A STUDY ON THE NONLOCAL SHALLOW-WATER MODEL ARISING FROM THE FULL WATER WAVES WITH THE CORIOLIS EFFECT

by

JUNWEI SUN

Presented to the Faculty of the Graduate School of
The University of Texas at Arlington in Partial Fulfillment
of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON
August 2017

Copyright © by Junwei Sun 2017

All Rights Reserved

To my mother Daiwei Wang and my father Daming Sun.

ACKNOWLEDGEMENTS

I would like to thank my supervising professor Dr. Yue Liu for his clairvoyant mathematical vision and cautious work attitude, like a lighthouse shining on my way forward. I wish to thank my academic advisors Dr. Jianzhong Su, Dr. Gaik Ambartsoumian, Dr. Guojun Liao for their interest in my research and for taking time to serve in my dissertation committee.

I would also like to extend my appreciation to Department of Mathematics, University of Texas at Arlington for providing financial support for my doctoral studies. I wish to thank Ting Luo and Emel Bolat. I have been enjoying the time we discussing questions and working like a family in past four years.

I am grateful to the teachers who taught me during the years I spent in school, first in China, then in the Unites States. I would like to thank Dr. Da-jun Zhang, my master's advisor, for encouraging and inspiring me to pursue graduate studies.

Last but not least, thanks to my parents, who love me, support me, listen to me selflessly.

May 31, 2017

ABSTRACT

A STUDY ON THE NONLOCAL SHALLOW-WATER MODEL ARISING FROM THE FULL WATER WAVES WITH THE CORIOLIS EFFECT

Junwei Sun, Ph.D.

The University of Texas at Arlington, 2017

Supervising Professor: Yue Liu

The Equatorial Undercurrent is a significant feature of the geophysical waves near the equator, which is one of the key factors to explain El Niño phenomenon. However, based on β-plane approximation, the classical theory of geophysical waves ignored the vertical structure of the Equatorial Undercurrent.

To obtain a better description of the equatorial waves, in this dissertation, I study the rotational-Camassa-Holm (R-CH) equation, which is a mathematical model of long-crested water waves near the equator, propagating mainly in one direction with the effect of Earth's rotation under the f-plane approximation. R-CH equation can be derived by following the formal asymptotic procedures. Such a model equation is analogous to the Camassa-Holm approximation of the two-dimensional incompressible and irrotational Euler equations and has a formal bi-Hamiltonian structure. Its solutions corresponding to physically relevant initial perturbations is more accurate on a much longer time scale. It is shown that the deviation of the free surface can be determined by the horizontal velocity at a certain depth in the second-order approximation. The effects of the Coriolis force caused by the Earth rotation and nonlocal

higher nonlinearities on blow-up criteria and wave-breaking phenomena are also investigated.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

CHAPTER 1

BACKGROUND

1.1 El Niño and equatorial waves

El Niño is associated with a band of warm ocean water that develops in the central and east-central equatorial Pacific (between approximately the International Date Line and 120°W), including off the Pacific coast of South America (See Figure 1.1). Fishermen off the west coast of South America were the first to notice appearances of unusually warm water that occurred at year's end. The phenomenon became known as El Niño because of its tendency to occur around Christmas time. El Niño is Spanish for "the boy child" and is named after the baby Jesus.

Most of early El Niño conditions were too weak to attracted people's attention. From the 19th century, some strong El Niño events were recorded gradually because of their effect on those enterprises that depend on biological productivity of the sea. Charles Todd, in 1888, suggested droughts in India and Australia tended to occur at the same time [18]; Norman Lockyer noted the same in 1904 [29]. An El Niño connection with flooding was reported in 1894 by Vctor Eguiguren (1852-1919) and in 1895 by Federico Alfonso Pezet (1859-1929) [17, 31]. In 1924, Gilbert Walker coined the term "Southern Oscillation" [37]. He and others are generally credited with identifying the El Niño effect.

Now, El Niño is widely regarded as a phenomenon affecting the global climate and disrupts normal weather patterns, which as a result can lead to intense storms in some places and droughts in others. Currently, a number of studies show that the sea surface temperature near the equator, especially Niño 3.4 and Niño 3 (See Figure 1.1), play an important role in constituting an El Niño event.

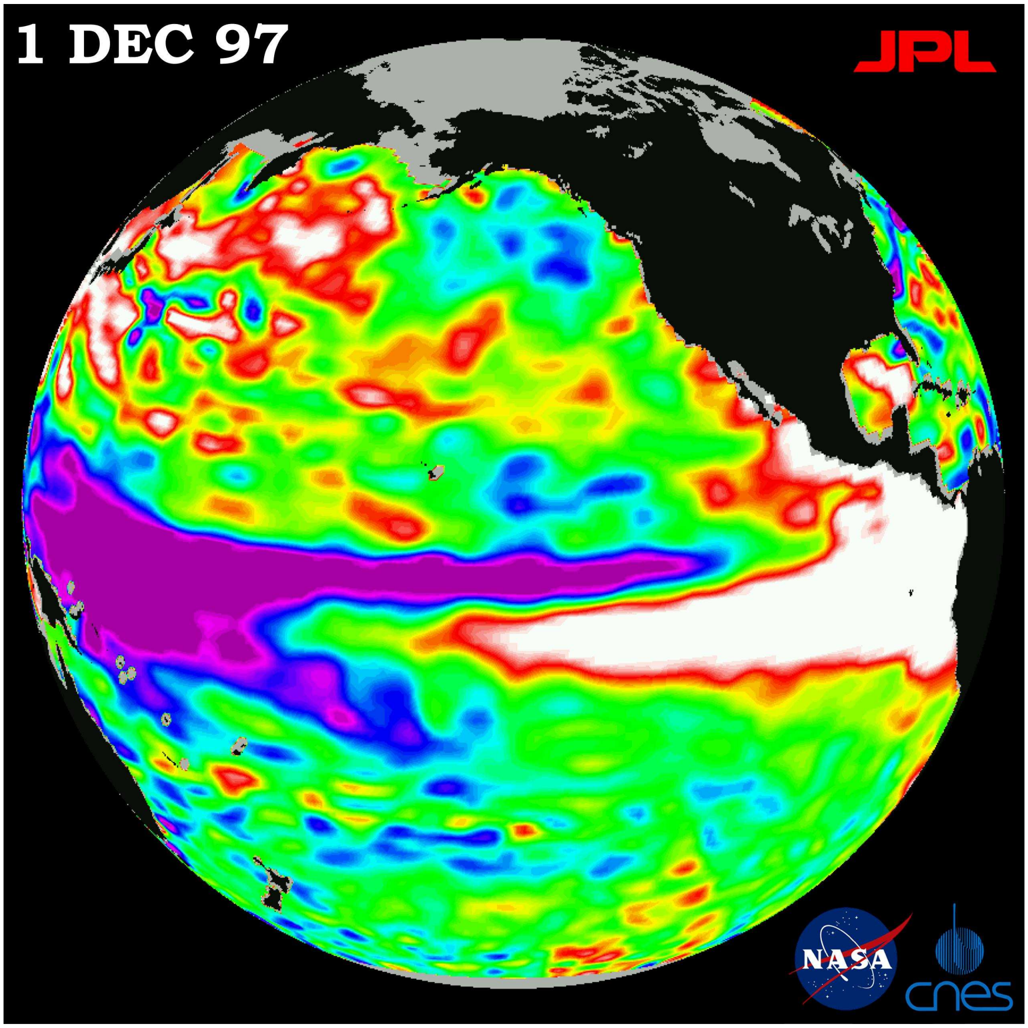

Figure 1.1. The 1997-98 El Niño observed by TOPEX/Poseidon. The white areas indicate the pool of warm water off the Tropical Western coasts of northern South and all Central America as well as along the Central-eastern equatorial and Southeastern Pacific Ocean.

The United States Climate Prediction Center and the International Research Institute for Climate and Society claims that an El Niño event is under way when the sea surface temperatures index in the central Pacific (the Niño 3.4 region) equal or exceed +0.5°C for several seasons in a row [19]. While in another research, the Japan Meteorological Agency declares that an El Niño event has started when the average 5 month sea surface temperature deviation for the Niño 3 region, is over +0.5°C warmer for six consecutive months or longer [25]. In these El Niño regions, an important feature is that the Equatorial Undercurrent. First discovered in 1951 [32], the flow reverses at a depth of several tens of meters while the surface flow is generally directed westward because of the prevalence of winds that blow westward. Because of the Equatorial Undercurrent, the inherent ocean adjustment influence the surface ocean temperature. Therefore, one of the keys in explaining El Niño is to model the Pacific Equatorial Undercurrent. Today, lots of researches shows that the Pacific Equatorial Undercurrent is thin (less than 200 m deep [32]), symmetric (from 5°S latitude to 5°N latitude) and remarkable long (about 13000 km, extending nearly the whole ocean basin [24]) such that it can be taken as a shallow water layer.

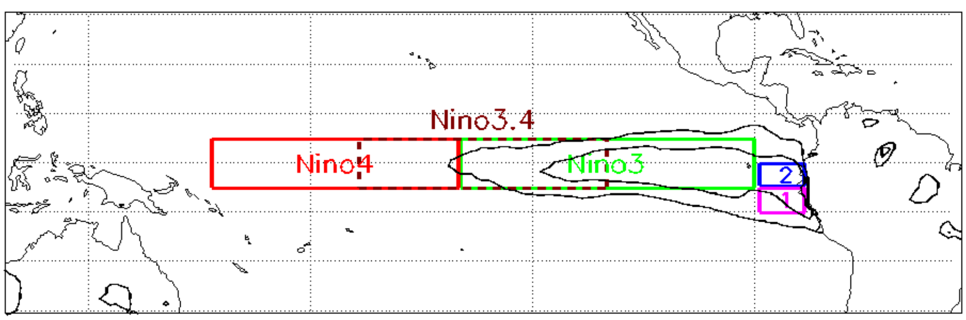

Figure 1.2. Map showing Niño 3.4 and other index regions.

The classical model of equatorial waves are eastward propagating Kelvin waves, which predicted theoretically by using a β-plane approximation to the governing equations in the shallow water regime of a one-layer reduced-gravity model [15]. Unfortunately, this theory ignored vertical variations of the flow but the fact is that the vertical stratification of the Coriolis force on the ocean is greater than anywhere else. Meanwhile, the Coriolis parameter along the Equator vanish and the β-plane effect on the planetary vorticity of the flow in these area amounts to less than 1.75%, which implies f-plane approximation is reasonable. Recently, a shallow water model with the f-plane approximation proposed by A. Constantin succeeded to capture these features of the Equatorial Undercurrent [10]. However, it is just the simplest approximation to the Euler dynamics such that more mathematical work is needed in the future.

1.2 Early Developments of solitary shallow waves

In this section we review the historical development of nonlinear shallow water wave theory following Ablowitz and Clarkson [1].

In 1834, a young engineer named John Scott Russel (1808-1882) made a remarkable discovery when he was riding on horseback along a narrow canal near Edinburgh, Scotland. He described it in his "Report on Waves" [33, 34].

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

To confirm and study this phenomenon, he build an experimental tank, 20 feet long and 1 foot wide, in his garden in August, 1837. He wanted to repeat the solitary waves in his tank and he succeeded. Based on the data from these experiments, he found that the speed of propagation c of the solitary wave in a channel of depth h to be

$$c = \sqrt{g(h + \alpha)},$$

where α is the amplitude of the wave and g the force due to gravity.

Unfortunately, Russel's contemporaries, containing famous fluid mechanic George Gabriel Stokes, ill-understood and doubted this discovery. And solitary waves was forgotten by the times.

Until 1895, D. J. Korteweg and his student, G. de Vries, deduced the now famous Korteweg-de Vries (KdV) equation

$$u_t + u_x + \frac{3}{2} u u_x + \frac{1}{6} u_{xxx} = 0, \tag{1.1}$$

under the long-wave and small-amplitude assumption[28]. And the traveling wave solution they obtained from KdV equation, also named one-soliton solution later, explain Russel's phenomenon. In 1965, Kruskal and Zabusky gave Russel's wave a new name, soliton, after using digital computer and numerical simulation method to research two-soliton solution[39]. Now, soliton, as a nonlinear mathematical and physical theory, can describe so many phenomena from fiber optics, quantum mechanics to biology. To commemorate Russel's contribution to soliton discovery, on 12 July 1995, an international gathering of scientists witnessed a re-creation of the famous 1834 'first' sighting of a soliton or solitary wave on the Union Canal near Edinburgh (see Figure 1.2).

Figure 1.3. Soliton on the Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, 12 July 1995.

1.3 The Camassa-Holm equation

In this section, we review the historical development of the Camssa-Holm equation.

It is known that many of the shallow water models as approximations to the full Euler dynamics are only valid in the weakly nonlinear regime, for instance, the classical Korteweg-de Vries (KdV) equation. However, the more interesting physical phenomena, such as wave breaking, waves of maxima height [2, 36], require a transition to full *nonlinearity*. The KdV equation is a simple mathematical model for gravity waves in shallow water, but it fails to model fundamental physical phenomena such as the extreme wave of Stokes [35] and does not include breaking waves (i.e. wave profile remains bounded while its slope becomes unbounded in finite time). The failure of weakly nonlinear shallow-water wave equations to model observed wave phenomena in nature is prime motivation in the search for alternative models for nonlinear shallow-water waves [34, 38]. The long-wave regime is usually characterized by presumptions of long wavelength λ and small amplitude a with the amplitude parameter ε and the shallowness parameter μ respectively by

$$\varepsilon = \frac{a}{h_0} \ll 1, \qquad \mu = \frac{h_0^2}{\lambda^2} \ll 1.$$

It is well understood that the KdV model provides a good asymptotic approximations of unidirectional solutions of the irrotational two-dimensional water waves problem on the Boussinesq regime $\mu \ll 1$, $\varepsilon = O(\mu)$ [7, 14]. To describe more accurately the motion of these unidirectional waves, it was shown in [13] that the Camassa-Holm (CH) equation [8, 20] in the CH scaling, $\mu \ll 1$, $\varepsilon = O(\sqrt{\mu})$, could be valid higher order approximations to the governing equation for full water waves in the long time scaling $O(\frac{1}{\varepsilon})$. Like the KdV, the CH equation is integrable and have solitons, while the CH equation models breaking waves and has peaked solitary waves [8, 12, 30]. It is also found that the Euler equation has breaking waves [4] and a traveling-wave solution with the greatest height which has a corner at its crest [36].

The Camassa-Holm equation inspired the search for various generalization of this equation with interesting properties and applications. Note that all nonlinear terms in the CH equation is quadratic. It is then of great interest to find those integrable equations with higher-power nonlinear terms.

CHAPTER 2

THE ROTATION-CAMASSA-HOLM EQUATION

2.1 Introduction

Our first main aim of this chapter is to formally derive a model equation with the Coriolis effect from the incompressible and irrotational two-dimensional shallow water in the equatorial region. This new model equation called the rotation-Camassa-Holm (R-CH) equation has a cubic and even quartic nonlinearities and a formal Hamiltonian structure. More precisely, the motion of the fluid is described by the scalar equation in the form

$$\begin{aligned} \partial_t u - \beta\mu\partial_t u_{xx} + cu_x + 3\alpha\varepsilon u u_x - \beta_0\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x \\ = \alpha\beta\varepsilon\mu(2u_x u_{xx} + u u_{xxx}), \end{aligned} \tag{2.1}$$

where the parameter Ω is the constant rotational frequency due to the Coriolis effect. The other constants appearing in the equation are defined by $c = \sqrt{1+\Omega^2} - \Omega$, $\alpha \stackrel{\text{def}}{=} \frac{c^2}{1+c^2}$, $\beta_0 \stackrel{\text{def}}{=} \frac{c(c^4+6c^2-1)}{6(c^2+1)^2}$, $\beta \stackrel{\text{def}}{=} \frac{3c^4+8c^2-1}{6(c^2+1)^2}$, $\omega_1 \stackrel{\text{def}}{=} \frac{-3c(c^2-1)(c^2-2)}{2(1+c^2)^3}$, and $\omega_2 \stackrel{\text{def}}{=} \frac{(c^2-2)(c^2-1)^2(8c^2-1)}{2(1+c^2)^5}$ satisfying $c \to 1$, $\beta \to \frac{5}{12}$, $\beta_0 \to \frac{1}{4}$, ω_1, $\omega_2 \to 0$ and $\alpha \to \frac{1}{2}$ when $\Omega \to 0$.

The solution u of (2.1) represents the horizontal velocity field at height z_0, and after the re-scaling, it is required that $0 \le z_0 \le 1$, where

$$z_0^2 = \frac{1}{2} - \frac{2}{3}\frac{1}{(c^2+1)} + \frac{4}{3}\frac{1}{(c^2+1)^2}. \tag{2.2}$$

Since it is also natural to require that the constant $\beta > 0$, it must be the case

$$0 \le \Omega < \sqrt{\frac{1}{6}(1+2\sqrt{19})} \approx 1.273,$$

and

$$\frac{1}{\sqrt{2}} \le z_0 < \sqrt{\frac{61-2\sqrt{19}}{54}} \approx 0.984.$$

In particular, when $\Omega = 0$, $z_0 = \frac{1}{\sqrt{2}}$ is corresponding to the case of classical CH equation.

The starting point of our derivation of the R-CH model in (2.1) is the paper [26] where the classical CH equation was derived. The R-CH equation in (2.1) is established by showing that after a double asymptotic expansion with respect to ε and μ, the free surface $\eta = \eta(\tau, \xi)$ under the field variable (η, ξ) defined in (2.5) in 2D Euler's dynamics (2.6) (see Section 2), is governed by the equation

$$2(\Omega + c)\eta_\tau + 3c^2\eta\eta_\xi + \frac{c^2}{3}\mu\eta_{\xi\xi\xi} + A_1\varepsilon\eta^2\eta_\xi + A_2\varepsilon^2\eta^3\eta_\xi + A_5\varepsilon^3\eta^4\eta_\xi$$
$$= \varepsilon\mu\Big[A_3\eta_\xi\eta_{\xi\xi} + A_4\eta\eta_{\xi\xi\xi}\Big] + O(\varepsilon^4, \mu^2),$$

where the constants $A_1 \stackrel{\text{def}}{=} \frac{3c^2(c^2-2)}{(c^2+1)^2}$, $A_2 \stackrel{\text{def}}{=} -\frac{c^2(2-c^2)(c^6-7c^4+5c^2-5)}{(c^2+1)^4}$, $A_3 \stackrel{\text{def}}{=} \frac{-c^2(9c^4+16c^2-2)}{3(c^2+1)^2}$, $A_4 \stackrel{\text{def}}{=} \frac{-c^2(3c^4+8c^2-1)}{3(c^2+1)^2}$, $A_5 \stackrel{\text{def}}{=} \frac{c^2(c^2-2)(3c^{10}+228c^8-540c^6-180c^4-13c^2+42)}{12(c^2+1)^6}$. The free surface η with respect to the horizontal component of the velocity u at $z = z_0$ under the CH regime $\varepsilon = O(\sqrt{\mu})$ is also given by

$$\eta = \frac{1}{c}u + \gamma_1\varepsilon u^2 + \gamma_2\varepsilon^2u^3 + \gamma_3\varepsilon^3u^4 + \gamma_4\varepsilon\mu u_{\xi\xi} + O(\varepsilon^4, \mu^2),$$

where the constants in the expression are given by $\gamma_1 = \frac{2-c^2}{2c^2(c^2+1)}$, $\gamma_2 = \frac{(c^2-1)(c^2-2)(2c^2+1)}{2c^3(c^2+1)^3}$, $\gamma_3 = -\frac{(c^2-1)^2(c^2-2)(21c^4+16c^2+4)}{8c^4(c^2+1)^5}$, and $\gamma_4 = \frac{z_0^2}{2c} - \frac{3c^2+1}{6c(c^2+1)} = \frac{-(3c^4+6c^2-5)}{12c(c^2+1)^2}$ (here the height parameter z_0 is determined by (2.2)).

Denote $m \stackrel{\text{def}}{=} (1 - \beta\mu\partial_x^2)u$, one can rewrite the above equation in terms of the evolution of the momentum density m, namely,

$$\partial_t m + \alpha\varepsilon(um_x + 2mu_x) + cu_x - \beta_0\mu u_{xxx} + \omega_1\varepsilon^2u^2u_x + \omega_2\varepsilon^3u^3u_x = 0. \tag{2.3}$$

2.2 Derivation of RCH equation

The formal derivation of the Camassa-Holm model equation with the Coriolis effect is the topic of the present section. Attention is given here is the so-called long-wave limit. in this setting, it is assumed that water flows are incompressible and inviscid with a constant density ρ and no surface tension, and the interface between the air and the water is a free surface. Then such a motion of water flow occupying a domain Ω_t in $\mathbb{R}^3$ under the influence of the gravity g and the Coriolis force due to the Earth's rotation can be described by the Euler equations [21], *viz.*

$$\begin{cases} \vec{u}_t + (\vec{u} \cdot \nabla)\, \vec{u} + 2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho} \nabla P + \vec{g}, \quad x \in \Omega_t, \\ \nabla \cdot \vec{u} = 0, \quad x \in \Omega_t, \\ \vec{u}|_{t=0} = \vec{u_0}, \quad x \in \Omega_0, \end{cases}$$

where $\vec{u} = (u, v, w)^T$ is the fluid velocity, $P(t, x, y, z)$ is the pressure in the fluid, $\vec{g} = (0, 0, -g)^T$ with $g \approx 9.8 m/s^2$ the constant gravitational acceleration at the Earth's surface, and $\vec{\Omega} = (0,\ \Omega_0 \cos\phi,\ \Omega_0 \sin\phi)^T$, with the rotational frequency $\Omega_0 \approx 73 \cdot 10^{-6}$rad/s and the local latitude ϕ, is the angular velocity vector which is directed along the axis of rotation of the rotating reference frame. We adopt a rotating framework with the origin located at a point on the Earth's surface, with the x-axis chosen horizontally due east, the y-axis horizontally due north and the z-axis upward. We consider here waves at the surface of water with a flat bed, and assume that $\Omega_t = \{(x, y, z) : 0 < z < h_0 + \eta(t, x, y)\}$, where h_0 is the typical depth of the water and $\eta(t, x, y)$ measures the deviation from the average level. Under the f-plane approximation ($\sin\phi \approx 0,\ \phi \ll 1$), the motion of inviscid irrotational fluid near the Equator in the region $0 < z < h_0 + \eta(t, x, y)$ with a constant density ρ is described by the Euler equations [10, 21] in the form

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\Omega_0 w = -\frac{1}{\rho} P_x, \\ v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho} P_y, \\ w_t + uw_x + vw_y + ww_z - 2\Omega_0 u = -\frac{1}{\rho} P_z - g, \end{cases}$$

the incompressibility of the fluid,

$$u_x + v_y + w_z = 0,$$

and the irrotational condition,

$$(w_y - v_z, u_z - w_x, v_x - u_y)^T = (0, 0, 0)^T.$$

The pressure is written as

$$P(t, x, z) = P_a + \rho g(h_0 - z) + p(t, x, y, z),$$

where P_a is the constant atmosphere pressure, and p is a pressure variable measuring the hydrostatic pressure distribution.

The dynamic condition posed on the surface $z = h_0 + \eta$ yields $P = P_a$. Then there appears that

$$p = \rho g \eta.$$

Meanwhile, the kinematic condition on the surface is given by

$$w = \eta_t + u\eta_x + v\eta_y, \quad \text{when} \quad z = h_0 + \eta(t, x, y).$$

Finally, we pose "no-flow" condition at the flat bottom $z = 0$, that is,

$$w|_{z=0} = 0.$$

Consider the two-dimensional flows, moving in the zonal direction along the equator independent of the y-coordinate, in other words, $v \equiv 0$ throughout the flow, the irrotational condition will be simplified as $u_z - w_x = 0$. According to the magnitude of the physical quantities, we introduce dimensionless quantities as follows

$$x \to \lambda x, \quad z \to h_0 z, \quad \eta \to a\eta, \quad t \to \frac{\lambda}{\sqrt{gh_0}} t,$$

which implies

$$u \to \sqrt{gh_0}u, \quad w \to \sqrt{\mu g h_0}w, \quad p \to \rho g h_0 p.$$

And under the influence of the Earth rotation, we introduce

$$\Omega = \sqrt{\frac{h_0}{g}}\Omega_0.$$

Furthermore, considering whenever $\varepsilon \to 0$,

$$u \to 0, \quad w \to 0, \quad p \to 0,$$

that is, u, w and p are proportional to the wave amplitude so that we require a scaling

$$u \to \varepsilon u, \quad w \to \varepsilon w, \quad p \to \varepsilon p.$$

Therefore the governing equations become

$$\begin{cases} u_t + \varepsilon(uu_x + wu_z) + 2\Omega w = -p_x & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ \mu\{w_t + \varepsilon(uw_x + ww_z)\} - 2\Omega u = -p_z & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ u_x + w_z = 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ u_z - \mu w_x = 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ p = \eta & \text{on} \quad z = 1 + \varepsilon\eta(t,x), \\ w = \eta_t + \varepsilon u\eta_x & \text{on} \quad z = 1 + \varepsilon\eta(t,x), \\ w = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.4}$$

To derive the R-CH equation for shallow water waves,, we first introduce a suitable scale and a double asymptotic expansion to get equations in groups with respect to ε and μ independent on each other, where $\varepsilon, \mu \ll 1$.

