}$$

Comparing (3.1.40) with (3.1.32) we obtain

$$\begin{bmatrix} i\lambda & p(x) \\ s(x) & i\lambda \end{bmatrix} = \begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix}^{-1} \left(- \begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix}' + \begin{bmatrix} -i\lambda & \lambda\, q(x) \\ r(x) & i\lambda \end{bmatrix} \begin{bmatrix} \tilde A & \tilde B \\ \tilde C & \tilde D \end{bmatrix} \right),$$

or equivalently

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} i\lambda & p(x) \\ s(x) & i\lambda \end{bmatrix} = \left(- \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}' + \begin{bmatrix} -i\lambda & \lambda\, q(x) \\ r(x) & i\lambda \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \right). \tag{3.1.41}$$

Since $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are assumed not to depend on λ, each side of (3.1.41) contains λ at most linearly. By equating the terms that do not contain λ, and also equating the coefficients of λ on each side of (3.1.41), we get

$$\begin{cases} \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} 0 & p(x) \\ s(x) & 0 \end{bmatrix} = - \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ r(x) & 0 \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}, \\ \\ \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{bmatrix} = \begin{bmatrix} -i\lambda & \lambda\, q(x) \\ 0 & i\lambda \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}, \end{cases}$$

or equivalently

$$\begin{cases} \begin{bmatrix} \tilde{B}\, s(x) & \tilde{A}\, p(x) \\ \tilde{D}\, s(x) & \tilde{C}\, p(x) \end{bmatrix} = \begin{bmatrix} -\tilde{A}' & -\tilde{B}' \\ -\tilde{C}' + \tilde{A}\, r(x) & -\tilde{D}' + r(x)\, \tilde{B} \end{bmatrix}, \\ \\ \begin{bmatrix} -i\,\tilde{A} & i\,\tilde{B} \\ -i\,\tilde{C} & i\,\tilde{D} \end{bmatrix} = \begin{bmatrix} -i\,\tilde{A} + q(x)\,\tilde{C} & -i\,\tilde{B} + q(x)\,\tilde{D} \\ i\,\tilde{C} & i\,\tilde{D} \end{bmatrix}. \end{cases} \tag{3.1.42}$$

From the second line of the second equality in (3.1.42) we obtain $\tilde{C} = 0$. Hence, we can rewrite (3.1.42) as

$$\begin{cases} \begin{bmatrix} \tilde{B}\, s(x) & \tilde{A}\, p(x) \\ D\, s(x) & 0 \end{bmatrix} = \begin{bmatrix} -\tilde{A}' & -\tilde{B}' \\ \tilde{A}\, r(x) & -\tilde{D}' + r(x)\, \tilde{B} \end{bmatrix}, \\ \\ \begin{bmatrix} 0 & i\, \tilde{B} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i\, \tilde{B} + q(x)\, \tilde{D} \\ 0 & 0 \end{bmatrix}. \end{cases} \tag{3.1.43}$$

From (3.1.43), by comparing the corresponding matrix entries, we obtain

$$\tilde{A}' = -s(x)\, \tilde{B}, \tag{3.1.44}$$

$$\tilde{A}\, p(x) = -\tilde{B}', \tag{3.1.45}$$

$$\tilde{D}\, s(x) = \tilde{A}\, r(x), \tag{3.1.46}$$

$$\tilde{D}' = r(x)\, \tilde{B}, \tag{3.1.47}$$

$$2i\, \tilde{B} = q(x)\, \tilde{D}. \tag{3.1.48}$$

Using (3.1.47) and (3.1.48) we have

$$\begin{cases} \tilde{B} = -\dfrac{i\, q(x)}{2}\, \tilde{D}, \\ \tilde{D}' = -\dfrac{i\, q(x)\, r(x)}{2}\, \tilde{D}. \end{cases} \tag{3.1.49}$$

Note that the second equality in (3.1.49) is a first-order linear homogeneous ordinary differential equation. The solution to (3.1.49) is given by

$$\begin{cases} \tilde{B} = -\dfrac{i\, q(x)}{2}\, \tilde{\delta}_1 \,\exp\left(-\dfrac{i}{2} \displaystyle\int_{-\infty}^{x} q\, r \right), \\ \tilde{D} = \tilde{\delta}_1 \,\exp\left(-\dfrac{i}{2} \displaystyle\int_{-\infty}^{x} q\, r \right), \end{cases} \tag{3.1.50}$$

where $\tilde{\delta}_1$ is a constant and for simplicity we have suppressed the dummy integration variables. Using (3.1.6) we can write (3.1.50) as

$$\begin{cases} \tilde{B} = -\dfrac{i\,q(x)}{2}\tilde{\delta}_1\,E^{-1}, \\ \tilde{D} = \tilde{\delta}_1\,E^{-1}. \end{cases} \tag{3.1.51}$$

Similarly, using (3.1.44), (3.1.45), and (3.1.46), respectively we obtain

$$\begin{cases} s(x) = -\dfrac{\tilde{A}\,r(x)}{\tilde{D}}, \\ p(x) = -\dfrac{\tilde{B}'}{\tilde{A}}, \\ \tilde{A}' = -\dfrac{r(x)\,\tilde{A}\,\tilde{B}}{\tilde{D}}. \end{cases} \tag{3.1.52}$$

Using (3.1.51) in the third line of (3.1.52) we get

$$\tilde{A}' = \frac{i\,q(x)\,r(x)}{2}\,\tilde{A}, \tag{3.1.53}$$

which is a first-order linear homogeneous ordinary differential equation. The solution to (3.1.53) is given by

$$\tilde{A} = \tilde{\delta}_2\,\exp\left(\frac{i}{2}\int_{-\infty}^{x} q\,r\right), \tag{3.1.54}$$

where $\tilde{\delta}_2$ is a constant. Using the definition of E given in (3.1.6), we write (3.1.54) as

$$\tilde{A} = \tilde{\delta}_2\,E. \tag{3.1.55}$$

Using (3.1.51) and (3.1.55) in (3.1.52) we obtain

$$\begin{cases} s(x) = \dfrac{\tilde{\delta}_2}{\tilde{\delta}_1}\,r(x)\,E^2, \\ p(x) = \dfrac{\tilde{\delta}_1}{\tilde{\delta}_2}\left(\dfrac{i\,q'(x)}{2} - \dfrac{q(x)^2\,r(x)}{4}\right)E^{-2}. \end{cases} \tag{3.1.56}$$

Let us choose $\tilde{\delta}_1 := \tilde{\delta}_2$ and use $\tilde{\delta}$ to denote their common value. Hence, using (3.1.51), (3.1.55), and (3.1.56) we get

$$\begin{cases} \tilde{A} = \tilde{\delta}\, E, \\ \tilde{B} = -\dfrac{i\, q(x)}{2} \tilde{\delta}\, E^{-1}, \\ C = 0, \\ \tilde{D} = \tilde{\delta}\, E^{-1}, \\ p(x) = \left(\dfrac{i\, q'(x)}{2} + \dfrac{r(x)^2\, q(x)}{4} \right) E^2, \\ s(x) = r(x)\, E^2. \end{cases} \tag{3.1.57}$$

With the help of (3.1.57), we can write (3.1.39) as

$$\begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix} = \tilde{\delta} \begin{bmatrix} E & \frac{-i\, q(x)}{2}\, E \\ 0 & E^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\xi} \\ \tilde{\eta} \end{bmatrix},$$

which completes the proof of (3.1.34). □

3.2 Jost Solutions

There are four particular column-vector solutions to (3.1.1), known as the Jost solutions and denoted by ϕ, ψ, $\bar{\phi}$, $\bar{\psi}$, respectively, which are uniquely determined by imposing the asymptotic conditions

$$\phi(\zeta, x) = \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1), \quad \bar{\phi}(\zeta, x) = \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.1}$$

$$\psi(\zeta, x) = \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + o(1), \quad \bar{\psi}(\zeta, x) = \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty. \tag{3.2.2}$$

Note that there are five first-order system we deal with, and they are given in (2.1.1), (3.1.1), (3.1.3),(3.1.31), and (3.1.32). Each of these five systems have their own four

Jost solutions, six scattering coefficients, and a pair of potentials. In order to avoid any confusion, we will label the Jost solutions and the scattering coefficients for each of the five system by using the superscripts $(\zeta q, \zeta r)$, $(q, \lambda r)$, $(\lambda q, r)$, (u, v), and (p, s), which identically the corresponding potentials. For example, for the system (3.1.1) we use $\phi^{(\zeta q,\zeta r)}, \psi^{(\zeta q,\zeta r)}, \bar{\phi}^{(\zeta q,\zeta r)}, \bar{\psi}^{(\zeta q,\zeta r)}$ to denote the corresponding Jost solutions and we use $T^{(\zeta q,\zeta r)}$, $R^{(\zeta q,\zeta r)}$, $L^{(\zeta q,\zeta r)}$, $\overline{T}^{(\zeta q,\zeta r)}$, $\overline{R}^{(\zeta q,\zeta r)}$, $\overline{L}^{(\zeta q,\zeta r)}$ to denote the corresponding scattering coefficients.

The Jost solutions to the system (3.1.3) are determined by using the asymptotics

$$\phi^{(q,\lambda r)} = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad \bar{\phi}^{(q,\lambda r)} = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.3}$$

$$\psi^{(q,\lambda r)} = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad \bar{\psi}^{(q,\lambda r)} = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty. \tag{3.2.4}$$

The Jost solutions to the system (3.1.31) are determined by using the asymptotics

$$\phi^{(\lambda q,r)} = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad \bar{\phi}^{(\lambda q,r)} = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.5}$$

$$\psi^{(\lambda q,r)} = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad \bar{\psi}^{(\lambda q,r)} = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty. \tag{3.2.6}$$

The Jost solutions to the system (3.1.32) are determined by using asymptotics

$$\phi^{(p,s)} = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad \bar{\phi}^{(p,s)} = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.7}$$

$$\psi^{(p,s)} = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad \bar{\psi}^{(p,s)} = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty. \tag{3.2.8}$$

In the next proposition, we show the relationship between the corresponding Jost solutions for the first-order system (3.1.3) and the standard system (2.1.1).

Proposition 3.1. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.3) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the Jost solutions $\phi^{(u,v)}, \psi^{(u,v)}, \bar\phi^{(u,v)}, \bar\psi^{(u,v)}$ to (2.1.1), and the Jost solutions $\phi^{(q,\lambda r)}, \psi^{(q,\lambda r)}, \bar\phi^{(q,\lambda r)}, \bar\psi^{(q,\lambda r)}$ to (3.1.3) are related to each other as*

$$\phi^{(q,\lambda r)} = \begin{bmatrix} E & 0 \\ \frac{i\, r(x)}{2}\, E & E^{-1} \end{bmatrix} \phi^{(u,v)}, \tag{3.2.9}$$

$$\psi^{(q,\lambda r)} = e^{i\mu/2} \begin{bmatrix} E & 0 \\ \frac{i\, r(x)}{2}\, E & E^{-1} \end{bmatrix} \psi^{(u,v)}, \tag{3.2.10}$$

$$\bar\phi^{(q,\lambda r)} = \begin{bmatrix} E & 0 \\ \frac{i\, r(x)}{2}\, E & E^{-1} \end{bmatrix} \bar\phi^{(u,v)}, \tag{3.2.11}$$

$$\bar\psi^{(q,\lambda r)} = e^{-i\mu/2} \begin{bmatrix} E & 0 \\ \frac{i\, r(x)}{2}\, E & E^{-1} \end{bmatrix} \bar\psi^{(u,v)}, \tag{3.2.12}$$

where E is the quantity defined in (3.1.6),

$$\mu := \int_{-\infty}^{\infty} q\, r. \tag{3.2.13}$$

Proof. Note that from (3.1.6) and (3.2.13) we have

$$E \to 1, \qquad x \to -\infty; \quad E \to e^{i\mu/2}, \qquad x \to +\infty. \tag{3.2.14}$$

First, let us relate the Jost solution $\phi^{(q,\lambda r)}$ to (3.1.3) and the Jost solution $\phi^{(u,v)}$ to (2.1.1) to each other. From (3.1.5) we have

$$\phi^{(q,\lambda r)} = \delta \begin{bmatrix} E & 0 \\ \frac{i\, r(x)}{2}\, E & E^{-1} \end{bmatrix} \phi^{(u,v)}. \tag{3.2.15}$$

Using the first equalities of (2.1.3), (3.2.3), and (3.2.14) in (3.2.15) we get

$$\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1) = \delta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.16}$$

and hence $\delta = 1$. Using $\delta = 1$ in (3.2.15) we obtain

$$\phi^{(q,\lambda r)} = \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \phi^{(u,v)}, \tag{3.2.17}$$

which establishes (3.2.9). Similarly, let us relate the Jost solution $\psi^{q,\lambda r}$ to (3.1.3) and the Jost solution $\psi^{(u,v)}$ to (2.1.1) to each other. From (3.1.5) we have

$$\psi^{(q,\lambda r)} = \delta \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \psi^{(u,v)}. \tag{3.2.18}$$

With the help of the first equalities of (2.1.2), (3.2.4), and the second equality in (3.2.14), from (3.2.18) we obtain

$$\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1) = \delta \begin{bmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{3.2.19}$$

and hence $\delta = e^{i\mu/2}$. Using $\delta = e^{i\mu/2}$ in (3.2.18) we have

$$\psi^{(q,\lambda r)} = e^{i\mu/2} \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \psi^{(u,v)}, \tag{3.2.20}$$

which establishes (3.2.10). In the same manner, let us relate the Jost solution $\bar{\phi}^{(q,\lambda r)}$ to (3.1.3) and the Jost solution $\bar{\phi}^{(u,v)}$ to (2.1.1) to each other. With the help of (3.1.5) we have

$$\bar{\phi}^{(q,\lambda r)} = \delta \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \bar{\phi}^{(u,v)}. \tag{3.2.21}$$

Using the second equalities of (2.1.3), (3.2.3), and the first equality of (3.2.14) in (3.2.21) we get

$$\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1) = \delta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.22}$$

and hence $\delta = 1$. Using $\delta = 1$ in (3.2.21) we can rewrite (3.2.21) as

$$\bar{\phi}^{(q,\lambda r)} = \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \bar{\phi}^{(u,v)}, \tag{3.2.23}$$

which establishes (3.2.11). Now, let us relate the Jost solution $\bar{\psi}^{(q,\lambda r)}$ to (3.1.3) and the Jost solution $\bar{\psi}^{(u,v)}$ to (2.1.1) to each other. From (3.1.5) we have

$$\bar{\psi}^{(q,\lambda r)} = \delta \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \bar{\psi}^{(u,v)}. \tag{3.2.24}$$

Substituting the second equalities of (2.1.2), (3.2.4), and (3.2.14) in (3.2.24) we obtain

$$\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1) = \delta \begin{bmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{bmatrix} \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{3.2.25}$$

and hence $\delta = e^{-i\mu/2}$. Using this value of δ in (3.2.24) we get

$$\bar{\psi}^{(q,\lambda r)} = e^{-i\mu/2} \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \bar{\psi}^{(u,v)}, \tag{3.2.26}$$

which completes the prove of (3.2.12). □

In the next proposition, we show the relationships among the Jost solutions for the first-order system (3.1.1) and the Jost solutions for (3.1.3).

Proposition 3.2. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the Jost solutions $\phi^{(\zeta q,\zeta r)}$, $\psi^{(\zeta q,\zeta r)}$, $\bar{\phi}^{(\zeta q,\zeta r)}$, $\bar{\psi}^{(\zeta q,\zeta r)}$ to (3.1.1), and the Jost solutions $\phi^{(q,\lambda r)}$, $\psi^{(q,\lambda r)}$, $\bar{\phi}^{(q,\lambda r)}$, $\bar{\psi}^{(q,\lambda r)}$ to (3.1.3) are related to each other as*

$$\phi^{(q,\lambda r)} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \phi^{(\zeta q,\zeta r)}, \tag{3.2.27}$$

$$\psi^{(q,\lambda r)} = \frac{1}{\sqrt{\lambda}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \psi^{(\zeta q,\zeta r)}, \tag{3.2.28}$$

$$\bar{\phi}^{(q,\lambda r)} = \frac{1}{\sqrt{\lambda}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \bar{\phi}^{(\zeta q,\zeta r)}, \tag{3.2.29}$$

$$\bar{\psi}^{(q,\lambda r)} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \bar{\psi}^{(\zeta q,\zeta r)}. \tag{3.2.30}$$

Proof. First, let us relate the Jost solution $\phi^{(\zeta q,\zeta r)}$ of (3.1.1) to the Jost solution $\phi^{(q,\lambda r)}$ of (3.1.3). From (3.1.4) we have

$$\phi^{(q,\lambda r)} = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \phi^{(\zeta q,\zeta r)}. \tag{3.2.31}$$

Using the first equality of (2.1.3) and of (3.2.3) in (3.2.31) we get

$$\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1) = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.32}$$

and hence $\epsilon = 1$. Using $\epsilon = 1$ in (3.2.31) we obtain

$$\phi^{(q,\lambda r)} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \phi^{(\zeta q,\zeta r)}, \tag{3.2.33}$$

which establishes (3.2.27). Similarly, let us relate the Jost solution $\psi^{(q,\lambda r)}$ of (3.1.3) to the Jost solution $\psi^{(\zeta q,\zeta r)}$ of (3.1.1). From (3.1.4) we have

$$\psi^{(q,\lambda r)} = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \psi^{(\zeta q,\zeta r)}. \tag{3.2.34}$$

With the help of the first equalities of (2.1.2) and of (3.2.4), from (3.2.34) we obtain

$$\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1) = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{3.2.35}$$

and hence $\epsilon = \frac{1}{\sqrt{\lambda}}$. Using $\epsilon = \frac{1}{\sqrt{\lambda}}$ in (3.2.34) we have

$$\psi^{(q,\lambda r)} = \frac{1}{\sqrt{\lambda}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \psi^{(\zeta q,\zeta r)}, \tag{3.2.36}$$

which completes the proof of (3.2.28). In the same manner, let us relate Jost solution $\bar{\phi}^{(q,\lambda r)}$ of (3.1.3) to the Jost solution $\bar{\phi}^{(\zeta q,\zeta r)}$ of (3.1.1). With the help of (3.1.4) we have

$$\bar{\phi}^{(q,\lambda r)} = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \bar{\phi}^{(\zeta q,\zeta r)}. \tag{3.2.37}$$

Using the second equalities of (2.1.3) and of (3.2.3) in (3.2.37) we get

$$\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1) = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.38}$$

and hence $\epsilon = \frac{1}{\sqrt{\lambda}}$. Using $\epsilon = \frac{1}{\sqrt{\lambda}}$ in (3.2.37) we can write (3.2.37) as

$$\bar{\phi}^{(q,\lambda r)} = \frac{1}{\sqrt{\lambda}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \bar{\phi}^{(\zeta q,\zeta r)}, \tag{3.2.39}$$

which establishes (3.2.29). Now, let us relate the Jost solution $\bar{\psi}^{(q,\lambda r)}$ of (3.1.3) to the Jost solution $\bar{\psi}^{(\zeta q,\zeta r)}$ of (3.1.1). From (3.1.4) we have

$$\bar{\psi}^{(q,\lambda r)} = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \bar{\psi}^{(\zeta q,\zeta r)}. \tag{3.2.40}$$

Substituting the second equalities of (2.1.2) and of (3.2.4) in (3.2.40) we obtain

$$\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1) = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{3.2.41}$$

and hence $\epsilon = 1$. Using this value of ϵ in (3.2.40) we get

$$\bar{\psi}^{(q,\lambda r)} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \bar{\psi}^{(\zeta q,\zeta r)}, \tag{3.2.42}$$

which completes the proof of (3.2.30). □

In the next proposition, we show the relationship between the corresponding Jost solutions for the first-order system (3.1.1) and the standard system (2.1.1).

Proposition 3.3. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the Jost solutions $\phi^{(u,v)}$, $\psi^{(u,v)}$, $\bar{\phi}^{(u,v)}$, $\bar{\psi}^{(u,v)}$ to (2.1.1), and the Jost solutions $\phi^{(\zeta q,\zeta r)}$, $\psi^{(\zeta q,\zeta r)}$, $\bar{\phi}^{(\zeta q,\zeta r)}$, $\bar{\psi}^{(\zeta q,\zeta r)}$ to (3.1.1) are related to each other as*

$$\phi^{(\zeta q,\zeta r)} = \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2\sqrt{\lambda}}\,E & \frac{E^{-1}}{\sqrt{\lambda}} \end{bmatrix} \phi^{(u,v)}, \tag{3.2.43}$$

$$\psi^{(\zeta q,\zeta r)} = e^{i\mu/2} \begin{bmatrix} \sqrt{\lambda}\,E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \psi^{(u,v)}, \tag{3.2.44}$$

$$\bar{\phi}^{(\zeta q,\zeta r)} = \begin{bmatrix} \sqrt{\lambda}\,E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \bar{\phi}^{(u,v)}, \tag{3.2.45}$$

$$\bar{\psi}^{(\zeta q,\zeta r)} = e^{-i\mu/2} \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2\sqrt{\lambda}}\,E & \frac{E^{-1}}{\sqrt{\lambda}} \end{bmatrix} \bar{\psi}^{(u,v)}, \tag{3.2.46}$$

where E and μ are the quantities defined in (3.1.6) and (3.2.13), respectively.

Proof. By comparing (3.2.9) and (3.2.27) we get

$$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \phi^{(\zeta q,\zeta r)} = \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \phi^{(u,v)},$$

which can be written as

$$\phi^{(\zeta q,\zeta r)} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}^{-1} \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \phi^{(u,v)},$$

or equivalently

$$\phi^{(\zeta q,\zeta r)} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix} \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \phi^{(u,v)}. \tag{3.2.47}$$

Evaluating the right-hand side of (3.2.47) we obtain

$$\phi^{(\zeta q,\zeta r)} = \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2\sqrt{\lambda}}\,E & \frac{E^{-1}}{\sqrt{\lambda}} \end{bmatrix} \phi^{(u,v)}. \tag{3.2.48}$$

which establishes (3.2.43). Similarly, comparing (3.2.10) and (3.2.28) we have

$$\frac{1}{\sqrt{\lambda}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \phi^{(\zeta q,\zeta r)} = e^{i\mu/2} \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \psi^{(u,v)},$$

which can be written as

$$\phi^{(\zeta q,\zeta r)} = e^{i\mu/2}\sqrt{\lambda} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}^{-1} \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \psi^{(u,v)},$$

or equivalently

$$\phi^{(\zeta q,\zeta r)} = e^{i\mu/2} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \psi^{(u,v)},$$

which yields

$$\psi^{(\zeta q,\zeta r)} = e^{i\mu/2} \begin{bmatrix} \sqrt{\lambda}\,E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \psi^{(u,v)}, \tag{3.2.49}$$

which establishes (3.2.44). Again, comparing (3.2.11) and (3.2.29) we obtain

$$\frac{1}{\sqrt{\lambda}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \bar{\phi}^{(\zeta q,\zeta r)} = \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \bar{\phi}^{(u,v)}, \tag{3.2.50}$$

we can write (3.2.50) as

$$\bar{\phi}^{(\zeta q,\zeta r)} = \sqrt{\lambda} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}^{-1} \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \bar{\phi}^{(u,v)},$$

or equivalently

$$\bar{\phi}^{(\zeta q,\zeta r)} = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \bar{\phi}^{(u,v)},$$

which yields

$$\bar{\phi}^{(\zeta q,\zeta r)} = \begin{bmatrix} \sqrt{\lambda}\, E & 0 \\ \frac{i\, r(x)}{2}\, E & E^{-1} \end{bmatrix} \bar{\phi}^{(u,v)}, \tag{3.2.51}$$

which establishes (3.2.45). Finally, comparing (3.2.12) and (3.2.30) we get

$$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \bar{\psi}^{(\zeta q,\zeta r)} = e^{-i\mu/2} \begin{bmatrix} E & 0 \\ \frac{i\, r(x)}{2}\, E & E^{-1} \end{bmatrix} \bar{\psi}^{(u,v)},$$

which can be written as

$$\bar{\psi}^{(\zeta q,\zeta r)} = e^{-i\mu/2} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}^{-1} \begin{bmatrix} E & 0 \\ \frac{i\, r(x)}{2}\, E & E^{-1} \end{bmatrix} \bar{\psi}^{(u,v)},$$

or equivalently

$$\bar{\psi}^{(\zeta q,\zeta r)} = e^{-i\mu/2} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix} \begin{bmatrix} E & 0 \\ \frac{i\, r(x)}{2}\, E & E^{-1} \end{bmatrix} \bar{\psi}^{(u,v)},$$

which yields

$$\bar{\psi}^{(\zeta q,\zeta r)} = e^{-i\mu/2} \begin{bmatrix} E & 0 \\ \frac{i\, r(x)}{2\sqrt{\lambda}}\, E & \frac{E^{-1}}{\sqrt{\lambda}} \end{bmatrix} \bar{\psi}^{(u,v)}, \tag{3.2.52}$$

which completes the proof of (3.2.46). □

Next, we show the relationships among the Jost solutions for the first-order system (3.1.31) and the standard system (3.1.32).

Proposition 3.4. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.31) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the Jost solutions $\phi^{(\lambda q,r)}$, $\psi^{(\lambda q,r)}$, $\bar{\phi}^{(\lambda q,r)}$, $\bar{\psi}^{(\lambda q,r)}$ to (3.1.31) and the Jost solutions $\phi^{(p,s)}$, $\psi^{(p,s)}$, $\bar{\phi}^{(p,s)}$, $\bar{\psi}^{(p,s)}$ to (3.1.32) are related to each other as*

$$\phi^{(\lambda q,r)} = \begin{bmatrix} E & -\frac{i\, q(x)}{2}\, E^{-1} \\ 0 & E^{-1} \end{bmatrix} \phi^{(p,s)}, \tag{3.2.53}$$

$$\psi^{(\lambda q,r)} = e^{i\mu/2} \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \psi^{(p,s)}, \tag{3.2.54}$$

$$\bar{\phi}^{(\lambda q,r)} = \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \bar{\phi}^{(p,s)}, \tag{3.2.55}$$

$$\bar{\psi}^{(\lambda q,r)} = e^{-i\mu/2} \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \bar{\psi}^{(p,s)}, \tag{3.2.56}$$

where E and μ are the quantities defined in (3.1.6) *and* (3.2.13)*, respectively.*

Proof. First, let us relate the Jost solution $\phi^{(\lambda q,r)}$ to (3.1.31) and Jost solution $\phi^{(p,s)}$ to (3.1.32) to each other. From (3.1.34) we have

$$\phi^{(\lambda q,r)} = \tilde{\delta} \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \phi^{(p,s)}. \tag{3.2.57}$$

Using the first equalities of (3.2.5), (3.2.7), and (3.2.14) in (3.2.57) we get

$$\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1) = \tilde{\delta} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.58}$$

and hence $\tilde{\delta} = 1$. Using this value of $\tilde{\delta}$ in (3.2.57) we obtain

$$\phi^{(\lambda q,r)} = \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \phi^{(p,s)}, \tag{3.2.59}$$

which establishes (3.2.53). Similarly, let us relate the Jost solution $\psi^{(\lambda q,r)}$ to (3.1.31) and the Jost solution $\psi^{(p,s)}$ to (3.1.32) to each other. From (3.1.34) we have

$$\psi^{(\lambda q,r)} = \tilde{\delta} \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \psi^{(p,s)}. \tag{3.2.60}$$

With the help of the first equalities of (3.2.6), (3.2.8), and the second equality in (3.2.14), from (3.2.60) we obtain

$$\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1) = \tilde{\delta} \begin{bmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{3.2.61}$$

and hence $\tilde{\delta} = e^{i\mu/2}$. Using $\tilde{\delta} = e^{i\mu/2}$ in (3.2.60) we have

$$\psi^{(\lambda q,r)} = e^{i\mu/2} \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \psi^{(p,s)}, \tag{3.2.62}$$

which establishes (3.2.54). In the same manner, let us relate the Jost solution $\bar{\phi}^{(\lambda q,r)}$ to (3.1.31) and the Jost solution $\bar{\phi}^{(p,s)}$ to (3.1.32) to each other. With the help of (3.1.34) we have

$$\bar{\phi}^{(\lambda q,r)} = \delta \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \bar{\phi}^{(p,s)}. \tag{3.2.63}$$

Using the second equalities of (3.2.5), (3.2.7), and the first equality of (3.2.14) in (3.2.63) we get

$$\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1) = \tilde{\delta} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.64}$$

and hence $\tilde{\delta} = 1$. Using this value of $\tilde{\delta}$ in (3.2.63) we can rewrite (3.2.63) as

$$\bar{\phi}^{(\lambda q,r)} = \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \bar{\phi}^{(p,s)}, \tag{3.2.65}$$

which establishes (3.2.55). Now, let us relate the Jost solution $\bar{\psi}^{(\lambda q,r)}$ to (3.1.31) and the Jost solution $\bar{\psi}^{(p,s)}$ to (3.1.32) to each other. From (3.1.34) we have

$$\bar{\psi}^{(\lambda q,r)} = \tilde{\delta} \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \bar{\psi}^{(p,s)}. \tag{3.2.66}$$

Substituting second equalities of (3.2.6), (3.2.8), and (3.2.14) in (3.2.66) we obtain

$$\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1) = \tilde{\delta} \begin{bmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{bmatrix} \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{3.2.67}$$

and hence $\tilde{\delta} = e^{-i\mu/2}$. Using this value of $\tilde{\delta}$ in (3.2.66) we get

$$\bar{\psi}^{(\lambda q,r)} = e^{-i\mu/2} \begin{bmatrix} E & -\frac{i\,q(x)}{2}\,E^{-1} \\ 0 & E^{-1} \end{bmatrix} \bar{\psi}^{(p,s)}, \tag{3.2.68}$$

which completes the prove of (3.2.56). □

In the next proposition, we show the relationships among the Jost solutions for (3.1.1) and the Jost solutions for (3.1.31).

Proposition 3.5. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the Jost solutions $\phi^{(\zeta q,\zeta r)}$, $\psi^{(\zeta q,\zeta r)}$, $\bar{\phi}^{(\zeta q,\zeta r)}$, $\bar{\psi}^{(\zeta q,\zeta r)}$ to (3.1.1) and the Jost solutions $\phi^{(\lambda q,r)}$, $\psi^{(\lambda q,r)}$, $\bar{\phi}^{(\lambda q,r)}$, $\bar{\psi}^{(\lambda q,r)}$ to (3.1.31) are related to each other as*

$$\phi^{(\lambda q,r)} = \frac{1}{\sqrt{\lambda}} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \phi^{(\zeta q,\zeta r)}, \tag{3.2.69}$$

$$\psi^{(\lambda q,r)} = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \psi^{(\zeta q,\zeta r)}, \tag{3.2.70}$$

$$\bar{\phi}^{(\lambda q,r)} = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \bar{\phi}^{(\zeta q,\zeta r)}, \tag{3.2.71}$$

$$\bar{\psi}^{(\lambda q,r)} = \frac{1}{\sqrt{\lambda}} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \bar{\psi}^{(\zeta q,\zeta r)}. \tag{3.2.72}$$

Proof. First, let us relate the Jost solution $\phi^{(\zeta q,\zeta r)}$ of (3.1.1) to the Jost solution $\phi^{(\lambda q,r)}$ of (3.1.31). From (3.1.33) we have

$$\phi^{(\lambda q,r)} = \tilde{\epsilon} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \phi^{(\zeta q,\zeta r)}. \tag{3.2.73}$$

Using the first equalities of (3.2.1) and of (3.2.5) in (3.2.73) we get

$$\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1) = \tilde{\epsilon} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.74}$$

and hence $\tilde{\epsilon} = \frac{1}{\sqrt{\lambda}}$. Using this value of $\tilde{\epsilon}$ in (3.2.73) we obtain

$$\phi^{(\lambda q,r)} = \frac{1}{\sqrt{\lambda}} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \phi^{(\lambda q,\lambda r)}, \tag{3.2.75}$$

which establishes (3.2.69). Similarly, let us relate the Jost solution $\psi^{(\lambda q,r)}$ of (3.1.31) to the Jost solution $\psi^{(\zeta q,\zeta r)}$ of (3.1.1). From (3.1.33) we have

$$\psi^{(\lambda q,r)} = \tilde{\epsilon} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \psi^{(\zeta q,\zeta r)}. \tag{3.2.76}$$

With the help of the first equalities of (3.2.2) and of (3.2.6), from (3.2.76) we obtain

$$\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1) = \tilde{\epsilon} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{3.2.77}$$

and hence $\tilde{\epsilon} = 1$. Using $\tilde{\epsilon} = 1$ in (3.2.76) we can rewrite (3.2.76) as

$$\psi^{(\lambda q,r)} = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \psi^{(\zeta q,\zeta r)}, \tag{3.2.78}$$

which completes the proof of (3.2.70). In the same manner, let us relate the Jost solution $\bar{\phi}^{(\lambda q,r)}$ of (3.1.31) to the Jost solution $\bar{\phi}^{(\zeta q,\zeta r)}$ of (3.1.1). With the help of (3.1.33) we have

$$\bar{\phi}^{(\lambda q,r)} = \tilde{\epsilon} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \bar{\phi}^{(\zeta q,\zeta r)}. \tag{3.2.79}$$

Using the second equalities of (3.2.1) and of (3.2.5) in (3.2.79) we get

$$\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1) = \tilde{\epsilon} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{3.2.80}$$

and hence $\tilde{\epsilon} = 1$. Using $\tilde{\epsilon} = 1$ in (3.2.79) we can rewrite (3.2.79) as

$$\bar{\phi}^{(\lambda q, r)} = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \bar{\phi}^{(\lambda q, \lambda r)}, \tag{3.2.81}$$

which establishes (3.2.71). Now, let us relate the Jost solution $\bar{\psi}^{(\lambda q, r)}$ of (3.1.31) to the Jost solution $\bar{\psi}^{(\zeta q, \zeta r)}$ of (3.1.1). From (3.1.33) we have

$$\bar{\psi}^{(\lambda q, r)} = \tilde{\epsilon} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \bar{\psi}^{(\zeta q, \zeta r)}. \tag{3.2.82}$$

Substituting the second equalities of (3.2.2) and of (3.2.6) in (3.2.82) we obtain

$$\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1) = \tilde{\epsilon} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{3.2.83}$$

and hence $\tilde{\epsilon} = \frac{1}{\sqrt{\lambda}}$. Using this value of $\tilde{\epsilon}$ in (3.2.82) we get

$$\bar{\psi}^{(\lambda q, r)} = \frac{1}{\sqrt{\lambda}} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \bar{\psi}^{(\zeta q, \zeta r)}, \tag{3.2.84}$$

which completes the proof of (3.2.72). □

In the next proposition, we show the relationship between the corresponding Jost solutions for the first-order system (3.1.1) and the standard system (3.1.32).

Proposition 3.6. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the Jost solutions $\phi^{(\zeta q, \zeta r)}$, $\psi^{(\zeta q, \zeta r)}$,*

$\bar{\phi}^{(\zeta q,\zeta r)}$, $\bar{\psi}^{(\zeta q,\zeta r)}$ to (3.1.1), and the Jost solutions $\phi^{(p,s)}, \psi^{(p,s)}$, $\bar{\phi}^{(p,s)}$, $\bar{\psi}^{(p,s)}$ to (3.1.32) are related to each other as

$$\phi^{(\zeta q,\zeta r)} = \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & \sqrt{\lambda}\, E^{-1} \end{bmatrix} \phi^{(p,s)}, \tag{3.2.85}$$

$$\psi^{(\zeta q,\zeta r)} = e^{i\mu/2} \begin{bmatrix} \frac{E}{\sqrt{\lambda}} & -\frac{i\,q(x)}{2\sqrt{\lambda}} E \\ 0 & E^{-1} \end{bmatrix} \psi^{(p,s)}, \tag{3.2.86}$$

$$\bar{\phi}^{(\zeta q,\zeta r)} = \begin{bmatrix} \frac{E}{\sqrt{\lambda}} & -\frac{i\,q(x)}{2\sqrt{\lambda}} E \\ 0 & E^{-1} \end{bmatrix} \bar{\phi}^{(p,s)}, \tag{3.2.87}$$

$$\bar{\psi}^{(\zeta q,\zeta r)} = e^{-i\mu/2} \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & \sqrt{\lambda}\, E^{-1} \end{bmatrix} \bar{\psi}^{(p,s)}, \tag{3.2.88}$$

where E and μ are the quantities defined in (3.1.6) and (3.2.13), respectively.

Proof. By comparing (3.2.69) and (3.2.53) we get

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix} \phi^{(\zeta q,\zeta r)} = \begin{bmatrix} E & -\frac{i\,q(x)}{2}\, E^{-1} \\ 0 & E^{-1} \end{bmatrix} \phi^{(p,s)},$$

which can be written as

$$\phi^{(\zeta q,\zeta r)} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix}^{-1} \begin{bmatrix} E & -\frac{i\,q(x)}{2}\, E^{-1} \\ 0 & E^{-1} \end{bmatrix} \phi^{(p,s)},$$

or equivalently

$$\phi^{(\zeta q,\zeta r)} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} E & -\frac{i\,q(x)}{2}\, E^{-1} \\ 0 & E^{-1} \end{bmatrix} \phi^{(p,s)}. \tag{3.2.89}$$

Evaluating the right-hand side of (3.2.89) we obtain

$$\phi^{(\zeta q,\zeta r)} = \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & \sqrt{\lambda}\, E^{-1} \end{bmatrix} \phi^{(p,s)}, \tag{3.2.90}$$

which establishes (3.2.85). Similarly, comparing (3.2.70) and (3.2.54) we have

$$\begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \phi^{(\zeta q, \zeta r)} = e^{i\mu/2} \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & E^{-1} \end{bmatrix} \psi^{(p,s)},$$

which can be written as

$$\phi^{(\zeta q, \zeta r)} = e^{i\mu/2} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & E^{-1} \end{bmatrix} \psi^{(p,s)},$$

or equivalently

$$\phi^{(\zeta q, \zeta r)} = e^{i\mu/2} \begin{bmatrix} \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & E^{-1} \end{bmatrix} \psi^{(p,s)},$$

which yields

$$\psi^{(\zeta q, \zeta r)} = e^{i\mu/2} \begin{bmatrix} \frac{E}{\sqrt{\lambda}} & -\frac{i\,q(x)}{2\sqrt{\lambda}} E \\ 0 & E^{-1} \end{bmatrix} \psi^{(p,s)}, \tag{3.2.91}$$

which establishes (3.2.86). Again, comparing (3.2.71) and (3.2.55) we obtain

$$\begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \bar{\phi}^{(\zeta q, \zeta r)} = \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & E^{-1} \end{bmatrix} \bar{\phi}^{(p,s)}, \tag{3.2.92}$$

we can write (3.2.92) as

$$\bar{\phi}^{(\zeta q, \zeta r)} = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & E^{-1} \end{bmatrix} \bar{\phi}^{(p,s)},$$

or equivalently

$$\bar{\phi}^{(\zeta q, \zeta r)} = \begin{bmatrix} \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & E^{-1} \end{bmatrix} \bar{\phi}^{(p,s)},$$

which yields

$$\bar{\phi}^{(\zeta q, \zeta r)} = \begin{bmatrix} \frac{E}{\sqrt{\lambda}} & -\frac{i\,q(x)}{2\sqrt{\lambda}} E \\ 0 & E^{-1} \end{bmatrix} \bar{\phi}^{(p,s)}, \tag{3.2.93}$$

which establishes (3.2.87). Finally, comparing (3.2.72) and (3.2.56) we get

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix} \bar{\psi}^{(\zeta q, \zeta r)} = e^{-i\mu/2} \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & E^{-1} \end{bmatrix} \bar{\psi}^{(p,s)},$$

which can be written as

$$\bar{\psi}^{(\zeta q, \zeta r)} = e^{-i\mu/2} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix}^{-1} \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & E^{-1} \end{bmatrix} \bar{\psi}^{(p,s)},$$

or equivalently

$$\bar{\psi}^{(\zeta q, \zeta r)} = e^{-i\mu/2} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & E^{-1} \end{bmatrix} \bar{\psi}^{(p,s)},$$

which yields

$$\bar{\psi}^{(\zeta q, \zeta r)} = e^{-i\mu/2} \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & \sqrt{\lambda}\, E^{-1} \end{bmatrix} \bar{\psi}^{(p,s)}, \tag{3.2.94}$$

which completes the proof of (3.2.88). □

3.3 Scattering Coefficients

The scattering coefficients can be defined by using the x-asymptotics of the Jost solutions, or equivalently they can be obtained with the help of Wronskians of Jost solutions. Since the potentials $q(x)$ and $r(x)$ appearing in (3.1.1) belong to Schwartz class $\mathbb{S}(\mathbb{R})$. we have

$$\phi(\lambda, x)^{(\zeta q, \zeta r)} = \begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q, \zeta r)} + o(1), \qquad x \to +\infty, \tag{3.3.1}$$

$$\bar{\phi}(\lambda, x)^{(\zeta q, \zeta r)} = \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q, \zeta r)} + o(1), \qquad x \to +\infty, \tag{3.3.2}$$

$$\psi(\lambda,x)^{(\zeta q,\zeta r)} = \begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to -\infty, \tag{3.3.3}$$

$$\bar{\psi}(\zeta,x)^{(\zeta q,\lambda r)} = \begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to -\infty, \tag{3.3.4}$$

where we recall that T and $\bar{T}$ are the transmission coefficients, and L , $\bar{L}$ and R, $\bar{R}$ are the reflection coefficients from the left and from the right, respectively, and $\lambda = \zeta^2$. Note that there are five first-order system we deal with, and they are given in (2.1.1), (3.1.1), (3.1.3),(3.1.31), and (3.1.32). Each of these five systems have six scattering coefficients. In order avoid any confusion we will label the scattering coefficients for each of the five system by using the superscripts $(\zeta q, \zeta r)$, $(q, \lambda r)$, $(\lambda q, r)$, (u, v), and (p, s), which identically the corresponding potentials. For example, for the system (3.1.3) we use $T^{(q,\lambda r)}$, $R^{(q,\lambda r)}$, $L^{(q,\lambda r)}$, $\bar{T}^{(q,\lambda r)}$, $\bar{R}^{(q,\lambda r)}$, $\bar{L}^{(q,\lambda r)}$ to denote the corresponding scattering coefficients. Similarly, for the x-asymptotics of the Jost solutions to the linear system (3.1.3) we have

$$\phi(\lambda,x)^{(q,\lambda r)} = \begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1), \qquad x \to +\infty, \tag{3.3.5}$$

$$\bar{\phi}(\lambda,x)^{(q,\lambda r)} = \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1), \qquad x \to +\infty, \tag{3.3.6}$$

$$\psi(\lambda,x)^{(q,\lambda r)} = \begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1), \qquad x \to -\infty, \tag{3.3.7}$$

$$\bar{\psi}(\lambda,x)^{(q,\lambda r)} = \begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1), \qquad x \to -\infty. \tag{3.3.8}$$

The scattering coefficients for (3.1.31) are determined by using the x-asymptotics of the Jost solutions to (3.1.31)

$$\phi(\lambda,x)^{(\lambda q,r)} = \begin{bmatrix} \frac{1}{T(\lambda)}e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)}e^{i\lambda x} \end{bmatrix}^{(\lambda q,r)} + o(1), \qquad x \to +\infty, \tag{3.3.9}$$

$$\bar{\phi}(\lambda,x)^{(\lambda q,r)} = \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)}e^{i\lambda x} \end{bmatrix}^{(\lambda q,r)} + o(1), \qquad x \to +\infty, \tag{3.3.10}$$

$$\psi(\lambda,x)^{(\lambda q,r)} = \begin{bmatrix} \frac{L(\lambda)}{T(\lambda)}e^{-i\lambda x} \\ \frac{1}{T(\lambda)}e^{i\lambda x} \end{bmatrix}^{(\lambda q,r)} + o(1), \qquad x \to -\infty, \tag{3.3.11}$$

$$\bar{\psi}(\lambda,x)^{(\lambda q,r)} = \begin{bmatrix} \frac{1}{\bar{T}(\lambda)}e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}e^{i\lambda x} \end{bmatrix}^{(\lambda q,r)} + o(1), \qquad x \to -\infty. \tag{3.3.12}$$

The scattering coefficients for (3.1.32) are determined by using the x-asymptotics of the Jost solutions to (3.1.32)

$$\phi(\lambda,x)^{(p,s)} = \begin{bmatrix} \frac{1}{T(\lambda)}e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)}e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to +\infty, \tag{3.3.13}$$

$$\bar{\phi}(\lambda,x)^{(p,s)} = \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)}e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to +\infty, \tag{3.3.14}$$

$$\psi(\lambda,x)^{(p,s)} = \begin{bmatrix} \frac{L(\lambda)}{T(\lambda)}e^{-i\lambda x} \\ \frac{1}{T(\lambda)}e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to -\infty, \tag{3.3.15}$$

$$\bar{\psi}(\lambda,x)^{(p,s)} = \begin{bmatrix} \frac{1}{\bar{T}(\lambda)}e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to -\infty. \tag{3.3.16}$$

In the next proposition, we show the relationship among the scattering coefficients for the first-order system (3.1.3) and the standard system (2.1.1).

