Some New Results for Equilibria of N-Person Games

by

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To my parents and my brother

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Abstract

Some New Results for Equilibria of N-Person Games

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In this dissertation, we present four journal articles in the area of game theory. In the first article, we define a generalized equilibrium for $n-$person normal form games. We prove that the Nash equilibrium and the mixed Berge equilibrium are special cases of the generalized equilibrium. In the second article, we study the computational complexity of finding a mixed Berge equilibrium in $n-$person normal form games. In particular, we prove that the problem is an NP-complete problem for $n \geq 3$. In the third article, we give an interpretation of mixed strategies via resource allocation. Finally, in the fourth article, we extend the concept of the mixed Berge equilibrium to $n-$person extensive form games.

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Chapter 1

Introduction

Game theory is the study of competitive situations among rational players, who choose their strategies in order to maximize their expected utilities based on their expectations of other players' behaviors. Games can be in normal or extensive form. The most used solution concept in game theory is the Nash equilibrium (NE). It was introduced in [1] and [2]. The NE assumes that every player wants to maximize his own expected payoff. The other solution concept we consider in this dissertation is the Berge equilibrium (BE). The BE, a pure strategy concept, was introduced in [3] and formally defined in [4]. A BE strategy means that every player other than player i wants to maximize player i's expected payoff. The BE was extended to the dual equilibrium (DE) or the mixed Berge equilibrium (MBE) in [5]. In this dissertation, we present four journal articles to which the two authors contributed equally.

In Chapter 2, we present the first article. We define a generalized equilibrium for $n-$person normal form games. In this article, we address a very important issue in the study of game theory which is the computation of the equilibrium points. For example, a nonlinear programming approach was introduced in [6] and [7] to find an NE in $3-$person and $n-$person games respectively. We extend their approach to the generalized equilibrium. We prove that a generalized equilibrium exists if and only if the maximum of a nonlinear program is 0. We also prove that both the NE and the MBE are special cases of the generalized equilibrium.

In the Chapter 3, we present a second article on the computational complexity of finding an MBE.The computational complexity of finding a Nash equilibrium is a well-studied problem in literature. The computational complexity of finding a pure BE was studied in [8]. However, in our article, we prove that finding an MBE is a PPAD-complete problem in the case of a $2-$person game and it is an NP-complete problem when $n \geq 3$.

In Chapter 4, we present an article to deal with the difficulties associated with mixed strategies. The concept of mixed strategies is widely used in game theory. However, the concept of mixed strategy requires a randomization process. For example see [9]. We show that mixed strategies can be interpreted as a resource allocation strategy. In other words, we show that a mixed strategy at an equilibrium is equivalent to each player allocating some resource among different strategies.

In Chapter 5, we present our fourth article. In this article, we extend the concept of the MBE to the $n-$person games in extensive form. Furthermore, we define the concept of a subgame perfect Berge equilibrium.

In Chapter 6, we give our conclusions.

The references for Chapter 1 are given below, in addition those in the articles of Chapters 2-5.

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Chapter 2

A Nonlinear Programming Approach to Determine a Generalized Equilibrium for n-Person Normal Form Games

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Abstract

A generalized equilibrium for finite $n-$person normal form games is defined as a collection of mixed strategies with the following property: no player in some subset B of the players can achieve a better expected payoff if players in an associated set G change strategies unilaterally. A generalized equilibrium is proved to exist for a game if and only if the maximum objective function value of a certain nonlinear programming problem is zero, in which case the solution to the nonlinear program yields a generalized equilibrium.

2.1 Introduction

Game theory is the study of mathematical decision making among players who make individual choices according to their personal notions of rationality and according to their expectations of the other players actions. For example, a player may act selfishly for his own personal gain or cooperatively for the benefit of other players. [1] proved, using fixed point theorems, the existence of a mixed Nash equilibrium (*NE*) for all noncooperative games where no player can obtain a better payoff by unilaterally changing his strategy. In other words in an *NE* the players are only concerned with their own self interest. The computation of the *NE* has been an active area of research. [2] modeled the problem of finding an *NE* for bimatrix games as a quadratic programming problem. [3] modeled the problem of finding an *NE* for 3−person games as a nonlinear optimization problem, while [4] did the same for n−person games.

A different solution concept was proposed by [5]. He developed a pure strategy refinement for the *NE* that was formalized by [6]. In a Berge equilibrium (*BE*) a unilateral change of strategy by any one player cannot increase another player's payoff. Many researchers have studied the *BE*, for example see [7], [8], and [9]. Existence theorems for a *BE* were considered in [10], [11], [12], [13], and [14]. For existence conditions also see [15],[16], [17], and [18]. [19] extended the pure *BE* to a mixed Berge equilibrium (*MBE*) in normal form n−person games and showed an *MBE* need not exist for $n > 2$.

For computational approaches to finding a *BE*, see [20] where an algorithm was developed for computing all *BE* in normal form games, and [21] who presented an evolutionary approach for detecting Berge and Nash equilibrium.

In this paper, we present a definition for a mixed generalized equilibrium (*GE*) in finite normal form n−person games for which both the *NE* and the *MBE* are special cases. The *GE* is an extension of the P/K-equilibrium [5] to mixed strategies. We then extend the nonlinear programming approach in [4] for both proving the existence and finding a *GE* for finite n−person games where each of the players wants to maximize the expected payoff for one or more of the players, including the cases of only himself or all other players.

This paper is organized as follows. In Section 2, needed notation is given. In Section 3, the *GE* is formally defined and the *NE* and the *MBE* are shown to be special cases. In Section 4, we give a nonlinear program for obtaining a *GE* if one exists. Numerical examples are presented in Section 5.

2.2 Preliminaries

In this paper, let $\Gamma = (I, (S_i)_{i\in I}, (u_i)_{i\in I})$ be an n−person normal form game, where $I = \{1, \dots, n\}$ is the set of the n players, and $S_i = \left\{s_i^1, \dots, s_i^{m_i}\right\}$ is the set of m_i pure strategies available for player i. Player i chooses each strategy s_i^j with probability $\sigma_i(s_i^j)$. A mixed strategy for player i is a probability distribution over the player's pure strategies set, $\sigma_i = (\sigma_i^1, \dots, \sigma_i^{m_i})$, where $\sum_{j=1}^{m_i} \sigma_i(s_i^j) = 1$, and $\sigma_i(s_i^j) \geq 0, j = 1, \dots, m_i$. Denote the set of mixed strategies for player i by $\triangle S_i$. A pure strategy is a special case of a mixed strategy where a player chooses one strategy with probability 1 and the remaining strategies with probability 0. A strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$ is the $n-$tuple of the n players mixed strategies.

The set of joint pure strategies of all players other than player i, is $S_{-i} = \left\{s_{-i}^1, \dots, s_{-i}^{m_{-i}}\right\}$, where $m_{-i} = \prod_{j \in I-i} m_j$ is the number of joint pure strategies available for all the players other than player i. The joint probability $\sigma_{-i}(s_{-i}^k)$ is the probability that all players other than player i play the joint pure strategy s_{-i}^k and is the product of the probability that each player in $I - i$ chooses his corresponding strategy. Note that $\sigma_{-i} = (\sigma_{-i}^1, \dots, \sigma_{-i}^{m_{-i}})$, where $\sum_{j=1}^{m_{-i}} \sigma_{-i}(s_{-i}^j) = 1$ and $\sigma_{-i}(s_{-i}^j) \geq 0, j = 1, \dots, m_{-i}$. The set of mixed strategies for all players other than player i is $\triangle S_{-i}$. Let $u_i(\sigma_i, \sigma_{-i})$ be the expected payoff for player i when player i plays the mixed strategy σ_i and the rest of the players play the mixed strategy σ_{-i}.

The following identities represent the expected payoff for player i from [19]. If player i chooses the mixed strategy σ_i and the rest of players choose the mixed strategy σ_{-i}, then player's i expected payoff is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{j=1}^{m_i} \sum_{k=1}^{m_{-i}} \sigma_i(s_i^j) \sigma_{-i}(s_{-i}^k) u_i(s_i^j, s_{-i}^k), \forall i \in I.$$

If player i chooses a pure strategy s_i^j and the rest of players choose the mixed strategy σ_{-i}, then player's i expected payoff is

$$u_i(s_i^j, \sigma_{-i}) = \sum_{k=1}^{m_{-i}} \sigma_{-i}(s_{-i}^k) u_i(s_i^j, s_{-i}^k), \forall i \in I.$$

We extend the approach of [12] for the *BE* and the *NE* to a *GE*.

Definition 2.1. *Let D be an index set and let $\{G_d\}_{d \in D}$ be a family of nonempty proper and distinct subsets of the set of all players I such that $\cup_{d \in D} G_d = I$. Let $\{B_d\}_{d \in D}$ be a family of nonempty subsets of the set of all players I such that $\cup_{d \in D} B_d = I$. The players in each of the proper subsets G_d want to maximize the expected payoff for each individual player in the associated subset B_d.*

Note $\cup_{d \in D} G_d = I$ and $\cup_{d \in D} B_d = I$; otherwise the game is reduced to one with fewer number of players. G_d is a proper subset or the problem of finding a *GE* becomes a series of maximization problems. Define $-G_d = I - G_d$ to be the set of all players other than the players in the proper subset G_d. In the case of subsets of players G_d, the joint strategy set is the Cartesian product $S_{G_d} = \times_{i \in G_d} S_i$ of the individual players in G_d pure strategy sets. The number of joint pure strategies for the players in the proper subset G_d is denoted by m_{G_d}.

The probability that the players in G_d choose a joint pure strategy is the product of the probability that each individual player in G_d chooses his corresponding individual strategy. A mixed strategy for the proper subset G_d is given by the probability distribution $\sigma_{G_d} = (\sigma_{G_d}^1, \dots, \sigma_{G_d}^{m_{G_d}})$, where $\sum_{j=1}^{m_{G_d}} \sigma_{G_d}(s_{G_d}^j) = 1$, and $\sigma_{G_d}(s_{G_d}^j) \geq 0, j = 1, \dots, m_{G_d}$. The set of mixed strategies for players in G_d is denoted by $\triangle S_{G_d}$. Let S_{-G_d} be the set of the m_{-G_d} joint pure strategies for all players in $-G_d$. As an example, consider a four player game where each player has two strategies. $S_1 = \left\{s_1^1, s_1^2\right\}$,

$S_2 = \{s_2^1, s_2^2\}$, $S_3 = \{s_3^1, s_3^2\}$, and $S_4 = \{s_4^1, s_4^2\}$. Let $G_1 = \{1,2\}$, note that $-G_1 = I - G_1 = \{3,4\}$. The number of pure joint strategies for players in G_1 is $m_{G_1} = m_1 \times m_2 = 2 \times 2 = 4$, and the set of pure strategies for players in G_1 is $S_{G_1} = \{(s_1^1, s_2^1), (s_1^1, s_2^2), (s_1^2, s_2^1), (s_1^2, s_2^2)\}$. The probability that the players in G_1 choose their first strategy is $\sigma_{G_1}(s_{G_1}^1) = \sigma_1(s_1^1) \times \sigma_2(s_2^1)$.

With G_d and $-G_d$ as two individual players, the following identities can be derived from (2.1) and (2.1). Player's i expected payoff when players in G_d choose the mixed strategy σ_{G_d} and players in $-G_d$ choose the mixed strategy σ_{-G_d} is,

$$u_i(\sigma_{G_d}, \sigma_{-G_d}) = \sum_{j=1}^{m_{G_d}} \sum_{k=1}^{m_{-G_d}} \sigma_{G_d}(s_{G_d}^j)\sigma_{-G_d}(s_{-G_d}^k)u_i(s_{G_d}^j, s_{-G_d}^k).$$

Player's i expected payoff when players in G_d choose their pure joint strategy $s_{G_d}^j$ and players in $-G_d$ choose the mixed strategy σ_{-G_d} is,

$$u_i(s_{G_d}^j, \sigma_{-G_d}) = \sum_{k=1}^{m_{-G_d}} \sigma_{-G_d}(s_{-G_d}^k)u_i(s_{G_d}^j, s_{-G_d}^k).$$

Proposition 2.1. *The identities (2.1) and (2.1) are equivalent, as for (2.1) and (2.1).*

Proof. Since $G_d \cup -G_d = I$, then either $i \in G_d$ or $i \in -G_d$. We provide the proof for $i \in G_d$ and it is similar for $i \in -G_d$. From (2.1) and since $-G_d = I - G_d$, then $u_i(\sigma_{G_d}, \sigma_{-G_d}) = u_i(\sigma_i, \sigma_{G_d - i}, \sigma_{-G_d}) = u_i(\sigma_i, \sigma_{-i})$. Hence (2.1) and (2.1) are equivalent. Similarly, from (2.1), $u_i(s_{G_d}^j, \sigma_{-G_d}) = u_i(s_i^j, s_{G_d-i}^k, \sigma_{-G_d})$. But, a pure strategy for $G_d - i$ is a special case of a mixed strategy. Hence, $u_i(s_{G_d}^j, \sigma_{-G_d}) = u_i(s_i^j, \sigma_{-i})$, and (2.1) and (2.1) are equivalent. □

2.3 The Generalized Equilibrium

In this section we define a *GE* for a normal form n–person game. We then show that both the *NE* and the *MBE* are special cases of the *GE*.

Definition 2.2. *(GE) A strategy σ^* is a GE if and only if*

$$u_i(\sigma^*) \geq u_i(\sigma_{G_d}, \sigma_{-G_d}^*), \forall \sigma_{G_d} \in \triangle S_{G_d}, \forall i \in B_d, \forall d \in D.$$

In Definition 2.2, all players in G_d share the goal of maximizing the individual expected payoff for each player in the corresponding B_d. Moreover, the definition of a *GE* implies no player $i \in B_d$ for any $d \in D$ gets a better expected payoff if any player in the corresponding distinct proper subset G_d change his strategy unilaterally. For comparison with the *GE*, we formally define the *NE* and the *MBE*.

Definition 2.3. *(NE) A strategy σ^* is an NE if and only if*

$$u_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \forall \sigma_i \in \triangle S_i, \forall i \in I.$$

In an *NE*, no player with a unilateral change of strategy can increase his expected payoff.

Definition 2.4. *(MBE) A strategy σ^* is an MBE if and only if*

$$u_i(\sigma^*) \geq u_i(\sigma_i^*, \sigma_{-i}), \forall \sigma_{-i} \in \triangle S_{-i}, \forall i \in I.$$

In an *MBE*, no player with a unilateral change of strategy can increase another player's expected payoff. As opposed to the mixed *NE* which is guaranteed to exist, an *MBE* exists only when the intersection of the set of fixed points for n correspondences is not empty. See [19] for topolgical conditions for the *MBE* existence. We now show that the *NE* and the *MBE* are special cases of the *GE*.

Proposition 2.2. *Let $D = I$, $G_i = \{i\}$ and $B_i = \{i\}$ $\forall i \in I$, then a GE is an NE.*

Proof. From Definition 2.2 a strategy σ^* is a *GE* if and only if

$$u_i(\sigma^*) \geq u_i(\sigma_{G_i}, \sigma^*_{-G_i}), \forall \sigma_{G_i} \in \triangle S_{G_i}, \forall i \in I.$$

Since $G_i = \{i\}$, $-G_i = \{-i\}$, and $B_i = \{i\}$, then

$$u_i(\sigma^*) \geq u_i(\sigma_i, \sigma^*_{-i}), \forall \sigma_i \in \triangle S_i, \forall i \in I,$$

so the *GE* is an *NE* by Definition 4.1. □

Proposition 2.3. *Let $D = I$, $G_i = \{-i\}$ and $B_i = \{i\}$ $\forall i \in I$, then a GE is an MBE.*

Proof. From Definition 2.2 a strategy σ^* is a *GE* if and only if

$$u_i(\sigma^*) \geq u_i(\sigma_{G_i}, \sigma^*_{-G_i}), \forall \sigma_{G_i} \in \triangle S_{G_i}, \forall i \in I.$$

Since $G_i = \{-i\}$, $-G_i = \{i\}$, and $B_i = \{i\}$, then

$$u_i(\sigma^*) \geq u_i(\sigma_i^*, \sigma_{-i}), \forall \sigma_{-i} \in \triangle S_{-i}, \forall i \in I,$$

so the *GE* is an *MBE* by Definition 2.4. □

2.4 Existence and Computation

In this section we give necessary and sufficient conditions for the existence of a *GE*. We then present a nonlinear program that finds a *GE* if and only if the maximum of the nonlinear program is 0.

Lemma 2.1. *For a game Γ, let $\beta^* = (\beta_1^*, \ldots, \beta_n^*)$ be the expected payoffs for the n players and let*

$\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ *be a GE. Then*

$$u_i(s_{G_d}^j, \sigma_{-G_d}^*) = \sum_{k=1}^{m_{-G_d}} \sigma_{-G_d}^*(s_{-G_d}^k) u_i(s_{G_d}^j, s_{-G_d}^k) \leq \beta_i^*,$$
$$\forall s_{G_d}^j \in S_{G_d}, \ \forall i \in B_d, \ \forall d \in D.$$

Proof. Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a GE for the game. Then from Definition 2.2

$$u_i(\sigma^*) = \beta_i^* = \sum_{j=1}^{m_{G_d}} \sum_{k=1}^{m_{-G_d}} \sigma_{G_d}^*(s_{G_d}^j)\sigma_{-G_d}^*(s_{-G_d}^k) u_i(s_{G_d}^j, s_{-G_d}^k) \geq$$
$$\sum_{j=1}^{m_{G_d}} \sum_{k=1}^{m_{-G_d}} \sigma_{G_d}(s_{G_d}^j)\sigma_{-G_d}^*(s_{-G_d}^k) u_i(s_{G_d}^j, s_{-G_d}^k),$$
$$\forall \sigma_{G_d} \in \triangle S_{G_d}, \ \forall i \in B_d, \ \forall d \in D.$$

Assume players in G_d choose a pure strategy $s_{G_d}^j$. Then $\sigma(s_{G_d}) = (0, \ldots, 1, \ldots, 0)$, and

$$\beta_i^* = \sum_{j=1}^{m_{G_d}} \sum_{k=1}^{m_{-G_d}} \sigma_{G_d}^*(s_{G_d}^j)\sigma_{-G_d}^*(s_{-G_d}^k) u_i(s_{G_d}^j, s_{-G_d}^k) \geq \sum_{k=1}^{m_{-G_d}} \sigma_{-G_d}^*(s_{-G_d}^k) u_i(s_{G_d}^j, s_{-G_d}^k). \tag{2.1}$$

Since $\sum_{k=1}^{m_{-G_d}} \sigma_{-G_d}^*(s_{-G_d}^k) u_i(s_{G_d}^j, s_{-G_d}^k) = u_i(s_{G_d}^j, \sigma_{-G_d}^*)$, then

$$u_i(s_{G_d}^j, \sigma_{-G_d}^*) \leq \beta_i^*, \forall s_{G_d}^j \in S_{G_d}, \forall i \in B_d, \forall d \in D. \tag{2.2}$$

This completes the proof. □

We next prove necessary and sufficient conditions for the existence of a *GE*.