Let c be the group speed of water waves. We can apply a suitable far field variable together with a propagation problem [26, 27]

$$\xi = \varepsilon^{1/2}(x - ct), \quad \tau = \varepsilon^{3/2}t, \tag{2.5}$$

which implies, for consistency from the equation of mass conservation, that we also transform

$$w = \sqrt{\varepsilon} W.$$

Then the governing equations (2.4) become

$$\begin{aligned}
-cu_\xi + \varepsilon(u_\tau + uu_\xi + Wu_z) + 2\Omega W &= -p_\xi \quad \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\
\varepsilon\mu\{-cW_\xi + \varepsilon(W_\tau + uW_\xi + WW_z)\} - 2\Omega u &= -p_z \quad \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\
u_\xi + W_z &= 0 \quad \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\
u_z - \varepsilon\mu W_\xi &= 0 \quad \text{in} \quad 0 < z < 1 + \varepsilon\eta, \\
p &= \eta \quad \text{on} \quad z = 1 + \varepsilon\eta, \\
W &= -c\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi) \quad \text{on} \quad z = 1 + \varepsilon\eta, \\
W &= 0 \quad \text{on} \quad z = 0.
\end{aligned} \tag{2.6}$$

A double asymptotic expansion is introduced to seek a solution of the system (2.6),

$$q \sim \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varepsilon^n \mu^m q_{nm}$$

as $\varepsilon \to 0, \mu \to 0$, where q will be taken the scale functions u, W, p and η, and all the functions q_{nm} satisfiy the far field conditions $q_{nm} \to 0$ as $|\xi| \to \infty$ for every n, $m = 0, 1, 2, 3, ...$

Substituting the asymptotic expansions of u, W, p, η into (2.6), we check all the coefficients of the order $O(\varepsilon^i \mu^j)$ $(i,\ j = 0, 1, 2, 3, ...)$.

From the order $O(\varepsilon^0\mu^0)$ terms of (2.6) we obtain

$$\begin{cases} -cu_{00,\xi} + 2\Omega W_{00} = -p_{00,\xi} & \text{in} \quad 0 < z < 1, \\ 2\Omega u_{00} = p_{00,z} & \text{in} \quad 0 < z < 1, \\ u_{00,\xi} + W_{00,z} = 0 & \text{in} \quad 0 < z < 1, \\ u_{00,z} = 0 & \text{in} \quad 0 < z < 1, \\ p_{00} = \eta_{00} & \text{on} \quad z = 1, \\ W_{00} = -c\eta_{00,\xi} & \text{on} \quad z = 1, \\ W_{00} = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.7}$$

To solve the system (2.7), we first obtain from the fourth equation in (2.7) that u_{00} is independent of z, that is,

$$u_{00} = u_{00}(\tau, \xi).$$

Thanks to the third equation in (2.7) and the boundary condition of W on $z = 0$, we get

$$W_{00} = W_{00}|_{z=0} + \int_0^z W_{00,z'} dz' = -\int_0^z u_{00,\xi}\, dz' = -zu_{00,\xi}, \tag{2.8}$$

which along with the boundary condition of W on $z = 1$ implies

$$u_{00,\xi}(\tau, \xi) = c\eta_{00,\xi}(\tau, \xi). \tag{2.9}$$

Thereore, we have

$$u_{00}(\tau, \xi) = c\eta_{00}(\tau, \xi), \quad W_{00} = -cz\eta_{00,\xi}, \tag{2.10}$$

here use has been made of the far field conditions u_{00}, $\eta_{00} \to 0$ as $|\xi| \to \infty$.

On the other hand, from the second equation in (2.7), there appears that

$$p_{00} = p_{00}|_{z=1} + \int_1^z p_{00,z'}\, dz' = \eta_{00} + 2\Omega \int_1^z u_{00}\, dz' = \eta_{00} + 2\Omega(z-1)u_{00}, \tag{2.11}$$

which along with (2.9) implies

$$p_{00,\xi} = \big(\frac{1}{c} + 2\Omega(z-1)\big)u_{00,\xi}, \tag{2.12}$$

Combining (2.12) with (2.8) and the first equation in (2.7) gives rise to

$$(c^2 + 2\Omega c - 1)u_{00,\xi} = 0,$$

which follows that

$$c^2 + 2\Omega c - 1 = 0, \tag{2.13}$$

if we assume that u_{00} is an non-trivial velocity. Therefore, when consider the waves move towards to the right side, we may obtain

$$c = \sqrt{1+\Omega^2} - \Omega. \tag{2.14}$$

Vanishing the order $O(\varepsilon^1\mu^0)$ terms of (2.6), we obtain from the second equation in (2.16) and the Taylor expansion

$$f(z) = f(1) + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} f^{(n)}(1) \tag{2.15}$$

that

$$\begin{cases} -cu_{10,\xi} + u_{00,\tau} + u_{00}u_{00,\xi} + 2\Omega W_{10} = -p_{10,\xi} & \text{in} \quad 0 < z < 1, \\ 2\Omega u_{10} = p_{10,z} & \text{in} \quad 0 < z < 1, \\ u_{10,\xi} + W_{10,z} = 0 & \text{in} \quad 0 < z < 1, \\ u_{10,z} = 0 & \text{in} \quad 0 < z < 1, \\ p_{10} + p_{00,z}\eta_{00} = \eta_{10} & \text{on} \quad z = 1, \\ W_{10} + \eta_{00}W_{00,z} = -c\eta_{10,\xi} + \eta_{00,\tau} + u_{00}\eta_{00,\xi} & \text{on} \quad z = 1, \\ W_{10} = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.16}$$

From the fourth equation in (2.16), we know that u_{10} is independent to z, that is, $u_{10} = u_{10}(\tau, \xi)$. Thanks to the third equation in (2.16) and the boundary conditions of W on $z = 0$ and $z = 1$, we get

$$W_{10} = W_{10}|_{z=0} + \int_0^z W_{10,z'} dz' = -z u_{10,\xi} \tag{2.17}$$

and

$$W_{10}|_{z=1} = -c\eta_{10,\xi} + \eta_{00,\tau} + (u_{00}\eta_{00})_\xi.$$

Hence, we obtain from the third equation in (2.7) and (2.10) that

$$u_{10,\xi} = c\eta_{10,\xi} - \eta_{00,\tau} - (u_{00}\eta_{00})_\xi, \tag{2.18}$$

and then

$$W_{10} = z(\eta_{00,\tau} + 2c\eta_{00}\eta_{00,\xi} - c\eta_{10,\xi}).$$

On the other hand, thanks to the second equation in (2.16) and (2.10), we deduce that

$$p_{10} = p_{10}|_{z=1} + \int_1^z p_{10,z'} dz' = \eta_{10} - 2\Omega u_{00}\eta_{00} + 2\Omega(z-1)u_{10},$$

and then

$$p_{10,\xi} = \eta_{10,\xi} - 2\Omega(u_{00}\eta_{00})_\xi + 2\Omega(z-1)u_{10,\xi}. \tag{2.19}$$

Taking account of the first equation in (2.16), (2.10), and (2.10), it must be

$$-p_{10,\xi} = -cu_{10,\xi} + c\eta_{00,\tau} + c^2\eta_{00}\eta_{00,\xi} - 2\Omega z u_{10,\xi},$$

which along with (2.19) and (2.18) implies

$$\begin{aligned} 0 =& -(c+2\Omega)u_{10,\xi} + \eta_{10,\xi} + c\eta_{00,\tau} + c^2\eta_{00}\eta_{00,\xi} - 2\Omega(u_{00}\eta_{00})_\xi \\ =& c(u_{00}\eta_{00})_\xi - (c^2 + 2\Omega c - 1)\eta_{10,\xi} + 2(c+\Omega)\eta_{00,\tau} + c^2\eta_{00}\eta_{00,\xi}. \end{aligned}$$

Hence, it follows from (2.10) and (2.13) that

$$2(\Omega + c)\eta_{00,\tau} + 3c^2\eta_{00}\eta_{00,\xi} = 0. \tag{2.20}$$

Defining

$$c_1 \stackrel{\text{def}}{=} -\frac{3c^2}{4(\Omega + c)} = -\frac{3c^3}{2(c^2+1)}, \tag{2.21}$$

we may rewrite (2.20) as

$$\eta_{00,\tau} = c_1(\eta_{00}^2)_\xi, \tag{2.22}$$

which, together with (2.18), implies

$$u_{10,\xi} = \big(c\eta_{10} - (c + c_1)\eta_{00}^2\big)_\xi. \tag{2.23}$$

Therefore, we get from the far field conditions u_{10}, $\eta_{00}, \eta_{10} \to 0$ as $|\xi| \to \infty$ that

$$u_{10} = c\eta_{10} - (c + c_1)\eta_{00}^2, \tag{2.24}$$

which follows from (2.22) that

$$u_{10,\tau} = c\eta_{10,\tau} - 4(c + c_1)c_1\eta_{00}^2\eta_{00,\xi}. \tag{2.25}$$

Similarly, vanishing the order $O(\varepsilon^0\mu^1)$ terms of (2.6), we obtain from the second equation in (2.7) and the Taylor expansion (2.15) that

$$\begin{cases} -cu_{01,\xi} + 2\Omega W_{01} = -p_{01,\xi} & \text{in} \quad 0 < z < 1, \\ 2\Omega u_{01} = p_{01,z} & \text{in} \quad 0 < z < 1, \\ u_{01,\xi} + W_{01,z} = 0 & \text{in} \quad 0 < z < 1, \\ u_{01,z} = 0 & \text{in} \quad 0 < z < 1, \\ p_{01} = \eta_{01} & \text{on} \quad z = 1, \\ W_{01} = -c\eta_{01,\xi} & \text{on} \quad z = 1, \\ W_{01} = 0 & \text{on} \quad z = 0. \end{cases}$$

From this, we may readily get from the above argument that

$$u_{01} = c\eta_{01} = c\eta_{01}(\tau, \xi),\ W_{01} = -cz\eta_{01,\xi},\ p_{01} = [2\Omega c(z-1)+1]\eta_{01}. \tag{2.26}$$

For the order $O(\varepsilon^2\mu^0)$ terms of (2.6), we obtain from the Taylor expansion (2.15) that

$$\begin{cases} -cu_{20,\xi} + u_{10,\tau} + (u_{00}u_{10})_\xi + 2\Omega W_{20} = -p_{20,\xi} & \text{in} \quad 0<z<1, \\ -2\Omega u_{20} = -p_{20,z} & \text{in} \quad 0<z<1, \\ u_{20,\xi} + W_{20,z} = 0 & \text{in} \quad 0<z<1, \\ u_{20,z} = 0 & \text{in} \quad 0<z<1, \\ p_{20} + \eta_{00}p_{10,z} + \eta_{10}p_{00,z} = \eta_{20} & \text{on} \quad z=1, \\ W_{20} + \eta_{00}W_{10,z} + \eta_{10}W_{00,z} & \\ \qquad = -c\eta_{20,\xi} + \eta_{10,\tau} + u_{00}\eta_{10,\xi} + u_{10}\eta_{00,\xi} & \text{on} \quad z=1, \\ W_{20} = 0 & \text{on} \quad z=0. \end{cases} \tag{2.27}$$

From the fourth equation in (2.27), we know that u_{20} is independent of z, that is,

$$u_{20} = u_{20}(\tau, \xi),$$

which along with the third equation in (2.27) and the boundary condition of W_{20} at $z=0$ implies that

$$W_{20} = -zu_{20,\xi}. \tag{2.28}$$

Combining (2.28) with the boundary condition of W_{20} at $z=1$, we get from the equations of $W_{00,z}$ and $W_{10,z}$ that

$$u_{20,\xi} = c\eta_{20,\xi} - \eta_{10,\tau} - (u_{00}\eta_{10} + u_{10}\eta_{00})_\xi,$$

that is,

$$u_{20,\xi} = c\eta_{20,\xi} - \eta_{10,\tau} - 2c(\eta_{00}\eta_{10})_\xi + (c+c_1)(\eta_{00}^3)_\xi. \tag{2.29}$$

While from the second equation in (2.27) and the boundary condition of p_{20} at $z = 1$, we get

$$p_{20} = p_{20}|_{z=1} + \int_1^z p_{20,z'}\, dz' = \eta_{20} - (\eta_{00} p_{10,z} + \eta_{10} p_{00,z}) + 2\Omega \int_1^z u_{20}\, dz'$$
$$= \eta_{20} - 2\Omega(\eta_{00} u_{10} + \eta_{10} u_{00}) + 2\Omega(z-1) u_{20},$$

which leads to

$$p_{20,\xi} = \eta_{20,\xi} - 2\Omega(\eta_{00} u_{10} + \eta_{10} u_{00})_\xi + 2\Omega(z-1) u_{20,\xi}. \tag{2.30}$$

On the other hand, due to the first equation in (2.27), we deduce from (2.28) and (2.29) that

$$-p_{20,\xi} = -c u_{20,\xi} + u_{10,\tau} + (u_{00} u_{10})_\xi - 2\Omega z u_{20,\xi}. \tag{2.31}$$

Combining (2.30) with (2.31), we have

$$\eta_{20,\xi} - 2\Omega(\eta_{00} u_{10} + \eta_{10} u_{00})_\xi - (c + 2\Omega) u_{20,\xi} + u_{10,\tau} + (u_{00} u_{10})_\xi = 0.$$

Thanks to (2.9), (2.24), and (2.25), we obtain

$$2(c+\Omega)\eta_{10,\tau} + 3c^2(\eta_{00}\eta_{10})_\xi - (2c + \frac{4}{3}c_1)(c + c_1)(\eta_{00}^3)_\xi = 0, \tag{2.32}$$

which leads to

$$\eta_{10,\tau} = 2c_1(\eta_{00}\eta_{10})_\xi + \frac{2c_1 + 3c}{3(c+\Omega)}(c + c_1)(\eta_{00}^3)_\xi. \tag{2.33}$$

Therefore, we have

$$u_{20,\xi} = c\eta_{20,\xi} - 2(c + c_1)(\eta_{00}\eta_{10})_\xi - \frac{2c_1 - 3\Omega}{3(c+\Omega)}(c + c_1)(\eta_{00}^3)_\xi,$$

which along with the far field conditions η_{00}, η_{10}, $\eta_{20} \to 0$ as $|\xi| \to \infty$ gives

$$u_{20} = c\eta_{20} - 2(c + c_1)\eta_{00}\eta_{10} - \frac{2c_1 - 3\Omega}{3(c+\Omega)}(c + c_1)\eta_{00}^3. \tag{2.34}$$

Thanks to (2.22) and (2.33), we deduce that

$$u_{20,\tau} = c\eta_{20,\tau} - 4(c+c_1)c_1(\eta_{00}^2\eta_{10})_\xi - \frac{8cc_1 + 4c_1^2 + \frac{21}{4}c^2}{2(c+\Omega)}(c+c_1)(\eta_{00}^4)_\xi. \tag{2.35}$$

For the order $O(\varepsilon^1\mu^1)$ terms of (2.6), we obtain from the Taylor expansion (2.15) that

$$\begin{cases} -cu_{11,\xi} + u_{01,\tau} + u_{00}u_{01,\xi} + u_{10}u_{00,\xi} + W_{00}u_{01,z} & \\ \qquad\qquad + W_{10}u_{00,z} + 2\Omega W_{11} = -p_{11,\xi} & \text{in} \quad 0 < z < 1, \\ -cW_{00,\xi} - 2\Omega u_{11} = -p_{11,z} & \text{in} \quad 0 < z < 1, \\ u_{11,\xi} + W_{11,z} = 0 & \text{in} \quad 0 < z < 1, \\ u_{11,z} - W_{00,\xi} = 0 & \text{in} \quad 0 < z < 1, \\ p_{11} = \eta_{11} - (\eta_{00}p_{01,z} + \eta_{01}p_{00,z}) & \text{on} \quad z = 1, \\ W_{11} + W_{00,z}\eta_{01} + W_{01,z}\eta_{00} & \\ \qquad = -c\eta_{11,\xi} + \eta_{01,\tau} + u_{00}\eta_{01,\xi} + u_{01}\eta_{00,\xi} & \text{on} \quad z = 1, \\ W_{11} = 0 & \text{on} \quad z = 0. \end{cases} \tag{2.36}$$

Thanks to (2.10) and the fourth equation of (2.36), we have

$$u_{11,z} = -cz\eta_{00,\xi\xi},$$

and then

$$u_{11} = -\frac{c}{2}z^2\eta_{00,\xi\xi} + \Phi_{11}(\tau,\xi) \tag{2.37}$$

for some arbitrary smooth function $\Phi_{11}(\tau,\xi)$ independent of z. While from the third equation in (2.36) with $W_{11}|_{z=0} = 0$, it follows that

$$W_{11} = W_{11}|_{z=0} + \int_0^z W_{11,z'}\,dz' = \frac{c}{6}z^3\eta_{00,\xi\xi\xi} - z\partial_\xi\Phi_{11}(\tau,\xi), \tag{2.38}$$

which, along with the equations of $W_{00,z}$ and $W_{01,z}$, and the boundary condition of W_{11} on $\{z=1\}$, implies

$$-\partial_\xi \Phi_{11}(\tau,\xi) == -\frac{c}{6}\eta_{00,\xi\xi\xi} + (u_{00}\eta_{01}+\eta_{00}u_{01})_\xi - c\eta_{11,\xi} + \eta_{01,\tau}. \tag{2.39}$$

Hence, in view of (2.38), (2.10), (2.26), and (2.9), we obtain

$$W_{11} = \frac{c}{6} z(z^2-1)\eta_{00,\xi\xi\xi} + z\Big(- c\eta_{11,\xi} + \eta_{01,\tau} + (u_{00}\eta_{01}+\eta_{00}u_{01})_\xi \Big). \tag{2.40}$$

Due to (2.10), (2.26), (2.37), and the boundary condition of p_{11} in (2.36), we deduce from the second equation of (2.36) that

$$\begin{aligned} p_{11} &= p_{11}|_{z=1} + \int_1^z p_{11,z'}\,dz' = p_{11}|_{z=1} + \int_1^z (cW_{00,\xi} + 2\Omega u_{11})\,dz' \\ &= \eta_{11} - 2\Omega(u_{00}\eta_{01}+\eta_{00}u_{01}) - \Big(\frac{c^2}{2}(z^2-1) + \frac{\Omega c}{3}(z^3-1)\Big)\eta_{00,\xi\xi} + 2\Omega(z-1)\Phi_{11}, \end{aligned}$$

which implies

$$\begin{aligned} p_{11,\xi} = \eta_{11,\xi} - 2\Omega(u_{00}\eta_{01}+\eta_{00}u_{01})_\xi - \Big(\frac{c^2}{2}(z^2-1) + \frac{\Omega c}{3}(z^3-1)\Big)\eta_{00,\xi\xi\xi} \\ + 2\Omega(z-1)\partial_\xi\Phi_{11}. \end{aligned} \tag{2.41}$$

Combining (2.41) and the first equation in (2.36), it follows from (2.10), (2.26), and (2.37) that

$$\begin{aligned} &- cu_{11,\xi} + c\eta_{01,\tau} + c^2(\eta_{00}\eta_{01})_\xi + 2\Omega W_{11} + \eta_{11,\xi} - 4\Omega c(\eta_{00}\eta_{01})_\xi \\ &- \Big(\frac{c^2}{2}(z^2-1) + \frac{\Omega c}{3}(z^3-1)\Big)\eta_{00,\xi\xi\xi} + 2\Omega(z-1)\partial_\xi\Phi_{11} = 0. \end{aligned} \tag{2.42}$$

Substituting (2.37) and (2.39) into (2.42), we obtain

$$2(\Omega+c)\eta_{01,\tau} + 3c^2(\eta_{00}\eta_{01})_\xi + \frac{c^2}{3}\eta_{00,\xi\xi\xi} = 0, \tag{2.43}$$

that is,

$$\eta_{01,\tau} = 2c_1(\eta_{00}\eta_{01})_\xi + \frac{2c_1}{9}\eta_{00,\xi\xi\xi}, \tag{2.44}$$

which, together with (2.39), (2.40), and (2.37), leads to

$$-\partial_\xi \Phi_{11}(\tau,\xi) = (\frac{2c_1}{9} - \frac{c}{6})\eta_{00,\xi\xi\xi} + 2(c+c_1)(\eta_{00}\eta_{01})_\xi - c\eta_{11,\xi},$$

and then

$$W_{11} = \left(\frac{2c_1}{9} + \frac{c}{6}(z^2-1)\right) z\,\eta_{00,\xi\xi\xi} + 2(c+c_1)\,z\,(\eta_{00}\eta_{01})_\xi - c\,z\,\eta_{11,\xi}$$

and

$$u_{11} = \left(\frac{c}{6} - \frac{2c_1}{9} - \frac{c}{2}z^2\right)\eta_{00,\xi\xi} + c\eta_{11} - 2(c+c_1)\eta_{00}\eta_{01}, \tag{2.45}$$

where use has been made by the far field conditions u_{11}, $\eta_{00,\xi\xi}$, η_{00}, η_{01}, $\eta_{11} \to 0$ as $|\xi| \to \infty$.

