Image Reconstruction from Incomplete Radon Data and Generalized Principal Component Analysis

by

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To my mother and my father

who set the example and who made me who I am.

To my wife Tiffany

who supported me all the way.

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ABSTRACT

Image Reconstruction from Incomplete Radon Data and Generalized Principal Component Analysis

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Image reconstruction in various types of tomography requires inversion of the Radon transform and its generalizations. While there are many stable and robust algorithms for such inversions from reasonably well sampled data, most of these algorithms fail when applied to limited view data. In the dissertation we develop a new method of stable reconstruction from limited view data for functions, whose support is a union of finitely many circles. Such images, among other things, are good approximations of tomograms of certain types of tumors in lungs. Our method is based on a modified version of GPCA (Generalized Principal Component Analysis) and some results from algebraic geometry.

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INTRODUCTION

1.1 Computerized Tomography

The *computerized axial tomography* (CAT or CT) is recovering internal images of a patient by using X-ray beams measurements. When an X-ray beam passes through the patient, the CT scanner measures the input and output intensities. Using these intensities and Beer's law, one can obtain the integrals of X-ray attenuation coefficient of the patient's body along various lines corresponding to the trajectories of the beams. These integrals are then used to recover the attenuation coefficient, which in its turn is used to generate an internal image of the patient. Below we describe this procedure in detail.

Suppose an X-ray passes through some medium located between the position x and the position $x + \triangle x$ (here x is a vector). Let $f(x)$ be the attenuation coefficient of the medium located at x and let $I(x)$ be the intensity of the beam at x. Then the *Beer's Law* states that the loss of intensity from x to $x + \triangle x$ is

$$\triangle I \approx -f(x) \times I(x) \times \triangle x \tag{1.1}$$

and as $|\triangle x| \to 0$, we have the following differential equation:

$$\frac{dI}{dx} = -f(x) \times I(x). \tag{1.2}$$

Integrating both sides of this equation we obtain

$$-\int_l \frac{dI}{I} = \int_l f ds, \tag{1.3}$$

where l is the line corresponding to the X-ray beam and ds is the length measure along that line.

Moving the X-ray source and receiver around the medium one can measure integrals of $f(x)$ along different lines.

1.2 The Radon Transform and Related Concepts

Definition 1.2.1. For $t \in \mathbb{R}$ and $\theta \in [0, 2\pi]$, denote by $l_{t,\theta}$ the line that passes through the point $(t\cos\theta, t\sin\theta)$ and is perpendicular to the unit vector $\overrightarrow{n} = \langle \cos\theta, \sin\theta \rangle$.

Figure 1.1. Line l using t and θ.

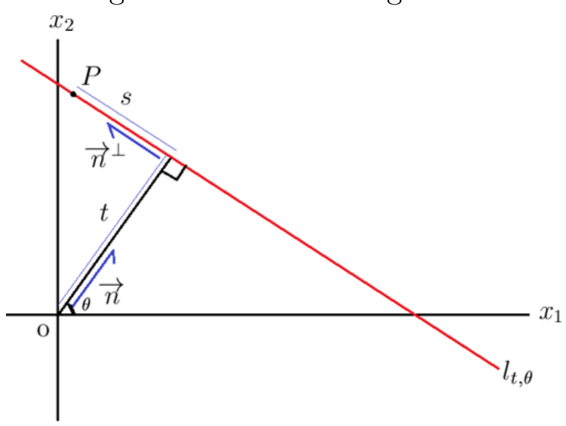
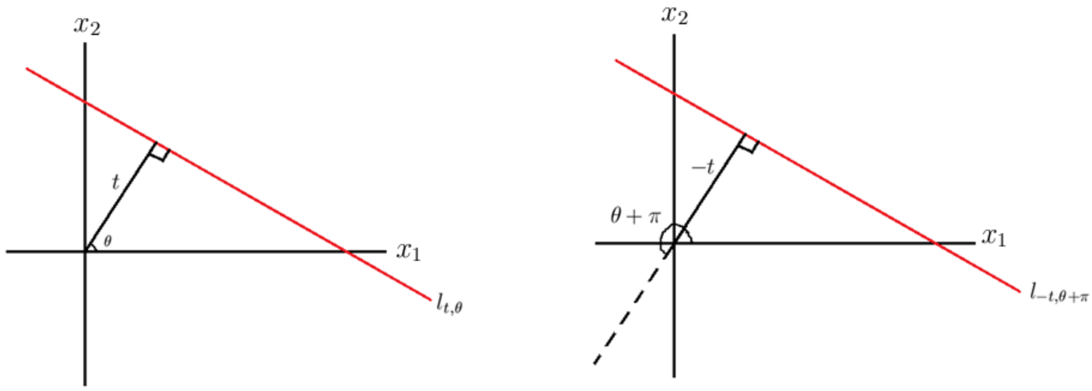


Figure 1.2. $l_{t,\theta}$ and $l_{-t,\theta+\pi}$.

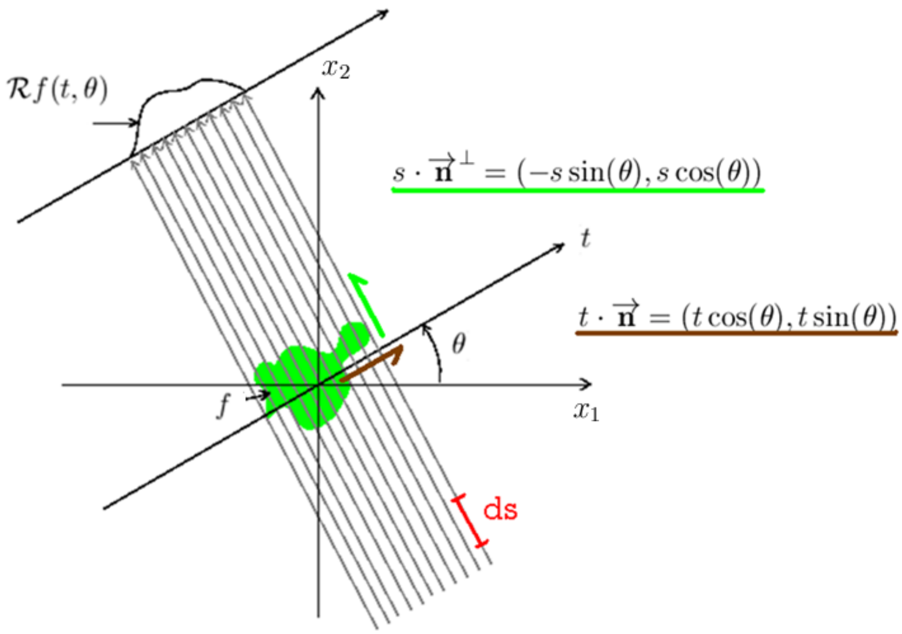


Using this definition, we parameterize each line by its signed distance t from the origin and the polar angle θ of the normal to the line. However, we have to keep in mind that, for each line there are two ways to parameterize it, $l_{t,\theta}$ and $l_{-t,\theta+\pi}$ (see Figure (1.2)).

Definition 1.2.2. For a given function f, whose domain is the plane, *the Radon transform* of f is defined, for real number t and $\theta \in [0, 2\pi]$ by

$$\mathcal{R}f(t,\theta) := \int_{l_{t,\theta}} f ds = \int_{s=-\infty}^{+\infty} f(t\cos(\theta) - s\sin(\theta), t\sin(\theta) + s\cos(\theta))ds. \quad (1.4)$$

Figure 1.3. Radon transform of f.



Notice from Figures (1.2) and (1.3), that any point P on $l_{t,\theta}$ can be represented in the form of $t \cdot \overrightarrow{n} + s \cdot \overrightarrow{n}^{\perp} = (t\cos(\theta) - s\sin(\theta), t\sin(\theta) + s\cos(\theta))$ by choosing appropriate value of $s \in (-\infty, +\infty)$, where $\overrightarrow{n}^{\perp}$ is the normal unit vector to $\overrightarrow{n}$.

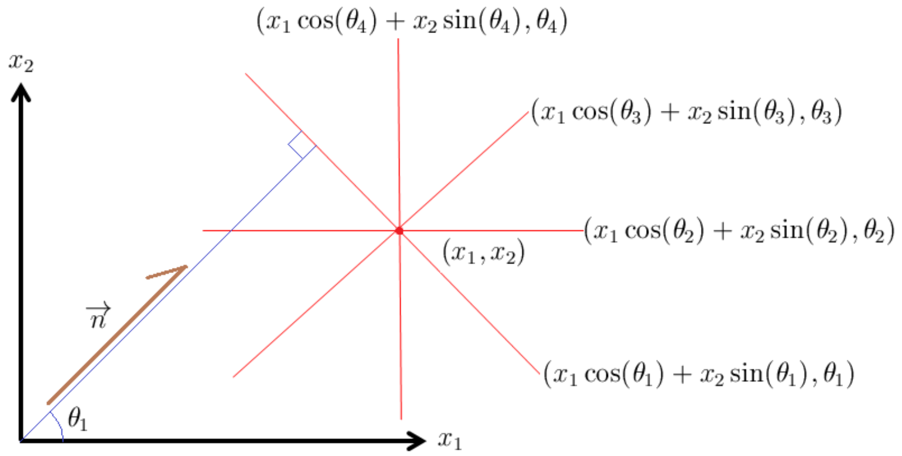
Using this definition, the data measured by CAT scanner corresponds to the Radon transform of the X-ray absorption coefficient. Hence, to recover the absorption coefficient and generate images using CAT, one needs to find an inverse of the Radon transform.

There are many ways to invert the Radon transform. One of the standard options is the so-called filtered back projection formula. To explain that formula we first need to define the concept of back projection.

Definition 1.2.3. Let $h = h(t,\theta)$ be a function defined on the set of lines in the plane, i.e. a function with independent variables $t \in \mathbb{R}$ and $\theta \in [0, 2\pi]$. The *back projection* of h at the point (x_1, x_2) is defined by

$$\mathcal{B}h(x_1, x_2) := \frac{1}{\pi} \int_{\theta=0}^{\pi} h(x_1 \cos(\theta) + x_2 \sin(\theta), \theta) d\theta. \tag{1.5}$$

Figure 1.4. Back projection of h at (x_1, x_2).



From Figure (1.4) it is easy to notice, that $\mathcal{B}h(x_1, x_2)$ is the average value of $h(x_1, x_2)$ over the set of all lines that are passing through the point (x_1, x_2).

Last we introduce the Fourier transform and inverse Fourier transform which are used in the inversion of the Radon transform.