Theorem 2.1. *For a game* Γ*, suppose there exists an* $n-$*tuple* $\beta^* = (\beta_1^*, \ldots, \beta_n^*)$ *and a mixed strategy profile* $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ *such that*

$$\sum_{k=1}^{m_{-G_d}} \sigma_{-G_d}^*(s_{-G_d}^k) u_i(s_{G_d}^j, s_{-G_d}^k) \leq \beta_i^*, \ \forall s_{G_d}^j \in S_{G_d}, \ \forall i \in B_d, \ \forall d \in D. \tag{2.3}$$

Then σ^* *is a GE and* $u_i(\sigma^*) = \beta_i^*, \ \forall i \in I$*, if and only if* $\sigma_{G_d}^*(s_{G_d}^j) = 0$ *whenever*

$$\sum_{k=1}^{m_{-G_d}} \sigma_{-G_d}^*(s_{-G_d}^k) u_i(s_{G_d}^j, s_{-G_d}^k) < \beta_i^*. \tag{2.4}$$

Proof. Let σ^* be a *GE* and $u_i(\sigma^*) = \beta_i^*,\ \forall i \in I$, then by Lemma 2.1,

$$\sum_{k=1}^{m_{-G_d}} \sigma^*_{-G_d}(s^k_{-G_d})u_i(s^j_{G_d}, s^k_{-G_d}) \le \beta_i^*, \forall s^j_{G_d} \in S_{G_d},\ \forall i \in B_d,\ \forall d \in D.$$

Suppose there exists a $d \in D$ such that for some $i \in B_d$, $\sigma^*_{G_d}(s^j_{G_d}) > 0$ and

$$\sum_{k=1}^{m_{-G_d}} \sigma^*_{-G_d}(s^k_{-G_d})u_i(s^j_{G_d}, s^k_{-G_d}) < \beta_i^*.$$

Then summing over gives

$$u_i(\sigma^*) = \sum_{j=1}^{m_{G_d}} \sum_{k=1}^{m_{-G_d}} \sigma^*_{G_d}(s^j_{G_d})\sigma^*_{-G_d}(s^k_{-G_d})u_i(s^j_{G_d}, s^k_{-G_d}) < \beta_i^*,$$

which is a contradiction. Hence $\sigma^*_{G_d}(s^j_{G_d}) = 0$.

Conversely, suppose $\sigma^*_{G_d}$ is a probability distribution over the set of the pure strategies for the players in the proper subset G_d such that (2.3) and (2.4) are satisfied. Then summing over gives

$$\beta_i^* = u_i(\sigma^*) = \sum_{j=1}^{m_{G_d}} \sum_{k=1}^{m_{-G_d}} \sigma^*_{G_d}(s^j_{G_d})\sigma^*_{-G_d}(s^k_{-G_d})u_i(s^j_{G_d}, s^k_{-G_d}) \ge$$
$$\sum_{j=1}^{m_{G_d}} \sum_{k=1}^{m_{-G_d}} \sigma_{G_d}(s^j_{G_d})\sigma^*_{-G_d}(s^k_{-G_d})u_i(s^j_{G_d}, s^k_{-G_d}),$$
$$\forall \sigma_{G_d} \in \triangle S_{G_d}, \forall i \in B_d, \forall d \in D,$$

which is the same as Definition 2.2. Hence σ^* is a *GE* and $u_i(\sigma^*) = \beta_i^*, \forall i \in I$. □

The following nonlinear program P obtains a *GE*, if one exists, in an $n-$person normal form game Γ. It seeks to

$$(P)\ maximize\ g(\sigma, \beta) = \sum_{i=1}^{N}[(\sum_{j=1}^{m_i}\sum_{k=1}^{m_{-i}} \sigma_i(s^j_i)\sigma_{-i}(s^k_{-i})u_i(s^j_i, s^k_{-i})) - \beta_i].$$

subject to

$$\sum_{k=1}^{m_{-G_d}} \sigma_{-G_d}(s^k_{-G_d})u_i(s^j_{G_d}, s^k_{-G_d}) \le \beta_i, \forall s^j_{G_d} \in S_{G_d},\ \forall i \in B_d,\ \forall d \in D.$$
$$\sum_{j=1}^{m_i} \sigma_i(s^j_i) = 1, \forall i \in I.$$
$$\sigma_i(s^j_i) \ge 0, \forall i \in I, j = 1, \ldots, m_i.$$

Lemma 2.2. *Let (σ^*, β^*) be a feasible point for the problem P. Then $g(\sigma^*, \beta^*) \le 0$.*

Proof. From (2.5),

$$\sum_{k=1}^{m_{-G_d}} \sigma^*_{-G_d}(s^k_{-G_d}) u_i(s^j_{G_d}, s^k_{-G_d}) \leq \beta^*_i, \forall s^j_{G_d} \in S_{G_d}, \ \forall i \in B_d, \ \forall d \in D.$$

and

$$\sum_{j=1}^{m_{G_d}} \sigma_{G_d}(s^j_{G_d}) = 1,$$

so

$$\sum_{j=1}^{m_{G_d}} \sum_{k=1}^{m_{-G_d}} \sigma^*_{G_d}(s^j_{G_d}) \sigma^*_{-G_d}(s^k_{-G_d}) u_i(s^j_{G_d}, s^k_{-G_d}) \leq \sum_{j=1}^{m_{G_d}} \sigma^*_{G_d}(s^j_{G_d}) \beta^*_i = \beta^*_i,$$
$$\forall i \in B_d, \ \forall d \in D.$$

Therefore

$$\sum_{j=1}^{m_{G_d}} \sum_{k=1}^{m_{-G_d}} \sigma^*_{G_d}(s^j_{G_d}) \sigma^*_{-G_d}(s^k_{-G_d}) u_i(s^j_{G_d}, s^k_{-G_d}) - \beta^*_i \leq 0, \forall i \in B_d, \ \forall d \in D. \tag{2.5}$$

Hence by (2.5), $g(\sigma^*, \beta^*) \leq 0$. □

Lemma 2.3. *If $g(\sigma, \beta) \neq 0$, $\forall(\sigma, \beta)$. Then there does not exist a GE for the game Γ.*

Proof. Assume $g(\sigma, \beta) \neq 0$. It follows from Lemma 2.2 that

$$g(\sigma, \beta) < 0, \ \forall(\sigma, \beta).$$

Thus there exists at least one player $i \in B_d$ for some $d \in D$ such that

$$\sum_{k=1}^{m_{-G_d}} \sigma_{-G_d}(s^k_{-G_d}) u_i(s^j_{G_d}, s^k_{-G_d}) < \beta_i$$

and

$$\sigma_{G_d}(s^j_{G_d}) > 0, \forall \sigma_{G_d} \in \triangle S_{G_d}.$$

Hence by Theorem 2.1 there does not exist a *GE* for the game Γ. □

Theorem 2.2. *Let $\sigma^* = (\sigma^*_1, \ldots, \sigma^*_n)$ be a strategy profile and $\beta^* = (\beta^*_1, \ldots, \beta^*_n)$ be the n players' expected payoffs. Then σ^* is a GE for the game Γ if and only if $g(\sigma^*, \beta^*) = 0$.*

Proof. Let $\sigma^* = (\sigma^*_1, \ldots, \sigma^*_n)$ be a *GE* and $\beta^* = (\beta^*_1, \ldots, \beta^*_n)$ be the n players expected payoffs. By Theorem 2.1,

$$g(\sigma^*, \beta^*) = \sum_{i}^{N} [(\sum_{j=1}^{m_i} \sum_{k=1}^{m_{-i}} \sigma^*_i(s^j_i) \sigma^*_{-i}(s^k_{-i}) u_i(s^j_i, s^k_{-i})) - \beta^*_i] = 0.$$

Furthermore, from Lemma 2.1 a *GE* satisfies constraints (2.5) so it is a feasible point for P.

Conversely let (σ^*, β^*) to be a feasible point for P, so (σ^*, β^*) satisfy constraints (2.5) and by Lemma 2.2 $g(\sigma^*, \beta^*) \leq 0$. We distinguish between two cases. The first case, $g(\sigma^*, \beta^*) < 0$. Hence by Lemma 2.3 there does not exist a *GE*. In the second case, let $g(\sigma^*, \beta^*) = 0$ so

$$\sum_{j=1}^{m_i} \sum_{k=1}^{m_{-i}} \sigma_i^*(s_i^j)\sigma_{-i}^*(s_{-i}^k)u_i(s_i^j, s_{-i}^k) - \beta_i^* = 0, \forall i \in I,$$

and the conditions from Theorem 2.1 are satisfied. Hence σ^* is a *GE*. □

In the following theorem, we prove if G_d, B_d are nonempty singleton subsets of all players then there always exists a *GE* for the game Γ.

Theorem 2.3. *Let $\{G_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be two sets of pairwise distinct singleton sets of the set of players I such that $\cup_{i\in I}\{G_i\} = I$ and $\cup_{i\in I}\{B_i\} = I$, then the game Γ has a GE.*

Proof. Let $f : I \to I$ be a bijection, and let $G_i = \{i\}, \forall i \in I$ and $B_i = \{j\}$ for some $j \in I$ such that $f(i) = j$. Therefore a *GE* σ^* reduces to an *NE* by replacing player i's payoffs with player j's payoffs,

$$u_j(\sigma^*) \geqslant u_j(\sigma_i, \sigma_{-i}^*), \ \forall \sigma_i \in \triangle S_i \tag{2.6}$$

However, an *NE* is always guaranteed to exist. Hence the game always has a *GE* to complete the proof. □

The *MBE* is shown in Proposition 2.3 to be a special case of the *GE*. Thus if $D = I$, $G_i = \{-i\}$, and $B_i = \{i\}$, then P for an *MBE* becomes

$$\begin{aligned}
&maximize\ g(\sigma,\beta) = \sum_{i=1}^{N}[(\sum_{j=1}^{m_i}\sum_{k=1}^{m_{-i}} \sigma_i(s_i^j)\sigma_{-i}(s_{-i}^k)u_i(s_i^j, s_{-i}^k)) - \beta_i] \\
&\qquad subject\ to \\
&\sum_{j=1}^{m_i} \sigma_i(s_i^j)u_i(s_i^j, s_{-i}^k) \le \beta_i, \forall s_{-i}^k \in S_{-i}, \forall i \in I. \\
&\sum_{j=1}^{m_i} \sigma_i(s_i^j) = 1, \forall i \in I. \\
&\sigma_i(s_i^j) \ge 0, \forall i \in I, j = 1, \ldots, m_i.
\end{aligned}$$

[4] showed that P obtains an *NE*. The *NE* is shown in Proposition 2.2 to be a special case of the

GE. Hence, for $D = I$, $G_i = \{i\}$, and $B_i = \{i\}$, P for an *NE* becomes

$$maximize\ g(\sigma,\beta) = \sum_{i=1}^{N}[(\sum_{j=1}^{m_i}\sum_{k=1}^{m_{-i}} \sigma_i(s_i^j)\sigma_{-i}(s_{-i}^k)u_i(s_i^j,s_{-i}^k)) - \beta_i]$$

$$subject\ to$$

$$\sum_{k=1}^{m_{-i}} \sigma_{-i}(s_{-i}^k)u_i(s_i^j,s_{-i}^k) \leq \beta_i, \forall s_i^j \in S_i, \forall i \in I.$$

$$\sum_{j=1}^{m_i} \sigma_i(s_i^j) = 1, \forall i \in I.$$

$$\sigma_i(s_i^j) \geq 0, \forall i \in I, j = 1,\ldots,m_i.$$

2.5 Examples

2.5.1 Example 1

In this example, we have a 3−person game, where each player has two strategies. For simplicity, we denote the strategies of the three players by p, q, r respectively.

Table 2.1: Example 1.

r_1	q_1	q_2	r_2	q_1	q_2
p_1	(9,1,9)	(4,9,6)	p_1	(8,2,1)	(7,8,4)
p_2	(1,4,2)	(6,6,3)	p_2	(2,3,8)	(3,7,7)

We first write P to find an *MBE*, and then we find an *NE*. For the *MBE*,

$$\underset{p,q,r,\beta}{\text{maximize}}\ 19p_1q_1r_1 + 19p_1q_2r_1 + 7p_2q_1r_1 + 15p_2q_2r_1$$

$$+ 11p_1q_1r_2 + 19p_1q_2r_2 + 13p_2q_1r_2 + 17p_2q_2r_2 - \beta_1 - \beta_2 - \beta_3$$

$$subject\ to$$

$$9p_1 + 1p_2 \leq \beta_1, 4p_1 + 6p_2 \leq \beta_1, 8p_1 + 2p_2 \leq \beta_1, 7p_1 + 3p_2 \leq \beta_1$$

$$1q_1 + 9q_2 \leq \beta_2, 4q_1 + 6q_2 \leq \beta_2, 2q_1 + 8q_2 \leq \beta_2, 3q_1 + 7q_2 \leq \beta_2$$

$$9r_1 + 1r_2 \leq \beta_3, 6r_1 + 4r_2 \leq \beta_3, 2r_1 + 8r_2 \leq \beta_3, 3r_1 + 7r_2 \leq \beta_3$$

$$p_1 + p_2 = 1, q_1 + q_2 = 1, r_1 + r_2 = 1, p_1, p_2, q_1, q_2, r_1, r_2 \geq 0.$$

Solving P gives a maximum of 0 with $p_1^* = p_2^* = q_1^* = q_2^* = r_1^* = r_2^* = 0.5$, as well as $\beta_1^* = \beta_2^* = \beta_3^* = 5$.

To obtain an *NE*, consider a special case of P.

$$\underset{p,q,r,\beta}{\text{maximize}}\ 19p_1q_1r_1 + 19p_1q_2r_1 + 7p_2q_1r_1 + 15p_2q_2r_1$$
$$+ 11p_1q_1r_2 + 19p_1q_2r_2 + 13p_2q_1r_2 + 17p_2q_2r_2 - \beta_1 - \beta_2 - \beta_3$$
$$\textit{subject to}$$
$$9q_1 * r_1 + 4q_2 * r_1 + 8q_1 * r2 + 7q_2 * r_2 \le \beta_1$$
$$1q_1 * r_1 + 6q_2 * r_1 + 2q_1 * r2 + 3q_2 * r_2 \le \beta_1$$
$$1p_1 * r_1 + 4p_2 * r_1 + 2p_1 * r2 + 3p_2 * r_2 \le \beta_2$$
$$9p_1 * r_1 + 6p_2 * r_1 + 8p_1 * r2 + 7p_2 * r_2 \le \beta_2$$
$$9p_1 * q_1 + 2p_2 * q_1 + 6p_1 * q2 + 3p_2 * q_2 \le \beta_3$$
$$1p_1 * q_1 + 8p_2 * q_1 + 4p_1 * q2 + 7p_2 * q_2 \le \beta_3$$
$$p_1 + p_2 = 1, q_1 + q_2 = 1, r_1 + r_2 = 1$$
$$p_1, p_2, q_1, q_2, r_1, r_2 \ge 0.$$

Solving P gives a maximum 0 with $p_1^* = 0.67, p_2^* = 0.33, q_1^* = 0, q_2^* = 1, r_1^* = 0.67, r_2^* = 0.33$, as well as $\beta_1^* = 5, \beta_2^* = 7.89, \beta_3^* = 5$.

2.5.2 Example 2

In the second example which was presented in [19], it was proven that there is not an *MBE* for the game. Consequently, the maximum of P is shown to be negative for any feasible solution.