Proposition 3.7. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.3) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the scattering coefficients $T^{(u,v)}, R^{(u,v)}, L^{(u,v)}, \bar{T}^{(u,v)}$, $\bar{R}^{(u,v)}$, $\bar{L}^{(u,v)}$ of (2.1.1) and the scattering coefficients $T^{(q,\lambda r)}$, $R^{(q,\lambda r)}$, $L^{(q,\lambda r)}$, $\bar{T}^{(q,\lambda r)}$, $\bar{R}^{(q,\lambda r)}$, $\bar{L}^{(q,\lambda r)}$ of (3.1.3) are related to each other as*

$$T^{(q,\lambda r)} = T^{(u,v)}\, e^{-i\mu/2}, \tag{3.3.17}$$

$$R^{(q,\lambda r)} = R^{(u,v)}\, e^{-i\mu}, \tag{3.3.18}$$

$$L^{(q,\lambda r)} = L^{(u,v)}, \tag{3.3.19}$$

$$\bar{T}^{(q,\lambda r)} = \bar{T}^{(u,v)}\, e^{i\mu/2}, \tag{3.3.20}$$

$$\bar{R}^{(q,\lambda r)} = \bar{R}^{(u,v)}\, e^{i\mu}, \tag{3.3.21}$$

$$\bar{L}^{(q,\lambda r)} = \bar{L}^{(u,v)}, \tag{3.3.22}$$

where μ is the quantity defined in (3.2.13).

Proof. By using (2.2.1), (3.3.5), and the second equality of (3.2.14) in (3.2.9) we have

$$\begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{bmatrix} \begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(u,v)} + o(1), \qquad x \to +\infty,$$

or equivalently

$$\begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} \frac{e^{i\mu/2}}{T(\lambda)} e^{-i\lambda x} \\ \frac{e^{-i\mu/2} R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(u,v)} + o(1), \qquad x \to +\infty. \tag{3.3.23}$$

From (3.3.23) we obtain

$$\left(\frac{1}{T(\lambda)}\right)^{(q,\lambda r)} = e^{i\mu/2} \left(\frac{1}{T(\lambda)}\right)^{(u,v)},$$

$$\left(\frac{R(\lambda)}{T(\lambda)}\right)^{(q,\lambda r)} = e^{-i\mu/2} \left(\frac{R(\lambda)}{T(\lambda)}\right)^{(u,v)},$$

which can be written as

$$T^{(q,\lambda r)} = e^{-i\mu/2} T^{(u,v)}, \tag{3.3.24}$$

$$R^{(q,\lambda r)} = e^{-i\mu} R^{(u,v)}, \tag{3.3.25}$$

which establish (3.3.17) and (3.3.18). Similarly, with the help of (2.2.3), (3.3.7), and the first equality of (3.2.14), from (3.2.10) we obtain

$$\begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} e^{i\mu/2} & 0 \\ 0 & e^{i\mu/2} \end{bmatrix} \begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(u,v)} + o(1), \qquad x \to -\infty,$$

or equivalently

$$\begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} \frac{e^{i\mu/2}\, L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{e^{i\mu/2}}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(u,v)} + o(1), \qquad x \to -\infty. \tag{3.3.26}$$

From (3.3.26) we get

$$\left(\frac{1}{T(\lambda)}\right)^{(q,\lambda r)} = e^{i\mu/2} \left(\frac{1}{T(\lambda)}\right)^{(u,v)},$$

$$\left(\frac{L(\lambda)}{T(\lambda)}\right)^{(q,\lambda r)} = e^{i\mu/2} \left(\frac{L(\lambda)}{T(\lambda)}\right)^{(u,v)},$$

which can be written as

$$T^{(q,\lambda r)} = e^{-i\mu/2} T^{(u,v)},$$

$$L^{(q,\lambda r)} = L^{(u,v)}, \tag{3.3.27}$$

which establishes (3.3.19). Now, let us relate the scattering coefficients $\bar{T}^{(q,\lambda r)}$ and $\bar{R}^{(q,\lambda r)}$ of (3.1.3) to the scattering coefficients $\bar{T}^{(u,v)}$ and $\bar{R}^{(u,v)}$ of (2.1.1), respectively. With the help of (2.2.2), (3.3.6), and the second equality of (3.2.14), from (3.2.11) we have

$$\begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{bmatrix} \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(u,v)} + o(1), \qquad x \to +\infty,$$

or equivalently

$$\begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} \frac{e^{i\mu/2}\,\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{e^{-i\mu/2}}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(u,v)} + o(1), \qquad x \to +\infty. \qquad (3.3.28)$$

Hence, from (3.3.28) we obtain

$$\left(\frac{1}{\bar{T}(\lambda)}\right)^{(q,\lambda r)} = e^{-i\mu/2} \left(\frac{1}{\bar{T}(\lambda)}\right)^{(u,v)},$$

$$\left(\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}\right)^{(q,\lambda r)} = e^{i\mu/2} \left(\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}\right)^{(u,v)},$$

which can be written as

$$\bar{T}^{(q,\lambda r)} = e^{i\mu/2} \bar{T}^{(u,v)},$$

$$\bar{R}^{(q,\lambda r)} = e^{i\mu/2} \bar{R}^{(u,v)},$$

which establish (3.3.20) and (3.3.21). In the same manner, let us relate the scattering coefficient $L^{(q,\lambda r)}$ of (3.1.3) to the scattering coefficient $L^{(u,v)}$ of (2.1.1). Using (2.2.4), (3.3.8), and the first equality of (3.2.14) in (3.2.12) we get

$$\begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} e^{-i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{bmatrix} \begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(u,v)} + o(1), \qquad x \to -\infty,$$

or equivalently

$$\begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} \frac{e^{-i\mu/2}}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{e^{-i\mu/2}\,\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(u,v)} + o(1), \qquad x \to -\infty. \qquad (3.3.29)$$

From (3.3.29) we obtain

$$\left(\frac{1}{\bar{T}(\lambda)}\right)^{(q,\lambda r)} = e^{-i\mu/2} \left(\frac{1}{\bar{T}(\lambda)}\right)^{(u,v)},$$

$$\left(\frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}\right)^{(q,\lambda r)} = e^{-i\mu/2} \left(\frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}\right)^{(u,v)},$$

which can be written as

$$\bar{T}^{(q,\lambda r)} = e^{i\mu/2}\bar{T}^{(u,v)},$$

$$\bar{L}^{(q,\lambda r)} = \bar{L}^{(u,v)},$$

which completes proof of (3.3.22). □

The next proposition is the analog of Proposition 3.7, we show relationship among the corresponding scattering coefficients for the first-order system (3.1.1) and (3.1.3).

Proposition 3.8. *Assume that the potentials $q(x)$ and $q(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the scattering coefficients $T^{(\zeta q,\zeta r)}, R^{(\zeta q,\zeta r)}, L^{(\zeta q,\zeta r)}, \bar{T}^{(\zeta q,\zeta r)}, \bar{R}^{(\zeta q,\zeta r)}$ of (3.1.1) are related to the scattering coefficients $T^{(q,\lambda r)}$, $R^{(q,\lambda r)}$, $L^{(q,\lambda r)}$, $\bar{T}^{(q,\lambda r)}$, $\bar{R}^{(q,\lambda r)}$, $\bar{L}^{(q,\lambda r)}$ of (3.1.3) as*

$$T^{(q,\lambda r)} = T^{(\zeta q,\zeta r)}, \tag{3.3.30}$$

$$R^{(q,\lambda r)} = \sqrt{\lambda}\, R^{(\zeta q,\zeta r)}, \tag{3.3.31}$$

$$L^{(q,\lambda r)} = \frac{1}{\sqrt{\lambda}}\, L^{(\zeta q,\zeta r)}, \tag{3.3.32}$$

$$\bar{T}^{(q,\lambda r)} = \bar{T}^{(\zeta q,\zeta r)}, \tag{3.3.33}$$

$$\bar{R}^{(q,\lambda r)} = \frac{1}{\sqrt{\lambda}}\, \bar{R}^{(\zeta q,\zeta r)}, \tag{3.3.34}$$

$$\bar{L}^{(q,\lambda r)} = \sqrt{\lambda}\, \bar{L}^{(\zeta q,\zeta r)}, \tag{3.3.35}$$

where $\zeta = \sqrt{\lambda}$.

Proof. By using (3.3.1) and (3.3.5) in (3.2.27) we have

$$\begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to +\infty,$$

or equivalently

$$\begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{\sqrt{\lambda}\, R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to +\infty. \tag{3.3.36}$$

Hence, from (3.3.36) we obtain

$$\left(\frac{1}{T(\lambda)}\right)^{(q,\lambda r)} = \left(\frac{1}{T(\lambda)}\right)^{(\zeta q,\zeta r)},$$

$$\left(\frac{R(\lambda)}{T(\lambda)}\right)^{(q,\lambda r)} = \sqrt{\lambda}\left(\frac{R(\lambda)}{T(\lambda)}\right)^{(\zeta q,\zeta r)},$$

which can be written as

$$T^{(q,\lambda r)} = T^{(\zeta q,\zeta r)}, \tag{3.3.37}$$

$$R^{(q,\lambda r)} = \sqrt{\lambda}\, R^{(\zeta q,\zeta r)}, \tag{3.3.38}$$

which establish (3.3.30) and (3.3.31). Similarly, substituting (3.3.3) and (3.3.7) in (3.2.28) we obtain

$$\begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to -\infty,$$

or equivalently

$$\begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} \frac{L(\lambda)}{\sqrt{\lambda}\, T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to -\infty. \tag{3.3.39}$$

Thus, from (3.3.39) we get

$$\left(\frac{1}{T(\lambda)}\right)^{(q,\lambda r)} = \left(\frac{1}{T(\lambda)}\right)^{(\zeta q,\zeta r)},$$

$$\left(\frac{L(\lambda)}{T(\lambda)}\right)^{(q,\lambda r)} = \frac{1}{\sqrt{\lambda}}\left(\frac{L(\lambda)}{T(\lambda)}\right)^{(\zeta q,\zeta r)},$$

which can be written as

$$T^{(q,\lambda r)} = T^{(\zeta q,\zeta r)},$$

$$L^{(q,\lambda r)} = \frac{1}{\sqrt{\lambda}} L^{(\zeta q,\zeta r)}, \tag{3.3.40}$$

which establishes (3.3.32). With the help of (3.3.2) and (3.3.6), from (3.2.29) we have

$$\begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to +\infty,$$

or equivalently

$$\begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} \frac{\bar{R}(\lambda)}{\sqrt{\lambda}\,\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to +\infty. \tag{3.3.41}$$

Hence, from (3.3.41) we obtain

$$\left(\frac{1}{\bar{T}(\lambda)}\right)_{(q,\lambda r)} = \left(\frac{1}{\bar{T}(\lambda)}\right)^{(\lambda q,\lambda r)},$$

$$\left(\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}\right)^{(q,\lambda r)} = \frac{1}{\sqrt{\lambda}} \left(\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}\right)^{(\lambda q,\lambda r)},$$

which can be written as

$$\bar{T}^{(q,\lambda r)} = \bar{T}^{(\zeta q,\zeta r)},$$

$$\bar{R}^{(q,\lambda r)} = \frac{1}{\sqrt{\lambda}} \bar{R}^{(\zeta q,\zeta r)},$$

which establish (3.3.33) and (3.3.34). In the same manner, using (2.2.4) and (3.3.8) in (3.2.12) we get

$$\begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to -\infty,$$

or equivalently

$$\begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(q,\lambda r)} + o(1) = \begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\sqrt{\lambda}\,\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to -\infty. \tag{3.3.42}$$

Thus, from (3.3.42) we obtain

$$\left(\frac{1}{\bar{T}(\lambda)}\right)^{(q,\lambda r)} = \left(\frac{1}{\bar{T}(\lambda)}\right)^{(\zeta q,\zeta r)},$$

$$\left(\frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}\right)^{(q,\lambda r)} = \sqrt{\lambda}\left(\frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}\right)^{(\zeta q,\zeta r)},$$

which can be written as

$$\bar{T}^{(q,\lambda r)} = \bar{T}^{(\zeta q,\zeta r)},$$

$$\bar{L}^{(q,\lambda r)} = \sqrt{\lambda}\,\bar{L}^{(\zeta q,\zeta r)},$$

which completes proof of (3.3.35). □

Next, we show the relationship among the scattering coefficients for the first-order system (3.1.1) and the standard system (2.1.1).

Proposition 3.9. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the scattering coefficients $T^{(u,v)}, R^{(u,v)}, L^{(u,v)}, \bar{T}^{(u,v)}$, $\bar{R}^{(u,v)}$, $\bar{L}^{(u,v)}$ of (2.1.1) are related to the scattering coefficients $T^{(\zeta q,\zeta r)}$, $R^{(\zeta q,\zeta r)}$, $L^{(\zeta q,\zeta r)}$, $\bar{T}^{(\zeta q,\zeta r)}$, $\bar{R}^{(\zeta q,\zeta r)}$, $\bar{L}^{(\zeta q,\zeta r)}$ of (3.1.1) as*

$$T^{(\zeta q,\zeta r)} = e^{-i\mu/2}\,T^{(u,v)}, \tag{3.3.43}$$

$$R^{(\zeta q,\zeta r)} = \frac{e^{-i\mu}}{\sqrt{\lambda}}R^{(u,v)}, \tag{3.3.44}$$

$$L^{(\zeta q,\zeta r)} = \sqrt{\lambda}\,L^{(u,v)}, \tag{3.3.45}$$

$$\bar{T}^{(\zeta q,\zeta r)} = e^{i\mu/2}\,\bar{T}^{(u,v)}, \tag{3.3.46}$$

$$\bar{R}^{(\zeta q,\zeta r)} = \sqrt{\lambda}\,e^{i\mu}\,\bar{R}^{(u,v)}, \tag{3.3.47}$$

$$\bar{L}^{(\zeta q,\zeta r)} = \frac{1}{\sqrt{\lambda}}\,\bar{L}^{(u,v)}, \tag{3.3.48}$$

where μ is the quantity defined in (3.2.13).

Proof. By comparing (3.3.17) and (3.3.30) we have

$$T^{(\zeta q,\zeta r)} = e^{-i\mu/2}\, T^{(u,v)}, \tag{3.3.49}$$

which establishes (3.3.43). Similarly, comparing (3.3.18) and (3.3.31) we get

$$\sqrt{\lambda}\, R^{(\zeta q,\zeta r)} = e^{-i\mu}\, R^{(u,v)},$$

or equivalently

$$R^{(\zeta q,\zeta r)} = \frac{e^{-i\mu}}{\sqrt{\lambda}}\, R^{(u,v)},$$

which establishes (3.3.44). In the same manner, comparing (3.3.19) and (3.3.32) we obtain

$$\frac{1}{\sqrt{\lambda}}\, L^{(\zeta q,\zeta r)} = L^{(u,v)},$$

or equivalently

$$L^{(\zeta q,\zeta r)} = \sqrt{\lambda}\, L^{(u,v)},$$

which establishes (3.3.45). Comparing (3.3.20) and (3.3.33) we have

$$\bar{T}^{(\zeta q,\zeta r)} = e^{i\mu/2}\, \bar{T}^{(u,v)},$$

which establishes (3.3.46). Again, comparing (3.3.21) and (3.3.34) we get

$$\frac{1}{\sqrt{\lambda}}\, \bar{R}^{(\zeta q,\zeta r)} = e^{i\mu}\, \bar{R}^{(u,v)},$$

or equivalently

$$\bar{R}^{(\zeta q,\zeta r)} = \sqrt{\lambda}\, e^{i\mu}\, \bar{R}^{(u,v)},$$

which establishes (3.3.47). Finally, let us prove (3.3.48). By comparing (3.3.22) and (3.3.35) we obtain

$$\sqrt{\lambda}\, \bar{L}^{(\zeta q,\zeta r)} = \bar{L}^{(u,v)},$$

or equivalently

$$\bar{L}^{(\zeta q,\zeta r)} = \frac{1}{\sqrt{\lambda}}\, \bar{L}^{(u,v)},$$

which completes the poof of (3.3.48). □

In the next Proposition, we show relationship among the scattering coefficients for the first-order system (3.1.31) and the standard system (3.1.32).

Proposition 3.10. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.31) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the scattering coefficients $T^{(\lambda q,r)}$, $R^{(\lambda q,r)}$, $L^{(\lambda q,r)}$, $\bar{T}^{(\lambda q,r)}$, $\bar{R}^{(\lambda q,r)}$, $\bar{L}^{(\lambda q,r)}$ of (3.1.31), and the scattering coefficients $T^{(p,s)}$, $R^{(p,s)}$, $L^{(p,s)}$, $\bar{T}^{(p,s)}$, $\bar{R}^{(p,s)}$, $\bar{L}^{(p,s)}$ of (3.1.32) are related to each other as*

$$T^{(\lambda q,r)} = T^{(p,s)}\, e^{-i\mu/2}, \tag{3.3.50}$$

$$R^{(\lambda q,r)} = R^{(p,s)}\, e^{-i\mu}, \tag{3.3.51}$$

$$L^{(\lambda q,r)} = L^{(p,s)}, \tag{3.3.52}$$

$$\bar{T}^{(\lambda q,r)} = \bar{T}^{(p,s)}\, e^{i\mu/2}, \tag{3.3.53}$$

$$\bar{R}^{(\lambda q,r)} = \bar{R}^{(p,s)}\, e^{i\mu}, \tag{3.3.54}$$

$$\bar{L}^{(\lambda q,r)} = \bar{L}^{(p,s)}, \tag{3.3.55}$$

where μ is the quantity defined in (3.2.13) and $\zeta = \sqrt{\lambda}$.

Proof. By using (3.3.9), (3.3.13), and the second equality of (3.2.14) in (3.2.53) we have

$$\begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q,r)} + o(1) = \begin{bmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{bmatrix} \begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to +\infty,$$

or equivalently

$$\begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q,r)} + o(1) = \begin{bmatrix} \frac{e^{i\mu/2}}{T(\lambda)} e^{-i\lambda x} \\ \frac{e^{-i\mu/2} R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to +\infty. \tag{3.3.56}$$

From (3.3.56) we obtain

$$\left(\frac{1}{T(\lambda)}\right)^{(\lambda q,r)} = e^{i\mu/2} \left(\frac{1}{T(\lambda)}\right)^{(p,s)},$$

$$\left(\frac{R(\lambda)}{T(\lambda)}\right)^{(q,\lambda r)} = e^{-i\mu/2}\left(\frac{R(\lambda)}{T(\lambda)}\right)^{(p,s)},$$

which can be written as

$$T^{(\lambda q,r)} = e^{-i\mu/2}\, T^{(p,s)}, \tag{3.3.57}$$

$$R^{(\lambda q,r)} = e^{-i\mu}\, R^{(p,s)}, \tag{3.3.58}$$

which establish (3.3.50) and (3.3.51). Similarly, with the help of (3.3.11), (3.3.15), and the first equality in (3.2.14), from (3.2.54) we have

$$\begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q,r)} + o(1) = \begin{bmatrix} e^{i\mu/2} & 0 \\ 0 & e^{i\mu/2} \end{bmatrix} \begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to -\infty,$$

or equivalently

$$\begin{bmatrix} \frac{L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{1}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q,r)} + o(1) = \begin{bmatrix} \frac{e^{i\mu/2}\, L(\lambda)}{T(\lambda)} e^{-i\lambda x} \\ \frac{e^{i\mu/2}}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to -\infty. \tag{3.3.59}$$

Thus, from (3.3.59) we get

$$\left(\frac{1}{T(\lambda)}\right)^{(\lambda q,r)} = e^{i\mu/2}\left(\frac{1}{T(\lambda)}\right)^{(p,s)},$$

$$\left(\frac{L(\lambda)}{T(\lambda)}\right)^{(\lambda q,r)} = e^{i\mu/2}\left(\frac{L(\lambda)}{T(\lambda)}\right)^{(p,s)},$$

which can be written as

$$T^{(\lambda q,r)} = e^{-i\mu/2}\, T^{(p,s)}, \tag{3.3.60}$$

$$L^{(\lambda q,r)} = L^{(p,s)}, \tag{3.3.61}$$

which establishes (3.3.52). With the help of (3.3.10), (3.3.14), and the second equality in (3.2.14), from (3.2.55) we have

$$\begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q,r)} + o(1) = \begin{bmatrix} e^{i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{bmatrix} \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to +\infty,$$

or equivalently

$$\begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q, r)} + o(1) = \begin{bmatrix} \frac{e^{i\mu/2}\, \bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{e^{-i\mu/2}}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to +\infty. \qquad (3.3.62)$$

From (3.3.62) we obtain

$$\left(\frac{1}{\bar{T}(\lambda)}\right)^{(\lambda q, r)} = e^{-i\mu/2} \left(\frac{1}{\bar{T}(\lambda)}\right)^{(p,s)},$$

$$\left(\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}\right)^{(\lambda q, r)} = e^{i\mu/2} \left(\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)}\right)^{(p,s)},$$

which can be written as

$$\bar{T}^{(\lambda q, r)} = e^{i\mu/2}\, \bar{T}^{(p,s)},$$

$$\bar{R}^{(\lambda q, r)} = e^{i\mu/2}\, \bar{R}^{p,s)},$$

which establish (3.3.53) and (3.3.54). In the same manner, using (3.3.16), (3.3.12), and the first equality of (3.2.14) in (3.2.56) we get

$$\begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q, r)} + o(1) = \begin{bmatrix} e^{-i\mu/2} & 0 \\ 0 & e^{-i\mu/2} \end{bmatrix} \begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to -\infty,$$

or equivalently

$$\begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q, r)} + o(1) = \begin{bmatrix} \frac{e^{-i\mu/2}}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{e^{-i\mu/2}\, \bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(p,s)} + o(1), \qquad x \to -\infty. \qquad (3.3.63)$$

Thus, from (3.3.63) we obtain

$$\left(\frac{1}{\bar{T}(\lambda)}\right)^{(\lambda q, r)} = e^{-i\mu/2} \left(\frac{1}{\bar{T}(\lambda)}\right)^{(p,s)},$$

$$\left(\frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}\right)^{(\lambda q, r)} = e^{-i\mu/2} \left(\frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}\right)^{(p,s)},$$

which can be written as

$$\bar{T}^{(\lambda q, r)} = e^{i\mu/2} \bar{T}^{(p,s)},$$

$$\bar{L}^{(\lambda q,r)} = \bar{L}^{(p,s)},$$

which completes proof of (3.3.55). □

The next proposition is the analog of Proposition 3.8, we show relationship among the scattering coefficients for first-order system (3.1.1) and (3.1.31).

Proposition 3.11. *Assume that the potentials $q(x)$ and $q(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the scattering coefficients $T^{(\zeta q,\zeta r)}$, $R^{(\zeta q,\zeta r)}$, $L^{(\zeta q,\zeta r)}$, $\bar{T}^{(\zeta q,\zeta r)}$, $\bar{R}^{(\zeta q,\zeta r)}$ of (3.1.1), and the scattering coefficients $T^{(\lambda q,r)}$, $R^{(\lambda q,r)}$, $L^{(\lambda q,r)}$, $\bar{T}^{(\lambda q,r)}$, $\bar{R}^{(\lambda q,r)}$, $\bar{L}^{(\lambda q,r)}$ of (3.1.31) are related to each other as*

$$T^{(\lambda q,r)} = T^{(\zeta q,\zeta r)}, \tag{3.3.64}$$

$$R^{(\lambda q,r)} = \frac{1}{\sqrt{\lambda}}\, R^{(\zeta q,\zeta r)}, \tag{3.3.65}$$

$$L^{(\lambda q,r)} = \sqrt{\lambda}\, L^{(\zeta q,\zeta r)}, \tag{3.3.66}$$

$$\bar{T}^{(\lambda q,r)} = \bar{T}^{(\zeta q,\zeta r)}, \tag{3.3.67}$$

$$\bar{R}^{(\lambda q,r)} = \sqrt{\lambda}\, \bar{R}^{(\zeta q,\zeta r)}, \tag{3.3.68}$$

$$\bar{L}^{(\lambda q,r)} = \frac{1}{\sqrt{\lambda}}\, \bar{L}^{(\zeta q,\zeta r)}, \tag{3.3.69}$$

where μ is the quantity defined in (3.2.13) and $\zeta = \sqrt{\lambda}$.

Proof. By using (3.3.1) and (3.3.9) in (3.2.69) we have

$$\begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q,r)} + o(1) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix} \begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to +\infty,$$

or equivalently

$$\begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q,r)} + o(1) = \begin{bmatrix} \frac{1}{T(\lambda)} e^{-i\lambda x} \\ \frac{R(\lambda)}{\sqrt{\lambda}\, T(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to +\infty. \tag{3.3.70}$$

Hence, from (3.3.70) we obtain

$$\left(\frac{1}{T(\lambda)}\right)^{(\lambda q,r)}=\left(\frac{1}{T(\lambda)}\right)^{(\zeta q,\zeta r)},$$

$$\left(\frac{R(\lambda)}{T(\lambda)}\right)^{(\lambda q,r)}=\frac{1}{\sqrt{\lambda}}\left(\frac{R(\lambda)}{T(\lambda)}\right)^{(\zeta q,\zeta r)},$$

which can be written as

$$T^{(q,\lambda r)}=T^{(\zeta q,\zeta r)}, \tag{3.3.71}$$

$$R^{(q,\lambda r)}=\frac{1}{\sqrt{\lambda}}R^{(\zeta q,\zeta r)}, \tag{3.3.72}$$

which establish (3.3.64) and (3.3.65). Similarly, substituting (3.3.3) and (3.3.11) in (3.2.70) we obtain

$$\begin{bmatrix}\frac{L(\lambda)}{T(\lambda)}e^{-i\lambda x}\\ \frac{1}{T(\lambda)}e^{i\lambda x}\end{bmatrix}^{(\lambda q,r)}+o(1)=\begin{bmatrix}\sqrt{\lambda}&0\\0&1\end{bmatrix}\begin{bmatrix}\frac{L(\lambda)}{T(\lambda)}e^{-i\lambda x}\\ \frac{1}{T(\lambda)}e^{i\lambda x}\end{bmatrix}^{(\zeta q,\zeta r)}+o(1),\qquad x\to-\infty,$$

or equivalently

$$\begin{bmatrix}\frac{L(\lambda)}{T(\lambda)}e^{-i\lambda x}\\ \frac{1}{T(\lambda)}e^{i\lambda x}\end{bmatrix}^{(\lambda q,r)}+o(1)=\begin{bmatrix}\frac{\sqrt{\lambda}\,L(\lambda)}{T(\lambda)}e^{-i\lambda x}\\ \frac{1}{T(\lambda)}e^{i\lambda x}\end{bmatrix}^{(\zeta q,\zeta r)}+o(1),\qquad x\to-\infty. \tag{3.3.73}$$

Thus, from (3.3.73) we get

$$\left(\frac{1}{T(\lambda)}\right)^{(\lambda q,r)}=\left(\frac{1}{T(\lambda)}\right)^{(\zeta q,\zeta r)},$$

$$\left(\frac{L(\lambda)}{T(\lambda)}\right)^{(\lambda q,r)}=\sqrt{\lambda}\left(\frac{L(\lambda)}{T(\lambda)}\right)^{(\zeta q,\zeta r)},$$

which can be written as

$$T^{(\lambda q,r)}=T^{(\zeta q,\zeta r)}, \tag{3.3.74}$$

$$L^{(\lambda q,r)}=\sqrt{\lambda}\,L^{(\zeta q,\zeta r)}, \tag{3.3.75}$$

which establishes (3.3.66). With the help of (3.3.2) and (3.3.10), from (3.2.71) we have

$$\begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q, r)} + o(1) = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q, \zeta r)} + o(1), \qquad x \to +\infty,$$

or equivalently

$$\begin{bmatrix} \frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q, r)} + o(1) = \begin{bmatrix} \frac{\sqrt{\lambda}\, \bar{R}(\lambda)}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q, \zeta r)} + o(1), \qquad x \to +\infty. \qquad (3.3.76)$$

Hence, from (3.3.76) we obtain

$$\left(\frac{1}{\bar{T}(\lambda)} \right)^{(\lambda q, r)} = \left(\frac{1}{\bar{T}(\lambda)} \right)^{(\zeta q, \zeta r)},$$

$$\left(\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} \right)^{(\lambda q, r)} = \sqrt{\lambda} \left(\frac{\bar{R}(\lambda)}{\bar{T}(\lambda)} \right)^{(\zeta q, \zeta r)},$$

which can be written as

$$\bar{T}^{(\lambda q, r)} = \bar{T}^{(\zeta q, \zeta r)},$$

$$\bar{R}^{(\lambda q, r)} = \sqrt{\lambda}\, \bar{R}^{(\zeta q, \zeta r)},$$

which establish (3.3.67) and (3.3.68). In the same manner, using (3.3.4) and (3.3.12) in (3.2.72) we get

$$\begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q, r)} + o(1) = \frac{1}{\sqrt{\lambda}} \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q, \zeta r)} + o(1), \qquad x \to -\infty,$$

or equivalently

$$\begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\lambda q, r)} + o(1) = \begin{bmatrix} \frac{1}{\bar{T}(\lambda)} e^{-i\lambda x} \\ \frac{\bar{L}(\lambda)}{\sqrt{\lambda}\, \bar{T}(\lambda)} e^{i\lambda x} \end{bmatrix}^{(\zeta q, \zeta r)} + o(1), \qquad x \to -\infty. \qquad (3.3.77)$$

From (3.3.77) we obtain

$$\left(\frac{1}{\bar{T}(\lambda)} \right)^{(\lambda q, r)} = \left(\frac{1}{\bar{T}(\lambda)} \right)^{(\zeta q, \zeta r)},$$

$$\left(\frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}\right)^{(\lambda q,r)} = \frac{1}{\sqrt{\lambda}}\left(\frac{\bar{L}(\lambda)}{\bar{T}(\lambda)}\right)^{(\zeta q,\zeta r)},$$

which can be written as

$$\bar{T}^{(\lambda q,r)} = \bar{T}^{(\zeta q,\zeta r)},$$
$$\bar{L}^{(\lambda q,r)} = \frac{1}{\sqrt{\lambda}}\,\bar{L}^{(\zeta q,\zeta r)},$$

which completes proof of (3.3.69). □

Next, we show the relationship among the scattering coefficients for the first-order system (3.1.1) and the standard system (3.1.32).

Proposition 3.12. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.3) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the scattering coefficients $T^{(\zeta q,\zeta r)}$, $R^{(\zeta q,\zeta r)}$, $L^{(\zeta q,\zeta r)}$, $\bar{T}^{(\zeta q,\zeta r)}$, $\bar{R}^{(\zeta q,\zeta r)}$, $\bar{L}^{(\zeta q,\zeta r)}$ of (3.1.1) and the scattering coefficients $T^{(p,s)}, R^{(p,s)}, L^{(p,s)}, \bar{T}^{(p,s)}$, $\bar{R}^{(p,s)}$, $\bar{L}^{(p,s)}$ of (3.1.32) are related to each other as*

$$T^{(\zeta q,\zeta r)} = e^{-i\mu/2}\,T^{(p,s)}, \tag{3.3.78}$$

$$R^{(\zeta q,\zeta r)} = e^{-i\mu}\sqrt{\lambda}\,R^{(p,s)}, \tag{3.3.79}$$

$$L^{(\zeta q,\zeta r)} = \frac{1}{\sqrt{\lambda}}\,L^{(u,v)}, \tag{3.3.80}$$

$$\bar{T}^{(\zeta q,\zeta r)} = e^{i\mu/2}\,\bar{T}^{(p,s)}, \tag{3.3.81}$$

$$\bar{R}^{(\zeta q,\zeta r)} = \frac{e^{i\mu}}{\sqrt{\lambda}}\,\bar{R}^{(p,s)}, \tag{3.3.82}$$

$$\bar{L}^{(\zeta q,\zeta r)} = \sqrt{\lambda}\,\bar{L}^{(p,s)}, \tag{3.3.83}$$

where μ is the quantity defined in (3.2.13) and $\zeta = \sqrt{\lambda}$.

Proof. By comparing (3.3.50) and (3.3.64) we have

$$T^{(\zeta q,\zeta r)} = e^{-i\mu/2}\,T^{(p,s)}, \tag{3.3.84}$$

which establishes (3.3.78). Similarly, comparing (3.3.51) and (3.3.65) we get

$$\frac{1}{\sqrt{\lambda}}\, R^{(\zeta q, \zeta r)} = e^{-i\mu}\, R^{(p,s)},$$

or equivalently

$$R^{(\zeta q, \zeta r)} = e^{-i\mu}\, \sqrt{\lambda}\, R^{(p,s)},$$

which establishes (3.3.79). In the same manner, comparing (3.3.52) and (3.3.66) we obtain

$$\sqrt{\lambda}\, L^{(\zeta q, \zeta r)} = L^{(p,s)},$$

or equivalently

$$L^{(\zeta q, \zeta r)} = \frac{1}{\sqrt{\lambda}}\, L^{(p,s)},$$

which establishes (3.3.80). Comparing (3.3.53) and (3.3.67) we have

$$\bar{T}^{(\zeta q, \zeta r)} = e^{i\mu/2}\, \bar{T}^{(p,s)},$$

which establishes (3.3.81). Again, comparing (3.3.54) and (3.3.68) we get

$$\sqrt{\lambda}\, \bar{R}^{(\zeta q, \zeta r)} = e^{i\mu}\, \bar{R}^{(p,s)},$$

or equivalently

$$\bar{R}^{(\zeta q, \zeta r)} = \frac{1}{\sqrt{\lambda}}\, e^{i\mu}\, \bar{R}^{(p,s)},$$

which establishes (3.3.82). Finally, let us prove (3.3.83). By comparing (3.3.55) and (3.3.69) we obtain

$$\frac{1}{\sqrt{\lambda}}\, \bar{L}^{(\zeta q, \zeta r)} = \bar{L}^{(p,s)},$$

or equivalently

$$\bar{L}^{(\zeta q, \zeta r)} = \sqrt{\lambda}\, \bar{L}^{(p,s)},$$

which completes the proof of (3.3.83). □

3.4 Bound-State Solutions to the Energy-Dependent System

A bound-state solution to (3.1.1) is a square-integrable column-vector solution in $x \in \mathbb{R}$. From (3.3.43) we know that the transmission coefficient $T^{(\zeta q,\zeta r)}$ for (3.1.1) is a function of λ, where $\lambda = \zeta^2$. Similarly, from (3.3.46) we know that the transmission coefficient $\bar{T}^{(\zeta q,\zeta r)}$ is also a function of λ. As we show in this section, a bound-state solution for (3.1.1) occurs at a λ-value at which $T^{(\zeta q,\zeta r)}(\lambda)$ has a pole in the upper-half complex λ-plane or at a λ-value at which $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ has a pole in the lower-half complex λ-plane. Note that $T^{(\zeta q,\zeta r)}(\lambda)$ and $T^{(u,v)}(\lambda)$ have common poles in $\mathbb{C}^+$ because from (3.3.43) we know $T^{(\zeta q,\zeta r)}(\lambda) = e^{-i\mu/2}\, T^{(u,v)}(\lambda)$. Since we have denoted the poles of $T^{(u,v)}(\lambda)$ in $\mathbb{C}^+$ by λ_j and assumed that there are N such poles, the poles of $T^{(\zeta q,\zeta r)}(\lambda)$ in $\mathbb{C}^+$ also denoted by λ_j and assume that there are N such poles. Similarly, $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ and $\bar{T}^{(u,v)}(\lambda)$ have coincide poles since from (3.3.46) we know $\bar{T}^{(\zeta q,\zeta r)}(\lambda) = e^{i\mu/2}\, \bar{T}^{(u,v)}(\lambda)$. Since we have denoted the poles of $\bar{T}^{(u,v)}(\lambda)$ in $\mathbb{C}^-$ by $\bar{\lambda}_j$ and assumed that there are $\bar{N}$ such poles, the poles of $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ in $\mathbb{C}^-$ also denoted by $\bar{\lambda}_j$ and assume that there are $\bar{N}$ such poles. It is possible that $N = 0$ or $\bar{N} = 0$. Let us use $c_j^{(\zeta q,\zeta r)}$ to denote bound-state norming constant if λ_j is a simple pole. Similarly, let us use $\bar{c}_j^{(\zeta q,\zeta r)}$ to denote the corresponding bound-state norming constant if $\bar{\lambda}_j$ is a simple pole. From (2.3.22) and (2.3.23) we know that the bound-state norming constants $c_j^{(\zeta q,\zeta r)}$ and $\bar{c}_j^{(\zeta q,\zeta r)}$ are related to the residues of $T^{(\zeta q,\zeta r)}(\lambda)$ and $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ at poles λ_j and $\bar{\lambda}_j$, respectively.

It may be possible that each bound state is not a simple one, i.e. the corresponding pole has a multiplicity greater than one. Let us assume that the multiplicity of the pole λ_j of $T^{(\zeta q,\zeta r)}(\lambda)$ in $\mathbb{C}^+$ is equal to the positive integer m_j and assume that the multiplicity of the pole $\bar{\lambda}_j$ of $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ in $\mathbb{C}^-$ is equal to the positive integer $\bar{m}_j$. Hence, for each λ_j, there are m_j norming constants $c_{jk}^{(\zeta q,\zeta r)}$ for $k = 0, 1, ..., m_j - 1$. Similarly, for each $\bar{\lambda}_j$ there are $\bar{m}_j$ norming constants $\bar{c}_{jk}^{(\zeta q,\zeta r)}$ for $k = 0, 1, ..., \bar{m}_j - 1$.