Definition 1.2.4. For a given function F such that $\int_{-\infty}^{\infty} |F(t)|dt < \infty$, the *Fourier transform* of f is defined, for each real number ω, by

$$\mathcal{F}F(\omega) := \int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt \tag{1.6}$$

Definition 1.2.5. For a function G for which $\int_{-\infty}^{\infty} |G(\omega)|d\omega < \infty$, the *inverse Fourier transform* of G is defined, for each real number t, by

$$\mathcal{F}^{-1}G(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)e^{i\omega t}d\omega \tag{1.7}$$

1.3 The Filtered Back Projection and Local Tomography Formulas

There are many different formulas for inversion of the Radon transform. The most famous and commonly used one is the so-called *Filtered Back Projection (FBP) formula.* FBP is formed by the Radon transform, Fourier transform, inverse Fourier transform and back projection as follows:

Theorem 1.3.1. *[6][9] For a suitable function f defined in the plane and real numbers x_1 and x_2,*

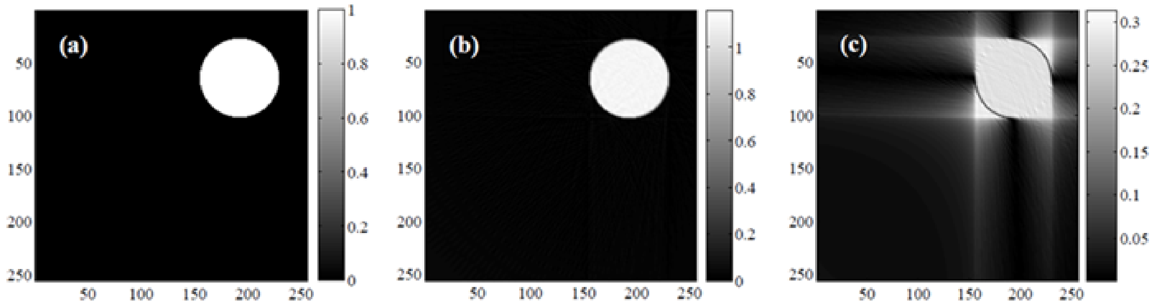
$$f(x_1, x_2) = \frac{1}{2}\mathcal{B}[\mathcal{F}^{-1}[|\omega|\mathcal{F}(\mathcal{R}f)(\omega, \theta)]](x_1, x_2). \tag{1.8}$$

Remark 1. Here the Fourier transform is applied to $\mathcal{R}f$ with respect to the first variable t, while the second variable θ is treated as a parameter. Similarly, the inverse Fourier is applied with respect to the first variable.

Remark 2. Notice, that before applying the back projection operator one has to switch from polar to Cartesian coordinates.

Remark 3. In the above theorem, the expression "suitable function" means that the integrals in the Radon, Fourier and inverse Fourier transforms are convergent.

Figure 1.5. [2] Image reconstruction using the filtered back projection formula. (a) is the original image $f(x)$, (b) is the reconstruction of $f(x)$ using discretized version of full data $\theta \in [0, 2\pi]$ and $t \in [-\sqrt{2}, \sqrt{2}]$ and (c) is the reconstruction of $f(x)$ using discretized partial data $\theta \in [0, \frac{\pi}{2}]$ and $t \in [-\sqrt{2}, \sqrt{2}]$.



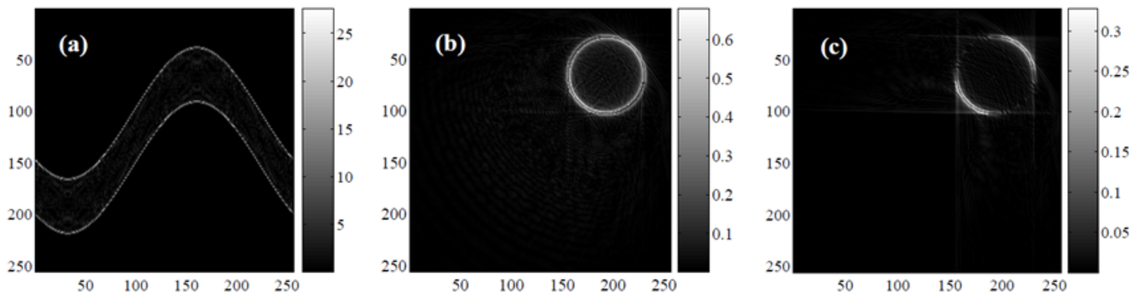
Filtered back projection formula can recover the image from full Radon data, i.e. the Radon transform of the function should be known for a reasonably well sampled set of $\theta \in [0, 2\pi]$ and $t \in [-\sqrt{2}, \sqrt{2}]$. With partial data, e.g. when the Radon transform is not known for a large interval of angles θ, (see Figure (1.5) part (c)), the back projection formula will give us an image with high distortion and it will be far different from the original image (e.g. see [11]). Therefore, we need another method for the image reconstruction in the case, when only limited data is available. Such cases are common in medical imaging, e.g. due to presence of medical implants in the body, which obstruct measurements. In some other cases one may deliberately avoid sending x-rays through certain parts of the body sensitive to radiation.

In many applications typically it suffices to locate the boundaries (discontinuities) of $f(x)$ instead of recovering the whole $f(x)$. One possible method to recover the boundary of $f(x)$ is the use of the so-called *local tomography formula*

$$\Lambda f(x_1, x_2) = -\frac{1}{2}\mathcal{B}[\frac{\partial^2(\mathcal{R}f)}{\partial t^2}(t, \theta)](x_1, x_2). \tag{1.9}$$

It is well known, that while Λf is not equal to f, it is however a fairly good approximation of f (e.g. see [2, 8, 10, 12]). Namely, when applied to full Radon data, Λf has singularities exactly in the same locations where f does, hence it correctly catches the major features of the image, including the shapes and edges. Moreover, the edges in Λf are more emphasized in comparison to those of f.

Figure 1.6. [2] Image reconstruction using local tomography formula.



In Figure (1.6) (a) is $-\frac{\partial^2(\mathcal{R}f)}{\partial t^2}$ which emphasizes the boundary of $\mathcal{R}f$. (b) is the graph of $\Lambda f(x_1, x_2)$, when we use local tomography formula with full Radon data $\theta \in [0, 2\pi]$ and $t \in [-\sqrt{2}, \sqrt{2}]$. We can see the boundary of $f(x)$ is emphasized. (c) is the graph of $\Lambda f(x_1, x_2)$, when we use local tomography formula with partial Radon data $\theta \in [0, \frac{\pi}{2}]$ and $t \in [-\sqrt{2}, \sqrt{2}]$.

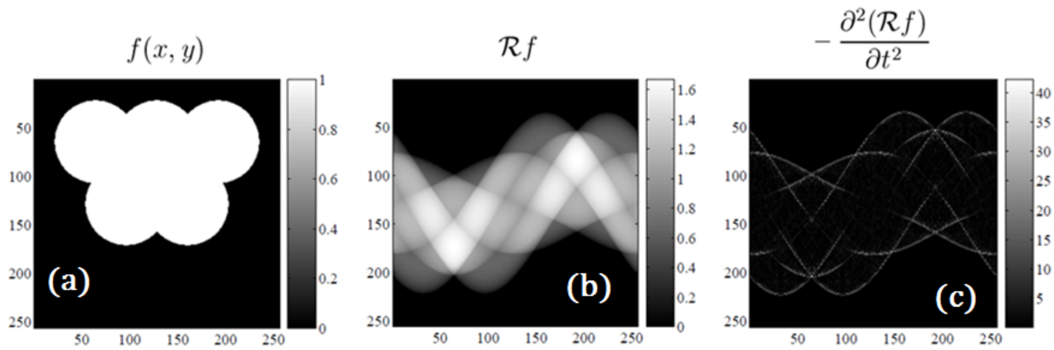
CHAPTER 2

FORMULATION OF THE PROBLEM AND MAIN RESULTS

2.1 Problems of Limited View Reconstruction of Nodular Images

One of the major problems in modern image reconstruction is related to usage of prior information to improve the inversion process. In this dissertation we study the problem of using the partial Radon data to recover so-called nodular images. These types of images are represented by characteristic functions of finite unions of circles, which among other things, are good approximations for certain types of tumors in lungs.

Figure 2.1. [2] (a) The original image of f, (b) the Radon transform of f, (c) the negative 2-nd derivative of $\mathcal{R}f$. It is easy to notice that in (b) the boundary of the sinogram does not stand out (has poor contrast), while in (c) the boundary of the sinogram is emphasized (has excellent contrast).



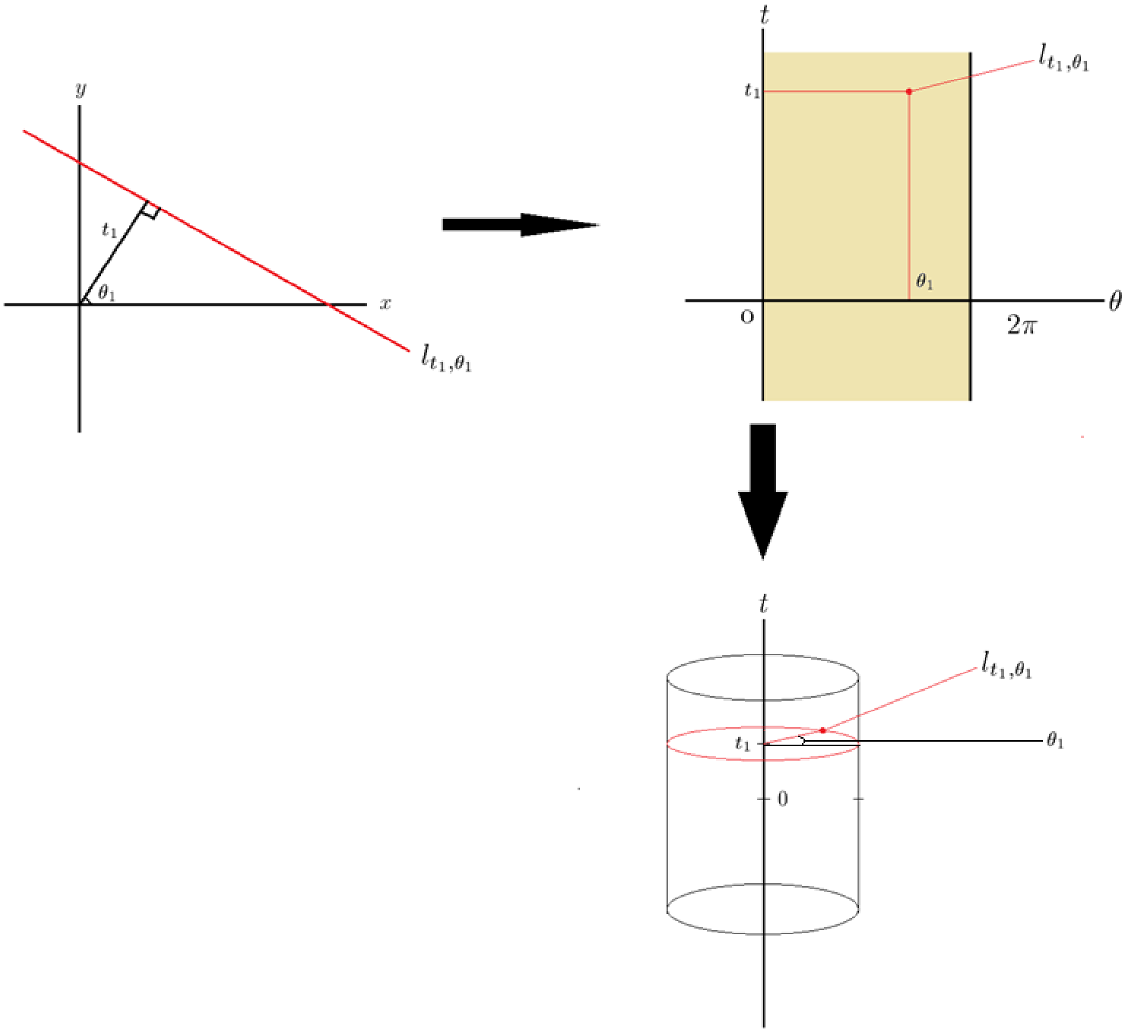
In the case of nodular images, even when using partial data, a part of the boundary of $f(x)$ can be reconstructed without much distortion. However, if we try to recover the boundary of $f(x)$ using sample points from $\Lambda f(x_1, x_2)$, it will require non-linear clustering and regression. Namely, the algorithm should be able to organize the points of the boundary of the domain into groups according to the circles to which they belong (non-linear clustering). Then after this separation has been accomplished, one would have to generate the circles from sample points corresponding to each one of them (non-linear regression). These tasks are nontrivial, unstable and very time consuming (e.g. see [1, 5] and the references there).