Thanks to (2.22) and (2.44), we obtain

$$\begin{aligned} u_{11,\tau} =& c\eta_{11,\tau} + \left(\frac{cc_1}{6} - \frac{2c_1^2}{9} - \frac{cc_1}{2}z^2\right)(\eta_{00}^2)_{\xi\xi\xi} \\ & - 2(c+c_1)\left(2c_1(\eta_{00}^2\eta_{01})_\xi + \frac{2c_1}{9}\eta_{00}\eta_{00,\xi\xi\xi}\right). \end{aligned} \tag{2.46}$$

For the order $O(\varepsilon^3\mu^0)$ terms of (2.6), we obtain from the Taylor expansion (2.15) that

$$\begin{cases} -cu_{30,\xi} + u_{20,\tau} + (u_{00}u_{20} + \frac{1}{2}u_{10}^2)_\xi + 2\Omega W_{30} = -p_{30,\xi} & \text{in} \quad 0<z<1, \\ -2\Omega u_{30} = -p_{30,z} & \text{in} \quad 0<z<1, \\ u_{30,\xi} + W_{30,z} = 0 & \text{in} \quad 0<z<1, \\ u_{30,z} = 0 & \text{in} \quad 0<z<1, \\ p_{30} + \eta_{00}p_{20,z} + \eta_{10}p_{10,z} + \eta_{20}p_{00,z} = \eta_{30} & \text{on} \quad z=1, \\ W_{30} + \eta_{00}W_{20,z} + \eta_{10}W_{10,z} + \eta_{20}W_{00,z} & \\ \qquad = -c\eta_{30,\xi} + \eta_{20,\tau} + u_{00}\eta_{20,\xi} + u_{10}\eta_{10,\xi} + u_{20}\eta_{00,\xi} & \text{on} \quad z=1, \\ W_{30} = 0 & \text{on} \quad z=0. \end{cases} \tag{2.47}$$

From the fourth equation in (2.47), we know that u_{30} is independent of z, that is,

$$u_{30} = u_{30}(\tau, \xi),$$

which along with the third equation in (2.47) and the boundary condition of W_{30} at $z = 0$ implies that

$$W_{30} = -z u_{30,\xi}.$$

Combining (2.28) with the boundary condition of W_{20} at $z = 1$, we have

$$u_{30,\xi} = c\eta_{30,\xi} - \eta_{20,\tau} - (u_{00}\eta_{20} + u_{10}\eta_{10} + u_{20}\eta_{00})_\xi. \tag{2.48}$$

While from the second equation in (2.47) and the boundary condition of p_{30} at $z = 1$, we get

$$\begin{aligned} p_{30} &= p_{30}|_{z=1} + \int_1^z p_{30,z'}\, dz' \\ &= \eta_{30} - (\eta_{00}p_{20,z} + \eta_{10}p_{10,z} + \eta_{20}p_{00,z}) + 2\Omega \int_1^z u_{30}\, dz' \\ &= \eta_{30} - 2\Omega(u_{00}\eta_{20} + u_{10}\eta_{10} + u_{20}\eta_{00}) + 2\Omega(z-1)u_{30}, \end{aligned}$$

which leads to

$$p_{30,\xi} = \eta_{30,\xi} - 2\Omega(u_{00}\eta_{20} + u_{10}\eta_{10} + u_{20}\eta_{00})_\xi + 2\Omega(z-1)u_{30,\xi}. \tag{2.49}$$

On the other hand, from the first equation in (2.47), we have

$$-p_{30,\xi} = -cu_{30,\xi} + u_{20,\tau} + (u_{00}u_{20} + \frac{1}{2}u_{10}^2)_\xi - 2\Omega z u_{30,\xi}. \tag{2.50}$$

Combining (2.49) with (2.50), we get

$$0 = \eta_{30,\xi} - 2\Omega(u_{00}\eta_{20} + u_{10}\eta_{10} + u_{20}\eta_{00})_\xi - (c + 2\Omega)u_{30,\xi} + u_{20,\tau} + (u_{00}u_{20} + \frac{1}{2}u_{10}^2)_\xi. \tag{2.51}$$

Substituting (2.48) and (2.35) into (2.51), we obtain

$$
\begin{aligned}
2(c+\Omega)\eta_{20,\tau} + 3c^2(\eta_{00}\eta_{20})_\xi + \frac{3c^2}{2}(\eta_{10}^2)_\xi - 2(2c_1+3c)(c+c_1)(\eta_{00}^2\eta_{10})_\xi \\
- \frac{(64cc_1+24c_1^2+45c^2-15)}{12(c+\Omega)}(c+c_1)(\eta_{00}^4)_\xi = 0,
\end{aligned}
\tag{2.52}
$$

that is,

$$
\begin{aligned}
\eta_{20,\tau} = 2c_1(\eta_{00}\eta_{20})_\xi + c_1(\eta_{10}^2)_\xi + \frac{2c_1+3c}{\Omega+c}(c+c_1)(\eta_{00}^2\eta_{10})_\xi \\
+ \frac{(64cc_1+24c_1^2+45c^2-15)}{24(c+\Omega)^2}(c+c_1)(\eta_{00}^4)_\xi.
\end{aligned}
\tag{2.53}
$$

Thanks to (2.48) again, we have

$$
\begin{aligned}
u_{30,\xi} =& c\eta_{30,\xi} - 2(c+c_1)(\eta_{00}\eta_{20})_\xi - (c+c_1)(\eta_{10}^2)_\xi - \frac{2c_1-3\Omega}{\Omega+c}(c+c_1)(\eta_{00}^2\eta_{10})_\xi \\
&- \frac{(64cc_1+24c_1^2+45c^2+24\Omega^2-3)}{24(c+\Omega)^2}(c+c_1)(\eta_{00}^4)_\xi,
\end{aligned}
$$

which implies

$$
\begin{aligned}
u_{30} =& c\eta_{30} - 2(c+c_1)(\eta_{00}\eta_{20}) - (c+c_1)(\eta_{10}^2) - \frac{2c_1-3\Omega}{\Omega+c}(c+c_1)(\eta_{00}^2\eta_{10}) \\
&- \frac{(64cc_1+24c_1^2+45c^2+24\Omega^2-3)}{24(c+\Omega)^2}(c+c_1)(\eta_{00}^4).
\end{aligned}
\tag{2.54}
$$

Therefore, due to (2.22), (2.33), and (2.53), we have

$$
\begin{aligned}
u_{30,\tau} =& c\eta_{30,\tau} - \frac{2(3c^2+5cc_1+4c_1^2-3\Omega c_1)}{\Omega+c}(c+c_1)(\eta_{00}^3\eta_{10})_\xi \\
&- 4c_1(c+c_1)(\eta_{00}\eta_{10}^2)_\xi - 4c_1(c+c_1)(\eta_{00}^2\eta_{20})_\xi - B_1\eta_{00}^4\eta_{00,\xi}
\end{aligned}
\tag{2.55}
$$

with

$$
\begin{aligned}
B_1 \overset{\text{def}}{=} & \frac{(c+c_1)^2(82cc_1+36c_1^2+45c^2-18\Omega c_1-27\Omega c-15)}{3(\Omega+c)^2} \\
&+ \frac{c_1(c+c_1)(64cc_1+24c_1^2+45c^2+24\Omega^2-3)}{3(\Omega+c)^2}.
\end{aligned}
$$

For the terms of (2.6) at order $O(\varepsilon^4\mu^0)$, it is inferred from the Taylor expansion (2.15) that

$$
\begin{cases}
-cu_{40,\xi} + u_{30,\tau} + (u_{00}u_{30} + u_{10}u_{20})_\xi + 2\Omega W_{40} = -p_{40,\xi} & \text{in} \quad 0 < z < 1, \\
-2\Omega u_{40} = -p_{40,z} & \text{in} \quad 0 < z < 1, \\
u_{40,\xi} + W_{40,z} = 0 & \text{in} \quad 0 < z < 1, \\
u_{40,z} = 0 & \text{in} \quad 0 < z < 1, \\
p_{40} + \eta_{00}p_{30,z} + \eta_{10}p_{20,z} + \eta_{20}p_{10,z} + \eta_{30}p_{00,z} = \eta_{40} & \text{on} \quad z = 1, \\
W_{40} + \eta_{00}W_{30,z} + \eta_{10}W_{20,z} + \eta_{20}W_{10,z} + \eta_{30}W_{00,z} & \\
\quad = -c\eta_{40,\xi} + \eta_{30,\tau} + u_{00}\eta_{30,\xi} + u_{10}\eta_{20,\xi} + u_{20}\eta_{10,\xi} + u_{30}\eta_{00,\xi} & \text{on} \quad z = 1, \\
W_{40} = 0 & \text{on} \quad z = 0.
\end{cases}
\tag{2.56}
$$

From the fourth equation in (2.47), we know that u_{40} is independent of z, that is,

$$u_{40} = u_{40}(\tau, \xi),$$

which along with the third equation in (2.56) and the boundary condition of W_{40} at $z = 0$ implies that

$$W_{40} = -zu_{40,\xi}. \tag{2.57}$$

Combining (2.57) with the boundary condition of W_{40} at $z = 1$, we have

$$u_{40,\xi} = c\eta_{40,\xi} - \eta_{30,\tau} - (u_{00}\eta_{30} + u_{10}\eta_{20} + u_{20}\eta_{10} + u_{30}\eta_{00})_\xi, \tag{2.58}$$

From the second equation in (2.56) and the boundary condition of p_{30} at $z=1$, we get

$$
\begin{aligned}
p_{40} &= p_{40}|_{z=1} + \int_1^z p_{40,z'}\,dz' \\
&= \eta_{40} - (\eta_{00}p_{30,z} + \eta_{10}p_{20,z} + \eta_{20}p_{10,z} + \eta_{30}p_{00,z}) + 2\Omega\int_1^z u_{40}\,dz' \\
&= \eta_{40} - 2\Omega(u_{00}\eta_{30} + u_{10}\eta_{20} + u_{20}\eta_{10} + u_{30}\eta_{00}) + 2\Omega(z-1)u_{40},
\end{aligned}
$$

which implies

$$
p_{40,\xi} = -\eta_{40,\xi} - 2\Omega(u_{00}\eta_{30} + u_{10}\eta_{20} + u_{20}\eta_{10} + u_{30}\eta_{00})_\xi + 2\Omega(z-1)u_{40,\xi}. \tag{2.59}
$$

On the other hand, from the first equation in (2.56), we have

$$
-p_{40,\xi} = -cu_{40,\xi} + u_{30,\tau} + (u_{00}u_{30} + u_{10}u_{20})_\xi + 2\Omega W_{40},
$$

which along with (2.57) and (2.59) gives rise to

$$
\begin{aligned}
0 = &-(c+2\Omega)u_{40,\xi} + u_{30,\tau} + (u_{00}u_{30} + u_{10}u_{20})_\xi \\
&+ \eta_{40,\xi} - 2\Omega(u_{00}\eta_{30} + u_{10}\eta_{20} + u_{20}\eta_{10} + u_{30}\eta_{00})_\xi
\end{aligned} \tag{2.60}
$$

Substituting (2.58) and (2.55) into (2.60), we obtain

$$
\begin{aligned}
&2(c+\Omega)\eta_{30,\tau} + 3c^2(\eta_{00}\eta_{30} + \eta_{10}\eta_{20})_\xi - 2(3c+2c_1)(c+c_1)(\eta_{00}^2\eta_{20} + \eta_{00}\eta_{10}^2)_\xi \\
&\quad - \frac{(64cc_1 + 24c_1^2 + 45c^2 - 15)}{3(c+\Omega)}(c+c_1)(\eta_{00}^3\eta_{10})_\xi - B_2(\eta_{00}^5)_\xi = 0
\end{aligned} \tag{2.61}
$$

with

$$
\begin{aligned}
B_2 &\stackrel{\text{def}}{=} \frac{1}{5}B_1 - \frac{(c+c_1)^2(2c_1 - 3\Omega)}{3(\Omega+c)} + \frac{2c(c+c_1)(64cc_1 + 24c_1^2 + 45c^2 + 24\Omega^2 - 3)}{12(\Omega+c)^2} \\
&= \frac{c^2(2-c^2)(3c^{10} + 228c^8 - 540c^6 - 180c^4 - 13c^2 + 42)}{60(c^2+1)^6}.
\end{aligned}
$$

For the terms in (2.6) at order $O(\varepsilon^2\mu^1)$, we have

$$\begin{cases} -cu_{21,\xi}+u_{11,\tau}+(u_{00}u_{11}+u_{10}u_{01})_\xi+W_{00}u_{11,z}+2\Omega W_{21}=-p_{21,\xi} & \text{in} \quad 0<z<1,\\ -cW_{10,\xi}+W_{00,\tau}+u_{00}W_{00,\xi}+W_{00}W_{00,z}-2\Omega u_{21}=-p_{21,z} & \text{in} \quad 0<z<1,\\ u_{21,\xi}+W_{21,z}=0 & \text{in} \quad 0<z<1,\\ u_{21,z}-W_{10,\xi}=0 & \text{in} \quad 0<z<1,\\ p_{21}+\eta_{10}p_{01,z}+\eta_{01}p_{10,z}+\eta_{00}p_{11,z}+\eta_{11}p_{00,z}=\eta_{21} & \text{on} \quad z=1,\\ W_{21}+\eta_{10}W_{01,z}+\eta_{01}W_{10,z}+\eta_{00}W_{11,z}+\eta_{11}W_{00,z} & \\ =-c\eta_{21,\xi}+\eta_{11,\tau}+u_{00}\eta_{11,\xi}+u_{11}\eta_{00,\xi}+u_{10}\eta_{01,\xi}+u_{01}\eta_{10,\xi} & \text{on} \quad z=1,\\ W_{21}=0 & \text{on} \quad z=0. \end{cases} \tag{2.62}$$

We now first derive from (2.17), (2.23), and the fourth equation in (2.62) that

$$u_{21,z}=W_{10,\xi}=z\bigg(2(c+c_1)(\eta_{00,\xi}^2+\eta_{00}\eta_{00,\xi\xi})-c\eta_{10,\xi\xi}\bigg),$$

which gives

$$u_{21}=\frac{z^2}{2}\bigg(2(c+c_1)(\eta_{00,\xi}^2+\eta_{00}\eta_{00,\xi\xi})-c\eta_{10,\xi\xi}\bigg)+\Phi_{21}(\tau,\xi)=\frac{z^2}{2}H_1+\Phi_{21}(\tau,\xi)$$

for some smooth function $\Phi_{21}(\tau,\xi)$ independent of z, where we denote

$$H_1\overset{\text{def}}{=}2(c+c_1)(\eta_{00,\xi}^2+\eta_{00}\eta_{00,\xi\xi})-c\eta_{10,\xi\xi}.$$

Hence, we have

$$u_{21,\xi}=\frac{z^2}{2}H_{1,\xi}+\partial_\xi\Phi_{21}(\tau,\xi).$$

On the other hand, thanks to the third equation in (2.62) and the boundary condition of W_{21} on $\{z=0\}$, we get

$$W_{21}=W_{21}|_{z=0}+\int_0^z W_{21,z'}\,dz'=-\int_0^z u_{21,\xi}\,dz'=-\frac{z^3}{6}H_{1,\xi}-z\partial_\xi\Phi_{21}(\tau,\xi),$$

which along with the boundary condition of W_{21} on $\{z=1\}$ leads to

$$
\begin{aligned}
-\frac{1}{6}H_{1,\xi}-\partial_\xi\Phi_{21}(\tau,\xi) &= -c\eta_{21,\xi}+\eta_{11,\tau}+(u_{00}\eta_{11}+u_{11}\eta_{00}+u_{10}\eta_{01}+u_{01}\eta_{10})_\xi|_{z=1}\\
&= -c\eta_{21,\xi}+\eta_{11,\tau}+H_{2,\xi}|_{z=1},
\end{aligned}
$$

where we denote

$$H_2 \stackrel{\text{def}}{=} u_{00}\eta_{11}+u_{11}\eta_{00}+u_{10}\eta_{01}+u_{01}\eta_{10}.$$

It then follows that

$$\partial_\xi\Phi_{21}(\tau,\xi)=c\eta_{21,\xi}-\eta_{11,\tau}-\frac{1}{6}H_{1,\xi}-H_{2,\xi}|_{z=1}, \tag{2.63}$$

which implies

$$u_{21,\xi}=c\eta_{21,\xi}-\eta_{11,\tau}+(\frac{z^2}{2}-\frac{1}{6})H_{1,\xi}-H_{2,\xi}|_{z=1} \tag{2.64}$$

and

$$W_{21}=\frac{z(1-z^2)}{6}H_{1,\xi}-cz\eta_{21,\xi}+z\eta_{11,\tau}+z(H_{2,\xi}|_{z=1}). \tag{2.65}$$

Substituting the expressions of $W_{00,\tau}$, u_{00}, $W_{00,\xi}$, W_{00}, $W_{00,z}$, and $W_{10,\xi}$ into the second equation in (2.62), we obtain

$$p_{21,z}=2\Omega u_{21}-c^2z\eta_{10,\xi\xi}+c(c+4c_1)z\eta_{00,\xi}^2+c(3c+4c_1)z\eta_{00}\eta_{00,\xi\xi}. \tag{2.66}$$

While from the boundary condition of p_{21} on $z=1$, we have

$$p_{21}|_{z=1}=\eta_{21}+c^2\eta_{00}\eta_{00,\xi\xi}-2\Omega H_2|_{z=1},$$

which along with (2.66) leads to

$$
\begin{aligned}
p_{21} &= p_{21}|_{z=1}+\int_1^z p_{21,z'}\,dz'\\
&= \eta_{21}-2\Omega H_2|_{z=1}+2\Omega\int_1^z u_{21}\,dz'-\frac{c^2}{2}(z^2-1)\eta_{10,\xi\xi}\\
&\quad+\frac{c(c+4c_1)}{2}(z^2-1)\eta_{00,\xi}^2+\left(c^2+\frac{c(3c+4c_1)}{2}(z^2-1)\right)\eta_{00}\eta_{00,\xi\xi},
\end{aligned}
\tag{2.67}
$$

and then

$$
\begin{aligned}
p_{21,\xi} &= \eta_{21,\xi} - 2\Omega H_{2,\xi}|_{z=1} + 2\Omega \int_1^z u_{21,\xi}\,dz' - \frac{c^2}{2}(z^2-1)\eta_{10,\xi\xi\xi} \\
&\quad + \frac{c(c+4c_1)}{2}(z^2-1)(\eta_{00,\xi}^2)_\xi + \left(c^2 + \frac{c(3c+4c_1)}{2}(z^2-1)\right)(\eta_{00}\eta_{00,\xi\xi})_\xi \\
&= -2\Omega z H_{2,\xi}|_{z=1} + 2\Omega(z-1)\left(c\eta_{21,\xi} - \eta_{11,\tau}\right) + \frac{z(z^2-1)}{6}H_{1,\xi} - \frac{c^2}{2}(z^2-1)\eta_{10,\xi\xi\xi} \\
&\quad + \eta_{21,\xi} + \frac{c(c+4c_1)}{2}(z^2-1)(\eta_{00,\xi}^2)_\xi + \left(c^2 + \frac{c(3c+4c_1)}{2}(z^2-1)\right)(\eta_{00}\eta_{00,\xi\xi})_\xi .
\end{aligned}
\tag{2.68}
$$

Thanks to the first equation in (2.62), (2.65), and (2.10), we get

$$
\begin{aligned}
-p_{21,\xi} =& - cu_{21,\xi} + u_{11,\tau} + (u_{00}u_{11} + u_{10}u_{01})_\xi + c^2 z^2 \eta_{00,\xi}\eta_{00,\xi\xi} \\
&+ \frac{\Omega}{3} z(1-z^2)H_{1,\xi} - 2\Omega cz\eta_{21,\xi} + 2\Omega z\eta_{11,\tau} + 2\Omega z H_{2,\xi}|_{z=1}.
\end{aligned}
\tag{2.69}
$$

Combining (2.69) with (2.67), we get

$$
\begin{aligned}
0 =& -cu_{21,\xi} + u_{11,\tau} + (u_{00}u_{11} + u_{10}u_{01})_\xi + \left(\frac{c^2}{2}z^2 + \frac{c(c+4c_1)}{2}(z^2-1)\right)(\eta_{00,\xi}^2)_\xi \\
&+ \frac{\Omega}{3}z(1-z^2)H_{1,\xi} + (1-2\Omega c)\eta_{21,\xi} + 2\Omega\eta_{11,\tau} + \frac{z(z^2-1)}{6}H_{1,\xi} - \frac{c^2}{2}(z^2-1)\eta_{10,\xi\xi\xi} \\
&+ \left(c^2 + \frac{c(3c+4c_1)}{2}(z^2-1)\right)(\eta_{00}\eta_{00,\xi\xi})_\xi .
\end{aligned}
\tag{2.70}
$$

Notice that

$$
\begin{aligned}
&(u_{01}u_{10} + u_{00}u_{11})_\xi \\
&\quad = c^2(\eta_{01}\eta_{10} + \eta_{00}\eta_{11})_\xi + \left(\frac{c^2}{6} - \frac{2cc_1}{9} - \frac{c^2z^2}{2}\right)(\eta_{00}\eta_{00,\xi\xi})_\xi - 3c(c+c_1)(\eta_{00}^2\eta_{01})_\xi
\end{aligned}
$$

and

$$
H_{2,\xi}|_{z=1} = 3c^2(\eta_{01}\eta_{10} + \eta_{00}\eta_{11})_\xi - \left(\frac{c^2}{3} + \frac{2cc_1}{9}\right)(\eta_{00}\eta_{00,\xi\xi})_\xi - 3c(c+c_1)(\eta_{00}^2\eta_{01})_\xi .
$$

We substitute (2.64) and (2.46) into (2.70) to get

$$
\begin{aligned}
&2(\Omega + c)\eta_{11,\tau} + 3c^2(\eta_{00}\eta_{11} + \eta_{10}\eta_{01})_\xi - 2(c+c_1)(3c+2c_1)(\eta_{00}^2\eta_{01})_\xi + \frac{c^2}{3}\eta_{10,\xi\xi\xi} \\
&- \left(\frac{c^2}{6} + \frac{10cc_1}{9} + \frac{2c_1^2}{9}\right)(\eta_{00,\xi}^2)_\xi - \left(\frac{c^2}{3} + \frac{20cc_1}{9} + \frac{8c_1^2}{9}\right)(\eta_{00}\eta_{00,\xi\xi})_\xi = 0.
\end{aligned}
\tag{2.71}
$$

Taking $\eta := \eta_{00} + \varepsilon\eta_{10} + \varepsilon^2\eta_{20} + \varepsilon^3\eta_{30} + \mu\eta_{01} + \varepsilon\mu\eta_{11} + O(\varepsilon^4, \mu^2)$. Multiplying the equations (2.20), (2.32), (2.43), (2.52), (2.61), and (2.71) by 1, ε, μ, ε^2, ε^3, and $\varepsilon\mu$, respectively, and then summating the results, we get the equation of η up to the order $O(\varepsilon^4, \mu^2)$ that

$$\begin{aligned} &2(\Omega + c)\eta_\tau + 3c^2\eta\eta_\xi + \frac{c^2}{3}\mu\eta_{\xi\xi\xi} + \varepsilon A_1\eta^2\eta_\xi + \varepsilon^2 A_2\eta^3\eta_\xi - 5B_0\varepsilon^3\eta^4\eta_\xi \\ &= \varepsilon\mu\bigg(A_3\eta_\xi\eta_{\xi\xi} + A_4\eta\eta_{\xi\xi\xi}\bigg) + O(\varepsilon^4, \mu^2), \end{aligned} \tag{2.72}$$

where $c_1 = -\frac{3c^3}{2(c^2+1)}$ is defined in (2.21), $A_1 \stackrel{\text{def}}{=} -2(3c + 2c_1)(c + c_1) = \frac{3c^2(c^2-2)}{(c^2+1)^2}$, $A_2 \stackrel{\text{def}}{=} -\frac{(64cc_1+24c_1^2+45c^2-15)}{3(c+\Omega)}(c + c_1) = -\frac{c^2(2-c^2)(c^6-7c^4+5c^2-5)}{(c^2+1)^4}$, $A_3 \stackrel{\text{def}}{=} \frac{2c^2}{3} + \frac{40cc_1}{9} + \frac{4c_1^2}{3} = \frac{-c^2(9c^4+16c^2-2)}{3(c^2+1)^2}$, $A_4 \stackrel{\text{def}}{=} \frac{c^2}{3} + \frac{20cc_1}{9} + \frac{8c_1^2}{9} = \frac{-c^2(3c^4+8c^2-1)}{3(c^2+1)^2}$.

On the other hand, notice that $u_{00} = c\eta_{00}$, $u_{10} = c\eta_{10} - (c_1 + c)\eta_{00}^2$, $u_{01} = c\eta_{01}$, $u_{11} = c\eta_{11} - 2(c_1 + c)\eta_{00}\eta_{01} + \left(\frac{c}{6} - \frac{2c_1}{9} - \frac{cz^2}{2}\right)\eta_{00,\xi\xi}$, $u_{20} = c\eta_{20} - 2(c + c_1)(\eta_{00}\eta_{10}) - \frac{2c_1-3\Omega}{3(c+\Omega)}(c + c_1)(\eta_{00}^3)$, and

$$\begin{aligned} u_{30} = c\eta_{30} - 2(c + c_1)(\eta_{00}\eta_{20}) - (c + c_1)(\eta_{10}^2) - \frac{2c_1 - 3\Omega}{\Omega + c}(c + c_1)(\eta_{00}^2\eta_{10}) \\ - \frac{(64cc_1 + 24c_1^2 + 45c^2 + 24\Omega^2 - 3)}{24(c + \Omega)^2}(c + c_1)(\eta_{00}^4), \end{aligned}$$

we obtain

$$\begin{aligned} &\eta_{00} = \frac{1}{c}u_{00},\ \eta_{10} = \frac{1}{c}u_{10} + \gamma_1 u_{00}^2,\ \eta_{01} = \frac{1}{c}u_{01},\ \eta_{20} = \frac{1}{c}u_{20} + 2\gamma_1 u_{00}u_{10} + \gamma_2 u_{00}^3, \\ &\eta_{30} = \frac{1}{c}u_{30} + \gamma_1 u_{10}^2 + 2\gamma_1 u_{00}u_{20} + 3\gamma_2 u_{00}^2 u_{10} + \gamma_3 u_{00}^4, \\ &\eta_{11} = \frac{1}{c}u_{11} + 2\gamma_1 u_{00}u_{01} + \gamma_4 u_{00,\xi\xi}, \end{aligned}$$

where $\gamma_1 \stackrel{\text{def}}{=} \frac{c_1+c}{c^3}$, $\gamma_2 \stackrel{\text{def}}{=} \frac{2(c+c_1)^2}{c^5} + \frac{(2c_1-3\Omega)(c+c_1)}{3c^4(c+\Omega)}$, $\gamma_3 \stackrel{\text{def}}{=} \frac{5(c+c_1)^3}{c^7} + \frac{5(2c_1-3\Omega)(c+c_1)^2}{3c^6(c+\Omega)} + \frac{(64cc_1+24c_1^2+45c^2+24\Omega^2-3)}{24c^5(c+\Omega)^2}(c + c_1)$, $\gamma_4 \stackrel{\text{def}}{=} -\left(\frac{1}{6c} - \frac{2c_1}{9c^2} - \frac{z^2}{2c}\right)$, or it is the same,

$$\begin{aligned} &\gamma_1 = -\frac{3}{2(c^2 + 1)}, \quad \gamma_2 = \frac{(c^2 - 1)(c^2 - 2)(2c^2 + 1)}{2c^3(c^2 + 1)^3}, \\ &\gamma_3 = -\frac{(c^2 - 1)^2(c^2 - 2)(21c^4 + 16c^2 + 4)}{8c^4(c^2 + 1)^5}, \quad \gamma_4 = \frac{z^2}{2c} - \frac{3c^2 + 1}{6c(c^2 + 1)}. \end{aligned} \tag{2.73}$$

Therefore, it follows that

$$
\begin{aligned}
\eta &= \eta_{00} + \varepsilon\eta_{10} + \varepsilon^2\eta_{20} + \mu\eta_{01} + \varepsilon^3\eta_{30} + \varepsilon\mu\eta_{11} + O(\varepsilon^4, \mu^2) \\
&= \frac{1}{c}u_{00} + \varepsilon\left(\frac{1}{c}u_{10} + \gamma_1 u_{00}^2\right) + \varepsilon^2\left(\frac{1}{c}u_{20} + 2\gamma_1 u_{00}u_{10} + \gamma_2 u_{00}^3\right) \\
&+ \mu\frac{1}{c}u_{01} + \varepsilon\mu\left(\frac{1}{c}u_{11} + 2\gamma_1 u_{00}u_{01} + \gamma_4 u_{00,\xi\xi}\right) \\
&+ \varepsilon^3\left(\frac{1}{c}u_{30} + \gamma_1 u_{10}^2 + 2\gamma_1 u_{00}u_{20} + 3\gamma_2 u_{00}^2 u_{10} + \gamma_3 u_{00}^4\right) + O(\varepsilon^4, \mu^2).
\end{aligned}
$$

which along with $u = u_{00} + \varepsilon u_{10} + \varepsilon^2 u_{20} + \mu u_{01} + \varepsilon^3 u_{30} + \varepsilon\mu u_{11} + O(\varepsilon^4, \mu^2)$ yields

$$
\eta = \frac{1}{c}u + \frac{c_1 + c}{c^3}\varepsilon u^2 + \gamma_2\varepsilon^2 u^3 + \gamma_3\varepsilon^3 u^4 + \gamma_4\varepsilon\mu u_{\xi\xi} + O(\varepsilon^4, \mu^2), \tag{2.74}
$$

where $c_1 = -\frac{3c^3}{2(c^2+1)}$ and γ_i $(i = 1, 2, 3, 4)$ are defined in (2.73) and the parameter $z \in [0, 1]$.