Let us first consider the case of simple bound states, i.e. when the poles of $T^{(\zeta q,\zeta r)}(\lambda)$ in $\mathbb{C}^+$ and the poles of $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ in $\mathbb{C}^-$ are all simple. With the help of the first equality in (3.2.2) and (3.3.1), from the Wronskian of the Jost solutions $\phi^{(\zeta q,\zeta r)}$ and $\psi^{(\zeta q,\zeta r)}$ for (3.1.1) we have

$$\left[\phi^{(\zeta q,\zeta r)};\psi^{(\zeta q,\zeta r)}\right] = \begin{vmatrix} \frac{1}{T(\zeta)}e^{-i\zeta^2 x} & 0 \\ \frac{R(\zeta)}{T(\zeta)}e^{i\zeta^2 x} & e^{i\zeta^2 x} \end{vmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to +\infty,$$

which yields

$$\left[\phi^{(\zeta q,\zeta r)};\psi^{(\zeta q,\zeta r)}\right] = \frac{1}{T^{(\zeta q,\zeta r)}}. \tag{3.4.1}$$

Hence, if $T^{(\zeta q,\zeta r)}(\lambda)$ has a pole at $\lambda_j \in \mathbb{C}^+$, the Jost solutions $\phi^{(\zeta q,\zeta r)}(\lambda,x)$ and $\psi^{(\zeta q,\zeta r)}(\lambda,x)$ are linearly dependent at that λ_j-value. Thus, there exist scalar constant $\tilde{\gamma}_j$-values such that

$$\phi^{(\zeta q,\zeta r)}(\lambda_j,x) = \tilde{\gamma}_j\, \psi^{(\zeta q,\zeta r)}(\lambda_j,x), \qquad j=1,...,N. \tag{3.4.2}$$

Similarly, with the help of the second equality in (3.2.2) and (3.3.2), from the Wronskian of the Jost solutions $\bar{\phi}^{(\zeta q,\zeta r)}$ and $\bar{\psi}^{(\zeta q,\zeta r)}$ for (3.1.1) we have

$$\left[\bar{\phi}^{(\zeta q,\zeta r)};\bar{\psi}^{(\zeta q,\zeta r)}\right] = \begin{vmatrix} \frac{\bar{R}(\zeta)}{\bar{T}(\zeta)}e^{-i\zeta^2 x} & e^{-i\zeta^2 x} \\ \frac{1}{\bar{T}(\zeta)}e^{i\zeta^2 x} & 0 \end{vmatrix}^{(\zeta q,\zeta r)} + o(1), \qquad x \to +\infty,$$

which yields

$$\left[\bar{\psi}^{(\zeta q,\zeta r)};\bar{\phi}^{(\zeta q,\zeta r)}\right] = \frac{1}{\bar{T}^{(\zeta q,\zeta r)}}. \tag{3.4.3}$$

Therefore, if $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ has a pole at $\bar{\lambda}_j$, the Jost solutions $\bar{\phi}^{(\zeta q,\zeta r)}(\lambda,x)$ and $\bar{\psi}^{(\zeta q,\zeta r)}(\lambda,x)$ are linearly dependent at that $\bar{\lambda}_j$-value. Thus, there exist scalar constant $\bar{\tilde{\gamma}}_j$-values such that

$$\bar{\phi}^{(\zeta q,\zeta r)}(\bar{\lambda}_j,x) = \bar{\tilde{\gamma}}_j\, \bar{\psi}^{(\zeta q,\zeta r)}(\bar{\lambda}_j,x), \qquad j=1,...,\bar{N}. \tag{3.4.4}$$

The constants $\tilde{\gamma}_j$ and $\bar{\tilde{\gamma}}_j$ are usually called the dependency constants because (3.4.2) and (3.4.4) indicate the linear dependence of the corresponding Jost solutions. Let us now explain why the bound state occurs at the poles of $T(\lambda)^{(\zeta q,\zeta r)}$ in $\mathbb{C}^+$ and at the poles of $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ in $\mathbb{C}^-$. With the help of (3.2.9), (3.2.10), and (3.2.14), from (2.1.81) and (2.1.82) we get

$$\phi^{(\zeta q,\zeta r)}(\lambda_j,x) = e^{-i\lambda_j x}\left(\begin{bmatrix}1\\0\end{bmatrix}+o(1)\right), \qquad x\to -\infty, \tag{3.4.5}$$

$$\psi(\lambda_j,x)^{(\zeta q,\zeta r)} = e^{i\lambda_j x}\left(\begin{bmatrix}0\\1\end{bmatrix}+o(1)\right), \qquad x\to +\infty, \tag{3.4.6}$$

where $\lambda_j \in \mathbb{C}^+$. From (3.4.5) we observe that $\phi^{(\zeta q,\zeta r)}(\lambda_j,x)$ decays exponentially as $x \to -\infty$, and from (3.4.6) we see that $\psi^{(\zeta q,\zeta r)}(\lambda_j,x)$ decays exponentially as $x \to +\infty$. Similarly, with the help of (3.2.11), (3.2.12), and (3.2.14), from (2.1.83) and (2.1.84) we obtain

$$\bar{\phi}^{(\zeta q,\zeta r)}(\lambda_j,x) = e^{i\lambda_j x}\left(\begin{bmatrix}0\\1\end{bmatrix}+o(1)\right), \qquad x\to -\infty, \tag{3.4.7}$$

$$\bar{\psi}^{(\zeta q,\zeta r)}(\lambda_j,x) = e^{-i\lambda_j x}\left(\begin{bmatrix}1\\0\end{bmatrix}+o(1)\right), \qquad x\to +\infty, \tag{3.4.8}$$

where $\lambda_j \in \mathbb{C}^-$. From (3.4.7) we observe that $\bar{\phi}^{(\zeta q,\zeta r)}(\lambda_j,x)$ decays exponentially as $x \to -\infty$, and from (3.4.8) we see that $\bar{\psi}^{(\zeta q,\zeta r)}(\lambda_j,x)$ decays exponentially as $x \to +\infty$.

From (3.4.6) we know that the right-hand of (3.4.2) decays exponentially as $x \to +\infty$ and from (3.4.5) we know that the left-hand side of (3.4.2) decays exponentially as $x \to -\infty$. Thus, from (3.4.2) we conclude that both $\phi^{(\zeta q,\zeta r)}(\lambda_j,x)$ and $\psi^{(\zeta q,\zeta r)}(\lambda_j,x)$ decay exponentially as $x \to \pm\infty$. Similarly, from (3.4.8) we know that the right-hand of (3.4.4) decays exponentially as $x \to +\infty$ and from (3.4.8) we know that the left-hand side of (3.4.4) decays exponentially as $x \to -\infty$. Thus, from (3.4.4) we conclude that both $\bar{\phi}^{(\zeta q,\zeta r)}(\bar{\lambda}_j,x)$ and $\bar{\psi}^{(\zeta q,\zeta r)}(\bar{\lambda}_j,x)$ decay exponentially as $x \to \pm\infty$.

With the help of (3.2.9), (3.2.10), (3.2.14), and theorem 2.3(c), we also know that $\phi^{(\zeta q,\zeta r)}(\lambda_j,x)$ and $\psi^{(\zeta q,\zeta r)}(\lambda_j,x)$ are continuous in $x\in\mathbb{R}$. Thus, each of $\phi^{(\zeta q,\zeta r)}(\lambda_j,x)$ and $\psi^{(\zeta q,\zeta r)}(\lambda_j,x)$ is square integrable in $x\in\mathbb{R}$, and hence each can be used as a bound-state solutions at $\lambda=\lambda_j$. Similarly, with the help of (3.2.11), (3.2.12), (3.2.14), and theorem 2.3(d) we know that $\bar\phi^{(\zeta q,\zeta r)}(\bar\lambda_j,x)$ and $\bar\psi^{(\zeta q,\zeta r)}(\bar\lambda_j,x)$ are continuous in $x\in\mathbb{R}$. Thus, each of $\bar\phi^{(\zeta q,\zeta r)}(\bar\lambda_j,x)$ and $\bar\psi^{(\zeta q,\zeta r)}(\bar\lambda_j,x)$ is square integrable in $x\in\mathbb{R}$, and hence each can be used as a bound-state solutions at $\bar\lambda=\bar\lambda_j$.

3.5 Direct Problem for the Energy-Dependent System

Recall that we have defined the scattering data set for (2.1.1) as in (2.4.1) and that such a scattering data set can be constructed from the corresponding scattering coefficients and the bound-state data. In this section we define the scattering data set for (3.1.1) in a similar manner. The direct problem for (3.1.1) consists of determining the scattering data set when the pair of potentials $\zeta q(x)$ and $\zeta r(x)$ are given.

We have already evaluated the scattering coefficients for the first-order system (3.1.1) with the help of the transformations we have established between the linear system (3.1.1) and linear system (2.1.1). In section 3.2, we have determined the Jost solutions $\phi^{(\zeta q,\zeta r)}$, $\bar\phi^{(\zeta q,\zeta r)}$, $\psi^{(\zeta q,\zeta r)}$, and $\bar\psi^{(\zeta q,\zeta r)}$ uniquely in terms of the corresponding Jost solutions for a pair of associated standard systems. Then in section 3.3, we have expressed the scattering coefficients $T^{(\zeta q,\zeta r)}$, $R^{(\zeta q,\zeta r)}$, $L^{(\zeta q,\zeta r)}$,$\bar T^{(\zeta q,\zeta r)}$, $\bar R^{(\zeta q,\zeta r)}$, and $\bar L^{(\zeta q,\zeta r)}$ in terms of the scattering coefficients for the relevant pair of standard systems. From (2.2.18) we know that the left reflection coefficient $\bar L^{(u,v)}$ can be expressed in terms of $\bar R^{(u,v)}$,$T^{(u,v)}$, and $\bar T^{(u,v)}$. Hence, from proposition 3.9, we know that $\bar L^{(\zeta q,\zeta r)}$ can also be expressed in terms of $\bar R^{(\zeta q,\zeta r)}$,$T^{(\zeta q,\zeta r)}$, and $\bar T^{(\zeta q,\zeta r)}$. Similarly, from (2.2.23) we know that $L^{(u,v)}$ can be expressed in terms of $\bar R^{(u,v)}$,$T^{(u,v)}$, and $\bar T^{(u,v)}$. Since the scattering coefficients for (3.1.1) can be expressed in terms of the corresponding scattering coefficients for (2.1.1), this is turn implies that $L^{(\zeta q,\zeta r)}$ can also be expressed in terms of $\bar{R}^{(\zeta q,\zeta r)}, T^{(\zeta q,\zeta r)}$, and $\bar{T}^{(\zeta q,\zeta r)}$. Hence, instead of using the six scattering coefficients $T^{(\zeta q,\zeta r)}$, $R^{(\zeta q,\zeta r)}$, $L^{(\zeta q,\zeta r)}$,$\bar{T}^{(\zeta q,\zeta r)}$, $\bar{R}^{(\zeta q,\zeta r)}$, and $\bar{L}^{(\zeta q,\zeta r)}$ in the scattering data set for (3.1.1), it is enough to use the four scattering coefficients $T^{(\zeta q,\zeta r)}, R^{(\zeta q,\zeta r)}, \bar{T}^{(\zeta q,\zeta r)}$, and $\bar{R}^{(\zeta q,\zeta r)}$. Recall that the scattering data set for (2.1.1) also includes the bound state data, and similarly the scattering data set for (3.1.1) needs to include the bound-state data as well.

A bound-state solution to (3.1.1) is a square-integrable column-vector solution in $x \in \mathbb{R}$. From (3.3.43) we know that the transmission coefficient $T^{(\zeta q,\zeta r)}$ for (3.1.1) is a function of λ, where $\lambda = \zeta^2$. Similarly, from (3.3.46) we know that the transmission coefficient $\bar{T}^{(\zeta q,\zeta r)}$ is also a function of λ. A bound-state solution for (3.1.1) occurs at a λ-value at which $T^{(\zeta q,\zeta r)}(\lambda)$ has a pole in the upper-half complex λ-plane or at a λ-value at which $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ has a pole in the lower-half complex λ-plane. Note that $T^{(\zeta q,\zeta r)}(\lambda)$ and $T^{(u,v)}(\lambda)$ have common poles in $\mathbb{C}^+$ because from (3.3.43) we know $T^{(\zeta q,\zeta r)}(\lambda) = e^{-i\mu/2}\, T^{(u,v)}(\lambda)$. Similarly, $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ and $\bar{T}^{(u,v)}(\lambda)$ have coincide poles since from (3.3.46) we know $\bar{T}^{(\zeta q,\zeta r)}(\lambda) = e^{i\mu/2}\, \bar{T}^{(u,v)}(\lambda)$. We denote the poles of $T^{(\zeta q,\zeta r)}(\lambda)$ in $\mathbf{C}^+$ by λ_j and assume that there are N such poles. Similarly, we denote the poles of $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ in $\mathbf{C}^-$ by $\bar{\lambda}_j$ and assume that there are $\bar{N}$ such poles. It is possible that $N = 0$ or $\bar{N} = 0$. Let us use $c_j^{(\zeta q,\zeta r)}$ to denote bound-state norming constant if λ_j is a simple pole. Similarly, let us use $\bar{c}_j^{(\zeta q,\zeta r)}$ to denote the corresponding bound-state norming constant if $\bar{\lambda}_j$ is a simple pole. From (2.3.22) and (2.3.23) we know that the bound-state norming constants $c_j^{(\zeta q,\zeta r)}$ and $\bar{c}_j^{(\zeta q,\zeta r)}$ are related to the residues of $T^{(\zeta q,\zeta r)}(\lambda)$ and $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ at poles λ_j and $\bar{\lambda}_j$, respectively.

It may be possible that each bound state is not a simple one, i.e. the corresponding pole has a multiplicity greater than one. Let us assume that the multiplicity of the pole λ_j of $T^{(\zeta q,\zeta r)}(\lambda)$ in $\mathbb{C}^+$ is equal to the positive integer m_j and assume that the multiplicity of the pole $\bar{\lambda}_j$ of $\bar{T}^{(\zeta q,\zeta r)}(\lambda)$ in $\mathbb{C}^-$ is equal to the positive integer $\bar{m}_j$. Hence, for each λ_j, there are m_j norming constants $c_{jk}^{(\zeta q,\zeta r)}$ for $k = 0, 1, ..., m_j - 1$. Similarly, for each $\bar{\lambda}_j$ there are $\bar{m}_j$ norming constants $\bar{c}_{jk}^{(\zeta q,\zeta r)}$ for $k = 0, 1, ..., \bar{m}_j - 1$. Let us write the scattering data set for (3.1.1) as

$$\boldsymbol{S}^{(\zeta q,\zeta r)} := \left\{ R, \bar{R}, \left\{ \lambda_j, \left\{ c_{jk} \right\}_{k=0}^{m_j-1} \right\}_{j=1}^{N}, \left\{ \bar{\lambda}_j, \left\{ \bar{c}_{jk} \right\}_{k=0}^{\bar{m}_j-1} \right\}_{j=1}^{\bar{N}} \right\}^{(\zeta q,\zeta r)}. \tag{3.5.1}$$

The scattering data set $\boldsymbol{S}^{(\zeta q,\zeta r)}$ is uniquely determined by the pair of potentials $\zeta q(x)$ and $\zeta r(x)$. We can outline the direct problem as the mapping given by

$$\{\zeta q(x), \zeta r(x)\} \; \mapsto \boldsymbol{S}^{(\zeta q,\zeta r)}. \tag{3.5.2}$$

In order to solve the direct problem in (3.5.2), we can exploit the transformations given between the linear system (3.1.1) and the standard systems (2.1.1) and (3.1.32). The procedure to solve the direct problem is indicated in the diagram given below:

$$\begin{array}{ccccc} & & \{u(x), v(x)\} \mapsto \boldsymbol{S}^{(u,v)} & & \\ & \nearrow & & \searrow & \\ \{\zeta q(x), \zeta r(x)\} & & & & \boldsymbol{S}^{(\zeta q,\zeta r)} \\ & \searrow & & \nearrow & \\ & & \{p(x), s(x)\} \mapsto \boldsymbol{S}^{(p,s)} & & \end{array}$$

Based on the above diagram, there are two ways to solve the direct problem for (3.1.1). The first way consists of the following steps:

- From the given the pair of potentials $\{\zeta q(x), \zeta r(x)\}$, we determine the pair of potentials $\{u(x), v(x)\}$ for the standard system (2.1.1) by using (3.1.6), (3.1.7), and (3.1.8).

- Having $u(x)$ and $v(x)$ at hand, we solve the direct problem for (2.1.1) and determine the scattering data set $\boldsymbol{S}^{(u,v)}$ for (2.1.1) by using $u(x)$ and $v(x)$ as input.
- Having $\boldsymbol{S}^{(u,v)}$ at hand, we determine the scattering data set $\boldsymbol{S}^{(\zeta q,\zeta r)}$ for (3.1.1) with the help of Propositions 3.3 and 3.9.

The second way consists of the following steps:

- From the given the pair of potentials $\{\zeta q(x), \zeta r(x)\}$, we determine the pair of potentials $\{p(x), s(x)\}$ for the standard system (3.1.32) by using (3.1.6), (3.1.35), and (3.1.36).
- Having the potentials $p(x)$ and $s(x)$ at hand, we solve the direct problem for (3.1.32) and determine the scattering data set $\boldsymbol{S}^{(p,s)}$.
- Having $\boldsymbol{S}^{(p,s)}$ at hand, we determine the scattering data set $\boldsymbol{S}^{(\zeta q,\zeta r)}$ for (3.1.1) with the help of the Propositions 3.6 and 3.12.

3.6 Inverse Problem for the Energy-Dependent System

The inverse problem for (3.1.1) consists of the determination of the potentials $q(x)$ and $r(x)$ from the scattering data set given in (3.5.1). We can exploit the transformations given between the linear system (3.1.1) and the standard systems (2.1.1) and (3.1.32) to solve the inverse problem as in the following diagram:

$$
\begin{array}{ccccc}
 & & \boldsymbol{S}^{(u,v)} \mapsto \{u(x), v(x)\} & & \\
 & \nearrow & & \searrow & \\
\boldsymbol{S}^{(\zeta q,\zeta r)} & & & & \{\zeta q(x), \zeta r(x)\} \\
 & \searrow & & \nearrow & \\
 & & \boldsymbol{S}^{(p,s)} \mapsto \{p(x), s(x)\} & &
\end{array}
$$

As the above diagram indicates, we can solve the inverse problem for (3.1.1) by the following steps:

- With the help of propositions 3.3 and 3.9, we construct the scattering data set $\boldsymbol{S}^{(u,v)}$ for (2.1.1) from the given scattering data set $\boldsymbol{S}^{(\zeta q,\zeta r)}$ for (3.1.1).
- Having the scattering data set $\boldsymbol{S}^{(u,v)}$ at the hand, we construct the Marchenko kernels $\Omega^{(u,v)}$ and $\bar{\Omega}^{(u,v)}$ defined in (2.5.26) and (2.5.27), respectively.
- As in theorem 2.7, we use the constructed $\Omega^{(u,v)}$ and $\bar{\Omega}^{(u,v)}$ in the Marchenko equation given by

$$\begin{aligned}K_1^{(u,v)}(x,y) &+ \bar{\Omega}^{(u,v)}(x+y) \\ &- \int_x^\infty dz \int_x^\infty dt\, K_1^{(u,v)}(x,t)\,\Omega^{(u,v)}(t+z)\,\bar{\Omega}^{(u,v)}(z+y) = 0, \quad y > x,\end{aligned} \tag{3.6.1}$$

$$\bar{K}_1^{(u,v)}(x,y) + \int_x^\infty dz\, K_1^{(u,v)}(x,z)\,\Omega(z+y)^{(u,v)} = 0, \qquad y > x, \tag{3.6.2}$$

$$\begin{aligned}\bar{K}_2^{(u,v)}(x,y) &+ \Omega^{(u,v)}(x+y) \\ &+ \int_x^\infty dz \int_x^\infty dt\, \bar{K}_2^{(u,v)}(x,t)\,\bar{\Omega}^{(u,v)}(t+z)\,\Omega^{(u,v)}(z+y) = 0, \quad y > x,\end{aligned} \tag{3.6.3}$$

$$K_2^{(u,v)}(x,y) + \int_x^\infty dz\, \bar{K}_2^{(u,v)}(x,z)\,\bar{\Omega}(z+y)^{(u,v)} = 0, \qquad y > x. \tag{3.6.4}$$

  and uniquely determine $K_1^{(u,v)}(x,y)$, $\bar{K}_1^{(u,v)}(x,y)$, $\bar{K}_2^{(u,v)}(x,y)$, $K_2^{(u,v)}(x,y)$.
- Having obtained $K_1^{(u,v)}(x,y)$ and $\bar{K}_2^{(u,v)}(x,y)$ at the hand, as in theorem 2.8 we recover the potentials $u(x)$ and $v(x)$ via

$$u(x) = -2\, K_1^{(u,v)}(x,x), \tag{3.6.5}$$

$$v(x) = -2\, \bar{K}_2^{(u,v)}(x,x). \tag{3.6.6}$$

- With the help of propositions 3.6 and 3.12, we construct the scattering data set $\boldsymbol{S}^{(p,s)}$ for (3.1.32) from the given scattering data set $\boldsymbol{S}^{(\zeta q,\zeta r)}$ for (3.1.1).

- Having the scattering data set $\boldsymbol{S}^{(p,s)}$ at hand, we construct the Marchenko kernels $\Omega^{(p,s)}$ and $\bar{\Omega}^{(p,s)}$ defined in (2.5.26) and (2.5.27), respectively.
- As in the theorem 2.7, we use the constructed $\Omega^{(p,s)}$ and $\bar{\Omega}^{(p,s)}$ in the Marchenko equation given by

$$K_1^{(p,s)}(x,y) + \bar{\Omega}^{(p,s)}(x+y) - \int_x^\infty dz \int_x^\infty dt\, K_1^{(p,s)}(x,t)\, \Omega^{(p,s)}(t+z)\, \bar{\Omega}^{(p,s)}(z+y) = 0, \quad y > x, \tag{3.6.7}$$

$$\bar{K}_1^{(p,s)}(x,y) + \int_x^\infty dz\, K_1^{(p,s)}(x,z)\, \Omega(z+y)^{(p,s)} = 0, \qquad y > x, \tag{3.6.8}$$

$$\bar{K}_2^{(p,s)}(x,y) + \Omega^{(p,s)}(x+y) + \int_x^\infty dz \int_x^\infty dt\, \bar{K}_2^{(p,s)}(x,t)\, \bar{\Omega}^{(p,s)}(t+z)\, \Omega^{(p,s)}(z+y) = 0, \quad y > x, \tag{3.6.9}$$

$$K_2(x,y)^{(p,s)} + \int_x^\infty dz\, \bar{K}_2^{(p,s)}(x,z)\, \bar{\Omega}(z+y)^{(p,s)} = 0, \qquad y > x. \tag{3.6.10}$$

and uniquely determine $K_1^{(p,s)}(x,y)$, $\bar{K}_1^{(p,s)}(x,y)$, $\bar{K}_2^{(p,s)}(x,y)$, $K_2^{(p,s)}(x,y)$.

- Having obtained $K_1^{(p,s)}(x,y)$, $\bar{K}_2^{(p,s)}(x,y)$ at the hand, as in theorem 2.8 we recover the potentials $p(x)$ and $s(x)$ via

$$p(x) = -2\, K_1^{(p,s)}(x,x), \tag{3.6.11}$$

$$s(x) = -2\, \bar{K}_2^{(p,s)}(x,x). \tag{3.6.12}$$

- Having obtained $K_2^{(u,v)}(x,y)$, $K_2^{(p,s)}(x,y)$ at hand, as to be seen in proposition 3.15 in this section, we construct the key quantity E appearing in (3.1.6) as

$$E = \exp\left(2 \int_x^\infty dt \left[K_2^{(p,s)}(t,t) - K_2^{(u,v)}(t,t)\right]\right) \left[\lim_{\lambda\to\pm\infty} T(\lambda)^{(\zeta q, \zeta r)}\right]^{-1}. \tag{3.6.13}$$

- Having $u(x)$, $s(x)$, E and the connection between the potential pair $(u(x), s(x))$ and the potential pair $(q(x), r(x)$ at hand, we recover the potentials $q(x)$ and $r(x)$ via

$$q(x) = -2\, K_1^{(u,v)}(x,x)\, E^2, \tag{3.6.14}$$

$$r(x) = -2\, \bar{K}_2^{(p,s)}(x,x)\, E^{-2}. \tag{3.6.15}$$

- Alternatively, having $v(x)$ from (3.6.6), $p(x)$ from (3.6.12), E from (3.6.13) at hand, as to be seen in proposition 3.16 in this section, we can determine $q(x)$ and $r(x)$ via

$$q(x) = \frac{2i}{E(x)} \int_{-\infty}^{x} dt\, p(t)\, E(t)^3, \tag{3.6.16}$$

$$r(x) = \frac{2i}{E(x)} \int_{-\infty}^{x} dt\, \frac{v(t)}{E(t)}, \tag{3.6.17}$$

where we write $E(x)$ for E to emphasize its dependence on x.

In the next two propositions, we show how to construct the key quantity E from the solutions $K_2^{(u,v)}(x,y)$, $K_2^{(p,s)}(x,y)$ to the Marchenko equations given in (3.6.4) and (3.6.10), respectively.

Proposition 3.13. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, we have*

$$\frac{ir(x)q(x)}{2} = 2[K_2^{(u,v)}(x,x) - K_2^{(p,s)}(x,x)], \tag{3.6.18}$$

where $K_2^{(u,v)}(x,y)$ and $K_2^{(p,s)}(x,y)$ are the quantities appearing in the Marchenko equations (3.6.4) and (3.6.10).

Proof. Multiplying (3.1.7) and (3.1.8), we have

$$u(x)\, v(x) = -\frac{ir'(x)\, q(x)}{2} + \frac{q(x)^2\, r(x)^2}{4}. \tag{3.6.19}$$

Similarly, multiplying (3.1.35) and (3.1.36) we obtain

$$p(x)\,s(x)=\frac{ir(x)\,q'(x)}{2}+\frac{q(x)^2\,r(x)^2}{4}. \tag{3.6.20}$$

Subtracting (3.6.19) from (3.6.20) we get

$$p(x)\,s(x)-u(x)\,v(x)=\frac{i}{2}\Big[r(x)\,q'(x)+r'(x)\,q(x)\Big],$$

or equivalently

$$p(x)\,s(x)-u(x)\,v(x)=\frac{i}{2}\Big[r(x)\,q(x)\Big]'. \tag{3.6.21}$$

From the analog of the first equality in (2.5.48) written for the potential pair $p(x)$ and $s(x)$, we know that

$$\int_x^\infty dz\,p(z)\,s(z)=2\,K_2^{(p,s)}(x,x). \tag{3.6.22}$$

Taking the derivative of both sides of (3.6.22) we obtain

$$p(x)\,s(x)=-2\frac{d\,K_2^{(p,s)}(x,x)}{dx}. \tag{3.6.23}$$

Similarly, from the first equality in (2.5.48) we know that

$$\int_x^\infty dz\,u(z)\,v(z)=2\,K_2^{(u,v)}(x,x). \tag{3.6.24}$$

Taking the derivative of both sides of (3.6.24) we have

$$-\,u(x)\,v(x)=2\frac{d\,K_2^{(u,v)}(x,x)}{dx}. \tag{3.6.25}$$

Adding (3.6.23) and (3.6.25) we get

$$p(x)\,s(x)-u(x)\,v(x)=2\frac{d}{dx}\left[K_2^{(u,v)}(x,x)-K_2^{(p,s)}(x,x)\right]. \tag{3.6.26}$$

Comparing (3.6.21) and (3.6.26) we obtain

$$\frac{i}{2}\Big(r(x)\,q(x)\Big)'=2\frac{d}{dx}\left[K_2^{(u,v)}(x,x)-K_2^{(p,s)}(x,x)\right]. \tag{3.6.27}$$

Integrating (3.6.27) on the interval $(x, +\infty)$ and by using the fact that $q(x)$, $r(x)$, $K_2^{(u,v)}(x,x)$, and $K_2^{(p,s)}(x,x)$ all vanish as $x \to +\infty$, we obtain

$$\frac{i}{2}\Big(r(x)\, q(x)\Big) = 2\left[K_2^{(u,v)}(x,x) - K_2^{(p,s)}(x,x)\right], \tag{3.6.28}$$

which completes the proof. □

Proposition 3.14. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, we have*

$$e^{-i\mu/2} = \lim_{\lambda \to \pm\infty} T^{(\zeta q, \zeta r)}(\lambda), \tag{3.6.29}$$

where $T^{(\zeta q, \zeta r)}(\lambda)$ is the transmission coefficient appearing in (3.3.43) and μ is the quantity defined in (3.2.13).

Proof. From (2.2.44), we have

$$T^{(u,v)}(\lambda) = 1 + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^+}. \tag{3.6.30}$$

Using (3.3.43) in (3.6.30) we obtain

$$T^{(\zeta q, \zeta r)}(\lambda) = e^{-i\mu/2}\left[1 + O\left(\frac{1}{\lambda}\right)\right], \qquad \lambda \to \infty \quad \text{in} \quad \overline{\mathbb{C}^+}, \tag{3.6.31}$$

which implies (3.6.29). □

Proposition 3.15. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the quantity E appearing in (3.1.6) satisfies (3.6.13).*

Proof. Let us copy (3.1.6) for the convenience of the reader as

$$E := \exp\left(\frac{i}{2}\int_{-\infty}^{x} dt\, q(t)\, r(t)\right). \tag{3.6.32}$$

With the help of (3.2.13), we can rewrite (3.6.32) as

$$E = e^{i\mu/2}\, e^{-i/2 \int_x^\infty dt\, q(t)\, r(t)}. \tag{3.6.33}$$

Using (3.6.18) and (3.6.29) in (3.6.33) we get

$$E = \exp\left(2\int_x^\infty dt\left[K_2^{(p,s)}(t,t) - K_2^{(u,v)}(t,t)\right]\right)\left[\lim_{\lambda\to\pm\infty} T(\lambda)^{(\zeta q,\zeta r)}\right]^{-1}, \tag{3.6.34}$$

which completes the proof. □

Proposition 3.16. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, we have*

$$q(x) = \frac{2i}{E(x)}\int_{-\infty}^x dt\, p(t)\, E(t)^3, \tag{3.6.35}$$

$$r(x) = \frac{2i}{E(x)}\int_{-\infty}^x dt\, \frac{v(t)}{E(t)}, \tag{3.6.36}$$

where E is the quantity defined in (3.1.6), $p(x)$ and $v(x)$ are the potentials appearing in (3.1.32) and (2.1.1), respectively.

Proof. Let us copy (3.1.35) for the convenience of the reader as

$$p(x) = \left(-\frac{i\, q'(x)}{2} + \frac{q(x)^2\, r(x)}{4}\right)[E(x)]^{-2}. \tag{3.6.37}$$

Multiplying both sides of (3.6.37) by $2i\,[E(x)]^2$, we have

$$2i\,[E(x)]^2\, p(x) = q'(x) + \left(\frac{i\, q(x)\, r(x)}{2}\right) q(x), \tag{3.6.38}$$

which is a first-order ordinary differential equation for the unknown $q(x)$. From (3.1.6) we see that $E(x)$ is an integrating factor for the first-order linear ordinary differential equation in (3.6.38), and multiplying both sides of (3.6.38) with $E(x)$, we get

$$2i\,[E(x)]^3\, p(x) = E(x)\, q'(x) + E(x)'\, q(x), \tag{3.6.39}$$

or equivalently

$$2i\,[E(x)]^3\, p(x) = \frac{d}{dx}\left[E(x)\, q(x)\right]. \tag{3.6.40}$$

Integrating (3.6.40) on the interval $(-\infty, x)$ and using the fact that $q(x)$ vanish as $x \to +\infty$ and $E(x)$ is bounded on $x \in \mathbb{R}$, we get

$$E(x)\, q(x) = 2i \int_{-\infty}^{x} dt\, p(t)\, [E(t)]^3\,, \tag{3.6.41}$$

from which (3.6.35) follows. Similarly, let us copy (3.1.8) for the convenience of the reader as

$$v(x) = \left(-\frac{i\, r'(x)}{2} + \frac{q(x)\, r(x)^2}{4} \right) [E(x)]^2\,. \tag{3.6.42}$$

Multiplying both sides of (3.6.42) by $2i\, [E(x)]^{-2}$, we have

$$2i\, [E(x)]^{-2}\, v(x) = r'(x) + \left(\frac{q(x)\, r(x)}{2} \right) r(x), \tag{3.6.43}$$

which is a first-order ordinary differential equation for the unknown $r(x)$. From (3.1.6) we see that $E(x)$ is an integrating factor for the first-order ordinary differential equation in (3.6.43), and multiplying both sides of (3.6.43) with $E(x)$, we obtain

$$2i\, [E(x)]^{-1}\, v(x) = E(x)\, r'(x) + E'(x)\, r(x), \tag{3.6.44}$$

or equivalently

$$2i\, [E(x)]^{-1}\, v(x) = \frac{d}{dx} [E(x)\, r(x)]\,. \tag{3.6.45}$$

Integrating (3.6.45) on the interval $(-\infty, x)$ and using the fact that $r(x)$ vanish as $x \to +\infty$ and $E(x)$ is bounded on $x \in \mathbb{R}$, we get

$$E(x)\, r(x) = 2i \int_{-\infty}^{x} dt\, \frac{v(t)}{E(t)}, \tag{3.6.46}$$

from which (3.6.36) follows. □

Chapter 4

The Alternate Marchenko Method

4.1 The Zero-Energy Wave Functions

In this chapter we analysis the inverse problem for (3.1.1) by using a different method. We determine the potentials $q(x)$ and $r(x)$ in (3.1.1) by formulating a Marchenko system, which we call the alternate Marchenko system. The motivation behind our method came from the analysis by Tsuchida in [9]. Tsuchida mainly used his analysis in relation to solutions to certain integrable nonlinear partial differential equations, in particular the derivative nonlinear Schrödinger equation [1, 9, 14] and its variances. As indicated in [9, 14], Tsuchida's formulation is not easy to comprehend because it is unclear how the scattering theory is used and it is unclear how his gauge transformation is implemented. As a result, Tsuchida's formulation is not intuitive and his derivation is difficult to follow. Our work given here clarifies the idea behind Tsuchida's formulation by providing a clearer presentation of the alternate Marchenko method. Our Marchenko equations, although somehow similar to those derived by Tsuchida, are slightly different and resemble more like the standard Marchenko equations [10, 13, 15, 16] .

The standard form of the first-order system with energy-dependent potentials is given in (3.1.1). As in chapter 3, we can transform (3.1.1) into (3.1.3). Let us start with the system (3.1.3) and set the value of $\lambda = 0$ in (3.1.3). Showing the dependence of the wave functions on the parameter λ as $\theta(\lambda, x)$ and $\omega(\lambda, x)$ in (3.1.3), setting $\lambda = 0$ there we obtain

$$\begin{bmatrix} \theta(0,x) \\ \omega(0,x) \end{bmatrix}' = \begin{bmatrix} 0 & q(x) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(0,x) \\ \omega(0,x) \end{bmatrix}, \qquad x \in \mathbb{R}. \tag{4.1.1}$$

We see that (4.1.1) is equivalent to

$$\begin{cases} \theta'(0,x) = q(x)\,\omega(0,x), \\ \omega'(0,x) = 0. \end{cases} \tag{4.1.2}$$

From the second equality in (4.1.2), we see that the quantity $\omega(0, x)$ is independent of x and hence it is a constant. Then, from the first equality in (4.1.2) we get

$$q(x) = \frac{\theta'(0'x)}{\omega(0,x)}, \tag{4.1.3}$$

or equivalently

$$q(x) = \left(\frac{\theta(0,x)}{\omega(0,x)}\right)', \tag{4.1.4}$$

because the quantity $\omega(0, x)$ is independent of x. Recall that the superscripts for the Jost functions are used to denote the potentials in the corresponding first-order system. Let us copy (3.2.10) for the convenience of the reader as

$$\psi^{(q,\lambda r)} = e^{i\mu/2} \begin{bmatrix} E & 0 \\ \frac{i\,r(x)}{2}\,E & E^{-1} \end{bmatrix} \psi^{(u,v)}, \tag{4.1.5}$$

where we recall that E is the quantity defined in (3.1.6) and depends on x. Let us write $\psi^{(q,\lambda r)}$ and $\psi^{(u,v)}$ in the component form as

$$\psi^{(q,\lambda r)} = \begin{bmatrix} \psi_1^{(q,\lambda r)}(\lambda,x) \\ \psi_2^{(q,\lambda r)}(\lambda,x) \end{bmatrix}, \quad \psi^{(u,v)} = \begin{bmatrix} \psi_1^{(u,v)}(\lambda,x) \\ \psi_2^{(u,v)}(\lambda,x) \end{bmatrix}. \tag{4.1.6}$$

Similarly, we can write $\bar{\psi}^{(q,\lambda r)}$ and $\bar{\psi}^{(u,v)}$ in the component form as

$$\bar{\psi}^{(q,\lambda r)} = \begin{bmatrix} \bar{\psi}_1^{(q,\lambda r)}(\lambda,x) \\ \bar{\psi}_2^{(q,\lambda r)}(\lambda,x) \end{bmatrix}, \quad \bar{\psi}^{(u,v)} = \begin{bmatrix} \bar{\psi}_1^{(u,v)}(\lambda,x) \\ \bar{\psi}_2^{(u,v)}(\lambda,x) \end{bmatrix}. \tag{4.1.7}$$

Using (4.1.6) in (4.1.5) we have

$$\begin{cases} \psi_1^{(q,\lambda r)}(\lambda,x) = e^{i\mu/2}\, E\, \psi_1^{(u,v)}(\lambda,x), \\ \psi_2^{(q,\lambda r)}(\lambda,x) = e^{i\mu/2}\, \left[\frac{i\,r(x)\,E}{2}\, \psi_1^{(u,v)}(\lambda,x) + E^{-1}\, \psi_2^{(u,v)}(\lambda,x)\right], \end{cases}$$

which can be written at $\lambda = 0$ as

$$\begin{cases} \psi_1^{(q,\lambda r)}(0,x) = e^{i\mu/2}\, E\, \psi_1^{(u,v)}(0,x), \\ \psi_2^{(q,\lambda r)}(0,x) = e^{i\mu/2}\, \left[\frac{i\,r(x)\,E}{2}\, \psi_1^{(u,v)}(0,x) + E^{-1}\, \psi_2^{(u,v)}(0,x)\right]. \end{cases} \tag{4.1.8}$$

Since $\psi^{(q,\lambda r)}(0,x)$ satisfies (4.1.1), we have

$$\psi'^{(q,\lambda r)}(0,x) = \begin{bmatrix} 0 & q(x) \\ 0 & 0 \end{bmatrix} \psi^{(q,\lambda r)}(0,x), \qquad x \in \mathbb{R}. \tag{4.1.9}$$

With the help of the first equality in (4.1.6), from (4.1.9) we get

$$\begin{cases} \psi'^{(q,\lambda r)}_1(0,x) = q(x)\, \psi_2^{(q,\lambda r)}(0,x), \\ \psi'^{(q,\lambda r)}_2(0,x) = 0. \end{cases} \tag{4.1.10}$$

From the second equality in (4.1.10), it is seen that the quantity $\psi'^{(q,\lambda r)}_2(0,x)$ is independent of x. Hence, the right-hand side of the second equality in (4.1.8) is independent of x and its value can be evaluated as $x \to +\infty$: We get

$$\psi_2^{(q,\lambda r)}(0,x) = \lim_{x\to+\infty} \left(e^{i\mu/2} \left[\frac{i\,r(x)\,E}{2}\, \psi_1^{(u,v)}(0,x) + E^{-1}\, \psi_2^{(u,v)}(0,x)\right]\right), \tag{4.1.11}$$

where we recall that $q(x)$ and $r(x)$ belong to the Schwartz class $\mathbb{S}(\mathbb{R})$ and hence $r(x) \to 0$ as $x \to +\infty$. With the help of the first equality in (2.1.2), (3.2.13), the second equality in (3.2.14), and the second equality in (4.1.6), we determine that the right-hand side of (4.1.11) is equal to 1 and hence we get

$$\psi_2^{(q,\lambda r)}(0,x) \equiv 1. \tag{4.1.12}$$

Using the first equality of (4.1.8) and (4.1.12) in the first equality of (4.1.10) we obtain

$$q(x) = e^{i\mu/2} \left[E\, \psi_1^{(u,v)}(0,x) \right]'. \tag{4.1.13}$$

In the next proposition, we show how to express the zero-energy Jost solutions $\bar{\psi}_1^{(u,v)}(0,x)$ and $\bar{\psi}_2^{(u,v)}(0,x)$ to the standard system (2.1.1) in terms of the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1).

Proposition 4.1. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system* (3.1.3) *belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, we have*

$$\bar{\psi}_1^{(u,v)}(0,x) = e^{i\mu/2}\, E^{-1}, \tag{4.1.14}$$

$$\bar{\psi}_2^{(u,v)}(0,x) = -\frac{i\, r(x)}{2}\, e^{i\mu/2}\, E, \tag{4.1.15}$$

where E and μ are the quantities defined in (3.1.6) *and* (3.2.13)*, respectively.*

Proof. Using (3.2.13), the second equality in (3.2.14), and $r(x) \to 0$ as $x \to +\infty$, from the left-hand side of (4.1.14) and of (4.1.15) we have

$$\begin{bmatrix} e^{i\mu/2}\, E^{-1} \\ -\frac{i\, r(x)}{2}\, e^{i\mu/2}\, E \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad x \to +\infty.$$

As the superscript (u,v) indicates, the column vector $\begin{bmatrix} \bar{\psi}_1^{(u,v)}(0,x) \\ \bar{\psi}_2^{(u,v)}(0,x) \end{bmatrix}$ satisfies the first-order system (2.1.1) at $\lambda = 0$. Furthermore as seen from the second equality in

(2.1.2), that column vector $\begin{bmatrix}\bar\psi_1^{(u,v)}(0,x)\\ \bar\psi_2^{(u,v)}(0,x)\end{bmatrix}$ has the asymptotics $\begin{bmatrix}1\\0\end{bmatrix}$ as $x\to+\infty$. In other words, we need to show that

$$\begin{bmatrix} e^{i\mu/2}\,E^{-1} \\ -\frac{i\,r(x)}{2}\,e^{i\mu/2}\,E \end{bmatrix}' = \begin{bmatrix} 0 & u(x) \\ v(x) & 0 \end{bmatrix} \begin{bmatrix} e^{i\mu/2}\,E^{-1} \\ -\frac{i\,r(x)}{2}\,e^{i\mu/2}\,E \end{bmatrix}. \tag{4.1.16}$$

Using (3.1.7) and (3.1.8) in (4.1.16) we get

$$\begin{bmatrix} e^{i\mu/2}\,E^{-1} \\ -\frac{i\,r(x)}{2}\,e^{i\mu/2}\,E \end{bmatrix}' = \begin{bmatrix} 0 & q(x)\,E^{-2} \\ \left(-\frac{i\,r'(x)}{2}+\frac{q(x)\,r^2(x)}{4}\right)E^2 & 0 \end{bmatrix} \begin{bmatrix} e^{i\mu/2}\,E^{-1} \\ -\frac{i\,r(x)}{2}\,e^{i\mu/2}\,E \end{bmatrix},$$

which can be written as

$$\begin{aligned}&\begin{bmatrix} -\frac{ir(x)\,q(x)}{2}e^{i\mu/2}\,E^{-1} \\ \left(-\frac{i\,r'(x)}{2}+\frac{q(x)\,r^2(x)}{4}\right)e^{i\mu/2}\,E \end{bmatrix}\\ &\qquad = \begin{bmatrix} 0 & q(x)\,E^{-2} \\ \left(-\frac{i\,r'(x)}{2}+\frac{q(x)\,r^2(x)}{4}\right)E^2 & 0 \end{bmatrix} \begin{bmatrix} e^{i\mu/2}\,E^{-1} \\ -\frac{i\,r(x)}{2}\,e^{i\mu/2}\,E \end{bmatrix},\end{aligned} \tag{4.1.17}$$

which can be directly verified by multiplying the two matrices on the right-hand side of (4.1.17). □

In the next proposition, we show how to express the zero-energy Jost solutions $\psi_1^{(u,v)}(0,x)$ and $\psi_2^{(u,v)}(0,x)$ to the standard system (2.1.1) in terms of the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1).