However, if instead of working in the image domain of (x_1, x_2) one does the work in the Radon data domain of (t, θ) these tasks can be handled in a linear fashion. First let us recall that the set of lines on the plane parameterized by (t, θ) can be naturally represented by points on a cylinder $[0, 2\pi] \times \mathbb{R}$ in $\mathbb{R}^3$.

Figure (2.2) shows the correspondence of a line $l_{t,\theta}$ in the (x_1, x_2) plane and a point on the cylinder $[0, 2\pi] \times \mathbb{R}$. We first match a line $l_{t,\theta}$ to a point (t, θ) on the (θ, t)-plane and then we "glue" the two vertical lines $t = 0$ and $t = 2\pi$. Then the point (θ, t) will be on a cylinder with new coordinates $(\cos\theta, \sin\theta, t)$. This cylinder has radius 1 and infinite height $t \in (-\infty, \infty)$. Since the line $l_{t,\theta}$ can be also represented by $l_{-t,\theta+\pi}$, one line $l_{t,\theta}$ will now correspond to two points on this cylinder.

Theorem 2.1.1. *[2] The set of lines tangentially touching a circle in the plane is represented on the manifold of lines by points of intersections of a cylinder and two parallel planes.*

Figure 2.2. Correspondence of a line on x_1x_2-plane and a point on the cylinder.



Proof. Consider a circle with a center $P(a,b)$ and radius r. Let $\omega = (a\cos\theta, b\sin\theta)$ be the unit normal vector to tangent line L, t is the distance from the origin to the line L (see Figure (2.3)). Then we have the following two equations.

$$r + \langle P, \omega \rangle \doteq r + a\cos\theta + b\sin\theta = t \tag{2.1}$$

when using (θ, t) for line L and

$$r - \langle P, \omega \rangle \doteq r - a\cos\theta - b\sin\theta = -t \tag{2.2}$$

when using $(\theta + \pi, -t)$ for line L.

Based on equations (2.1) and (2.2), we will have intersection of two parallel planes

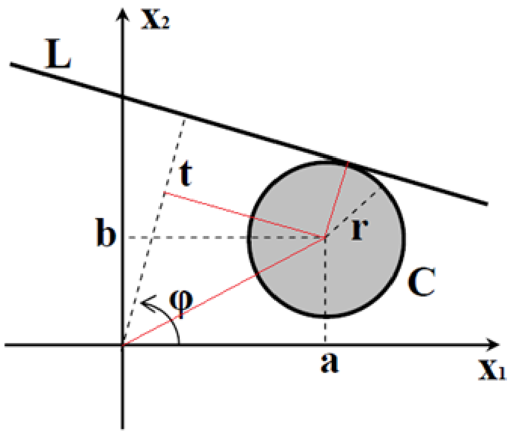
$$a\tilde{x} + b\tilde{y} - \tilde{z} = -r$$

and

$$a\tilde{x} + b\tilde{y} - \tilde{z} = r$$

and the cylinder $\tilde{x} = \cos\theta$, $\tilde{y} = \sin\theta$ and $\tilde{z} = t$. □

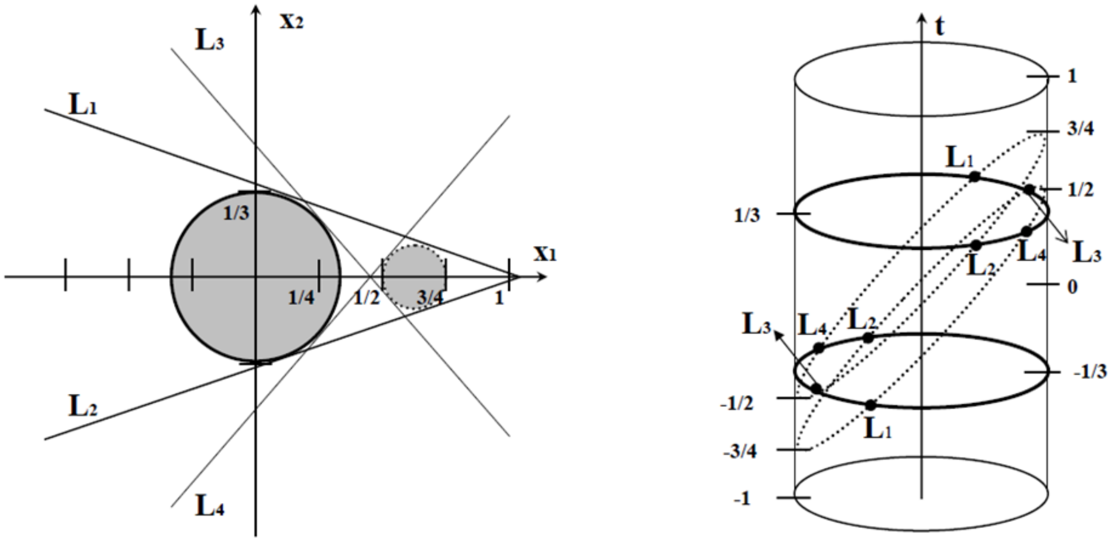
Figure 2.3. [2] Parametrization of lines tangentially touching a circle.



Based on Theorem (2.1.1), when we convert the lines tangent to a circle to points on the cylinder, those points are located on the intersection of the cylinder with two parallel planes. If we can recover an equation of one of these planes, then we will immediately get the parameters (a, b, r) of the circle, hence uniquely recover $f(x)$.

Figure (2.6) shows the process of reconstructing $f(x)$. We first use the points on the cylinder to reconstruct the equation of the plane. This is done using a modified version of the GPCA (Generalized Principal Component Analysis). Using Theorem

Figure 2.4. [2] Tangent lines of circles and representation on the cylinder.



(2.1.1) and the equations of parallel planes, we obtian the parameters (a, b, r) of the circle, hence recovering $f(x)$ on x_1x_2-plane.

In the case when $f(x)$ is a characteristic function of a union of more than one circle, we will have to recover multiple planes at the same time. This process will require linear regression and linear clustering.

2.2 Circular Radon Transform

Definition 2.2.1. For a given function f, whose domain is the plane, *the circular Radon transform* of f is defined, for real numbers $t \in [0, 2]$ and $\theta \in [0, 2\pi]$ by

$$\mathcal{R}_C f(t, \theta) := \int_{C_{t,\theta}} f\, ds, \tag{2.3}$$

where $C_{t,\theta}$ is the circle of radius t centered at $(\cos\theta, \sin\theta)$ and ds is the elementary arc length.

Figure 2.5. A tangent line corresponds to two points on the cylinder coordinate system.