Remark 2.2.1. *From the above derivation, we know that, in the free-surface incompressible irrotational Euler equations, the relation between the free surface η and the horizontal velocity u formally obeys the equation (2.74), with or without Coriollis effect. It also illustrates that, all the classical models, such as the classical KdV equation, the BBM equation, or the (improved) Boussinesq equation, can be also formally derived from relation (2.74) in the KdV regime $\varepsilon = O(\mu)$.*

In the following steps, we will derive the equation for u from express (2.72). In view of (2.74), we have

$$
\begin{aligned}
2(\Omega + c)\eta_\tau =& \frac{2(\Omega + c)}{c}u_\tau + \frac{2(\Omega + c)(c_1 + c)}{c^3}\varepsilon(u^2)_\tau + 2(\Omega + c)\gamma_2\varepsilon^2(u^3)_\tau \\
&+ 2(\Omega + c)\gamma_3\varepsilon^3(u^4)_\tau + 2(\Omega + c)\gamma_4\varepsilon\mu u_{\tau\xi\xi} + O(\varepsilon^4, \mu^2),
\end{aligned} \tag{2.75}
$$

and

$$
\begin{aligned}
3c^2\eta\eta_\xi &= \frac{3c^2}{2}\left((\frac{1}{c}u + \frac{c_1 + c}{c^3}\varepsilon u^2 + \gamma_2\varepsilon^2 u^3 + \gamma_3\varepsilon^3 u^4)^2 + \gamma_4\varepsilon\mu u_{\xi\xi}\right)_\xi + O(\varepsilon^4, \mu^2) \\
&= \frac{3c^2}{2}\left(\frac{1}{c^2}u^2 + \frac{2(c_1 + c)}{c^4}\varepsilon u^3 + (\frac{(c_1 + c)^2}{c^6} + \frac{2}{c}\gamma_2)\varepsilon^2 u^4 + \frac{2}{c}\gamma_4\mu\varepsilon u u_{\xi\xi}\right. \\
&\qquad \left. + (\frac{2}{c}\gamma_3 + \frac{2(c_1 + c)}{c^3}\gamma_2)\varepsilon^3 u^5\right)_\xi + O(\varepsilon^4, \mu^2).
\end{aligned}
$$

Similarly, we may get

$$\frac{c^2}{3}\mu\eta_{\xi\xi\xi} = \frac{c^2}{3}\mu(\frac{1}{c}u + \frac{c_1+c}{c^3}\varepsilon u^2)_{\xi\xi\xi} + O(\varepsilon^4,\mu^2),$$

$$\varepsilon\mu\left(A_3\eta_\xi\eta_{\xi\xi} + A_4\eta\eta_{\xi\xi\xi}\right) = \varepsilon\mu\left(\frac{A_3}{c^2}u_\xi u_{\xi\xi} + \frac{A_4}{c^2}uu_{\xi\xi\xi}\right) + O(\varepsilon^4,\mu^2),$$

$$A_1\varepsilon\eta^2\eta_\xi = \frac{A_1}{3}\varepsilon\left[\frac{1}{c^3}u^3 + \frac{3(c_1+c)}{c^5}\varepsilon u^4 + (\frac{3(c_1+c)^2}{c^7} + \frac{3}{c^2}\gamma_2)\varepsilon^2u^5\right]_\xi + O(\varepsilon^4,\mu^2),$$

$$A_2\varepsilon^2\eta^3\eta_\xi = \frac{A_2}{4c^4}\varepsilon^2(u^4)_\xi + \frac{A_2(c_1+c)}{c^6}\varepsilon^3(u^5)_\xi + O(\varepsilon^4,\mu^2),$$

and

$$-5B_2\varepsilon^3\eta^4\eta_\xi = -\frac{B_2}{c^5}\varepsilon^3(u^5)_\xi + O(\varepsilon^4,\mu^2).$$

Hence, we deduce from the equation (2.72) that

$$\begin{aligned} u_\tau\ &+ \frac{2(c_1+c)}{c^2}\varepsilon uu_\tau + 3\gamma_2c\varepsilon^2u^2u_\tau + \gamma_4c\varepsilon\mu u_{\tau\xi\xi} + 4\gamma_3c\varepsilon^3u^3u_\tau + \frac{3c}{2(\Omega+c)}uu_\xi \\ &+ \frac{cA_5}{2(\Omega+c)}\varepsilon^2u^3u_\xi + \frac{cA_6}{2(\Omega+c)}\varepsilon u^2u_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi} + \frac{cA_7}{2(\Omega+c)}\varepsilon^3u^4u_\xi \\ &+ (\frac{cA_8}{2(\Omega+c)}u_\xi u_{\xi\xi} + \frac{cA_9}{2(\Omega+c)}uu_{\xi\xi\xi})\varepsilon\mu = O(\varepsilon^4,\varepsilon^2\mu,\mu^2), \end{aligned} \tag{2.76}$$

where $A_5 := \frac{6(c_1+c)^2}{c^4} + 12c\gamma_2 + \frac{4A_1(c_1+c)}{c^5}\varepsilon^2 + \frac{A_2}{c^4}$, $A_6 := \frac{9(c_1+c)}{c^2} + \frac{A_1}{c^3}$, $A_8 := 3c\gamma_4 + \frac{2(c_1+c)}{c} - \frac{A_3}{c^2}$, $A_9 := 3c\gamma_4 + \frac{2(c_1+c)}{3c} - \frac{A_4}{c^2}$, and $A_7 := 5\left[\frac{3}{2}c^2(\frac{2}{c}\gamma_3 + \frac{2(c_1+c)}{c^3}\gamma_2) + \frac{A_1}{3}(\frac{3}{c^7}(c_1+c)^2 + \frac{3}{c^2}\gamma_2) + \frac{A_2(c_1+c)}{c^6} - \frac{B_2}{c^5}\right]$.

Hence, we obtain

$$\begin{aligned} \varepsilon uu_\tau\ &= -\varepsilon u\left(\frac{2(c_1+c)}{c^2}\varepsilon uu_\tau + 3\gamma_2c\varepsilon^2u^2u_\tau + \frac{3c}{2(\Omega+c)}uu_\xi + \frac{cA_5}{2(\Omega+c)}\varepsilon^2u^3u_\xi\right. \\ &\qquad \left. + \frac{cA_6}{2(\Omega+c)}\varepsilon u^2u_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi}\right) + O(\varepsilon^4,\varepsilon^2\mu,\mu^2), \end{aligned}$$

which implies

$$\begin{aligned} \varepsilon u\left(1 + \frac{2(c_1+c)}{c^2}\varepsilon u\ + 3\gamma_2c\varepsilon^2u^2\right)u_\tau &= -\varepsilon u\left(\frac{3c}{2(\Omega+c)}uu_\xi + \frac{cA_5}{2(\Omega+c)}\varepsilon^2u^3u_\xi\right. \\ &\qquad \left. + \frac{cA_6}{2(\Omega+c)}\varepsilon u^2u_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi}\right) + O(\varepsilon^4,\varepsilon^2\mu,\mu^2). \end{aligned}$$

This follows that

$$\varepsilon u u_\tau = -\varepsilon u\Big[1-(\frac{2(c_1+c)}{c^2}\varepsilon u + 3\gamma_2 c\varepsilon^2 u^2)+(\frac{2(c_1+c)}{c^2}\varepsilon u)^2\Big]\Big[\frac{3c}{2(\Omega+c)}uu_\xi + \frac{cA_5}{2(\Omega+c)}\varepsilon^2 u^3 u_\xi + \frac{cA_6}{2(\Omega+c)}\varepsilon u^2 u_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi}\Big] + O(\varepsilon^4,\mu^2),$$

and then

$$\begin{aligned}\varepsilon u u_\tau &= -\varepsilon u\Big[\frac{3c}{2(\Omega+c)}uu_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi} + \frac{c^2A_6-6(c_1+c)}{2c(\Omega+c)}\varepsilon u^2 u_\xi \\ &\quad + \frac{c^2A_5-2A_6(c_1+c)+3c^2(\frac{4(c_1+c)^2}{c^4}-3\gamma_2 c)}{2c(\Omega+c)}\varepsilon^2 u^3 u_\xi\Big] + O(\varepsilon^4,\mu^2),\end{aligned} \tag{2.77}$$

$$\begin{aligned}\varepsilon^2 u^2 u_\tau &= -\varepsilon^2 u^2\Big[\frac{3c}{2(\Omega+c)}uu_\xi + \frac{c^2A_6-6(c_1+c)}{2c(\Omega+c)}\varepsilon u^2 u_\xi\Big] + O(\varepsilon^4,\varepsilon^2\mu,\mu^2),\\ \varepsilon^3 u^3 u_\tau &= -\frac{3c}{2(\Omega+c)}\varepsilon^3 u^4 u_\xi + O(\varepsilon^4,\mu^2), \quad \varepsilon\mu u_{\tau\xi\xi} = -\frac{3c}{2(\Omega+c)}\varepsilon\mu(uu_\xi)_{\xi\xi} + O(\varepsilon^4,\mu^2)\end{aligned} \tag{2.78}$$

Decompose $\varepsilon\mu u_{\tau\xi\xi}$ into $\varepsilon\mu(1-\nu)u_{\tau\xi\xi}+\varepsilon\mu\nu u_{\tau\xi\xi}$ for some constant ν (to be determined later), we may get from (2.78) that

$$\varepsilon\mu u_{\tau\xi\xi} = \varepsilon\mu(1-\nu)u_{\tau\xi\xi} - \frac{3c\nu}{2(\Omega+c)}\varepsilon\mu(uu_\xi)_{\xi\xi} + O(\varepsilon^4,\mu^2). \tag{2.79}$$

Substituting (2.77)-(2.79) into (2.76), we obtain that

$$\begin{aligned}u_\tau &+ c\gamma_4(1-\nu)\mu\varepsilon u_{\tau\xi\xi} + \frac{3c}{2(\Omega+c)}uu_\xi + \frac{c^2}{6(\Omega+c)}\mu u_{\xi\xi\xi} - \frac{9c^2\gamma_2}{2(\Omega+c)}\varepsilon^2 u^3 u_\xi \\ &- \frac{3c^2\gamma_4\nu}{2(\Omega+c)}\mu\varepsilon(uu_\xi)_{\xi\xi} + \frac{2(c_1+c)}{c^2}\varepsilon\Big[\frac{3c}{2(\Omega+c)}u^2u_\xi + \frac{c^2}{6(\Omega+c)}\mu u u_{\xi\xi\xi} \\ &+ \frac{c^2A_6-6(c_1+c)}{2c(\Omega+c)}\varepsilon u^3 u_\xi\Big] + \frac{cA_5}{2(\Omega+c)}\varepsilon^2 u^3 u_\xi + \frac{cA_6}{2(\Omega+c)}\varepsilon u^2 u_\xi \\ &+ \mu\varepsilon(\frac{cA_8}{2(\Omega+c)}u_\xi u_{\xi\xi} + \frac{cA_9}{2(\Omega+c)}uu_{\xi\xi\xi}) + A_{10}\varepsilon^3 u^4 u_\xi = +O(\varepsilon^4,\mu^2),\end{aligned}$$

where

$$\begin{aligned}A_{10} := \frac{cA_7}{2(\Omega+c)} &- \frac{(c_1+c)\Big(c^2A_5-2A_6(c_1+c)+3c^2(\frac{4(c_1+c)^2}{c^4}-3\gamma_2 c)\Big)}{c^3(\Omega+c)} \\ &- \frac{3\gamma_2(c^2A_6-6(c_1+c))+12c^2\gamma_3}{2(\Omega+c)},\end{aligned}$$

which implies

$$
\begin{aligned}
u_\tau\ &+\frac{3c^2}{c^2+1}uu_\xi+\frac{c^3}{3(c^2+1)}\mu u_{\xi\xi\xi}+c\gamma_4(1-\nu)\mu\varepsilon u_{\tau\xi\xi}+A_{11}\varepsilon u^2u_\xi\\
&+A_{12}\varepsilon^2u^3u_\xi+A_{10}\varepsilon^3u^4u_\xi+\mu\varepsilon\Big[A_{13}uu_{\xi\xi\xi}+A_{14}u_\xi u_{\xi\xi}\Big]=O(\varepsilon^4,\varepsilon^2\mu,\mu^2).
\end{aligned}
\tag{2.80}
$$

where $A_{11}:=\frac{c^2A_6-6(c_1+c)}{2c(\Omega+c)}=\frac{-3c(c^2-1)(c^2-2)}{2(c^2+1)^3}$, $A_{12}:=\frac{cA_5}{2(\Omega+c)}-\frac{9c^2\gamma_2}{2(\Omega+c)}-\frac{2(c_1+c)}{c^2}\frac{c^2A_6-6(c_1+c)}{2c(\Omega+c)}=\frac{(c^2-1)^2(c^2-2)(8c^2-1)}{2(c^2+1)^5}$, $A_{13}:=\frac{cA_9}{2(\Omega+c)}-\frac{3c^2\gamma_4\nu}{2(\Omega+c)}-\frac{c_1+c}{3(\Omega+c)}=\frac{3c^3\gamma_4}{(c^2+1)}(1-\nu)+\frac{c^2(3c^4+8c^2-1)}{3(c^2+1)^3}$, $A_{14}:=\frac{cA_8}{2(\Omega+c)}-\frac{9c^2\gamma_4\nu}{2(\Omega+c)}=\frac{3c^3}{(c^2+1)}\gamma_4(1-3\nu)+\frac{c^2(6c^4+19c^2+4)}{3(c^2+1)^3}$.

Consider the transformation $x=\varepsilon^{-\frac{1}{2}}\xi+c\varepsilon^{-\frac{3}{2}}\tau,\quad t=\varepsilon^{-\frac{3}{2}}\tau$, we have

$$
\frac{\partial}{\partial\xi}=\varepsilon^{-\frac{1}{2}}\partial_x,\quad \frac{\partial}{\partial\tau}=\varepsilon^{-\frac{3}{2}}(c\partial_x+\partial_t).
$$

Hence, according to this transformation, the equation (2.80) can be written as

$$
\begin{aligned}
&\partial_tu+c\partial_xu+\frac{3c^2}{c^2+1}\varepsilon uu_x+A_{11}\varepsilon^2u^2u_x+A_{12}\varepsilon^3u^3u_x+c\gamma_4(1-\nu)\mu u_{txx}\\
&+\Big(\frac{c^3}{3(c^2+1)}-c^2\gamma_4(1-\nu)\Big)\mu u_{xxx}+\mu\varepsilon\Big(A_{13}uu_{xxx}+A_{14}u_xu_{xx}\Big)=O(\varepsilon^4,\mu^2).
\end{aligned}
$$

In order to get the R-CH equation, we need

$$
\frac{2c^2}{(c^2+1)}c\gamma_4(1-\nu)=2A_{13}=A_{14},
$$

which yields

$$
\frac{2c^3}{(c^2+1)}\gamma_4=\frac{-c^2(3c^4+6c^2-5)}{6(c^2+1)^3}
\tag{2.81}
$$

and then

$$
\frac{2c^2}{(c^2+1)}c\gamma_4(1-\nu)=2A_{13}=A_{14}=\frac{-c^2(3c^4+8c^2-1)}{3(c^2+1)^3}.
$$

Therefore, it enables us to derive the R-CH equation in the form

$$
\begin{aligned}
\partial_tu-\beta\mu\partial_tu_{xx}+cu_x+3\alpha\varepsilon uu_x-\beta_0\mu u_{xxx}+\omega_1\varepsilon^2u^2u_x+\omega_2\varepsilon^3u^3u_x&\\
=\alpha\beta\varepsilon\mu(2u_xu_{xx}+uu_{xxx}).&
\end{aligned}
$$

Combining (2.81) and (2.73), it is found that the height parameter z in γ_4 may take the value

$$z_0 = \frac{1}{2} - \frac{2}{3}\frac{1}{(c^2+1)} + \frac{4}{3}\frac{1}{(c^2+1)^2}. \tag{2.82}$$

2.3 Some other forms of R-CH equation

In this section, I state some more forms of R-CH equation, which will help us in later analysis.

2.3.1 Weak forms of R-CH equation

Recall rotation-Camassa-Holm equation as follow,

$$\begin{aligned} u_t - \beta\mu u_{xxt} + cu_x + 3\alpha\varepsilon u u_x - \beta'\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x \\ = \alpha\beta\varepsilon\mu(2u_x u_{xx} + u u_{xxx}). \end{aligned}$$

We may rewrite it as follow,

$$\begin{aligned} u_t - \beta\mu u_{xxt} + \alpha\varepsilon u u_x - \alpha\beta\varepsilon\mu(3u_x u_{xx} + u u_{xxx}) + \frac{\beta'}{\beta}u_x - \beta'\mu u_{xxx} \\ + cu_x - \frac{\beta'}{\beta}u_x + 2\alpha\varepsilon u u_x + \alpha\beta\varepsilon\mu u_x u_{xx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x. \end{aligned}$$

Since $(uu_x)_{xx} = 3u_x u_{xx} + u u_{xxx}$, we get

$$\begin{aligned} (1-\beta\mu\partial_x^2)u_t + (1-\beta\mu\partial_x^2)\alpha\varepsilon u u_x + (1-\beta\mu\partial_x^2)\frac{\beta'}{\beta}u_x \\ + \left(c - \frac{\beta'}{\beta}\right)u_x + 2\alpha\varepsilon u u_x + \alpha\beta\varepsilon\mu u_x u_{xx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x = 0. \end{aligned}$$

Denote $p_\mu(x) \overset{\text{def}}{=} \frac{1}{2\sqrt{\beta\mu}}e^{-\frac{|x|}{\sqrt{\beta\mu}}}$, $x \in \mathbb{R}$, then $(1-\beta\mu\partial_x^2)^{-1}f = p_\mu * f$ for all $f \in L^2(\mathbb{R})$ and $p_\mu * (u - \beta\mu u_{xx}) = u$, where $*$ denotes convolution with respect to the spatial variable

x. With this notation, Therefore, equation (2.1) can also be equivalently rewritten as the following nonlocal form:

$$
\begin{aligned}
&u_t + \alpha\varepsilon u u_x + \frac{\beta'}{\beta} u_x \\
&\quad + p_\mu * \partial_x \left\{ \left(c - \frac{\beta'}{\beta} \right) u + \alpha\varepsilon \left(u^2 + \frac{1}{2}\beta\mu u_x^2 \right) + \frac{1}{3}\omega_1 \varepsilon^2 u^3 + \frac{1}{4}\omega_2 \varepsilon^3 u^4 \right\} = 0. \quad (2.83)
\end{aligned}
$$

or what is the same,

$$
\begin{cases}
u_t + \frac{\beta_0}{\beta} u_x + \alpha\varepsilon u u_x + \partial_x P = 0, \\
(1 - \beta\mu\partial_x^2) P = (c - \frac{\beta_0}{\beta}) u + \alpha\varepsilon u^2 + \frac{1}{2}\alpha\beta\varepsilon\mu u_x^2 + \frac{\omega_1}{3}\varepsilon^2 u^3 + \frac{\omega_2}{4}\varepsilon^3 u^4.
\end{cases}
$$

2.3.2 A simplified form of R-CH equation

To get some convenience in mathematical analysis, We eliminate ε and μ in R-CH equation (2.1) by a transformation

$$
u'(t', x') = \alpha\varepsilon u(\sqrt{\beta\mu}\, t, \sqrt{\beta\mu}\, x). \quad (2.84)
$$

In fact, choosing point $(\sqrt{\beta\mu}\, t, \sqrt{\beta\mu}\, x)$ in (2.1), this transformation gives us that

$$
\begin{aligned}
&\frac{1}{\alpha\varepsilon\sqrt{\beta\mu}} u'_{t'}(t', x') + \frac{c}{\alpha\varepsilon\sqrt{\beta\mu}} u'_{x'}(t', x') + \frac{3\alpha\varepsilon}{\alpha^2\varepsilon^2\sqrt{\beta\mu}} u'(t', x') u'_{x'}(t', x') - \frac{\beta'\mu}{\alpha\varepsilon\sqrt{\beta\mu}^3} u'_{x'x'x'}(t', x') \\
&- \frac{\beta\mu}{\alpha\varepsilon\sqrt{\beta\mu}^3} u'_{x'x't'}(t', x') + \frac{\omega_1\varepsilon^2}{\alpha^3\varepsilon^3\sqrt{\beta\mu}} u'^2(t', x') u'_{x'}(t', x') + \frac{\omega_2\varepsilon^3}{\alpha^4\varepsilon^4\sqrt{\beta\mu}} u'^3(t', x') u'_{x'}(t', x') \\
&\qquad = \frac{\alpha\beta\varepsilon\mu}{\alpha^2\varepsilon^2\sqrt{\beta\mu}^2} (2u'_{x'}(t', x') u'_{x'x'}(t', x') + u'(t', x') u'_{x'x'x'}(t', x'))
\end{aligned}
$$

After some simplification, we have

$$
u'_{t'} + c u'_{x'} + 3u' u'_{x'} - \frac{\beta'}{\beta} u'_{x'x'x'} - u'_{x'x't'} + \frac{\omega_1}{\alpha^2} u'^2 u'_{x'} + \frac{\omega_2}{\alpha^3} u'^3 u'_{x'} = 2u'_{x'} u'_{x'x'} + u' u'_{x'x'x'},
$$

where we take equation at point (t', x'). For convenience, we still use $u(t, x)$ instead of $u'(t', x')$, then we obtain a simplified R-CH, that is,

$$
u_t + c u_x + 3u u_x - \frac{\beta'}{\beta} u_{xxx} - u_{xxt} + \frac{\omega_1}{\alpha^2} u^2 u_x + \frac{\omega_2}{\alpha} u^3 u_x = 2u_x u_{xx} + u u_{xxx}. \quad (2.85)
$$

2.3.3 Some more discussions

In the case that the Coriolis effect vanishes ($\Omega = 0$), the coefficients in the higher-power nonlinearities $\omega_1 = 0$ and $\omega_2 = 0$. Using the scaling transformation $u(t,x) \to \alpha\varepsilon u(\sqrt{\beta\mu}\, t, \sqrt{\beta\mu}\, x)$ and then the Galilean transformation $u(t,x) \to u(t, x - \frac{3}{4}t) + \frac{1}{4}$, the R-CH equation (2.3) is then reduced to the classical CH equation

$$u - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

On the other hand, if we take formally $\beta = 0$ and $\omega_2 = 0$ in (2.3), then we get the following integrable Gardner equation [22]

$$u_t + cu_x + 3\alpha\varepsilon uu_x - \beta_0\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x = 0.$$

CHAPTER 3

LOCAL WELL POSEDNESS OF R-CH EQUATIONS

3.1 Introduction

Our attention in this chapter is now turned to the local-posedness issue for the R-CH equation. Recall the R-CH equation (2.1) in terms of the evolution of m, namely, the equation (2.3). Applying the transformation $u_{\varepsilon,\mu}(t,x) = \alpha\varepsilon u(\sqrt{\beta\mu}\,t, \sqrt{\beta\mu}\,x)$ to (2.3), we know that $u_{\varepsilon,\mu}(t,x)$ solves

$$u_t - u_{xxt} + cu_x + 3uu_x - \frac{\beta_0}{\beta}u_{xxx} + \frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x = 2u_xu_{xx} + uu_{xxx},$$

and its corresponding three conserved quantities (still denoted by $I(u)$, $E(u)$, and $F(u)$) are as follows

$$I(u) = \int_{\mathbb{R}} u\,dx, \quad E(u) = \frac{1}{2}\int_{\mathbb{R}} u^2 + u_x^2\,dx,$$

and

$$F(u) = \frac{1}{2}\int_{\mathbb{R}} cu^2 + u^3 + \frac{\beta_0}{\beta}u_x^2 + \frac{\omega_1}{6\alpha^2}u^4 + \frac{\omega_2}{10\alpha^3}u^5 + uu_x^2\,dx.$$

And we also have two more forms of equations,

$$\begin{cases} m_t + um_x + 2u_xm + cu_x - \frac{\beta_0}{\beta}u_{xxx} + \frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x = 0, \\ m = u - u_{xx}, \end{cases}$$

and

$$u_t + uu_x + \frac{\beta_0}{\beta}u_x + p * \partial_x\left\{\left(c - \frac{\beta_0}{\beta}\right)u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right\} = 0. \quad (3.1)$$

where $p = \frac{1}{2}e^{-|x|}$.