Proposition 4.2. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (2.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, we have*

$$\psi_1^{(u,v)}(0,x) = -e^{-i\mu/2}\,E^{-1}\int_x^\infty dy\,q(y), \tag{4.1.18}$$

$$\psi_2^{(u,v)}(0,x) = \left[1+\frac{i\,r(x)}{2}\int_x^\infty dy\,q(y)\right]e^{-i\mu/2}\,E, \tag{4.1.19}$$

where E and μ are the quantities defined in (3.1.6) and (3.2.13), respectively.

Proof. The Wronskian of the Jost solutions $\bar{\psi}^{(u,v)}(0,x)$ and $\psi^{(u,v)}(0,x)$ for the linear system (2.1.1) at $\lambda = 0$ is given by

$$\left[\bar{\psi}^{(u,v)}(0,x); \psi^{(u,v)}(0,x)\right] = \begin{vmatrix} \bar{\psi}_1^{(u,v)}(0,x) & \psi_1^{(u,v)}(0,x) \\ \bar{\psi}_2^{(u,v)}(0,x) & \psi_2^{(u,v)}(0,x) \end{vmatrix}. \tag{4.1.20}$$

With the help of (2.1.2), the second equality in (4.1.6) and in (4.1.7), from (4.1.20) we get

$$\left[\bar{\psi}^{(u,v)}(0,x); \psi^{(u,v)}(0,x)\right] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix},$$

which yields

$$\left[\bar{\psi}^{(u,v)}(0,x); \psi^{(u,v)}(0,x)\right] = 1. \tag{4.1.21}$$

Hence, using (4.1.14) and (4.1.15) in (4.1.20) we have

$$\begin{vmatrix} e^{i\mu/2}\, E^{-1} & \psi_1^{(u,v)}(0,x) \\ -\frac{i\, r(x)}{2}\, e^{i\mu/2}\, E & \psi_2^{(u,v)}(0,x) \end{vmatrix} = 1. \tag{4.1.22}$$

Evaluating the left-hand side of (4.1.22) we get

$$e^{i\mu/2}\, E^{-1}\, \psi_2^{(u,v)}(0,x) + \frac{i\, r(x)}{2}\, e^{i\mu/2}\, E\, \psi_1^{(u,v)}(0,x) = 1. \tag{4.1.23}$$

Since $\begin{bmatrix} \psi_1^{(u,v)}(0,x) \\ \psi_2^{(u,v)}(0,x) \end{bmatrix}$ is a solution to (2.1.1) at $\lambda = 0$ we have

$$\begin{bmatrix} \psi_1^{(u,v)}(0,x) \\ \psi_2^{(u,v)}(0,x) \end{bmatrix}' = \begin{bmatrix} 0 & u(x) \\ v(x) & 0 \end{bmatrix} \begin{bmatrix} \psi_1^{(u,v)}(0,x) \\ \psi_2^{(u,v)}(0,x) \end{bmatrix}. \tag{4.1.24}$$

From (4.1.24) we obtain

$$\begin{cases} {\psi'}_1^{(u,v)}(0,x) = u(x)\, \psi_2^{(u,v)}(0,x), \\ {\psi'}_2^{(u,v)}(0,x) = v(x)\, \psi_1^{(u,v)}(0,x). \end{cases} \tag{4.1.25}$$

Using (3.1.7) in the first equality of (4.1.25) we get

$$\psi'^{(u,v)}_1(0,x) = q(x)\,E^{-2}\,\psi^{(u,v)}_2(0,x). \tag{4.1.26}$$

With the help of (4.1.26), from (4.1.23) we have

$$e^{i\mu/2}\,E^{-1}\,\frac{\psi'^{(u,v)}_1(0,x)}{q(x)\,E^{-2}} + \frac{i\,r(x)}{2}\,E\,e^{i\mu/2}\,\psi^{(u,v)}_1(0,x) = 1,$$

or equivalently

$$\psi'^{(u,v)}_1(0,x) + \frac{i\,q(x)\,r(x)}{2}\,\psi^{(u,v)}_1(0,x) = q(x)\,E^{-1}\,e^{-i\mu/2}. \tag{4.1.27}$$

Note that (4.1.27) is a first-order linear ordinary differential equation and an integrating factor for it is E. We can write (4.1.27) as

$$E\,\psi'^{(u,v)}_1(0,x) + E'\,\psi^{(u,v)}_1(0,x) = q(x)\,e^{i\mu/2}, \tag{4.1.28}$$

which yields

$$\frac{d}{dx}\left[E\,\psi^{(u,v)}_1(0,x)\right] = q(x)\,e^{-i\mu/2}. \tag{4.1.29}$$

With the help of the first equality of (2.1.2) and the second equality of (4.1.6), integrating both-sides of (4.1.29) we obtain

$$E\,\psi^{(u,v)}_1(0,x) = e^{-i\mu/2}\int_\infty^x dy\,q(y),$$

or equivalently

$$\psi^{(u,v)}_1(0,x) = -e^{-i\mu/2}\,E^{-1}\int_x^\infty dy\,q(y), \tag{4.1.30}$$

which establishes (4.1.18). Now, let us prove (4.1.19). Using (4.1.30) in (4.1.23) we get

$$e^{i\mu/2}\,E^{-1}\,\psi^{(u,v)}_2(0,x) - \frac{i\,r(x)}{2}\int_x^\infty dy\,q(y) = 1,$$

or equivalently

$$\psi^{(u,v)}_2(0,x) = \left[1 + \frac{i\,r(x)}{2}\int_x^\infty dy\,q(y)\right]\,E\,e^{-i\mu/2}, \tag{4.1.31}$$

which completes the proof of (4.1.19). □

The standard form of the first-order system with energy-dependent potentials is given in (3.1.1). As in section 3, we can transform (3.1.1) into (3.1.31). Let us start with the system (3.1.31) and set the value of $\lambda = 0$ in (3.1.31). Showing the dependence of the wave functions on the parameter λ as $\tilde{\theta}(\lambda, x)$ and $\tilde{\omega}(\lambda, x)$ in (3.1.31), setting $\lambda = 0$ there we obtain

$$\begin{bmatrix} \tilde{\theta}(0,x) \\ \tilde{\omega}(0,x) \end{bmatrix}' = \begin{bmatrix} 0 & 0 \\ r(x) & 0 \end{bmatrix} \begin{bmatrix} \tilde{\theta}(0,x) \\ \tilde{\omega}(0,x) \end{bmatrix}, \qquad x \in \mathbb{R}. \tag{4.1.32}$$

We see that (4.1.32) is equivalent to

$$\begin{cases} \tilde{\theta}'(0,x) = 0, \\ \tilde{\omega}'(0,x) = r(x)\, \tilde{\theta}(0,x). \end{cases} \tag{4.1.33}$$

From the first equality in (4.1.33), we see that the quantity $\tilde{\theta}(0,x)$ is independent of x and hence it is a constant. Thus, from the second equality in (4.1.33) we obtain

$$r(x) = \frac{\tilde{\omega}'(0,x)}{\tilde{\theta}(0,x)}, \tag{4.1.34}$$

or equivalently

$$r(x) = \left(\frac{\tilde{\omega}(0,x)}{\tilde{\theta}(0,x)} \right)', \tag{4.1.35}$$

because the quantity $\tilde{\theta}(0,x)$ is independent of x. Recall that a superscript for the Jost function is used to denote the potentials in the corresponding first-order system. Let us copy (3.2.56) for the convenience of the reader as

$$\bar{\psi}^{(\lambda q,r)} = e^{-i\mu/2} \begin{bmatrix} E & -\frac{i\, q(x)}{2}\, E^{-1} \\ 0 & E^{-1} \end{bmatrix} \bar{\psi}^{(p,s)}, \tag{4.1.36}$$

where we recall that E is the quantity defined in (3.1.6) and depends on x. Let us write $\bar{\psi}^{(\lambda q,r)}$ and $\bar{\psi}^{(p,s)}$ in the component form as

$$\bar{\psi}^{(\lambda q,r)} = \begin{bmatrix} \bar{\psi}_1^{(\lambda q,r)}(\lambda,x) \\ \bar{\psi}_2^{(\lambda q,r)}(\lambda,x) \end{bmatrix}, \quad \bar{\psi}^{(p,s)} = \begin{bmatrix} \bar{\psi}_1^{(p,s)}(\lambda,x) \\ \bar{\psi}_2^{(p,s)}(\lambda,x) \end{bmatrix}. \tag{4.1.37}$$

Similarly, let us write $\psi^{(\lambda q,r)}$ and $\psi^{(p,s)}$ in the component form as

$$\psi^{(\lambda q,r)} = \begin{bmatrix} \psi_1^{(\lambda q,r)}(\lambda,x) \\ \psi_2^{(\lambda q,r)}(\lambda,x) \end{bmatrix}, \quad \psi^{(p,s)} = \begin{bmatrix} \psi_1^{(p,s)}(\lambda,x) \\ \psi_2^{(p,s)}(\lambda,x) \end{bmatrix}. \tag{4.1.38}$$

Using (4.1.37) in (4.1.36) we have

$$\begin{cases} \bar{\psi}_1^{(\lambda q,r)}(\lambda,x) = e^{-i\mu/2} \left[E\, \bar{\psi}_1^{(p,s)}(\lambda,x) - \frac{i\, q(x)\, E^{-1}}{2}\, \bar{\psi}_2^{(p,s)}(\lambda,x) \right], \\ \bar{\psi}_2^{(\lambda q,r)}(\lambda,x) = e^{-i\mu/2}\, E^{-1}\, \bar{\psi}_2^{(p,s)}(\lambda,x), \end{cases}$$

which can be written at $\lambda = 0$ as

$$\begin{cases} \bar{\psi}_1^{(\lambda q,r)}(0,x) = e^{-i\mu/2} \left[E\, \bar{\psi}_1^{(p,s)}(0,x) - \frac{i\, q(x)\, E^{-1}}{2}\, \bar{\psi}_2^{(p,s)}(0,x) \right], \\ \bar{\psi}_2^{(\lambda q,r)}(0,x) = e^{-i\mu/2}\, E^{-1}\, \bar{\psi}_2^{(p,s)}(0,x). \end{cases} \tag{4.1.39}$$

Since $\bar{\psi}^{(\lambda q,r)}(0,x)$ satisfies (4.1.32), we have

$$\bar{\psi}'_{(\lambda q,r)} = \begin{bmatrix} 0 & 0 \\ r(x) & 0 \end{bmatrix} \bar{\psi}_{(\lambda q,r)}, \qquad x \in \mathbb{R}. \tag{4.1.40}$$

With the help of the first equality in (4.1.37), from (4.1.40) we get

$$\begin{cases} \bar{\psi}'^{(\lambda q,r)}_1(0,x) = 0, \\ \bar{\psi}'^{(\lambda q,r)}_2(0,x) = r(x)\, \bar{\psi}_1^{(\lambda q,r)}(0,x). \end{cases} \tag{4.1.41}$$

From the first equality in (4.1.41), it is seen that the quantity $\bar{\psi}'^{(\lambda q,r)}_1(0,x)$ is independent of x. Hence, the right-hand side of the first equality in (4.1.39) is independent of x and its value can be evaluated as $x \to +\infty$. We have

$$\bar{\psi}_1^{(\lambda q,r)}(0,x) = \lim_{x\to+\infty} \left(e^{-i\mu/2} \left[E\, \bar{\psi}_1^{(p,s)}(0,x) - \frac{i\, q(x)\, E^{-1}}{2}\, \bar{\psi}_2^{(p,s)}(0,x) \right] \right), \tag{4.1.42}$$

where we recall that $q(x)$ and $r(x)$ belong to the Schwartz class $\mathbb{S}(\mathbb{R})$ and hence $q(x) \to 0$ as $x \to +\infty$. With the help of the second equality in (2.1.2), (2.5.7), the second equality in (3.2.14), and the second equality in (4.1.37), we determine that the right-hand side of (4.1.42) is equal to 1 and hence we have

$$\bar{\psi}_1^{(\lambda q,r)}(0,x) \equiv 1. \tag{4.1.43}$$

Using the second equality of (4.1.39) and (4.1.43) in the second equality of (4.1.41) we obtain

$$r(x) = e^{-i\mu/2} \left[E^{-1}\, \bar{\psi}_2^{(p,s)}(0,x) \right]'. \tag{4.1.44}$$

In the next proposition, we show how to express the zero-energy Jost solutions $\psi_1^{(p,s)}(0,x)$ and $\psi_2^{(p,s)}(0,x)$ to the standard system (3.1.32) in terms of the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1).

Proposition 4.3. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.31) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, we have*

$$\psi_1^{(p,s)}(0,x) = \frac{i\, q(x)}{2}\, e^{-i\mu/2}\, E^{-1}, \tag{4.1.45}$$

$$\psi_2^{(p,s)}(0,x) = e^{-i\mu/2}\, E, \tag{4.1.46}$$

where E and μ are the quantities defined in (3.1.6) and (3.2.13), respectively.

Proof. Using (3.2.13), the second equality in (3.2.14), and $q(x) \to 0$ as $x \to +\infty$, from (4.1.45) and (4.1.46) we have

$$\begin{bmatrix} \frac{i\,q(x)}{2}\, e^{-i\mu/2}\, E^{-1} \\ e^{-i\mu/2}\, E \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad x \to +\infty.$$

As the superscript (p,s) indicates, the column vector $\begin{bmatrix} \psi_1^{(p,s)}(0,x) \\ \psi_2^{(p,s)}(0,x) \end{bmatrix}$ satisfies the first-order system (3.1.32) at $\lambda = 0$. Furthermore, as seen from the first equality in

(3.2.8), that column vector $\begin{bmatrix}\psi_1^{(p,s)}(0,x)\\ \psi_2^{(p,s)}(0,x)\end{bmatrix}$ has the asymptotic $\begin{bmatrix}0\\ 1\end{bmatrix}$ as $x \to +\infty$. In other words, we need to show that

$$\begin{bmatrix}\frac{i\,q(x)}{2}\, e^{-i\mu/2}\, E^{-1}\\ e^{-i\mu/2}\, E\end{bmatrix}' = \begin{bmatrix}0 & p(x)\\ s(x) & 0\end{bmatrix}\begin{bmatrix}\frac{i\,q(x)}{2}\, e^{-i\mu/2}\, E^{-1}\\ e^{-i\mu/2}\, E\end{bmatrix}. \tag{4.1.47}$$

Using (3.1.35) and (3.1.36) in (4.1.47) we get

$$\begin{bmatrix}\frac{i\,q(x)}{2}\, e^{-i\mu/2}\, E^{-1}\\ e^{-i\mu/2}\, E\end{bmatrix}' = \begin{bmatrix}0 & \left(\frac{i\,q'(x)}{2}+\frac{r(x)\,q^2(x)}{4}\right)E^{-2}\\ r(x)\,E^2 & 0\end{bmatrix}\begin{bmatrix}\frac{i\,q(x)}{2}\, e^{-i\mu/2}\, E^{-1}\\ e^{-i\mu/2}\, E\end{bmatrix},$$

which can be written as

$$\begin{bmatrix}\left(\frac{i\,q'(x)}{2}+\frac{r(x)\,q^2(x)}{4}\right) e^{-i\mu/2}\, E^{-1}\\ \frac{ir(x)\,q(x)}{2} e^{-i\mu/2}\, E\end{bmatrix} = \begin{bmatrix}0 & \left(\frac{i\,q'(x)}{2}+\frac{r(x)\,q^2(x)}{4}\right)E^{-2}\\ r(x)\,E^2 & 0\end{bmatrix}\begin{bmatrix}\frac{i\,q(x)}{2}\, e^{-i\mu/2}\, E^{-1}\\ e^{-i\mu/2}\, E\end{bmatrix}, \tag{4.1.48}$$

which can be verified directly by multiplying the two matrices on the right-hand side of (4.1.48). □

In the next proposition, we show how to express the zero-energy Jost solutions $\bar\psi_1^{(p,s)}(0,x)$ and $\bar\psi_2^{(p,s)}(0,x)$ to the standard system (3.1.32) in terms of the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1).

Proposition 4.4. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, we have*

$$\bar\psi_1^{(p,s)}(0,x) = \left[1 - \frac{i\,q(x)}{2}\int_x^\infty dy\, r(y)\right] e^{i\mu/2}\, E^{-1}, \tag{4.1.49}$$

$$\bar\psi_2^{(p,s)}(0,x) = -e^{i\mu/2}\, E\int_x^\infty dy\, r(y), \tag{4.1.50}$$

where E and μ are the quantities defined in (3.1.6) and (3.2.13), respectively.

Proof. The Wronskian of the Jost solutions $\bar{\psi}_1^{(p,s)}(0,x)$ and $\bar{\psi}_2^{(p,s)}(0,x)$ for the linear system (3.1.32) at $\lambda = 0$ is given by

$$\left[\bar{\psi}^{(p,s)}(0,x); \psi^{(p,s)}(0,x)\right] = \begin{vmatrix} \bar{\psi}_1^{(p,s)}(0,x) & \psi_1^{(p,s)}(0,x) \\ \bar{\psi}_2^{(p,s)}(0,x) & \psi_2^{(p,s)}(0,x) \end{vmatrix}. \tag{4.1.51}$$

With the help of (3.2.8), the second equality in (4.1.37) and in (4.1.38), from (4.1.51) we get

$$\left[\bar{\psi}^{(p,s)}(0,x); \psi^{(p,s)}(0,x)\right] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix},$$

which yields

$$\left[\bar{\psi}^{(p,s)}(0,x); \psi^{(p,s)}(0,x)\right] = 1. \tag{4.1.52}$$

Using (4.1.45) and (4.1.46) in (4.1.51) we have

$$\begin{vmatrix} \bar{\psi}_1^{(p,s)}(0,x) & \frac{i\,q(x)}{2}\, e^{-i\mu/2}\, E^{-1} \\ \bar{\psi}_2^{(p,s)}(0,x) & e^{-\mu/2}\, E \end{vmatrix} = 1. \tag{4.1.53}$$

Evaluating the left-hand side of (4.1.53) we get

$$e^{-i\mu/2}\, E\, \bar{\psi}_1^{(p,s)}(0,x) - \frac{i\,q(x)}{2}\, e^{-i\mu/2}\, E^{-1}\, \bar{\psi}_2^{(p,s)}(0,x) = 1. \tag{4.1.54}$$

Since $\begin{bmatrix} \bar{\psi}_1^{(p,s)}(0,x) \\ \bar{\psi}_2^{(p,s)}(0,x) \end{bmatrix}$ is a solution to (3.1.32) at $\lambda = 0$ we have

$$\begin{bmatrix} \bar{\psi}_1^{(p,s)}(0,x) \\ \bar{\psi}_2^{(p,s)}(0,x) \end{bmatrix}' = \begin{bmatrix} 0 & p(x) \\ s(x) & 0 \end{bmatrix} \begin{bmatrix} \bar{\psi}_1^{(p,s)}(0,x) \\ \bar{\psi}_2^{(p,s)}(0,x) \end{bmatrix}. \tag{4.1.55}$$

From (4.1.55) we obtain

$$\begin{cases} \bar{\psi}'^{(p,s)}_1(0,x) = p(x)\, \bar{\psi}_2^{(p,s)}(0,x), \\ \bar{\psi}'^{(p,s)}_2(0,x) = s(x)\, \bar{\psi}_1^{(p,s)}(0,x). \end{cases} \tag{4.1.56}$$

Using (3.1.36) in the second equality of (4.1.56) we get

$$\bar{\psi}'^{(p,s)}_2(0,x) = r(x)\,E^2\,\bar{\psi}^{(p,s)}_1(0,x). \tag{4.1.57}$$

With the help of (4.1.57), from (4.1.54) we have

$$e^{-i\mu/2}\,E\,\frac{\bar{\psi}'^{(p,s)}_2(0,x)}{r(x)\,E^2} - \frac{i\,q(x)}{2}\,E^{-1}\,e^{-i\mu/2}\,\bar{\psi}^{(p,s)}_2(0,x) = 1,$$

or equivalently

$$\bar{\psi}'^{(p,s)}_2(0,x) - \frac{i\,q(x)\,r(x)}{2}\,\bar{\psi}^{(p,s)}_2(0,x) = r(x)\,E\,e^{i\mu/2}. \tag{4.1.58}$$

Note that (4.1.58) is a first-order linear ordinary differential equation and an integrating factor for it is E. We can rewrite (4.1.58) as

$$E^{-1}\,\bar{\psi}'^{(p,s)}_2(0,x) - \frac{i\,q(x)\,r(x)}{2}\,E^{-1}\,\bar{\psi}^{(p,s)}_2(0,x) = r(x)\,e^{i\mu/2}, \tag{4.1.59}$$

which yields

$$\frac{d}{dx}\left[E^{-1}\,\bar{\psi}^{(p,s)}_2(0,x)\right] = r(x)\,e^{i\mu/2}. \tag{4.1.60}$$

With the help of the second equality of (2.1.2) and the second equality of (4.1.37), integrating both-sides of (4.1.60) we obtain

$$E^{-1}\,\bar{\psi}^{(p,s)}_2(0,x) = e^{i\mu/2}\int_{\infty}^{x} dy\,r(y),$$

or equivalently

$$\bar{\psi}^{(p,s)}_2(0,x) = -e^{i\mu/2}\,E\int_x^{\infty} dy\,r(y), \tag{4.1.61}$$

which establishes (4.1.50). Now, let us prove (4.1.49). Using (4.1.61) in (4.1.54) we get

$$e^{-i\mu/2}\,E\,\bar{\psi}^{(p,s)}_1(0,x) + \frac{i\,q(x)}{2}\int_x^{\infty} dy\,r(y) = 1,$$

or equivalently

$$\bar{\psi}^{(p,s)}_1(0,x) = \left[1 - \frac{i\,q(x)}{2}\int_x^{\infty} dy\,r(y)\right]E^{-1}\,e^{i\mu/2}, \tag{4.1.62}$$

which completes the proof of (4.1.49). □

We would like to express $q(x)$ and $r(x)$ appearing in the energy-dependent system (3.1.1) in terms of the zero-energy Jost solutions for the energy-independent systems (2.1.1) and (3.1.32). That is, we would like to express $q(x)$ in terms of $\psi^{(u,v)}(0,x)$ and $\bar{\psi}^{(u,v)}(0,x)$. Similarly, we would like to express $r(x)$ in terms of $\psi^{(p,s)}(0,x)$ and $\bar{\psi}^{(p,s)}(0,x)$. Using (4.1.14) and (4.1.15) we obtain

$$-\frac{i\,r(x)}{2}\,e^{i\mu/2} = \bar{\psi}_1^{(u,v)}(0,x)\,\bar{\psi}_2^{(u,v)}(0,x),$$

which yields

$$r(x) = 2i\,e^{-i\mu/2}\,\bar{\psi}_1^{(u,v)}(0,x)\,\bar{\psi}_2^{(u,v)}(0,x). \tag{4.1.63}$$

Similarly, using (4.1.45) and (4.1.46) we get

$$\frac{i\,q(x)}{2}\,e^{-i\mu/2} = \psi_1^{(p,s)}(0,x)\,\psi_2^{(p,s)}(0,x),$$

or equivalently

$$q(x) = -2i\,e^{i\mu/2}\,\psi_1^{(p,s)}(0,x)\,\psi_2^{(p,s)}(0,x). \tag{4.1.64}$$

And also using (4.1.14) in (4.1.18) we obtain

$$\psi_1^{(u,v)}(0,x) = -e^{-i\mu/2}\left(e^{-i\mu/2}\,\bar{\psi}_1^{(u,v)}(0,x)\right)\int_x^\infty dy\,q(y),$$

or equivalently

$$-\int_x^\infty dy\,q(y) = e^{i\mu}\left(\frac{\psi_1^{(u,v)}(0,x)}{\bar{\psi}_1^{(u,v)}(0,x)}\right). \tag{4.1.65}$$

From (4.1.65) we get

$$q(x) = e^{i\mu}\,\frac{d}{dx}\left[\frac{\psi_1^{(u,v)}(0,x)}{\bar{\psi}_1^{(u,v)}(0,x)}\right]. \tag{4.1.66}$$

In the same manner, we can recover $r(x)$. Using (4.1.46) in (4.1.50) we obtain

$$\bar{\psi}_2^{(p,s)}(0,x) = -e^{i\mu/2}\left(\frac{\psi_2^{(p,s)}(0,x)}{e^{-i\mu/2}}\right)\int_x^\infty dy\,r(y),$$

or equivalently

$$-\int_x^\infty dy\,r(y) = e^{-i\mu}\left(\frac{\bar{\psi}_2^{(p,s)}(0,x)}{\psi_2^{(p,s)}(0,x)}\right),$$

which can be written as

$$r(x)=e^{-i\mu}\,\frac{d}{dx}\left[\frac{\bar{\psi}_2^{(p,s)}(0,x)}{\psi_2^{(p,s)}(0,x)}\right]. \tag{4.1.67}$$

4.2 The Alternate Marchenko Equation

In (4.1.66) we have related the potential $q(x)$ of (3.1.1) to the first components of the zero-energy Jost solutions $\psi^{(u,v)}(0,x)$ and $\bar{\psi}^{(u,v)}(0,x)$. Similarly, in (4.1.67) we have related the potential $r(x)$ of (3.1.1) to the second components of the Jost solutions $\psi^{(p,s)}(0,x)$ and $\bar{\psi}^{(p,s)}(0,x)$. In this section by relating those zero-energy Jost solutions to their Fourier transform, we express $q(x)$ and $r(x)$ in terms of the scalar quantities $\boldsymbol{K}(x,y)$ and $\bar{\boldsymbol{K}}(x,y)$ defined in (4.2.2) and (4.2.6), respectively. We then obtain Marchenko equations (4.2.9) and (4.2.10) satisfied by $\boldsymbol{K}(x,y)$ and $\bar{\boldsymbol{K}}(x,y)$, respectively. Those Marchenko equations use input $G^{(u,v)}(y)$, $\bar{G}^{(u,v)}(y)$, $G^{(p,s)}(y)$, $\bar{G}^{(u,v)}(y)$ defined in (4.2.11) and (4.2.12). This procedure allows us to develop a Marchenko method to solve inverse problem for (3.1.1) as in the following steps. For clarity we present the procedure by assuming that there are no bound-states, and later on we show the modification when the bound states are present. Recall that the inverse problem for (3.1.1) consist of recovery of the potentials $q(x)$ and $r(x)$ from the scattering data given $\boldsymbol{S}^{(\zeta q,\zeta r)}$ in (3.5.1).

- From the scattering data set $\boldsymbol{S}^{(\zeta q,\zeta r)}$ we obtain the scattering data sets $\boldsymbol{S}^{(u,v)}$ and $\boldsymbol{S}^{(p,s)}$ defined in (2.4.1). This is done by relations in the Propotions 3.3, 3.6, 3.9 and 3.9.
- Using the scattering data sets $\boldsymbol{S}^{(u,v)}$ and $\boldsymbol{S}^{(p,s)}$, we form the Marchenko kernels $G^{(u,v)}(y)$, $\bar{G}^{(u,v)}(y)$, $G^{(p,s)}(y)$, $\bar{G}^{(u,v)}(y)$ defined in (4.2.11) and (4.2.12).
- Using $G^{(u,v)}(y)$, $\bar{G}^{(u,v)}(y)$, $G^{(p,s)}(y)$, $\bar{G}^{(u,v)}(y)$ as input to the Marchenko equations (4.2.9) and (4.2.10), we obtain $\boldsymbol{K}(x,y)$ and $\bar{\boldsymbol{K}}(x,y)$.

- We finally recover $q(x)$ and $r(x)$ from $\boldsymbol{K}(x,y)$ and $\bar{\boldsymbol{K}}(x,y)$ as in (4.2.4) and (4.2.19).

Using (2.5.6), (2.5.7), (2.5.28), and the second equality of (4.1.6) and (4.1.7) in (4.1.66) we obtain

$$q(x) = e^{i\mu} \frac{d}{dx} \left[\frac{\int_x^\infty dz\, K_1^{(u,v)}(x,z)}{1 + \int_x^\infty dz\, \bar{K}_1^{(u,v)}(x,z)} \right]. \tag{4.2.1}$$

Now, let us define

$$\boldsymbol{K}(x,y) := \frac{\int_y^\infty dz_1\, K_1^{(u,v)}(x,z_1)}{1 + \int_x^\infty dz\, \bar{K}_1^{(u,v)}(x,z)}, \qquad y > x, \tag{4.2.2}$$

where we note that the y-dependence appears only in the numerator of (4.2.2). Taking the y- derivative of both sides of (4.2.2) we have

$$\frac{\partial}{\partial y} \boldsymbol{K}(x,y) = -\frac{K_1^{(u,v)}(x,y)}{1 + \int_x^\infty dz\, \bar{K}_1^{(u,v)}(x,z)}, \qquad y > x. \tag{4.2.3}$$

Using (4.2.2) in (4.2.1) we get

$$q(x) = e^{i\mu} \frac{d\boldsymbol{K}(x,x)}{dx}. \tag{4.2.4}$$

Similarly, Using (2.5.6), (2.5.7), (2.5.28), and the second equality of (4.1.37) and of (4.1.38) in (4.1.67) we obtain

$$r(x) = e^{-i\mu} \frac{d}{dx} \left[\frac{\int_x^\infty dz\, \bar{K}_2^{(p,s)}(x,z)}{1 + \int_x^\infty dz\, K_2^{(p,s)}(x,z)} \right]. \tag{4.2.5}$$

Let us define

$$\bar{\boldsymbol{K}}(x,y) := \frac{\int_y^\infty dz_1\, \bar{K}_2^{(p,s)}(x,z_1)}{1 + \int_x^\infty dz\, K_2^{(p,s)}(x,z)}, \qquad y > x, \tag{4.2.6}$$

where we note that the y-dependence appears only in the numerator of (4.2.6). By taking the derivative of (4.2.6) with respect to y we have

$$\frac{\partial}{\partial y} \bar{\boldsymbol{K}}(x,y) = -\frac{\bar{K}_2^{(p,s)}(x,y)}{1 + \int_x^\infty ds K_2^{(p,s)}(x,s)}, \qquad y > x. \tag{4.2.7}$$

With the help of (4.2.6), from (4.2.5) we have

$$r(x) = e^{-i\mu}\,\frac{d\bar{\boldsymbol{K}}(x,x)}{dx}. \tag{4.2.8}$$

From (4.2.4) and (4.2.8), we see that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) can be expressed explicitly in terms of $\boldsymbol{K}(x,y)$ and $\bar{\boldsymbol{K}}(x,y)$. In the next theorem, we prove that $\boldsymbol{K}(x,y)$ and $\bar{\boldsymbol{K}}(x,y)$ are the solution to the Marchenko integral equation associated with (3.1.1).

Theorem 4.1. *The Marchenko integral equations associated with* (3.1.1) *are given by*

$$\begin{aligned}&\boldsymbol{K}(x,y) + \bar{G}^{(u,v)}(x+y)\\ &\quad+\int_x^\infty dz \int_x^\infty dt\, \frac{\partial \boldsymbol{K}(x,t)}{\partial t}\, G^{(u,v)}(t+z)\, \frac{\partial\, \bar{G}^{(u,v)}(z+y)}{\partial z} = 0, \qquad y > x,\end{aligned} \tag{4.2.9}$$

$$\begin{aligned}&\bar{\boldsymbol{K}}(x,y) + G^{(p,s)}(x+y)\\ &\quad+\int_x^\infty dz \int_x^\infty dt\, \frac{\partial \bar{\boldsymbol{K}}(x,t)}{\partial t}\, \bar{G}^{(p,s)}(t+z)\, \frac{\partial\, G^{(p,s)}(z+y)}{\partial z} = 0, \qquad y > x,\end{aligned} \tag{4.2.10}$$

where

$$G^{(u,v)}(y) := \int_y^\infty dz\, \hat{R}^{(u,v)}(z), \quad \bar{G}^{(u,v)}(y) := \int_y^\infty dz\, \hat{\bar{R}}^{(u,v)}(z), \tag{4.2.11}$$

$$G^{(p,s)}(y) := \int_y^\infty dz\, \hat{R}^{(p,s)}(z), \quad \bar{G}^{(p,s)}(y) := \int_y^\infty dz\, \hat{\bar{R}}^{(p,s)}(z), \tag{4.2.12}$$

with

$$R^{(u,v)} = e^{i\mu}\sqrt{\lambda}\, R^{(\zeta q,\zeta r)}, \quad R^{(p,s)} = \frac{e^{i\mu}}{\sqrt{\lambda}}\, R^{(\zeta q,\zeta r)}, \tag{4.2.13}$$

$$\bar{R}^{(u,v)} = \frac{e^{-i\mu}}{\sqrt{\lambda}}\, \bar{R}^{(\zeta q,\zeta r)}, \quad \bar{R}^{(p,s)} = e^{-i\mu}\sqrt{\lambda}\, \bar{R}^{(\zeta q,\zeta r)}. \tag{4.2.14}$$

Proof. In the absence of bound states, from (2.5.11) and (2.5.31) we have

$$K_1^{(u,v)}(x,y) + \hat{\bar{R}}^{(u,v)}(x+y) + \int_x^\infty dz\, \bar{K}_1^{(u,v)}(x,z)\, \hat{\bar{R}}^{(u,v)}(z+y) = 0, \qquad y > x. \tag{4.2.15}$$

Similarly, in the absence of bound states, from (2.5.10) and (2.5.29) we get

$$\bar{K}_1^{(u,v)}(x,y) + \int_x^\infty dz\, K_1^{(u,v)}(x,z)\,\hat{R}^{(u,v)}(z+y) = 0, \qquad y > x. \tag{4.2.16}$$

We modify the integral term in (4.2.15) as

$$\int_x^\infty dz\, \bar{K}_1^{(u,v)}(x,z)\,\hat{\bar{R}}^{(u,v)}(z+y)$$
$$= -\int_x^\infty dz \left[\frac{\partial}{\partial z}\int_z^\infty dz_1\, \bar{K}_1^{(u,v)}(x,z_1)\right] \hat{\bar{R}}^{(u,v)}(z+y),$$

which can be written as

$$\begin{aligned}\int_x^\infty dz\, \bar{K}_1^{(u,v)}(x,z)\,\hat{\bar{R}}^{(u,v)}(z+y) &\\ = -\int_x^\infty dz\, \frac{\partial}{\partial z}\left(\left[\int_z^\infty dz_1\, \bar{K}_1^{(u,v)}(x,z_1)\right] \hat{\bar{R}}^{(u,v)}(z+y)\right) &\\ + \int_x^\infty dz\, \left(\int_z^\infty dz_1\, \bar{K}_1^{(u,v)}(x,z_1)\right) \frac{\partial \hat{\bar{R}}^{(u,v)}(z+y)}{\partial z}.&\end{aligned} \tag{4.2.17}$$

We simplify the first integral on the right-hand side of (4.2.17) and get

$$\int_x^\infty dz\, \bar{K}_1^{(u,v)}(x,z)\,\hat{\bar{R}}^{(u,v)}(z+y)$$
$$= -\int_z^\infty dz_1\, \bar{K}_1^{(u,v)}(x,z_1)\,\hat{\bar{R}}^{(u,v)}(z+y)\bigg|_{z=x}^{\infty}$$
$$+ \int_x^\infty dz \int_z^\infty dz_1 \bar{K}_1^{(u,v)}(x,z_1) \frac{\partial \hat{\bar{R}}^{(u,v)}(z+y)}{\partial z},$$

which can also be written as

$$\begin{aligned}\int_x^\infty dz\, \bar{K}_1^{(u,v)}(x,z)\,\hat{\bar{R}}^{(u,v)}(z+y) &\\ = \int_x^\infty dz_1\, \bar{K}_1^{(u,v)}(x,z_1)\,\hat{\bar{R}}^{(u,v)}(x+y) &\\ + \int_x^\infty dz \int_z^\infty dz_1 \bar{K}_1^{(u,v)}(x,z_1) \frac{\partial \hat{\bar{R}}^{(u,v)}(z+y)}{\partial z}.&\end{aligned} \tag{4.2.18}$$

Let us consider $\bar{K}_1^{(u,v)}(x, z_1)$ appearing in the second integral term on the right-hand side of (4.2.18). We replace that term by its equivalent obtained from (4.2.16). Then, from (4.2.18) we get

$$\begin{aligned}\int_x^\infty dz\, \bar{K}_1^{(u,v)}(x,z)\, \hat{\bar{R}}^{(u,v)}(z+y) \\ = \int_x^\infty dz_1\, \bar{K}_1^{(u,v)}(x,z_1)\, \hat{\bar{R}}^{(u,v)}(x+y) \\ - \int_x^\infty dz \int_z^\infty dz_1 \int_x^\infty dt\, K_1^{(u,v)}(x,t)\, \hat{R}^{(u,v)}(t+z_1)\, \frac{\partial \hat{\bar{R}}^{(u,v)}(z+y)}{\partial z}.\end{aligned} \tag{4.2.19}$$

Replacing the left-hand side of (4.2.19) by its equivalent obtained from (4.2.15) we get

$$\begin{aligned}K_1^{(u,v)}(x,y) + \left[1 + \int_x^\infty dz_1\, \bar{K}_1^{(u,v)}(x,z_1)\right] \hat{\bar{R}}^{(u,v)}(x+y) \\ - \int_x^\infty dz \int_z^\infty dz_1 \int_x^\infty dt\, K_1^{(u,v)}(x,t)\, \hat{R}^{(u,v)}(t+z_1)\, \frac{\partial \hat{\bar{R}}^{(u,v)}(z+y)}{\partial z} = 0.\end{aligned} \tag{4.2.20}$$

Replacing y by z_2 in (4.2.20) and integrating the resulting equation over $z_2 \in (y, +\infty)$ we get

$$\begin{aligned}\int_y^\infty dz_2\, K_1^{(u,v)}(x,z_2) + \left[1 + \int_x^\infty ds\, \bar{K}_1^{(u,v)}(x,s)\right] \int_y^\infty dz_2\, \hat{\bar{R}}^{(u,v)}(x+z_2) \\ - \int_y^\infty dz_2 \int_x^\infty dz \int_z^\infty dz_1 \int_x^\infty dt\, K_1^{(u,v)}(x,t)\, \hat{R}^{(u,v)}(t+z_1)\, \frac{\partial \hat{\bar{R}}^{(u,v)}(z+z_2)}{\partial z_2} = 0,\end{aligned}$$

which can be written as

$$\begin{aligned}\int_y^\infty dz_2\, K_1^{(u,v)}(x,z_2) + \left[1 + \int_x^\infty dz_1\, \bar{K}_1^{(u,v)}(x,z_1)\right] \int_y^\infty dz_2\, \hat{\bar{R}}^{(u,v)}(x+z_2) \\ + \int_x^\infty dz \int_z^\infty dz_1 \int_x^\infty dt\, K_1^{(u,v)}(x,t)\, \hat{R}^{(u,v)}(t+z_1)\, \hat{\bar{R}}^{(u,v)}(z+y) = 0.\end{aligned} \tag{4.2.21}$$

We divide each term of (4.2.21) with $\left[1+\int_x^\infty dz_1\,\bar{K}_1^{(u,v)}(x,z_1)\right]$. Then, using (4.2.2) and (4.2.3) in the resulting equation we obtain

$$\boldsymbol{K}(x,y)+\int_y^\infty dz_2\,\hat{\bar{R}}^{(u,v)}(x+z_2)$$
$$-\int_x^\infty dz\int_z^\infty dz_1\int_x^\infty dt\,\frac{\partial\boldsymbol{K}(x,t)}{\partial t}\,\hat{R}^{(u,v)}(t+z_1)\,\hat{\bar{R}}^{(u,v)}(z+y)=0,$$

or equivalently

$$\begin{aligned}&\boldsymbol{K}(x,y)+\int_y^\infty dz_2\,\hat{\bar{R}}^{(u,v)}(x+z_2)\\&-\int_x^\infty dz\int_x^\infty dt\,\frac{\partial\boldsymbol{K}(x,t)}{\partial t}\left[\int_z^\infty dz_1\hat{R}^{(u,v)}(t+z_1)\right]\hat{\bar{R}}^{(u,v)}(z+y)=0.\end{aligned}\tag{4.2.22}$$

With the help of (4.2.11), from (4.2.22) we get

$$\boldsymbol{K}(x,y)+\bar{G}^{(u,v)}(x+y)-\int_x^\infty dz\int_x^\infty dt\,\frac{\partial\boldsymbol{K}(x,t)}{\partial t}\,G^{(u,v)}(t+z)\,\hat{\bar{R}}^{(u,v)}(z+y)=0.\tag{4.2.23}$$

Taking y-derivative of the second equality in (4.2.11) we have

$$\hat{\bar{R}}^{(u,v)}(y)=-\frac{\partial\,\bar{G}^{(u,v)}(y)}{\partial y}.\tag{4.2.24}$$