Table 2.2: Example 2.

r_1	q_1	q_2	r_2	q_1	q_2
p_1	(1,1,0)	(0,0,0)	p_1	(0,0,1)	(0,0,0)
p_2	(0,0,0)	(0,0,1)	p_2	(0,0,0)	(1,1,0)

We write P for an *MBE*.

$$\underset{p,q,r,\beta}{\text{maximize}}\ 2p_1q_1r_1 + 0p_1q_2r_1 + 0p_2q_1r_1$$
$$+ 1p_2q_2r_1 + 1p_1q_1r_2 + 0p_1q_2r_2 + 0p_2q_1r_2 + 2p_2q_2r_2 - \beta_1 - \beta_2 - \beta_3$$
$$\textit{subject to}$$
$$1p_1 + 0p_2 \le \beta_1, 0p_1 + 0p_2 \le \beta_1, 0p_1 + 0p_2 \le \beta_1, 0p_1 + 1p_2 \le \beta_1$$
$$1q_1 + 0q_2 \le \beta_2, 0q_1 + 0q_2 \le \beta_2, 0q_1 + 0q_2 \le \beta_2, 0q_1 + 1q_2 \le \beta_2$$
$$1r_1 + 0r_2 \le \beta_3, 0r_1 + 0r_2 \le \beta_3, 0r_1 + 0r_2 \le \beta_3, 0r_1 + 1r_2 \le \beta_3$$
$$p_1 + p_2 = 1, q_1 + q_2 = 1, r_1 + r_2 = 1$$
$$p_1, p_2, q_1, q_2, r_1, r_2 \ge 0.$$

Proposition 2.4. *The game in example 2 does not have an MBE.*

Proof. Suppose that any player i chooses a pure strategy. We will prove the case of player 1 and the same argument applies to the other two players. By constraints (2.7) if player 1 chooses $p_1 = 1$, then $\beta_1 = 1$. Hence by Theorem (2.1), $q_1 r_1 = 1$ and by constraints (2.7) $\beta_2 = \beta_3 = 1$. On the other hand, by the same argument, if $p_2 = 1$, then $q_2 r_2 = 1$ and by constraints (2.7) $\beta_2 = \beta_3 = 1$. Therefore, if any player chooses a pure strategy, then the other two players must choose a pure strategy by Theorem (2.1). However, there is no pure strategy such that each player gets a payoff 1. Therefore, for an *MBE* to exist, every player must use a fully mixed strategy. However, each player has only two strategies and note $u_1(\sigma_1, q_1 r_1) > u_1(\sigma_1, q_2 r_1)$ for any σ_1, but both $q_1 r_1, q_2 r_1 > 0$ by assumption. Hence by Theorem (2.1) there does not exist an *MBE* to complete the proof. □

We can find a *GE* for example 2 such that each player wants to maximize the expected payoffs for the remaining two players. That is,

$$u_j(\sigma^*) \geq u_j(\sigma_i, \sigma^*_{-i}), \forall \sigma_i \in \triangle S_i, \forall j \in I - i, \forall i \in I.$$

Let $D = \{1,2,3\}$, $G_1 = \{1\}$, $G_2 = \{2\}$, and $G_3 = \{3\}$. Let $B_1 = \{2,3\}$, $B_2 = \{1,3\}$, and $B_3 = \{1,2\}$. We write P as

$$\begin{aligned}
&\underset{p,q,r,\beta}{\text{maximize}}\ 2p_1q_1r_1 + 0p_1q_2r_1 + 0p_2q_1r_1 \\
&+ 1p_2q_2r_1 + 1p_1q_1r_2 + 0p_1q_2r_2 + 0p_2q_1r_2 + 2p_2q_2r_2 - \beta_1 - \beta_2 - \beta_3 \\
&\quad \textit{subject to} \\
&1p_1 * q_1 + 0p_1 * q_2 + 0p_2 * q_1 + 0p_2 * q_2 \leq \beta_1 \\
&0p_1 * q_1 + 0p_1 * q_2 + 0p_2 * q_1 + 1p_2 * q_2 \leq \beta_1 \\
&1p_1 * q_1 + 0p_1 * q_2 + 0p_2 * q_1 + 0p_2 * q_2 \leq \beta_2 \\
&0p_1 * q_1 + 0p_1 * q_2 + 0p_2 * q_1 + 1p_2 * q_2 \leq \beta_2 \\
&1r_1 * q_1 + 0r_1 * q_2 + 0r_2 * q_1 + 0r_2 * q_2 \leq \beta_2 \\
&0r_1 * q_1 + 0r_1 * q_2 + 0r_2 * q_1 + 1r_2 * q_2 \leq \beta_2 \\
&0r_1 * q_1 + 0r_1 * q_2 + 1r_2 * q_1 + 0r_2 * q_2 \leq \beta_3 \\
&0r_1 * q_1 + 1r_1 * q_2 + 0r_2 * q_1 + 0r_2 * q_2 \leq \beta_3 \\
&1r_1 * p_1 + 0r_1 * p_2 + 0r_2 * p_1 + 0r_2 * p_2 \leq \beta_1 \\
&0r_1 * p_1 + 0r_1 * p_2 + 0r_2 * p_1 + 1r_2 * p_2 \leq \beta_1 \\
&0r_1 * p_1 + 0r_1 * p_2 + 1r_2 * p_1 + 0r_2 * p_2 \leq \beta_3 \\
&0r_1 * p_1 + 1r_1 * p_2 + 0r_2 * p_1 + 0r_2 * p_2 \leq \beta_3 \\
&p_1 + p_2 = 1, q_1 + q_2 = 1, r_1 + r_2 = 1 \\
&p_1, p_2, q_1, q_2, r_1, r_2 \geq 0.
\end{aligned}$$

Solving P gives a maximum 0 with $p_1^* = p_2^* = q_1^* = q_2^* = r_1^* = r_2^* = 0.5$, as well as $\beta_1^* = \beta_2^* = \beta_3^* = 0.25$. Note that the *GE* in this example coincidentally is an *NE*.

2.5.3 Example 3

We now present a third example in which there is no *MBE* for the game. The analysis is identical to the analysis of example 2. Thus, the maximum of P is negative for any feasible solution.

Table 2.3: Example 3.

r_1	q_1	q_2	r_2	q_1	q_2
p_1	(1,2,1)	(1,1,1)	p_1	(1,1,2)	(2,1,1)
p_2	(2,1,1)	(1,1,2)	p_2	(1,1,1)	(1,2,1)

We write P for finding a *GE*, as we did in Example 2, where each player wants to maximize the expected payoffs for all other players,

$$\begin{aligned}
&\underset{p,q,r,\beta}{\text{maximize}}\ 4p_1q_1r_1 + 3p_1q_2r_1 + 4p_2q_1r_1\\
&+ 4p_2q_2r_1 + 4p_1q_1r_2 + 4p_1q_2r_2 + 3p_2q_1r_2 + 4p_2q_2r_2 - \beta_1 - \beta_2 - \beta_3\\
&\quad \textit{subject to}\\
&1p_1 * q_1 + 1p_1 * q_2 + 2p_2 * q_1 + 1p_2 * q_2 \leq \beta_1\\
&1p_1 * q_1 + 2p_1 * q_2 + 1p_2 * q_1 + 1p_2 * q_2 \leq \beta_1\\
&2p_1 * q_1 + 1p_1 * q_2 + 1p_2 * q_1 + 1p_2 * q_2 \leq \beta_2\\
&1p_1 * q_1 + 1p_1 * q_2 + 1p_2 * q_1 + 2p_2 * q_2 \leq \beta_2\\
&2r_1 * q_1 + 1r_1 * q_2 + 1r_2 * q_1 + 1r_2 * q_2 \leq \beta_2\\
&1r_1 * q_1 + 1r_1 * q_2 + 1r_2 * q_1 + 2r_2 * q_2 \leq \beta_2\\
&1r_1 * q_1 + 1r_1 * q_2 + 2r_2 * q_1 + 1r_2 * q_2 \leq \beta_3\\
&1r_1 * q_1 + 2r_1 * q_2 + 1r_2 * q_1 + 1r_2 * q_2 \leq \beta_3\\
&1r_1 * p_1 + 2r_1 * p_2 + 1r_2 * p_1 + 1r_2 * p_2 \leq \beta_1\\
&1r_1 * p_1 + 1r_1 * p_2 + 2r_2 * p_1 + 1r_2 * p_2 \leq \beta_1\\
&1r_1 * p_1 + 1r_1 * p_2 + 2r_2 * p_1 + 1r_2 * p_2 \leq \beta_3\\
&1r_1 * p_1 + 2r_1 * p_2 + 1r_2 * p_1 + 1r_2 * p_2 \leq \beta_3\\
&p_1 + p_2 = 1, q_1 + q_2 = 1, r_1 + r_2 = 1\\
&p_1, p_2, q_1, q_2, r_1, r_2 \geq 0.
\end{aligned}$$

Solving P gives a maximum 0 with $p_1^* = 1, p_2^* = 0, q_1^* = 1, q_2^* = 0, r_1^* = 1, r_2^* = 0$, as well as

$\beta_1^* = 1, \beta_2^* = 2, \beta_3^* = 1$, which is not an *NE*. An *NE* can be obtained by solving.

$$\underset{p,q,r,\beta}{\text{maximize}}\ 4p_1q_1r_1 + 3p_1q_2r_1 + 4p_2q_1r_1$$

$$4p_2q_2r_1 + 4p_1q_1r_2 + 4p_1q_2r_2 + 3p_2q_1r_2 + 4p_2q_2r_2 - \beta_1 - \beta_2 - \beta_3$$

$$\textit{subject to}$$

$$1q_1 * r_1 + 1q_2 * r_1 + 1q_1 * r_2 + 2q_2 * r_2 \leq \beta_1$$

$$2q_1 * r_1 + 1q_2 * r_1 + 1q_1 * r_2 + 1q_2 * r_2 \leq \beta_1$$

$$2p_1 * r_1 + 1p_2 * r_1 + 1p_1 * r_2 + 1p_2 * r_2 \leq \beta_2$$

$$1p_1 * r_1 + 1p_2 * r_1 + 1p_1 * r_2 + 2p_2 * r_2 \leq \beta_2$$

$$1p_1 * q_1 + 1p_2 * q_1 + 1p_1 * q_2 + 2p_2 * q_2 \leq \beta_3$$

$$2p_1 * q_1 + 1p_2 * q_1 + 1p_1 * q_2 + 1p_2 * q_2 \leq \beta_3$$

$$p1 + p2 = 1, q1 + q2 = 1, r1 + r2 = 1$$

$$p1 \geq 0, p2 \geq 0, q1 \geq 0, q2 \geq 0, r1 \geq 0, r2 \geq 0.$$

Solving P gives a maximum 0 with $p_1^* = 0, p_2^* = 1, q_1^* = 0.68, q_2^* = 0.32, r_1^* = 1, r_2^* = 0$, as well as $\beta_1^* = 1.68, \beta_2^* = 1, \beta_3^* = 1.32$.

2.6 Conclusion

A generalized equilibrium (GE) for finite n–person normal form games has been defined here as a collection of mixed strategies such that each player in some subset B of all players cannot achieve a better expected payoff if players in an associated proper subset of all players G change their strategies unilaterally. Special cases of *GE* include the *NE* and the *MBE*. We have also developed a nonlinear program that determines whether a *GE* exists and obtains one if so.

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Chapter 3

The Computational Complexity of Finding a Mixed Berge Equilibrium for a k-Person Noncooperative Game in Normal Form

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Abstract

The mixed Berge equilibrium (MBE) is an extension of the Berge equilibrium (BE) to mixed strategies. The MBE models mutual support in a k-person noncooperative game in normal form. An MBE is a mixed-strategy profile for the k players in which every $k-1$ players have mixed strategies that maximize the expected payoff for the remaining player equilibrium strategy. In this paper, we study the computational complexity of the existence of an MBE in a k-person normal form game. For a 2-person game, an MBE always exists and the problem of finding an MBE is PPAD-complete. In contrast to the mixed Nash equilibrium (NE), the MBE is not guaranteed to exist in games with 3 or more players. Here we prove that determining if an MBE exists in a $k \geq 3$-person normal-form game is an NP-complete decision problem.

3.1 Introduction

The Nash equilibrium (NE) is a solution concept in game theory introduced by [1], who used fixed-point theorems to prove that an NE always exists for a finite normal-form game. The computational complexity of finding an NE in normal-form games has been extensively studied. [2] introduced the class Polynomial Parity Arguments on Directed graphs (PPAD). The problem of finding an NE in 2 or 3-person games is shown to be PPAD-complete by [3], [4], [5], and [6]. NP-completeness has been established for more restricted games. For example, [7] showed the problem of determining whether an NE exists with certain natural properties is NP-complete. [8] showed the problem of deciding whether $(0,1)$ bimatrix games has more than one NE is NP-complete.

On the other hand, the computational complexity for other solution concepts for noncooperative games is less well developed. In particular, we consider here the mixed Berge equilibrium (MBE), which is a mixed-strategy extension of the Berge equilibrium (BE) defined intuitively by [9] as a refinement for the pure NE. The BE was formally developed by [10]. For more on the Berge equilibrium, see [11] and [12]. [13] developed polynomial-time algorithm to find all Berge equilibria in a $k-$person normal form game. [14] extended the BE, a pure-strategy concept, to a mixed Berge equilibrium in k-person normal-form games and proved that the MBE may not exist for $k \geq 3$. He also related the NE to the MBE.

The organization of this paper is as follows. In Section 2, we summarize the required notation. In Section 3, we introduce a reduction from any k-SAT instance, with $k \geq 3$, to a k-person game. In Section 4, we prove that for 3 or more players the problem of finding an MBE is NP-complete, as opposed to the 2 players case where finding an MBE is a PPAD-complete problem.

3.2 Preliminaries

The following notation is used. Let $\Gamma = (I, (S_p)_{p \in I}, (u_p)_{p \in I})$ be a k-person normal-form game. The set $I = \{1, \dots, k\}$ is the set of the k players. S_p is the finite set of the pure strategies available for player p. Let $\sigma_p(s_p)$ be the probability that player p chooses the strategy $s_p \in S_p$. A mixed strategy for player p is given by the probability distribution $\boldsymbol{\sigma_p}$, where

$$\sum_{s_p \in S_p} \sigma_p(s_p) = 1,$$

and $\sigma_p(s_p) \geq 0, \forall s_p \in S_p$. Define the set of mixed strategies for player p by $\triangle S_p$.

Let S_{-p} be the set of pure-strategy profiles for all players other than player p. Similarly, let S_{-p-q} be the set of pure-strategy profiles for all players other than players p, q. The joint probability $\sigma_{-p}(\boldsymbol{s_{-p}})$ is the probability that all the players other than player p choose the joint pure strategy $\boldsymbol{s_{-p}}$. It is the product of the probability that each player in $I - p$ choosing his corresponding strategy. Note that

$$\sum_{\boldsymbol{s_{-p}} \in S_{-p}} \sigma_{-p}(\boldsymbol{s_{-p}}) = 1,$$

and $\sigma_{-p}(\boldsymbol{s_{-p}}) \geq 0, \forall \boldsymbol{s_{-p}} \in S_{-p}$. A mixed-strategy profile is the $k-$tuple $\boldsymbol{\sigma} = (\boldsymbol{\sigma_1}, \ldots, \boldsymbol{\sigma_k})$ of the mixed strategies for the $k-$players. The utility function $\boldsymbol{u_i} : S \to R$ assigns each player a payoff for each of the pure strategies. The support for player p's mixed strategy $\boldsymbol{\sigma_p}$, denoted by $supp_p(\boldsymbol{\sigma_p}) = \{s_p \in S_p | \sigma_p(s_p) > 0\}$, is the set of player p's strategies that has a positive probability in the mixed strategy $\boldsymbol{\sigma_p}$. The same definition applies for $-p$ and $-p-q$.

The following identities were given by [14] and represent the expected payoff for player p. When player p's mixed strategy is $\boldsymbol{\sigma_p}$ and the mixed strategy for the remaining players is $\boldsymbol{\sigma_{-p}}$, then the expected payoff for player p is

$$u_p(\boldsymbol{\sigma}) = u_p(\boldsymbol{\sigma_p}, \boldsymbol{\sigma_{-p}}) = \sum_{s_p \in S_p} \sum_{\boldsymbol{s_{-p}} \in S_{-p}} \sigma_p(s_p)\sigma_{-p}(\boldsymbol{s_{-p}})u_p(s_p, \boldsymbol{s_{-p}}). \tag{3.1}$$

When player p's mixed strategy is $\boldsymbol{\sigma_p}$ and the pure strategy for the remaining players is $\boldsymbol{s_{-p}}$, then the expected payoff for player p is

$$u_p(\boldsymbol{\sigma_p}, \boldsymbol{s_{-p}}) = \sum_{s_p \in S_p} \sigma_p(s_p)u_p(s_p, \boldsymbol{s_{-p}}).$$

With this notation, the mixed Berge equilibrium is now defined.

Definition 3.1. *The strategy profile $\boldsymbol{\sigma^*}$ is an MBE for Γ if and only if*

$$u_p(\boldsymbol{\sigma^*}) = \max_{\boldsymbol{\sigma_{-p}} \in \triangle S_{-p}} u_p(\boldsymbol{\sigma^*_p}, \boldsymbol{\sigma_{-p}}), \;\; \forall p \in I.$$

The decision problem of this paper is now stated as follows.

Definition 3.2. *Given a k-person normal-form game, is there a mixed-strategy profile $\boldsymbol{\sigma^*}$ satisfying Definition 3.1?*

It should be noted that for a $\boldsymbol{\sigma^*}$ to be an MBE, the $I-p$ players strategies should maximize that expected payoff for player p's mixed strategy $\boldsymbol{\sigma_p}$. The following alternative definition was proved in [14].

Lemma 3.1. *The strategy profile $\boldsymbol{\sigma^*}$ is an MBE for Γ if and only if*

$$u_p(\boldsymbol{\sigma^*}) = \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma^*}, \boldsymbol{s_{-p}}), \;\; \forall p \in I.$$

Lemma 3.2. *The mixed strategy $\boldsymbol{\sigma^*}$ is an MBE for Γ if and only if for each $\boldsymbol{s_{-p}} \in S_{-p}$ with $\sigma^*_{-p}(\boldsymbol{s_{-p}}) > 0$, then $u_p(\boldsymbol{\sigma^*_p}, \boldsymbol{s_{-p}}) = \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma^*}, \boldsymbol{s_{-p}}), \;\; \forall p \in I$.*

Proof. The sufficiency is first established. Suppose that for each $\boldsymbol{s_{-p}} \in S_{-p}$, if $\sigma^*_{-p}(\boldsymbol{s_{-p}}) > 0$, then

$$u_p(\boldsymbol{\sigma^*_p}, \boldsymbol{s_{-p}}) = \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma^*}, \boldsymbol{s_{-p}}), \;\; \forall p \in I.$$

Therefore,

$$u_p(\boldsymbol{\sigma}^*) = \max_{\boldsymbol{\sigma_{-p}} \in \triangle S_{-p}} u_p(\boldsymbol{\sigma_p^*}, \boldsymbol{\sigma_{-p}}), \ \forall p \in I.$$

Hence $\boldsymbol{\sigma}^*$ is an MBE by Definition 3.1.

Conversely, let $\boldsymbol{\sigma}^*$ be an MBE. By Lemma 3.1

$$u_p(\boldsymbol{\sigma}^*) = \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma}^*, \boldsymbol{s_{-p}}), \ \forall p \in I.$$

However, by (4.2)

$$u_p(\boldsymbol{\sigma}^*) = u_p(\boldsymbol{\sigma_p^*}, \boldsymbol{\sigma_{-p}^*}) = \sum_{s_p \in S_p} \sum_{\boldsymbol{s_{-p}} \in S_{-p}} \sigma_p^*(s_p)\sigma_{-p}^*(\boldsymbol{s_{-p}})u_p(s_p, \boldsymbol{s_{-p}}).$$

But if $\sigma_{-p}^*(s_{-p}) > 0$, then

$$u_p(\boldsymbol{\sigma_p^*}, \boldsymbol{s_{-p}}) = \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma_p^*}, \boldsymbol{s_{-p}}), \ \forall p \in I.$$

Otherwise, $\boldsymbol{\sigma}^*$ is not an MBE by Definition 3.1 to complete the proof. □

3.3 Reduction

In this section, we develop a reduction from any instance of the k-SAT problem in normal conjunctive form, with $k \geq 3$, to a k-person normal-form game. The reduction will be used in Section 4 in the proof of the NP-completeness of finding an MBE in normal-form k-person games with $k \geq 3$.

The k-SAT problem in the conjunctive normal-form consists of a finite number m of clauses. Each clause consists of exactly k boolean variables called literals or negations of literals. Either the literal l or its negation $\neg l$ is $True$. The negation of the negation of a literal is the literal itself $\neg\neg l = l$. Define $\neg C$ to be the negation of each of the k literals in clause C; i.e., assign a $True$ value for each negation of the k literals in clause C. The k-SAT problem determines whether there is a truth assignment ($True$ or $False$) for each of the literals such that all clauses are $True$.