Define that

$$B_1 \stackrel{\text{def}}{=} \partial_x(1-\beta\mu\partial_x^2), \qquad \text{and}$$

$$B_2 \stackrel{\text{def}}{=} \partial_x((\alpha\varepsilon m+\frac{c}{2})\cdot)+(\alpha\varepsilon m+\frac{c}{2})\partial_x-\beta_0\mu\partial_x^3+\frac{2}{3}\omega_1\varepsilon^2\partial_x(u\partial_x^{-1}(u\partial_x\cdot)) + \frac{5}{8}\omega_2\varepsilon^3\partial_x(u^{\frac{3}{2}}\partial_x^{-1}(u^{\frac{3}{2}}\partial_x\cdot)).$$

A simple calculation then reveals that the R-CH equation (2.1) has the formal bi-Hamiltonian structure, that is,

$$m_t = -B_1\frac{\delta F}{\delta m} = -B_2\frac{\delta E}{\delta m},$$

where B_1 and B_2 are two Hamiltonian operators, which provides the recursion operator $\mathrm{R} = B_2 \circ B_1^{-1}$.

The class of evolution equations (2.1) are all formally models for small amplitude, long waves on the surface of water over a flat bottom. It is our expectation that these equations approximate solutions of the full water-wave problem with the Coriolis effect for an ideal fluid with an error that is of order $O(\mu^2 t)$ over a CH time scale at least of order $O(\varepsilon^{-1})$. Rigorous theory to this effect is available in [9, 23] (see also [13] for the case without the Coriolis effect).

It is also found that the consideration of the Coriolis effect gives rise to a higher power nonlinear term into the R-CH model, which has interesting implications for the fluid motion, particular in the relation to the wave breaking phenomena and the permanent waves. On the other hand, it is also our goal in the present paper to investigate from this model how the Coriolis forcing due to the Earth rotation with the higher power nonlinearities affects the wave breaking phenomena and what conditions can ensure the occurrence of the wave-breaking phenomena or permanent waves.

The dynamics of the blow-up quantity along the characteristics in the R-CH equation actually involves the interaction among three parts: a local nonlinearity, a nonlocal term, and a term stemming from the weak Coriolis forcing. It is observed that the nonlocal (smoothing) effect can help maintain the regularity while waves propagate and hence prevent them from blowing up, even when dispersion is weak or absent. See, for example, the Benjamin-Bona-Mahoney (BBM) equation [3]. As the local nonlinearity becomes stronger and dominates over the dispersion and nonlocal effects singularities may occur in the sense of *wave-breaking*. Examples can be found in the Whitham equation [12, 38], Camassa-Holm (CH) equation [8, 13, 20]. It is also found that the Coriolis effect will spread out waves and make them decay in time, delaying the onset of wave-breaking. Understanding the wave-breaking mechanism such as when a singularity can form and what the nature of it is not only presents fundamental importance from mathematical point of view but also is of great physical interest, since it would help provide a key-mechanism for localizing energy in conservative systems by forming one or several small-scale spots. For instance, in fluid dynamics, the possible phenomenon of finite time breakdown for the incompressible Euler equations signifies the onset of turbulence in high Reynolds number flows.

The R-CH equation with a nonlocal structure can be reformulated in a weak form of nonlinear nonlocal transport type. From the transport theory, the blow-up criteria assert that singularities are caused by the focusing of characteristics, which involve the information on the gradient u_x. The dynamics of the wave-breaking quantity along the characteristics is established by the Riccati-type differential inequality. The argument is then approached by a refined analysis on evolution of the solution u and its gradient u_x. Recently Brandolese and Cortez [5] introduced a new type of blow-up criteria in the study of the classical CH equation. It is shown how local structure of the solution affects the blow-ups. Their argument relies heavily on the fact that the convolution terms are quadratic and positively definite. As for the R-CH equation, the convolution contains cubic even quartic nonlinearities which do not have a lower bound in terms of the local terms. Hence the higher-power nonlinearities in the equation makes it difficult to obtain a purely local condition on the initial data can generate finite-time wave-breaking. In our case, the blow-up can be deduced by the interplay between u and u_x. More precisely, this motivates us to carry out a refined analysis of the characteristic dynamics of $M = u - u_x + c_1$ and $N = u + u_x + c_2$. The estimates of M and N can be closed in the form of

$$M'(t) \geq -cMN + \mathcal{N}_1, \quad N'(t) \leq cMN + \mathcal{N}_2,$$

where the nonlocal terms $\mathcal{N}_i$ $(i = 1, 2)$ can be bounded in terms of certain order conservation laws. From these Riccati-type differential inequalities the monotonicity of M and N can be established, and hence the finite-time wave-breaking follows.

The present contribution proceeds in the following. In the next section, the R-CH model equation is formally derived from the incompressible and irrotational full water wave equations with the Coriolis effect considered, which is an asymptotic model in the CH regime to the f-plane geophysical governing equations in the equatorial region. Sections 3.5.1 is devoted to the local well-posedness and blow-up criteria. In the last section, Section 3.6, the wave-breaking criteria are established in Theorem 3.6.1 and the breakdown mechanisms are set up in Theorem 3.6.2.

Notation. In the sequel, we denote by $*$ the convolution. For $1 \leq p < \infty$, the norms in the Lebesgue space $L^p(\mathbb{R})$ is $\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}}$, the space $L^\infty(\mathbb{R})$ consists of all essentially bounded, Lebesgue measurable functions f equipped with the norm $\|f\|_\infty = \inf\limits_{\mu(e)=0} \sup\limits_{x \in \mathbb{R} \setminus e} |f(x)|$. For a function f in the classical Sobolev spaces $H^s(\mathbb{R})$ $(s \geq$

0) the norm is denoted by $\|f\|_{H^s}$. We denote $p(x) = \frac{1}{2}e^{-|x|}$ the fundamental solution of $1 - \partial_x^2$ on $\mathbb{R}$, and define the two convolution operators $p_+,\ p_-$ as

$$p_+ * f(x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^y f(y)dy$$
$$p_- * f(x) = \frac{e^{x}}{2} \int_{x}^{\infty} e^{-y} f(y)dy.$$

Then we have the relations $p = p_+ + p_-, \quad p_x = p_- - p_+$.

3.2 First three conservation laws

The first conservation law of R-CH is

$$I(u) = \int_{\mathbb{R}} u \, \mathrm{d}x. \tag{3.2}$$

In fact, we rewrite (2.83) as

$$\frac{\partial}{\partial t}u = -\frac{\partial}{\partial x}\left\{\frac{1}{2}\alpha\varepsilon u^2 + \frac{\beta'}{\beta}u \right.$$
$$\left. + p_\mu * \left[\left(c - \frac{\beta'}{\beta}\right)u + \alpha\varepsilon(u^2 + \frac{1}{2}\beta\mu u_x^2) + \frac{1}{3}\omega_1\varepsilon^2u^3 + \frac{1}{4}\omega_2\varepsilon^3u^4\right]\right\}.$$

Because we assume that for any $n \in \mathbb{N}$, $|\partial_x^n u| \to 0$ as $|x| \to \infty$, integrating this equation on $\mathbb{R}$, we find that

$$\int_{\mathbb{R}} u \, \mathrm{d}x$$
$$= -\int_{\mathbb{R}} \frac{\partial}{\partial x}\left\{\frac{1}{2}\alpha\varepsilon u^2 + \frac{\beta'}{\beta}u + p_\mu * \left[\left(c - \frac{\beta'}{\beta}\right)u + \alpha\varepsilon(u^2 + \frac{1}{2}\beta\mu u_x^2) + \frac{1}{3}\omega_1\varepsilon^2u^3 + \frac{1}{4}\omega_2\varepsilon^3u^4\right]\right\} \mathrm{d}x$$
$$= 0,$$

which means $I(u)$ is a conservation law.

The second conservation law of R-CH is

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} u^2 + \beta\mu u_x^2 \, dx. \tag{3.3}$$

To show $E(u)$ is a conservation law, first, we multiply u on (2.1), we obtain

$$u u_t - \beta\mu u u_{xxt} = -\{c u u_x + 3\alpha\varepsilon u^2 u_x - \beta'\mu u u_{xxx} + \omega_1\varepsilon^2 u^3 u_x + \omega_2\varepsilon^3 u^4 u_x - \alpha\beta\varepsilon\mu(2u u_x u_{xx} + u^2 u_{xxx})\}$$

Then we integrate the equation on $\mathbb{R}$, we get

$$\int_{\mathbb{R}} u u_t - \beta\mu u u_{xxt}\,\mathrm{d}x = -\int_{\mathbb{R}} c u u_x + 3\alpha\varepsilon u^2 u_x - \beta'\mu u u_{xxx} + \omega_1\varepsilon^2 u^3 u_x + \omega_2\varepsilon^3 u^4 u_x - \alpha\beta\varepsilon\mu(2u u_x u_{xx} + u^2 u_{xxx})\,\mathrm{d}x.$$

Since $u u_{xxx} = \partial_x\left(u u_{xx} - \frac{1}{2}u_x^2\right)$ and $2u u_x u_{xx} + u^2 u_{xxx} = \partial_x(u^2 u_{xx})$, we have

$$\begin{aligned}&\int_{\mathbb{R}} c u u_x + 3\alpha\varepsilon u^2 u_x - \beta'\mu u u_{xxx} + \omega_1\varepsilon^2 u^3 u_x + \omega_2\varepsilon^3 u^4 u_x - \alpha\beta\varepsilon\mu(2u u_x u_{xx} + u^2 u_{xxx})\,\mathrm{d}x\\ &= \int_{\mathbb{R}} \frac{\partial}{\partial x}\left\{\frac{1}{2}c u^2 + \alpha\varepsilon u^3 - \beta'\mu\left(u u_{xx} - \frac{1}{2}u_x^2\right) + \frac{1}{4}\varepsilon^2 u^4 + \frac{1}{5}\varepsilon^4 u^5 - \alpha\beta\varepsilon\mu u^2 u_{xx}\right\}\mathrm{d}x\end{aligned}$$

As we assume that for any $n \in \mathbb{N}$, $|\partial_x^n u| \to 0$ as $|x| \to \infty$, we get

$$\int_{\mathbb{R}} \frac{\partial}{\partial x}\left\{\frac{1}{2}c u^2 + \alpha\varepsilon u^3 - \beta'\mu\left(u u_{xx} - \frac{1}{2}u_x^2\right) + \frac{1}{4}\varepsilon^2 u^4 + \frac{1}{5}\varepsilon^4 u^5 - \alpha\beta\varepsilon\mu u^2 u_{xx}\right\}\mathrm{d}x = 0.$$

So with the integration by parts, we obtain

$$\begin{aligned}&\int_{\mathbb{R}} u u_t - \beta\mu u u_{xxt}\,\mathrm{d}x\\ =&\int_{\mathbb{R}} u u_t + \beta\mu u_x u_{xt}\mathrm{d}x\\ =&\frac{\partial}{\partial t}\left\{\frac{1}{2}\int_{\mathbb{R}} u^2 + \beta\mu u_x^2\,\mathrm{d}x\right\}\\ =&\frac{\partial}{\partial t}E(u)\\ =&0.\end{aligned}$$

Therefore, we know $E(u)$ is a conservation law.

The third conservation law is

$$F(u) = \frac{1}{2}\int_{\mathbb{R}} cu^2 + \alpha\varepsilon u^3 + \beta'\mu u_x^2 + \frac{1}{6}\omega_1\varepsilon^2 u^4 + \frac{1}{10}\omega_2\varepsilon^3 u^5 + \alpha\beta\varepsilon\mu u u_x^2 \,\mathrm{d}x \tag{3.4}$$

To show $F(u)$ is a conservation law, we rewrite (2.1) as

$$\begin{aligned}(1-\beta\mu\partial_x^2)u_t + cu_x + 3\alpha\varepsilon u u_x - \beta'\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x \\ + \omega_2\varepsilon^3 u^3 u_x - \alpha\beta\varepsilon\mu(2u_x u_{xx} + u u_{xxx}) = 0.\end{aligned}$$

Since $2u_x u_{xx} + u u_{xxx} = \left(\frac{1}{2}u_x^2 + u u_{xxx}\right)_x$, we have

$$\begin{aligned}(1-\beta\mu\partial_x^2)u_t + \partial_x \Bigg\{ cu + \frac{3}{2}\alpha\varepsilon u^2 - \beta'\mu u_{xx} + \frac{1}{3}\omega_1\varepsilon^2 u^3 \\ + \frac{1}{4}\omega_2\varepsilon^3 u^4 - \alpha\beta\varepsilon\mu\left(\frac{1}{2}u_x^2 + u u_{xx}\right)\Bigg\} = 0,\end{aligned}$$

or

$$u_t + p_\mu * \partial_x \left\{ cu + \frac{3}{2}\alpha\varepsilon u^2 - \beta'\mu u_{xx} + \frac{1}{3}\omega_1\varepsilon^2 u^3 + \frac{1}{4}\omega_2\varepsilon^3 u^4 - \alpha\beta\varepsilon\mu\left(\frac{1}{2}u_x^2 + u u_{xx}\right)\right\} = 0.$$

For convenience, we define

$$F_1 = cu + \frac{3}{2}\alpha\varepsilon u^2 - \beta'\mu u_{xx} + \frac{1}{3}\omega_1\varepsilon^2 u^3 + \frac{1}{4}\omega_2\varepsilon^3 u^4 - \alpha\beta\varepsilon\mu\left(\frac{1}{2}u_x^2 + u u_{xx}\right),$$

thus we have $u_t = -p_\mu * \partial_x F_1$.

In fact, ∂_x is a skew-symmetric operator but p_μ is a symmetric operator, we have

$$\langle F_1, p_\mu * \partial_x F_1\rangle = 0,$$

which implies $\langle F_1, u_t\rangle = 0$, that is,

$$\begin{aligned}\int_{\mathbb{R}} cuu_t + \frac{3}{2}\alpha\varepsilon u^2 u_t - \beta'\mu u_{xx} u_t + \frac{1}{3}\omega_1\varepsilon^2 u^3 u_t \\ + \frac{1}{4}\omega_2\varepsilon^3 u^4 u_t - \alpha\beta\varepsilon\mu\left(\frac{1}{2}u_x^2 u_t + u u_{xx} u_t\right)\mathrm{d}x = 0.\end{aligned} \tag{3.5}$$

By the integration by parts, we have

$$-\int_{\mathbb{R}} \beta' \mu u_{xx} u_t \,\mathrm{d}x = \int_{\mathbb{R}} \beta' \mu u_x u_{xt} \,\mathrm{d}x = \frac{\partial}{\partial t}\frac{1}{2}\int_{\mathbb{R}} \beta' \mu u_x^2 \,\mathrm{d}x,$$

and

$$\begin{aligned}
&-\alpha\beta\varepsilon\mu \int_{\mathbb{R}} \frac{1}{2} u_x^2 u_t + u u_{xx} u_t \,\mathrm{d}x \\
=&-\alpha\beta\varepsilon\mu \left(-\int_{\mathbb{R}} u_x^2 u_t + u u_{xx} u_t \,\mathrm{d}x + \int_{\mathbb{R}} \frac{1}{2} u_x^2 u_t \,\mathrm{d}x \right) \\
=&\frac{1}{2}\alpha\beta\varepsilon\mu \int_{\mathbb{R}} -(u^2)_{xx} u_t + u_x^2 u_t \,\mathrm{d}x \\
=&\frac{1}{2}\alpha\beta\varepsilon\mu \int_{\mathbb{R}} (u^2)_x u_{xt} + u_x^2 u_t \,\mathrm{d}x \\
=&\alpha\beta\varepsilon\mu \int_{\mathbb{R}} u u_x u_{xt} + \frac{1}{2} u_x^2 u_t \,\mathrm{d}x \\
=&\frac{1}{2}\alpha\beta\varepsilon\mu \frac{\partial}{\partial t} \int_{\mathbb{R}} u u_x^2 \,\mathrm{d}x.
\end{aligned}$$

Therefore, from (3.5), we have

$$\frac{\partial}{\partial t}\frac{1}{2}\int_{\mathbb{R}} c u^2 + \alpha\varepsilon u^3 + \beta' \mu u_x^2 + \frac{1}{6}\omega_1 \varepsilon^2 u^4 + \frac{1}{10}\omega_2 \varepsilon^3 u^5 + \alpha\beta\varepsilon\mu u u_x^2 \,\mathrm{d}x = 0,$$

which means $F(u)$ is a conservation law.

3.3 Hamiltonian format

Here I show that $E(u)$ and $F(u)$ satisfy Hamiltonian format, that is,

$$\frac{\partial}{\partial t}\frac{\delta E}{\delta u} = -\frac{\partial}{\partial x}\frac{\delta F}{\delta u}. \tag{3.6}$$

In fact, directly computation shows that

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\delta E}{\delta u} &= \frac{\partial}{\partial t}\frac{\delta}{\delta u}\left(\frac{1}{2}\int_{\mathbb{R}} u^2 + \beta\mu u_x^2 \mathrm{d}x \right) \\
&= \frac{\partial}{\partial t}\frac{1}{2}(2u - 2\beta\mu u_{xx}) \\
&= \frac{\partial}{\partial t}(u - \beta\mu u_{xx}) \\
&= u_t - \beta\mu u_{xxt},
\end{aligned}$$

and

$$
\begin{aligned}
&-\frac{\partial}{\partial x}\frac{\delta E}{\delta u}\\
&=-\frac{\partial}{\partial x}\frac{\delta}{\delta u}\left(\frac{1}{2}\int_{\mathbb{R}} cu^2+\alpha\varepsilon u^3+\beta'\mu u_x^2+\frac{1}{6}\omega_1\varepsilon^2u^4+\frac{1}{10}\omega_2\varepsilon^3u^5+\alpha\beta\varepsilon\mu u u_x^2\,\mathrm{d}x\right)\\
&=-\frac{\partial}{\partial x}\frac{1}{2}\left\{2cu+3\alpha\varepsilon u^2-2\beta'\mu u_{xx}+\frac{2}{3}\omega_1\varepsilon^2u^3+\frac{1}{2}\omega_2\varepsilon^3u^4+\alpha\beta\varepsilon\mu\left[u_x^2-2(uu_x)_x\right]\right\}\\
&=-\frac{\partial}{\partial x}\left\{cu+\frac{3}{2}\alpha\varepsilon u^2-\beta'\mu u_{xx}+\frac{1}{3}\omega_1\varepsilon^2u^3+\frac{1}{4}\omega_2\varepsilon^3u^4-\alpha\beta\varepsilon\mu\left(\frac{1}{2}u_x^2+uu_{xx}\right)\right\}\\
&=-cu_x-3\alpha\varepsilon uu_x+\beta'\mu u_{xxx}-\omega_1\varepsilon^2u^2u_x-\omega_2\varepsilon^3u^3u_x+\alpha\beta\varepsilon\mu(2u_xu_{xx}+uu_{xxx}).
\end{aligned}
$$

Therefore (2.1) implies

$$
\frac{\partial}{\partial t}\frac{\delta E}{\delta \hat{u}}=-\frac{\partial}{\partial x}\frac{\delta F}{\delta \hat{u}}.
$$

3.4 Preliminaries

For convenience, we recall some useful properties of Sobolev space.

Lemma 3.4.1 (Commutator estimate)**.** *For all $s>3/2$ and some $C>0$, if f and g are smooth enough, then*

$$
\|[\Lambda^s,f]g\|_{L^2}\le C\left(\|f\|_{H^s}\|g\|_{L^\infty}+\|\partial_x f\|_{L^\infty}\|g\|_{H^{s-1}}\right), \tag{3.7}
$$

Lemma 3.4.2. *For all $s>3/2$ and some $C>0$, if f and g are smooth enough, then*

$$
\left|\left\langle\Lambda^s\left(f\partial_x g\right),\Lambda^s g\right\rangle\right|\le C\|f\|_{H^s}\|g\|_{H^s}^2. \tag{3.8}
$$

Proof. In fact, for all constant coefficient skew-symmetric differential polynomial P, and f,g smooth enough, a commutator process gives

$$
\Lambda^s\left(fPg\right)=fP\Lambda^s g+[\Lambda^s,f]Pg. \tag{3.9}
$$

Taking $P=\partial_x$ and integration by parts we have

$$
\begin{aligned}
\left\langle\Lambda^s\left(f\partial_x g\right),\Lambda^s g\right\rangle&=\int_{\mathbb{R}} f\partial_x\Lambda^s g\cdot\Lambda^s g\,dx+\int_{\mathbb{R}}[\Lambda^s,f]\partial_x g\cdot\Lambda^s g\,dx\\
&=-\frac{1}{2}\int_{\mathbb{R}} f_x\left(\Lambda^s g\right)^2\,dx+\int_{\mathbb{R}}[\Lambda^s,f]\partial_x g\cdot\Lambda^s g\,dx.
\end{aligned}
$$

On one hand,

$$\left|\frac{1}{2}\int_{\mathbb{R}} f_x\left(\Lambda^s g\right)^2\, dx\right| \le \frac{1}{2}\|f_x\|_{L^\infty}\|\Lambda^s g\|_{L^2}^2, \\ = \frac{1}{2}\|f_x\|_{L^\infty}\|g\|_{H^s}^2 \tag{3.10}$$

On the other hand, by Lemma 3.4.1, we infer that

$$\begin{aligned}|\langle[\Lambda^s, f]\partial_x g, \Lambda^s g\rangle| &\le \|\Lambda^s g\|_{L^2}\|[\Lambda^s, f]\partial_x g\|_{L^2} \\ &\le C\|g\|_{H^s}\left(\|f\|_{H^s}\|\partial_x g\|_{L^\infty} + \|f_x\|_{L^\infty}\|\partial_x g\|_{H^{s-1}}\right), \\ &\le C\|g\|_{H^s}\left(\|f\|_{H^s}\|\partial_x g\|_{L^\infty} + \|f_x\|_{L^\infty}\|g\|_{H^s}\right)\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we obtain that

$$|\left\langle\Lambda^s\left(f\partial_x g\right), \Lambda^s g\right\rangle| \le C(\|f_x\|_{L^\infty}\|g\|_{H^s}^2 + \|f\|_{H^s}\|g\|_{H^s}\|\partial_x g\|_{L^\infty}). \tag{3.12}$$

Thanks to the Sobolev embedding theorem, $\forall s > 1/2$, $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we know

$$\|\partial_x g\| \le C\|u\|_{H^s}, s > \frac{3}{2}.$$

Therefore, (3.12) implies that

$$|\left\langle\Lambda^s\left(f\partial_x g\right), \Lambda^s g\right\rangle| \le C\|f\|_{H^s}\|g\|_{H^s}^2.$$

□

Lemma 3.4.3. *Let $\phi(t), \psi(t)$ be two positive function on $[0, T]$, for a non-negative, absolutely continuous function $\eta(\cdot)$ on $[0, T]$, if the differential inequality*

$$\frac{d}{dt}(\eta^2)(t) \le \phi(t)\eta^2 + \psi(t)\eta,$$

holds for a.e. t, then

$$\eta(t) \le e^{\frac{1}{2}\int_0^t \phi(\tau)d\tau}\left[\eta(0) + \frac{1}{2}\int_0^t \psi(\tau)d\tau\right]. \tag{3.13}$$

3.5 Local well-posedness

Now we are in a position to state the local well-posedness result of the following Cauchy problem, which may be similarly obtained as in [16] (up to a slight modification).

$$\begin{cases} u_t - u_{xxt} + cu_x + 3uu_x - \frac{\beta_0}{\beta}u_{xxx} + \frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x = 2u_xu_{xx} + uu_{xxx}, \\ u|_{t=0} = u_0. \end{cases} \tag{3.14}$$

Theorem 3.5.1. *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Then there exist a positive time $T > 0$ and a unique solution $u \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$ to the Cauchy problem (3.14) with $u(0) = u_0$. Moreover, the solution u depends continuously on the initial value u_0.*

Proof. We prove the Theorem 3.5.1 by six steps. In fact, the Cauchy problem has a weak form as

$$\begin{cases} u_t + uu_x + \frac{\beta_0}{\beta}u_x + p * \partial_x \left\{ \left(c - \frac{\beta_0}{\beta}\right) u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \right\} = 0. \\ u|_{t=0} = u_0. \end{cases} \tag{3.15}$$

Step 1: Approximation solution. We use a standard iterative process to construct a solution of (3.15). Not lose the generation, we define $u^{(0)} = 0$, then a sequence of approximation solutions $\left\{u^{(n)}\right\}_{n\in\mathbb{N}}$ is build as solving the following linear transport equation,

$$\begin{cases} \partial_t u^{(n+1)} + u^{(n)}\partial_x u^{(n+1)} + \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right) p * \right] \partial_x u^{(n+1)} \\ \qquad\qquad = -p * \partial_x \left\{ u^{(n)^2} + \frac{1}{2}u_x^{(n)^2} + \frac{\omega_1}{3\alpha^2}u^{(n)^3} + \frac{\omega_2}{4\alpha^3}u^{(n)^4} \right\}, \\ u^{(n+1)}(0,x) = u_0^{(n+1)}(x) = S_{n+1}u_0, \end{cases} \tag{3.16}$$

where

$$\widehat{S_{n+1}u_0}(\xi) := 1_{|\xi|<2\times 2^{n+2}}(\xi)\hat{u_0}(\xi).$$

By the theory of the linear evolution, $\forall n \in \mathbb{N}$, there is a unique smooth solution $u^{(n+1)}$ of (3.16) and $u^{(n+1)} \in C^1(\mathbb{R}; H^\infty(\mathbb{R}))$.