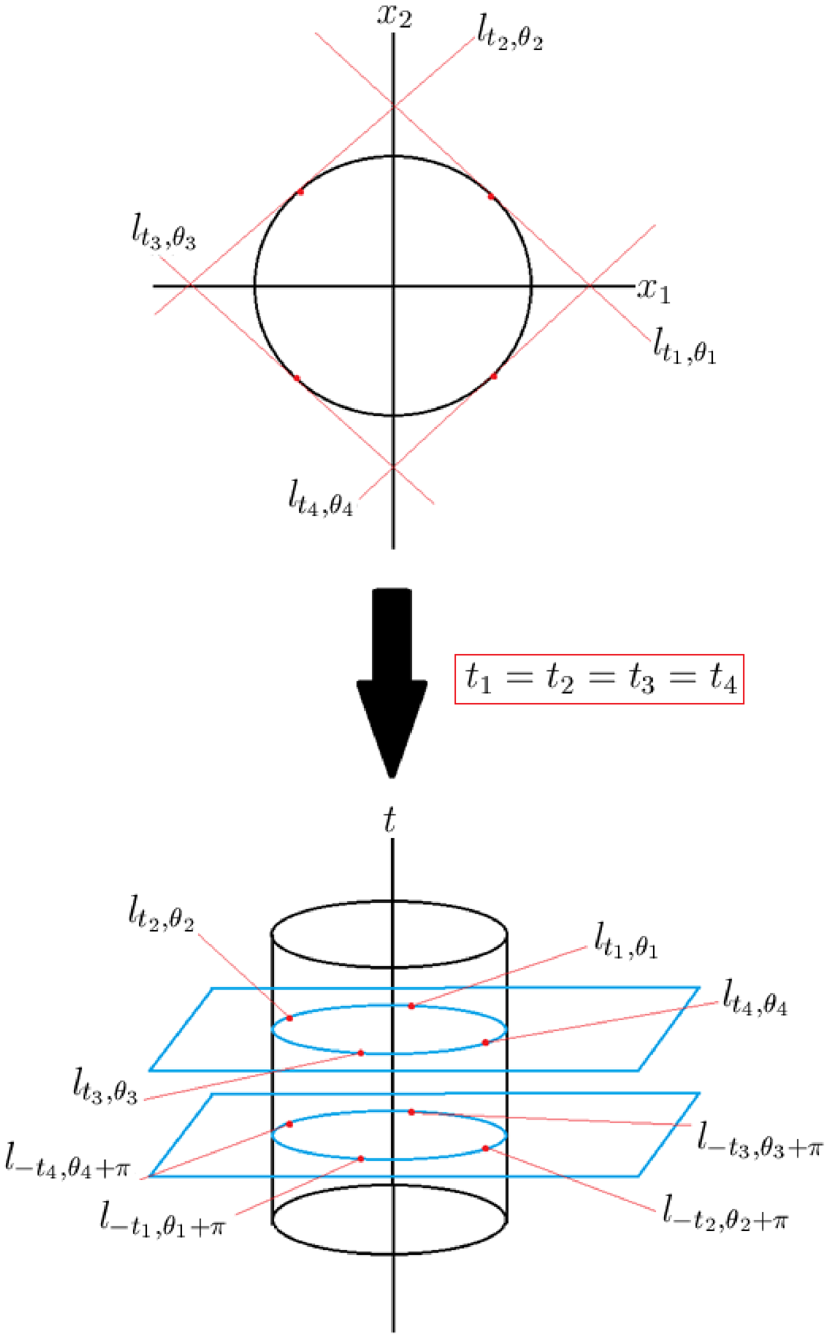
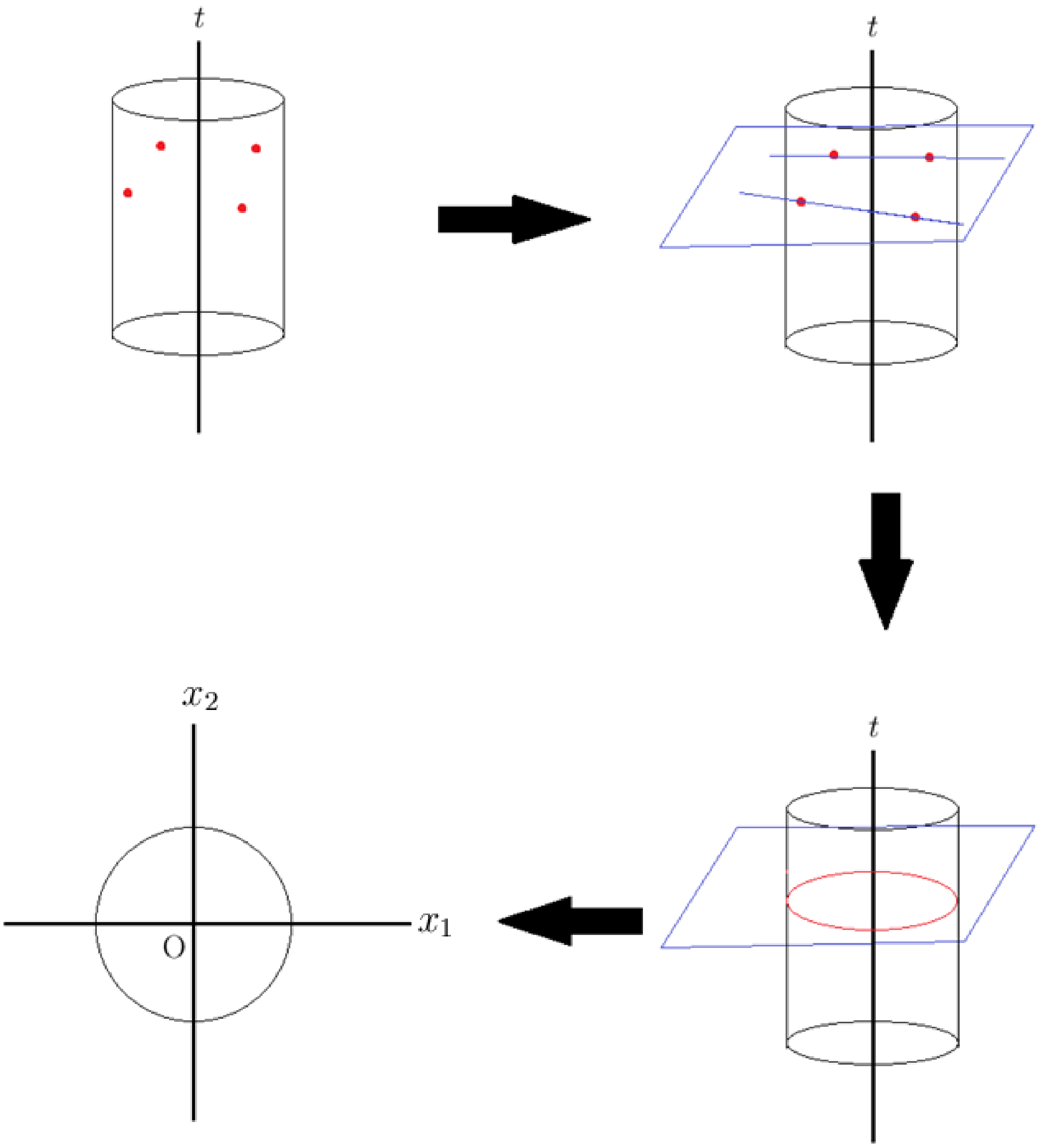


Figure 2.6. Reconstruction process.



Theorem 2.2.1. *Let S be the set of circles centered on the boundary of the unit disc, which tangentially touch another circle located inside the unit disc. Then S can be represented in a 4-dimensional Euclidean space by points located on intersections of 2 hyper-planes and a fixed smooth surface of co-dimension 2.*

Notice, that although these hyper-planes are not parallel, their intersection happens only at $\tilde{z} = 0$, i.e. for $t = 0$, which is outside of our region of interest. We assume that all phantoms are supported away from the unit circle.

Proof. Consider a circle P with a center (a, b) and radius r. Let θ be the angle of the unit vector to the center of the tangent circle C from x_1-axis. t is the radius of the circle C. (see Figure (2.7) and (2.8)). Then we have the following two equations.

$$(\cos\theta - a)^2 + (\sin\theta - b)^2 = (t + r)^2 \tag{2.4}$$

when using a circle C that does not contain circle P and

$$(\cos\theta - a)^2 + (\sin\theta - b)^2 = (t - r)^2 \tag{2.5}$$

when using a circle C that contains circle P.

Based on equations (2.4) and (2.5), we will have 2 hyper-planes that intersect with a fixed smooth surface of co-dimension 2 such as

$$-2a\tilde{x} - 2b\tilde{y} - \tilde{\gamma} + 2r\tilde{z} = r^2 - a^2 - b^2 - 1$$

and

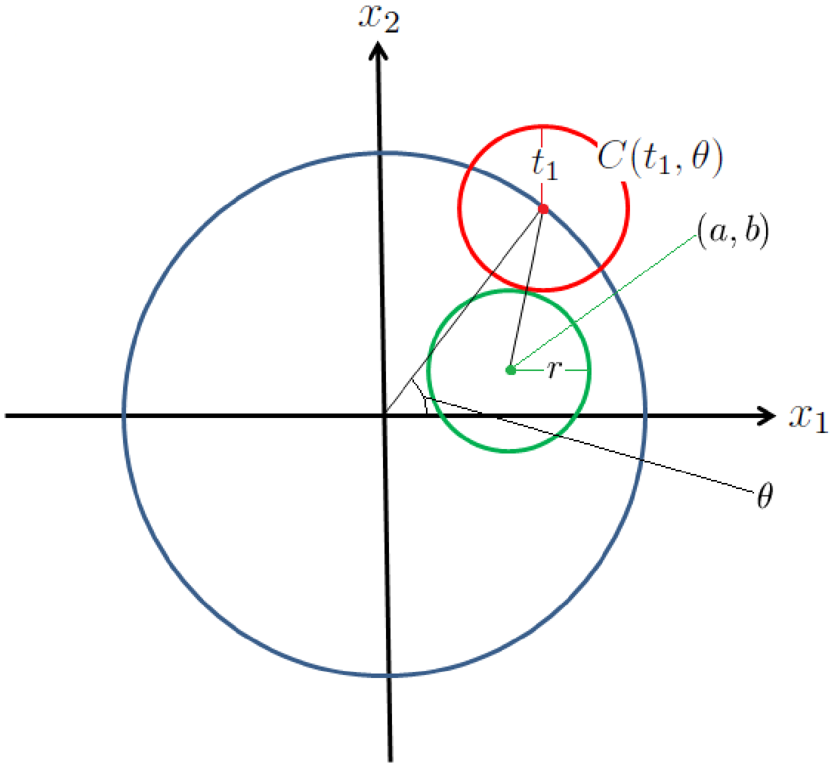
$$-2a\tilde{x} - 2b\tilde{y} - \tilde{\gamma} - 2r\tilde{z} = r^2 - a^2 - b^2 - 1$$

where $\tilde{x} = \cos\theta$, $\tilde{y} = \sin\theta$, $\tilde{z} = t$ and $\tilde{\gamma} = t^2$. □

2.3 Generalized Principal Component Analysis

In order to recover the equations of planes from the points on the cylinder, we use a modified version of the *Generalized Principal Component Analysis*(GPCA).

Figure 2.7. Circle C that does not contain circle P.



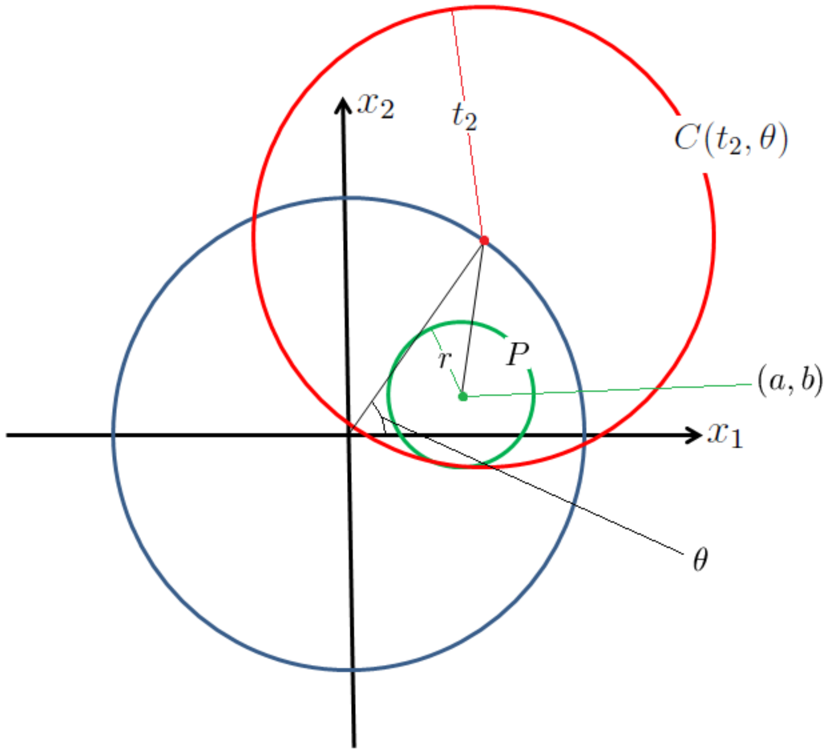
In the description of the standard GPCA below we closely follow the exposition of the material in dissertation [13]. At the same time we explain which parts of the standard method do not work for our problem and what modifications we introduce to overcome those limitations.

Suppose $X = \mathbf{x}^j \in \mathbb{R}^{K}{}_{j=1}^{N}$ is a collection of points on the cylinder. We will call x^j's as sample points. Let the (unknown) number of hyper-planes be $n > 1$ and S_i's denote these hyper-planes in $\mathbb{R}^3$ for $i = 1, \ldots, n$. GPCA consists of the three following steps.

1. Identify the number of subspaces n.

2. Identify a basis (or a set of principal components) for each hyper-plane S_i (or equivalently $S_i^{\perp}$).

Figure 2.8. Circle C that contains circle P.