Definition 3.3. *A satisfiable assignment for a k-SAT instance is a truth assignment for all the literals such that all clauses are True; that is,* $C_1 \wedge \ldots \wedge C_m = True$.

A k-SAT instance can be reduced to a k-person game called here the literal game Γ_l. We assume, without a loss of generality, that the m clauses are combinations of literals or negations of literals indexed $1, 2, \ldots, N$. Let $S_p = \{l_1, \neg l_1, l_2, \neg l_2, \ldots, l_N, \neg l_N\}, p = 1, \ldots, k$, be the set of the $2N$ pure strategies available for the k players. Each literal and negation of a literal represent a pure strategy for each of the players. A literal in the support of a mixed strategy of a player means assigning a $True$ value to that literal. Each literal and its negation have the same index. For example, l_1 and $\neg l_1$ have the same index 1. Define i_p to be the index of the literal that represents player p's pure strategy s_p. In this paper, we ignore the clauses that have both a literal and the negation of that literal since such a clause is satisfied no matter how the $True$ values are assigned to the literals. In a 3-SAT instance, for example, the clause $(l_1 \vee l_2 \vee \neg l_1)$ is $True$ for any truth assignment. We assume that $N > 1$, because if $N = 1$, then all the k literals in the m clauses will be l_1 or $\neg l_1$, which is a trivial case.

Definition 3.4. *We assign each player a payoff for all strategies $\boldsymbol{s} = (s_1, \ldots, s_k) \in S$ using the following method. If $\boldsymbol{s} = \neg C_d, d = 1, \ldots, m$, or if any player's strategy is a negation of any other player's strategy, then at least one player gets a payoff 0 as follows.*

1. *If the player $p-1$ is odd-numbered and his strategy is a literal, then player p gets a payoff 0.*
2. *If the player $p-1$ is even-numbered and his strategy is a negation of a literal, then player p gets a payoff 0.*
3. *If all even-numbered players' strategies are literals and all the odd-numbered players' strategies are negations of literals, then all players get a payoff 0.*

For all other payoffs, let $\boldsymbol{u_{initial}} = (N, 1, N, 1, \ldots, N)$ if k is odd, and $\boldsymbol{u_{initial}} = (N, 1, N, 1, \ldots, 1)$ if k is even. Let $\boldsymbol{u}(\boldsymbol{s}) = \boldsymbol{u_{initial}} + ((i_1 - 1) + \cdots + (i_k - 1))(1, \ldots, 1)$. Therefore for any strategy $\boldsymbol{s}$, to get the payoffs for k players, add a $k-$vector of $1's$ to the initial payoff vector for $((i_1 - 1) + \cdots + (i_k - 1))$ times such that $N + 1 = 1$. Note that player $p's$ payoff is the p^{th} element in the vector $\boldsymbol{u}(\boldsymbol{s})$.

Remarks 3.1, 3.2, and 3.3 follow immediately from Definition 3.4.

Remark 3.1. *For any pure-strategy profile $\boldsymbol{s}$, $u_p(\boldsymbol{s}) = r$ if p is an odd-numbered player and $u_p(\boldsymbol{s}) = r + 1$ if p is even-numbered player, where $r = ((i_1 - 1) + (i_2 - 1) + \cdots + (i_k - 1))$ modulo N. When $r = 0$, then r is updated to N.*

Remark 3.2. *Whenever each odd-numbered player gets a payoff N, each even-numbered player gets a payoff 1. Whenever each even-numbered player gets a payoff N, each odd-numbered player gets a payoff $N - 1$.*

Remark 3.3. *For any player p, there exists a pure strategy $\boldsymbol{s_{-p}}$ such that $u_p(s_p, \boldsymbol{s_{-p}}) > 0, \forall s_p \in S_p$. Furthermore, for any $s_p \in S_p$, there exists a pure strategy $\boldsymbol{s_{-p}}$ such that $u_p(s_p, \boldsymbol{s_{-p}}) = a$ for any $a = 1, \ldots, N$.*

3.4 Computational complexity of the mixed Berge equilibrium

In this section, we study the computational complexity of the existence of an MBE in a k-person normal-form game. For $k = 2$, the complexity of finding an MBE is easily shown to be PPAD-complete. [14] showed that for $k = 2$, there is a one-to-one correspondence between an MBE and a corresponding NE by simply interchanging the payoffs. Hence, finding an MBE is computationally equivalent to finding an NE, a problem that [6] showed was PPAD-complete. Thus the following remark follows immediately.

Remark 3.4. *For $k = 2$, finding an MBE is PPAD-complete.*

The proof of the NP-completeness of finding an MBE in a k-person normal-form game with $k \geq 3$ is based on the conventional approach of [15]. We first prove that the finding an MBE for $k \geq 3$ is in NP.

Theorem 3.1. *Let $k \geq 3$, the problem of finding an MBE in k-person games is in NP.*

Proof. Let $\boldsymbol{\sigma}^*$ be a strategy profile. From Definition 3.1, $\boldsymbol{\sigma}^*$ is an MBE for Γ if and only if

$$u_p(\boldsymbol{\sigma}^*) = \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma_p^*}, \boldsymbol{s_{-p}}), \forall p \in I. \tag{3.2}$$

The equations (3.2) clearly can be checked in polynomial time, so finding an MBE in k-person games with $k \geq 3$ is in NP. □

We now establish a sequence of lemmas to show that an MBE for a k-person game reduced from any k-SAT instance is a satisfactory assignment for the k-SAT problem. We also show that any satisfactory assignment for a k-SAT instance is an MBE for the reduced k-person game.

Lemma 3.3. *Let $\boldsymbol{\sigma}^*$ be an MBE for Γ_l, if $\sigma_p^*(s_p) > 0$ and $u_p(s_p, \boldsymbol{s_{-p}}) = 0$, then $\sigma_{-p}^*(\boldsymbol{s_{-p}}) = 0$.*

Proof. Let $\boldsymbol{\sigma}^*$ be an MBE. Suppose $u_p(s_p, \boldsymbol{s_{-p}}) = 0$ and $\sigma_p^*(s_p) > 0$. By Remark 3.3 there exists a strategy $\boldsymbol{s'_{-p}}$ such that $u_p(s_p, \boldsymbol{s'_{-p}}) > 0, \forall s_p \in S_p$. Therefore, $u_p(\boldsymbol{\sigma_p^*}, \boldsymbol{s'_{-p}}) > u_p(\boldsymbol{\sigma_p^*}, \boldsymbol{s_{-p}})$. However, for $\boldsymbol{\sigma}^*$ to be an MBE, $u_p(\boldsymbol{\sigma_p^*}) = \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma}^*, \boldsymbol{s_{-p}}),\ \forall p \in I$, and $\sigma_{-p}^*(\boldsymbol{s_{-p}}) = 0$ by Lemma 3.2. □

Lemma 3.4. *For $\boldsymbol{\sigma}^*$ to be an MBE for Γ_l, if $l_i \in supp_p(\boldsymbol{\sigma_p^*})$ for some $i = 1, \ldots, N$, then $\neg l_i \notin supp_q(\boldsymbol{\sigma_q^*})$ for any $q \in I - p$.*

Proof. From Definition 3.4, at least one player gets a payoff 0 if the strategy of any player is a negation of any other player's strategy no matter what other players' strategies are. Hence by Lemma 3.3, in order for $\boldsymbol{\sigma}^*$ to be an MBE for Γ_l, we have either a literal or a negation of a literal – but not both – in $supp_p(\boldsymbol{\sigma_p^*})$ and $supp_q(\boldsymbol{\sigma_q^*})$ for any $p, q \in I$. □

Lemma 3.5. *Let $\boldsymbol{\sigma}^*$ be an MBE for Γ_l, then no player chooses just one pure strategy and hence there is no pure BE, except for the trivial case where $N = 1$.*

Proof. Let $\boldsymbol{\sigma}^*$ be an MBE for Γ_l. If any player p chooses one pure strategy, then by Remark 3.3, $u_p(\boldsymbol{\sigma}^*) = \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma_p^*}, \boldsymbol{s_{-p}}) = N$. On the other hand, also by Remark 3.3, players $I - p$ must choose a pure strategy such that p gets a payoff N, but from Remark 3.2 there is no strategy such that all odd-numbered and even-numbered players get a payoff N except when $N = 1$. Therefore, there exists a player q where

$$u_q(\boldsymbol{\sigma}^*) \neq \max_{\boldsymbol{s_{-q}} \in S_{-q}} u_q(\boldsymbol{\sigma_q^*}, \boldsymbol{s_{-q}}) = N. \tag{3.3}$$

Hence $\boldsymbol{\sigma}^*$ is not an MBE as a contradictory result. □

Lemma 3.6. *If $\boldsymbol{\sigma^*}$ is an MBE for Γ_l, then $\sigma_p^*(s_p) = \sigma_p^*(s_p^{'})$ for any $s_p, s_p^{'}$ with different indices such that $s_p, s_p^{'} \in supp_p(\boldsymbol{\sigma_p^*})$.*

Proof. Let $\boldsymbol{\sigma^*}$ be an MBE for Γ_l. Suppose player p chooses some strategies with non-equal positive probabilities. Therefore there exists $s_p, s_p^{'} \in supp_p(\boldsymbol{\sigma_p^*})$ such that $\sigma_p^*(s_p) > \sigma_p^*(s_p^{'})$. Hence by Definition 3.1 $u_p(s_p, \boldsymbol{s_{-p}}) > u_p(s_p^{'}, \boldsymbol{s_{-p}}), \forall s_{-p} \in supp_{-p}(\boldsymbol{\sigma_{-p}^*})$. Pick q to be any even-numbered player if p is an odd-numbered player and vice versa. Remark 3.2 implies that $u_q(s_p^{'}, \boldsymbol{\sigma_{-p}^*}) > u_q(s_p, \boldsymbol{\sigma_{-p}^*})$. Hence by Definition 3.1 $s_p \notin supp_p(\boldsymbol{\sigma_p^*})$ to yield a contradiction. □

We have established that for $\boldsymbol{\sigma^*}$ to be an MBE, each player p's strategies in the support of $\boldsymbol{\sigma_p^*}$ have an equal probability. We next show that each player chooses N strategies, in which case if $\boldsymbol{\sigma^*}$ is an MBE, then each player assigns a literal or its negation a probability of $\frac{1}{N}$. The intuition behind our proof is that in an MBE (or any equilibrium) each player chooses a strategy that makes the other players indifferent about which strategies they use.

Lemma 3.7. *If σ^* is an MBE for Γ_l, then for $i = 1, \ldots, N, \forall p \in I$, either*

$$\sigma_p^*(l_i) = \frac{1}{N} \text{ and } \sigma_p^*(\neg l_i) = 0 \tag{3.4}$$

or

$$\sigma_p^*(l_i) = 0 \text{ and } \sigma_p^*(\neg l_i) = \frac{1}{N}. \tag{3.5}$$

Hence each player chooses the same literals or negations of literals with a $\frac{1}{N}$ probability.

Proof. Let $\boldsymbol{\sigma^*}$ be an MBE for Γ_l. Suppose player q is any even-numbered player and he chooses a strategy such that for some $i = 1, 2, \ldots, N$, both $+l_i, \neg l_i \notin supp_q(\boldsymbol{\sigma_q^*})$. Hence for any $\boldsymbol{s_{-q}} \in supp_{-q}(\boldsymbol{\sigma_{-q}^*})$, $u_q(+l_i, \boldsymbol{s_{-q}}) = 1$ and $u_q(\neg l_i, \boldsymbol{s_{-q}}) = 1$. Otherwise there exists a $\boldsymbol{s_{-q}}$ that gives player q a higher expected payoff for his strategy $\boldsymbol{\sigma_q^*}$. But from Remark 3.2 any odd-numbered player p gets a payoff N only when even-numbered players get a payoff 1. Therefore $u_p(\boldsymbol{\sigma^*}) \neq \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma_p^*}, \boldsymbol{s_{-p}})$, a result that means $\boldsymbol{\sigma^*}$ is not an MBE to give a contradiction. The same argument applies if q is an odd-numbered player and p is an even-numbered player.

From Lemma 3.4, on the other hand, if $\boldsymbol{\sigma^*}$ is an MBE, then no two players choose a literal and the negation of that literal with a positive probability. Hence each player chooses with equal probability the same N strategies from the $2N$ strategies of S_p defined following Definition 3.3. Therefore either (3.4) or (3.5) holds since each player chooses either l_i or $\neg l_i, i = 1, \ldots, N$ with a $\frac{1}{N}$ probability. □

Theorem 3.2. *For $k \geq 3$, a k-SAT instance is satisfiable if and only if there is an MBE for the reduced k-person game Γ_l. Therefore, finding an MBE for $k \geq 3$-person games is NP-hard.*

Proof. Let σ^* be an MBE for Γ_l. From Lemma 3.6 and Lemma 3.7 each player chooses the same N literals or negations of literals with a probability $\frac{1}{N}$. Furthermore, σ^* is an MBE. Hence by Lemma 3.3 no player gets a payoff 0 for any strategy chosen with a positive probability. It follows that no strategy in the support of the k players is a negation of any clause and k-SAT is satisfiable. On the other hand, suppose there is no MBE for the reduced game and every player chooses the same strategies with a probability $\frac{1}{N}$, then there exists at least one $p \in I$ such that for any strategy profile $\boldsymbol{\sigma}$

$$u_p(\boldsymbol{\sigma}) \neq \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma_p}, \boldsymbol{s_{-p}}).$$

Hence for at least one strategy that is chosen with a positive probability by the k players, player p gets a payoff 0. Therefore, all strategies result in at least one unsatisfied clause, so there is no satisfiable truth assignment for the k-SAT instance.

Conversely, any satisfiable truth assignment guarantees that all clauses are $True$. Moreover, only the literal or its negation is assigned a $True$ value. Hence, to obtain an MBE the literals or negations of literals that are $True$ can be assigned a probability $\frac{1}{N}$, and

$$u_p(\boldsymbol{\sigma^*}) = \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma^*_p}, \boldsymbol{s_{-p}}) = \frac{1}{N}(1 + \cdots + N) = \frac{N+1}{2}, \forall p \in I.$$

Therefore $\boldsymbol{\sigma^*}$ is an MBE for Γ_l. □

Lemma 3.8. *For $k \geq 3$, any instance of the k-SAT problem can be reduced to a k-person game in polynomial time in respect to the input.*

Proof. Any instance of the k-SAT problem with m clauses has a size of km. Since each player has $2N$ strategies, it is clear to see that $m \leq (2N)^k$. Furthermore, $N \leq km$. For the reduction to a k-person game we check $(2N)^k$ cases and compare them with the m clauses and for each case there are $k(\frac{k-1}{2})$ steps to check if any two players strategies' are negations of each other. Hence for any k-SAT problem the reduction has a time complexity of $O((2N)^{2k})$, but $N \leq km$ so the reduction can be done in $O((2km))^{2k}$. However, k is a constant for any k-SAT problem and does not change among instances where the size of an instance changes only with m. Thus a k-SAT instance is reducible to a k-person game in polynomial time with respect to the input. □

Theorem 3.3. *For $k \geq 3$, the decision problem of finding an MBE is NP-complete.*

Proof. The problem is in NP by Theorem 3.1. Moreover, by Theorem 3.2 and Lemma 3.8, the problem is NP-hard. Hence it is NP-complete. □

Theorem 3.3 obviously refers to the worst-case scenario since a problem with a pure BE may be solvable in polynomial time as shown in [13].

Example 1. Consider the following 3-SAT instance consisting of 2 clauses: $(l_1 \vee l_2 \vee l_3) \wedge (\neg l_1 \vee \neg l_2 \vee \neg l_3)$. In this example, $S_1 = S_2 = S_3 = \{l_1, \neg l_1, l_2, \neg l_2, l_3, \neg l_3\}$. The payoffs for the three players are shown in Table 3.1.

An MBE can be attained by assigning a probability $\frac{1}{3}$ for any combination of the literals and negations of literals such that not all literals nor all negations of literals have a positive probability.

For example, one MBE is,

$$\sigma_p(\neg l_1) = \sigma_p(l_2) = \sigma_p(l_3) = \frac{1}{3}, p = 1, 2, 3$$

and

$$u_p(\sigma^*) = 2, p = 1, 2, 3.$$

Hence, $l_1 = False, l_2 = l_3 = True$, which represents a satisfiable assignment for the 3−SAT instance.