Then, we show that the sequence $\left\{u^{(n)}\right\}_{n\in\mathbb{N}}$ converges. Because of the compactness of Sobolev space, we show two facts to release the convergence of $\left\{u^{(n)}\right\}_{n\in\mathbb{N}}$, that $\left\{u^{(n)}\right\}_{n\in\mathbb{N}}$ is uniformly bounded and that $\left\{u^{(n)}\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Step 2: Uniform bounds. In this step, we show that the sequence $\{u^{(n)}\}_{n\in\mathbb{N}}$ is uniformly bounded in some Sobolev space. Applying the operator Λ^s to the (3.16), we get

$$\begin{aligned}\partial_t \Lambda^s u^{(n+1)} = &-\Lambda^s \left(u^{(n)}\partial_x u^{(n+1)}\right) - \frac{\beta_0}{\beta}\Lambda^s \partial_x u^{(n+1)} - \left(c - \frac{\beta_0}{\beta}\right)\Lambda^s\left(p * \partial_x u^{(n+1)}\right)\\ &- \Lambda^s\left(p * \partial_x\left\{u^{(n)^2} + \frac{1}{2}u_x^{(n)^2} + \frac{\omega_1}{3\alpha^2}u^{(n)^3} + \frac{\omega_2}{4\alpha^3}u^{(n)^4}\right\}\right).\end{aligned}$$

Multiply $\Lambda^s u^{(n+1)}$ and integrate on $\mathbb{R}$, then we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\Lambda^s u^{(n+1)}\right\|_{L^2}^2 = \left\langle \partial_t \Lambda^s u^{(n+1)}, \Lambda^s u^{(n+1)}\right\rangle\\ =& -\left\langle \Lambda^s\left(u^{(n)}\partial_x u^{(n+1)}\right), \Lambda^s u^{(n+1)}\right\rangle - \frac{\beta_0}{\beta}\left\langle \Lambda^s \partial_x u^{(n+1)}, \Lambda^s u^{(n+1)}\right\rangle\\ &- \left(c - \frac{\beta_0}{\beta}\right)\left\langle \Lambda^{s-2}\partial_x u^{(n+1)}, \Lambda^s u^{(n+1)}\right\rangle\\ &- \left\langle \Lambda^{s-2}\partial_x\left(u^{(n)^2}\right), \Lambda^s u^{(n+1)}\right\rangle - \frac{1}{2}\left\langle \Lambda^{s-2}\partial_x\left(u_x^{(n)^2}\right), \Lambda^s u^{(n+1)}\right\rangle\\ &- \frac{\omega_1}{3\alpha^2}\left\langle \Lambda^{s-2}\partial_x\left(u^{(n)^3}\right), \Lambda^s u^{(n+1)}\right\rangle - \frac{\omega_2}{4\alpha^3}\left\langle \Lambda^{s-2}\partial_x\left(u^{(n)^4}\right), \Lambda^s u^{(n+1)}\right\rangle.\end{aligned}$$

In fact, Λ^s is symmetric operator and ∂_x is skew-symmetric, so for all h smooth enough, we have

$$\langle \Lambda^s \partial_x h, \Lambda^s h\rangle = \left\langle \Lambda^{s-2}\partial_x h, \Lambda^s h\right\rangle = 0. \tag{3.17}$$

Taking $h = u^{(n+1)}$, we get

$$
\begin{aligned}
&\frac{d}{dt}\left\|\Lambda^s u^{(n+1)}\right\|_{L^2}^2 \\
=& -2\left\langle \Lambda^s\left(u^{(n)}\partial_x u^{(n+1)}\right), \Lambda^s u^{(n+1)}\right\rangle - 2\left\langle \Lambda^{s-2}\partial_x\left(u^{(n)^2}\right), \Lambda^s u^{(n+1)}\right\rangle \\
&- \left\langle \Lambda^{s-2}\partial_x\left(u_x^{(n)^2}\right), \Lambda^s u^{(n+1)}\right\rangle - \frac{2\omega_1}{3\alpha^2}\left\langle \Lambda^{s-2}\partial_x\left(u^{(n)^3}\right), \Lambda^s u^{(n+1)}\right\rangle \\
&- \frac{\omega_2}{2\alpha^3}\left\langle \Lambda^{s-2}\partial_x\left(u^{(n)^4}\right), \Lambda^s u^{(n+1)}\right\rangle \\
=& I_{11} + I_{12} + I_{13} + I_{14} + I_{15}.
\end{aligned}
$$

From Lemma 3.4.2, we have that

$$
|\, I_{11} \,| \lesssim \|u^{(n)}\|_{H^s}\|u^{(n+1)}\|_{H^s}^2. \tag{3.18}
$$

Furthermore, we know that

$$
\begin{aligned}
|I_{12}| &= \left|2\left\langle \Lambda^{s-2}\partial_x\left(u^{(n)^2}\right), \Lambda^s u^{(n+1)}\right\rangle\right| \\
&\lesssim \|\Lambda^{s-2}\partial_x(u^{(n)^2})\|_{L^2}\|\Lambda^s u^{(n+1)}\|_{L^2} \\
&\lesssim \|\partial_x(u^{(n)^2})\|_{H^{s-2}}\|u^{(n+1)}\|_{H^s} \\
&\lesssim \|u^{(n)}\|_{H^{s-1}}^2\|u^{(n+1)}\|_{H^s} \\
&\lesssim \|u^{(n)}\|_{H^s}^2\|u^{(n+1)}\|_{H^s},
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
|I_{13}| &= \left|\left\langle \Lambda^{s-2}\partial_x\left(u_x^{(n)^2}\right), \Lambda^s u^{(n+1)}\right\rangle\right| \\
&\lesssim \|\Lambda^{s-2}\partial_x(u_x^{(n)^2})\|_{L^2}\|\Lambda^s u^{(n+1)}\|_{L^2} \\
&\lesssim \|\partial_x u_x^{(n)^2}\|_{H^{s-2}}\|u^{(n+1)}\|_{H^s} \\
&\lesssim \|u_x^{(n)}\|_{H^{s-1}}^2\|u^{(n+1)}\|_{H^s} \\
&\lesssim \|u^{(n)}\|_{H^s}^2\|u^{(n+1)}\|_{H^s},
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
|I_{14}| &= \left| \frac{2\omega_1}{3\alpha^2} \left\langle \Lambda^{s-2}\partial_x \left(u^{(n)^3} \right), \Lambda^s u^{(n+1)} \right\rangle \right| \\
&\lesssim \|\Lambda^{s-2}\partial_x(u^{(n)^3})\|_{L^2} \|\Lambda^s u^{(n+1)}\|_{L^2} \\
&\lesssim \|\partial_x u^{(n)^3}\|_{H^{s-2}} \|u^{(n+1)}\|_{H^s} \\
&\lesssim \|u^{(n)}\|_{H^{s-1}}^3 \|u^{(n+1)}\|_{H^s} \\
&\lesssim \|u^{(n)}\|_{H^s}^3 \|u^{(n+1)}\|_{H^s},
\end{aligned} \tag{3.21}
$$

and

$$
\begin{aligned}
|I_{15}| &= \left| \frac{\omega_2}{2\alpha^3} \left\langle \Lambda^{s-2}\partial_x \left(u^{(n)^4} \right), \Lambda^s u^{(n+1)} \right\rangle \right| \\
&\lesssim \|\Lambda^{s-2}\partial_x(u^{(n)^4})\|_{L^2} \|\Lambda^s u^{(n+1)}\|_{L^2} \\
&\lesssim \|\partial_x u^{(n)^4}\|_{H^{s-2}} \|u^{(n+1)}\|_{H^s} \\
&\lesssim \|u^{(n)}\|_{H^{s-1}}^4 \|u^{(n+1)}\|_{H^s} \\
&\lesssim \|u^{(n)}\|_{H^s}^4 \|u^{(n+1)}\|_{H^s}.
\end{aligned} \tag{3.22}
$$

Combining these equaitons, (3.18) - (3.22), we get

$$
\frac{d}{dt}\|u^{(n+1)}\|_{H^s}^2 \le C_0 \left\{ \|u^{(n)}\|_{H^s} \|u^{(n+1)}\|_{H^s}^2 + \sum_{j=2}^{4} \|u^{(n)}\|_{H^s}^j \|u^{(n+1)}\|_{H^s} \right\}. \tag{3.23}
$$

Without loss of generation, we may assume $C_0 \ge 2$. Let's fix a $T > 0$ such that

$$
C_0^5 \max(\|u_0\|_{H^s}^3, 1)T < 1. \tag{3.24}
$$

Now we claim that $\forall n \in \mathbb{N}$,

$$
\|u^{(n)}(t)\|_{H^s} \le C_0 \|u_0\|_{H^s}, \quad \forall t \in [0, T]. \tag{3.25}
$$

We prove this claim by an inductive argument.

i) For $n = 0$, from (3.16), noting $u^{(0)} = 0$, we have

$$
\partial_t u^{(1)} + \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta} \right) p * \right] \partial_x u^{(1)} = 0,
$$

which implies that

$$\begin{aligned}
&\frac{d}{dt}\|u^{(1)}\|_{H_s}^2 = 2\langle \partial_t \Lambda^s u^{(1)}, \Lambda^s u^{(1)}\rangle \\
&= -\frac{2\beta_0}{\beta}\langle \Lambda^s \partial_x u^{(1)}, \Lambda^s u^{(1)}\rangle - 2\left(c - \frac{\beta_0}{\beta}\right)\langle \Lambda^{s-2}\partial_x u^{(1)}, \Lambda^s u^{(1)}\rangle.
\end{aligned}$$

Recalling (3.17), we get

$$\langle \Lambda^s \partial_x u^{(1)}, \Lambda^s u^{(1)}\rangle = \langle \Lambda^{s-2}\partial_x u^{(1)}, \Lambda^s u^{(1)}\rangle = 0,$$

which infers that

$$\frac{d}{dt}\|u^{(1)}\|_{H_s}^2 = 0.$$

Along with $u^{(1)}(0,x) = S_1 u_0(x)$, we get

$$u^{(1)}(t,x) = S_1 u_0(x), \quad \forall t \in \mathbb{R}.$$

In fact, $\forall s \geq 0$, as the definition of $S_n u_0$, for all $n \in \mathbb{R}$, we have

$$\begin{aligned}
\|S_n u_0\|_{H^s}^2 &= \int_{\mathbb{R}} (1+|\xi|^2)^s 1_{|\xi|\leq 2\cdot 2^{n+2}}(\xi)|\hat{u_0}(\xi)|^2 \mathrm{d}\xi \\
&= \int_{|\xi|\leq 2\cdot 2^{n+2}} (1+|\xi|^2)^s |\hat{u_0}(\xi)|^2 \mathrm{d}\xi \\
&\leq \int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{u_0}(\xi)|^2 \mathrm{d}\xi \\
&= \|u_0\|_{H^s}^2.
\end{aligned} \tag{3.26}$$

Therefore, we have

$$\|u^{(1)}\|_{H^s} = \|S_1 u_0\|_{H^s} \leq \|u_0\|_{H^s} \leq C_0 \|u_0\|_{H^s}.$$

ii) For some fixed n, $n \in \mathbb{N}$, we assume

$$\sup_{0\leq t\leq T} \|u^{(n)}\|_{H^s} \leq C_0 \|u_0\|_{H^s}. \tag{3.27}$$

Applying this assumption on (3.23), we have

$$\|u^{(n+1)}\|_{H^s} \leq e^{\frac{C_0}{2}\int_0^t \|u^{(n)}\|_{H^s} d\tau} \left(\|u_0^{(n+1)}\|_{H^s} + \frac{C_0}{2}\int_0^t \sum_{j=2}^4 \|u^{(n)}\|_{H^s}^j d\tau \right)$$

By our assumption on $u^{(n)}$, (3.27), we have

$$\|u^{(n+1)}\|_{H^s} \le e^{\frac{C_0^2}{2}\|u_0\|_{H^s}T}\left(\|u_0^{(n+1)}\|_{H^s} + \frac{T}{2}\sum_{j=2}^{4} C_0^{j+1}\|u_0\|_{H^s}^{j}\right)$$

Since $\|u_0^{(n+1)}\|_{H^s} = \|S_{n+1}u_0\|_{H^s} \le \|u_0\|_{H^s}$, we have

$$\|u^{(n+1)}\|_{H^s} \le e^{\frac{C_0^2}{2}\|u_0\|_{H^s}T}\left(1 + \frac{T}{2}\sum_{j=1}^{3} C_0^{j+2}\|u_0\|_{H^s}^{j}\right)\|u_0\|_{H^s}. \tag{3.28}$$

When $\|u_0\|_{H^s} \le 1$, we have

$$\|u^{(n+1)}\|_{H^s} \le e^{\frac{C_0^2}{2}T}\left(1 + \frac{T}{2}\sum_{j=1}^{3} C_0^{j+2}\right)\|u_0\|_{H^s}.$$

Thanks to (3.24), we have

$$\|u^{(n+1)}\|_{H^s} < e^{\frac{1}{2C_0^3}}\left(1 + \frac{1}{2}\sum_{j=0}^{2} C_0^{j-2}\right)\|u_0\|_{H^s}.$$

Recalling $C_0 \ge 2$, we get

$$\|u^{(n+1)}\|_{H^s} < \frac{15}{8}e^{\frac{1}{16}}\|u_0\|_{H^s} < 2\|u_0\|_{H^s} \le C_0\|u_0\|_{H^s}.$$

On the other hand, when $\|u_0\|_{H^s} > 1$, from (3.28), we have

$$\|u^{(n+1)}\|_{H^s} \le e^{\frac{C_0^2}{2}\|u_0\|_{H^s}^3T}\left(1 + \frac{1}{2}C_0^5\|u_0\|_{H^s}^3 T\sum_{j=0}^{2} C_0^{j-2}\right)\|u_0\|_{H^s}.$$

Thanks to (3.24), we have

$$\begin{aligned}\|u^{(n+1)}\|_{H^s} &< e^{\frac{1}{2C_0^3}}\left(1 + \frac{1}{2}\sum_{j=0}^{2} C_0^{j-2}\right)\|u_0\|_{H^s} \\ &< \frac{15}{8}e^{\frac{1}{16}}\|u_0\|_{H^s} < 2\|u_0\|_{H^s} \le C_0\|u_0\|_{H^s}\end{aligned}$$

Therefore, we get

$$\|u^{(n+1)}\|_{H^s} \le C_0\|u_0\|_{H^s},$$

which complete our induction. Hence, the sequence $\{u^{(n)}\}$ is uniformly bounded in $C([0,T];H^s)$. Thanks to (3.16), we get that $\{\partial_t u^{(n+1)}\}_{n\in\mathbb{N}}$ is uniformly bounded in

$C([0,T];H^{s-1})$.

Step 3 In this step, we want to prove that $\{u^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];H^{s-1})$. From (3.16), for any $n,m\in\mathbb{N}$, we have

$$\begin{aligned}\partial_t u^{(m+n+1)} &+ u^{(m+n)}\partial_x u^{(m+n+1)} + \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right) p*\right]\partial_x u^{(m+n+1)} \\ &= -p*\partial_x\left\{u^{(m+n)^2} + \frac{1}{2}u_x^{(m+n)^2} + \frac{\omega_1}{3\alpha^2}u^{(m+n)^3} + \frac{\omega_2}{4\alpha^3}u^{(m+n)^4}\right\}, \quad (3.29)\end{aligned}$$

and

$$\begin{aligned}\partial_t u^{(n+1)} &+ u^{(n)}\partial_x u^{(n+1)} + \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right) p*\right]\partial_x u^{(n+1)} \\ &= -p*\partial_x\left\{u^{(n)^2} + \frac{1}{2}u_x^{(n)^2} + \frac{\omega_1}{3\alpha^2}u^{(n)^3} + \frac{\omega_2}{4\alpha^3}u^{(n)^4}\right\}. \quad (3.30)\end{aligned}$$

Taking difference between (3.29) and (3.30), and adopting

$$\begin{aligned}u^{(m+n)}\partial_x u^{(m+n+1)} - u^{(n)}\partial_x u^{(n+1)} &= u^{(m+n)}\partial_x(u^{(m+n+1)} - u^{(n+1)}) \\ &\quad + (u^{(m+n)} - u^{(n)})\partial_x u^{(n+1)},\end{aligned}$$

we have

$$\begin{aligned}\partial_t(u^{(m+n+1)} - u^{(n+1)}) =& - u^{(m+n)}\partial_x(u^{(m+n+1)} - u^{(n+1)}) \\ & - (u^{(m+n)} - u^{(n)})\partial_x u^{(n+1)} \\ & - \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right) p*\right]\partial_x(u^{(m+n+1)} - u^{(n+1)}) \\ & - p*\partial_x\left\{(u^{(m+n)^2} - u^{(n)^2}) + \frac{1}{2}(u_x^{(m+n)^2} - u_x^{(n)^2})\right. \\ & \left. + \frac{\omega_1}{3\alpha^2}(u^{(m+n)^3} - u^{(n)^3}) + \frac{\omega_2}{4\alpha^3}(u^{(m+n)^4} - u^{(n)^4})\right\},\end{aligned}$$

which implies that $u^{(m+n+1)} - u^{(n+1)}$ solves

$$\begin{cases} \partial_t(u^{(m+n+1)} - u^{(n+1)}) = -u^{(m+n)}\partial_x(u^{(m+n+1)} - u^{(n+1)}) \\ \qquad -(u^{(m+n)} - u^{(n)})\partial_x u^{(n+1)} \\ \qquad - \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right) p* \right] \partial_x(u^{(m+n+1)} - u^{(n+1)}) \\ \qquad -p * \partial_x \left\{ (u^{(m+n)^2} - u^{(n)^2}) + \frac{1}{2}(u_x^{(m+n)^2} - u_x^{(n)^2}) \right. \\ \qquad \left. + \frac{\omega_1}{3\alpha^2}(u^{(m+n)^3} - u^{(n)^3}) + \frac{\omega_2}{4\alpha^3}(u^{(m+n)^4} - u^{(n)^4}) \right\} \\ (u^{(m+n+1)} - u^{n+1})|_{t=0} = S_{m+n+1}u_0 - S_{n+1}u_0. \end{cases} \tag{3.31}$$

We adopt Λ^{s-1} on (3.31), multiply $\Lambda^{s-1}(u^{m+n+1} - u^{(n+1)})$ and integrate on $\mathbb{R}$, we have

$$\begin{aligned} &\langle \partial_t \Lambda^{s-1}(u^{m+n+1} - u^{(n+1)}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle \\ = & -\langle \Lambda^{s-1}(u^{(m+n)}\partial_x(u^{(m+n+1)} - u^{(n+1)})), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle \\ & - \langle \Lambda^{s-1}((u^{(m+n)} - u^{(n)})\partial_x u^{(n+1)}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle \\ & - \langle \Lambda^{s-3}\partial_x(u^{(m+n)^2} - u^{(n)^2}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle \\ & - \frac{1}{2}\langle \Lambda^{s-3}\partial_x(u_x^{(m+n)^2} - u_x^{(n)^2}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle \\ & - \frac{\omega_1}{3\alpha^2}\langle \Lambda^{s-3}\partial_x(u^{(m+n)^3} - u^{(n)^3}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle \\ & - \frac{\omega_2}{4\alpha^3}\langle \Lambda^{s-3}\partial_x(u^{(m+n)^4} - u^{(n)^4}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle \\ = & I_{21} + I_{22} + I_{23} + I_{24} + I_{25} + I_{26}, \end{aligned} \tag{3.32}$$

where we ignore two terms since we know that by (3.17),

$$-\frac{\beta_0}{\beta}\langle \Lambda^{s-1}\partial_x(u^{(m+n+1)} - u^{(n+1)}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle = 0,$$

and

$$-\left(c - \frac{\beta_0}{\beta}\right)\langle \Lambda^{s-3}\partial_x(u^{(m+n+1)} - u^{(n+1)}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle = 0,$$

Thanks to Lemma 3.4.2, we get

$$| \, I_{21} \, | \lesssim \|u^{(m+n)}\|_{H^{s-1}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}^2. \tag{3.33}$$

Furthermore, a commutator process,

$$\begin{aligned}\Lambda^{s-1}((u^{(m+n)} - u^{(n)})\partial_x u^{(n+1)}) =&(u^{(m+n)} - u^{(n)})\Lambda^{s-1}\partial_x u^{(n+1)} \\ &+ [\Lambda^{s-1}, u^{(m+n)} - u^{(n)}]\partial_x u^{(n+1)},\end{aligned}$$

implies that

$$\begin{aligned}|I_{22}| &= \left|\langle \Lambda^{s-1}((u^{(m+n)} - u^{(n)})\partial_x u^{(n+1)}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle\right| \\ &\lesssim \left|\langle (u^{(m+n)} - u^{(n)})\Lambda^{s-1}\partial_x u^{(n+1)}, \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle\right| \\ &+ \left|\langle [\Lambda^{s-1}, u^{(m+n)} - u^{(n)}]\partial_x u^{(n+1)}, \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle\right|.\end{aligned}$$

On one hand, using the integration by parts and a fact from Soblev embedding theorem, that is,

$$\|\partial_x u\|_{L^\infty} \lesssim \|u\|_{H^\sigma}, \forall \sigma > 3/2,$$

we get

$$\begin{aligned}&\left|\langle (u^{(m+n)} - u^{(n)})\Lambda^{s-1}\partial_x u^{(n+1)}, \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle\right| \\ &\lesssim \|(u^{(m+n)} - u^{(n)})\Lambda^{s-1}\partial_x u^{(n+1)}\|_{L^2} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\ &\lesssim \|\partial_x(u^{(m+n)} - u^{(n)})\|_{L^\infty} \|u^{(n+1)}\|_{H^{s-1}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\ &\lesssim \|\partial_x(u^{(m+n)} - u^{(n)})\|_{H^{s-2}} \|u^{(n+1)}\|_{H^{s-1}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\ &\lesssim \|u^{(m+n)} - u^{(n)}\|_{H^{s-1}} \|u^{(n+1)}\|_{H^{s-1}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}},\end{aligned}$$

where $s > 3/2$.