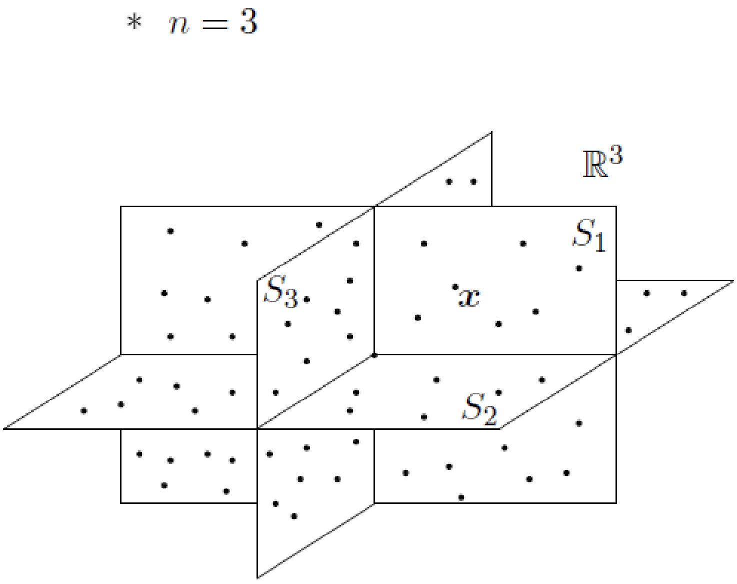
3. Group or segment the given N data points into the subspace(s) to which they belong.

Parts (1) and (3) are essentially equivalent to linear clustering, while (2) corresponds to linear regression.

To explain the essence of our method, in the text that follows we will consider a few examples, one with $K = 3$ and another with $K = 4$. For part (2) we will identify a normal vector $S_i^{\perp}$ for each hyper-plane S_i.

Figure (2.9) illustrates GPCA for 3 hyper-planes in $\mathbb{R}^3$.

Figure 2.9. [13] K=3 and n=3.



We will look at the details of three steps of GPCA in order to apply this in our case. First, finding number of hyperplanes. Let $\mathbf{b}_i$ be a normal vector for each hyper-plane S_i. For each S_i, we can describe using $\mathbf{b}_i$.

$$S_i = \{\mathbf{x} \in \mathbb{R}^K : \mathbf{b}_i^T \mathbf{x} = 0\}.$$

Therefore, if $\mathbf{x}$ is in one of the hyper-planes S_i, then it should satisfy the following

$$(\mathbf{b}_1^T \mathbf{x} = 0) \bigvee (\mathbf{b}_2^T \mathbf{x} = 0) \bigvee \cdots \bigvee (\mathbf{b}_n^T \mathbf{x} = 0)$$

or

$$p_n(\mathbf{x}) = \prod_{i=1}^{n} (\mathbf{b}_i^T \mathbf{x}) = 0.$$

Definition 2.3.1. [13] Given n and K, the Veronese map of degree n, $\nu_n : \mathbb{R}^K \to \mathbb{R}^{M_n}$, is defined as:

$$\nu_n : [x_1, x_2, \ldots, x_K]^T \mapsto [\ldots, \mathbf{x}^n, \ldots]^T, \tag{2.6}$$

where $\mathbf{x}^n$ is a monomial of the form $x_1^{n_1}x_2^{n_2}\cdots x_K^{n_K}$ with n chosen in the degree-lexicographic order and $M_n = \binom{n+K-1}{n}$.

Here we will give an order to the $p_n(\mathbf{x})$ using *Veronese map.*

$$p_n(\mathbf{x}) = \nu_n(\mathbf{x})^T\mathbf{c}_n = \sum c_{n_1,n_2,\ldots,n_K} x_1^{n_1}x_2^{n_2}\cdots x_K^{n_K} = 0.$$

We consider a matrix $L_n\mathbf{c}_n$ where j-th row is $p_n(\mathbf{x}^j)$.

$$L_n\mathbf{c}_n = \begin{bmatrix} \nu_n(\mathbf{x}^1)^T \\ \nu_n(\mathbf{x}^2)^T \\ \vdots \\ \nu_n(\mathbf{x}^N)^T \end{bmatrix} \mathbf{c}_n = \mathbf{0}, \tag{2.7}$$

where $L_n \in \mathbb{R}^{N\times M_n}$. Recovering the coefficients of $p_n(\mathbf{x})$, $c_{n_1,n_2,\ldots,n_K}$ is directly related to recovering the normal vectors $\mathbf{b}_i$'s. But before we recover $c_{n_1,n_2,\ldots,n_K}$ from $L_n\mathbf{c}_n = \mathbf{0}$, we need to know the number of hyperplanes, n. By applying the following lemma, we can derive n.

Lemma 2.3.1. *[13] Assume that a collection of $N \geq M_n - 1$ sample points $\{x^j\}_{j=1}^N$ on n different $K-1$ dimensional subspaces of $\mathbb{R}^K$ is given. Let $L_i \in \mathbb{R}^{N\times M_i}$ be the matrix defined in* (2.3)*, but computed with the Veronese map $\nu_i(\boldsymbol{x})$ of degree i. If the sample points are in general position and at least $K-1$ points correspond to each hyperplane, then:*

$$\operatorname{rank}(L_i) = \begin{cases} > M_i - 1, & i < n, \\ = M_i - 1, & i = n, \\ < M_i - 1, & i > n. \end{cases} \tag{2.8}$$

Here general position is when $N \geq \sum_{i=1}^n k_i$ sample points are in $\bigcup_{i=1}^n S_i$ and k_i points span S_i.

Once we find the number of hyper-planes, we will know which L_n to use. We can compute $\mathbf{c}_n$ from $L_n\mathbf{c}_n = \mathbf{0}$ and we should have a unique solution $\mathbf{c}_n$, if we can find

n that satisfies Lemma (2.3.1). The second step of GPCA is recovering the normal vectors $\mathbf{b}_i$'s from $\mathbf{c}_n$. We will first recover the last two entries of each vector $\mathbf{b}_i$ and then the remaining $K-2$ entries. We consider all the coefficients of $p_n(\mathbf{x})$ that are related to last two entries of $\mathbf{b}_i$'s which are last $n+1$ coefficients of $p_n(\mathbf{x})$ or entries of $\mathbf{c}_n$.

$$[c_{0,\ldots,0,n,0}, c_{0,\ldots,0,n-1,1}, \cdots, c_{0,\ldots,0,0,n}]^T \in \mathbb{R}^{n+1}.$$

Add all terms that use that coefficient then,

$$\sum c_{0,\ldots,0,n_{K-1},n_K} x_{K-1}^{n_{K-1}} x_K^{n_K} = \prod_{i=1}^{n} (b_{iK-1}x_{K-1} + b_{iK}x_K).$$

Where $n_{K-1} + n_K = n$. Here, we divide by x_K^n and let $t = \frac{x_{K-1}}{x_K}$ then we have

$$q_n(t) = c_{0,\ldots,0,n,0}t^n + c_{0,\ldots,0,n-1,1}t^{n-1} + \cdots + c_{0,\ldots,0,0,n} = \prod_{i=1}^{n} (b_{iK-1}t + b_K).$$

Based on the roots of $q_n(t)$, we can derive (b_{iK-1}, b_{iK})'s. Roots of $q_n(t)$ depend on the first coefficients of $q_n(t)$. There are three possible cases for the coefficient and for each we can find (b_{iK-1}, b_{iK})'s in the following way.

1. If $c_{0,\ldots,0,n,0} \neq 0$ then we will have n roots form $\frac{q_n(t)}{c_{0,\ldots,0,n,0}}$ and we can conclude as $(b_{iK-1}, b_{iK}) = (1, -t_i)$ for each root t_i where $i = 1, ldots, n$.

2. If $c_{0,\ldots,0,n,0} = 0$ then let $l+1$ to be the first coefficients of $q_n(t)$ that is not zero and t_i's are the roots of $q_n(t)$ divided by the $(l+1)$-st coefficient where $i = l+1, \ldots, n$. Then we conclude as $(b_{iK-1}, b_{iK}) = (0, 1)$ for $i = 1, ldots, l$ and $(b_{iK-1}, b_{iK}) = (1, -t_i)$ for each root t_i where $i = l+1, \ldots, n$.

3. If all the coefficients of $q_n(t)$ are zero then we conclude as $(b_{iK-1}, b_{iK}) = (0, 0)$ for all $i = 1, \ldots, n$.

Using (b_{iK-1}, b_{iK})'s, we will derive the rest of the $K-2$ entries of $\mathbf{b}_i$'s. Suppose we know b_{ij} for $i = 1, \ldots, n$, $j = J+1, \ldots, K$ where J starts from $K-2$. Here we consider the partial derivatives of $p_n(\mathbf{x})$ with respect to x_J evaluated at $x_1 = x_2 = \cdots = x_J = 0$ then

$$\sum c_{0,\ldots,0,n_{J+1},\ldots,n_K} x_{J+1}^{n_{J+1}} \cdots x_K^{n_K} = \sum_{i=1}^{n} b_{iJ} g_i^J(\mathbf{x}), \tag{2.9}$$

where

$$g_i^J(\mathbf{x}) \prod_{l=1}^{i-1} \left(\sum_{j=J+1}^{K} b_{lj} x_j \right) \prod_{l=i+1}^{n} \left(\sum_{j=J+1}^{K} b_{lj} x_j \right). \tag{2.10}$$

Let $\mathcal{V}_i^J$ be the coefficient vector of $g_i^J(\mathbf{x})$. By (2.9), we get

$$\begin{bmatrix} \mathcal{V}_1^J & \mathcal{V}_2^J & \cdots & \mathcal{V}_n^J \end{bmatrix} \begin{bmatrix} b_{1J} \\ b_{2J} \\ \vdots \\ b_{nJ} \end{bmatrix} = \begin{bmatrix} c_{0,\ldots,0,1,n-1,0,\ldots,0} \\ c_{0,\ldots,0,1,n-2,1,\ldots,0} \\ \vdots \\ c_{0,\ldots,0,1,0,0,\ldots,n-1} \end{bmatrix}$$

Since we know b_{ij} for $i = 1, \ldots, n$, $j = J+1 \ldots, K$, we also know the vectors $\{\mathcal{V}_i^J\}_{i=1}^n$ and we can derive $\{b_{iJ}\}_{i=1}^n$. This concludes the finding of normal vectors $\mathbf{b}_i$'s.

Our main purpose of using the GPCA is for finding the hyper-planes equations, which is done by finding the normal vectors. However, there are two problems in the process of applying GPCA to our case.

Problem (1) Our hyper-planes never pass through the origin.

Problem (2) Sample points must satisfy the so-called Brill's equations in order to apply Lemma (2.3.1).

Problem (1) can be solved by increasing the dimension of the sample points and hyper-planes. First, increase the dimension of sample points by adding 1 at the last element. For example, our sample points are in $\mathbb{R}^3$ and are in the form of $\mathbf{x}^J = \{x_1, x_2, x_3\}$. For each sample point $\mathbf{x}^J$, we consider new sample point $\mathbf{x}_1^J = \{x_1, x_2, x_3, 1\}$ that are from $\mathbb{R}^4$. We apply our new sample points $X_1 = \{\mathbf{x}_1^J \in \mathbb{R}^4\}_{j=1}^N$ at GPCA and we will have hyper-planes $\{S_i'\}_{i=1}^n$ from $\mathbb{R}^4$. Each S_i' contains S_i and if we project S_i' on $\mathbb{R}^3$, we can compute S_i. For example, let $a_1x_1+a_2x_2+a_3x_3+a_4x_4=0$ be the hyper-plane equation of S_i' then $a_1x_1+a_2x_2+a_3x_3+a_4=0$ will be the hyper-plane equation of S_i.