Using (4.2.24) in (4.2.23) we obtain

$$\begin{aligned}&\boldsymbol{K}(x,y)+\bar{G}^{(u,v)}(x+y)\\&+\int_x^\infty dz\int_x^\infty dt\,\frac{\partial\boldsymbol{K}(x,t)}{\partial t}\,G^{(u,v)}(t+z)\,\frac{\partial\,\bar{G}^{(u,v)}(z+y)}{\partial z}=0,\qquad y>x,\end{aligned}\tag{4.2.25}$$

which establishes (4.2.9). In the same manner, we can prove (4.2.10). In the absence of bound states , from (2.5.10), (2.5.11), (2.5.30) and (2.5.32) we have

$$\bar{K}_2^{(p,s)}(x,y)+\hat{R}^{(p,s)}(x+y)+\int_x^\infty dz\,K_2^{(p,s)}(x,z)\,\hat{R}^{(p,s)}(z+y)=0,\quad y>x,\tag{4.2.26}$$

$$K_2^{(p,s)}(x,y)+\int_x^\infty dz\,\bar{K}_2^{(p,s)}(x,z)\,\hat{\bar{R}}^{(p,s)}(z+y)=0,\quad y>x.\tag{4.2.27}$$

We modify the integral term in (4.2.26) as

$$\int_x^\infty dz\, K_2^{(p,s)}(x,z)\, \hat{R}^{(p,s)}(z+y) = -\int_x^\infty dz \left[\frac{\partial}{\partial z}\int_z^\infty dz_1\, K_2^{(p,s)}(x,z_1)\right] \hat{R}^{(p,s)}(z+y),$$

which can be written as

$$\begin{aligned}\int_x^\infty dz\, K_2^{(p,s)}(x,z)\, \hat{R}^{(p,s)}(z+y) &= -\int_x^\infty dz\, \frac{\partial}{\partial z}\left(\left[\int_z^\infty dz_1\, K_2^{(p,s)}(x,z_1)\right] \hat{R}^{(p,s)}(z+y)\right) \\ &\quad + \int_x^\infty dz \left(\int_z^\infty dz_1\, K_2^{(p,s)}(x,z_1)\right) \frac{\partial \hat{R}^{(p,s)}(z+y)}{\partial z}.\end{aligned} \tag{4.2.28}$$

We simplify the first integral on the right-hand side of (4.2.28) and get

$$\begin{aligned}\int_x^\infty dz\, K_2^{(p,s)}(x,z)\, \hat{R}^{(p,s)}(z+y) &= -\int_z^\infty dz_1\, K_2^{(p,s)}(x,z_1)\, \hat{R}^{(p,s)}(z+y)\Big|_{z=x}^\infty \\ &\quad + \int_x^\infty dz \int_z^\infty dz_1\, K_2^{(p,s)}(x,z_1) \frac{\partial \hat{R}^{(p,s)}(z+y)}{\partial z},\end{aligned}$$

which can also be written as

$$\begin{aligned}\int_x^\infty dz\, K_2^{(p,s)}(x,z)\, \hat{R}^{(p,s)}(z+y) &= \int_x^\infty dz_1\, K_2^{(p,s)}(x,z_1)\, \hat{R}^{(p,s)}(z+y) \\ &\quad + \int_x^\infty dz \int_z^\infty dz_1\, K_2^{(p,s)}(x,z_1) \frac{\partial \hat{R}^{(p,s)}(z+y)}{\partial z}.\end{aligned} \tag{4.2.29}$$

Let us consider $K_2^{(p,s)}(x,z_1)$ appearing in the second integral on the right-hand side of (4.2.29). We replace that term by its equivalent obtained from (4.2.27). Then, from (4.2.29) we have

$$\begin{aligned}\int_x^\infty dz\, K_2^{(p,s)}(x,z)\, \hat{R}^{(p,s)}(z+y) &= \int_x^\infty dz_1\, K_2^{(p,s)}(x,z_1)\, \hat{R}^{(p,s)}(z+y) \\ &\quad - \int_x^\infty dz \int_z^\infty dz_1 \int_x^\infty dt\, \bar{K}_2^{(p,z_1)}(x,t)\, \hat{\bar{R}}^{(p,s)}(t+z_1)\, \frac{\partial \hat{R}^{(p,s)}(z+y)}{\partial z}.\end{aligned} \tag{4.2.30}$$

Replacing the left-hand side of (4.2.30) by its equivalent obtained from (4.2.26) we get

$$\begin{aligned} &\bar{K}_2^{(p,s)}(x,y) + \left[1 + \int_x^\infty dz_1\, K_2^{(p,s)}(x,z_1)\right] \hat{R}^{(p,s)}(x+y) \\ &\quad - \int_x^\infty dz \int_z^\infty dz_1 \int_x^\infty dt\, \bar{K}_2^{(p,s)}(x,t)\, \hat{\bar{R}}^{(p,s)}(t+z_1)\, \frac{\partial \hat{R}^{(p,s)}(z+y)}{\partial z} = 0. \end{aligned} \tag{4.2.31}$$

Replacing y by z_2 in (4.2.31) and integrating the resulting equation over $z_2 \in (y, +\infty)$ we get

$$\begin{aligned} &\int_y^\infty dz_2\, \bar{K}_2^{(p,s)}(x,z_2) + \left[1 + \int_x^\infty dz_1\, K_2^{(p,s)}(x,z_1)\right] \int_y^\infty dz_2\, \hat{R}^{(p,s)}(x+z_2) \\ &\quad - \int_y^\infty dz_2 \int_x^\infty dz \int_z^\infty dz_1 \int_x^\infty dt\, \bar{K}_2^{(p,s)}(x,t)\, \hat{\bar{R}}^{(p,s)}(t+z_1)\, \frac{\partial \hat{R}^{(p,s)}(z+z_2)}{\partial z_2} = 0, \end{aligned}$$

which can be written as

$$\begin{aligned} &\int_y^\infty dz_2\, \bar{K}_2^{(p,s)}(x,z_2) + \left[1 + \int_x^\infty dz_1\, K_2^{(p,s)}(x,z_1)\right] \int_y^\infty dz_2\, \hat{R}^{(p,s)}(x+z_2) \\ &\quad + \int_x^\infty dz \int_z^\infty dz_1 \int_x^\infty dt\, \bar{K}_2^{(p,s)}(x,t)\, \hat{\bar{R}}^{(u,v)}(t+z_1)\, \hat{R}^{(p,s)}(z+y) = 0. \end{aligned} \tag{4.2.32}$$

We divide each term of (4.2.32) with $\left[1 + \int_x^\infty dz_1\, K_2^{(p,s)}(x,z_1)\right]$. Then, using (4.2.6) and (4.2.7) in the resulting equation we obtain

$$\begin{aligned} &\bar{\boldsymbol{K}}(x,y) + \int_y^\infty dz_2\, \hat{R}^{(p,s)}(x+z_2) \\ &\quad - \int_x^\infty dz \int_z^\infty dz_1 \int_x^\infty dt\, \frac{\partial \bar{\boldsymbol{K}}(x,t)}{\partial t}\, \hat{\bar{R}}^{(u,v)}(t+z_1)\, \hat{R}^{(u,v)}(z+y) = 0, \end{aligned}$$

or equivalently

$$\begin{aligned} &\bar{\boldsymbol{K}}(x,y) + \int_y^\infty dz_2\, \hat{R}^{(u,v)}(x+z_2) \\ &\quad - \int_x^\infty dz \int_x^\infty dt\, \frac{\partial \bar{\boldsymbol{K}}(x,t)}{\partial t} \left[\int_z^\infty dz_1 \hat{\bar{R}}^{(u,v)}(t+z_1)\right] \hat{R}^{(u,v)}(z+y) = 0. \end{aligned} \tag{4.2.33}$$

With the help of (4.2.12), from (4.2.33) we get

$$\begin{aligned} &\bar{\boldsymbol{K}}(x,y) + G^{(p,s)}(x+y) \\ &\quad - \int_x^\infty dz \int_x^\infty dt\, \frac{\partial \bar{\boldsymbol{K}}(x,t)}{\partial t}\, \bar{G}^{(p,s)}(t+z)\, \hat{R}^{(p,s)}(z+y) = 0. \qquad y > x. \end{aligned} \tag{4.2.34}$$

Taking y-derivative of the first equality in (4.2.12) we have

$$\hat{R}^{(p,s)}(y) = -\frac{d\,G^{(p,s)}(y)}{dy}. \tag{4.2.35}$$

Using (4.2.35) in (4.2.34), we obtain

$$\begin{aligned} &\bar{\boldsymbol{K}}(x,y) + G^{(p,s)}(x+y) \\ &\quad + \int_x^\infty dz \int_x^\infty dt\, \frac{\partial \bar{\boldsymbol{K}}(x,t)}{\partial t}\, \bar{G}^{(p,s)}(t+z)\, \frac{\partial G^{(p,s)}(z+y)}{\partial z} = 0, \qquad y > x, \end{aligned} \tag{4.2.36}$$

which completes the proof of (4.2.10). □

In the presence of bound states, the Marchenko kernels $G^{(u,v)}(y)$ and $\bar{G}^{(u,v)}(y)$ given in (4.2.11) need to be modified to include the bound-states. From the scattering data set $\boldsymbol{S}^{(u,v)}$ given in (2.4.1). The Marchenko kernel $G^{(u,v)}(y)$ and $\bar{G}^{(u,v)}(y)$ appearing in (4.2.9) can be written as

$$G^{(u,v)}(y) := \int_y^\infty dz \left[\hat{R}^{(u,v)}(z) + \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} c_{jk}^{(u,v)}\, \frac{z^k}{k!}\, e^{i\lambda_j z} \right], \tag{4.2.37}$$

$$\bar{G}^{(u,v)}(y) := \int_y^\infty dz \left[\hat{\bar{R}}^{(u,v)}(z) + \sum_{j=1}^{\bar{N}} \sum_{k=0}^{\bar{m}_j-1} \bar{c}_{js}^{(u,v)}\, \frac{z^k}{k!}\, e^{-i\bar{\lambda}_j z} \right]. \tag{4.2.38}$$

Similarly, in the presence of bound states, the Marchenko kernels $G^{(p,s)}(y)$ and $\bar{G}^{(p,s)}(y)$ given in (4.2.12) need to be modified to include the bound-states. Let $\boldsymbol{S}^{(p,s)}$ be the scattering data set for (3.1.32). As in (2.4.1) we have

$$\boldsymbol{S}^{(p,s)} = \left\{ R, \bar{R}, \left\{ \lambda_j, \{c_{jk}\}_{k=0}^{m_j-1} \right\}_{j=1}^{N}, \left\{ \bar{\lambda}_j, \{\bar{c}_{jk}\}_{k=0}^{\bar{m}_j-1} \right\}_{j=1}^{\bar{N}} \right\}^{(p,s)}$$

The Marchenko kernel $G^{(p,s)}(y)$ and $\bar{G}^{(p,s)}(y)$ appearing in (4.2.10) can be written as.

$$G^{(p,s)}(y) := \int_y^\infty dz \left[\hat{R}^{(p,s)}(z) + \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} c_{jk}^{(p,s)}\, \frac{z^k}{k!}\, e^{i\lambda_j z} \right], \tag{4.2.39}$$

$$\bar{G}^{(p,s)}(y) := \int_y^\infty dz \left[\hat{\bar{R}}^{(p,s)}(z) + \sum_{j=1}^{\bar{N}} \sum_{k=0}^{\bar{m}_j-1} \bar{c}_{jk}^{(p,s)}\, \frac{y^k}{k!}\, e^{-i\bar{\lambda}_j y} \right]. \tag{4.2.40}$$

Chapter 5

The Kaup-Newell Method

Kaup and Newell [1] used the inverse scattering transform method [1, 4, 12] to solve the initial-value problem for (1.0.3). Recall that the initial-value problem (IVP) for certain nonlinear partial differential equation (NPDE), known as integrable evolution equation, can be solved by the inverse scattering transform method. This method associated the corresponding integrable nonlinear partial differential equation with a linear ordinary differential equations (LODEs) with a spectral problem. The inverse problem for the integrable partial differential equations can be solved as indicated in the following diagram.

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{direct problem for the LODE at t=0}} & \boldsymbol{S}(\lambda;\, 0) \\ \text{solution to IVP}\Big\downarrow & & \Big\downarrow\text{time evolution} \\ u(x,t) & \xleftarrow[\text{inverse problem for the LODE at time t}]{} & \boldsymbol{S}(\lambda;\, t) \end{array} \tag{5.0.1}$$

The inverse scattering transform method use the following steps:

(1) The initial value of the solution $u(x,t)$ to the integrable nonlinear partial differential equation is given by $u(x,0)$, as we used $t=0$ as the initial time.

(2) In the corresponding linear ordinary differential equation, the quantity $u(x,0)$ appears as the potential, i.e. appears as a coefficient in the corresponding linear ordinary differential equation, which also contains the spectral parameter λ. Since the potential $u(x,0)$ is assumed to vanish as $x \to \pm\infty$, it is possible to develop the scattering theory for the ordinary differential equation with the spectral parameters. The corresponding scattering data set usually consists of the scattering coefficients (i.e. the transmission coefficients and the reflection coefficients from the left and from the

right) as well as the bound-state data. Since the potential $u(x,0)$ does not contain t, the corresponding scattering data set $\boldsymbol{S}(\lambda,0)$ does not contain the parameter t either.
(3) One can view the solution to the inverse problem for the integrable evolution equation the evolvement of the evolution of $u(x,0)$ into $u(x,t)$, and this time evolution is governed by the integrable nonlinear partial differential equation, respectively to the integrable evolution equations.
(4) One can view $u(x,t)$ as the potential in the corresponding linear ordinary differential equation with the spectral parameter. That linear ordinary differential equation is identical to the linear ordinary differential equation involving $u(x,0)$; the only difference is that $u(x,0)$ is now replaced by $u(x,t)$, where t is only a parameter and not an independent variable in the linear ordinary differential equation. We still have $u(x,t)$ vanishing as $x\to\pm\infty$ for each fixed value of the parameter t. Hence, it is possible to develop the scattering theory corresponding to the potential $u(x,t)$. The corresponding scattering data now depends on the parameter t as well as the spectral parameter λ. We denote that scattering data set by $\boldsymbol{S}(\lambda,t)$.
(5) The connection between $\boldsymbol{S}(\lambda,0)$ and $\boldsymbol{S}(\lambda,t)$ can be described as the time evolution of the scattering data. That time evolution $\boldsymbol{S}(\lambda,0)\mapsto\boldsymbol{S}(\lambda,t)$ is determined by the corresponding evolution of the potentials $u(x,0)\mapsto u(x,t)$. The attractiveness of the inverse scattering method is that, although the time evolution $u(x,0)\mapsto u(x,t)$ is usually very complicated, the time evolution $\boldsymbol{S}(\lambda,0)\mapsto\boldsymbol{S}(\lambda,t)$ is rather simple. For each integrable evolution equation governed by an integrable nonlinear partial differential equation, the time evolution of the scattering data is uniquely determined.
(6) Once the time evolution of the scattering data is specified, then one needs to solve the inverse problem for the corresponding linear ordinary differential equation and recover the time evolved potential $u(x,t)$ from the time evolved scattering data $\boldsymbol{S}(\lambda,t)$. Normally, one uses a Marchenko method[1, 3, 4] or one uses its variants to

solve inverse problem $\boldsymbol{S}(\lambda,t) \mapsto\ u(x,t)$.

(7) It is amazing that $u(x,t)$ constructed from $s(\lambda,t)$ is the solution to the inverse problem for the corresponding integrable nonlinear partial differential equation, i.e. $u(x,t)$ satisfies the integrable nonlinear partial differential equation, $u(x,0)$ at $t=0$ coincides with the initial value $u(x,0)$ appearing in the corresponding left corner in diagram 5.1.

Kaup and Newell applied the inverse scattering transform to solve initial value problem for (1.0.3). They associated (1.0.3) with the linear ordinary differential equation appearing in (3.1.1) in the special case $r(x) = \pm q(x)^*$. They used a Marchenko method, i.e. they set up the linear Marchenko equation and recovered $q(x)$ from the solution to the Marchenko equation. Since their goal was to develop the inverse scattering transform method for the derivative nonlinear Schrödinger equation given in (1.0.3), they were less interested in developing the scattering theory for (3.1.1). It is not quite clear how the scattering data used by Kaup and Newell are related to the scattering coefficients for (3.1.1).

Our goal in this chapter is to clarify this issue and indicate how the Kaup-Newell data set is related to the scattering coefficients for (3.1.1). Since the Kaup and Newell [1] considered only the special case $r(x) = \pm q(x)^*$, we extend the results in [1] by using two potentials $q(x)$ and $r(x)$ without relating those two potentials to each other. In order clearly indicate how the scattering data set of Kaup and Newell is related to the scattering theory for (3.1.1), we use the original notation of the Kaup and Newell for the quantities used in [1] and we also use our own notation introduced earlier, namely we use the superscript on quantities to indicate the two potentials to identify appropriate linear system. For example, we use $\psi^{(\zeta q,\zeta r)}$ for the Jost solution for (3.1.1), $T^{(\zeta q,\zeta r)}$ and $R^{(\zeta q,\zeta r)}$ for the transmission and right-reflection coefficients associated with (3.1.1).

We would like to express the quantities appearing in Kaup-Newell formulation in terms of the corresponding quantities for the systems (3.1.1). The Jost solutions defined in (5) and (6) of [1] can be express in term of Jost solution defined for (3.1.1) as

$$\phi = \phi^{(\zeta q,\zeta r)} = \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1), \quad \bar{\phi} = \bar{\phi}^{\zeta q,\zeta r} = \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + o(1), \quad x \to -\infty, \quad (5.0.2)$$

$$\psi = \psi^{\zeta q,\zeta r} = \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + o(1), \quad \bar{\psi} = \bar{\psi}^{\zeta q,\zeta r} = \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty. \quad (5.0.3)$$

Similarly, the scattering coefficients defined as in (7a) and (7b) in [1] can be express in term of the scattering coefficients for (3.1.1) as

$$a = \frac{1}{T^{(\zeta q,\zeta r)}}, \quad b = \left(\frac{R}{T}\right)^{(\zeta q,\zeta r)}, \quad \bar{a} = \frac{1}{\bar{T}^{(\zeta q,\zeta r)}}, \quad \bar{b} = \left(\frac{\bar{R}}{\bar{T}}\right)^{(\zeta q,\zeta r)}. \quad (5.0.4)$$

And the integral representation for the Jost solutions ψ and $\bar{\psi}$ are defined in (19) of [1] as

$$\psi = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\lambda x + \frac{i}{2}\int_x^\infty rq} + \int_x^\infty dy\, e^{i\lambda y} \begin{bmatrix} \sqrt{\lambda}\, K_1(x,y)\, e^{\frac{-i}{2}\int_x^\infty rq} \\ K_2(x,y)\, e^{\frac{i}{2}\int_x^\infty rq} \end{bmatrix}, \quad (5.0.5)$$

$$\bar{\psi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i\lambda x - \frac{i}{2}\int_x^\infty rq} + \int_x^\infty dy\, e^{-i\lambda y} \begin{bmatrix} \bar{K}_1(x,y)\, e^{\frac{-i}{2}\int_x^\infty rq} \\ \frac{1}{\sqrt{\lambda}}\bar{K}_2(x,y)\, e^{\frac{i}{2}\int_x^\infty rq} \end{bmatrix}. \quad (5.0.6)$$

As seen in (5.0.3), the Jost solutions defined in [1] are equal to the Jost solutions for the system (3.1.1). Hence, by using this fact we can express the quantities appearing in (5.0.5) and (5.0.6) such as $K_1(x,y)$, $K_2(x,y)$, $\bar{K}_1(x,y)$, and $\bar{K}_2(x,y)$ in terms of quantities appearing in the integral representation for Jost solutions to (3.1.1). Using (2.5.6) and the second equality of (2.5.28) in (3.2.44) we obtain

$$\psi^{(\zeta q,\zeta r)} = e^{i\mu/2} \begin{bmatrix} \sqrt{\lambda}\, E & 0 \\ \frac{i\, r(x)}{2}\, E & E^{-1} \end{bmatrix} \left(\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + \int_x^\infty dy\, e^{i\lambda y} \begin{bmatrix} K_1^{(u,v)}(x,y) \\ K_2^{(u,v)}(x,y) \end{bmatrix} \right). \quad (5.0.7)$$

Evaluating the right-hand side of (5.0.7) we get

$$\psi^{(\zeta q,\zeta r)} = \begin{bmatrix} \sqrt{\lambda}\, e^{i\mu/2}\, E \int_x^\infty dy\, K_1^{(u,v)}(x,y)\, e^{i\lambda y} \\ \\ e^{i\mu/2}\, E^{-1}\, e^{i\lambda x} + \frac{i\,r(x)}{2}\, e^{i\mu/2}\, E\, \int_x^\infty dy\, K_1^{(u,v)}(x,y)\, e^{i\lambda y} \\ +e^{i\mu/2}\, E^{-1}\, \int_x^\infty dy\, K_2^{(u,v)}(x,y)\, e^{i\lambda y} \end{bmatrix}. \tag{5.0.8}$$

Comparing the first entry of (5.0.5) and (5.0.8) we have

$$\int_x^\infty dy\, e^{i\lambda y}\, \sqrt{\lambda}\, K_1(x,y)\, e^{-\frac{i}{2}\int_x^\infty qr} = \sqrt{\lambda}\, e^{i\mu/2}\, E \int_x^\infty dy\, K_1^{(u,v)}(x,y)\, e^{i\lambda y}, \tag{5.0.9}$$

where we recall that $K_1(x,y)$ without the superscript refer to the quantity used in (20c) of [1] and $K_1^{(u,v)}(x,y)$ is the quantity used in (2.5.46). From (5.0.9) we obtain

$$K_1(x,y)\, e^{-\frac{i}{2}\int_x^\infty qr} = e^{i\mu/2}\, E\, K_1^{(u,v)}(x,y). \tag{5.0.10}$$

Using (3.1.6) and (3.2.13) in (5.0.10) we get

$$K_1(x,y)\, e^{-\frac{i}{2}\int_x^\infty qr} = e^{\frac{i}{2}\int_{-\infty}^\infty qr}\, e^{\frac{i}{2}\int_{-\infty}^x qr}\, K_1^{(u,v)}(x,y),$$

or equivalently

$$K_1(x,y) = e^{i\mu}\, K_1^{(u,v)}(x,y). \tag{5.0.11}$$

Using (2.5.46) and (3.1.7) in (5.0.11) we have

$$K_1(x,x) = e^{i\mu}\left[-\frac{q(x)}{2}\, E^{-2}\right], \tag{5.0.12}$$

or equivalently

$$q(x) = -2\, K_1(x,x)\, e^{i\int_x^\infty qr}. \tag{5.0.13}$$

Similarly, comparing the second entry of (5.0.5) and (5.0.8) we have

$$\begin{aligned} e^{i\lambda x}\, e^{\frac{i}{2}\int_x^\infty} + \int_x^\infty dy\, K_2(x,y)\, e^{\frac{i}{2}\int_x^\infty qr}\, e^{i\lambda y} = e^{i\mu/2}\, E^{-1}\, e^{i\lambda x} \\ + \frac{i\,r(x)}{2}\, e^{i\mu/2}\, E \int_x^\infty dy\, K_1^{(u,v)}(x,y)\, e^{i\lambda y} \\ + e^{i\mu/2}\, E^{-1} \int_x^\infty dy\, K_2^{(u,v)}(x,y)\, e^{i\lambda y}. \end{aligned} \tag{5.0.14}$$

Using (3.1.6) and (3.2.13) in (5.0.14) we get

$$\begin{aligned}\int_x^\infty dy\, K_2(x,y)\, e^{\frac{i}{2}\int_x^\infty qr}\, e^{i\lambda y} \\ &= \frac{i\, r(x)}{2}\, e^{\frac{i}{2}\int_{-\infty}^\infty qr}\, e^{\frac{i}{2}\int_{-\infty}^x qr} \int_x^\infty dy\, K_1^{(u,v)}(x,y)\, e^{i\lambda y} \\ &\quad + e^{\frac{i}{2}\int_x^\infty qr} \int_x^\infty dy\, K_2^{(u,v)}(x,y)\, e^{i\lambda y}.\end{aligned} \tag{5.0.15}$$

From (5.0.15) we have

$$\begin{aligned}K_2(x,y)\, e^{\frac{i}{2}\int_x^\infty qr} = \frac{i\, r(x)}{2}\, e^{\frac{i}{2}\int_{-\infty}^\infty qr}\, e^{\frac{i}{2}\int_{-\infty}^x qr}\, K_1^{(u,v)}(x,y) \\ + e^{\frac{i}{2}\int_x^\infty qr}\, K_2^{(u,v)}(x,y),\end{aligned} \tag{5.0.16}$$

or equivalently

$$K_2(x,y) = \frac{i\, r(x)}{2}\, e^{i\int_{-\infty}^x qr}\, K_1^{(u,v)}(x,y) + K_2^{(u,v)}(x,y), \tag{5.0.17}$$

Using (3.1.6) in (5.0.17), we obtain

$$K_2(x,y) = \frac{i\, r(x)}{2}\, E^2\, K_1^{(u,v)}(x,y) + K_2^{(u,v)}(x,y). \tag{5.0.18}$$

Next, we would like to express $L(x,y)$ appearing in Kaup-Newell formulation in term of the corresponding quantities for standard system (3.1.1). The $L(x,y)$ given in (22) of [1] as

$$L(x,y) = K_2(x,y) - \frac{i\, r(x)}{2}\, K_1(x,y)\, e^{-i\int_x^\infty qr}. \tag{5.0.19}$$

With the help of (3.2.13), (5.0.11), and (5.0.17), from (5.0.19) we have

$$\begin{aligned}L(x,y) =& \frac{i\, r(x)}{2}\, e^{i\int_{-\infty}^x qr}\, K_1^{(u,v)}(x,y) + K_2^{(u,v)}(x,y) \\ &- \frac{i\, r(x)}{2}\, K_1^{(u,v)}(x,y)\, e^{i\int_{-\infty}^\infty qr}\, e^{-i\int_x^\infty qr},\end{aligned} \tag{5.0.20}$$

which can be written as

$$L(x,y) = K_2^{(u,v)}(x,y). \tag{5.0.21}$$

Similarly, using (2.5.7) and the first equality of (2.5.7) in (3.2.88) we obtain

$$\bar{\psi}^{(\zeta q,\zeta r)} = e^{-i\mu/2} \begin{bmatrix} E & -\frac{i\,q(x)}{2} E \\ 0 & \sqrt{\lambda}\, E^{-1} \end{bmatrix} \left(\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + \int_x^\infty dy\, e^{-i\lambda y} \begin{bmatrix} \bar{K}_1^{(p,s)}(x,y) \\ \bar{K}_2^{(p,s)}(x,y) \end{bmatrix} \right). \quad (5.0.22)$$

Evaluating the right-hand side of (5.0.22) we get

$$\bar{\psi}^{(\zeta q,\zeta r)} = \begin{bmatrix} e^{-i\mu/2}\, E\, e^{-i\lambda x} + e^{-i\mu/2}\, E\, \int_x^\infty dy\, \bar{K}_1^{(p,s)}(x,y)\, e^{-i\lambda y} \\ -\frac{i\,q(x)}{2}\, e^{-i\mu/2}\, E^{-1}\, \int_x^\infty dy\, \bar{K}_2^{(p,s)}(x,y)\, e^{-i\lambda y} \\ \\ \frac{e^{-i\mu/2}}{\sqrt{\lambda}}\, E^{-1}\, \int_x^\infty dy\, \bar{K}_2^{(p,s)}(x,y)\, e^{-i\lambda y} \end{bmatrix}. \quad (5.0.23)$$

Comparing the first entry of (5.0.6) and (5.0.23) we have

$$\begin{aligned} e^{-i\lambda x}\, e^{-\frac{i}{2}\int_x^\infty} + \int_x^\infty dy\, \bar{K}_1(x,y)\, e^{-\frac{i}{2}\int_x^\infty qr}\, e^{-i\lambda y} \\ &= e^{-i\mu/2}\, E\, e^{-i\lambda x} \\ &\quad - \frac{i\,q(x)}{2}\, e^{-i\mu/2}\, E^{-1} \int_x^\infty dy\, \bar{K}_2^{(p,s)}(x,y)\, e^{-i\lambda y} \\ &\quad + e^{-i\mu/2}\, E \int_x^\infty dy\, \bar{K}_1^{(p,s)}(x,y)\, e^{-i\lambda y}. \end{aligned} \quad (5.0.24)$$

Using (3.1.6) and (3.2.13) in (5.0.24) we get

$$\begin{aligned} \int_x^\infty dy\, \bar{K}_1(x,y)\, e^{-\frac{i}{2}\int_x^\infty qr}\, e^{-i\lambda y} \\ &= -\frac{i\,q(x)}{2}\, e^{-\frac{i}{2}\int_{-\infty}^\infty qr}\, e^{-\frac{i}{2}\int_{-\infty}^x qr} \int_x^\infty dy\, \bar{K}_2^{(p,s)}(x,y)\, e^{i\lambda y} \\ &\quad + e^{-\frac{i}{2}\int_x^\infty qr} \int_x^\infty dy\, \bar{K}_1^{(p,s)}(x,y)\, e^{i\lambda y}. \end{aligned} \quad (5.0.25)$$

From (5.0.25) we have

$$\begin{aligned} \bar{K}_1(x,y)\, e^{-\frac{i}{2}\int_x^\infty qr} = -\frac{i\,q(x)}{2}\, e^{-\frac{i}{2}\int_{-\infty}^\infty qr}\, e^{-\frac{i}{2}\int_{-\infty}^x qr}\, \bar{K}_2^{(p,s)}(x,y) \\ + e^{-\frac{i}{2}\int_x^\infty qr}\, \bar{K}_1^{(p,s)}(x,y), \end{aligned} \quad (5.0.26)$$

or equivalently

$$\bar{K}_1(x,y) = -\frac{i\,q(x)}{2}\, e^{-i\int_{-\infty}^x qr}\, \bar{K}_2^{(p,s)}(x,y) + \bar{K}_1^{(p,s)}(x,y). \quad (5.0.27)$$

Comparing the second entry of (5.0.6) and (5.0.23) we obtain

$$\int_x^\infty dy\, \frac{e^{-i\lambda y}}{\sqrt{\lambda}}\, \bar{K}_2(x,y)\, e^{\frac{i}{2}\int_x^\infty q\, r} = \frac{e^{-i\mu/2}}{\sqrt{\lambda}}\, E^{-1} \int_x^\infty dy\, \bar{K}_2^{(p,s)}(x,y)\, e^{-i\lambda y}. \tag{5.0.28}$$

Using (3.1.6)and (3.2.13) in (5.0.28) we have

$$\bar{K}_2(x,y)\, e^{\frac{i}{2}\int_x^\infty qr} = e^{-\frac{i}{2}\int_{-\infty}^\infty qr}\, e^{\frac{-i}{2}\int_{-\infty}^x qr}\, \bar{K}_2^{(p,s)}(x,y),$$

or equivalently

$$\bar{K}_2(x,y) = e^{-i\mu}\, \bar{K}_2^{(p,s)}(x,y). \tag{5.0.29}$$

With the help of (2.5.47) and (3.1.36), from (5.0.29) we get

$$\bar{K}_2(x,x) = e^{-i\mu}\left[-\frac{r(x)}{2}\, E^2\right], \tag{5.0.30}$$

or equivalently

$$r(x) = -2\, \bar{K}_2(x,x)\, e^{i\int_x^\infty qr}. \tag{5.0.31}$$

From (5.0.13) and (5.0.31), we see that the potentials $q(x)$ and $r(x)$ appearing in the system (3.1.1) can be expressed explicitly in terms of $K_(x,y)$ and $\bar{K}_2(x,y)$. In the next theorem, we prove that $K_(x,y)$ and $\bar{K}_2(x,y)$ are the solution to the Marchenko integral equation associated with (3.1.1).

Theorem 5.1. *The Marchenko integral equations associated with* (3.1.1) *are given by*

$$K_1(x,y) + \bar{F}(x+y) + i\int_x^\infty dz \int_x^\infty dt\, K_1(x,t)\, \bar{F}(t+z)\, \frac{\partial F(z+y)}{\partial z} = 0, \quad y > x, \tag{5.0.32}$$

$$\bar{K}_2(x,y) + F(x+y) - i\int_x^\infty dz \int_x^\infty dt\, \bar{K}_2(x,y)\, F(t+z)\, \frac{\partial \bar{F}(z+y)}{\partial z} = 0, \quad y > x, \tag{5.0.33}$$

where

$$F(y) := \frac{1}{2\pi}\int_{-\infty}^\infty \frac{d\lambda}{\sqrt{\lambda}}\, R^{(\zeta q,\zeta r)}(\lambda)\, e^{i\lambda y}, \tag{5.0.34}$$

$$\bar{F}(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{\lambda}}\, \bar{R}^{(\zeta q,\zeta r)}(\lambda)\, e^{-i\lambda y}, \tag{5.0.35}$$

with

$$\frac{dF(y)}{dy} := \frac{i}{2\pi} \int_{-\infty}^{\infty} d\lambda\, \sqrt{\lambda}\, R^{(\zeta q,\zeta r)}(\lambda)\, e^{i\lambda y}, \tag{5.0.36}$$

$$\frac{d\bar{F}(y)}{dy} := -\frac{i}{2\pi} \int_{-\infty}^{\infty} d\lambda\, \sqrt{\lambda}\, \bar{R}^{(\zeta q,\zeta r)}(\lambda)\, e^{-i\lambda y}, \tag{5.0.37}$$

and

$$R^{(\zeta q,\zeta r)}(\lambda) = \frac{1}{\sqrt{\lambda}}\, R^{(u,v)}(\lambda) = \sqrt{\lambda}\, R^{(p.s)}(\lambda), \tag{5.0.38}$$

$$\bar{R}^{(\zeta q,\zeta r)}(\lambda) = \sqrt{\lambda}\, \bar{R}^{(u,v)}(\lambda) = \frac{1}{\sqrt{\lambda}}\, \bar{R}^{(u,v)}(\lambda), \tag{5.0.39}$$

where we recall that $\lambda = \zeta^2$*.*

Proof. In the absence of bound states, from (2.5.11) and (2.5.31) we have

$$K_1^{(u,v)}(x,y) + \hat{\bar{R}}^{(u,v)}(x+y) + \int_x^{\infty} dz\, \bar{K}_1^{(u,v)}(x,z)\, \hat{\bar{R}}^{(u,v)}(z+y) = 0, \qquad y > x. \tag{5.0.40}$$

Similarly, in the absence of bound states, from (2.5.10) and (2.5.29) we get

$$\bar{K}_1^{(u,v)}(x,y) + \int_x^{\infty} dz\, K_1^{(u,v)}(x,z)\, \hat{R}^{(u,v)}(z+y) = 0, \qquad y > x. \tag{5.0.41}$$

From (2.5.12) and (2.5.13) we have

$$\hat{R}^{(u,v)}(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, R^{(u,v)}(\lambda)\, e^{i\lambda y}, \tag{5.0.42}$$

$$\hat{\bar{R}}^{(u,v)}(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, \bar{R}^{(u,v)}(\lambda)\, e^{-i\lambda y}. \tag{5.0.43}$$

Using (3.3.44) in (5.0.42) we get

$$\hat{R}^{(u,v)}(y) = \frac{e^{i\mu}}{2\pi} \int_{-\infty}^{\infty} d\lambda\, \sqrt{\lambda}\, R^{(\zeta q,\zeta r)}(\lambda)\, e^{i\lambda y}. \tag{5.0.44}$$

With the help of (3.3.47), from (5.0.43), we obtain

$$\hat{\bar{R}}^{(u,v)}(y) = \frac{e^{-i\mu}}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{\lambda}} \bar{R}^{(\zeta q,\zeta r)}(\lambda)\, e^{-i\lambda y}. \tag{5.0.45}$$

Using (5.0.11) and (5.0.45) in (5.0.40) we have

$$\begin{aligned} e^{-i\mu} K_1(x,y) + \frac{e^{-i\mu}}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{\lambda}} \bar{R}^{(\zeta q, \zeta r)}(\lambda)\, e^{-i\lambda(x+y)} \\ + \frac{1}{2\pi} \int_x^{\infty} dz\, \bar{K}_1^{(u,v)}(x,z)\, e^{-i\mu} \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{\lambda}} \bar{R}^{(\zeta q, \zeta r)}(\lambda)\, e^{-i\lambda(z+y)} = 0, \end{aligned} \tag{5.0.46}$$

which yields

$$\begin{aligned} K_1(x,y) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{\lambda}} \bar{R}^{(\zeta q, \zeta r)}(\lambda)\, e^{-i\lambda(x+y)} \\ + \frac{1}{2\pi} \int_x^{\infty} dz\, \bar{K}_1^{(u,v)}(x,z) \int_{-\infty}^{\infty} \frac{d\lambda}{\sqrt{\lambda}} \bar{R}^{(\zeta q, \zeta r)}(\lambda)\, e^{-i\lambda(z+y)} = 0. \end{aligned} \tag{5.0.47}$$

Using (5.0.35) in (5.0.47) we get

$$K_1(x,y) + \bar{F}(x+y) + \int_x^{\infty} dz\, \bar{K}_1^{(u,v)}(x,z)\, \bar{F}(z+y) = 0. \tag{5.0.48}$$

With the help of (5.0.11) and (5.0.44), from (5.0.41) we have

$$\bar{K}_1^{(u,v)}(x,y) + \int_x^{\infty} dz\, e^{-i\mu}\, K_1(x,z) \left[\frac{e^{i\mu}}{2\pi} \int_{-\infty}^{\infty} d\lambda\, \sqrt{\lambda}\, R^{(\zeta q, \zeta r)}(\lambda)\, e^{i\lambda(z+y)} \right] = 0, \tag{5.0.49}$$

which yields

$$\bar{K}_1^{(u,v)}(x,y) + \frac{1}{2\pi} \int_x^{\infty} dz\, K_1(x,z) \int_{-\infty}^{\infty} d\lambda \sqrt{\lambda}\, R^{(\zeta q, \zeta r)}(\lambda)\, e^{i\lambda(z+y)} = 0. \tag{5.0.50}$$

Using (5.0.36) in (5.0.50) we obtain

$$\bar{K}_1^{(u,v)}(x,y) = i \int_x^{\infty} dz\, K_1(x,z)\, \frac{\partial F(z+y)}{\partial z} = 0. \tag{5.0.51}$$

With help of (5.0.51), from (5.0.48) we get

$$K_1(x,y) + \bar{F}(x+y) - i \int_x^{\infty} dz \int_x^{\infty} dt\, K_1(x,t)\, \bar{F}(t+z)\, \frac{\partial F(z+y)}{\partial z} = 0, \qquad y > x,$$

which establishes (5.0.32). In the same manner, we can prove (5.0.33). In the absence of bound states , from (2.5.10), (2.5.11), (2.5.30) and (2.5.32) we have

$$\bar{K}_2^{(p,s)}(x,y) + \hat{R}^{(p,s)}(x+y) + \int_x^{\infty} dz\, K_2^{(p,s)}(x,z)\, \hat{R}^{(p,s)}(z+y) = 0, \quad y > x, \tag{5.0.52}$$

$$K_2^{(p,s)}(x,y)+\int_x^\infty dz\,\bar{K}_2^{(p,s)}(x,z)\,\hat{\bar{R}}^{(p,s)}(z+y)=0,\qquad y>x. \tag{5.0.53}$$

From (2.5.12) and (2.5.13) we have

$$\hat{R}^{(p,s)}(y):=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,R^{(p,s)}(\lambda)\,e^{i\lambda y}, \tag{5.0.54}$$

$$\hat{\bar{R}}^{(p,s)}(y):=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,\bar{R}^{(p,s)}(\lambda)\,e^{-i\lambda y}. \tag{5.0.55}$$

Using (3.3.79) in (5.0.54) we get

$$\hat{R}^{(p,s)}(y)=\frac{e^{i\mu}}{2\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{\sqrt{\lambda}}\,R^{(\zeta q,\zeta r)}(\lambda)\,e^{i\lambda y}. \tag{5.0.56}$$

With the help of (3.3.82), from (5.0.55), we obtain

$$\hat{\bar{R}}^{(p,s)}(y)=\frac{e^{-i\mu}}{2\pi}\int_{-\infty}^{\infty}d\lambda\sqrt{\lambda}\bar{R}^{(\zeta q,\zeta r)}(\lambda)\,e^{-i\lambda y}. \tag{5.0.57}$$

Using (5.0.29) and (5.0.56) in (5.0.52) we have

$$\begin{aligned}&e^{i\mu}\,\bar{K}_2(x,y)+\frac{e^{i\mu}}{2\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{\sqrt{\lambda}}R^{(\zeta q,\zeta r)}(\lambda)\,e^{-i\lambda(x+y)}\\&\quad+\frac{1}{2\pi}\int_x^\infty dz\,K_2^{(p,s)}(x,z)\,e^{i\mu}\int_{-\infty}^{\infty}\frac{d\lambda}{\sqrt{\lambda}}\,R^{(\zeta q,\zeta r)}(\lambda)\,e^{-i\lambda(z+y)}=0,\end{aligned} \tag{5.0.58}$$

which yields

$$\begin{aligned}&\bar{K}_2(x,y)+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{\sqrt{\lambda}}R^{(\zeta q,\zeta r)}(\lambda)\,e^{-i\lambda(x+y)}\\&\qquad+\frac{1}{2\pi}\int_x^\infty dz\,K_2^{(p,s)}(x,z)\int_{-\infty}^{\infty}\frac{d\lambda}{\sqrt{\lambda}}\,R^{(\zeta q,\zeta r)}(\lambda)\,e^{-i\lambda(z+y)}=0.\end{aligned} \tag{5.0.59}$$

Using (5.0.34) in (5.0.59) we get

$$\bar{K}_2(x,y)+F(x+y)+\int_x^\infty dz\,K_2^{(p,s)}(x,z)\,F(z+y)=0. \tag{5.0.60}$$

With the help of (5.0.29) and (5.0.57), from (5.0.53) we have

$$K_2^{(p,s)}(x,y)+\int_x^\infty dz\,e^{i\mu}\,\bar{K}_2(x,z)\left[\frac{e^{-i\mu}}{2\pi}\int_{-\infty}^{\infty}d\lambda\,\sqrt{\lambda}\,\bar{R}^{(\zeta q,\zeta r)}(\lambda)\,e^{-i\lambda(z+y)}\right]=0, \tag{5.0.61}$$

which yields

$$K_2^{(p,s)}(x,y) + \frac{1}{2\pi}\int_x^\infty dz\, \bar{K}_2(x,z)\int_{-\infty}^\infty d\lambda\, \sqrt{\lambda}\, \bar{R}^{(\zeta q,\zeta r)}(\lambda)\, e^{-i\lambda(z+y)} = 0. \tag{5.0.62}$$

Using (5.0.37) in (5.0.62) we obtain

$$K_2^{(p,s)}(x,y) = -i\int_x^\infty dz\, \bar{K}_2(x,z)\, \frac{\partial \bar{F}(z+y)}{\partial z} = 0. \tag{5.0.63}$$

Substituting (5.0.63) in (5.0.60) we get

$$\bar{K}_2(x,y) + F(x+y) - i\int_x^\infty dz \int_x^\infty dt \bar{K}_2(x,y)\, F(t+z)\, \frac{\partial \bar{F}(z+y)}{\partial z} = 0, \qquad y > x,$$

which completes the proof of (5.0.33). □

Chapter 6

Applications to Integrable Systems

In the analysis of the scattering and inverse scattering for (1.0.1) the potentials q and r have been functions of x alone. In this chapter such potentials also depend on the parameter t and hence q and r denote $q(x,t)$ and $r(x,t)$, respectively. The same dependence is valid for other potentials as well.