Table 3.1: Example 1

l_1	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(3,1,3)	(0,0,0)	(1,2,1)	(1,2,1)	(2,3,2)	(2,3,2)
$\neg l_1$	(3,0,0)	(3,0,3)	(1,0,0)	(1,0,1)	(2,0,0)	(2,0,2)
l_2	(1,2,1)	(0,0,0)	(2,3,2)	(0,0,0)	(3,1,0)	(3,1,3)
$\neg l_2$	(1,2,1)	(1,0,1)	(2,0,0)	(2,3,2)	(3,1,3)	(3,1,3)
l_3	(2,3,2)	(0,0,0)	(3,1,0)	(3,1,3)	(1,2,1)	(0,0,0)
$\neg l_3$	(2,3,2)	(2,0,2)	(3,1,3)	(3,1,3)	(1,0,0)	(1,2,1)
$\neg l_1$	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(0,1,0)	(0,1,3)	(0,2,0)	(0,2,1)	(0,3,0)	(0,3,2)
$\neg l_1$	(0,0,0)	(3,1,3)	(1,2,1)	(1,2,1)	(2,3,2)	(2,3,2)
l_2	(0,2,0)	(1,2,1)	(2,3,2)	(0,3,2)	(3,1,3)	(3,1,3)
$\neg l_2$	(0,0,0)	(1,2,1)	(0,0,0)	(2,3,2)	(3,1,3)	(0,0,3)
l_3	(0,3,0)	(2,3,2)	(3,1,3)	(3,1,3)	(1,2,1)	(0,2,1)
$\neg l_3$	(0,0,0)	(2,3,2)	(3,1,3)	(0,0,3)	(0,0,0)	(1,2,1)
l_2	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(1,2,1)	(0,0,0)	(2,3,2)	(0,0,0)	(3,1,0)	(3,1,3)
$\neg l_1$	(1,0,0)	(1,2,1)	(2,3,2)	(2,0,2)	(3,1,3)	(3,1,3)
l_2	(2,3,2)	(2,3,2)	(3,1,3)	(0,0,0)	(1,2,1)	(1,2,1)
$\neg l_2$	(2,0,0)	(2,0,2)	(3,0,0)	(3,0,3)	(1,0,0)	(1,0,1)
l_3	(3,1,0)	(3,1,3)	(1,2,1)	(0,0,0)	(2,3,2)	(0,0,0)
$\neg l_3$	(3,1,3)	(3,1,3)	(1,2,1)	(1,0,1)	(2,0,0)	(2,3,2)
$\neg l_2$	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(1,2,1)	(0,2,1)	(0,3,0)	(2,3,2)	(3,1,3)	(3,1,3)
$\neg l_1$	(0,0,0)	(1,2,1)	(0,0,0)	(2,3,2)	(3,1,3)	(0,0,3)
l_2	(0,3,0)	(0,3,2)	(0,1,0)	(0,1,3)	(0,2,0)	(0,2,1)
$\neg l_2$	(2,3,2)	(2,3,2)	(0,0,0)	(3,1,3)	(1,2,1)	(1,2,1)
l_3	(3,1,3)	(3,1,3)	(0,2,0)	(1,2,1)	(2,3,2)	(0,3,2)
$\neg l_3$	(3,1,3)	(0,0,3)	(0,0,0)	(1,2,1)	(0,0,0)	(2,3,2)
l_3	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(2,3,2)	(0,0,0)	(3,1,0)	(3,1,3)	(1,2,1)	(0,0,0)
$\neg l_1$	(2,0,0)	(2,3,2)	(3,1,3)	(3,1,3)	(1,2,1)	(1,0,1)
l_2	(3,1,0)	(3,1,3)	(1,2,1)	(0,0,0)	(2,3,2)	(0,0,0)
$\neg l_2$	(3,1,3)	(3,1,3)	(1,0,0)	(1,2,1)	(2,3,2)	(2,0,2)
l_3	(1,2,1)	(1,2,1)	(2,3,2)	(2,3,2)	(3,1,3)	(0,0,0)
$\neg l_3$	(1,0,0)	(1,0,1)	(2,0,0)	(2,0,2)	(3,0,0)	(3,0,3)

Table 3.1: Example 1

$\neg l_3$	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(2,3,2)	(0,3,2)	(3,1,3)	(3,1,3)	(0,2,0)	(1,2,1)
$\neg l_1$	(0,0,0)	(2,3,2)	(3,1,3)	(0,0,3)	(0,0,0)	(1,2,1)
l_2	(3,1,3)	(3,1,3)	(1,2,1)	(0,2,1)	(0,3,0)	(2,3,2)
$\neg l_2$	(3,1,3)	(0,0,3)	(0,0,0)	(1,2,1)	(0,0,0)	(2,3,2)
l_3	(0,2,0)	(0,2,1)	(0,3,0)	(0,3,2)	(0,1,0)	(0,1,3)
$\neg l_3$	(1,2,1)	(1,2,1)	(2,3,2)	(2,3,2)	(0,0,0)	(3,1,3)

Example 2. Consider the following 3-SAT instance consisting of 8 clauses:

$$(l_1 \vee l_2 \vee l_3) \wedge (l_1 \vee l_2 \vee \neg l_3) \wedge (l_1 \vee \neg l_2 \vee l_3) \wedge (l_1 \vee \neg l_2 \vee \neg l_3) \wedge$$
$$(\neg l_1 \vee l_2 \vee l_3) \wedge (\neg l_1 \vee l_2 \vee \neg l_3) \wedge (\neg l_1 \vee \neg l_2 \vee l_3) \wedge (\neg l_1 \vee \neg l_2 \vee \neg l_3).$$

There is no satisfiable assignment for the 3-SAT instance since making any of the clauses $True$ would make one of the other clauses $False$.

In the reduced game shown in Table 3.2, $S_1 = S_2 = S_3 = \{l_1, \neg l_1, l_2, \neg l_2, l_3, \neg l_3\}$. Note that assigning positive probabilities for any combination of the three literals $1, 2,$ *and* 3 or their negations, results in at least one of the players getting a payoff 0 for some strategy $\boldsymbol{s}$ that has a positive probability. Therefore,

$$u_p(\boldsymbol{\sigma}^*) = \max_{\boldsymbol{s_{-p}} \in S_{-p}} u_p(\boldsymbol{\sigma}^*, \boldsymbol{s_{-p}}), \ \forall p \in I,$$

for at least one of the three players. Hence there is no MBE by Lemma 3.1.

Table 3.2: Example 2

l_1	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(3,1,3)	(0,0,0)	(1,2,1)	(1,2,1)	(2,3,2)	(2,3,2)
$\neg l_1$	(3,0,0)	(3,0,3)	(1,0,0)	(1,0,1)	(2,0,0)	(2,0,2)
l_2	(1,2,1)	(0,0,0)	(2,3,2)	(0,0,0)	(3,1,0)	(0,0,0)
$\neg l_2$	(1,2,1)	(1,0,1)	(2,0,0)	(2,3,2)	(3,0,0)	(3,0,3)
l_3	(2,3,2)	(0,0,0)	(3,1,0)	(0,0,0)	(1,2,1)	(0,0,0)
$\neg l_3$	(2,3,2)	(2,0,2)	(3,0,0)	(3,0,3)	(1,0,0)	(1,2,1)

$\neg l_1$	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(0,1,0)	(0,1,3)	(0,2,0)	(0,2,1)	(0,3,0)	(0,3,2)
$\neg l_1$	(0,0,0)	(3,1,3)	(1,2,1)	(1,2,1)	(2,3,2)	(2,3,2)
l_2	(0,2,0)	(1,2,1)	(2,3,2)	(0,3,2)	(0,1,0)	(0,1,3)
$\neg l_2$	(0,0,0)	(1,2,1)	(0,0,0)	(2,3,2)	(0,0,0)	(0,0,3)
l_3	(0,3,0)	(2,3,2)	(0,1,0)	(0,1,3)	(1,2,1)	(0,2,1)
$\neg l_3$	(0,0,0)	(2,3,2)	(0,0,0)	(0,0,3)	(0,0,0)	(1,2,1)

l_2	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(1,2,1)	(0,0,0)	(2,3,2)	(0,0,0)	(3,1,0)	(0,0,0)
$\neg l_1$	(1,0,0)	(1,2,1)	(2,3,2)	(2,0,2)	(3,0,0)	(3,0,3)
l_2	(2,3,2)	(2,3,2)	(3,1,3)	(0,0,0)	(1,2,1)	(1,2,1)
$\neg l_2$	(2,0,0)	(2,0,2)	(3,0,0)	(3,0,3)	(1,0,0)	(1,0,1)
l_3	(3,1,0)	(0,0,0)	(1,2,1)	(0,0,0)	(2,3,2)	(0,0,0)
$\neg l_3$	(3,0,0)	(3,0,3)	(1,2,1)	(1,0,1)	(2,0,0)	(2,3,2)

$\neg l_2$	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(1,2,1)	(0,2,1)	(0,3,0)	(2,3,2)	(0,1,0)	(0,1,3)
$\neg l_1$	(0,0,0)	(1,2,1)	(0,0,0)	(2,3,2)	(0,0,0)	(0,0,3)
l_2	(0,3,0)	(0,3,2)	(0,1,0)	(0,1,3)	(0,2,0)	(0,2,1)
$\neg l_2$	(2,3,2)	(2,3,2)	(0,0,0)	(3,1,3)	(1,2,1)	(1,2,1)
l_3	(0,1,0)	(0,1,3)	(0,2,0)	(1,2,1)	(2,3,2)	(0,3,2)
$\neg l_3$	(0,0,0)	(0,0,3)	(0,0,0)	(1,2,1)	(0,0,0)	(2,3,2)

l_3	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(2,3,2)	(0,0,0)	(3,1,0)	(0,0,0)	(1,2,1)	(0,0,0)
$\neg l_1$	(2,0,0)	(2,3,2)	(3,0,0)	(3,0,3)	(1,2,1)	(1,0,1)
l_2	(3,1,0)	(0,0,0)	(1,2,1)	(0,0,0)	(2,3,2)	(0,0,0)
$\neg l_2$	(3,0,0)	(3,0,3)	(1,0,0)	(1,2,1)	(2,3,2)	(2,0,2)
l_3	(1,2,1)	(1,2,1)	(2,3,2)	(2,3,2)	(3,1,3)	(0,0,0)
$\neg l_3$	(1,0,0)	(1,0,1)	(2,0,0)	(2,0,2)	(3,0,0)	(3,0,3)

Table 3.2: Example 2

$\neg l_3$	l_1	$\neg l_1$	l_2	$\neg l_2$	l_3	$\neg l_3$
l_1	(2,3,2)	(0,3,2)	(0,1,0)	(0,1,3)	(0,2,0)	(1,2,1)
$\neg l_1$	(0,0,0)	(2,3,2)	(0,0,0)	(0,0,3)	(0,0,0)	(1,2,1)
l_2	(0,1,0)	(0,1,3)	(1,2,1)	(0,2,1)	(0,3,0)	(2,3,2)
$\neg l_2$	(0,0,0)	(0,0,3)	(0,0,0)	(1,2,1)	(0,0,0)	(2,3,2)
l_3	(0,2,0)	(0,2,1)	(0,3,0)	(0,3,2)	(0,1,0)	(0,1,3)
$\neg l_3$	(1,2,1)	(1,2,1)	(2,3,2)	(2,3,2)	(0,0,0)	(3,1,3)

3.5 Conclusion

The MBE extends the BE to mixed strategies. In this paper, we study the computational complexity of finding an MBE for a k-person normal-form game. For a 2-person normal-form game, an MBE always exists, and the problem of finding an MBE is PPAD-complete. The MBE may not exist for games with $k \geq 3$ players. However, we proved in this paper that the problem of finding an MBE in $k \geq 3$-person normal-form games is an NP-complete problem. In other words, if in the worst-case scenario there exists a polynomial-time algorithm that finds an MBE, then P=NP. The proof of the NP-completeness was based on a polynomial-time reduction from the k-SAT problem.

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Chapter 4

An Alternative Interpretation of Mixed Strategies in n-Person Normal Form Games via Resource Allocation

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Abstract

In this paper we give an interpretation of mixed strategies in normal form games via resource allocation games. We define a game in normal form such that each player allocates to each of his pure strategies a fraction of the maximum resource he has available. However, the total amount he allocates does not necessarily equal to his maximum resource. The payoff functions in the resource allocation games vary with how each player allocates his resource. We prove that a Nash equilibrium always exists in mixed strategies for $n-$person resource allocation games. On the other hand, we show that a mixed Berge equilibrium may not exist in such games.

4.1 Introduction

Game theory is the study of mathematical decision making among multiple players. Each player makes an individual choice according to his notion of rationality and to his expectations of the other players' choices. The concept of the Nash equilibrium (NE) was introduced in [1] and [2]. The proof of the existence was based on the Kakutani and the Brouwer fixed point theorems [3]. Another solution concept, the Berge equilibrium, was introduced in [4] and formalized by [5]. A strategy is considered to be a Berge equilibrium if all players other than player i cannot increase the expected payoff for player i by changing their strategy. The Berge equilibrium was extended to mixed strategies in [6], where it was also shown that a mixed Berge equilibrium may not exist.

The computation of equilibria points is an essential component of game theory research and is well studied in literature. For example, a nonlinear programming approach to find an NE for three player games was developed in [7] and for $n-$person games in [8]. The nonlinear programming approach for finding an NE was extended in [9] to find a generalized equilibrium that includes the case of an MBE.

The purpose of this paper is to deal with the difficulties associated with mixed strategies. See [10] for an extensive literature review on the concept of mixed strategies, which require a randomizing process as described in [11] and [12]. According to [13], randomization lacks behavioral support. [14] gives two interpretations for mixed strategies. The first is based on the purification theorem of [15]. Purification refers to how mixed strategies reflect the player's lack of knowledge of other players' information and decision-making process. The second interpretation is that a mixed strategy represents the fraction of a large population that adapts each of the pure strategies.

In this paper we construct resource allocation games (RAGs) such that the equilibria strategies represent the fraction of a resource each player allocates to each of his pure strategies. In particular, we consider the NE and the MBE. The purpose of RAGs is to give an interpretation of the concept of mixed strategies. This interpretation is as follows. The probability that a player chooses a pure strategy equals the fraction of the resource the player allocates to that pure strategy over the total amount of the the resource the player allocates to all his pure strategies.

A related notion was studied in [16] for infinitely repeated noncooperative games played at discrete instants called stages. The payoffs in [16] were linear in the frequency that they had been played previously. Our approach differs significantly. For example, here a mixed strategy may or may not maximize the payoff functions for each player.

The organization of this paper is as following. In Section 2 we present the notation used. In Section 3 we prove the existence of an NE using Brouwer fixed point theorem. In Section 4 we present a nonlinear program to find an NE analytically. In Section 5 we consider the case of the MBE and present a nonlinear program to find one if one exists. In Section 6 we give some numerical examples and show that an MBE may not exist. In Section 7 we state our conclusions.

4.2 Preliminaries

In this section we define the notation used. Let the RAG $\Gamma =< I, (S_i)_{i\in I}, (f_i)_{i\in I} >$ be an $n-$person resource allocation game in normal form. The set $I = 1, ..., n$ is the set of the $n-$players. Let R_i be the resource available for player i and the $n-$tuple $\boldsymbol{R} = (R_1, \ldots, R_n)$ represents the amount of the resource each player has available. Define $R_i^{min} > 0$ to be the minimum amount of the resource R_i player i needs to allocate and $\alpha_i^{min} = \frac{R_i^{min}}{R_i}$. The set of the m_i pure strategies available for player i is $S_i = (s_i^1, \ldots, s_i^{m_i})$.

Each player player i allocates from his resource R_i the fraction α_i^j to his pure strategy $s_i^j, j = 1, \ldots, m_i$. The set of all possible allocations for each player i is

$$\triangle_i = \left\{ \boldsymbol{\alpha_i} = (\alpha_i^1, \ldots, \alpha_i^{m_i}) : \alpha_i^j \geq 0, j = 1, \ldots, m_i, \alpha_i^{min} \leq \sum_{j=1}^{m_i} \alpha_i^j \leq 1 \right\}.$$

Note that $\triangle_i$ is compact and convex for each player $i \in I$. Let $\triangle_{-i} = \triangle_1 \times \cdots \triangle_{i-1} \times \triangle_{i+1} \times \cdots \times \triangle_n$ and $\triangle = \triangle_1 \times \cdots \times \triangle_n$. The probability the player i chooses strategy s_i^j is $\frac{\alpha_i^j}{\sum_{j=1}^{m_i} \alpha_i^j}$. Hence A mixed strategy for player i is the m_i-tuple $(\frac{\alpha_i^1}{\sum_{j=1}^{m_i} \alpha_i^j}, \ldots, \frac{\alpha_i^{m_i}}{\sum_{j=1}^{m_i} \alpha_i^j})$, where $\alpha_i^{min} \leq \sum_{j=1}^{m_i} \alpha_i^j \leq 1$, and $\alpha_i^j \geq 0, j = 1, \ldots, m_i$. A pure strategy j is an allocation $\alpha_i^{min} \leq \alpha_i^j \leq 1$ where the player i allocates α_i^j to his pure strategy j and allocates 0 to the rest of his pure strategies. The payoff function for each player is $f_i^{j,k}(\boldsymbol{\alpha})$. Here we have $\boldsymbol{\alpha} = (\boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_n})$ and $\boldsymbol{\alpha_{-i}} = (\boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_{i-1}}, \boldsymbol{\alpha_{i+1}}, \ldots, \boldsymbol{\alpha_n})$. The payoff functions $f_i^{j,k}(\boldsymbol{\alpha}), j = 1, \ldots, m_i, k = 1, \ldots, m_{-i}$, are assumed to be continuous in $\alpha_i^j \in [0,1], j = 1, \ldots, m_i, \forall i \in I$.

The set of joint pure strategies of all players other than player i, is the Cartesian product of the sets of pure strategies of all players other than player i, $S_{-i} = \times_{j\in I-\{i\}}(S_j)$ and is denoted by $S_{-i} = \{s_{-i}^1, \ldots, s_{-i}^{m_{-i}}\}$, where $m_{-i} = \prod_{j\in I-\{i\}} m_j$. The joint probability $\alpha_{-i}^k = \prod_{p\in I-\{i\}} \frac{\alpha_p^k}{\sum_{j=1}^{m_p} \alpha_p^j}, k = 1, \ldots, m_{-i}$ is the probability that all the players other than player i choose the joint pure strategy s_{-i}^k. It is the product of the fraction that each player in $I - \{i\} = \{1, \ldots, i-1, i+1, \ldots, n\}$ allocates to his corresponding strategy.

We extend the identities proved in [6] to Γ. The following identities represent the expected payoff for player i. If player i allocates $\alpha_i^{min} \leq \alpha_i^j \leq 1$ to his strategy j and he allocates 0 to his other pure strategies while the rest of players choose the allocation $\boldsymbol{\alpha_{-i}}$ is

$$F_i^j(\boldsymbol{\alpha}) = \sum_{k=1}^{m_{-i}} \alpha_{-i}^k f_i^{j,k}(\boldsymbol{\alpha}). \tag{4.1}$$

If player i chooses the mixed allocation $\boldsymbol{\alpha_i}$ and the rest of players choose the allocation $\boldsymbol{\alpha_{-i}}$, then the expected payoff for player i is

$$F_i(\boldsymbol{\alpha}) = \sum_{j=1}^{m_i} \sum_{k=1}^{m_{-i}} \frac{\alpha_i^j}{\sum_{j=1}^{m_i} \alpha_i^j} \alpha_{-i}^k f_i^{j,k}(\boldsymbol{\alpha}). \tag{4.2}$$

Table 4.1 shows an example of a 2−person RAG.

Table 4.1: Example 1

	s_2^1	s_2^2
s_1^1	$f_1^{1,1}(\alpha_1^1 R_1, \alpha_2^1 R_2)$,$f_2^{1,1}(\alpha_1^1 R_1, \alpha_2^1 R_2)$	$f_1^{1,2}(\alpha_1^1 R_1, \alpha_2^2 R_2)$,$f_2^{1,2}(\alpha_1^1 R_1, \alpha_2^2 R_2)$
s_1^2	$f_1^{2,1}(\alpha_1^2 R_1, \alpha_2^1 R_2)$,$f_2^{2,1}(\alpha_1^2 R_1, \alpha_2^1 R_2)$	$f_1^{2,2}(\alpha_1^2 R_1, \alpha_2^2 R_2)$,$f_2^{2,2}(\alpha_1^2 R_1, \alpha_2^2 R_2)$

In this paper, we consider the following two cases.

1. Case 1. Each player i allocates all of his resource R_i. In other words, $\sum_{j=1}^{m_i} \alpha_i^j = 1, \forall i \in I$. In this case, each player i chooses strategy j with the probability α_i^j.