One of the main part is finding the number of hyper-planes by Lemma (2.3.1). However, one of the hypothesis of this lemma is that sample points needs to satisfy Brill's equations.

Example 2.3.1. Consider two circles C_1: $(x+1)^2+(y+1)^2=1$ and C_2: $(x-1)^2+(y+1)^2=1$. Then by (2.1.1) we will have two hyper-planes for each circle on the cylinder. Here we choose one hyper-plane for one circle using $a\tilde{x}+b\tilde{y}-\tilde{z}=r$. Then we will have S_1: $-x-y-z=1$ and S_2: $x-y-z=1$ two hyper-planes in $\mathbb{R}^3$. For sample points, suppose we have $\mathbf{x}^1(2,3,-6)$, $\mathbf{x}^2(3,5,-9)$, $\mathbf{x}^3(5,7,-13)$, $\mathbf{x}^4(7,11,-19)$ from S_1 and choose $\mathbf{x}^5(11,13,-3)$, $\mathbf{x}^6(13,17,-5)$, $\mathbf{x}^7(17,19,-3)$, $\mathbf{x}^8(19,23,-5)$, $\mathbf{x}^9(23,2,20)$ from S_2.

In what follows we assume only the knowledge of sample points, and describe our strategy to find the total number of hyper-planes to which the sample points belong, then find the normals to these hyper-planes and cluster the points.

We start with Lemma (2.3.1) to find the number of hyper-planes.

First we will check for which i we have $\text{rank}(L_i)=M_i(K)-1$ and that i will be the number of hyper-planes.

1) $i=1$ then

$$L_1 = \begin{bmatrix} \mathcal{V}_1(2,3,-6,1) \\ \mathcal{V}_1(3,5,-9,1) \\ \mathcal{V}_1(5,7,-13,1) \\ \mathcal{V}_1(7,11,-19,1) \\ \mathcal{V}_1(11,13,-3,1) \\ \mathcal{V}_1(13,17,-5,1) \\ \mathcal{V}_1(17,19,-3,1) \\ \mathcal{V}_1(19,23,-5,1) \\ \mathcal{V}_1(23,2,20,1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3-6 & 1 & \\ 3 & 5 & -9 & 1 \\ 5 & 7 & -13 & 1 \\ 7 & 11 & -19 & 1 \\ 11 & 13 & -3 & 1 \\ 13 & 17 & -5 & 1 \\ 17 & 19 & -3 & 1 \\ 19 & 23 & -5 & 1 \\ 23 & 2 & 20 & 1 \end{bmatrix}$$

where $\mathcal{V}_1(x) = (x_1, x_2, x_3, x_4)$

$$\therefore \text{rank}(L_1) = 4 > 3 = M_1 - 1$$

2) $i = 2$ then

$$L_2 = \begin{bmatrix} \mathcal{V}_2(2,3,-6,1) \\ \mathcal{V}_2(3,5,-9,1) \\ \mathcal{V}_2(5,7,-13,1) \\ \mathcal{V}_2(7,11,-19,1) \\ \mathcal{V}_2(11,13,-3,1) \\ \mathcal{V}_2(13,17,-5,1) \\ \mathcal{V}_2(17,19,-3,1) \\ \mathcal{V}_2(19,23,-5,1) \\ \mathcal{V}_2(23,2,20,1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & -12 & 2 & 9 & -18 & 3 & 36 & -6 & 1 \\ 9 & 15 & -27 & 3 & 25 & -45 & 5 & 81 & -9 & 1 \\ 25 & 35 & -65 & 5 & 49 & -91 & 7 & 169 & -13 & 1 \\ 49 & 77 & -133 & 7 & 121 & -209 & 11 & 361 & -19 & 1 \\ 121 & 143 & -33 & 11 & 169 & -39 & 13 & 9 & -3 & 1 \\ 169 & 221 & -65 & 13 & 289 & -85 & 17 & 25 & -5 & 1 \\ 289 & 323 & -51 & 17 & 361 & -57 & 19 & 9 & -3 & 1 \\ 361 & 437 & -95 & 19 & 529 & -115 & 23 & 25 & -5 & 1 \\ 529 & 46 & 460 & 23 & 4 & 40 & 2 & 400 & 20 & 1 \end{bmatrix}$$

where $\mathcal{V}_2(x) = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2)$

$$\therefore \text{rank}(L_2) = 9 = M_2 - 1$$

For $i = 2$, we have $\text{rank}(L_2) = 9 = M_2 - 1$. Therefore we have checked there are two hyper-planes and now we will use the basis of $rank(L_2)$ to find the normal vectors for each hyper-planes.

Here $\mathbf{C}$ is the basis of $rank(L_2)$.

$$\mathbf{C} = (-0.2500, 0.0000, 0.0000, 0.0000, 0.2500, 0.5000, 0.5000, 0.2500, 0.5000, 0.2500)$$

$$= (c_{2,0,0,0}, c_{1,1,0,0}, c_{1,0,1,0}, c_{1,0,0,1}, c_{0,2,0,0}, c_{0,1,1,0}, c_{0,1,0,1}, c_{0,0,2,0}, c_{0,0,1,1}, c_{0,0,0,2})$$

By considering the coefficient of $\frac{\partial p_n(x)}{\partial x_J}$ and using the last three elements of $\mathbf{C}$, we can find the last two elements of two normal vectors b_i's.

$$\begin{aligned}
&\frac{1}{{x_4}^2}(b_{13}x_3 + b_{14}x_4)(b_{23}x_3 + b_{24}x_4) \\
&= \frac{1}{{x_4}^2}(b_{13}b_{23}{x_3}^2 + (b_{13}b_{24} + b_{14}b_{23})x_3x_4 + b_{14}b_{24}x_4^2) \\
&= \frac{1}{{x_4}^2}(c_{0,0,2,0}{x_3}^2 + c_{0,0,1,1}x_3x_4 + c_{0,0,0,2}x_4^2) \\
&= (c_{0,0,2,0}t^2 + c_{0,0,1,1}t + c_{0,0,0,2}) \quad (\because t = \frac{x_3}{x_4}) \\
&= q_2(t) = 0.25t^2 + 0.5t + 0.25
\end{aligned}$$

Here $q_2(t)$ has one real roots $t_1 = -1$.

$$\therefore (b_{13}, b_{14}) = (1, 1), (b_{23}, b_{24}) = (1, 1)$$

Now we will use (b_{13}, b_{14}) and (b_{23}, b_{24}) to find rest of the first 3 entries of b_i's.

(1) J=2, (page 61)

$$g_1^2(x) = \prod_{l=2}^{2}(\sum_{j=3}^{4} b_{lj}x_j) = b_{23}x_3 + b_{24}x_4$$

$$g_2^2(x) = \prod_{l=1}^{1}(\sum_{j=3}^{4} b_{lj}x_j) = b_{13}x_3 + b_{14}x_4$$

$$\begin{bmatrix} b_{23} & b_{13} \\ b_{24} & b_{14} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} \frac{c_{0,1,1,0}}{c_{0,0,2,0}} \\ \frac{c_{0,1,0,1}}{c_{0,0,2,0}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\therefore b_{12} = 1, b_{22} = 1$$

(2) J=1, (page 61)

$$g_1^1(x) = \prod_{l=2}^{2}(\sum_{j=2}^{4} b_{lj}x_j) = b_{22}x_2 + b_{23}x_3 + b_{24}x_4$$

$$g_2^1(x) = \prod_{l=1}^{1}(\sum_{j=2}^{4} b_{lj}x_j) = b_{12}x_2 + b_{13}x_3 + b_{14}x_4$$

$$\begin{bmatrix} b_{22} & b_{12} \\ b_{23} & b_{13} \\ b_{24} & b_{14} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} \frac{c_{1,1,0,0}}{c_{0,0,2,0}} \\ \frac{c_{1,0,1,0}}{c_{0,0,2,0}} \\ \frac{c_{1,0,0,1}}{c_{0,0,2,0}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore b_{11} = 1, b_{21} = -1$$

The two normal of vector of hyperplane are $(1, 1, 1, 1)$ and $(-1, 1, 1, 1)$.

2.4 Brill's Equations

The question of whether a homogeneous polynomial of multiple variables can be factored into a product of first order polynomials has been studied by multiple authors, e.g. see [3][4][7]. The answer is non-trivial and requires a lot of tedious work. In short, such a factorization can be accomplished (over the field of complex numbers) if a special polynomial built using the original one is identically zero. This special polynomial is called Brill's covariant, and below we describe the algorithm of deriving the Brill's covariant.

Let $f(\mathbf{x})$ be a polynomial of degree n in K variables. We decompose $f(\mathbf{x}+\mathbf{y}) = \sum_{i=0}^{n} f^{(i)}(\mathbf{x};\mathbf{y})$, where $f^{(i)}(\mathbf{x};\mathbf{y})$ is the part that is homogeneous of degree i in x. Let $E(u) = \sum_{i=0}^{n} C_i u^i$ where its coefficients are $C_i(f;\mathbf{x};\mathbf{z}) = (-1)^i f^{(i)}(\mathbf{x};\mathbf{z}) f(\mathbf{z})^{i-1}$. We compute $P_{f,\mathbf{z}}(\mathbf{x}) = (C_n)^n \sum_{k=1}^{n} u_k^n$ where u_k's are the n roots of $E(u)$. We use $P_{f,\mathbf{z}}(\mathbf{x})$ to derive Brill's Covariant

$$B(f;\mathbf{x};\mathbf{y};\mathbf{z}) = \frac{1}{n+1} \sum_{i=0}^{n} (-1)^i i!(n-1)! f^{(i)}(\mathbf{x};\mathbf{y}) P_{f,\mathbf{z}}^{(i)}(\mathbf{y};\mathbf{x}).$$

Theorem 2.4.1. *[7] A form $f(\boldsymbol{x})$ is a product of linear forms if and only if the polynomial $B(f;\boldsymbol{x};\boldsymbol{y};\boldsymbol{z})$ is identically equal to* 0.

Based on theorem (2.4.1), we let each coefficient of Brill's Covariant to be 0 and we collect all non-trivial equations. These are the Brill's equations.

Example 2.4.1. Find Brill's equation(s) for $n = 2$ degree in $K = 3$ dimension.