On the other hand, from Lemma 3.4.1 we know

$$\begin{aligned}&\langle [\Lambda^{s-1}, u^{(m+n)} - u^{(n)}]\partial_x u^{(n+1)}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)})\rangle \\ &\lesssim \|u^{(m+n)} - u^{(n)}\|_{H^{s-1}} \|u^{(n+1)}\|_{H^{s-1}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}\end{aligned}$$

Therefore, we get

$$\|I_{22}\| \lesssim \|u^{(m+n)} - u^{(n)}\|_{H^{s-1}} \|u^{(n+1)}\|_{H^{s-1}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}. \tag{3.34}$$

Also we have these estimates,

$$\begin{aligned}
|I_{23}| &= \left| \langle \Lambda^{s-3}\partial_x (u^{(m+n)^2} - u^{(n)^2}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)}) \rangle \right| \\
&\lesssim \|\partial_x (u^{(m+n)^2} - u^{(n)^2})\|_{H^{s-3}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&\lesssim \|u^{(m+n)^2} - u^{(n)^2}\|_{H^{s-2}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&\lesssim \|u^{(m+n)} + u^{(n)}\|_{H^{s-2}} \|u^{(m+n)} - u^{(n)}\|_{H^{s-2}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
|I_{24}| &= \left| \frac{1}{2} \langle \Lambda^{s-3}\partial_x (u_x^{(m+n)^2} - u_x^{(n)^2}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)}) \rangle \right| \\
&\lesssim \|u_x^{(m+n)^2} - u_x^{(n)^2}\|_{H^{s-3}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&\lesssim \|u_x^{(m+n)} + u_x^{(n)}\|_{H^{s-3}} \|u_x^{(m+n)} - u_x^{(n)}\|_{H^{s-3}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&\lesssim \|u^{(m+n)} + u^{(n)}\|_{H^{s-2}} \|u^{(m+n)} - u^{(n)}\|_{H^{s-2}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
|I_{25}| &= \left| \frac{\omega_1}{3\alpha^2} \langle \Lambda^{s-3}\partial_x (u^{(m+n)^3} - u^{(n)^3}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)}) \rangle \right| \\
&\lesssim \|u^{(m+n)} - u^{(n)}\|_{H^{s-2}} \|u^{(m+n)^2} + u^{(m+n)}u^{(n)} + u^{(n)^2}\|_{H^{s-2}} \\
&\quad \cdot \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
|I_{26}| &= \left| \frac{\omega_2}{4\alpha^3} \langle \Lambda^{s-3}\partial_x (u^{(m+n)^4} - u^{(n)^4}), \Lambda^{s-1}(u^{(m+n+1)} - u^{(n+1)}) \rangle \right| \\
&\lesssim \|u^{(m+n)} - u^{(n)}\|_{H^{s-2}} \|u^{(m+n)} + u^{(n)}\|_{H^{s-2}} \\
&\quad \cdot \|u^{(m+n)^2} + u^{(n)^2}\|_{H^{s-2}} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}
\end{aligned} \tag{3.38}$$

Combining these estimates, (3.33) - (3.38), we get

$$
\begin{aligned}
&\frac{d}{dt}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}^2 \\
&\lesssim \|u^{(m+n)}\|_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}^2 \\
&+ \|(u^{(m+n)} - u^{(n)})\|_{H^{s-1}}\|u^{(n+1)}\|_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&+ \|u^{(m+n)} + u^{(n)}\|_{H^{s-2}}\|u^{(m+n)} - u^{(n)}\|_{H^{s-2}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&+ \|u^{(m+n)} + u^{(n)}\|_{H^{s-2}}\|u^{(m+n)} - u^{(n)}\|_{H^{s-2}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&+ \|u^{(m+n)} - u^{(n)}\|_{H^{s-2}}\|u^{(m+n)^2} + u^{(m+n)}u^{(n)} + u^{(n)^2}\|_{H^{s-2}} \\
&\quad \cdot \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&+ \|u^{(m+n)} - u^{(n)}\|_{H^{s-2}}\|u^{(m+n)} + u^{(n)}\|_{H^{s-2}} \\
&\quad \cdot \|u^{(m+n)^2} + u^{(n)^2}\|_{H^{s-2}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}
\end{aligned}
$$

Since H^s is Banach algebra, $H^{s-1} \hookrightarrow H^{s-2}$, we get

$$
\begin{aligned}
&\frac{d}{dt}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}^2 \\
&\lesssim \|u^{(m+n)}\|_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}^2 \\
&+ \|(u^{(m+n)} - u^{(n)})\|_{H^{s-1}}\|u^{(n+1)}\|_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&+ \|u^{(m+n)} + u^{(n)}\|_{H^{s-1}}\|u^{(m+n)} - u^{(n)}\|_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&+ \|u^{(m+n)} - u^{(n)}\|_{H^{s-1}}\|u^{(m+n)^2} + u^{(m+n)}u^{(n)} + u^{(n)^2}\|_{H^{s-1}} \\
&\quad \cdot \|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&+ \|u^{(m+n)} - u^{(n)}\|_{H^{s-1}}\|u^{(m+n)} + u^{(n)}\|_{H^{s-1}} \\
&\quad \cdot \|u^{(m+n)^2} + u^{(n)^2}\|_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}.
\end{aligned}
$$

Recalling $\|u^{(n)}(t)\|_{H^s} \leq C_0\|u_0\|_{H^s}$ for any $n \in \mathbb{N}$, we get

$$
\begin{aligned}
&\frac{d}{dt}\|u^{(m+n+1)} - u^{(n+1)}\|^2_{H^{s-1}} \\
&\lesssim \|u_0\|_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|^2_{H^{s-1}} \\
&+ \|(u^{(m+n)} - u^{(n)})\|_{H^{s-1}}\|u_0\|_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&+ \|u^{(m+n)} - u^{(n)}\|_{H^{s-1}}\|u_0\|^2_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&+ \|u^{(m+n)} - u^{(n)}\|_{H^{s-1}}\|u_0\|^3_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&= \|u_0\|_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|^2_{H^{s-1}} \\
&+ \sum_{j=1}^{3}\|u_0\|^j_{H^{s-1}}\|(u^{(m+n)} - u^{(n)})\|_{H^{s-1}}\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}
\end{aligned}
$$

Thanks to Lemma 3.4.3, there is some $C_1 > 0$ such that

$$
\begin{aligned}
&\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&\lesssim \left(\|u_0^{(m+n+1)} - u_0^{(n+1)}\|_{H^{s-1}} + \frac{C_1}{2}\sum_{j=1}^{3}\int_0^t \|u_0\|^j_{H^{s-1}}\|(u^{(m+n)} - u^{(n)})\|_{H^{s-1}}d\tau\right) \\
&\quad \cdot e^{\frac{C_1}{2}\int_0^t \|u_0\|_{H^{s-1}}d\tau}
\end{aligned}
$$

Let $M = \frac{3C_1}{2}\max(1, \|u_0\|^3_{H^s})$, we have

$$
\begin{aligned}
&\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
&\lesssim \left(\|u_0^{(m+n+1)} - u_0^{(n+1)}\|_{H^{s-1}} + M\int_0^t \|(u^{(m+n)} - u^{(n)})\|_{H^{s-1}}d\tau\right) \\
&\quad \cdot e^{\frac{C_1}{2}\int_0^t \|u_0\|_{H^{s-1}}d\tau} \\
&\lesssim C_T\left(\|u_0^{(m+n+1)} - u_0^{(n+1)}\|_{H^{s-1}} + \int_0^t \|(u^{(m+n)} - u^{(n)})\|_{H^{s-1}}d\tau\right),
\end{aligned}
\tag{3.39}
$$

where C_T depends on T and M.

Before I show the estimate of $\|u_0^{(m+n+1)} - u_0^{(n+1)}\|$, I use an iterative process to estimate $\|u^{(n+1)} - u^{(n)}\|$. Fixing $m = 1$ in (3.39), we have

$$
\|u^{(n+2)} - u^{(n+1)}\|_{H^{s-1}} \leq C_T\left(\|u_0^{(n+2)} - u_0^{(n+1)}\|_{H^{s-1}} + \int_0^t \|(u^{(n+1)} - u^{(n)})\|_{H^{s-1}}d\tau\right). \tag{3.40}
$$

Furthermore, as the definition in (3.16),

$$u_0^{(m+n+1)} - u_0^{(n+1)} = S_{n+m+1}u_0 - S_{n+1}u_0 = 1_{2\cdot 2^{n+2}\leq|\xi|\leq 2\cdot 2^{m+n+2}}(\xi)u_0(\xi), \tag{3.41}$$

we have the estimate,

$$\begin{aligned}
&\|u_0^{(m+n+1)} - u_0^{(m+n)}\|_{H^{s-1}}^2 \\
=&\int_{\mathbb{R}}(1+|\xi|^2)^{(s-1)}|\hat{u}_0(\xi)|1_{2^{n+3}\leq\xi\leq 2^{m+n+3}}(\xi)d\xi \\
=&\int_{2^{n+3}\leq|\xi|\leq 2^{m+n+3}}(1+|\xi|^2)^{s-1}|\hat{u}_0(\xi)|^2 d\xi \\
\leq&\frac{1}{2^{2(n+3)}}\int_{2^{n+3}\leq|\xi|\leq 2^{m+n+3}}|\xi|^2(1+|\xi|^2)^{s-1}|\hat{u}_0(\xi)|^2 d\xi \\
\leq&\frac{1}{2^{2(n+3)}}\int_{2^{n+3}\leq|\xi|\leq 2^{m+n+3}}(1+|\xi|^2)^{s}|\hat{u}_0(\xi)|^2 d\xi \\
\leq&\frac{1}{2^{2(n+3)}}\int_{\mathbb{R}}(1+|\xi|^2)^{s}|\hat{u}_0(\xi)|^2 d\xi \\
\leq&2^{-2(n+3)}\|u_0\|_{H^s}^2,
\end{aligned}$$

which implies that

$$\|u_0^{(m+n+1)} - u_0^{(m+n)}\|_{H^{s-1}} \leq \frac{1}{2^{n+1}}\|u_0\|_{H^s}. \tag{3.42}$$

Applying this estimate on (3.40), we have

$$\|u^{(n+2)} - u^{(n+1)}\|_{H^{s-1}} \leq C_T\left(\frac{1}{2^{n+1}}\|u_0\|_{H^s} + \int_0^t \|(u^{(n+1)} - u^{(n)})\|_{H^{s-1}}d\tau\right). \tag{3.43}$$

We claim that $\forall n \geq 1$,

$$\|u^{(n+1)} - u^{(n)}\| \leq \left(\sum_{k=0}^{n-1}\frac{1}{2^{n-k}}\frac{1}{(k+1)!}C_T^{k+1}T^k\right)\|u_0\|_{H^s} + \frac{C_0(C_T\cdot T)^n}{n!}\|u_0\|_{H^{s-1}}. \tag{3.44}$$

Let $n = 0$ in (3.43), recalling the fact $\forall t \in [0,T], \|u^{(1)}\|_{H^s} \leq C_0\|u_0\|_{H^s}$ and $u^{(0)} = 0$, we have

$$\begin{aligned}
\|u^{(2)} - u^{(1)}\|_{H^{s-1}} \leq& C_T\left(\frac{1}{2}\|u_0\|_{H^s} + \int_0^t \|(u^{(1)} - u^{(0)})\|_{H^{s-1}}d\tau\right) \\
=& C_T\left(\frac{1}{2}\|u_0\|_{H^s} + \int_0^t \|u^{(1)}\|_{H^{s-1}}d\tau\right) \\
\leq& C_T\left(\frac{1}{2}\|u_0\|_{H^s} + C_0\|u_0\|_{H^{s-1}}T\right),
\end{aligned} \tag{3.45}$$

which satisfy our claim.

Now we apply our assumption (3.44) on (3.43), we have

$$
\begin{aligned}
&\|u^{(n+2)} - u^{(n+1)}\| \\
\leq& C_T \left(\frac{1}{2^{n+1}} \|u_0\|_{H^s} + \int_0^t \|(u^{(n+1)} - u^{(n)})\|_{H^{s-1}} d\tau \right) \\
\leq& C_T \ \{ \frac{1}{2^{n+1}} \|u_0\|_{H^s} \\
&+ \int_0^t \left(\sum_{k=0}^{n-1} \frac{1}{2^{n-k}} \frac{1}{(k+1)!} C_T^{k+1} T^k \right) \|u_0\|_{H^s} + \frac{C_0 (C_T \cdot T)^n}{n!} \|u_0\|_{H^{s-1}} d\tau \Bigg\} \\
\leq& \frac{C_T}{2^{n+1}} \|u_0\|_{H^s} + \left(\sum_{k=0}^{n-1} \frac{1}{2^{n-k}} \frac{1}{(k+2)!} C_T^{k+2} T^k \cdot t \right) \|u_0\|_{H^s} + \frac{C_0 C_T^{n+1} T^n \cdot t}{(n+1)!} \|u_0\|_{H^{s-1}} \\
\leq& \frac{C_T}{2^{n+1}} \|u_0\|_{H^s} + \left(\sum_{k=0}^{n-1} \frac{1}{2^{n-k}} \frac{1}{(k+2)!} C_T^{k+2} T^{k+1} \right) \|u_0\|_{H^s} + \frac{C_0 (C_T \cdot T)^{n+1}}{(n+1)!} \|u_0\|_{H^{s-1}} \\
=& \left(\sum_{k=0}^{n} \frac{1}{2^{n-k}} \frac{1}{(k+1)!} C_T^{k+1} T^k \right) \|u_0\|_{H^s} + \frac{C_0 (C_T \cdot T)^{n+1}}{(n+1)!} \|u_0\|_{H^{s-1}},
\end{aligned}
\tag{3.46}
$$

which prove our claim completely.

Furthermore, recalling the fact that $\|u_0\|_{H^{s-1}} \lesssim \|u_0\|_{H^s}$, from (3.44), we infer that $\forall n \geq 1$,

$$
\begin{aligned}
\|u^{(n+1)} - u^{(n)}\| \leq& \left(\sum_{k=0}^{n-1} \frac{1}{2^{n-k}} \frac{1}{(k+1)!} C_T^{k+1} T^k \right) \|u_0\|_{H^s} + \frac{C_0 (C_T \cdot T)^n}{n!} \|u_0\|_{H^{s-1}} \\
\leq& \frac{1}{2^n} \left(\sum_{k=0}^{n-1} C_T \frac{(2TC_T)^k}{(k+1)!} + C_0 \frac{(2TC_T)^n}{n!} \right) \|u_0\|_{H^s} \\
\leq& \frac{C_T'}{2^n},
\end{aligned}
$$

where C_T' depends on T, C_T and $\|u_0\|_{H^s}$.

Now, let $m, n \in \mathbb{N}$, then

$$\begin{aligned}
&\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \\
=&\|u^{(m+n+1)} - u^{(m+n)} + u^{(m+n)} - u^{(m+n-1)} + \cdots + u^{(n+2)} - u^{(n+1)}\|_{H^{s-1}} \\
\leq&\|u^{(m+n+1)} - u^{(m+n)}\|_{H^{s-1}} + \cdots + \|u^{(n+2)} - u^{(n+1)}\|_{H^{s-1}} \\
\leq&\sum_{k=0}^{m} \frac{C'_T}{2^{n+k+1}} \\
\leq&\frac{C'_T}{2^{n+1}} \sum_{k=0}^{\infty} \frac{1}{2^k} \\
=&\frac{C'_T}{2^n},
\end{aligned} \tag{3.47}$$

which implies that as $n \to \infty$, $\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}} \to 0$. Hence, $\{u^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]; H^{s-1})$. By the completeness of the Banach Space $C([0,T]; H^{s-1})$, we get a limit u in $C([0,T]; H^{s-1})$ such that

$$u^{(n)} \to u \quad \text{in} \quad C([0,T]; H^{s-1}).$$

Step 4 Passing to the limit. For any $s' \in (s-1, s)$, thanks to the fact that $\{u^{(n)}\}_{n\in\mathbb{N}}^{\infty}$ is uniformly bounded is $C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$ and the interpolation inequality,

$$\|u\|_{H^{s'}} \leq C\|u\|_{H^{s-1}}^{\theta} \|u\|_{H^s}^{1-\theta},$$

where $\theta = s - s'$, we have

$$\begin{aligned}
\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s'}} \lesssim&\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}^{\theta} \|u^{(m+n+1)} - u^{(n+1)}\|_{H^s}^{1-\theta} \\
\lesssim&\|u^{(m+n+1)} - u^{(n+1)}\|_{H^{s-1}}^{\theta},
\end{aligned}$$

which with (3.47), implies that $\{u^{(n)}\}_{n\in\mathbb{N}}^{\infty}$ is a Cauchy sequence in $C([0,T]; H^{s'})$. So, by the uniqueness of the limit, we have

$$u^{(n)} \to u \quad \text{in} \quad C([0,T]; H^{s'}), \tag{3.48}$$

This yields that

$$u^{(n)}\partial_x u^{(n+1)} \to u\partial_x u \quad \text{in} \quad C([0,T];H^{s'-1}),$$

which requires $s' > 3/2$. Similarly, we have

$$\left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right) p*\right] \partial_x u^{(n+1)} \to \left[\frac{\beta_0}{\beta} + \left(c - \frac{\beta_0}{\beta}\right) p*\right] \partial_x u$$

in $C([0,T],H^{s'-1})$, where $s' > 3/2$;

$$p * \partial_x \left\{u^{(n)^2} + \frac{\omega_1}{3\alpha^2}u^{(n)^3} + \frac{\omega_2}{4\alpha^3}u^{(n)^4}\right\} \to p * \partial_x \left\{u^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right\}$$

in $C([0,T],H^{s'+1})$, where $s' > -1/2$;

$$p * \partial_x \frac{1}{2} u_x^{(n)^2} \to p * \partial_x \frac{1}{2} u_x^2$$

in $C([0,T],H^{s'})$, where $s' > 1/2$. By (3.16), we know that $\{\partial_t u^{(n)}\}_{n\in\mathbb{N}}^{\infty}$ is a Cauchy sequence in $C([0,T];H^{s'-1}), \forall s' > 3/2$ and $\exists v \in C([0,T];H^{s'-1})$ s.t.

$$\partial_t u^{(n)} \to v \tag{3.49}$$

in $C([0,T];H^{(s'-1)})$ for any $s' \in [3/2, s]$. On the other hand, from the fact that $\{u^{(n)}\}_{n\in N}^{\infty}$ is uniformly bounded in $C([0,T];H^s) \cap C^1([0,T];H^{s-1})$ and (3.48), we infer that

$$\partial_t u^{(n)} \to \partial_t u \tag{3.50}$$

in the sense of distribution, which along with (3.49) implies that

$$v = \partial_t u \quad \text{and} \quad \partial_t u^{(n)} \to \partial_t u \tag{3.51}$$

in $C([0,T];H^{s'-1}), \forall 3/2 < s' < s$.

On the other hand, by the Banach-Alaoglu theorem, there is a subsequence $\{u^{(n_j)}\}_{j\in N}$ of $\{u^{(n)}\}_{j\in N}$ such that

$$u^{(n_j)}(t) \to u^*(t)$$

weekly in $L^2([0,T];H^s)$ and $\forall t \in [0,T]$,

$$u^{(n_j)}(t) \to u^*(t) \quad \text{weekly in} \quad H^s$$

$$\partial_t u^{(n_j)}(t) \to \partial_t u^*(t) \quad \text{weekly in} \quad H^{s-1}$$

which gives rise to

$$u = u^* \quad \text{in} \quad L^\infty([0,T];H^s) \cap Lip([0,T];H^{s-1}).$$

Up to a subsequence, we get that for fixed $t \in [0,T]$, $\limsup_{n\to\infty} \|u^{(n)}(t)\|_{H^s} \geq \|u(t)\|_{H^s}$. Hence, we have

$$\limsup_{t\to 0^+} \|u(t)\|_{H^s} \geq \|u_0\|_{H^s}$$

On the other hand, from $u \in C_w([0,T];H^s)$ and the fact that

$$\|f\|_{H^s} = \sup_{\Psi \in H^{-s}} |\langle f, \Psi\rangle_{H^s \times H^{-s}}|,$$

we get that

$$\liminf_{t\to 0+} \|u(t)\| \geq \|u_0\|_{H^s}.$$

Therefore, we have

$$\liminf_{t\to 0+} \|u(t)\| = \|u_0\|_{H^s},$$

which means $\|u(t)\|_{H^s}$ is strongly right continuous at $t = 0$. Similarly, $\|u(t)\|_{H^s}$ is strongly left continuous at $t = 0$. So $\|u(t)\|_{H^s}$ is continuous strongly at $t = 0$. **Step 5 Uniqueness.** It is easy to prove the uniqueness of the solution of the R-CH equation. Assume that $u_1, u_2 \in C([0,T;H^{s'}]) \cap L^\infty([0,T];H^s)$ with $\partial_t u_1, \partial_t u_2 \in$

$C([0,T;H^{s'-1}])\cap L^\infty([0,T];H^s)$ where $3/2<s'<s$ and $u_1|_{t=0}=u_2|_{t=0}$. Solve the R-CH equation, then we have

$$\begin{aligned}\partial_t(u_1-u_2)=&-u_1\partial_x(u_1-u_2)-(u_1-u_2)\partial_x u_2\\&-\left[\frac{\beta_0}{\beta}+\left(c-\frac{\beta_0}{\beta}\right)p*\right]\partial_x(u_1-u_2)\\&-p*\partial_x\left\{(u_1^2-u_2^2)+\frac{1}{2}(u_{1,x}^2-u_{2,x}^2)\right.\\&\left.+\frac{\omega_1}{3\alpha^2}(u_1^3-u_2^3)+\frac{\omega_2}{4\alpha^3}(u_1^4-u_2^4)\right\},\end{aligned}$$

Then by the similar process to the one in Step 3, we can prove the uniqueness of the solution.

Step 6 Continuity. Back to the proof of the fact that $u\in C([0,T];H^s)$. We have known that $\|u(t)\|_{H^s}$ is continuous at $t=0$, then $\forall T_0\in[0,T]$ and the solution $u(\cdot,T_0)$, we obtain

$$\|u_0^{T_0}\|_{H^s}\leq\|u_0\|_{H^s}e^{\frac{C_0^2}{2}\|u_0\|_{H^s}T_0},$$

where $u(\cdot,T_0)\equiv u_0^{T_0}\in H^s(R)$ at a fixed time T_0. So we take $u_0^{T_0}$ as initial data and construct a forward and backward-in-time solution by solving (3.16). We obtain approximation solution $u^{(n)T_0}(t)$ and then its limit $u_{T_0}(t)$ which solves the R-CH equation with initial data $u_{T_0}(t)|_{t=0}=u_0^{T_0}$ for some positive $T_1>0$ and then $\|u_{T_0}(t)\|_{H^s}$ is continuous at $t=0$. By the uniqueness, we get that

$$u_{T_0}(t)=u(t+T_0)\quad\text{on}\quad[T_0-T_1,T_0+T_1],$$

which implies that $u(t)$ is continuous at $t=T_0$. Therefore, we obtain that $u\in C([0,T];H^s)$.

We have completed the proof of Theorem 3.5.1. □

Motivated to the method in [16], the following blow-up criterion can be also derived, and we omit details of its proof.

Theorem 3.5.2 (Blow-up criterion)**.** *Let $s > \frac{3}{2}$, $u_0 \in H^s$ and u be the corresponding solution to* (3.14) *as in Theorem 3.5.1. Assume $T^*_{u_0}$ is the maximal time of existence. Then*

$$T^*_{u_0} < \infty \quad \Rightarrow \quad \int_0^{T^*_{u_0}} \|\partial_x u(\tau)\|_{L^\infty} d\tau = \infty. \tag{3.52}$$

Remark 3.5.1. *The blow-up criterion* (3.52) *implies that the lifespan $T^*_{u_0}$ does not depend on the regularity index s of the initial data u_0.*

Now we return to the original R-CH (2.1), and let

$$\|u\|^2_{X^{s+1}_\mu} = \|u\|^2_{H^s} + \mu\beta\|\partial_x u\|^2_{H^s}.$$

For some $\mu_0 > 0$ and $M > 0$, we define the Camassa-Holm regime $\mathcal{P}_{\mu_0,M} := \{(\varepsilon, \mu) :< \mu \leq \mu_0, 0 < \varepsilon \leq M\sqrt{\mu}\}$. Then, we have the following corollary.