2. Case 2. Each player i does not necessarily allocate all of his resource R_i. Hence $\alpha_i^{min} \leq \sum_{j=1}^{m_i} \alpha_i^j \leq 1, \forall i \in I$. In this case, each player i chooses strategy j with the probability $\frac{\alpha_i^j}{\sum_{j=1}^{m_i} \alpha_i^j}, j = 1, \ldots, m_i$.

$R_i, \forall i \in I$, is considered fixed in these two cases. However, in the second case each player i may not use all of his resource. Note that the first case is a special case of the second case. In particular if $R_i^{min} = R_i$, then the second case becomes the first case. We formalize this previous statement as follows.

Lemma 4.1. *$\forall i \in I$ let $R_i^{min} = R_i$. Then Case 1 and Case 2 are equivalent.*

Proof. Let $R_i^{min} = R_i, \forall i \in I$. Hence $\alpha_i^{min} = 1$ and $\sum_{j=1}^{m_i} \alpha_i^j = 1$. It follows immediately that Case 2 reduces to Case 1. □

4.3 Existence

In this section we prove the existence of an NE in Case 1 and Case 2 above. Hence we seek to find a mixed strategy such that a player i chooses strategy j with a probability $\frac{\alpha_i^j}{\sum_{j=1}^{m_i} \alpha_i^j}, j = 1, \ldots, m_i, \forall i \in I$. We next restate the definition of an NE in terms of allocation.

Definition 4.1. (NE) *A strategy α^* is an NE if and only if*

$$F_i(\boldsymbol{\alpha}^*) = \max_{j=1,\ldots,m_i} F_i^j(\boldsymbol{\alpha}^*), \forall \boldsymbol{\alpha_i} \in \triangle_i, \forall i \in I. \quad (4.3)$$

In an NE for the game Γ, no player can improve his expected payoff with a unilateral change in strategy, i.e., a unilateral reallocation of his previously allocated resource level.

In the following theorem, we prove the existence of an NE in a finite n−person Γ. It suffices to prove the existence for case 2 since it subsumes case 1 by Lemma 4.1 when $R_i^{min} = R_i$.

The proof of the next theorem is similar to the proof of the existence of an equilibrium in [2]. Let $\triangle_i = \left\{\boldsymbol{\alpha_i} : \alpha_i^j \geq 0, j = 1, \ldots, m_i, \alpha_i^{min} \leq \sum_{j=1}^{m_i} \alpha_i^j \leq 1\right\}$ and $\triangle = \triangle_1 \times \cdots \times \triangle_n$. The set $\triangle$ is compact and convex since the number of player is finite and each player has a finite number of strategies. Define the function $\phi = (\phi_1, \ldots, \phi_n) : \triangle \to \triangle$ where $\phi_i = (\phi_i^1, \ldots, \phi_i^{m_i})$ and

$$\phi_i^j = \frac{\alpha_i^j + \max\left\{0, F_i^j(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha})\right\}}{1 + \sum_{j=1}^{m_i} max\left\{0, F_i^j(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha})\right\}}, i = 1, \ldots, n, j = 1, \ldots, m_i. \tag{4.4}$$

The functions ϕ_i^j are continuous since we assume that the $f_i^{j,k}(\boldsymbol{\alpha})$ are continuous in $\alpha_i^j \in [0,1], j = 1, \ldots, m_i, \forall i \in I$. Therefore by Brouwer fixed point theorem there exists fixed points

$$\alpha_i^j = \frac{\alpha_i^j + max\left\{0, F_i^j(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha})\right\}}{1 + \sum_{j=1}^{m_i} max\left\{0, F_i^j(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha})\right\}}, i = 1, \ldots, n, j = 1, \ldots, m_i. \tag{4.5}$$

We now prove the existence of an NE in every finite Γ.

Theorem 4.1. *Every finite RAG Γ has an NE in mixed strategies.*

Proof. Let $\boldsymbol{\alpha}$ be an NE. Then no player has an incentive to change his strategy based on the allocation $\boldsymbol{\alpha}$. Note that the function $max\left\{0, F_i^j(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha})\right\}$ represent player's i gain by choosing his pure strategy j given the previous allocation $\boldsymbol{\alpha}$. Hence $\max\left\{0, F_i^j(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha})\right\} = 0, j = 1, \ldots, m_i, \forall i \in I$. Thus $\boldsymbol{\alpha}$ is a fixed point.

Conversely, let $\boldsymbol{\alpha}$ be a fixed point. Then for each i let l be a pure strategy such that $\alpha_i^l > 0$, and $F_i^l(\boldsymbol{\alpha}) = \min_{j=1,\ldots,m_i} F_i^j(\boldsymbol{\alpha})$. Therefore, $max\left\{0, F_i^j(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha})\right\} = 0$, since $F_i^j(\boldsymbol{\alpha}) \le F_i(\boldsymbol{\alpha})$. Note that from Equation 4.5, the right hand side is α_i^j only when the denominator equals 1. Hence, $\sum_{j=1}^{m_i} max\left\{0, F_i^j(\boldsymbol{\alpha}) - F_i(\boldsymbol{\alpha})\right\} = 0$. Hence no player has an incentive to change his strategy, and so $\boldsymbol{\alpha}$ is an NE allocation to complete the proof. □

We now show how a standard $n-$person game in normal form with constant von Neumann-Morgenstern (VNM) utility functions is a special case of an allocation game as defined in this paper.

Theorem 4.2. *The payoff matrix for a standard normal form game is a special case of the payoff matrix for a normal form allocation game.*

Proof. Let $u_i(s_i^j, s_{-i}^k) = c_i^{j,k}, j = 1, \ldots, m_i, k = 1, \ldots, m_{-i}, \forall i \in I$, be constant VNM utilities for a normal form game. It suffices to show that for any player i the VNM utilities can be written as the payoffs for player i in an allocation game Γ. To do so simply let $f_i^{j,k}(\boldsymbol{\alpha}) = c_i^{j,k} \times R_i, R_i = 1, \forall i \in I$. It follows that a standard normal form game with constant VNM utilities is a special case of the game Γ to complete the proof. □

In other words, for $R_i = 1, \forall i \in I$, the payoff functions for each player i need not vary with the fraction each player allocates to each of his pure strategies. It follows that for any equilibrium, say an NE or an MBE, a normal form game with VNM utilities is a special case of an associated allocation game Γ. In the next section we consider the computation of an NE. The computation of an MBE will be considered in Section 5.

4.4 The Computation of an NE

In this section we extend the nonlinear program in [8] to find an NE for the game Γ. Therefore an allocation $\boldsymbol{\alpha}$ is an NE if and only the maximum of the following nonlinear program is zero.

Theorem 4.3. *$\boldsymbol{\alpha}^*$ is an NE for Γ if and only if the maximum of the following nonlinear program is* $0:$

$$\begin{aligned} &maximize\; g(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{i=1}^{n}[F_i(\boldsymbol{\alpha}) - \beta_i] \\ &\quad subject\ to \\ &\sum_{k=1}^{m_{-i}} \alpha_{-i}^k f_i^{j,k}(\boldsymbol{\alpha}) \le \beta_i, j = 1,\dots,m_i, \forall i \in I, \\ &\alpha_i^j \ge 0, j = 1,\dots,m_i, \forall i \in I, \\ &\alpha_i^{min} \le \sum_{j=1}^{m_i} \alpha_i^j \le 1, \forall i \in I. \end{aligned} \tag{4.6}$$

Proof. Let $\boldsymbol{\alpha}^*$ be an NE where each player i allocates $\sum_{j=1}^{m_i} \alpha_i^{j^*}$ of his total resource R_i. Then $F_i(\alpha^*) = \max_j F_i^j(\alpha^*) = \beta_i^*$. Therefore $g(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*) = 0$. Furthermore, all constraints (4.6) are satisfied since from Definition 4.1 $\beta_i^* = \max_{j=1,\dots,m_i} F_i^j(\boldsymbol{\alpha}^*)$.

Conversely, let $\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*$ be a feasible point such that $g(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*) = 0$. It can easily be checked by the constraints (4.6) and equation (4.2) that $F_i(\boldsymbol{\alpha}^*) \le \beta_i^*$. Hence it must be the case that $F_i(\boldsymbol{\alpha}^*) = \beta_i^*$. Otherwise $g(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*) \ne 0$ which yields a contradiction. Moreover, from the constraints 4.6 $\beta_i^* = \max_{j=1,\dots,m_i} F_i^j(\boldsymbol{\alpha}^*)$. Therefore $\boldsymbol{\alpha}^*$ is a NE by Definition 4.1. □

It is worth noting that the payoff functions at an NE may not be maximized. To maximize the payoff functions one needs to find an allocation such that the fraction each player i allocates to each strategy maximizes each of the player i's payoff functions. We discuss here the case where the payoff functions are monotonically nondecreasing functions in the fraction of the resource α_i^j. Hence each of the payoff functions is maximized when the total resource R_i is allocated to that strategy. In other words, $F_i^j(\alpha_i^j = 1, \boldsymbol{\alpha_{-i}}) \ge F_i^j(\boldsymbol{\alpha_i}, \boldsymbol{\alpha_{-i}}), j = 1,\dots,m_i, \forall \alpha_i \in \triangle_i, \forall i \in I$.

Lemma 4.2. *Let $f_i^{j,k}(\boldsymbol{\alpha})$ be monotonically increasing functions in $\alpha_i^j, j = 1\dots,m_i, \forall i \in I$. Then the optimal strategy for each player is an NE if and only if the maximum of the following nonlinear* *program is* 0.

$$\begin{aligned}
&maximize\ g(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{i=1}^{n}[F_i(\boldsymbol{\alpha}) - \beta_i] \\
&\quad subject\ to \\
&F_i^j(\alpha_i^j = 1, \boldsymbol{\alpha_{-i}}) \leq \beta_i, j = 1,\ldots,m_i, \forall i \in I, \\
&\alpha_i^j \geq 0, \forall i \in I, j = 1,\ldots,m_i, \\
&\sum_{j=1}^{m_i}\alpha_i^j = 1, \forall i \in I.
\end{aligned} \tag{4.7}$$

Proof. Let $\boldsymbol{\alpha^*}$ be an NE where the payoff is maximized. Then

$$\beta_i^* = \max_{j=1,\ldots,m_i} F_i^j(\alpha_i^{j^*} = 1, \boldsymbol{\alpha_{-i}}).$$

However, $\boldsymbol{\alpha^*}$ is an NE. Hence from Theorem 4.3 the maximum of the nonlinear program is 0.

Conversely, let $\boldsymbol{\alpha^*}, \boldsymbol{\beta^*}$ be a feasible point such that the maximum of (4.7) is 0. The functions f_i^j are monotonically nondecreasing in $\alpha_i^j, \forall i \in I, j = 1,\ldots,m_i$. Therefore, there exists a solution that zero-maximizes the objective function and satisfies all the conditions of the nonlinear program in Theorem 4.3. Hence the solution is an NE. Furthermore, the solution maximizes the expected payoff for each player of over all payoff functions and the proof is complete. □

4.5 The Computation of an MBE

In this section, we consider the MBE. We only present a computational approach similar to (4.6), since example 2 of section 6 illustrates that an MBE may not exist for $n \geq 3$. A strategy is an MBE when all players other than player i cannot increase player i's expected payoff. The following is the definition of an MBE.

Definition 4.2. *A strategy* α^* *is an MBE for* Γ *if and only if*

$$F_i(\boldsymbol{\alpha^*}) = \max_{k=1,\ldots,m_{-i}} \sum_{j=1}^{m_i} \frac{\alpha_i^{j^*}}{\sum_{j=1}^{m_i}\alpha_i^{j^*}} f_i^{j,k}(\boldsymbol{\alpha^*}), \forall \alpha_{-i} \in \triangle_{-i}, \forall i \in I. \tag{4.8}$$

In an MBE for the game Γ, no player has an incentive of a unilateral change of his strategy based on how he allocates his resource. In other words, any unilateral change of strategy results in a less expected payoff to at least one of the remaining players. We extend the nonlinear program presented in [9] to the game Γ.

Theorem 4.4. $\boldsymbol{\alpha^*}$ *is an MBE for* Γ *if and only if the maximum of the following nonlinear program* *is* 0 :

$$\begin{aligned}
&maximize\ h(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{i=1}^{n}[F_i(\boldsymbol{\alpha}) - \beta_i] \\
&\quad subject\ to \\
&\sum_{j=1}^{m_i} \frac{\alpha_i^j}{\sum_{j=1}^{m_i}\alpha_i^j} f_i^{j,k}(\boldsymbol{\alpha}) \le \beta_i, k = 1,\dots,m_{-i}, \forall i \in I, \\
&\alpha_i^j \ge 0, \forall i \in I, j = 1,\dots,m_i, \\
&\alpha_i^{min} \le \sum_{j=1}^{m_i}\alpha_i^j \le 1, \forall i \in I.
\end{aligned} \tag{4.9}$$

Proof. Let α^* be an MBE allocation. Then each player allocates to each strategy a fraction $\alpha_i^{j^*}$ of his resource that equals to the probability that the player uses that strategy. From Definition 4.2 one can check that $F_i(\boldsymbol{\alpha^*}) = \beta_i^* = \max_{k=1,\dots,m_{-i}} F_i^k(\boldsymbol{\alpha^*}), \forall i \in I$. Hence all constraints are satisfied. Moreover, $h(\boldsymbol{\beta^*},\alpha^*) = 0$.

Conversely, let $(\boldsymbol{\alpha^*},\boldsymbol{\beta^*})$ be a feasible solution such that $h(\boldsymbol{\alpha^*},\beta^*) = 0$. From (4.9), it is easy to see that $F_i(\boldsymbol{\alpha^*}) \le \beta_i^*, \forall i \in I$. But $h(\boldsymbol{\alpha^*},\boldsymbol{\beta^*}) = 0$, so it must be that $F_i(\boldsymbol{\alpha^*}) = \beta_i^*, \forall i \in I$ and $\beta_i^* = \max_{k=1,\dots,m_{-i}} F_i^k(\boldsymbol{\alpha})$. Therefore, $F_i(\boldsymbol{\alpha^*}) = \max_{k=1,\dots,m_{-i}} F_i^k(\boldsymbol{\alpha^*})$, and hence $\boldsymbol{\alpha^*}$ is an MBE by Definition 4.2. □

4.6 Examples

In this section we present three examples. The first example is a 2−person RAG, while the second and third are 3−person RAGs.

Example 1.

In this 2−person RAG each player has 2 strategies. Player 1 has a resource $R_1 = 30$ and player 2 has a resource $R_2 = 50$. The payoff matrix for each player is shown in Table 4.2. For this game,

Table 4.2: Example 1

	s_2^1	s_2^2
s_1^1	$(3 + \alpha_1^1 \times 30, 5 + \alpha_2^1 \times 50)$	$(2 + \alpha_1^1 \times 30, 8 + \alpha_2^2 \times 50)$
s_1^2	$(2 + \alpha_1^2 \times 30, 6 + \alpha_2^1 \times 50)$	$(5 + \alpha_1^2 \times 30, 4 + \alpha_2^2 \times 50)$

we consider Case 1 and Case 2 from section 2. In the first case, each player uses his maximum resource. The following NLP finds an NE for *Gamma* for Case 1.

$$
\begin{aligned}
(P1)\ maximize\ g(\boldsymbol{\alpha},\boldsymbol{\beta}) = & \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(3+\alpha_1^1\times 30+5+\alpha_2^1\times 50) \\
& + \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^1\times 30+8+\alpha_2^2\times 50) \\
& + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^2\times 30, 6+\alpha_2^1\times 50) \\
& + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(5+\alpha_1^2\times 30, 4+\alpha_2^2\times 50) - \beta_1 - \beta_2
\end{aligned}
$$

subject to

$$
\begin{aligned}
& \frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(3+\alpha_1^1\times 30 + \frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^1\times 30 \le \beta_1 \\
& \frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^2\times 30 + \frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(5+\alpha_1^2\times 30 \le \beta_1 \\
& \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}(5+\alpha_2^1\times 50) + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}(6+\alpha_2^1\times 50) \le \beta_2 \\
& \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}(8+\alpha_2^2\times 50) + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}(4+\alpha_2^2\times 50) \le \beta_2 \\
& \alpha_1^1+\alpha_1^2 = 1 \\
& \alpha_2^1+\alpha_2^2 = 1.
\end{aligned}
$$

One solution to $(P1)$ with $g(\alpha^*,\beta^*) = 0$ and hence an NE is $\alpha_1^{1^*} = 0.52, \alpha_1^{2^*} = 0.48, \alpha_2^{1^*} = 0.51, \alpha_2^{2^*} = 0.49, \beta_1^* = 17.99, \beta_2^* = 30.77$.

In Case 2 when each player allocates at least 0.4 of his resource, the following NLP finds an NE strategy for this problem.

$$
\begin{aligned}
(P2)\ maximize\ g(\boldsymbol{\alpha},\boldsymbol{\beta}) = & \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(3+\alpha_1^1\times 30+5+\alpha_2^1\times 50) \\
& + \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^1\times 30+8+\alpha_2^2\times 50) \\
& + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^2\times 30, 6+\alpha_2^1\times 50) \\
& + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(5+\alpha_1^2\times 30, 4+\alpha_2^2\times 50) - \beta_1 - \beta_2
\end{aligned}
$$

subject to

$$\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(3+\alpha_1^1\times 30+\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^1\times 30\leq\beta_1$$

$$\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^2\times 30+\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(5+\alpha_1^2\times 30\leq\beta_1$$

$$\frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}(5+\alpha_2^1\times 50)+\frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}(6+\alpha_2^1\times 50)\leq\beta_2$$

$$\frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}(8+\alpha_2^2\times 50)+\frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}(4+\alpha_2^2\times 50)\leq\beta_2$$

$$0.4\leq\alpha_1^1+\alpha_1^2\leq 1$$

$$0.4\leq\alpha_2^1+\alpha_2^2\leq 1.$$

One solution to $(P2)$ with $g(\alpha^*,\beta^*)=0$ and hence an NE is $\alpha_1^{1^*}=0.45,\alpha_1^{2^*}=0,\alpha_2^{1^*}=0.23,\alpha_2^{2^*}=0.17,\beta_1^*=15.94,\beta_2^*=16.5$.

However, the MBE may not exist as shown in [6]. The interpretation here is that there may not exist an allocation such that every player other than player i allocates to each strategy a fraction equals to the probability of using that strategy that maximizes player i's payoff. In the next example, an MBE does not exist, However, an NE exists by Theorem 4.1.