Here $f(\mathbf{x})$ is a polynomial of degree 2 in 3 variables. For convenience here we will denote the coefficients of the polynomial by single letters a, b, c, etc. instead of the notation with multiple indices.

$$f(\mathbf{x}) = c_{2,0,0}x_1^2 + c_{1,1,0}x_1x_2 + c_{1,0,1}x_1x_3 + c_{0,2,0}x_2^2 + c_{0,1,1}x_2x_3 + c_{0,0,2}x_3^2$$

$$= ax_1^2 + bx_1x_2 + cx_1x_3 + dx_2^2 + ex_2x_3 + fx_3^2$$

We will be using $f(\mathbf{x} + \mathbf{y})$.

$$f(\mathbf{x} + \mathbf{y}) = a(x_1 + y_1)^2 + b(x_1 + y_1)(x_2 + y_2) + c(x_1 + y_1)(x_3 + y_3)$$

$$+d(x_2 + y_2)^2 + e(x_2 + y_2)(x_3 + y_3) + f(x_3 + y_3)^2$$

$$= f(x)+(2ay_1x_1+by_1x_2+by_2x_1+cy_1x_3+cy_3x_1+2dy_2x_2+ey_2x_3+ey_3x_2+2fy_3x_3)+f(y)$$

Let $f^{(i)}(\mathbf{x}; \mathbf{y})$ be the part that is homogeneous of degree i in x. Then we can group $f(\mathbf{x} + \mathbf{y})$ into three parts: $f(0)$, $f(1)$ and $f(2)$.

$$f^{(0)}(\mathbf{x}; \mathbf{y}) = f(y),$$

$$f^{(1)}(\mathbf{x}; \mathbf{y}) = 2ay_1x_1 + by_1x_2 + by_2x_1 + cy_1x_3 + cy_3x_1 + 2dy_2x_2 + ey_2x_3 + ey_3x_2 + 2fy_3x_3$$

$$= (2ay_1 + by_2 + cy_3)x_1 + (by_1 + 2dy_2 + ey_3)x_2 + (cy_1 + ey_2 + 2fy_3)x_3$$

$$= (2ax_1 + bx_2 + cx_3)y_1 + (bx_1 + 2dx_2 + ex_3)y_2 + (cx_1 + ex_2 + 2fx_3)y_3$$

$$f^{(2)}(\mathbf{x};\mathbf{y}) = f(x)$$

Since our $f(\mathbf{x})$ is of degree 2, we consider quadratic polynomial

$$E(u) = C_2u^2 + C_1u + C_0$$

where

$$C_i(f;\mathbf{x};\mathbf{z}) = (-1)^i f^{(i)}(\mathbf{x};\mathbf{z})f(\mathbf{z})^{i-1}$$

Three coefficients C_i are

$C_1(f;\mathbf{x};\mathbf{z}) = -f^{(1)}(\mathbf{x};\mathbf{z})$, $C_0(f;\mathbf{x};\mathbf{z}) = 1$ and $C_2(f;\mathbf{x};\mathbf{z}) = f(\mathbf{x})f(\mathbf{z})$.

Let u_i's be the root of $E(u)$. Then we can derive p_k's.

$$p_2 = P_{f,\mathbf{z}} = (C_2)^2(u_1^2 + u_2^2) = (C_1)^2 - 2C_0C_2$$

We will express $P_{f,\mathbf{z}}(\mathbf{x})$ using $\mathbf{x}$ and $\mathbf{z}$ instead of using $C_i's$

$$P_{f,\mathbf{z}}(\mathbf{x}) = (f^{(1)}(\mathbf{x};\mathbf{z}))^2 - 2f(\mathbf{x})f(\mathbf{z})$$

We consider $P_{f,\mathbf{z}}(\mathbf{y}+\mathbf{x})$.

$$P_{f,\mathbf{z}}(\mathbf{y}+\mathbf{x}) = (f^{(1)}((\mathbf{y}+\mathbf{x});\mathbf{z}))^2 - 2f(\mathbf{y}+\mathbf{x})f(\mathbf{z})$$

$$= ((2az_1 + bz_2 + cz_3)(y_1 + x_1) + (bz_1 + 2dz_2 + ez_3)(y_2 + x_2)$$

$$+(cz_1 + ez_2 + 2fz_3)(y_3 + x_3))^2 - 2f(\mathbf{y}+\mathbf{x})f(\mathbf{z})$$

$$= (Z_1(y_1 + x_1) + Z_2(y_2 + x_2) + Z_3(y_3 + x_3))^2 - 2f(\mathbf{y}+\mathbf{x})f(\mathbf{z})$$

$$P^{(2)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) = Z_1^2y_1^2 + 2Z_1Z_2y_1y_2 + 2Z_1Z_3y_1y_3 + Z_2^2y_2^2 + 2Z_2Z_3y_2y_3 + Z_3^2y_3^2 - 2f(\mathbf{y})f(\mathbf{z}),$$

$$P^{(1)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) = 2Z_1Z_1y_1x_1 + 2Z_1Z_2y_1x_2 + 2Z_1Z_3y_1x_3 + 2Z_2Z_1y_2x_1 + 2Z_2Z_2y_2x_2$$

$$+2Z_2Z_3y_2x_3 + 2Z_3Z_1y_3x_1 + 2Z_3Z_2y_3x_2 + 2Z_3Z_3y_3x_3$$

$$-2((2ax_1 + bx_2 + cx_3)y_1 + (bx_1 + 2dx_2 + ex_3)y_2 + (cx_1 + ex_2 + 2fx_3)y_3)f(\mathbf{z}),$$

$$P^{(0)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) = Z_1^2x_1^2 + 2Z_1Z_2x_1x_2 + 2Z_1Z_3x_1x_3 + Z_2^2x_2^2 + 2Z_2Z_3x_2x_3 + Z_3^2x_3^2 - 2f(\mathbf{x})f(\mathbf{z})$$

Now we can write $B(f;\mathbf{x};\mathbf{y};\mathbf{z})$ using $f^{(i)}(\mathbf{x};\mathbf{y})$ and $P^(i)_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})$ and we can derive Brill's equation(s) by letting each coefficient of $B(f;\mathbf{x};\mathbf{y};\mathbf{z})$ to be zero.

$$B(f;\mathbf{x};\mathbf{y};\mathbf{z}) = \frac{1}{1+n}\sum_{i=0}^{n}(-1)^i i!(n-i)!f^{(i)}(\mathbf{x};\mathbf{y})P^{(i)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})$$

$$= \frac{1}{3}\left(2f^{(0)}(\mathbf{x};\mathbf{y})P^{(0)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) - f^{(1)}(\mathbf{x};\mathbf{y})P^{(1)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) + 2f^{(2)}(\mathbf{x};\mathbf{y})P^{(2)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})\right)$$

$$= \frac{1}{3}\left(2f(\mathbf{y})P^{(0)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) - f^{(1)}(\mathbf{x};\mathbf{y})P^{(1)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) + 2f(\mathbf{x})P^{(2)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})\right)$$

We need to compute the coefficients of $B(f;\mathbf{x};\mathbf{y};\mathbf{z})$ but there are too many terms to check. Therefore we will reduce the case by letting $\mathbf{z} = (1,0,0)$ (see [3]). Here are $P^{(i)}_{f,\mathbf{z}}(\mathbf{y}+\mathbf{x})$'s using $\mathbf{z} = (1,0,0)$.

$$P_{f,\mathbf{z}}(\mathbf{y}+\mathbf{x}) = (f^{(1)}((\mathbf{y}+\mathbf{x});\mathbf{z}))^2 - 2f(\mathbf{x}+\mathbf{y})f(\mathbf{z})$$

$$= ((2az_1 + bz_2 + cz_3)(y_1 + x_1) + (bz_1 + 2dz_2 + ez_3)(y_2 + x_2)$$

$$+(cz_1 + ez_2 + 2fz_3)(y_3 + x_3))^2 - 2f(\mathbf{x}+\mathbf{y})f(\mathbf{z})$$

$$= (2a(y_1 + x_1) + b(y_2 + x_2) + c(y_3 + x_3))^2 - 2af(\mathbf{x}+\mathbf{y})$$

$$P^{(2)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) = 4a^2y_1^2 + 4aby_1y_2 + 4acy_1y_3 + b^2y_2^2 + 2bcy_2y_3 + c^2y_3^2 - 2af(\mathbf{y})$$

$$= (4a^2 - 2a^2)y_1^2 + (4ab - 2ab)y_1y_2 + (4ac - 2ac)y_1y_3$$

$$+(b^2-2ad)y_2^2+(2bc-2ae)y_2y_3+(c^2-2af)y_3^2$$

$$P^{(1)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) = 8a^2y_1x_1+4aby_1x_2+4acy_1x_3+4aby_2x_1+2b^2y_2x_2$$

$$+2bcy_2x_3+4acy_3x_1+2bcy_3x_2+2c^2y_3x_3$$

$$-2a((2ax_1+bx_2+cx_3)y_1+(bx_1+2dx_2+ex_3)y_2+(cx_1+ex_2+2fx_3)y_3),$$

$$P^{(0)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) = 4a^2x_1^2+4abx_1x_2+4acx_1x_3+b^2x_2^2+2bcx_2x_3+c^2x_3^2-2af(\mathbf{x})$$

$$= (4a^2-2a^2)x_1^2+(4ab-2ab)x_1x_2+(4ac-2ac)x_1x_3$$

$$+(b^2-2ad)x_2^2+(2bc-2ae)x_2x_3+(c^2-2af)x_3^2$$

$$B(f;\mathbf{x};\mathbf{y};(1,0,0)) = \frac{1}{1+n}\sum_{i=0}^{n}(-1)^i i!(n-i)!f^{(i)}(\mathbf{x};\mathbf{y})P^{(i)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})$$

$$= \frac{1}{3}\left(2f^{(0)}(\mathbf{x};\mathbf{y})P^{(0)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) - f^{(1)}(\mathbf{x};\mathbf{y})P^{(1)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) + 2f^{(2)}(\mathbf{x};\mathbf{y})P^{(2)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})\right)$$

$$= \frac{1}{3}\left(2f(y)P^{(0)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) - f^{(1)}(\mathbf{x};\mathbf{y})P^{(1)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) + 2f(\mathbf{x})P^{(2)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})\right)$$

Here is $B(f;\mathbf{x};\mathbf{y};(1,0,0))$ after distribution.

$$= \frac{1}{3}((2ay_1^2+2by_1y_2+2cy_1y_3+2dy_2^2+2ey_2y_3+2fy_3^2)$$

$$((2a^2)x_1^2+(2ab)x_1x_2+(2ac)x_1x_3+(b^2-2ad)x_2^2+(2bc-2ae)x_2x_3+(c^2-2af)x_3^2)$$

$$-((2ax_1+bx_2+cx_3)y_1+(bx_1+2dx_2+ex_3)y_2+(cx_1+ex_2+2fx_3)y_3)$$

$$(4a^2y_1x_1+2aby_1x_2+2acy_1x_3+2aby_2x_1+(2b^2-4ad)y_2x_2$$

$$+(2bc-2ae)y_2x_3+2acy_3x_1+(2bc-2ae)y_3x_2+(2c^2-4af)y_3x_3)$$

$$+(2ax_1^2+2bx_1x_2+2cx_1x_3+2dx_2^2+2ex_2x_3+2fx_3^2)$$

$$((2a^2)y_1^2 + (2ab)y_1y_2 + (2ac)y_1y_3 + (b^2 - 2ad)y_2^2 + (2bc - 2ae)y_2y_3 + (c^2 - 2af)y_3^2)$$