Corollary 3.5.1. *([11]) Let $u_0 \in H^{s+1}(\mathbb{R})$, $\mu_0 > 0$ and $M > 0$, $s > \frac{3}{2}$. Then, there exist $T > 0$ and a unique family of solutions $(u_{\varepsilon,\mu})\,|_{(\varepsilon,\mu)\in\mathcal{P}_{\mu_0,M}}$ in $C\left(\left[0, \frac{T}{\varepsilon}\right]; X^{s+1}(\mathbb{R})\right) \cap C^1\left(\left[0, \frac{T}{\varepsilon}; X^s(\mathbb{R})\right]\right)$ to the Cauchy problem*

$$\begin{cases} \partial_t u - \beta\mu\partial_t u_{xx} + cu_x + 3\alpha\varepsilon u u_x - \beta_0\mu u_{xxx} + \omega_1\varepsilon^2 u^2 u_x + \omega_2\varepsilon^3 u^3 u_x \\ \qquad\qquad\qquad\qquad\qquad\qquad = \alpha\beta\varepsilon\mu(2u_x u_{xx} + u u_{xxx}), \\ u|_{t=0} = u_0. \end{cases}$$

3.6 Wake-breaking phenomena

Using the energy estimates, we can further obtain the following wave breaking criterion to the R-CH equation.

Theorem 3.6.1 (Wave breaking criterion)**.** *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, and $T^*_{u_0} > 0$ be the maximal existence time of the solution u to the system* (3.14) *with initial data*

u_0 as in Theorem 3.5.1. Then the corresponding solution blows up in finite time if and only if

$$\liminf_{t\uparrow T^*_{u_0},x\in\mathbb{R}} u_x(t,x) = -\infty. \tag{3.6.1}$$

Proof. Applying Theorem 3.5.1, Remark 3.5.1, and a simple density argument, we only need to show that Theorem 3.6.1 holds for some $s \geq 3$. Here we assume $s = 3$ to prove the above theorem.

Multiplying the first equation in (3.14) by u and integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{H^1} = 0, \tag{3.6.2}$$

and then for any $t \in (0, T^*_{u_0})$

$$\|u(t)\|_{H^1} = \|u_0\|_{H^1}. \tag{3.6.3}$$

On the other hand, multiplying the first equation in (3.14) by u_{xx} and integrating by parts again, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\|^2_{H^1} &= -\frac{3}{2}\int_{\mathbb{R}} u_x(u_x^2+u_{xx}^2)dx - \int_{\mathbb{R}}(\frac{\omega_1}{\alpha^2}u^2u_x + \frac{\omega_2}{\alpha^3}u^3u_x)u_{xx}\,dx \\ &= -\frac{3}{2}\int_{\mathbb{R}} u_x(u_x^2+u_{xx}^2)dx + \int_{\mathbb{R}}\big|\frac{\omega_1}{2\alpha^2}u^2 + \frac{\omega_2}{2\alpha^3}u^3\big|(u_x^2+u_{xx}^2)\,dx.\end{aligned} \tag{3.6.4}$$

Assume that $T^*_{u_0} < +\infty$ and there exists $M > 0$ such that

$$u_x(t,x) \geq -M, \quad \forall\,(t,\,x) \in [0, T^*_{u_0}) \times \mathbb{R}. \tag{3.6.5}$$

It then follows from (3.6.2), (3.6.3), and (3.6.4) that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}(u^2+2u_x^2+u_{xx}^2)dx &\leq (\frac{3}{2}M + \frac{|\omega_1|}{2\alpha^2}\|u\|^2_{L^\infty} + \frac{|\omega_2|}{2|\alpha|^3}\|u\|^3_{L^\infty})\int_{\mathbb{R}}(u_x^2+u_{xx}^2)\,dx \\ &\leq C(1+M+\|u\|^3_{H^1})\int_{\mathbb{R}}(u_x^2+u_{xx}^2)\,dx,\end{aligned} \tag{3.6.6}$$

where we used the Sobolev embedding theorem $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ (with $s > \frac{1}{2}$) in the last inequaity. Applying Gronwall's inequality to (3.6.6) yields for every $t \in [0, T^*_{u_0})$

$$\|u(t)\|^2_{H^2} \leq 2\|u_0\|^2_{H^2(\mathbb{R})} e^{Ct(1+M+\|u_0\|^3_{H^1})} \leq 2\|u_0\|^2_{H^2(\mathbb{R})} e^{CT^*_{u_0}(1+M+\|u_0\|^3_{H^1})}. \tag{3.6.7}$$

Differentiating the first equation in (3.14) with respect to x, and multiplying the result equation by u_{xxx}, then integrating by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (u^2_{xx} + u^2_{xxx})\,dx \\
&= -\frac{15}{2}\int_{\mathbb{R}} u_x u^2_{xx} dx - \frac{5}{2}\int_{\mathbb{R}} u_x u^2_{xxx} dx - \int_{\mathbb{R}} (\frac{\omega_1}{\alpha^2} u^2 u_x + \frac{\omega_2}{\alpha^3} u^3 u_x)_x u_{xxx}\,dx \\
&\leq C(1 + M + \|u\|^3_{L^\infty}) \int_{\mathbb{R}} (u^2_{xx} + u^2_{xxx})\,dx + C(\|u\|^2_{L^\infty} + \|u\|^4_{L^\infty})\|u_x\|^4_{L^4},
\end{aligned}$$

where we have used the assumption (3.6.5), which follows from the Sobolev embedding theorem and the interpolation inequality $\|f\|_{L^4(\mathbb{R})} \leq C\|f\|^{\frac{3}{4}}_{L^2(\mathbb{R})}\|f_x\|^{\frac{1}{4}}_{L^2(\mathbb{R})}$ that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} (u^2_{xx} + u^2_{xxx})\,dx &\leq C(1 + M + \|u_0\|^3_{H^1}) \int_{\mathbb{R}} (u^2_{xx} + u^2_{xxx})\,dx \\
&\quad + C\|u_0\|^5_{H^1}(1 + \|u_0\|^2_{H^1})\|u_{xx}\|_{L^2} \\
&\leq C(1 + M + \|u_0\|^{14}_{H^1}) \int_{\mathbb{R}} (u^2_{xx} + u^2_{xxx})\,dx.
\end{aligned}$$

Hence, Gronwall's inequality applied implies that for every $t \in [0, T^*_{u_0})$

$$\int_{\mathbb{R}} (u^2_{xx} + u^2_{xxx})dx \leq e^{C(1+M+\|u_0\|^{14}_{H^1})T^*_{u_0}} \int_{\mathbb{R}} (u^2_{0xx} + u^2_{0xxx})dx,$$

which, together with (3.6.7), yields that for every $t \in [0, T^*_{u_0})$,

$$\|u(t)\|^2_{H^3(\mathbb{R})} \leq 3\|u_0\|^2_{H^3(\mathbb{R})} e^{C(1+M+\|u_0\|^{14}_{H^1})T^*_{u_0}}.$$

This contradicts the assumption the maximal existence time $T^*_{u_0} < +\infty$.

Conversely, the Sobolev embedding theorem $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ (with $s > \frac{1}{2}$) implies that if (3.6.1) holds, the corresponding solution blows up in finite time, which completes the proof of Theorem 3.6.1. $\square$

Recall the R-CH equation (3.1), namely,

$$u_t + uu_x + \frac{\beta_0}{\beta} u_x + p_x * \left(\left(c - \frac{\beta_0}{\beta} \right) u + u^2 + \frac{1}{2} u_x^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \right) = 0,$$

where $p = \frac{1}{2} e^{-|x|}$. The wave breaking phenomena could be now illustrated by choosing certain the initial data.

Theorem 3.6.2 (Wave breaking data)**.** *Suppose $u_0 \in H^s$ with $s > 3/2$. Let $T > 0$ be the maximal time of existence of the corresponding solution $u(t, x)$ to* (3.1) *with the initial data u_0. Assume these is $x_0 \in \mathbb{R}$ such that*

$$u_{0,x}(x_0) < - \left| u_0(x_0) - \frac{1}{2} \left(\frac{\beta_0}{\beta} - c \right) \right| - \sqrt{2} C_0,$$

where $C_0 > 0$ is defined by

$$C_0^2 = \frac{|\omega_1|}{2\alpha^2} E_0^{\frac{3}{2}} + \frac{|\omega_2|}{2\alpha^3} E_0^2, \tag{3.6.8}$$

and

$$E_0 = \frac{1}{2} \int_{\mathbb{R}} \left(u_0^2 + (\partial_x u_0)^2 \right) dx.$$

Then the solution $u(t, x)$ breaks down at the time

$$T \leq \frac{2}{\sqrt{u_{0,x}^2(x_0) - \left(u_0(x_0) - \frac{1}{2} \left(\frac{\beta_0}{\beta} - c \right) \right)^2} - \sqrt{2} C_0}.$$

Remark 3.6.1. *In the case of the rotation frequency $\Omega = 0$, or the wave speed $c = 1$, the corresponding constant C_0 in* (3.6.8) *must be zero, because the parameters ω_1 and ω_2 vanish. The assumption on the wave breaking is then back to the case of the classical CH equation.*

Proof. Applying the translation $u(t, x) \mapsto u(t, x - \frac{\beta_0}{\beta} t)$ to equation (3.1) yields the equation in the form,

$$u_t + uu_x + p_x * \left(\left(c - \frac{\beta_0}{\beta} \right) u + u^2 + \frac{1}{2} u_x^2 + \frac{\omega_1}{3\alpha^2} u^3 + \frac{\omega_2}{4\alpha^3} u^4 \right) = 0. \tag{3.6.9}$$

Taking the derivative ∂_x to (3.6.9), we have

$$\begin{aligned} u_{xt} + uu_{xx} =& -\frac{1}{2}u_x^2 + u^2 + \left(c - \frac{\beta_0}{\beta}\right)u + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ & - p * \left(\left(c - \frac{\beta_0}{\beta}\right)u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right). \end{aligned} \tag{3.6.10}$$

We introduce the associated Lagrangian scales of (3.6.9) as

$$\begin{cases} \frac{\partial q}{\partial t} = u(t,q), & 0 < t < T, \\ q(0,x) = x, & x \in \mathbb{R}, \end{cases}$$

where $u \in C^1([0,T), H^{s-1})$ is the solution to equation (3.6.9) with initial data $u_0 \in H^s$, $s > 3/2$. Along with the trajectory of $q(t,x_))$, (3.6.9) and (3.6.10) become

$$\begin{aligned} \frac{\partial u(t,q)}{\partial t} &= -p_x * \left(\left(c - \frac{\beta_0}{\beta}\right)u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right), \\ \frac{\partial u_x(t,q)}{\partial t} &= -\frac{1}{2}u_x^2 + u^2 + \left(c - \frac{\beta_0}{\beta}\right)u + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\ & \quad - p * \left(\left(c - \frac{\beta_0}{\beta}\right)u + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right). \end{aligned}$$

Denote now at $(t, q(t,x_0))$,

$$M(t) = u(t,q) - \frac{k}{2} - u_x(t,q) \quad \text{and} \quad N(t) = u(t,q) - \frac{k}{2} + u_x(t,q),$$

where $k = \frac{\beta_0}{\beta} - c$. Recall the two convolution operators p_+, p_- as

$$\begin{aligned} p_+ * f(x) &= \frac{e^{-x}}{2}\int_{-\infty}^{x} e^y f(y)dy, \\ p_- * f(x) &= \frac{e^x}{2}\int_x^{\infty} e^{-y} f(y)dy \end{aligned}$$

and the relation

$$p = p_+ + p_-, \qquad p_x = p_- - p_+.$$

Applying [6, Lemma 3.1 (1)] with $m = -k^2/4$ and $K = 1$ we have the following convolution estimates

$$p_\pm * \left(u^2 - ku + \frac{1}{2}u_x^2\right) \geq \frac{1}{4}\left(u^2 - ku - \frac{k^2}{4}\right).$$

It then follows that at $(t, q(t, x_0))$,

$$\begin{aligned}
\frac{\partial M}{\partial t} =& \frac{1}{2}u_x^2 - u^2 + ku - \frac{\omega_1}{3\alpha^2}u^3 - \frac{\omega_2}{4\alpha^3}u^4 \\
&+ 2p_+ * \left(-ku + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right) \\
\geq& \frac{1}{2}\left(u_x^2 - \left(u - \frac{k}{2}\right)^2\right) - \frac{\omega_1}{3\alpha^2}u^3 - \frac{\omega_2}{4\alpha^3}u^4 + 2p_+ * \left(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right) \\
=& -\frac{1}{2}MN - \frac{\omega_1}{3\alpha^2}u^3 - \frac{\omega_2}{4\alpha^3}u^4 + 2p_+ * \left(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right) \\
\frac{\partial N}{\partial t} =& -\frac{1}{2}u_x^2 + u^2 - ku + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \\
&- 2p_- * \left(-ku + u^2 + \frac{1}{2}u_x^2 + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right) \\
\leq& -\frac{1}{2}\left(u_x^2 - \left(u - \frac{k}{2}\right)^2\right) + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 - 2p_- * \left(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right) \\
=& \frac{1}{2}MN + \frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 - 2p_- * \left(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right)
\end{aligned}$$

The terms with ω_1 and ω_2 in the right sides of the above estimates can be bounded by

$$\begin{aligned}
&\left|\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4 \mp 2p_\pm * \left(\frac{\omega_1}{3\alpha^2}u^3 + \frac{\omega_2}{4\alpha^3}u^4\right)\right| \\
&\leq \frac{|\omega_1|}{3\alpha^2}\|u\|_{L^\infty}^3 + \frac{|\omega_2|}{4\alpha^3}\|u\|_{L^\infty}^4 + \|u\|_{L^\infty}\left(\frac{|\omega_1|}{3\alpha^2}\|u\|_{L^2}^2\right) + \|u\|_{L^\infty}^2\left(\frac{|\omega_2|}{4\alpha^3}\|u\|_{L^2}^2\right) \\
&\leq \frac{|\omega_1|}{2\alpha^2}E_0^{\frac{3}{2}} + \frac{|\omega_2|}{2\alpha^3}E_0^2 = C_0^2 > 0,
\end{aligned}$$

where use has been made of the fact that

$$\|p_\pm\|_{L^\infty} = \frac{1}{2}, \quad \|p_\pm\|_{L^2} = \frac{1}{2\sqrt{2}}.$$

In consequence, we have

$$\begin{cases} \frac{dM}{dt} \geq -\frac{1}{2}MN - C_0^2, \\ \frac{dN}{dt} \leq \frac{1}{2}MN + C_0^2. \end{cases} \tag{3.6.11}$$

By the assumptions on $u_0(x_0)$, it is easy to see that

$$M(0) = u_0(x_0) - \frac{k}{2} - u_{0,x}(x_0) > 0,\ N(0) = u_0(x_0) - \frac{k}{2} + u_{0,x}(x_0) < 0,\ \frac{1}{2}M(0)N(0) + C_0^2 < 0.$$

By the continuity of $M(t)$ and $N(t)$, it then ensures that

$$\frac{dM}{dt} > 0, \quad \frac{dN}{dt} < 0, \quad \forall t \in [0, T).$$

This in turn implies that

$$M(t) > M(0) > 0, \quad N(t) < N(0) < 0, \quad \forall t \in [0, T).$$

Let $h(t) = \sqrt{-M(t)N(t)}$. It then follows from (3.6.11) that

$$\begin{aligned}\frac{dh}{dt} = \frac{-M'(t)N(t) - M(t)N'(t)}{2h} &\geq \frac{\left(-\frac{1}{2}MN - C_0^2\right)(-N) - M\left(\frac{1}{2}MN + C_0^2\right)}{2h} \\ &= \frac{M-N}{2h}\left(-\frac{1}{2}MN - C_0^2\right).\end{aligned}$$

Using the estimate $\frac{M-N}{2h} \geq 1$ and the fact that $h + \sqrt{2}C_0 > h - \sqrt{2}C_0 > 0$, we obtain the following differential inequalities

$$\frac{dh}{dt} \geq -\frac{1}{2}MN - C_0^2 = \frac{1}{2}(h - \sqrt{2}C_0)(h + \sqrt{2}C_0) \geq \frac{1}{2}(h - \sqrt{2}C_0)^2.$$

Solving this inequality gives

$$t \leq \frac{2}{\sqrt{u_{0,x}(x_0)^2 - (u_0(x_0) - \frac{k}{2})^2} - \sqrt{2}C_0} < \infty.$$

This in turn implies there exists $T < \infty$, such that

$$\liminf_{t \uparrow T_{u_0}, x \in \mathbb{R}} \partial_x u(t, x) = -\infty,$$

the desired result as indicated in Theorem 3.6.2. □

Remark 3.6.2. *Returning to the original scale, our assumption for the blow-up phenomena becomes*

$$\sqrt{\beta\mu}\, u_{0,x}(\sqrt{\beta\mu}x_0) + \left| u_0(\sqrt{\beta\mu}x_0) - \frac{1}{2\alpha\varepsilon}\left(\frac{\beta_0}{\beta} - c\right) \right| < -\frac{\sqrt{2}}{\alpha\varepsilon}C_1.$$

Note that when Ω increases, α and β decrease. It is then observed that with effect of the Earth rotation, a worse initial data $u_0(x_0)$ are required to make the breaking wave happen. On the other hand, in the original scale, we have

$$T \leq \frac{2}{\alpha\varepsilon\left(\sqrt{\beta\mu u_{0,x}^2(\sqrt{\beta\mu}x_0) - \left(u_0(\sqrt{\beta\mu}x_0) - \frac{1}{2\alpha\varepsilon}(\frac{\beta_0}{\beta} - c)\right)^2} - \frac{\sqrt{2}}{\alpha\varepsilon}C_1\right)}$$

where

$$C_1^2 = \frac{|\omega_1|\alpha\varepsilon^3}{2}E^{\frac{3}{2}} + \frac{|\omega_2|\varepsilon^2}{2\alpha}E^2 \quad \textit{with} \quad E(u_0) = \frac{1}{\alpha^2\varepsilon^2}E_0(\alpha\varepsilon u_0(\sqrt{\beta\mu}x_0)),$$

which also implies that a longer time is required for wave to break down when effect of the Earth rotation is considered.

REFERENCES

[1] M. Ablowitz and P. Clarkson, Solitons, nonlinear evolution equations and inverse scattering, *Cambridge University Press*, 2004.

[2] C. Amick and J. Toland, On solitary waves of finite amplitude, *Arch. Rat. Mech. Anal.,* **76** (1981), 9-95.

[3] T. Benjamin, J. Bona, and J. Mahony, Model equations for long waves in non-linear dispersive media, *Phil. Trans. Roy. Soc. Lond. A,* **272** (1972), 47-78.

[4] T. Benjamin and P. Olver, Hamiltonian structure, symmetries and conservation laws for water waves, *J. Fluid Mech.,* **125** (1982), 137-185.

[5] L. Brandolese, Local-in-space criteria for blowup in shallow water and dispersive rod equations, *Comm. Math. Phys.*, **330** (2014), 401–414.

[6] L. Brandolese and M. F. Cortez, Blowup issues for a class of nonlinear dispersive wave equations, *J. Differential Equations*, **256** (2014), 3981-3998.

[7] J. L. Bona, T. Colin, and D. Lannes, Long Wave Approximations for Water Waves, *Arch. Rational Mech. Anal.,* **178** (2005) 373-410.

[8] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.,* **71** (1993), 1661-1664.

[9] M. Chen, G. Gui, and Y. Liu, On a shallow-water approximation to the Green-Naghdi equations with the Coriolis effect, submitted, 2016.

[10] A. Constantin, On the modelling of Equatorial waves, *Geophys. Res. Lett.*, **39** (2012) L05602.

[11] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations, *Archive for Rational Mechanics and Analysis*, **192.1** (2009) 165-186.

[12] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, **181** (1998), 229–243.

[13] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations, *Arch. Ration. Mech. Anal.*, **192** (2009) 165-186.

[14] W. Craig, An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits, *Comm. Partial Differential Equations*, **10** (1985) 787-1003.

[15] B. Cushman-Roisin, and J M. Beckers, Introduction to geophysical fluid dynamics: physical and numerical aspects, *Comm. Academic Press*, (2011).

[16] R. Danchin, A few remarks on the Camassa-Holm equation, *Differential Integral Equations*, **14** (2001) 953-988.

[17] D. V. Eguiguren, The rains of Piura, *Boletn de la Sociedad Geogrfica de Lima*, **4** (1894) 241-258.

[18] R.L.J. Ellery, H.C. Russell, and C. Todd, Droughts in Australia: Their causes, duration, and effect: The views of three government astronomers, *The Australasian,* 29 December (1988), 1455-1456.

[19] B. Emily, March 2015 ENSO discussion: El Niño is here, *https://www.climate.gov/news-features/blogs/enso/march-2015-enso-discussion-el-niño-here,* March 5, 2015, Retrieved 20 April 2017.

[20] B. Fuchssteiner and A. S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Physica D,* **4** (1981/1982), 47-66.

[21] I. Gallagher and L. Saint-Raymond, On the influence of the Earth's rotation on geophysical flows, *Handbook of Mathematical Fluid Mechanics*, **4** (2007) 201-329.

[22] C. S. Gardner, M. D. Kruskal, and R. Miura, Korteweg-de Vries equation and generalizations, II. Existence of conservation laws and constants of motion, *J. Math. Phys.*, **9** (8) (1968) 1204-1209.

[23] G. Gui, Y. Liu, and T. Luo, Asymptotic analysis on the modelling of the shallow-water waves with the Coriolis effect, submitted, 2016.

[24] T. Izumo, The equatorial current, meridional overturning circulation, and their roles in mass and heat exchanges during the El Niño events in the tropical Pacific ocean, *Ocean Dyn.*, (2005) 110-123.

[25] Japan Meteorological Agency, Historical El Niño and La Niña Events, *http://ds.data.jma.go.jp/tcc/tcc/products/elnino/ensoevents.html*, Retrieved 20 April 2017

[26] R. S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *Journal of Fluid Mechanics*, **455** (2002) 63-82.

[27] R. S. Johnson, A Modern Introduction to the Mathematical Theory of Water Waves, *Cambridge University Press*, **19** (1997), 24-31.

[28] D. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular channel, and on a new type of long stationary waves, *Phil. Mag.,* **39**, No. 5 (1895), 422-442.

[29] N. Lockyer and J.S. Lockyer, The behavior of the short-period atmospheric pressure variation over the Earth's surface, *Proceedings of the Royal Society of London,* **73** (1904), 457-470.

[30] H. P. Mckean, Breakdown of the Camassa-Holm equation, *Comm. Pure Appl. Math.,***LVII** (2004), 0416-0418.

[31] F. A. Pezet, The counter current "El Niño," on the coast of northern Peru, *Report of the Sixth International Geographical Congress,* held in London, (1895), 603-606.

[32] S. G. H. Philander, The equatorial undercurrent revisited, *Annu. Rev. Earth Planet Sci.,* **8**, (1980), 191-204.

[33] J. S. Russell, Report of the committee on waves, *British Assoc. Report,* 1838, 417-496.

[34] J. S. Russell, Report on Water Waves, *British Assoc. Report,* (1844), 311-390.

[35] G. G. Stokes, On the theory of oscillatory waves, *Cambridge Univ. Press, Cambridge, UK*, **I**, (1880), 197-229.

[36] J. Toland, On the existence of a wave of greatest height and Stokes' conjecture, *Proc. Roy. Soc. London A,* **363** (1978), 469-485.

[37] G. T. Walker, Correlation in seasonal variations of weather. IX. A further study of world weather, *Memoirs of the Indian Meteorological Department,* **24** (1973), 275-332.

[38] G. Whitham, Linear and nonlinear waves, *John Wiley and Sons, New York,* 1973.

[39] N. J. Zabusky and M. D. Kruskal Iteration of solitons in a collisionless plasma and the recurrence of initial states, *Phys Rev Lett.,* **15** (1965), 240-243.

BIOGRAPHICAL STATEMENT

Junwei Sun was born in Shanghai, China, in 1988. He received his B.S. and M.S. degrees from Shanghai University, Shanghai, in 2010 and 2013 respectively, both in Mathematics and Applied Mathematics. In 2016, he joined the doctoral program in the Department of Mathematics, University of Texas at Arlington. His current research interest are the mathematical questions of well-posedness, wave-breaking phenomena and the physical derivation of the water wave models. Apart from the research in partial differential equations, he is interested in Artificial Intelligence, Macro Economics and Physics. Also, he is concerned with math education, especially youth education.