The Inverse Scattering Transform (IST) method is a method to solve initial-value problems for integrable nonlinear partial differential equations (PDEs). It was introduced in 1967 by Gardner, Greene, Kruskal, and Miura [2] to solve the initial-value problem for the Korteweg-deVries (KdV) equation $u_t - 6uu_x + u_{xxx} = 0$, where the subscripts signify the respective x- and t- derivatives. Since then this method has been applied to various integrable nonlinear PDEs. In this chapter we illustrate the use of the IST method on (1.0.2). Kaup and Newell[1] considered (1.0.2) in the special case with $r(x,t) = \pm q(x,t)^*$, where we recall that an asterisk denotes complex conjugation.

In the formulation of the IST method by Gardner, Greene, Kruskal, and Miura a pair of linear differential operators $\mathcal{L}$ and $\mathcal{A}$ are associated with the corresponding integrable nonlinear partial differential equation, i.e. with the KdV equation. The operator $\mathcal{L}$ and $\mathcal{A}$ are known as the Lax pair, and they describe the spatial and temporal evolution, respectively, of the wave function in the associated linear problem. Another formulation [6] of the inverse scattering transform can be given by the so-called AKNS pair, i.e. by using a pair of matrices $\mathcal{X}$ and $\mathcal{T}$, which describe the spatial and temporal evolution, respectively. The AKNS pair for (1.0.2) is already known [1] and given by

$$\mathcal{X} = \begin{bmatrix} -i\zeta^2 & \zeta q \\ \zeta r & i\zeta^2 \end{bmatrix}, \tag{6.0.1}$$

$$\mathcal{T} = \begin{bmatrix} -2i\zeta^4 - iqr\zeta^2 & 2q\zeta^3 + (iq_x + q^2 r)\zeta \\ 2r\zeta^3 + (-ir_x + q\,r^2)\zeta & 2i\zeta^4 + iqr\zeta^2 \end{bmatrix}. \tag{6.0.2}$$

For the benefit of the reader, we show below how the system (1.0.2) is derived from $\mathcal{X}$ and $\mathcal{T}$ given in (6.0.1) and (6.0.2), respectively. We write the linear system (1.0.1) as

$$\Psi_x = \mathcal{X}\,\Psi, \tag{6.0.3}$$

and require [12] that $\Psi_t - \mathcal{T}\,\Psi$ be a solution to (6.0.3). The nonlinear system (1.0.2) is obtained from (6.0.1) and (6.0.2) via the condition

$$\mathcal{X}_t - \mathcal{T}_x + \mathcal{X}\,\mathcal{T} - \mathcal{T}\,\mathcal{X} = 0, \tag{6.0.4}$$

where the zero on the right-hand side denotes the 2×2 zero matrix. Note that the spectral parameter ζ does not depend on x or t. Taking the t-derivative of (6.0.1) and x-derivative of (6.0.2) we have

$$\mathcal{X}_t = \begin{bmatrix} 0 & \zeta q_t \\ \zeta r_t & 0 \end{bmatrix}, \tag{6.0.5}$$

$$\mathcal{T}_x = \begin{bmatrix} -iq_x r\zeta^2 - iqr_x\zeta^2 & 2q_x\zeta^3 + (iq_{xx} + 2qq_x r + q^2 r_x)\zeta \\ 2r_x\zeta^3 + (-ir_{xx} + q_x\,r^2 + 2qrr_x)\zeta & iq_x r\zeta^2 + iqr_x\zeta^2 \end{bmatrix}. \tag{6.0.6}$$

From (6.0.1) and (6.0.2) we obtain

$$\mathcal{X}\mathcal{T} = \begin{bmatrix} -2\zeta^6 + qr\zeta^4 - i(qr_x + iq^2r^2)\zeta^2 & q_x\zeta^3 \\ r_x\zeta^3 & -2\zeta^6 + qr\zeta^4 + i(rq_x - iq^2r^2)\zeta^2 \end{bmatrix}, \tag{6.0.7}$$

$$\mathcal{T}\mathcal{X} = \begin{bmatrix} -2\zeta^6 + qr\zeta^4 + i(q_x r - iq^2r^2)\zeta^2 & -q_x\zeta^3 \\ -r_x\zeta^3 & -2\zeta^6 + rq\zeta^4 - i(qr_x + iq^2r^2)\zeta^2 \end{bmatrix}. \quad (6.0.8)$$

Using (6.0.5)-(6.0.8) in (6.0.4) and expanding the left-hand side of (6.0.4) in powers of ζ we obtain

$$i\zeta \begin{bmatrix} 0 & iq_t + q_{xx} - i(q^2r)_x \\ ir_t - r_{xx} - i(qr^2)_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (6.0.9)$$

Since (6.0.9) must hold for all $\zeta \in \mathbb{R}$, we get the derivation NLS system (1.0.2), which we quote below for the convenience of the reader as

$$\begin{cases} iq_t + q_{xx} - i(qrq)_x = 0, \\ ir_t - r_{xx} - i(rqr)_x = 0, \end{cases} \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}^+. \quad (6.0.10)$$

Choosing $r(x) = \pm q(x)^*$ in (6.0.10) we get (1.0.3).

There is another version of the derivative NLS system, known as the Chen-Lee-Liu system [9, 19], and it is given by

$$\begin{cases} i\tilde{q}_t + \tilde{q}_{xx} - i\tilde{q}\tilde{r}\tilde{q}_x = 0, \\ i\tilde{r}_t - \tilde{r}_{xx} - i\tilde{r}\tilde{q}\tilde{r}_x = 0, \end{cases} \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}^+. \quad (6.0.11)$$

The system (6.0.11) is integrable, and its corresponding AKNS pair $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{T}}$ are known [18] and we have

$$\tilde{\mathcal{X}} = \begin{bmatrix} -i\zeta^2 & \zeta\tilde{q} \\ \zeta\tilde{q} & i\zeta^2 + \frac{i\tilde{q}\tilde{r}}{2} \end{bmatrix}, \quad (6.0.12)$$

$$\tilde{\mathcal{T}} = \begin{bmatrix} -2i\zeta^4 - i\tilde{q}\tilde{r}\zeta^2 & 2\tilde{q}\zeta^3 + (i\tilde{q}_x + \frac{\tilde{q}^2\tilde{r}}{2})\zeta \\ 2\tilde{r}\zeta^3 + (-i\tilde{r}_x + \frac{\tilde{q}\tilde{r}^2}{2})\zeta & 2i\zeta^4 + i\tilde{r}\tilde{q}\zeta^2 + \frac{1}{2}(\tilde{r}_x\tilde{q} - \tilde{r}\tilde{q}_x) + \frac{i}{4}\tilde{r}^2\tilde{q}^2 \end{bmatrix}. \quad (6.0.13)$$

From (6.0.12) we see that the matrix operator $\tilde{\mathcal{X}}$ appears in the linear system $\tilde{\Psi}_x = \tilde{\mathcal{X}}\,\tilde{\Psi}$, which we write as

$$\begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & \zeta\tilde{q} \\ \zeta\tilde{r} & i\zeta^2 + \frac{i\tilde{q}\tilde{r}}{2} \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}, \tag{6.0.14}$$

where we recall that the prime denotes the x-derivative. Note that the matrix $\tilde{\mathcal{X}}$ given in (6.0.12) does not have its trace equal to zero. Hence, the Wronskian of two solutions to (6.0.14) in general depends on x. As a result, one cannot express a scattering coefficient as a Wronskian of two solutions to (6.0.14). In chapter 2 we have expressed the scattering coefficients for (1.0.4) in terms of Wronskians of certain solutions to (1.0.4). This was possible because the trace of the coefficient matrix in (1.0.4) is zero. One way to determine the scattering coefficients for (6.0.14) and to analyze the scattering and inverse scattering for (6.0.14) is to establish the connection between (1.0.1) and (6.0.14), as we have established a connection between (1.0.1) and (1.0.4).

In the next theorem, we present the transformation for the wave functions between the first-order systems (1.0.1) and (6.0.14). If the potentials q, r, $\tilde{q}$, and $\tilde{r}$ depend on bath x and t, then the wave functions in (1.0.1) and (6.0.14) also depend on x and t. On the other hand, t is a parameter, i.e. in the linear systems (1.0.1) and (6.0.14). We have the derivative with respect to x but not with respect to t. Thus, when we say a potential belongs to the Schwartz class $\mathbb{S}(\mathbb{R})$, we mean that for each fixed t, the potential as a function of x belongs to the Schwartz class.

Theorem 6.1. *Assume that the potentials $\tilde{q}$ and $\tilde{r}$ appearing in the system (6.0.11) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the wave functions for first order systems (1.0.1) and (6.0.14) are related to each other as*

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}, \tag{6.0.15}$$

where ε is scalar quantity depending on ζ but not on x, and

$$\tilde{E}(x,t) := \exp\left(\frac{i}{2} \int_{-\infty}^{x} dy\, \tilde{q}(y,t)\, \tilde{r}(y,t) \right), \tag{6.0.16}$$

$$q(x,t) = \tilde{q}(x,t)\, \tilde{E}(x,t), \tag{6.0.17}$$

$$r(x,t) = \tilde{r}(x,t) \left[\tilde{E}(x,t) \right]^{-1}. \tag{6.0.18}$$

Furthermore, we have

$$E(x,t) = \tilde{E}(x,t), \tag{6.0.19}$$

where $E(x,t)$ is the quantity defined in (3.1.6).

Proof. For convenience, we suppress the x- and t-dependence on the left-hand side of (6.0.16) and write $\tilde{E}$ instead of $\tilde{E}(x,t)$. It is enough to verify (6.0.15) by showing that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ in (6.0.15) satisfies (1.0.1) and $\begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}$ in (6.0.15) satisfies (6.0.14). Assuming that the wave function $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ satisfies (1.0.1), we show below that the wave function $\begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}$ obtained as in (6.0.15) satisfies (6.0.14). Isolating $\begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}$ from (6.0.15), we see that (6.0.14) yields

$$\frac{1}{\varepsilon} \left(\begin{bmatrix} 1 & 0 \\ 0 & \tilde{E} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right)' = \begin{bmatrix} -i\zeta^2 & \zeta\tilde{q} \\ \zeta\tilde{r} & i\zeta^2 + \frac{i\tilde{q}\tilde{r}}{2} \end{bmatrix} \frac{1}{\varepsilon} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{6.0.20}$$

From (6.0.20) we get

$$\begin{bmatrix} 0 & 0 \\ 0 & \frac{i\tilde{q}\tilde{r}}{2}\tilde{E} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & \zeta\tilde{q} \\ \zeta\tilde{r} & i\zeta^2 + \frac{i\tilde{q}\tilde{r}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

which can be written as

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \left(\begin{bmatrix} -i\zeta^2 & \zeta\tilde{q}\tilde{E} \\ \zeta\tilde{r} & \left(i\zeta^2 + \frac{i\tilde{q}\tilde{r}}{2}\right)\tilde{E} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \frac{i\tilde{q}\tilde{r}}{2}\tilde{E} \end{bmatrix} \right) \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \begin{bmatrix} -i\zeta^2 & \zeta\tilde{q}\tilde{E} \\ \zeta\tilde{r} & i\zeta^2\tilde{E} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{6.0.21}$$

Simplifying (6.0.21) through a matrix multiplication, we get

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}' = \begin{bmatrix} -i\zeta^2 & \zeta\tilde{q}\tilde{E} \\ \zeta\tilde{r}\tilde{E}^{-1} & i\zeta^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{6.0.22}$$

Comparing (6.0.22) and (1.0.1) we see that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ satisfies (1.0.1) if and only if (6.0.17) and (6.0.18) are satisfied. To complete the proof, we observe from (6.0.17) and (6.0.18) that

$$q(x,t)\, r(x,t) = \tilde{q}(x,t)\, \tilde{r}(x,t), \tag{6.0.23}$$

and hence from (3.1.6), (6.0.16), and (6.0.23) we obtain (6.0.19). □

The use of the scalar ε in (6.0.15) may seen unnecessary. We will see later for specific solutions to (6.0.14) the value of ε will be different. Let us also remark that from (6.0.17) and (6.0.18) it follows that the potential pair q and r of (1.0.1) belongs to the Schwartz class $\mathbb{S}(\mathbb{R})$ if and only if the potential pair $\tilde{q}$ and $\tilde{r}$ belongs to the Schwartz class.

For the convenience of the reader, we illustrate the derivation of the AKNS pair $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{T}}$ given in (6.0.12) and (6.0.13). This will be done by obtaining $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{T}}$ interms of the AKNS pair $\mathcal{X}$ and $\mathcal{T}$ associated with (1.0.1), and this done in the next proposition.

Proposition 6.1. *Assume that the potentials $\tilde{q}$ and $\tilde{r}$ appearing in the system* (6.0.11) *belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the AKNS pair $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{T}}$ associated with* (6.0.11) *is given by* (6.0.12) *and* (6.0.13)*, respectively.*

Proof. We assume that $\mathcal{X}$ and $\mathcal{T}$ given in (6.0.12) and (6.0.13) form the AKNS pair for the system (1.0.2). This means that we have

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_x = \mathcal{X}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}, \qquad \begin{bmatrix}\alpha\\ \beta\end{bmatrix}_t = \mathcal{T}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}, \tag{6.0.24}$$

Now, let us write a system of linear equations for (1.0.1) as

$$\begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix}_x = \tilde{\mathcal{X}}\begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix}, \qquad \begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix}_t = \tilde{\mathcal{T}}\begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix}, \tag{6.0.25}$$

where $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{T}}$ form the AKNS pair for (6.0.11). Using (6.0.15) in the first equality of (6.0.24) we have

$$\left(\varepsilon\begin{bmatrix}1 & 0\\ 0 & \tilde{E}^{-1}\end{bmatrix}\begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix}\right)_x = \mathcal{X}\left(\varepsilon\begin{bmatrix}1 & 0\\ 0 & \tilde{E}^{-1}\end{bmatrix}\begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix}\right). \tag{6.0.26}$$

Evaluating the left hand-side of (6.0.26) we get

$$\begin{bmatrix}0 & 0\\ 0 & -\frac{i\tilde{q}\tilde{r}}{2}\tilde{E}^{-1}\end{bmatrix}\begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & \tilde{E}^{-1}\end{bmatrix}\begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix}_x = \mathcal{X}\left(\begin{bmatrix}1 & 0\\ 0 & \tilde{E}^{-1}\end{bmatrix}\begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix}\right), \tag{6.0.27}$$

or equivalently

$$\begin{bmatrix}1 & 0\\ 0 & \tilde{E}^{-1}\end{bmatrix}\begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix}_x = \left(\mathcal{X}\begin{bmatrix}1 & 0\\ 0 & \tilde{E}^{-1}\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & -\frac{i\tilde{q}\tilde{r}}{2}\tilde{E}^{-1}\end{bmatrix}\right)\begin{bmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{bmatrix}. \tag{6.0.28}$$

Multiplying both sides of (6.0.28) with the inverse of first matrix on the left-hand of (6.0.28) we obtain

$$\begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}_x = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E} \end{bmatrix} \left(\mathcal{X} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & -\frac{i\tilde{q}\tilde{r}}{2}\tilde{E}^{-1} \end{bmatrix} \right) \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}. \tag{6.0.29}$$

Comparing (6.0.29) with the first equality in (6.0.25) we get

$$\tilde{\mathcal{X}} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E} \end{bmatrix} \left(\mathcal{X} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & -\frac{i\tilde{q}\tilde{r}}{2}\tilde{E}^{-1} \end{bmatrix} \right). \tag{6.0.30}$$

Using (6.0.1) in (6.0.30) we have

$$\tilde{\mathcal{X}} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E} \end{bmatrix} \left(\begin{bmatrix} -i\zeta^2 & \zeta q \\ \zeta r & i\zeta^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & -\frac{i\tilde{q}\tilde{r}}{2}\tilde{E}^{-1} \end{bmatrix} \right). \tag{6.0.31}$$

With the help of (6.0.17) and (6.0.18), from (6.0.31) we obtain

$$\tilde{\mathcal{X}} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E} \end{bmatrix} \left(\begin{bmatrix} -i\zeta^2 & \zeta\tilde{q}\,\tilde{E} \\ \zeta\tilde{r}\,\tilde{E}^{-1} & i\zeta^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & -\frac{i\tilde{q}\tilde{r}}{2}\tilde{E}^{-1} \end{bmatrix} \right). \tag{6.0.32}$$

Simplifying the right-hand side of (6.0.32) we have

$$\tilde{\mathcal{X}} = \begin{bmatrix} -i\zeta^2 & \zeta\tilde{q} \\ \zeta\tilde{q} & i\zeta^2 + \frac{i\tilde{q}\tilde{r}}{2} \end{bmatrix},$$

which establishes (6.0.12). Now, let us prove (6.0.13). Using (6.0.15) in the second equality of (6.0.24) we get

$$\left(\varepsilon \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix} \right)_t = \mathcal{T} \left(\varepsilon \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix} \right). \tag{6.0.33}$$

Evaluating the left hand-side of (6.0.33) we have

$$\begin{bmatrix} 0 & 0 \\ 0 & (\tilde{E}^{-1})_t \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}_t = \mathcal{T} \left(\begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix} \right),$$

which can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}_t = \left(\mathcal{T} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & (\tilde{E}^{-1})_t \end{bmatrix} \right) \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix},$$

which simplifies to

$$\begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}_t = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E} \end{bmatrix} \left(\mathcal{T} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & (\tilde{E}^{-1})_t \end{bmatrix} \right) \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}. \tag{6.0.34}$$

Comparing the second equality in (6.0.25) and (6.0.34) we obtain

$$\tilde{\mathcal{T}} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E} \end{bmatrix} \left(\mathcal{T} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & (\tilde{E}^{-1})_t \end{bmatrix} \right). \tag{6.0.35}$$

Now, let us find the t-derivative of $\tilde{E}^{-1}$. From (6.0.16)

$$\left[\tilde{E}(x,t)\right]^{-1} = \exp\left(-\frac{i}{2} \int_{-\infty}^{x} dy\, \tilde{q}(y,t)\, \tilde{r}(y,t) \right), \tag{6.0.36}$$

Taking t-derivative of (6.0.36) we get

$$\left(\tilde{E}^{-1}\right)_t = \left[-\frac{i}{2} \int_{-\infty}^{x} (\tilde{q}_t\, \tilde{r} + \tilde{q}\, \tilde{r}_t) \right] \tilde{E}^{-1},$$

or equivalently

$$\left(\tilde{E}^{-1}\right)_t = -\frac{\tilde{E}^{-1}}{2} \int_{-\infty}^{x} (i\tilde{q}_t\, \tilde{r} + \tilde{q}\, i\tilde{r}_t). \tag{6.0.37}$$

With the help of (6.0.11), from (6.0.37) we have

$$\left(\tilde{E}^{-1}\right)_t = -\frac{\tilde{E}^{-1}}{2} \int_{-\infty}^{x} \left[(-\tilde{q}_{yy} + i\tilde{q}\tilde{r}\tilde{q}_y)\tilde{r} + (\tilde{r}_{yy} + i\tilde{r}\tilde{q}\tilde{r}_y)\tilde{q} \right],$$

or equivalently

$$\left(\tilde{E}^{-1}\right)_t = -\frac{\tilde{E}^{-1}}{2} \int_{-\infty}^{x} \left[(-\tilde{q}_y\tilde{r} + \tilde{q}\tilde{r}_y)_y + \frac{i}{2}(\tilde{q}^2\tilde{r}^2)_y \right].$$

Thus, we obtain

$$\left(\tilde{E}^{-1}\right)_t = -\frac{\tilde{E}^{-1}}{2} \left[(-\tilde{q}_x\tilde{r} + \tilde{q}\tilde{r}_x) + \frac{i}{2}(\tilde{q}^2\tilde{r}^2) \right]. \tag{6.0.38}$$

Using (6.0.2) and (6.0.38) in (6.0.35) we get

$$\begin{aligned}\tilde{\mathcal{T}} = &\begin{bmatrix}1 & 0\\ 0 & \tilde{E}\end{bmatrix}\begin{bmatrix}-2i\zeta^4 - iqr\zeta^2 & 2q\zeta^3 + (iq_x + q^2 r)\zeta\\ 2r\zeta^3 + (-ir_x + q\,r^2)\zeta & 2i\zeta^4 + iqr\zeta^2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \tilde{E}^{-1}\end{bmatrix}\\ &- \begin{bmatrix}1 & 0\\ 0 & \tilde{E}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & -\frac{\tilde{E}^{-1}}{2}\left[(-\tilde{q}_x\tilde{r} + \tilde{q}\tilde{r}_x) + \frac{i}{2}(\tilde{q}^2\tilde{r}^2)\right]\end{bmatrix}.\end{aligned} \tag{6.0.39}$$

which simplifies to

$$\begin{aligned}\tilde{\mathcal{T}} = &\begin{bmatrix}-2i\zeta^4 - iqr\zeta^2 & 2q\tilde{E}^{-1}\zeta^3 + (iq_x + q^2 r)\tilde{E}^{-1}\zeta\\ 2r\tilde{E}\zeta^3 + (-ir_x + q\,r^2)\tilde{E}\zeta & 2i\zeta^4 + iqr\zeta^2\end{bmatrix}\\ &- \begin{bmatrix}0 & 0\\ 0 & -\frac{1}{2}\left[(-\tilde{q}_x\tilde{r} + \tilde{q}\tilde{r}_x) + \frac{i}{2}(\tilde{q}^2\tilde{r}^2)\right]\end{bmatrix}.\end{aligned} \tag{6.0.40}$$

Using (6.0.17) and (6.0.18) in (6.0.40) we obtain

$$\begin{aligned}\tilde{\mathcal{T}} = &\begin{bmatrix}-2i\zeta^4 - i\tilde{q}\tilde{r}\zeta^2 & 2\tilde{q}\zeta^3 + (i\tilde{q}_x - \frac{1}{2}\tilde{q}\tilde{r} + \tilde{q}^2\tilde{r})\zeta\\ 2\tilde{r}\zeta^3 + (-i\tilde{r}_x - \frac{1}{2}\tilde{q}\tilde{r} + \tilde{q}\tilde{r}^2)\zeta & 2i\zeta^4 + i\tilde{q}\tilde{r}\zeta^2\end{bmatrix}\\ &- \begin{bmatrix}0 & 0\\ 0 & -\frac{1}{2}\left[(-\tilde{q}_x\tilde{r} + \tilde{q}\tilde{r}_x) + \frac{i}{2}(\tilde{q}^2\tilde{r}^2)\right]\end{bmatrix},\end{aligned} \tag{6.0.41}$$

which yields

$$\tilde{\mathcal{T}} = \begin{bmatrix}-2i\zeta^4 - i\tilde{q}\tilde{r}\zeta^2 & 2\tilde{q}\zeta^3 + (i\tilde{q}_x - \frac{1}{2}\tilde{q}\tilde{r} + \tilde{q}^2\tilde{r})\zeta\\ 2\tilde{r}\zeta^3 + (-i\tilde{r}_x - \frac{1}{2}\tilde{q}\tilde{r} + \tilde{q}\tilde{r}^2)\zeta & 2i\zeta^4 + i\tilde{r}\tilde{q}\zeta^2 + \frac{1}{2}(\tilde{r}_x\tilde{q} - \tilde{r}\tilde{q}_x) + \frac{i}{4}\tilde{r}^2\tilde{q}^2\end{bmatrix}, \tag{6.0.42}$$

which completes the proof of (6.0.13). □

6.1 Jost Solutions

There are four particular column-vector solutions to (6.0.14), known as the Jost solutions and denoted by ϕ, ψ, $\bar{\phi}$, $\bar{\psi}$, respectively, which are uniquely determined by imposing the asymptotic conditions

$$\phi^{(\zeta\tilde{q},\zeta\tilde{r})} = \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1), \quad \bar{\phi}^{(\zeta\tilde{q},\zeta\tilde{r})} = \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{6.1.1}$$

$$\psi^{(\zeta\tilde{q},\zeta\tilde{r})} = \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + o(1), \quad \bar{\psi}^{(\zeta\tilde{q},\zeta\tilde{r})} = \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty. \tag{6.1.2}$$

Note that there are two first-order systems we deal with, and they are given in (1.0.1) and (6.0.14), respectively. Each of these two systems have their own four Jost solutions, scattering coefficients, and a pair of potentials. For (1.0.1) we know from (2.2.14) and (2.2.15) that left and right transmission coefficients are equal and we have used T and $\bar{T}$ to denote their common value. In this chapter we find that the left and right transmission coefficients for (6.0.14) are not equal to each other and hence we use $T_{\rm r}^{(\zeta\tilde{q},\zeta\tilde{r})}$ and $T_{\rm l}^{(\zeta\tilde{q},\zeta\tilde{r})}$, and similarly $\bar{T}_{\rm r}^{(\zeta\tilde{q},\zeta\tilde{r})}$ and $\bar{T}_{\rm l}^{(\zeta\tilde{q},\zeta\tilde{r})}$. In order to avoid any confusion, we will label the Jost solutions and the scattering coefficients for each of the two system by using the superscripts $(\zeta q, \zeta r)$,$(\zeta\tilde{q}, \zeta\tilde{r})$, which identically the corresponding potentials. For example, for the system (6.0.14) we use $\phi^{(\zeta\tilde{q},\zeta\tilde{r})}, \psi^{(\zeta\tilde{q},\zeta\tilde{r})}, \bar{\phi}^{(\zeta\tilde{q},\zeta\tilde{r})}, \bar{\psi}^{(\zeta\tilde{q},\zeta\tilde{r})}$ to denote the corresponding Jost solutions and we use $T_{\rm r}^{(\zeta\tilde{q},\zeta\tilde{r})}$, $T_{\rm l}^{(\zeta\tilde{q},\zeta\tilde{r})}$, $R^{(\zeta\tilde{q},\zeta\tilde{r})}$, $L^{(\zeta\tilde{q},\zeta\tilde{r})}$, $\bar{T}_{\rm r}^{(\zeta\tilde{q},\zeta\tilde{r})}$, $\bar{T}_{\rm l}^{(\zeta\tilde{q},\zeta\tilde{r})}$, $\bar{R}^{(\zeta\tilde{q},\zeta\tilde{r})}$, $\bar{L}^{(\zeta\tilde{q},\zeta\tilde{r})}$ to denote the corresponding scattering coefficients.

In the next proposition, we show the relationship between the corresponding Jost solutions for the first-order system (1.0.1) and the first-order system (6.0.14).

Proposition 6.2. *Assume that the potentials $\tilde{q}$ and $\tilde{r}$ appearing in the system* (6.0.14) *belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the Jost solutions $\phi^{(\zeta q,\zeta r)}$, $\psi^{(\zeta q,\zeta r)}$, $\bar{\phi}^{(\zeta q,\zeta r)}$, $\bar{\psi}^{(\zeta q,\zeta r)}$ to* (1.0.1)*, and the Jost solutions $\phi^{(\zeta\tilde{q},\zeta\tilde{r})}$, $\psi^{(\zeta\tilde{q},\zeta\tilde{r})}$, $\bar{\phi}^{(\zeta\tilde{q},\zeta\tilde{r})}$, $\bar{\psi}^{(\zeta\tilde{q},\zeta\tilde{r})}$ to* (6.0.14) *are related to each other as*

$$\phi^{(\zeta q,\zeta r)} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \phi^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.1.3}$$

$$\psi^{(\zeta q,\zeta r)} = e^{i\tilde{\mu}/2} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \psi^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.1.4}$$

$$\bar{\phi}^{(\zeta q,\zeta r)} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \bar{\phi}^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.1.5}$$

$$\bar{\psi}^{(\zeta q,\zeta r)} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \bar{\psi}^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.1.6}$$

where $\tilde{E}$ is the quantity defined in (6.0.16),

$$\tilde{\mu} := \int_{-\infty}^{\infty} \tilde{q}\,\tilde{r}. \tag{6.1.7}$$

Furthermore, the scalar quantity $\tilde{\mu}$ defined in (6.1.7) *is equal to the quantity μ defined in* (3.2.13).

Proof. Note that from (6.0.16) and (6.1.7) we have

$$\tilde{E} \to 1, \qquad x \to -\infty; \quad \tilde{E} \to e^{i\tilde{\mu}/2}, \qquad x \to +\infty. \tag{6.1.8}$$

First, let us relate the Jost solution $\phi^{(\zeta q,\zeta r)}$ to (1.0.1) and the Jost solution $\phi^{(\zeta\tilde{q},\zeta\tilde{r})}$ to (6.0.14) to each other. From (6.0.15) we have

$$\phi^{(\zeta q,\zeta r)} = \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \phi^{(\zeta\tilde{q},\zeta\tilde{r})}. \tag{6.1.9}$$

Using the first equalities of (3.2.1), (6.1.1), and (6.1.8) in (6.1.9) we get

$$\begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1) = \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{6.1.10}$$

and hence $\varepsilon = 1$. Using $\varepsilon = 1$ in (6.1.9) we obtain

$$\phi^{(\zeta q,\zeta r)} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \phi^{(\zeta\tilde{q},\zeta\tilde{r})}. \tag{6.1.11}$$

which establishes (6.1.3). Similarly, let us relate the Jost solution $\psi^{(\zeta q,\zeta r)}$ to (1.0.1) and the Jost solution $\psi^{(\zeta\tilde q,\zeta\tilde r)}$ to (6.0.14) to each other. From (6.0.15) we have

$$\psi^{(\zeta q,\zeta r)}=\varepsilon\begin{bmatrix}1&0\\0&\tilde E^{-1}\end{bmatrix}\psi^{(\zeta\tilde q,\zeta\tilde r)}. \qquad (6.1.12)$$

With the help of the first equalities of (3.2.2), (6.1.2), and the second equality in (6.1.8), from (6.1.12) we obtain

$$\begin{bmatrix}0\\e^{i\zeta^2x}\end{bmatrix}+o(1)=\varepsilon\begin{bmatrix}1&0\\0&e^{-i\tilde\mu/2}\end{bmatrix}\begin{bmatrix}0\\e^{i\zeta^2x}\end{bmatrix}+o(1),\qquad x\to+\infty, \qquad (6.1.13)$$

and hence $\varepsilon=e^{i\tilde\mu/2}$. Using $\varepsilon=e^{-i\tilde\mu/2}$ in (6.1.12) we have

$$\psi^{(\zeta q,\zeta r)}=e^{i\tilde\mu/2}\begin{bmatrix}1&0\\0&\tilde E^{-1}\end{bmatrix}\psi^{(\zeta\tilde q,\zeta\tilde r)}. \qquad (6.1.14)$$

which establishes (6.1.4). In the same manner, let us relate the Jost solution $\bar\phi^{(\zeta q,\zeta r)}$ to (1.0.1) and the Jost solution $\bar\phi^{(\zeta\tilde q,\zeta\tilde r)}$ to (6.0.14) to each other. With the help of (6.0.15) we have

$$\bar\phi^{(\zeta q,\zeta r)}=\varepsilon\begin{bmatrix}1&0\\0&\tilde E^{-1}\end{bmatrix}\bar\phi^{(\zeta\tilde q,\zeta\tilde r)}. \qquad (6.1.15)$$

Using the second equalities of (3.2.1), (6.1.1), and the first equality of (6.1.8) in (6.1.15) we get

$$\begin{bmatrix}0\\e^{i\zeta^2x}\end{bmatrix}+o(1)=\varepsilon\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0\\e^{i\zeta^2x}\end{bmatrix}+o(1),\qquad x\to-\infty, \qquad (6.1.16)$$

and hence $\varepsilon=1$. Using $\varepsilon=1$ in (6.1.15) we can rewrite (6.1.15) as

$$\bar\phi^{(\zeta q,\zeta r)}=\begin{bmatrix}1&0\\0&\tilde E^{-1}\end{bmatrix}\bar\phi^{(\zeta\tilde q,\zeta\tilde r)}. \qquad (6.1.17)$$

which establishes (6.1.5). Now, let us relate the Jost solution $\bar{\psi}^{(\zeta q,\zeta r)}$ to (1.0.1) and the Jost solution $\bar{\psi}^{(\zeta\tilde{q},\zeta\tilde{r})}$ to (6.0.14) to each other. From (6.0.15) we have

$$\bar{\psi}^{(\zeta q,\zeta r)} = \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \bar{\psi}^{(\zeta\tilde{q},\zeta\tilde{r})}. \qquad (6.1.18)$$

Substituting the second equalities of (3.2.2), (6.1.2), and (3.2.14) in (6.1.18) we obtain

$$\begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1) = \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\tilde{\mu}/2} \end{bmatrix} \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + o(1), \qquad x \to +\infty, \qquad (6.1.19)$$

and hence $\varepsilon = 1$. Using this value of ε in (6.1.18) we get

$$\bar{\psi}^{(\zeta q,\zeta r)} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{E}^{-1} \end{bmatrix} \bar{\psi}^{(\zeta\tilde{q},\zeta\tilde{r})}. \qquad (6.1.20)$$

which completes the proof of (6.1.6). The equivalence of μ and $\tilde{\mu}$ given in (3.2.13) and (6.1.7), respectively, follows directly from (6.0.23). □

6.2 Scattering Coefficients

The scattering coefficients can be defined by using the x-asymptotics of the Jost solutions. Since the potentials $\tilde{q}$ and $\tilde{r}$ appearing in (6.0.14) belong to Schwartz class $\mathbb{S}(\mathbb{R})$. we have

$$\phi^{(\zeta\tilde{q},\zeta\tilde{r})} = \begin{bmatrix} \frac{1}{T_{\rm r}(\zeta)} e^{-i\lambda x} \\ \frac{R(\zeta)}{T_{\rm r}(\zeta)} e^{i\lambda x} \end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})} + o(1), \qquad x \to +\infty, \qquad (6.2.1)$$

$$\bar{\phi}^{(\zeta\tilde{q},\zeta\tilde{r})} = \begin{bmatrix} \frac{\bar{R}(\zeta)}{\bar{T}_{\rm r}(\lambda)} e^{-i\lambda x} \\ \frac{1}{\bar{T}_{\rm r}(\zeta)} e^{i\lambda x} \end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})} + o(1), \qquad x \to +\infty, \qquad (6.2.2)$$

$$\psi^{(\zeta\tilde{q},\zeta\tilde{r})} = \begin{bmatrix} \frac{L(\zeta)}{T_{\rm l}(\zeta)} e^{-i\lambda x} \\ \frac{1}{T_{\rm l}(\zeta)} e^{i\lambda x} \end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})} + o(1), \qquad x \to -\infty, \qquad (6.2.3)$$

$$\bar{\psi}^{(\zeta\tilde{q},\zeta\tilde{r})} = \begin{bmatrix} \frac{1}{\bar{T}_{\rm l}(\zeta)} e^{-i\lambda x} \\ \frac{\bar{L}(\zeta)}{\bar{T}_{\rm l}(\zeta)} e^{i\lambda x} \end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})} + o(1), \qquad x \to -\infty, \tag{6.2.4}$$

where we recall that $T_{\rm l}$ and $T_{\rm r}$ are transmission coefficients from left and from right, respectively, and the pair L and $\bar{L}$ and the pair R and $\bar{R}$ are the reflection coefficients from the left and from the right, respectively. Note that there are two first-order systems we deal with, and they are given in (1.0.1) and (6.0.14), respectively. Each of these two systems have scattering coefficients. In order to avoid any confusion, we label the scattering coefficients for each of the two system by using the superscripts $(\zeta q, \zeta r)$ and $(\zeta\tilde{q}, \zeta\tilde{r})$, which identically the corresponding potentials. For example, for the system (1.0.1) we use $T^{(\zeta q,\zeta r)}$, $R^{(\zeta q,\zeta r)}$, $L^{(\zeta q,\zeta r)}$, $\bar{T}^{(\zeta q,\zeta r)}$, $\bar{R}^{(\zeta q,\zeta r)}$, $\bar{L}^{(\zeta q,\zeta r)}$ to denote the corresponding scattering coefficients.

In the next proposition, we show the relationship among the scattering coefficients for the first-order system (1.0.1) and the standard system (6.0.14).

Proposition 6.3. *Assume that the potentials $q(x)$ and $r(x)$ appearing in the system (1.0.1) belong to the Schwartz class $\mathbb{S}(\mathbb{R})$. Then, the scattering coefficients $T^{(\zeta q,\zeta r)}, R^{(\zeta q,\zeta r)}, L^{(\zeta q,\zeta r)}, \bar{T}^{(\zeta q,\zeta r)}$, $\bar{R}^{(\zeta q,\zeta r)}$, $\bar{L}^{(\zeta q,\zeta r)}$ of (1.0.1) and the scattering coefficients $T_{\rm r}^{(\zeta\tilde{q},\zeta\tilde{r})}$, $T_{\rm l}^{(\zeta\tilde{q},\zeta\tilde{r})}$, $R^{(\zeta\tilde{q},\zeta\tilde{r})}$, $L^{(\zeta\tilde{q},\zeta\tilde{r})}$, $\bar{T}_{\rm r}^{(\zeta\tilde{q},\zeta\tilde{r})}$, $\bar{T}_{\rm l}^{(\zeta\tilde{q},\zeta\tilde{r})}$, $\bar{R}^{(\zeta\tilde{q},\zeta\tilde{r})}$, $\bar{L}^{(\zeta\tilde{q},\zeta\tilde{r})}$ of (6.0.14) are related to each other as*

$$T^{(\zeta q,\zeta r)} = T_{\rm r}^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.5}$$

$$T^{(\zeta q,\zeta r)} = e^{-i\tilde{\mu}/2} T_{\rm l}^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.6}$$

$$R^{(\zeta q,\zeta r)} = e^{-i\tilde{\mu}/2} R^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.7}$$

$$L^{(\zeta q,\zeta r)} = L^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.8}$$

$$\bar{T}^{(\zeta q,\zeta r)} = e^{i\tilde{\mu}/2} \bar{T}_{\rm r}^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.9}$$

$$\bar{T}^{(\zeta q,\zeta r)} = \bar{T}_{\rm l}^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.10}$$

$$\bar{R}^{(\zeta q,\zeta r)} = e^{i\tilde{\mu}/2}\bar{R}^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.11}$$

$$\bar{L}^{(\zeta q,\zeta r)} = \bar{L}^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.12}$$

where $\tilde{\mu}$ is the quantity defined in (6.1.7).