Example 2.

In this 3−person RAG each player has 2 strategies with $R_1=R_2=R_3=1$ and needs to allocate at least 0.2 of his maximum resource. The payoff matrix for each player is shown in Table 4.3.

Table 4.3: Example 2

s_3^1	s_2^1	s_2^2
s_1^1	$(1+\alpha_2^1+\alpha_3^1,1+\alpha_1^1+\alpha_3^1,0)$	(0,0,0)
s_1^2	(0,0,0)	$(0,0,1+\alpha_1^2+\alpha_2^2)$
s_3^2	s_2^1	s_2^2
s_1^1	$(0,0,1+\alpha_1^1+\alpha_2^1)$	(0,0,0)
s_1^2	(0,0,0)	$(1+\alpha_2^2+\alpha_3^2,1+\alpha_1^2+\alpha_3^2,0)$

We now write the following NLP to find an MBE:

$$
\begin{aligned}
(P3)\ \textit{maximize}\ h(\boldsymbol{\alpha},\boldsymbol{\beta}) = & \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(1+\alpha_2^1+\alpha_3^1+1+\alpha_1^1+\alpha_3^1+0)\\
&+\frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^1+\alpha_2^1)\\
&+\frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^2+\alpha_2^2)\\
&+\frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(1+\alpha_2^2+\alpha_3^2+1+\alpha_1^2+\alpha_3^2)-\beta_1-\beta_2-\beta_3
\end{aligned}
$$

subject to

$$
\begin{aligned}
&\frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}(1+\alpha_2^1+\alpha_3^1)+\frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}(0)\le\beta_1\\
&\frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}(0)+\frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}(0)\le\beta_1\\
&\frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}(0)+\frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}(1+\alpha_2^2+\alpha_3^2)\le\beta_1\\
&\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(1+\alpha_1^1+\alpha_3^1)+\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(0)\le\beta_2\\
&\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(0)+\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(0)\le\beta_2\\
&\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(0)+\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(1+\alpha_1^2+\alpha_3^2)\le\beta_2\\
&\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^2+\alpha_2^2)+\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(0)\le\beta_3\\
&\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(0)+\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(0)\le\beta_3\\
&\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(0)+\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^1+\alpha_2^1)\le\beta_3\\
&\alpha_i^j\ge 0,\ \forall i\in I,\ j=1,\dots,m_i\\
&0.2\le\sum_{j=1}^{m_i}\alpha_i^j\le 1,\ \forall i\in I.
\end{aligned}
$$

In this problem, an MBE does not exist. Note that there is not any pure Berge equilibrium because whenever players 1 and 2 gets a positive payoff, player 3 gets a payoff 0 and vice versa. Furthermore, if any mixed strategy is used then for at least one player i, the players $-i$ will choose with a positive probability a strategy where at least one player i gets a payoff 0. Hence the maximum of $(P3)$ cannot be 0, and there is no MBE by Theorem 4.4. In contrast to the MBE, an NE always exists by Theorem 4.1. The following is the nonlinear program to find an NE for this game.

$$
\begin{aligned}
(P4)\ maximize\ g(\boldsymbol{\alpha},\boldsymbol{\beta}) = {} & \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(1+\alpha_2^1+\alpha_3^1+1+\alpha_1^1+\alpha_3^1+0) \\
& + \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^1+\alpha_2^1) \\
& + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^2+\alpha_2^2) \\
& + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(1+\alpha_2^2+\alpha_3^2+1+\alpha_1^2+\alpha_3^2) - \beta_1 - \beta_2 - \beta_3
\end{aligned}
$$

subject to

$$
\begin{aligned}
& \frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(1+\alpha_2^1+\alpha_3^1) \le \beta_1 \\
& \frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(1+\alpha_2^2+\alpha_3^2) \le \beta_1 \\
& \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^1+\alpha_3^1) \le \beta_2 \\
& \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^2+\alpha_3^2) \le \beta_2 \\
& \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(1+\alpha_1^2+\alpha_2^2) \le \beta_3 \\
& \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(1+\alpha_1^1+\alpha_2^1) \le \beta_3
\end{aligned}
$$

$$
\begin{aligned}
& \alpha_1^1, \alpha_1^2, \alpha_2^1, \alpha_2^2, \alpha_3^1, \alpha_3^2 \ge 0, \forall i \in I, j = 1, \dots, m_i \\
& 0.2 \le \alpha_1^1 + \alpha_1^2 \le 1 \\
& 0.2 \le \alpha_2^1 + \alpha_2^2 \le 1 \\
& 0.2 \le \alpha_3^1 + \alpha_3^2 \le 1.
\end{aligned}
$$

One solution to $(P4)$ with $g(\alpha^*, \beta^*) = 0$ and hence an NE is $\alpha_1^{1^*} = \alpha_1^{2^*} = \alpha_2^{1^*} = \alpha_2^{2^*} = \alpha_3^{1^*} = \alpha_3^{2^*} = 0.1, \beta_1^* = \beta_2^*, \beta_3^* = 0.3$.

Example 3.

In this 3–person RAG each player has 2 pure strategies with $R_1 = R_2 = R_3 = 1$ and each player needs to allocate at least 0.2 of his maximum resource. The payoff matrices are shown in Table 4.4.

Table 4.4: Example 3

s_3^1	s_2^1	s_2^2
s_1^1	$(2+\alpha_2^1+\alpha_3^1, 1+\alpha_1^1+\alpha_3^1, 2+\alpha_1^1+\alpha_2^1)$	$(1+\alpha_2^2+\alpha_3^1, 2+\alpha_1^1+\alpha_3^1, 1+\alpha_1^1+\alpha_2^2)$
s_1^2	$(1+\alpha_2^1+\alpha_3^1, 2+\alpha_1^2+\alpha_3^1, 1+\alpha_1^2+\alpha_2^1)$	$(2+\alpha_2^2+\alpha_3^1, 1+\alpha_1^2+\alpha_3^1, 2+\alpha_1^2+\alpha_2^2)$
s_3^2	s_2^1	s_2^2
s_1^1	$(1+\alpha_2^1+\alpha_3^2, 2+\alpha_1^1+\alpha_3^2, 1+\alpha_1^1+\alpha_2^1)$	$(2+\alpha_2^2+\alpha_3^2, 1+\alpha_1^1+\alpha_3^2, 2+\alpha_1^1+\alpha_2^2)$
s_1^2	$(2+\alpha_2^1+\alpha_3^2, 1+\alpha_1^2+\alpha_3^2, 2+\alpha_1^2+\alpha_2^1)$	$(1+\alpha_2^2+\alpha_3^2, 2+\alpha_1^2+\alpha_3^2, 1+\alpha_1^2+\alpha_2^2)$

This example has an MBE. The following NLP finds an MBE.

$$
\begin{aligned}
(P5)\ \mathit{maximize}\ h(\boldsymbol{\alpha},\boldsymbol{\beta}) = {} & \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(2+\alpha_2^1+\alpha_3^1+1+\alpha_1^1+\alpha_3^1+2+\alpha_1^1+\alpha_2^1)\\
& + \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(1+\alpha_2^2+\alpha_3^1+2+\alpha_1^1+\alpha_3^1+1+\alpha_1^1+\alpha_2^2)\\
& + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(1+\alpha_2^1+\alpha_3^1+2+\alpha_1^2+\alpha_3^1+1+\alpha_1^2+\alpha_2^1)\\
& + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(2+\alpha_2^2+\alpha_3^1+1+\alpha_1^2+\alpha_3^1+2+\alpha_1^2+\alpha_2^2)\\
& + \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(1+\alpha_2^1+\alpha_3^2+2+\alpha_1^1+\alpha_3^2+1+\alpha_1^1+\alpha_2^1)\\
& + \frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(2+\alpha_2^2+\alpha_3^2+1+\alpha_1^1+\alpha_3^2+2+\alpha_1^1+\alpha_2^2)\\
& + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(2+\alpha_2^1+\alpha_3^2+1+\alpha_1^2+\alpha_3^2+2+\alpha_1^2+\alpha_2^1)\\
& + \frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(1+\alpha_2^2+\alpha_3^2+2+\alpha_1^2+\alpha_3^2+1+\alpha_1^2+\alpha_2^2)-\beta_1-\beta_2-\beta_3
\end{aligned}
$$

subject to

$$\frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}(2+\alpha_2^1+\alpha_3^1)+\frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}(1+\alpha_2^1+\alpha_3^1)\le\beta_1$$
$$\frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}(1+\alpha_2^2+\alpha_3^1)+\frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}(2+\alpha_2^2+\alpha_3^1)\le\beta_1$$
$$\frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}(1+\alpha_2^1+\alpha_3^2)+\frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}(2+\alpha_2^1+\alpha_3^2)\le\beta_1$$
$$\frac{\alpha_1^1}{\alpha_1^1+\alpha_1^2}(2+\alpha_2^2+\alpha_3^2)+\frac{\alpha_1^2}{\alpha_1^1+\alpha_1^2}(1+\alpha_2^2+\alpha_3^2)\le\beta_1$$
$$\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(1+\alpha_1^1+\alpha_3^1)+\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^1+\alpha_3^1)\le\beta_2$$
$$\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^2+\alpha_3^1)+\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(1+\alpha_1^2+\alpha_3^1)\le\beta_2$$
$$\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^1+\alpha_3^2)+\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(1+\alpha_1^1+\alpha_3^2)\le\beta_2$$
$$\frac{\alpha_2^1}{\alpha_2^1+\alpha_2^2}(1+\alpha_1^2+\alpha_3^2)+\frac{\alpha_2^2}{\alpha_2^1+\alpha_2^2}(2+\alpha_1^2+\alpha_3^2)\le\beta_2$$
$$\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(2+\alpha_1^1+\alpha_2^1)+\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^1+\alpha_2^1)\le\beta_3$$
$$\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^1+\alpha_2^2)+\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(2+\alpha_1^1+\alpha_2^2)\le\beta_3$$
$$\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^2+\alpha_2^1)+\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(2+\alpha_1^2+\alpha_2^1)\le\beta_3$$
$$\frac{\alpha_3^1}{\alpha_3^1+\alpha_3^2}(2+\alpha_1^2+\alpha_2^2)+\frac{\alpha_3^2}{\alpha_3^1+\alpha_3^2}(1+\alpha_1^2+\alpha_2^2)\le\beta_3$$
$$\alpha_i^j\ge 0, \forall i\in I, j=1,\ldots,m_i$$
$$0.2\le\sum_{j=1}^{m_i}\alpha_i^j\le 1, \forall i\in I.$$

One solution to $(P5)$ with $h(\alpha^*,\beta^*)=0$ and hence an MBE is $\alpha_1^{1^*}=\alpha_1^{2^*}=\alpha_2^{1^*}=\alpha_2^{2^*}=\alpha_3^{1^*}=\alpha_3^{2^*}=0.125, \beta_1^*=\beta_2^*, \beta_3^*=1.75$.

4.7 Conclusion

In this paper we gave an interpretation for the mixed via resource allocation games in normal form. In these games, a mixed strategy is an allocation. Each player chooses a pure strategy with a probability that equals to the fraction of the maximum available resource allocated to that pure strategy over the total fraction of the the resource the player allocates to all his pure strategies. We proved by Brouwer fixed point theorem the existence of an NE in these games. Furthermore, we showed that an MBE may not exist in a resource allocation game unless there exists a strategy yielding zero for the associated nonlinear program.

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Chapter 5

The Mixed Berge Equilibrium in Extensive Form Games

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Abstract

In this paper we apply the concept of a mixed Berge equilibrium to finite n−person games in extensive form. We study the mixed Berge equilibrium in both perfect and imperfect information finite games. In addition, we define the notion of a subgame perfect mixed Berge equilibrium and show that for a 2−person game, there always exists a subgame perfect Berge equilibrium. Thus there exists a mixed Berge equilibrium for any 2−person game in extensive form. For games with 3 or more players, however, a mixed Berge equilibrium and a subgame perfect mixed Berge equilibrium may not exist. In summary, this paper extends extensive form games to include players acting altruistically.

5.1 Introduction

The Berge equilibrium (BE) is a solution concept in game theory introduced in [1] and formally defined in [2]. It was extended to mixed strategies (MBE) in [3]. The Berge equilibrium represents a strategy that is mutually cooperative. In other words, at a Berge equilibrium player i cannot gain a better payoff if any other player changes his strategy unilaterally. In effect, an MBE represents the situation where every $n-1$ players choose the best joint mixed strategy for the remaining player. In this paper, we apply the concept of an MBE to finite extensive form games, where players make decisions sequentially. We consider here finite $n-$person extensive form games both with complete information and incomplete information. In a complete information game, each player is aware of the actions of the other players. In imperfect information games, however, players are not aware of the actions that other players choose.

The paper is organized as follows. In Section 2, we give the needed notation and definitions. In Section 3, we study the existence of an MBE in extensive form games. In Section 4, we give examples, and then give conclusions in Section 5.

5.2 Preliminaries

We use here a notation similar to that of [4]. An extensive form game G is written as $G = (N, H, P, I)$, where N is the set of the players, H is the set of histories, P is a function assigning a player to each non-terminal history, and I represents an information set.

Each history h is a sequence of actions $(a^k)_{k=1,\dots,K}$. In this paper, we assume that all the sequences of actions are finite. Hence the game is finite. Each history $h \in H$ ends with a terminal node which gives the utility value for each of the $n-$players. Each non-terminal node belongs to an information set I_i for a player i such that $P(h) = i$. The set of all information sets for player i is τ_i. If each information set has only one node, then the game is a perfect information game. In an imperfect information game, two or more nodes belong to some information set. If two or more nodes belong to the same information set, then they are connected with a dotted line. The idea is that if an information set includes only one decision node, then a player knows the actions that led to that node so the game is a perfect information game. An example of a $2-$person extensive form game is show in Figure 5.1. In this game, player 1 makes a decision s_1 or t_1. Next, player 2 makes a decision. After that, either the game is finished or player 1 makes a decision with imperfect information. The label above a node $(i:j)$ means information set j for player i.

Each game in extensive form can be represented as a game in normal form [5]. The set of pure strategies $S_i = \{\times A_i | A_i \in I_i, I_i \in \tau_i\}$ for each player i is the Cartesian product over the actions player i has at each of his information sets. A mixed strategy σ_i for a player i is a probability distribution over his set of pure strategies. The set of all mixed strategies for player i is $\triangle S_i$. The support of a mixed strategy for player i is $supp(\sigma_i) = \{s_i \in S_i | \sigma_i(s_i) > 0\}$. A pure strategy for player i is a special case of the mixed strategy where a player chooses exactly one action at each of his information sets.

Similarly, a mixed strategy for all players other than player i is a probability distribution σ_{-i}

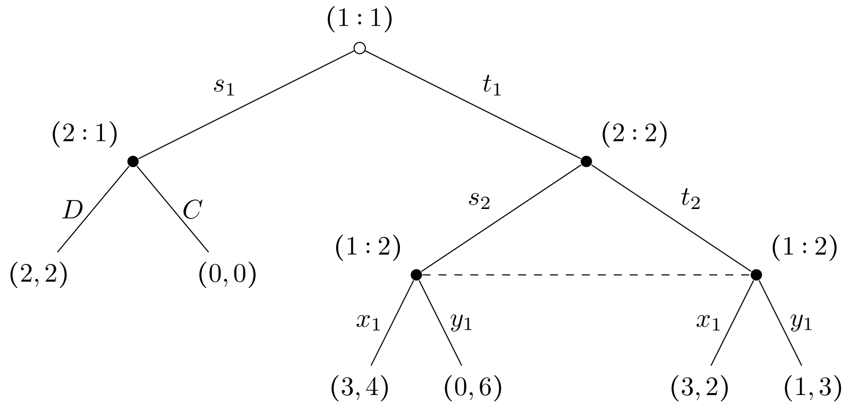

Figure 5.1: Example of a two-person extensive form game

over the set of the Cartesian product of all the pure strategies for all players other than player i. Hence $\sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) = 1, \sigma_{-i}(s_{-i}) \geq 0$, where $\sigma_{-i}(s_{-i})$ is the that product of the probabilities that each player other than player i chooses the strategy s_{-i}. The set of all mixed strategies for all players other than player i is $\triangle S_{-i}$. The support of a mixed strategy for all players other than player i is $supp(\sigma_{-i}) = \{s_{-i} \in S_{-i} | \sigma_{-i}(s_{-i}) > 0\}$.

The following identities were derived in [3]. Player i's expected payoff for strategy s_i for player i and strategy σ_{-i} for the remaining players is

$$u_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i}). \tag{5.1}$$

Player i's expected payoff for strategy σ_i for player i and strategy s_{-i} for the remaining players is

$$u_i(\sigma_i, s_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, s_{-i}). \tag{5.2}$$

Player i's expected payoff for strategy σ_i for player i and strategy σ_{-i} for the remaining players is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i) \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i}). \tag{5.3}$$

We now define the NE.

Definition 5.1. *A strategy σ^* is an NE if and only if*

$$\max_{s_i \in S_i} u_i(s_i, \sigma_i^*) = u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \forall \sigma_i \in \triangle S_i, \forall i \in N. \tag{5.4}$$

In an NE, no player can increase his expected payoff by changing his strategy unilaterally. We can similarly define an MBE.

Definition 5.2. *A strategy σ^* is an MBE if and only if*

$$\max_{s_{-i}\in S_{-i}} u_i(\sigma_i^*, s_{-i}) = u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i^*, \sigma_{-i}), \forall \sigma_{-i} \in \triangle S_{-i}, \forall i \in N. \tag{5.5}$$

In an MBE all players other than player i cannot increase the expected payoff for player i by changing their strategies. Hence no player can increase other player's expected payoff by changing his strategy unilaterally.

The subgame perfect Nash equilibrium (SPNE) is an important concept in extensive games since it always exists. An SPNE can be obtained using backward induction. The following definition of a subgame is from [4].

Definition 5.3. *An extensive form subgame is a sequence of actions h' after a history h such that $(h, h') \in H$.*

We extend the concept of the SPNE to a subgame perfect MBE (SPMBE). We prove that one exists for every 2−person game. However, we show that one may not exist for $n \geq 3$.

Definition 5.4. *A strategy σ^* is an SPNE if and only if for every nonterminal history h with $P(h) = i$, then*

$$u_i(\sigma_i^*|_h, \sigma_{-i}^*|_h) \geq u_i(\sigma_i, \sigma_{-i}^*|_h), \forall \sigma_i \in \triangle S_i, \forall i \in N. \tag{5.6}$$

An SPNE, is an NE for some subgame. Furthermore, no player can increase his expected payoff by changing his strategy unilaterally at any information node and history h such that $P(h) = i$.