Proof. By using (3.3.1), (6.2.1), and the second equality of (6.1.8) in (6.1.3) we have

$$\begin{bmatrix} \frac{1}{T(\zeta)}e^{-i\zeta^2 x} \\ \frac{R(\zeta)}{T(\zeta)}e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\tilde{\mu}/2} \end{bmatrix} \begin{bmatrix} \frac{1}{T_{\mathrm{r}}(\zeta)}e^{-i\zeta^2 x} \\ \frac{R(\zeta)}{T_{\mathrm{r}}(\zeta)}e^{i\zeta^2 x} \end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})} + o(1), \qquad x \to +\infty,$$

or equivalently

$$\begin{bmatrix} \frac{1}{T(\zeta)}e^{-i\zeta^2 x} \\ \frac{R(\zeta)}{T(\zeta)}e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1) = \begin{bmatrix} \frac{1}{T_{\mathrm{r}}(\zeta)}e^{-i\zeta^2 x} \\ \frac{e^{-i\tilde{\mu}/2}R(\zeta)}{T_{\mathrm{r}}(\zeta)}e^{i\zeta^2 x} \end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})} + o(1), \qquad x \to +\infty. \tag{6.2.13}$$

From (6.2.13) we obtain

$$\left(\frac{1}{T(\zeta)}\right)^{(\zeta q,\zeta r)} = \left(\frac{1}{T_{\mathrm{r}}(\zeta)}\right)^{(\zeta\tilde{q},\zeta\tilde{r})},$$

$$\left(\frac{R(\zeta)}{T(\zeta)}\right)^{(\zeta q,\zeta r)} = e^{-i\tilde{\mu}/2}\left(\frac{R(\zeta)}{T_{\mathrm{r}}(\zeta)}\right)^{(\zeta\tilde{q},\zeta\tilde{r})},$$

which can be written as

$$T^{(\zeta q,\zeta r)} = T_{\mathrm{r}}^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.14}$$

$$R^{(\zeta q,\zeta r)} = e^{-i\tilde{\mu}/2}R^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.15}$$

which establish (6.2.5) and (6.2.7). Similarly, with the help of (3.3.3), (6.2.3), and the first equality of (6.1.8), from (6.1.4) we obtain

$$\begin{bmatrix} \frac{L(\zeta)}{T(\zeta)}e^{-i\zeta^2 x} \\ \frac{1}{T(\zeta)}e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1) = \begin{bmatrix} e^{i\tilde{\mu}/2} & 0 \\ 0 & e^{i\tilde{\mu}/2} \end{bmatrix} \begin{bmatrix} \frac{L(\zeta)}{T_{\mathrm{l}}(\zeta)}e^{-i\zeta^2 x} \\ \frac{1}{T_{\mathrm{l}}(\zeta)}e^{i\zeta^2 x} \end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})} + o(1), \qquad x \to -\infty,$$

or equivalently

$$\begin{bmatrix} \frac{L(\zeta)}{T(\zeta)}e^{-i\zeta^2 x} \\ \frac{1}{T(\zeta)}e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1) = \begin{bmatrix} \frac{e^{i\tilde{\mu}/2}\,L(\zeta)}{T_{\mathrm{l}}(\zeta)}e^{-i\zeta^2 x} \\ \frac{e^{i\tilde{\mu}/2}}{T_{\mathrm{l}}(\zeta)}e^{i\zeta^2 x} \end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})} + o(1), \qquad x \to -\infty. \tag{6.2.16}$$

From (6.2.16) we get

$$\left(\frac{1}{T(\zeta)}\right)^{(\zeta q,\zeta r)}=e^{i\tilde{\mu}/2}\left(\frac{1}{T_{\rm l}(\zeta)}\right)^{(\zeta\tilde{q},\zeta\tilde{r})},$$

$$\left(\frac{L(\zeta)}{T(\zeta)}\right)^{(\zeta q,\zeta r)}=e^{i\tilde{\mu}/2}\left(\frac{L(\zeta)}{T_{\rm r}(\zeta)}\right)^{(\zeta\tilde{q},\zeta\tilde{r})},$$

which can be written as

$$T^{(\zeta q,\zeta r)}=e^{-i\tilde{\mu}/2}T_{\rm l}^{(\zeta\tilde{q},\zeta\tilde{r})},\tag{6.2.17}$$

$$L^{(\zeta q,\zeta r)}=L^{(\zeta\tilde{q},\zeta\tilde{r})},\tag{6.2.18}$$

which establish (6.2.6) and (6.2.8). Now, let us relate the scattering coefficients $\bar{T}^{(\zeta q,\zeta r)}$ and $\bar{R}^{(\zeta q,\zeta r)}$ of (1.0.1) to the scattering coefficients $\bar{T}_{\rm r}^{(\zeta\tilde{q},\zeta\tilde{r})}$ and $\bar{R}^{(\zeta\tilde{q},\zeta\tilde{r})}$ of (6.0.14), respectively. With the help of (3.3.2), (6.2.2), and the second equality of (6.1.8), from (6.1.5) we have

$$\begin{bmatrix}\frac{\bar{R}(\zeta)}{\bar{T}(\zeta)}e^{-i\zeta^2x}\\ \frac{1}{\bar{T}(\zeta)}e^{i\zeta^2x}\end{bmatrix}^{(\zeta q,\zeta r)}+o(1)=\begin{bmatrix}1&0\\0&e^{-i\tilde{\mu}/2}\end{bmatrix}\begin{bmatrix}\frac{\bar{R}(\zeta)}{\bar{T}_{\rm r}(\zeta)}e^{-i\zeta^2x}\\ \frac{1}{\bar{T}_{\rm r}(\zeta)}e^{i\zeta^2x}\end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})}+o(1),\qquad x\to+\infty,$$

or equivalently

$$\begin{bmatrix}\frac{\bar{R}(\zeta)}{\bar{T}(\zeta)}e^{-i\zeta^2x}\\ \frac{1}{\bar{T}(\zeta)}e^{i\zeta^2x}\end{bmatrix}^{(\zeta q,\zeta r)}+o(1)=\begin{bmatrix}\frac{\bar{R}(\zeta)}{\bar{T}_{\rm r}(\zeta)}e^{-i\zeta^2x}\\ \frac{e^{-i\tilde{\mu}/2}}{\bar{T}_{\rm r}(\zeta)}e^{i\zeta^2x}\end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})}+o(1),\qquad x\to+\infty.\tag{6.2.19}$$

Hence, from (6.2.19) we obtain

$$\left(\frac{1}{\bar{T}(\zeta)}\right)^{(\zeta q,\zeta r)}=e^{-i\tilde{\mu}/2}\left(\frac{1}{\bar{T}_{\rm r}(\zeta)}\right)^{(\zeta\tilde{q},\zeta\tilde{r})},$$

$$\left(\frac{\bar{R}(\zeta)}{\bar{T}(\zeta)}\right)^{(\zeta q,\zeta r)}=\left(\frac{\bar{R}(\zeta)}{\bar{T}_{\rm r}(\zeta)}\right)^{(\zeta\tilde{q},\zeta\tilde{r})},$$

which can be written as

$$\bar{T}^{(\zeta q,\zeta r)}=e^{i\tilde{\mu}/2}\bar{T}_{\rm r}^{(\zeta\tilde{q},\zeta\tilde{r})},\tag{6.2.20}$$

$$\bar{R}^{(\zeta q,\zeta r)}=e^{i\tilde{\mu}/2}\bar{R}^{(\zeta\tilde{q},\zeta\tilde{r})},\tag{6.2.21}$$

which establish (6.2.9) and (6.2.11). In the same manner, let us relate the scattering coefficient $L^{(\zeta q,\zeta r)}$ of (1.0.1) to the scattering coefficient $L^{(\zeta\tilde{q},\zeta\tilde{r})}$ of (6.0.14). Using (3.3.4), (6.2.4), and the first equality of (6.1.8) in (6.1.6) we get

$$\begin{bmatrix} \frac{1}{\bar{T}(\zeta)} e^{-i\zeta^2 x} \\ \frac{\bar{L}(\zeta)}{\bar{T}(\zeta)} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\bar{T}_1(\zeta)} e^{-i\zeta^2 x} \\ \frac{\bar{L}(\zeta)}{\bar{T}_1(\zeta)} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})} + o(1), \qquad x \to -\infty,$$

or equivalently

$$\begin{bmatrix} \frac{1}{\bar{T}(\zeta)} e^{-i\zeta^2 x} \\ \frac{\bar{L}(\zeta)}{\bar{T}(\zeta)} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)} + o(1) = \begin{bmatrix} \frac{1}{\bar{T}_1(\zeta)} e^{-i\zeta^2 x} \\ \frac{\bar{L}(\zeta)}{\bar{T}_1(\zeta)} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta\tilde{q},\zeta\tilde{r})} + o(1), \qquad x \to -\infty. \tag{6.2.22}$$

From (6.2.22) we obtain

$$\left(\frac{1}{\bar{T}(\zeta)}\right)^{(\zeta q,\zeta r)} = \left(\frac{1}{\bar{T}_1(\zeta)}\right)^{(\zeta\tilde{q},\zeta\tilde{r})},$$

$$\left(\frac{\bar{L}(\zeta)}{\bar{T}(\zeta)}\right)^{(\zeta q,\zeta r)} = \left(\frac{\bar{L}(\zeta)}{\bar{T}_1(\zeta)}\right)^{(\zeta\tilde{q},\zeta\tilde{r})},$$

which can be written as

$$\bar{T}^{(\zeta q,\zeta r)} = \bar{T}_1^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.23}$$

$$\bar{L}^{(\zeta q,\zeta r)} = \bar{L}^{(\zeta\tilde{q},\zeta\tilde{r})}, \tag{6.2.24}$$

which completes proof of (6.2.12). □

6.3 Time Evolution of the Scattering Data for the Standard System

In this section we determine the time evolution of the scattering data associated with (1.0.4) when the potentials u and v in (1.0.4) contain t as a parameter. In that case the potentials u and v satisfy the integrable nonlinear system given in (6.3.4). Although the results presented in this section are already known [3, 4, 6, 7], we provide the proofs with some details for the benefit of the reader. For the convenience of the reader, we illustrate the derivation of the time evolution of scattering data for (1.0.4). With the help of the AKNS pair associated with the NLS system, we obtain the time evolution of the scattering data by using the description given in [12, 13]. The AKNS pair for the NLS system already known [3, 4, 7] and is given by

$$\mathcal{X}^{(u,v)} = \begin{bmatrix} -i\lambda & u \\ v & i\lambda \end{bmatrix}, \quad \mathcal{T}^{(u,v)} = \begin{bmatrix} -2i\lambda^2 - iuv & 2\lambda u + iu_x \\ 2\lambda v - iv_x & 2i\lambda^2 + iuv \end{bmatrix}. \tag{6.3.1}$$

We write the linear system (1.0.4) as

$$\Psi_x^{(u,v)} = \mathcal{X}^{(u,v)}\, \Psi^{(u,v)}, \tag{6.3.2}$$

and require [12, 13] that $\Psi_t^{(u,v)} - \mathcal{T}^{(u,v)}\, \Psi^{(u,v)}$ is a solution to (6.3.2). In the literature it is usually but incorrectly stated [1, 3, 4, 6, 18] that the time evolution of the wave function is described by the linear system $\Psi_t^{(u,v)} = \mathcal{T}^{(u,v)}\Psi^{(u,v)}$. The correct time evolution is given in [12, 13] by saying that $\Psi_t^{(u,v)} - \mathcal{T}^{(u,v)}\Psi^{(u,v)}$ must satisfy the linear system $\Psi_x^{(u,v)} = \mathcal{X}^{(u,v)}\, \Psi^{(u,v)}$. In other words, the correct time evolution for $\Psi^{(u,v)}$ is obtained by requiring that $\Psi_t^{(u,v)} - \mathcal{T}^{(u,v)}\Psi^{(u,v)}$ can be written as a linear combination of two linearly independent solutions to $\Psi_x^{(u,v)} = \mathcal{X}^{(u,v)}\, \Psi^{(u,v)}$. For further details we refer the reader to [12, 13]. The NLS system is obtained from (6.3.1) via the condition

$$\mathcal{X}_t^{(u,v)} - \mathcal{T}_x^{(u,v)} + \mathcal{X}^{(u,v)}\, \mathcal{T}^{(u,v)} - \mathcal{T}^{(u,v)}\, \mathcal{X}^{(u,v)} = 0, \tag{6.3.3}$$

which yields the NLS system

$$\begin{cases} iu_t + u_{xx} - 2uvu = 0, \\ iv_t - v_{xx} + 2vuv = 0, \end{cases} \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}^+. \tag{6.3.4}$$

Choosing $v(x) = \pm u(x)^*$ in (6.3.4) we get the NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}^+. \tag{6.3.5}$$

In order to determine the time evolution of the scattering data for (1.0.4) we use the AKNS pair $\mathcal{X}^{(u,v)}$ and $\mathcal{T}^{(u,v)}$ given in (6.3.1) respectively, and require that $\phi_t^{(u,v)} - \mathcal{T}^{(u,v)}\,\phi^{(u,v)}$ is a solution to (6.3.2), i.e.

$$\left(\phi_t^{(u,v)} - \mathcal{T}^{(u,v)}\,\phi^{(u,v)}\right)_x = \mathcal{X}^{(u,v)}\Phi^{(u,v)}. \tag{6.3.6}$$

This is equivalent to the requirement that $\phi_t^{(u,v)} - \mathcal{T}^{(u,v)}\,\phi^{(u,v)}$ is written as a linear combination of the Jost solutions $\phi^{(u,v)}$ and $\psi^{(u,v)}$ to (1.0.4). Hence, we can write $\phi_t^{(u,v)} - \mathcal{T}(u,v)\,\phi^{(u,v)}$ as a linear combination of $\phi^{(u,v)}$ and $\psi^{(u,v)}$ as

$$\phi_t^{(u,v)} - \mathcal{T}(u,v)\,\phi^{(u,v)} = a_1(\lambda,t)\,\phi^{(u,v)} + a_2(\lambda,t)\,\psi^{(u,v)}, \tag{6.3.7}$$

where $a_1(\lambda,t)$ and $a_2(\lambda,t)$ are some scalar coefficients to be determined. Since u and v belong to the Schwartz class, we know that u, v, u_x, v_x vanish rapidly as $x \to \pm\infty$. Hence, from the second equality in (6.3.1) we have

$$\mathcal{T}^{(u,v)} = \begin{bmatrix} -2i\lambda^2 & 0 \\ 0 & 2i\lambda^2 \end{bmatrix} + o(1), \qquad x \to \pm\infty. \tag{6.3.8}$$

With the help of first equality in (2.1.3), (2.2.3), and (6.3.8), from (6.3.7) as $x \to -\infty$, we have

$$\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}_t - \begin{bmatrix} -2i\lambda^2 & 0 \\ 0 & 2i\lambda^2 \end{bmatrix}\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} = a_1(\lambda,t)\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + a_2(\lambda,t)\begin{bmatrix} \frac{L}{T}e^{-i\lambda x} \\ \frac{1}{T}e^{i\lambda x} \end{bmatrix}^{(u,v)}. \tag{6.3.9}$$

The spectral parameter λ does not change in time, i.e. we have $\lambda_t = 0$. Hence, we can simplify (6.3.9) to

$$\begin{bmatrix} 2i\lambda^2 e^{-i\lambda x} \\ 0 \end{bmatrix} = a_1(\lambda,t)\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + a_2(\lambda,t)\begin{bmatrix} \frac{L}{T}e^{-i\lambda x} \\ \frac{1}{T}e^{i\lambda x} \end{bmatrix}^{(u,v)}. \tag{6.3.10}$$

Thus, we have $a_1(\lambda,t) = 2i\lambda^2$ and $a_2(\lambda,t) = 0$. Hence, we have the time evolution for the Jost solution $\phi^{(\zeta q,\zeta r)}$ as

$$\phi_t^{(u,v)} - \mathcal{T}^{(u,v)}\,\phi^{(u,v)} = 2i\lambda^2\,\phi^{(u,v)}. \tag{6.3.11}$$

Using (2.2.1) and (6.3.8) in (6.3.11) as $x \to +\infty$, we get

$$\begin{bmatrix} \frac{1}{T}e^{-i\lambda x} \\ \frac{R}{T}e^{i\lambda x} \end{bmatrix}_t^{(u,v)} - \begin{bmatrix} -2i\lambda^2 & 0 \\ 0 & 2i\lambda^2 \end{bmatrix} \begin{bmatrix} \frac{1}{T}e^{-i\lambda x} \\ \frac{R}{T}e^{i\lambda x} \end{bmatrix}^{(u,v)} = 2i\lambda^2 \begin{bmatrix} \frac{1}{T}e^{-i\lambda x} \\ \frac{R}{T}e^{i\lambda x} \end{bmatrix}^{(u,v)}, \tag{6.3.12}$$

which yields

$$\left(\frac{1}{T}\right)_t^{(u,v)} + 2i\lambda^2\left(\frac{1}{T}\right)^{(u,v)} = 2i\lambda^2\left(\frac{1}{T}\right)^{(u,v)}, \tag{6.3.13}$$

$$\left(\frac{R}{T}\right)_t^{(u,v)} - 2i\lambda^2\left(\frac{R}{T}\right)^{(u,v)} = 2i\lambda^2\left(\frac{R}{T}\right)^{(u,v)}. \tag{6.3.14}$$

From (6.3.13) we get

$$\left(\frac{1}{T}\right)_t^{(u,v)} = 0, \tag{6.3.15}$$

which implies

$$T_t^{(u,v)} = 0. \tag{6.3.16}$$

Hence, we obtain

$$T^{(u,v)}(\lambda,t) = T^{(u,v)}(\lambda,0). \tag{6.3.17}$$

From (6.3.14) we have

$$\left(\frac{R}{T}\right)_t^{(u,v)} = 4i\lambda^2\left(\frac{R}{T}\right)^{(u,v)}. \tag{6.3.18}$$

With the help of (6.3.16), from (6.3.18) we get

$$R_t^{(u,v)} = 4i\lambda^2 R^{(u,v)},$$

which yields

$$R^{(u,v)}(\lambda,t) = e^{4i\lambda^2 t}R^{(u,v)}(\lambda,0). \tag{6.3.19}$$

Similarly, we know that $\psi_t^{(u,v)} - \mathcal{T}^{(u,v)}\,\psi^{(u,v)}$ must be a solution to (6.3.2). Hence, we can write it as linear combination of $\phi^{(u,v)}$ and $\psi^{(u,v)}$ as

$$\psi_t^{(u,v)} - \mathcal{T}(u,v)\,\psi^{(u,v)} = a_3(\lambda,t)\,\psi^{(u,v)} + a_4(\zeta,t)\,\phi^{(u,v)}, \tag{6.3.20}$$

for some scalar coefficients $a_3(\lambda,t)$ and $a_4(\lambda,t)$, which are to be determined. With the help of the first equality in (2.1.2), (2.2.1), and (6.3.8), from (6.3.20) as $x \to +\infty$, we have

$$\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}_t - \begin{bmatrix} -2i\lambda^2 & 0 \\ 0 & 2i\lambda^2 \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} = a_3(\lambda,t) \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + a_4(\lambda,t) \begin{bmatrix} \frac{1}{T} e^{-i\lambda x} \\ \frac{R}{T} e^{i\lambda x} \end{bmatrix}^{(u,v)}, \tag{6.3.21}$$

which simplifies to

$$\begin{bmatrix} 0 \\ -2i\lambda^2 e^{-i\lambda x} \end{bmatrix} = a_3(\lambda,t) \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + a_4(\lambda,t) \begin{bmatrix} \frac{1}{T} e^{-i\lambda x} \\ \frac{R}{T} e^{i\lambda x} \end{bmatrix}^{(u,v)}. \tag{6.3.22}$$

Thus, we have $a_3(\lambda,t) = -2i\lambda^2$ and $a_4(\lambda,t) = 0$. Hence, we have the time evolution for the Jost solution $\psi^{(u,v)}$ given by

$$\psi_t^{(u,v)} - \mathcal{T}^{(u,v)}\,\psi^{(u,v)} = -2i\lambda^2\,\psi^{(u,v)}. \tag{6.3.23}$$

Using (2.2.3) and (6.3.8) in (6.4.18), as $x \to -\infty$ we get

$$\begin{bmatrix} \frac{L}{T} e^{-i\lambda x} \\ \frac{1}{T} e^{i\lambda x} \end{bmatrix}_t^{(u,v)} - \begin{bmatrix} -2i\lambda^2 & 0 \\ 0 & 2i\lambda^2 \end{bmatrix} \begin{bmatrix} \frac{L}{T} e^{-i\lambda x} \\ \frac{1}{T} e^{i\lambda x} \end{bmatrix}^{(u,v)} = -2i\lambda^2 \begin{bmatrix} \frac{L}{T} e^{-i\lambda x} \\ \frac{1}{T} e^{i\lambda x} \end{bmatrix}^{(u,v)}. \tag{6.3.24}$$

Thus, we obtain

$$\left(\frac{1}{T}\right)_t^{(u,v)} - 2i\lambda^2 \left(\frac{1}{T}\right)^{(u,v)} = -2i\lambda^2 \left(\frac{1}{T}\right)^{(u,v)}, \tag{6.3.25}$$

$$\left(\frac{L}{T}\right)_t^{(u,v)} + 2i\lambda^2 \left(\frac{L}{T}\right)^{(u,v)} = -2i\lambda^2 \left(\frac{L}{T}\right)^{(u,v)}. \tag{6.3.26}$$

From (6.3.25) we get

$$\left(\frac{1}{T}\right)_t^{(u,v)} = 0, \tag{6.3.27}$$

which implies

$$T_t^{(u,v)} = 0. \tag{6.3.28}$$

Hence, we obtain

$$T^{(u,v)}(\lambda, t) = T^{(u,v)}(\lambda, 0). \tag{6.3.29}$$

From (6.3.26) we have

$$\left(\frac{L}{T}\right)_t^{(u,v)} = -4i\lambda^2 \left(\frac{L}{T}\right)^{(u,v)}. \tag{6.3.30}$$

With the help of (6.3.28), from (6.3.30) we get

$$L_t^{(u,v)} = -4i\lambda^2 L^{(u,v)},$$

which yields

$$L^{(u,v)}(\lambda, t) = e^{-4i\lambda^2 t} L^{(u,v)}(\lambda, 0). \tag{6.3.31}$$

In the same manner, we know that $\bar{\psi}_t^{(u,v)} - \mathcal{T}^{(u,v)}\, \bar{\psi}^{(u,v)}$ must be a solution to (6.3.2). Hence, we can write it as a linear combination of $\bar{\phi}^{(u,v)}$ and $\bar{\psi}^{(u,v)}$ as

$$\bar{\psi}_t^{(u,v)} - \mathcal{T}^{(u,v)}\, \bar{\psi}^{(u,v)} = a_5(\lambda, t)\, \bar{\psi}^{(u,v)} + c_6(\lambda, t)\, \bar{\phi}^{(u,v)}, \tag{6.3.32}$$

where the scalar coefficients $a_5(\lambda, t)$ and $a_5(\lambda, t)$ are to be determined. With the help of the second equality in (2.1.2), (2.2.2), and (6.3.8), from (6.3.32), as $x \to +\infty$ we have

$$\begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix}_t - \begin{bmatrix} -2i\lambda^2 & 0 \\ 0 & 2i\lambda^2 \end{bmatrix} \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} = a_5(\lambda, t) \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + a_6(\lambda, t) \begin{bmatrix} \frac{\bar{R}}{\bar{T}} e^{-i\lambda x} \\ \frac{1}{\bar{T}} e^{i\lambda x} \end{bmatrix}^{(u,v)}, \tag{6.3.33}$$

which simplifies to

$$\begin{bmatrix} 2i\lambda^2 e^{-i\lambda x} \\ 0 \end{bmatrix} = a_5(\lambda,t) \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + a_6(\lambda,t) \begin{bmatrix} \frac{\bar{R}}{\bar{T}} e^{-i\lambda x} \\ \frac{1}{\bar{T}} e^{i\lambda x} \end{bmatrix}^{(u,v)}. \tag{6.3.34}$$

Thus, we have $a_5(\lambda,t) = 2i\lambda^2$ and $a_6(\lambda,t) = 0$. Hence, we have the time evolution for the Jost solution $\bar{\psi}^{(u,v)}$ given by

$$\bar{\psi}_t^{(u,v)} - \mathcal{T}^{(u,v)}\,\bar{\psi}^{(u,v)} = 2i\lambda^2\,\bar{\psi}^{(u,v)}. \tag{6.3.35}$$

Using (2.2.4) and (6.3.8) in (6.3.35) as $x \to -\infty$, we get

$$\begin{bmatrix} \frac{1}{\bar{T}} e^{-i\lambda x} \\ \frac{\bar{L}}{\bar{T}} e^{i\lambda x} \end{bmatrix}_t^{(u,v)} - \begin{bmatrix} -2i\lambda^2 & 0 \\ 0 & 2i\lambda^2 \end{bmatrix} \begin{bmatrix} \frac{1}{\bar{T}} e^{-i\lambda x} \\ \frac{\bar{L}}{\bar{T}} e^{i\lambda x} \end{bmatrix}^{(u,v)} = 2i\lambda^2 \begin{bmatrix} \frac{1}{\bar{T}} e^{-i\lambda x} \\ \frac{\bar{L}}{\bar{T}} e^{i\lambda x} \end{bmatrix}^{(u,v)}. \tag{6.3.36}$$

Thus, we obtain

$$\left(\frac{1}{\bar{T}}\right)_t^{(u,v)} + 2i\lambda^2 \left(\frac{1}{\bar{T}}\right)^{(u,v)} = 2i\lambda^2 \left(\frac{1}{\bar{T}}\right)^{(u,v)}, \tag{6.3.37}$$

$$\left(\frac{\bar{L}}{\bar{T}}\right)_t^{(u,v)} - 2i\lambda^2 \left(\frac{\bar{L}}{\bar{T}}\right)^{(u,v)} = 2i\lambda^2 \left(\frac{\bar{L}}{\bar{T}}\right)^{(u,v)}. \tag{6.3.38}$$

From (6.3.37) we get

$$\left(\frac{1}{\bar{T}}\right)_t^{(u,v)} = 0, \tag{6.3.39}$$

which implies

$$\bar{T}_t^{(u,v)} = 0. \tag{6.3.40}$$

Hence, we obtain

$$\bar{T}^{(u,v)}(\zeta,t) = \bar{T}^{(u,v)}(\lambda,0). \tag{6.3.41}$$

From (6.3.38) we have

$$\left(\frac{\bar{L}}{\bar{T}}\right)_t^{(u,v)} = 4i\lambda^2 \left(\frac{\bar{L}}{\bar{T}}\right)^{(u,v)}. \tag{6.3.42}$$

With the help of (6.3.40), from (6.3.42) we get

$$\bar{L}_t^{(u,v)} = 4i\lambda^2 \bar{L}^{(u,v)},$$

which yields

$$\bar{L}^{(u,v)}(\lambda,t) = e^{4i\lambda^2 t}\bar{L}^{(u,v)}(\lambda,0). \tag{6.3.43}$$

Again, the quantity $\bar{\phi}_t^{(u,v)} - \mathcal{T}^{(u,v)}\,\bar{\phi}^{(u,v)}$ must be a solution to (6.3.2). Hence, we can write it as a linear combination of $\bar{\phi}^{(u,v)}$ and $\bar{\psi}^{(u,v)}$ as

$$\bar{\phi}_t^{(u,v)} - \mathcal{T}^{(u,v)}\,\bar{\phi}^{(u,v)} = a_7(\lambda,t)\,\bar{\phi}^{(u,v)} + a_8(\lambda,t)\,\bar{\psi}^{(u,v)}, \tag{6.3.44}$$

for some scalar coefficients $a_7(\lambda,t)$ and $a_8(\lambda,t)$, which are to be determined. With the help of the second equality in (2.1.3), (2.2.4), and (6.3.8), from (6.3.44), $x \to -\infty$ we have

$$\begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}_t - \begin{bmatrix} -2i\lambda^2 & 0 \\ 0 & 2i\lambda^2 \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} = a_7(\lambda,t) \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + a_8(\lambda,t) \begin{bmatrix} \frac{1}{\bar{T}} e^{-i\lambda x} \\ \frac{\bar{L}}{\bar{T}} e^{i\lambda x} \end{bmatrix}^{(u,v)}, \tag{6.3.45}$$

which simplifies to

$$\begin{bmatrix} 0 \\ -2i\lambda^2 e^{i\lambda x} \end{bmatrix} = a_7(\lambda,t) \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + a_8(\lambda,t) \begin{bmatrix} \frac{1}{\bar{T}} e^{-i\lambda x} \\ \frac{\bar{L}}{\bar{T}} e^{i\lambda x} \end{bmatrix}^{(u,v)}. \tag{6.3.46}$$

Thus, we have $a_7(\lambda,t) = -2i\lambda^2$ and $a_8(\lambda,t) = 0$. Hence, we have the time evolution for the Jost solution $\bar{\phi}^{(u,v)}$ as

$$\bar{\phi}_t^{(u,v)} - \mathcal{T}^{(u,v)}\,\bar{\phi}^{(u,v)} = -2i\lambda^2\,\bar{\phi}^{(u,v)}. \tag{6.3.47}$$

Using (2.2.2) and (6.3.8) in (6.3.47), as $x \to +\infty$ we get

$$\begin{bmatrix} \frac{\bar{R}}{\bar{T}} e^{-i\lambda x} \\ \frac{1}{\bar{T}} e^{i\lambda x} \end{bmatrix}_t^{(u,v)} - \begin{bmatrix} -2i\lambda^2 & 0 \\ 0 & 2i\lambda^2 \end{bmatrix} \begin{bmatrix} \frac{\bar{R}}{\bar{T}} e^{-i\lambda x} \\ \frac{1}{\bar{T}} e^{i\lambda x} \end{bmatrix}^{(u,v)} = -2i\lambda^2 \begin{bmatrix} \frac{\bar{R}}{\bar{T}} e^{-i\lambda x} \\ \frac{1}{\bar{T}} e^{i\lambda x} \end{bmatrix}_t^{(u,v)}. \tag{6.3.48}$$

Thus, we obtain

$$\left(\frac{1}{\bar{T}}\right)_t^{(u,v)} - 2i\lambda^2 \left(\frac{1}{\bar{T}}\right)^{(u,v)} = -2i\zeta^4 \left(\frac{1}{\bar{T}}\right)^{(u,v)}, \tag{6.3.49}$$

$$\left(\frac{\bar{R}}{\bar{T}}\right)_t^{(u,v)} + 2i\lambda^2 \left(\frac{\bar{R}}{\bar{T}}\right)^{(u,v)} = -2i\lambda^2 \left(\frac{\bar{R}}{\bar{T}}\right)^{(u,v)}. \tag{6.3.50}$$

From (6.3.49) we get

$$\left(\frac{1}{\bar{T}}\right)_t^{(u,v)} = 0, \tag{6.3.51}$$

which implies

$$\bar{T}_t^{(u,v)} = 0. \tag{6.3.52}$$

Hence, we obtain

$$\bar{T}^{(u,v)}(\lambda, t) = \bar{T}^{(u,v)}(\lambda, 0). \tag{6.3.53}$$

From (6.3.50) we have

$$\left(\frac{\bar{R}}{\bar{T}}\right)_t^{(u,v)} = -4i\lambda^2 \left(\frac{\bar{R}}{\bar{R}}\right)^{(u,v)}. \tag{6.3.54}$$

With the help of (6.3.52), from (6.3.54) we get

$$\bar{R}_t^{(u,v)} = -4i\lambda^2 \bar{R}^{(u,v)},$$

which yields

$$\bar{R}^{(u,v)}(\lambda, t) = e^{-4i\lambda^2 t} \bar{R}^{(u,v)}(\lambda, 0). \tag{6.3.55}$$

Now, let us evaluate the time evolution of the dependency constants $\gamma_j(t)$ which are defined in (2.3.2), by assuming the bound-state poles are all simple. Let us copy (2.3.2) for the convenience of the reader as

$$\phi^{(u,v)}(\lambda_j, x) = \gamma_j \, \psi^{(u,v)}(\lambda_j, x), \qquad j = 1, ..., N. \tag{6.3.56}$$

Using (6.3.56) in (6.3.11) we get

$$\left[\gamma_j(t)\, \psi^{(u,v)}\right]_t - \mathcal{T}^{(u,v)} \gamma_j(t)\, \psi^{(u,v)} = 2i\lambda_j^2 \, \gamma_j(t)\, \psi^{(u,v)},$$

which can be written as

$$\partial_t \, \gamma_j(t) \, \psi^{(u,v)} + \gamma_j(t) \, \psi_t^{(u,v)} - \mathcal{T}^{(u,v)} \, \gamma_j(t) \, \psi^{(u,v)} = 2i\lambda_j^2 \, \gamma_j(t) \, \psi^{(u,v)}.$$

or equivalently

$$\partial_t \gamma_j(t) \, \psi^{(u,v)} + \gamma_j(t) \left[\psi_t^{(u,v)} - \mathcal{T}^{(u,v)} \, \psi^{(u,v)} \right] = 2i\lambda_j^2 \, \gamma_j(t) \, \psi^{(u,v)}. \tag{6.3.57}$$

Using (6.3.23) in (6.3.57) we obtain

$$\partial_t \gamma_j(t) \, \psi^{(u,v)} + \gamma_j(t) \left[-2i\lambda_j^2 \, \psi^{(u,v)} \right] = 2i\lambda_j^2 \, \gamma_j(t) \, \psi^{(u,v)}.$$

which yields

$$\partial_t \gamma_j(t) = 4i\lambda_j^2 \, \gamma_j(t).$$

Thus, we obtain

$$\gamma_j(t) = e^{4i\lambda_j^2 t} \, \gamma_j(0). \tag{6.3.58}$$

Similarly, let us evaluate the time evolution of the dependency constants $\bar{\gamma}_j(t)$ which are defined in (2.3.4), by assuming the bound-state poles are all simple. Let us copy (2.3.4) for the convenience of the reader as

$$\bar{\phi}^{(u,v)}(\bar{\lambda}_j, x) = \bar{\gamma}_j \, \bar{\psi}^{(u,v)}(\bar{\lambda}_j, x), \qquad j = 1, ..., \bar{N}. \tag{6.3.59}$$

Using (6.3.59) in (6.3.47) we get

$$\left[\bar{\gamma}_j(t) \, \bar{\psi}^{(u,v)} \right]_t - \mathcal{T}^{(u,v)} \, \bar{\gamma}_j(t) \, \bar{\psi}^{(u,v)} = -2i\bar{\lambda}_j^2 \, \bar{\gamma}_j(t) \, \bar{\psi}^{(u,v)},$$

which can be written as

$$\partial_t \, \bar{\gamma}_j(t) \, \bar{\psi}^{(u,v)} + \bar{\gamma}_j(t) \, \bar{\psi}_t^{(u,v)} - \mathcal{T}^{(u,v)} \, \bar{\gamma}_j(t) \, \bar{\psi}^{(u,v)} = -2i\bar{\lambda}_j^2 \, \bar{\gamma}_j(t) \, \bar{\psi}^{(u,v)}.$$

or equivalently

$$\partial_t \bar{\gamma}_j(t) \, \bar{\psi}^{(u,v)} + \bar{\gamma}_j(t) \left[\bar{\psi}_t^{(u,v)} - \mathcal{T}^{(u,v)} \, \bar{\psi}^{(u,v)} \right] = -2i\bar{\lambda}_j^2 \, \bar{\gamma}_j(t) \, \bar{\psi}^{(u,v)}. \tag{6.3.60}$$

Using (6.3.35) in (6.3.60) we obtain

$$\partial_t \bar{\gamma}_j(t)\, \bar{\psi}^{(u,v)} + \bar{\gamma}_j(t) \left[2i\bar{\lambda}_j^2\, \bar{\psi}^{(u,v)}\right] = -2i\bar{\lambda}_j^2\, \bar{\gamma}_j(t)\, \bar{\psi}^{(u,v)}.$$

which yields

$$\partial_t \bar{\gamma}_j(t) = -4i\bar{\lambda}_j^2\, \bar{\gamma}_j(t).$$

Thus, we obtain

$$\bar{\gamma}_j(t) = e^{-4i\bar{\lambda}_j^2 t}\, \bar{\gamma}_j(0). \tag{6.3.61}$$

Now, let us evaluate the time evolution of the norming constants $c_j^{(u,v)}$ by assuming that the bound-state poles are all simple. With the help of (2.3.22), (6.3.17), and (6.3.58), we determine the time evolution of the norming constant $c_j^{(u,v)}$ as

$$c_j^{(u,v)}(t) = c_j^{(u,v)}(0)\, e^{4i\lambda_j^2 t}, \qquad j = 1, ..., N. \tag{6.3.62}$$

Similarly, let us evaluate the time evolution of the norming constants $\bar{c}_j^{(u,v)}$ by assuming that the bound-state poles are all simple. With the help of (2.3.23), (6.3.41), and (6.3.61), we determine the time evolution of the norming constant $\bar{c}_j^{(u,v)}$ as

$$\bar{c}_j^{(u,v)}(t) = \bar{c}_j^{(u,v)}(0)\, e^{-4i\bar{\lambda}_j^2 t}, \qquad j = 1, ..., \bar{N}. \tag{6.3.63}$$

In case the bound states have multiplicities, we can use the method of [10, 20, 21] and express the bound-state data $\left\{\lambda_j, \left\{c_{jk}^{(u,v)}\right\}_{k=0}^{m_j-1}\right\}_{j=1}^{N}$ in terms of three matrices A, B, and C in the form $C\, e^{-Ay+f(A)t}\, B$, with appropriate matrix sizes and with an appropriate choice of $f(A)$ and similarly express the bound-state data $\left\{\bar{\lambda}_j, \left\{\bar{c}_{jk}^{(u,v)}\right\}_{k=0}^{m_j-1}\right\}_{j=1}^{\bar{N}}$ in terms of three matrices $\bar{A}$, $\bar{B}$, and $\bar{C}$ in the form $\bar{C}\, e^{-\bar{A}y+\bar{f}(\bar{A})t}\, \bar{B}$, with appropriate matrix sizes and with an appropriate choice of $\bar{f}(\bar{A})$. Assume that the bound states for (1.0.4) occur at $\lambda = \lambda_j$ for $j = 1, ..., N$ and at $\bar{\lambda} = \bar{\lambda}_j$ for $j = 1, ..., \bar{N}$. Assume that the multiplicity of λ_j is m_j and the multiplicity of $\bar{\lambda}_j$ is $\bar{m}_j$. Proceeding as in [10, 20, 21] we can form the row vectors C_j and $\bar{C}_j$ as

$$C_j := \begin{bmatrix} c_{j(m_j-1)}^{(u,v)} & c_{j(m_j-2)}^{(u,v)} & \dots & c_{j1}^{(u,v)} & c_{j0}^{(u,v)} \end{bmatrix}, \tag{6.3.64}$$

$$\bar{C}_j := \begin{bmatrix} \bar{c}_{j(\bar{m}_j-1)}^{(u,v)} & \bar{c}_{j(\bar{m}_j-2)}^{(u,v)} & \dots & \bar{c}_{j1}^{(u,v)} & \bar{c}_{j0}^{(u,v)} \end{bmatrix}, \tag{6.3.65}$$

where we see that C_j is a row vector with m_j components and $\bar{C}_j$ is a row vector with $\bar{m}_j$ components. We also form the square matrices A_j and $\bar{A}_j$, with respective sizes of $m_j \times m_j$ and $\bar{m}_j \times \bar{m}_j$, as

$$A_j := \begin{bmatrix} -i\lambda_j & -1 & 0 & \dots & 0 & 0 \\ 0 & -i\lambda_j & -1 & \dots & 0 & 0 \\ 0 & 0 & -i\lambda_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -i\lambda_j & -1 \\ 0 & 0 & 0 & \dots & 0 & -i\lambda_j \end{bmatrix}, \tag{6.3.66}$$

$$\bar{A}_j := \begin{bmatrix} -i\bar{\lambda}_j & -1 & 0 & \dots & 0 & 0 \\ 0 & -i\bar{\lambda}_j & -1 & \dots & 0 & 0 \\ 0 & 0 & -i\bar{\lambda}_j^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -i\bar{\lambda}_j & -1 \\ 0 & 0 & 0 & \dots & 0 & -i\bar{\lambda}_j \end{bmatrix}. \tag{6.3.67}$$

We note that A_j and $\bar{A}_j$ are in Jordan canonical forms. Let us also define the column vectors B_j and $\bar{B}_j$, where B_j has m_j components that are equal to zero except for the last component being equal to one and $\bar{B}_j$ has $\bar{m}_j$ components that are equal to zero except the last component being equal to one.

$$B_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad j = 1, \dots, N, \tag{6.3.68}$$

$$\bar{B}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad j = 1, \dots, \bar{N}. \tag{6.3.69}$$

We then form the block matrices A and $\bar{A}$, the block row vectors C and $\bar{C}$, and the block column vectors B and $\bar{B}$ as

$$A := \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & a \\ 0 & 0 & \dots & A_N \end{bmatrix}, \quad \bar{A} := \begin{bmatrix} \bar{A}_1 & 0 & \dots & 0 \\ 0 & \bar{A}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & a \\ 0 & 0 & \dots & \bar{A}_{\bar{N}} \end{bmatrix}, \tag{6.3.70}$$

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_N \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \vdots \\ B_{\bar{N}} \end{bmatrix}, \tag{6.3.71}$$

$$C := \begin{bmatrix} C_1 & C_2 & \dots & C_N \end{bmatrix}, \quad \bar{C} := \begin{bmatrix} \bar{C}_1 & \bar{C}_2 & \dots & \bar{C}_{\bar{N}} \end{bmatrix}. \tag{6.3.72}$$

Now, let us write the Marchenko kernel $\Omega(y)$ defined in (2.5.10) in terms of matrices A, B, and C when $t = 0$. In the absence of multiplicities in the bound states, from (2.5.10) we have

$$\Omega(y) = \hat{R}^{(u,v)}(y) + \sum_{j=1}^{N} c_j^{(u,v)}\, e^{i\lambda_j y}, \tag{6.3.73}$$

where

$$\hat{R}^{(u,v)}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, R^{(u,v)}(\lambda)\, e^{i\lambda y}. \tag{6.3.74}$$

In the Marchenko kernel $\Omega(y)$ given in (6.3.73) in order to account for bound states with the multiplicities with the help of the first equalities in (6.3.70)-(6.3.72), we replace the summation term in (6.3.73) with $C\,e^{-Ay}\,B$, i.e. we get

$$\Omega(y) = \hat{R}^{(u,v)}(y) + C\,e^{-Ay}\,B, \tag{6.3.75}$$

or equivalently

$$\Omega(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, R^{(u,v)}(\lambda)\, e^{i\lambda y} + C\,e^{-Ay}\,B. \tag{6.3.76}$$

Now, let us consider the time-evolved scattering data and determine the corresponding time-evolved Marchenko kernel. Using (6.3.19) and (6.3.62) in (6.3.76) we obtain

$$\Omega(y;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, R^{(u,v)}(\lambda;0)\, e^{4i\lambda^2 t}\, e^{i\lambda y} + C\,e^{-4iA^2 t}\, e^{-Ay}\,B, \tag{6.3.77}$$

which can be written as

$$\Omega(y;t) = \hat{R}^{(u,v)}(y;0)\, e^{4i\lambda^2 t} + C\,e^{-4iA^2 t - Ay}\,B. \tag{6.3.78}$$

Since A^2 and A commute with each other, from (6.3.78) we obtain

$$\Omega(x+y;t) = \hat{R}^{(u,v)}(x+y;0)\, e^{4i\lambda^2 t} + C\,e^{-4iA^2 t - Ax - Ay}\,B. \tag{6.3.79}$$

Similarly, let us write the Marchenko kernel $\bar{\Omega}(y)$ defined in (2.5.11) in terms of the matrices $\bar{A}$, $\bar{B}$, and $\bar{C}$. In the absence of multiplicities for the bound states, from (2.5.11) we have

$$\bar{\Omega}(y) = \hat{\bar{R}}^{(u,v)}(y) + \sum_{j=1}^{\bar{N}} \bar{c}_j^{(u,v)}\, e^{-i\bar{\lambda}_j y}, \tag{6.3.80}$$

where

$$\hat{\bar{R}}^{(u,v)}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \bar{R}^{(u,v)}(\lambda)\, e^{-i\lambda y}. \tag{6.3.81}$$

With the help of the second equalities in (6.3.70)-(6.3.72), we replace the summation term in (6.3.80) with $\bar{C}\,e^{\bar{A}y}\,\bar{B}$ and get

$$\bar{\Omega}(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \bar{R}^{(u,v)}(\lambda) e^{-i\lambda y} + \bar{C}\,e^{\bar{A}y}\,\bar{B}, \tag{6.3.82}$$

With the help of (6.3.55), (6.3.63), and the second equalities in (6.3.70)-(6.3.72), from (6.3.82) we obtain

$$\bar{\Omega}(y;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \bar{R}^{(u,v)}(\lambda;0)\, e^{-4i\lambda^2 t} e^{-i\lambda y} + \bar{C}\, e^{4i\bar{A}^2 t}\, e^{\bar{A}y}\, \bar{B}, \tag{6.3.83}$$

which can be written as

$$\bar{\Omega}(y;t) = \hat{\bar{R}}^{(u,v)}(y;0)\, e^{-4i\lambda^2 t} + \bar{C}\, e^{4i\bar{A}^2 t + \bar{A}y}\, \bar{B}. \tag{6.3.84}$$

Since $\bar{A}^2$ and $\bar{A}$ commute, from (6.3.84) we get

$$\bar{\Omega}(x+y;t) = \hat{\bar{R}}^{(u,v)}(x+y;0)\, e^{-4i\lambda^2 t} + \bar{C}\, e^{4i\bar{A}^2 t + \bar{A}x + \bar{A}y}\, \bar{B}. \tag{6.3.85}$$

6.4 Time Evolution of the Scattering Data for the Energy-Dependent System

In this section we determine the time evolution of the scattering data associated with (1.0.1) when the potentials q and r in (1.0.1) contain t as a parameter. In that case the potentials satisfy the integrable nonlinear system given in (1.0.2).