We now give the definition of an SPMBE. Note the difference in history as opposed to Definition 5.4.

Definition 5.5. *A strategy σ^* is an SPMBE if and only if for every non-terminal history h with $P(h) \neq i$, then*

$$u_i(\sigma_i^*|_h, \sigma_{-i}^*|_h) \geq u_i(\sigma_i^*|_h, \sigma_{-i}), \forall h \in H, P(h) \neq i, \forall \sigma_{-i} \in \triangle S_{-i}, \forall i \in N. \tag{5.7}$$

Thus an SPMBE is a subgame concept where a strategy is an MBE for some subgame. Furthermore, players other than player i cannot increase player i's expected payoff by unilaterally changing their strategies at any information node with a non-terminal history h for which $P(h) \neq i$.

5.3 MBE Existence in Extensive Form Games

We now consider the existence of an MBE in an extensive form game. The following theorem from [5] is used.

Theorem 5.1. *Every game in extensive form has a subgame perfect NE.*

Next we define a 2−person game G' in extensive form. Each player has the same set of actions as he has in the game G. However, the two players payoffs are swapped.

Definition 5.6. *The game $G^{'}$ is a 2−person game where each player has the same actions as in the game G. The payoffs for player 1 in G are the payoffs for player 2 in $G^{'}$ and vice versa.*

Lemma 5.1. *Let G be a 2−person normal form game. Then any NE for the game $G^{'}$ is an MBE for G.*

Proof. Let σ^* be an NE for $G^{'}$. Then,

$$u_1(\sigma_1^*, \sigma_2^*) \geq u_1(\sigma_1^*, \sigma_2), \forall \sigma_2 \in \triangle S_2, \tag{5.8}$$

and

$$u_1(\sigma_1^*, \sigma_2^*) \geq u_1(\sigma_1^*, \sigma_2), \forall \sigma_2 \in \triangle S_2. \tag{5.9}$$

Thus σ^* is an MBE by Definition 5.5 to complete the proof. □

The following remark follows immediately from Theorem 5.1 and Lemma 5.1.

Remark 5.1. *Every 2−person game G has an SPMBE. Hence every 2−person game in extensive form has an MBE.*

Proof. Let G be a 2−person game and $G^{'}$ is the game with the swapped payoffs for the two players. By Theorem 5.1, $G^{'}$ always has an SPNE σ^* for some subgame in $G^{'}$. Therefore by Lemma 5.1, σ^* is an MBE for the same subgame in G. Hence the game G has an SPMBE. Moreover, by Definition 5.5 an SPMBE is an MBE for the game G, and the proof is complete. □

In the following lemma, we give necessary and sufficient conditions for the existence on an MBE.

Lemma 5.2. *A strategy σ^* is an MBE for G if and only if $\sigma_{-i}^*(s_{-i}) = 0$ when*

$$u_i(\sigma_i^*, s_{-i}) < \max_{s_{-i} \in S_{-i}} u_i(\sigma_i^*, s_{-i}). \tag{5.10}$$

Proof. Let σ^* be an MBE for G. Suppose that there exists a strategy s_{-i} such that $u_i(\sigma_i^*, s_{-i}) < \max_{s_{-i} \in S_{-i}} u_i(\sigma_i^*, s_{-i})$ and $\sigma_{-i}(s_{-i}) > 0$. Hence by Equation 5.3 $u_i(\sigma_i^*, \sigma_{-i}^*) < max_{s_{-i} \in S_{-i}} u_i(\sigma_i^*, s_{-i})$. Therefore, by Definition 5.5 the strateg σ^* is not an MBE to yield a contradiction.

Conversely, suppose σ^* is a strategy such that if

$$u_i(\sigma_i^*, s_{-i}) < \max_{s_{-i} \in S_{-i}} u_i(\sigma_i^*, s_{-i}), \tag{5.11}$$

then $\sigma_{-i}^*(s_{-i}) = 0$. Hence

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{s_{-i} \in S_{-i}} u_i(\sigma_i^*, s_{-i}), \forall i \in I. \tag{5.12}$$

Thus σ^* is an MBE by Definition 5.5. □

We now use a counterexample to prove that an MBE may not exist in n−person extensive form games with $n \geq 3$.

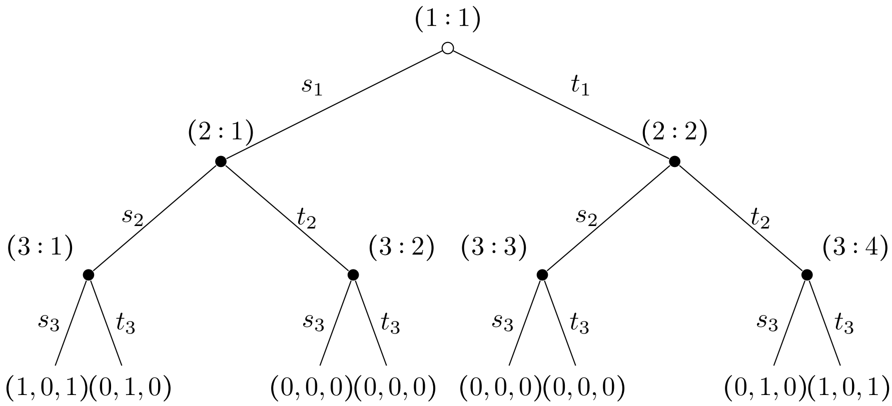

Figure 5.2: Three-Person Game with No MBE

Theorem 5.2. *An MBE may not exist when* $n \geq 3$.

Proof. The proof of this theorem is by a counterexample. Consider the following example of Figure 5.2. We claim that there is not an MBE for this game. Suppose there exists an MBE σ^* for the game. Let σ_1^* be the strategy of player 1. Note that from Figure 5.2

$$\max_{s_{-1} \in S_{-1}} u_1(\sigma_1^*, s_{-1}) = 1. \tag{5.13}$$

Moreover, σ^* is an MBE. Hence players 2 and 3 choose with positive probabilities their pure strategies that gives player 1 a payoff 1. Hence player 2 would only choose strategy s_2, t_2. However, whenever player 2 wants to maximize player $1'$s payoff there exists a pure strategy for player 3 such that for some pure strategy for player 1 in $supp(\sigma_1^*)$,

$$\max_{s_{-2} \in S_{-2}} u_2(\sigma_2^*, s_{-2}) = 1. \tag{5.14}$$

Any strategy chosen by player 3 can only maximize either player $1'$s or player $2's$ expected payoff, but not both. Hence σ^* cannot be an MBE by Lemma 5.2 to yield a contradiction. □

5.4 Examples

In this section we give two examples. In the first example we consider a 3−person game with imperfect information. We show that the game does not have an MBE. If we consider the same game with perfect information, then it has an MBE. However, the game does not have an SPMBE even with perfect information.

Example 1

We now show an example of an imperfect information game. Consider the 3−person game shown in Figure 5.3. The game shown in Table 5.1 is the normal form representation for the game in Figure 5.3. However, it was proven in [3] that an MBE does not exist for this game. We now consider the

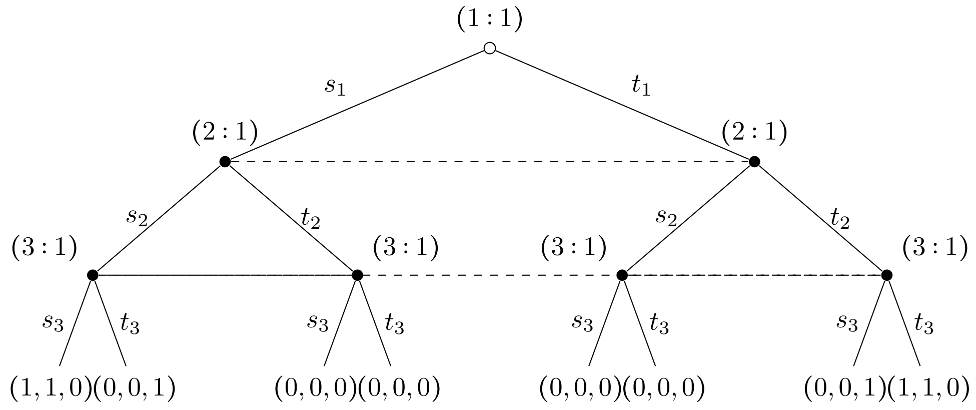

Figure 5.3: Three-Person Game with Imperfect Information

Table 5.1: Normal Form Representation

s_3	s_2	t_2	t_3	s_2	t_2
s_1	(1,1,0)	(0,0,0)	s_1	(0,0,1)	(0,0,0)
t_1	(0,0,0)	(0,0,1)	t_1	(0,0,0)	(1,1,0)

same game but with perfect information as shown in Figure 5.4. An interesting result is that the game has multiple MBEs in the case of perfect information.

The strategies for player 1 are simply s_1 and t_1. However, player 2 has 4 pure strategies and player 3 has 16 pure strategies, as shown in Tables 5.2 and 5.3 respectively.

For this game, player 3 has 16 different strategies as shown in Table 5.3. For example strategy 1 means that if player 1 chooses s_1, and player 2 chooses s_2, then player 3 chooses s_3. If player 1 chooses s_1, and player 2 chooses t_2, then player 3 chooses s_3. If player 1 chooses t_1, and player 2 chooses s_2, then player 3 chooses s_3. If player 1 chooses t_1, and player 2 chooses t_2, then player 3 chooses s_3.

One BE for this game is that player 1 chooses s_1, player 2 chooses s_2, s_2, and player 3 chooses strategy s_3, s_3, t_3, t_3. Note that for this BE, player 3 gets a payoff 0. However, players 1 and 2 cannot increase player $3'$s payoff regardless of their strategies. Moreover, they want to maximize

Table 5.2: Player 2's Strategies

Player $2'$s pure strategies	s_1	t_1
Strategy 1	If player 1 chooses s_1, then s_2.	If player 1 chooses t_1, then s_2.
Strategy 2	If player 1 chooses s_1, then s_2.	If player 1 chooses t_1, then t_2.
Strategy 3	If player 1 chooses s_1, then t_2.	If player 1 chooses t_1, then s_2.
Strategy 4	If player 1 chooses s_1, then t_2.	If player 1 chooses t_1, then t_2.

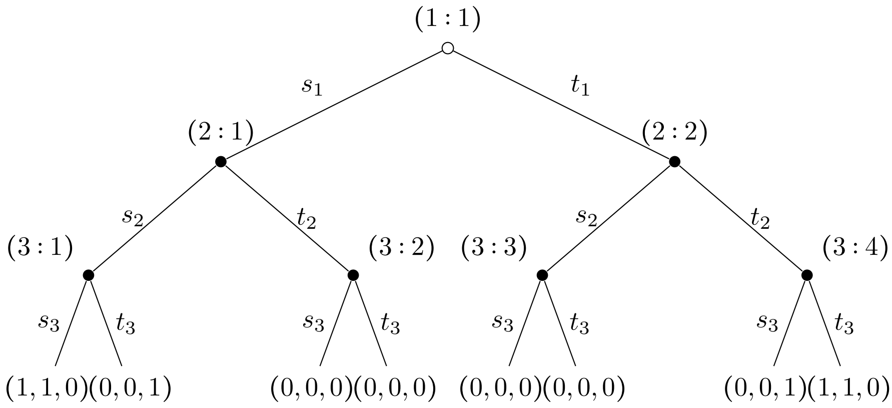

Figure 5.4: Three-Person Game with Perfect Information

the expected payoff for each other. Hence the strategy is an MBE.

Note that even with perfect information, the game does not have an SPMBE. Using backward induction, regardless of whether player 3 chooses s_3 or t_3, then player 1 alone can increase player $3'$s expected payoff. However that would result in reducing player $2'$s expected payoff. A symmetric result holds for player 1 if player 2 increases player $3'$s payoff. Hence the game cannot have an SPMBE. The next remark follows immediately.

Remark 5.2. *An MBE for the game G is not necessarily an SPMBE.*

Example 2

We now a give an example of a 2−person Bayesian game. Bayesian games with different types have been considered in literature; e.g., see [5]. In this example, we consider a 2−person game in extensive form. Each player has two strategies cooperate (C) and defect (D). We assume that there is a probability distribution over the types of player 1. The first type is an altruistic type. This type wants to maximize player $2'$s expected payoff. The second type chooses the strategy Tit-for-Tat of [6]. The second type will cooperate with player 2 only if player 2 chooses to cooperate with player 1. The third type is selfish and wants to maximize his own expected payoff. In this example, let the probability of each type be $P[Type\ 1] = p_1, P[Type\ 2] = p_2$, and $P[Type\ 3] = p_3$. The game is an imperfect information game. We assume that the game is repeated and not a one-stage game. At each stage, player 1 can be from any type and player 2 only knows the probability distribution over the types. Player 2 next chooses his action. Then player 1 chooses his action without knowing what action player 2 chose. The payoffs for each are shown in Figure 5.5.

Table 5.3: Player 3′s Strategies

Player 3′s pure strategies	s_1, s_2	s_1, t_2	t_1, s_2	t_1, t_2
Strategy 1	s_3	s_3	s_3	s_3
Strategy 2	s_3	s_3	s_3	t_3
Strategy 3	s_3	s_3	t_3	s_3
Strategy 4	s_3	t_3	s_3	s_3
Strategy 5	s_3	s_3	t_3	t_3
Strategy 6	s_3	t_3	t_3	s_3
Strategy 7	s_3	t_3	s_3	t_3
Strategy 8	s_3	t_3	t_3	t_3
Strategy 9	t_3	s_3	s_3	s_3
Strategy 10	t_3	s_3	s_3	t_3
Strategy 11	t_3	s_3	t_3	s_3
Strategy 12	t_3	t_3	s_3	s_3
Strategy 13	t_3	s_3	t_3	t_3
Strategy 14	t_3	t_3	t_3	s_3
Strategy 15	t_3	t_3	s_3	t_3
Strategy 16	t_3	t_3	t_3	t_3

Table 5.4: Two-Person Bayesian Game in Normal Form

	s_2	t_2
s_1	$(4 \times p_1 + 8 \times p_2 + 4 \times p_3, 4)$	$(5 \times p_1 + 0 \times p_2 + 1 \times p_3, 5)$
t_1	$(1 \times p_1 + 0 \times p_2 + 5 \times p_3, 1)$	$(2, 2)$

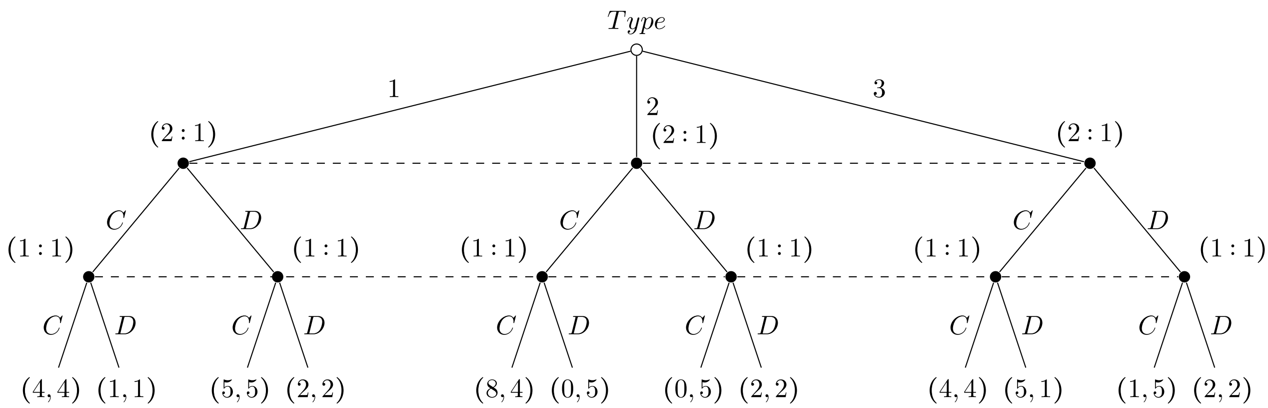

Figure 5.5: Two-Person Bayesian Game in Extensive Form

The normal form representation of the game in Figure 5.5 is shown in Table 5.4.
Note that for $p_1 \geq 0.9$ an NE for the game would be (C, D). Hence player 2 would always defect.

In the case that $p_3 \geq 0.9$, an NE would be (D, D). However, when $p_2 \geq 0.9$, then an NE for the game is (C, C). In all three cases, player 2 is selfish and concerned with his own payoff. Hence he would rather defect unless there is a high probability for the Tit-for-Tat type where player 2 can maximize his expected payoff by cooperating if the game is repeated.

5.5 Conclusion

The MBE is a solution concept in game theory that represents mutual cooperation among players and extends the BE to mixed strategies. In this paper, we extended extensive form games to include players acting altruistically. In particular, we applied the concept of an MBE to finite $n-$person games in extensive form. We showed how an MBE always exists for $2-$person games. However, we showed that an MBE may not exist in an $n-$person extensive form games with $n \geq 3$. We extended the definition of the subgame perfect equilibrium to include the case of the MBE. Moreover we proved that an SPMBE may not exist for $n \geq 3$.

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Chapter 6

Conclusion

Game theory is the study of competitive situations among rational players, who choose their strategies in order to maximize their expected utilities based on their expectations of other players' behaviors. In this dissertation, we explored some theoretical, computational, and practical aspects of equilibria in $n-$person games. We presented four journal articles.

In the first article, we defined a generalized equilibrium for $n-$person games in normal form. The Nash equilibrium and the Mixed Berge equilibrium are both special cases of the generalized equilibrium. We proved that the generalized equilibrium exists if and only if the maximum of a nonlinear program is zero.

In the second article, we studied the computational complexity of finding a mixed Berge equilibrium in normal form $n-$person games. We proved that for the $2-$person games, finding a mixed Berge equilibrium is a PPAD-complete problem. However, for games with 3 or more players, we proved that finding a mixed Berge equilibrium is an np-complete problem.

In the third paper, we gave a new interpretation of mixed strategies for the Nash and the mixed Berge equilibria. The interpretation is that a mixed strategy represents an allocation of the single resource of each player. The purpose of our approach is to avoid the ambiguities associated with the standard approach to mixed strategies.

In the fourth article, we extended the concept of a mixed Berge equilibrium to $n-$person games in extensive form. We defined a subgame perfect mixed Berge equilibrium and proved that it always exists in $2-$person games. However, a mixed Berge equilibrium may not exist in games with 3 or more players.

6.1 Appendix

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