We recall that an integrable nonlinear system can be derived from the corresponding AKNS pair $\mathcal{X}$ and $\mathcal{T}$ via (6.0.4). In the literature it is usually but incorrectly stated [1, 3, 4, 6, 18] that the time evolution of the wave function is described by the linear system $\Psi_t^{(\zeta q,\zeta r)} = \mathcal{T}\Psi^{(\zeta q,\zeta r)}$. The correct time evolution is given [12, 13] by saying that $\Psi_t^{(\zeta q,\zeta r)} - \mathcal{T}\Psi^{(\zeta q,\zeta r)}$ must satisfy the linear system $\Psi_x^{(\zeta q,\zeta r)} = \mathcal{X}\Psi^{(\zeta q,\zeta r)}$. In other words, the correct time evolution for $\Psi^{(\zeta q,\zeta r)}$ is obtained by requiring that $\Psi_t^{(\zeta q,\zeta r)} - \mathcal{T}\Psi^{(\zeta q,\zeta r)}$ can be written as a linear combination of two linearly independent solutions to $\Psi_x^{(\zeta q,\zeta r)} = \mathcal{X}\Psi^{(\zeta q,\zeta r)}$. For further details we refer the reader to [12, 13].

In order to determine the time evolution of the scattering data for (1.0.1) we use the AKNS pair ($\mathcal{X}$ and $\mathcal{T}$) given in (6.0.1) and (6.0.2) respectively, and require that $\Psi_t^{(\zeta q,\zeta r)} - \mathcal{T}\,\Psi^{(\zeta q,\zeta r)}$ is a solution to (6.0.3), i.e.

$$\left(\Psi_t^{(\zeta q,\zeta r)} - \mathcal{T}\,\Psi^{(\zeta q,\zeta r)}\right)_x = \mathcal{X}\,\Psi^{(\zeta q,\zeta r)}. \tag{6.4.1}$$

This is equivalent to the requirement that $\Psi_t^{(\zeta q,\zeta r)} - \mathcal{T}\,\Psi^{(\zeta q,\zeta r)}$ is written as a linear combination of the Jost solutions $\phi^{(\zeta q,\zeta r)}$ and $\psi^{(\zeta q,\zeta r)}$ for (1.0.1). Hence, we can write $\phi_t^{(\zeta q,\zeta r)} - \mathcal{T}\,\phi^{(\zeta q,\zeta r)}$ linear combination of $\phi^{(\zeta q,\zeta r)}$ and $\psi^{(\zeta q,\zeta r)}$ as

$$\phi_t^{(\zeta q,\zeta r)} - \mathcal{T}\,\phi^{(\zeta q,\zeta r)} = c_1(\zeta,t)\,\phi^{(\zeta q,\zeta r)}(\zeta,x,t) + c_2(\zeta,t)\,\psi^{(\zeta q,\zeta r)}(\zeta,x,t), \tag{6.4.2}$$

where $c_1(\zeta,t)$ and $c_2(\zeta,t)$ are some scalar coefficients to be determine. For notational simplicity, we suppress the arguments of the Jost solutions and write $\phi^{(\zeta q,\zeta r)}$ for $\phi^{(\zeta q,\zeta r)}(\zeta,x,t)$, $\phi^{(\zeta q,\zeta r)}$ for $\psi^{(\zeta q,\zeta r)}(\zeta,x,t)$, $\bar{\phi}^{(\zeta q,\zeta r)}$ for $\bar{\phi}^{(\zeta q,\zeta r)}(\zeta,x,t)$, and $\bar{\psi}^{(\zeta q,\zeta r)}$ for $\bar{\psi}^{(\zeta q,\zeta r)}(\zeta,x,t)$. Since q and r belong to the Schwartz class, we know that q, r, q_x, and r_x vanish rapidly as $x \to \pm\infty$. Hence, from (6.0.2) we have

$$\mathcal{T} = \begin{bmatrix} -2i\zeta^4 & 0 \\ 0 & 2i\zeta^4 \end{bmatrix} + o(1), \qquad x \to \pm\infty. \tag{6.4.3}$$

With the help of first equality in (3.2.1), (3.1.20), and (6.4.3), from (6.4.2) as $x \to -\infty$, we have

$$\begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix}_t - \begin{bmatrix} -2i\zeta^4 & 0 \\ 0 & 2i\zeta^4 \end{bmatrix} \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} = c_1(\zeta,t) \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + c_2(\zeta,t) \begin{bmatrix} \frac{L}{T} e^{-i\zeta^2 x} \\ \frac{1}{T} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)}. \tag{6.4.4}$$

The spectral parameter ζ does not change in time, i.e. we have $\zeta_t = 0$. Hence, we can simplify (6.4.4) to

$$\begin{bmatrix} 2i\zeta^4 e^{-i\zeta^2 x} \\ 0 \end{bmatrix} = c_1(\zeta,t) \begin{bmatrix} e^{-i\zeta^2 x} \\ 0 \end{bmatrix} + c_2(\zeta,t) \begin{bmatrix} \frac{L}{T} e^{-i\zeta^2 x} \\ \frac{1}{T} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)}. \tag{6.4.5}$$

Thus, we have $c_1(\zeta,t) = 2i\zeta^4$ and $c_2(\zeta,t) = 0$. Hence, we have the time evolution for the Jost solution $\phi^{(\zeta q,\zeta r)}$ as

$$\phi_t^{(\zeta q,\zeta r)} - \mathcal{T}\,\phi^{(\zeta q,\zeta r)} = 2i\zeta^4\,\phi^{(\zeta q,\zeta r)}. \tag{6.4.6}$$

Using (3.3.1) and (6.4.3) in (6.4.6) as $x \to +\infty$, we get

$$\begin{bmatrix} \frac{1}{T}e^{-i\zeta^2 x} \\ \frac{R}{T}e^{i\zeta^2 x} \end{bmatrix}_t^{(\zeta q,\zeta r)} - \begin{bmatrix} -2i\zeta^4 & 0 \\ 0 & 2i\zeta^4 \end{bmatrix} \begin{bmatrix} \frac{1}{T}e^{-i\zeta^2 x} \\ \frac{R}{T}e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)} = 2i\zeta^4 \begin{bmatrix} \frac{1}{T}e^{-i\zeta^2 x} \\ \frac{R}{T}e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)}, \quad (6.4.7)$$

which yields

$$\left(\frac{1}{T}\right)_t^{(\zeta q,\zeta r)} + 2i\zeta^4 \left(\frac{1}{T}\right)^{(\zeta q,\zeta r)} = 2i\zeta^4 \left(\frac{1}{T}\right)^{(\zeta q,\zeta r)}, \quad (6.4.8)$$

$$\left(\frac{R}{T}\right)_t^{(\zeta q,\zeta r)} - 2i\zeta^4 \left(\frac{R}{T}\right)^{(\zeta q,\zeta r)} = 2i\zeta^4 \left(\frac{R}{T}\right)^{(\zeta q,\zeta r)}. \quad (6.4.9)$$

From (6.4.8) we get

$$\left(\frac{1}{T}\right)_t^{(\zeta q,\zeta r)} = 0, \quad (6.4.10)$$

which implies

$$T_t^{(\zeta q,\zeta r)} = 0. \quad (6.4.11)$$

Hence, we obtain

$$T^{(\zeta q,\zeta r)}(\zeta,t) = T^{(\zeta q,\zeta r)}(\zeta,0). \quad (6.4.12)$$

From (6.4.9) we have

$$\left(\frac{R}{T}\right)_t^{(\zeta q,\zeta r)} = 4i\zeta^4 \left(\frac{R}{T}\right)^{(\zeta q,\zeta r)}. \quad (6.4.13)$$

With the help of (6.4.11), from (6.4.13) we get

$$R_t^{(\zeta q,\zeta r)} = 4i\zeta^4 R^{(\zeta q,\zeta r)},$$

which yields

$$R^{(\zeta q,\zeta r)}(\zeta,t) = e^{4i\zeta^4 t} R^{(\zeta q,\zeta r)}(\zeta,0). \quad (6.4.14)$$

Similarly, we know that $\psi_t^{(\zeta q,\zeta r)} - \mathcal{T}\,\psi^{(\zeta q,\zeta r)}$ must be a solution to (6.0.3). Hence, we can write it as linear combination of $\phi^{(\zeta q,\zeta r)}$ and $\psi^{(\zeta q,\zeta r)}$ as

$$\psi_t^{(\zeta q,\zeta r)} - \mathcal{T}\,\psi^{(\zeta q,\zeta r)} = c_3(\zeta,t)\,\psi^{(\zeta q,\zeta r)} + c_4(\zeta,t)\,\phi^{(\zeta q,\zeta r)}, \quad (6.4.15)$$

for some scalar coefficients $c_3(\zeta,t)$ and $c_4(\zeta,t)$, which are to be determined. With the help of the first equality in (3.2.2), (3.3.1), and (6.4.3), from (6.4.15) as $x\to+\infty$, we have

$$\begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix}_t - \begin{bmatrix} -2i\zeta^4 & 0 \\ 0 & 2i\zeta^4 \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} = c_3(\zeta,t) \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + c_4(\zeta,t) \begin{bmatrix} \frac{1}{T} e^{-i\zeta^2 x} \\ \frac{R}{T} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)}, \tag{6.4.16}$$

which simplifies to

$$\begin{bmatrix} 0 \\ -2i\zeta^4 e^{-i\zeta^2 x} \end{bmatrix} = c_3(\zeta,t) \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + c_4(\zeta,t) \begin{bmatrix} \frac{1}{T} e^{-i\zeta^2 x} \\ \frac{R}{T} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)}. \tag{6.4.17}$$

Thus, we have $c_3(\zeta,t) = -2i\zeta^4$ and $c_4(\zeta,t) = 0$. Hence, we have the time evolution for the Jost solution $\psi^{(\zeta q,\zeta r)}$ given by

$$\psi_t^{(\zeta q,\zeta r)} - \mathcal{T}\,\psi^{(\zeta q,\zeta r)} = -2i\zeta^4\,\psi^{(\zeta q,\zeta r)}. \tag{6.4.18}$$

Using (3.3.3) and (6.4.3) in (6.4.18), as $x\to-\infty$ we get

$$\begin{bmatrix} \frac{L}{T} e^{-i\zeta^2 x} \\ \frac{1}{T} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)}_t - \begin{bmatrix} -2i\zeta^4 & 0 \\ 0 & 2i\zeta^4 \end{bmatrix} \begin{bmatrix} \frac{L}{T} e^{-i\zeta^2 x} \\ \frac{1}{T} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)} = -2i\zeta^4 \begin{bmatrix} \frac{L}{T} e^{-i\zeta^2 x} \\ \frac{1}{T} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q,\zeta r)}. \tag{6.4.19}$$

Thus, we obtain

$$\left(\frac{1}{T}\right)_t^{(\zeta q,\zeta r)} - 2i\zeta^4 \left(\frac{1}{T}\right)^{(\zeta q,\zeta r)} = -2i\zeta^4 \left(\frac{1}{T}\right)^{(\zeta q,\zeta r)}, \tag{6.4.20}$$

$$\left(\frac{L}{T}\right)_t^{(\zeta q,\zeta r)} + 2i\zeta^4 \left(\frac{L}{T}\right)^{(\zeta q,\zeta r)} = -2i\zeta^4 \left(\frac{L}{T}\right)^{(\zeta q,\zeta r)}. \tag{6.4.21}$$

From (6.4.20) we get

$$\left(\frac{1}{T}\right)_t^{(\zeta q,\zeta r)} = 0, \tag{6.4.22}$$

which implies

$$T_t^{(\zeta q,\zeta r)} = 0. \tag{6.4.23}$$

Hence, we obtain

$$T^{(\zeta q,\zeta r)}(\zeta,t)=T^{(\zeta q,\zeta r)}(\zeta,0). \tag{6.4.24}$$

From (6.4.21) we have

$$\left(\frac{L}{T}\right)_t^{(\zeta q,\zeta r)}=-4i\zeta^4\left(\frac{L}{T}\right)^{(\zeta q,\zeta r)}. \tag{6.4.25}$$

With the help of (6.4.23), from (6.4.25) we get

$$L_t^{(\zeta q,\zeta r)}=-4i\zeta^4L^{(\zeta q,\zeta r)},$$

which yields

$$L^{(\zeta q,\zeta r)}(\zeta,t)=e^{-4i\zeta^4t}L^{(\zeta q,\zeta r)}(\zeta,0). \tag{6.4.26}$$

In the same manner, we know that $\bar\psi_t^{(\zeta q,\zeta r)}-\mathcal{T}\,\bar\psi^{(\zeta q,\zeta r)}$ must be a solution to (6.0.3). Hence, we can write it as a linear combination of $\bar\phi^{(\zeta q,\zeta r)}$ and $\bar\psi^{(\zeta q,\zeta r)}$ as

$$\bar\psi_t^{(\zeta q,\zeta r)}-\mathcal{T}\,\bar\psi^{(\zeta q,\zeta r)}=c_5(\zeta,t)\,\bar\psi^{(\zeta q,\zeta r)}+c_6(\zeta,t)\,\bar\phi^{(\zeta q,\zeta r)}, \tag{6.4.27}$$

where the scalar coefficients $c_5(\zeta,t)$ and $c_5(\zeta,t)$ are to be determined. With the help of the second equality in (3.2.2), (3.3.2), and (6.4.3), from (6.4.27), as $x\to+\infty$ we have

$$\begin{bmatrix}e^{-i\zeta^2x}\\ 0\end{bmatrix}_t-\begin{bmatrix}-2i\zeta^4 & 0\\ 0 & 2i\zeta^4\end{bmatrix}\begin{bmatrix}e^{-i\zeta^2x}\\ 0\end{bmatrix}=c_5(\zeta,t)\begin{bmatrix}e^{-i\zeta^2x}\\ 0\end{bmatrix}+c_6(\zeta,t)\begin{bmatrix}\frac{\bar R}{\bar T}e^{-i\zeta^2x}\\ \frac{1}{\bar T}e^{i\zeta^2x}\end{bmatrix}^{(\zeta q,\zeta r)}, \tag{6.4.28}$$

which simplifies to

$$\begin{bmatrix}2i\zeta^4e^{-i\zeta^2x}\\ 0\end{bmatrix}=c_5(\zeta,t)\begin{bmatrix}e^{-i\zeta^2x}\\ 0\end{bmatrix}+c_6(\zeta,t)\begin{bmatrix}\frac{\bar R}{\bar T}e^{-i\zeta^2x}\\ \frac{1}{\bar T}e^{i\zeta^2x}\end{bmatrix}^{(\zeta q,\zeta r)}. \tag{6.4.29}$$

Thus, we have $c_5(\zeta,t)=2i\zeta^4$ and $c_6(\zeta,t)=0$. Hence, we have the time evolution for the Jost solution $\bar\psi^{(\zeta q,\zeta r)}$ given by

$$\bar\psi_t^{(\zeta q,\zeta r)}-\mathcal{T}\,\bar\psi^{(\zeta q,\zeta r)}=2i\zeta^4\,\bar\psi^{(\zeta q,\zeta r)}. \tag{6.4.30}$$

Using (3.3.4) and (6.4.3) in (6.4.30) as $x \to -\infty$, we get

$$\begin{bmatrix} \frac{1}{\bar{T}} e^{-i\zeta^2 x} \\ \frac{\bar{L}}{\bar{T}} e^{i\zeta^2 x} \end{bmatrix}_t^{(\zeta q, \zeta r)} - \begin{bmatrix} -2i\zeta^4 & 0 \\ 0 & 2i\zeta^4 \end{bmatrix} \begin{bmatrix} \frac{1}{\bar{T}} e^{-i\zeta^2 x} \\ \frac{\bar{L}}{\bar{T}} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q, \zeta r)} = 2i\zeta^4 \begin{bmatrix} \frac{1}{\bar{T}} e^{-i\zeta^2 x} \\ \frac{\bar{L}}{\bar{T}} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q, \zeta r)}. \quad (6.4.31)$$

Thus, we obtain

$$\left(\frac{1}{\bar{T}}\right)_t^{(\zeta q, \zeta r)} + 2i\zeta^4 \left(\frac{1}{\bar{T}}\right)^{(\zeta q, \zeta r)} = 2i\zeta^4 \left(\frac{1}{\bar{T}}\right)^{(\zeta q, \zeta r)}, \quad (6.4.32)$$

$$\left(\frac{\bar{L}}{\bar{T}}\right)_t^{(\zeta q, \zeta r)} - 2i\zeta^4 \left(\frac{\bar{L}}{\bar{T}}\right)^{(\zeta q, \zeta r)} = 2i\zeta^4 \left(\frac{\bar{L}}{\bar{T}}\right)^{(\zeta q, \zeta r)}. \quad (6.4.33)$$

From (6.4.32) we get

$$\left(\frac{1}{\bar{T}}\right)_t^{(\zeta q, \zeta r)} = 0, \quad (6.4.34)$$

which implies

$$\bar{T}_t^{(\zeta q, \zeta r)} = 0. \quad (6.4.35)$$

Hence, we obtain

$$\bar{T}^{(\zeta q, \zeta r)}(\zeta, t) = \bar{T}^{(\zeta q, \zeta r)}(\zeta, 0). \quad (6.4.36)$$

From (6.4.33) we have

$$\left(\frac{\bar{L}}{\bar{T}}\right)_t^{(\zeta q, \zeta r)} = 4i\zeta^4 \left(\frac{\bar{L}}{\bar{T}}\right)^{(\zeta q, \zeta r)}. \quad (6.4.37)$$

With the help of (6.4.35), from (6.4.37) we get

$$\bar{L}_t^{(\zeta q, \zeta r)} = 4i\zeta^4 \bar{L}^{(\zeta q, \zeta r)},$$

which yields

$$\bar{L}^{(\zeta q, \zeta r)}(\zeta, t) = e^{4i\zeta^4 t} \bar{L}^{(\zeta q, \zeta r)}(\zeta, 0). \quad (6.4.38)$$

Again, the quantity $\bar{\phi}_t^{(\zeta q, \zeta r)} - \mathcal{T}\, \bar{\phi}^{(\zeta q, \zeta r)}$ must be a solution to (6.0.3). Hence, we can write it as a linear combination of $\bar{\phi}^{(\zeta q, \zeta r)}$ and $\bar{\psi}^{(\zeta q, \zeta r)}$ as

$$\bar{\phi}_t^{(\zeta q, \zeta r)} - \mathcal{T}\, \bar{\phi}^{(\zeta q, \zeta r)} = c_7(\zeta, t)\, \bar{\phi}^{(\zeta q, \zeta r)} + c_8(\zeta, t)\, \bar{\psi}^{(\zeta q, \zeta r)}, \quad (6.4.39)$$

for some scalar coefficients $c_7(\zeta, t)$ and $c_8(\zeta, t)$, which are to be determined. With the help of the second equality in (3.2.1), (3.3.4), and (6.4.3), from (6.4.39), $x \to -\infty$ we have

$$\begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix}_t - \begin{bmatrix} -2i\zeta^4 & 0 \\ 0 & 2i\zeta^4 \end{bmatrix} \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} = c_7(\zeta, t) \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + c_8(\zeta, t) \begin{bmatrix} \frac{1}{\bar{T}} e^{-i\zeta^2 x} \\ \frac{\bar{L}}{\bar{T}} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q, \zeta r)}, \tag{6.4.40}$$

which simplifies to

$$\begin{bmatrix} 0 \\ -2i\zeta^4 e^{i\zeta^2 x} \end{bmatrix} = c_7(\zeta, t) \begin{bmatrix} 0 \\ e^{i\zeta^2 x} \end{bmatrix} + c_8(\zeta, t) \begin{bmatrix} \frac{1}{\bar{T}} e^{-i\zeta^2 x} \\ \frac{\bar{L}}{\bar{T}} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q, \zeta r)}. \tag{6.4.41}$$

Thus, we have $c_7(\zeta, t) = -2i\zeta^4$ and $c_8(\zeta, t) = 0$. Hence, we have the time evolution for the Jost solution $\bar{\phi}^{(\zeta q, \zeta r)}$ as

$$\bar{\phi}_t^{(\zeta q, \zeta r)} - \mathcal{T}\, \bar{\phi}^{(\zeta q, \zeta r)} = -2i\zeta^4\, \bar{\phi}^{(\zeta q, \zeta r)}. \tag{6.4.42}$$

Using (3.3.2) and (6.4.3) in (6.4.42) as $x \to +\infty$, we get

$$\begin{bmatrix} \frac{\bar{R}}{\bar{T}} e^{-i\zeta^2 x} \\ \frac{1}{\bar{T}} e^{i\zeta^2 x} \end{bmatrix}_t^{(\zeta q, \zeta r)} - \begin{bmatrix} -2i\zeta^4 & 0 \\ 0 & 2i\zeta^4 \end{bmatrix} \begin{bmatrix} \frac{\bar{R}}{\bar{T}} e^{-i\zeta^2 x} \\ \frac{1}{\bar{T}} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q, \zeta r)} = -2i\zeta^4 \begin{bmatrix} \frac{\bar{R}}{\bar{T}} e^{-i\zeta^2 x} \\ \frac{1}{\bar{T}} e^{i\zeta^2 x} \end{bmatrix}^{(\zeta q, \zeta r)}. \tag{6.4.43}$$

Thus, we obtain

$$\left(\frac{1}{\bar{T}}\right)_t^{(\zeta q, \zeta r)} - 2i\zeta^4 \left(\frac{1}{\bar{T}}\right)^{(\zeta q, \zeta r)} = -2i\zeta^4 \left(\frac{1}{\bar{T}}\right)^{(\zeta q, \zeta r)}, \tag{6.4.44}$$

$$\left(\frac{\bar{R}}{\bar{T}}\right)_t^{(\zeta q, \zeta r)} + 2i\zeta^4 \left(\frac{\bar{R}}{\bar{T}}\right)^{(\zeta q, \zeta r)} = -2i\zeta^4 \left(\frac{\bar{R}}{\bar{T}}\right)^{(\zeta q, \zeta r)}. \tag{6.4.45}$$

From (6.4.44) we get

$$\left(\frac{1}{\bar{T}}\right)_t^{(\zeta q, \zeta r)} = 0, \tag{6.4.46}$$

which implies

$$\bar{T}_t^{(\zeta q, \zeta r)} = 0. \tag{6.4.47}$$

Hence, we obtain

$$\bar{T}^{(\zeta q,\zeta r)}(\zeta,t)=\bar{T}^{(\zeta q,\zeta r)}(\zeta,0). \tag{6.4.48}$$

From (6.4.45) we have

$$\left(\frac{\bar{R}}{\bar{T}}\right)_t^{(\zeta q,\zeta r)}=-4i\zeta^4\left(\frac{\bar{R}}{\bar{R}}\right)^{(\zeta q,\zeta r)}. \tag{6.4.49}$$

With the help of (6.4.47), from (6.4.49) we get

$$\bar{R}_t^{(\zeta q,\zeta r)}=-4i\zeta^4\bar{R}^{(\zeta q,\zeta r)},$$

which yields

$$\bar{R}^{(\zeta q,\zeta r)}(\zeta,t)=e^{-4i\zeta^4 t}\bar{R}^{(\zeta q,\zeta r)}(\zeta,0). \tag{6.4.50}$$

Now, let us evaluate the time evaluation of the dependency constants $\tilde{\gamma}_j(t)$ which are defined (3.4.2), by assuming that the bound-state poles are all simple. Let us copy (3.4.2) for the convenience of the reader as

$$\phi^{(\zeta q,\zeta r)}(\lambda_j,x)=\tilde{\gamma}_j\,\psi^{(\zeta q,\zeta r)}(\lambda_j,x),\qquad j=1,...,N, \tag{6.4.51}$$

where $\lambda=\zeta^2$ and $\lambda_j=\zeta_j^2$. Using (6.4.51) in (6.4.6) we get

$$\left[\tilde{\gamma}_j(t)\,\psi^{(\zeta q,\zeta r)}\right]_t-\mathcal{T}\,\tilde{\gamma}_j(t)\,\psi^{(\zeta q,\zeta r)}=2i\zeta_j^4\,\tilde{\gamma}_j(t)\,\psi^{(\zeta q,\zeta r)},$$

which can be written as

$$\partial_t\tilde{\gamma}_j(t)\,\psi^{(\zeta q,\zeta r)}+\tilde{\gamma}_j(t)\,\psi_t^{(\zeta q,\zeta r)}-\mathcal{T}\,\tilde{\gamma}_j(t)\,\psi^{(\zeta q,\zeta r)}=2i\zeta_j^4\,\tilde{\gamma}_j(t)\,\psi^{(\zeta q,\zeta r)}.$$

or equivalently

$$\partial_t\tilde{\gamma}_j(t)\,\psi^{(\zeta q,\zeta r)}+\tilde{\gamma}_j(t)\left[\psi_t^{(\zeta q,\zeta r)}-\mathcal{T}\,\psi^{(\zeta q,\zeta r)}\right]=2i\zeta_j^4\,\tilde{\gamma}_j(t)\,\psi^{(\zeta q,\zeta r)}. \tag{6.4.52}$$

Using (6.4.18) in (6.4.52) we obtain

$$\partial_t\tilde{\gamma}_j(t)\,\psi^{(\zeta q,\zeta r)}+\tilde{\gamma}_j(t)\left[-2i\zeta^4\,\psi^{(\zeta q,\zeta r)}\right]=2i\zeta_j^4\,\tilde{\gamma}_j(t)\,\psi^{(\zeta q,\zeta r)}.$$

which yields

$$\partial_t \tilde{\gamma}_j(t) = 4i\zeta_j^4\, \tilde{\gamma}_j(t).$$

Thus, we obtain

$$\tilde{\gamma}_j(t) = e^{4i\zeta_j^4 t}\, \tilde{\gamma}_j(0). \tag{6.4.53}$$

Similarly, let us evaluate the time evaluation of the dependency constants $\tilde{\bar{\gamma}}_j(t)$ which are defined in (3.4.4), by assuming that the bound-state poles are all simple. Let us copy (3.4.4) for the convenience of the reader as

$$\bar{\phi}^{(\zeta q, \zeta r)}(\bar{\lambda}_j, x) = \tilde{\bar{\gamma}}_j\, \bar{\psi}^{(\zeta q, \zeta r)}(\bar{\lambda}_j, x), \qquad j = 1, ..., \bar{N}. \tag{6.4.54}$$

where $\lambda = \zeta^2$ and $\lambda_j = \zeta_j^2$. Using (6.4.54) in (6.4.42) we get

$$\left[\tilde{\bar{\gamma}}_j(t)\, \bar{\psi}^{(\zeta q, \zeta r)}\right]_t - \mathcal{T}\, \tilde{\bar{\gamma}}_j(t)\, \bar{\psi}^{(\zeta q, \zeta r)} = -2i\bar{\zeta}_j^4\, \tilde{\bar{\gamma}}_j(t)\, \bar{\psi}^{(\zeta q, \zeta r)},$$

which can be written as

$$\partial_t \tilde{\bar{\gamma}}_j(t)\, \bar{\psi}^{(\zeta q, \zeta r)} + \tilde{\bar{\gamma}}_j(t)\, \bar{\psi}_t^{(\zeta q, \zeta r)} - \mathcal{T}\, \tilde{\bar{\gamma}}_j(t)\, \bar{\psi}^{(\zeta q, \zeta r)} = -2i\bar{\zeta}_j^4 \tilde{\bar{\gamma}}_j(t)\, \bar{\psi}^{(\zeta q, \zeta r)}.$$

or equivalently

$$\partial_t \tilde{\bar{\gamma}}_j(t)\, \bar{\psi}^{(\zeta q, \zeta r)} + \tilde{\bar{\gamma}}_j(t) \left[\bar{\psi}_t^{(\zeta q, \zeta r)} - \mathcal{T}\, \bar{\psi}^{(\zeta q, \zeta r)}\right] = -2i\bar{\zeta}_j^4\, \tilde{\bar{\gamma}}_j(t)\, \bar{\psi}^{(\zeta q, \zeta r)}. \tag{6.4.55}$$

Using (6.4.30) in (6.4.55) we obtain

$$\partial_t \tilde{\bar{\gamma}}_j(t)\, \bar{\psi}^{(\zeta q, \zeta r)} + \tilde{\bar{\gamma}}_j(t) \left[2i\zeta^4\, \bar{\psi}^{(\zeta q, \zeta r)}\right] = -2i\bar{\zeta}_j^4\, \tilde{\bar{\gamma}}_j(t)\, \bar{\psi}^{(\zeta q, \zeta r)}.$$

which yields

$$\partial_t \tilde{\bar{\gamma}}_j(t) = -4i\bar{\zeta}_j^4\, \tilde{\bar{\gamma}}_j(t).$$

Thus, we obtain

$$\tilde{\bar{\gamma}}_j(t) = e^{-4i\bar{\zeta}_j^4 t}\, \tilde{\bar{\gamma}}_j(0). \tag{6.4.56}$$

Now, let us evaluate the time evolution of the norming constants $c_j^{(\zeta q,\zeta r)}$ by assuming that the bound-state poles are all simple. With the help of (2.3.22), (6.4.12), and (6.4.53), we determine the time evolution of the norming constant $\bar{c}_j^{(\zeta q,\zeta r)}$ as

$$c_j^{(\zeta q,\zeta r)}(t) = c_j^{(\zeta q,\zeta r)}(0)\, e^{4i\zeta_j^4 t}, \qquad j = 1, ..., N. \tag{6.4.57}$$

In the same manner, let us evaluate the time evolution of the norming constants $\bar{c}_j^{(\zeta q,\zeta r)}$ by assuming that the bound-state poles are all simple. With the help of (2.3.23), (6.4.36), and (6.4.56), we determine the time evolution of the norming constant $c_j^{(\zeta q,\zeta r)}$ as

$$\bar{c}_j^{(\zeta q,\zeta r)}(t) = \bar{c}_j^{(\zeta q,\zeta r)}(0)\, e^{-4i\bar{\zeta}_j^4 t}, \qquad j = 1, ..., \bar{N}. \tag{6.4.58}$$

In case the bound states have multiplicities, we can use the method of [10, 20, 21] and express the bound-state data $\left\{\lambda_j, \left\{c_{jk}^{(u,v)}\right\}_{k=0}^{m_j-1}\right\}_{j=1}^{N}$ in terms of three matrices A, B, and C in the form $C\, e^{-Ay+f(A)t}\, B$, with appropriate matrix sizes and with an appropriate choice of $f(A)$ and similarly express the bound-state data $\left\{\bar{\lambda}_j, \left\{\bar{c}_{jk}^{(u,v)}\right\}_{k=0}^{m_j-1}\right\}_{j=1}^{\bar{N}}$ in terms of three matrices $\bar{A}$, $\bar{B}$, and $\bar{C}$ in the form $\bar{C}\, e^{-\bar{A}y+\bar{f}(\bar{A})t}\, \bar{B}$, with appropriate matrix sizes and with an appropriate choice of $\bar{f}(\bar{A})$. In case the bound states do not have multiplicities, the alternate Merchenko kernels defined in (4.2.11) can be written as

$$G^{(u,v)}(y) := \int_y^\infty dz \left[\hat{R}^{(u,v)}(z) + \sum_{j=1}^{N} c_j^{(u,v)}\, e^{i\lambda_j z}\right], \tag{6.4.59}$$

$$\bar{G}^{(u,v)}(y) := \int_y^\infty dz \left[\hat{\bar{R}}^{(u,v)}(z) + \sum_{j=1}^{\bar{N}} \bar{c}_j^{(u,v)}\, e^{-i\bar{\lambda}_j z}\right]. \tag{6.4.60}$$

In order to account for bound states with multiplicities, with the help of the first equalities in (6.3.70)-(6.3.72), we replace the summation term in (6.4.59) with $C\, e^{-Az}\, B$ and obtain

$$G^{(u,v)}(y) = \int_y^\infty dz \left[\hat{R}^{(u,v)}(z) + C\, e^{-Az}\, B\right], \tag{6.4.61}$$

which simplifies to

$$G^{(u,v)}(y) = \int_y^\infty dz \hat{R}^{(u,v)}(z) + C\, A^{-1} e^{-Ay}\, B. \tag{6.4.62}$$

Now, let us consider the time-evolved scattering data and determine the corresponding time-evolved alternate Marchenko kernel. Using (6.3.19) and (6.3.62) in (6.4.62) we obtain

$$G^{(u,v)}(y;t) = \int_y^\infty dz \hat{R}^{(u,v)}(z;t) + C\, A^{-1}\, e^{-4iA^2 t} e^{-Ay}\, B, \tag{6.4.63}$$

where

$$\hat{R}^{(u,v)}(z;t) = \frac{1}{2\pi} \int_{-\infty}^\infty d\lambda\, R^{(u,v)}(\lambda;0)\, e^{4i\lambda^2 t}\, e^{i\lambda z}, \tag{6.4.64}$$

or equivalently

$$\hat{R}^{(u,v)}(z;t) = \hat{R}^{(u,v)}(z;0)\, e^{4i\lambda^2 t}. \tag{6.4.65}$$

Using (6.4.65) in (6.4.63) we have

$$G^{(u,v)}(y;t) = \int_y^\infty dz \hat{R}^{(u,v)}(z;0)\, e^{4i\lambda^2 t} + C\, A^{-1}\, e^{-4iA^2 t - Ay}\, B. \tag{6.4.66}$$

Similarly, in case the bound states with multiplicities, with the help of the second equalities in (6.3.70)-(6.3.72), we replace the summation term in (6.4.60) with $\bar{C}\, e^{\bar{A}z}\, \bar{B}$ and obtain

$$\bar{G}^{(u,v)}(y) = \int_y^\infty dz \left[\hat{\bar{R}}^{(u,v)}(z) + \bar{C}\, e^{\bar{A}z}\, \bar{B} \right], \tag{6.4.67}$$

which can be written as

$$\bar{G}^{(u,v)}(y) = \int_y^\infty dz \hat{\bar{R}}^{(u,v)}(z) - \bar{C}\, \bar{A}^{-1}\, e^{\bar{A}y}\, \bar{B}. \tag{6.4.68}$$

Now, let us consider the time-evolved scattering data and determine the corresponding alternate Marchenko kernel. Using (6.3.55) and (6.3.63) in (6.4.68) we obtain

$$\bar{G}^{(u,v)}(y;t) = \int_y^\infty dz\, \hat{\bar{R}}^{(u,v)}(z;t) - \bar{C}\, \bar{A}^{-1}\, e^{4i\bar{A}^2 t}\, e^{\bar{A}y}\, \bar{B}, \tag{6.4.69}$$

where

$$\hat{\bar{R}}^{(u,v)}(z;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \bar{R}^{(u,v)}(\lambda;0)\, e^{-4i\lambda^2 t}\, e^{-i\lambda z}, \tag{6.4.70}$$

or equivalently

$$\hat{\bar{R}}^{(u,v)}(z;t) = \hat{\bar{R}}^{(u,v)}(z;0)\, e^{-4i\lambda^2 t}. \tag{6.4.71}$$

Using (6.4.71) in (6.4.69) we have

$$\bar{G}^{(u,v)}(y;t) := \int_y^{\infty} dz\, \hat{\bar{R}}^{(u,v)}(z;0)\, e^{-4i\lambda^2 t} - \bar{C}\, \bar{A}^{-1}\, e^{4i\bar{A}^2 t + \bar{A}y}\, \bar{B}. \tag{6.4.72}$$

Note that $\lambda_j \in \mathbb{C}^+$ and $\bar{\lambda}_j \in \mathbb{C}^-$ and hence e^{-Az} appearing in (6.4.61) is integrable in $z \in (y,\infty)$ and $e^{\bar{A}z}$ appearing in (6.4.67) is integrable in $z \in (y,\infty)$. In case the bound states do not have multiplicities, the alternate Merchenko kernels defined in (4.2.12) can be written as

$$G^{(p,s)}(y) := \int_y^{\infty} dz \left[\hat{R}^{(p,s)}(z) + \sum_{j=1}^{N} c_j^{(p,s)}\, e^{i\lambda_j z}\right], \tag{6.4.73}$$

$$\bar{G}^{(u,v)}(y) := \int_y^{\infty} dz \left[\hat{\bar{R}}^{(u,v)}(z) + \sum_{j=1}^{\bar{N}} \bar{c}_j^{(u,v)}\, e^{-i\bar{\lambda}_j z}\right]. \tag{6.4.74}$$

In order to account for bound states with multiplicities, with the help of the first equalities in (6.3.70)-(6.3.72), we replace the summation term in (6.4.73) with $C\, e^{-Az}\, B$ and obtain

$$G^{(p,s)}(y) = \int_y^{\infty} dz \left[\hat{R}^{(p,s)}(z) + C\, e^{-Az}\, B\right], \tag{6.4.75}$$

which simplifies to

$$G^{(p,s)}(y) = \int_y^{\infty} dz \hat{R}^{(p,s)}(z) + C\, A^{-1} e^{-Ay}\, B. \tag{6.4.76}$$

Now, let us consider the time-evolved scattering data and determine the corresponding time-evolved alternate Marchenko kernel. Using (6.3.19) and (6.3.62) in (6.4.76) we obtain

$$G^{(p,s)}(y;t) = \int_y^{\infty} dz \hat{R}^{(p,s)}(z;t) + C\, A^{-1}\, e^{-4iA^2 t} e^{-Ay}\, B, \tag{6.4.77}$$

where

$$\hat{R}^{(p,s)}(z;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, R^{(p,s)}(\lambda;0) \, e^{4i\lambda^2 t} \, e^{i\lambda z}, \tag{6.4.78}$$

or equivalently

$$\hat{R}^{(p,s)}(z;t) = \hat{R}^{(p,s)}(z;0) \, e^{4i\lambda^2 t}. \tag{6.4.79}$$

Using (6.4.79) in (6.4.77) we have

$$G^{(p,s)}(y;t) = \int_{y}^{\infty} dz \hat{R}^{(p,s)}(z;0) \, e^{4i\lambda^2 t} + C \, A^{-1} \, e^{-4iA^2 t - Ay} \, B. \tag{6.4.80}$$

Similarly, in case the bound states with multiplicities, with the help of the second equalities in (6.3.70)-(6.3.72), we replace the summation term in (6.4.74) with $\bar{C} \, e^{\bar{A}z} \, \bar{B}$

$$\bar{G}^{(p,s)}(y) = \int_{y}^{\infty} dz \left[\hat{\bar{R}}^{(p,s)}(z) + \bar{C} \, e^{\bar{A}z} \, \bar{B} \right], \tag{6.4.81}$$

which can be written as

$$\bar{G}^{(p,s)}(y) = \int_{y}^{\infty} dz \hat{\bar{R}}^{(p,s)}(z) - \bar{C} \, \bar{A}^{-1} \, e^{\bar{A}y} \, \bar{B}. \tag{6.4.82}$$

Now, let us consider the time-evolved scattering data and determine the corresponding time-evolved alternate Marchenko kernel. Using (6.3.55) and (6.3.63) in (6.4.82) we obtain

$$\bar{G}^{(p,s)}(y;t) := \int_{y}^{\infty} dz \, \hat{\bar{R}}^{(p,s)}(z;t) - \bar{C} \, \bar{A}^{-1} \, e^{4iA^2 t} \, e^{\bar{A}y} \, \bar{B}. \tag{6.4.83}$$

where

$$\hat{\bar{R}}^{(p,s)}(z;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \bar{R}^{(p,s)}(\lambda;0) \, e^{-4i\lambda^2 t} \, e^{-i\lambda z}, \tag{6.4.84}$$

or equivalently

$$\hat{\bar{R}}^{(p,s)}(z;t) = \hat{\bar{R}}^{(p,s)}(z;0) \, e^{-4i\lambda^2 t}. \tag{6.4.85}$$

Using (6.4.85) in (6.4.83) we have

$$\bar{G}^{(p,s)}(y;t) = \int_{y}^{\infty} dz \, \hat{\bar{R}}^{(p,s)}(z;0) \, e^{-4i\lambda^2 t} - \bar{C} \, \bar{A}^{-1} \, e^{4i\bar{A}^2 t + \bar{A}y} \, \bar{B}. \tag{6.4.86}$$

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Biographical Information

Ramazan Ercan was born in Malatya, Turkey, in 1984. He received his B.S. degree in Mathematics from Erciyes University, Turkey, in 2008 and his M.S. degree in Mathematics from the University of Texas at Arlington in 2013. He started the Ph.D. program in Mathematics at the University of Texas at Arlington in 2013. He completed his Ph.D. under the supervision of Prof. T. Aktosun in May 2018. His research involves studies of inverse problems, scattering and integrable nonlinear evolution equations.