By using the above Brill's covariant $B(f; \mathbf{x}; \mathbf{y}; (1,0,0))$, we can check the Brill's equations corresponding to terms in Brill's covariant with z_1^2. In order to check the rest of the Brill's equations, one must use recursive equations (see [3]). Let $b_{\alpha,\beta,\gamma}$ be the coefficient of the $\mathbf{x}^\alpha\mathbf{y}^\beta\mathbf{z}^\gamma$ term from Brill's covariant. α, β, γ are the vectors in $\mathbb{R}^K$ for the total degree of $\mathbf{x}$ $\mathbf{y}$ $\mathbf{z}$. For example, in $n = 2$, $K = 3$ case, $b_{(1,1,0),(0,2,0),(1,0,1)} = x_1x_2y_2^2z_1z_3$. Let us define a differential operator

$$\triangle_j = \sum_{\omega}(1+\omega_j)\, C_{\omega+\xi_j-\xi_1}\frac{\partial}{\partial C_\omega},$$

where ξ_j is a vector in $\mathbb{R}^K$ with j-th element 1 and rest of the elements 0. Then one can find all remaining Brill's equations from a recursive equations [3]

$$(1+\gamma_j)b_{\alpha,\beta,\gamma} = \triangle_j b_{\alpha,\beta,\gamma} - (1+\alpha_j)b_{\alpha+\xi_j-\xi_1,\beta,\gamma} - (1+\beta_j)b_{\alpha,\beta+\xi_j-\xi_1,\gamma}. \qquad (2.11)$$

The tables below show all equations obtained from making the coefficients of the corresponding terms of Brill's covariant equal to zero. Here 0 implies that the Brill's equation is $0 = 0$ and * is when the Brill's equations is $c^2d - 4adf - bce + ae^2 + b^2f = 0$

1) z_1^2 case

	y_1^2	y_1y_2	y_1y_3	y_2^2	y_2y_3	y_3^2
x_1^2	0	0	0	0	0	0
x_1x_2	0	0	0	0	0	0
x_1x_3	0	0	0	0	0	0
x_2^2	0	0	0	0	0	*
x_2x_3	0	0	0	0	*	0
x_3^2	0	0	0	*	0	0

2) $z_1 z_2$ case

	y_1^2	y_1y_2	y_1y_3	y_2^2	y_2y_3	y_3^2
x_1^2	0	0	0	0	0	0
x_1x_2	0	0	0	0	0	*
x_1x_3	0	0	0	0	*	0
x_2^2	0	0	0	0	0	0
x_2x_3	0	0	*	0	0	0
x_3^2	0	*	0	0	0	0

3) $z_1 z_3$ case

	y_1^2	y_1y_2	y_1y_3	y_2^2	y_2y_3	y_3^2
x_1^2	0	0	0	0	0	0
x_1x_2	0	0	0	0	*	0
x_1x_3	0	0	0	*	0	0
x_2^2	0	0	*	0	0	0
x_2x_3	0	*	0	0	0	0
x_3^2	0	0	0	0	0	0

4) z_2^2 case

	y_1^2	y_1y_2	y_1y_3	y_2^2	y_2y_3	y_3^2
x_1^2	0	0	0	0	0	$*$
x_1x_2	0	0	0	0	0	0
x_1x_3	0	0	$*$	0	0	0
x_2^2	0	0	0	0	0	0
x_2x_3	0	0	0	0	0	0
x_3^2	$*$	0	0	0	0	0

5) z_2z_3 case

	y_1^2	y_1y_2	y_1y_3	y_2^2	y_2y_3	y_3^2
x_1^2	0	0	0	0	$*$	0
x_1x_2	0	0	$*$	0	0	0
x_1x_3	0	$*$	0	0	0	0
x_2^2	0	0	0	0	0	0
x_2x_3	$*$	0	0	0	0	0
x_3^2	0	0	0	0	0	0

6) z_3^2 case

	y_1^2	y_1y_2	y_1y_3	y_2^2	y_2y_3	y_3^2
x_1^2	0	0	0	*	0	0
x_1x_2	0	*	0	0	0	0
x_1x_3	0	0	0	0	0	0
x_2^2	*	0	0	0	0	0
x_2x_3	0	0	0	0	0	0
x_3^2	0	0	0	0	0	0

In other words, for the case when n=2, K=3 one gets only a single Brill's equation, namely:

$$c^2d - 4adf - bce + ae^2 + b^2f = 0 \tag{2.12}$$

Example 2.4.2. Find Brill's equation(s) for $n = 2$ degree in $K = 4$ dimension.

Here $f(\mathbf{x})$ is a polynomial of degree 2 in 4 variables. For convenience here we will denote the coefficients of the polynomial by single letters a_1, a_2, a_3, etc. instead of the notation with multiple indices.

$$f(\mathbf{x}) = C_{2,0,0,0}x_1^2 + C_{1,1,0,0}x_1x_2 + C_{1,0,1,0}x_1x_3 + C_{1,0,0,1}x_1x_4 + C_{0,2,0,0}x_2^2 + C_{0,1,1,0}x_2x_3$$

$$+C_{0,1,0,1}x_2x_4 + C_{0,0,2,0}x_3^2 + C_{0,0,1,1}x_3x_4 + C_{0,0,0,2}x_4^2$$

$$= a_1x^2 + a_2x_1x_2 + a_3x_1x_3 + a_4x_1x_4 + a_5x_2^2 + a_6x_2x_3 + a_7x_2x_4 + a_8x_3^2 + a_9x_3x_4 + a_{10}x_4^2$$

We will be using $f(\mathbf{x}+\mathbf{y})$.

$$f(\mathbf{x}+\mathbf{y}) = a_1(x_1+y_1)^2+a_2(x_1+y_1)(x_2+y_2)+a_3(x_1+y_1)(x_3+y_3)+a_4(x_1+y_1)(x_4+y_4)$$

$$+a_5(x_2+y_2)^2+a_6(x_2+y_2)(x_3+y_3)+a_7(x_2+y_2)(x_4+y_4)+a_8(x_3+y_3)^2$$

$$+a_9(x_3+y_3)(x_4+y_4)+a_{10}(x_4+y_4)^2$$

Let $f^{(i)}(\mathbf{x};\mathbf{y})$ be the part that is homogeneous of degree i in x. Then we can group $f(\mathbf{x}+\mathbf{y})$ into three parts: $f(0)$, $f(1)$ and $f(2)$.

$$f^{(0)}(\mathbf{x};\mathbf{y}) = f(y)$$

$$f^{(1)}(\mathbf{x};\mathbf{y}) = 2a_1x_1y_1+a_2x_1y_2+a_2x_2y_1+a_3x_1y_3+a_3x_3y_1+a_4x_1y_4+a_4x_4y_1+2a_5x_2y_2$$

$$+a_6x_2y_3+a_6x_3y_2+a_7x_2y_4+a_7x_4y_2+2a_8x_3y_3+a_9x_3y_4+a_9x_4y_3+2a_{10}x_4y_4$$

$$=(2a_1y_1+a_2y_2+a_3y_3+a_4y_4)x_1+(a_2y_1+2a_5y_2+a_6y_3+a_7y_4)x_2$$

$$+(a_3y_1+a_6y_2+2a_8y_3+a_9y_4)x_3+(a_4y_1+a_7y_2+a_9y_3+2a_{10}y_4)x_4$$

$$f^{(2)}(\mathbf{x};\mathbf{y}) = f(\mathbf{x})$$

Since our $f(\mathbf{x})$ is of degree 2, we consider quadratic polynomial

$$E(u) = C_2u^2+C_1u+C_0$$

where

$$C_i(f;\mathbf{x};\mathbf{z}) = (-1)^i f^{(i)}(\mathbf{x};\mathbf{z}) f(\mathbf{z})^{i-1}$$

Three coefficients C_i are

$C_1(f;\mathbf{x};\mathbf{z}) = -f^{(1)}(\mathbf{x};\mathbf{z})$, $C_0(f;\mathbf{x};\mathbf{z}) = 1$ and $C_2(f;\mathbf{x};\mathbf{z}) = f(\mathbf{x})f(\mathbf{z})$.

Let u_i's be the root of $E(u)$. Then we can derive p_k's.

$$p_2 = P_{f,\mathbf{z}} = (C_2)^2(u_1^2+u_2^2) = (C_1)^2 - 2C_0C_2$$

We will express $P_{f,\mathbf{z}}(\mathbf{x})$ using $\mathbf{x}$ and $\mathbf{z}$ instead of using $C_i's$

$$P_{f,\mathbf{z}}(\mathbf{x}) = (f^{(1)}(\mathbf{x};\mathbf{z}))^2 - 2f(\mathbf{x})f(\mathbf{z})$$

We consider $P_{f,\mathbf{z}}(\mathbf{y}+\mathbf{x})$.

$$P_{f,\mathbf{z}}(\mathbf{y}+\mathbf{x}) = (f^{(1)}((\mathbf{y}+\mathbf{x});\mathbf{z}))^2 - 2f(\mathbf{y}+\mathbf{x})f(\mathbf{z})$$

$$= ((2a_1z_1 + a_2z_2 + a_3z_3 + a_4z_4)(y_1 + x_1) + (a_2z_1 + 2a_5z_2 + a_6z_3 + a_7z_4)(y_2 + x_2)$$

$$+(a_3z_1 + a_6z_2 + 2a_8z_3 + a_9z_4)(y_3 + x_3) + (a_4z_1 + a_7z_2 + a_9z_3 + 2a_{10}z_4)(y_4 + x_4))^2$$

$$-2f(\mathbf{y}+\mathbf{x})f(\mathbf{z})$$

Here we let $2a_1z_1 + a_2z_2 + a_3z_3 + a_4z_4 = Z_1$, $a_2z_1 + 2a_5z_2 + a_6z_3 + a_7z_4 = Z_2$, $a_3z_1 + a_6z_2 + 2a_8z_3 + a_9z_4 = Z_3$ and $a_4z_1 + a_7z_2 + a_9z_3 + 2a_{10}z_4 = Z_4$ then

$$P_{f,\mathbf{z}}(\mathbf{y}+\mathbf{x}) = (Z_1(y_1+x_1) + Z_2(y_2+x_2) + Z_3(y_3+x_3) + Z_4(y_4+x_4))^2 - 2f(\mathbf{y}+\mathbf{x})f(\mathbf{z})$$

$$P^{(2)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) = Z_1^2y_1^2 + 2Z_1Z_2y_1y_2 + 2Z_1Z_3y_1y_3 + 2Z_1Z_4y_1y_4 + Z_2^2y_2^2$$

$$+2Z_2Z_3y_2y_3 + 2Z_2Z_4y_2y_4 + Z_3^2y_3^2 + 2Z_3Z_4y_3y_4 + Z_4^2y_4^2 - 2f(\mathbf{y})f(\mathbf{z})$$

$P^{(1)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})$

$$= 2Z_1^2y_1x_1 + 2Z_1Z_2y_1x_2 + 2Z_1Z_3y_1x_3 + 2Z_1Z_4y_1x_4$$

$$+2Z_2Z_1y_2x_1 + 2Z_2^2y_2x_2 + 2Z_2Z_3y_2x_3 + 2Z_2Z_4y_2x_4$$

$$+2Z_3Z_1y_3x_1 + 2Z_3Z_2y_3x_2 + 2Z_3^2y_3x_3 + 2Z_3Z_4y_3x_4$$

$$+2Z_4Z_1y_4x_1 + 2Z_4Z_2y_4x_2 + 2Z_4Z_3y_4x_3 + 2Z_4^2y_4x_4$$

$$-2((2a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4)x_1 + (a_2y_1 + 2a_5y_2 + a_6y_3 + a_7y_4)x_2$$

$$+(a_3y_1 + a_6y_2 + 2a_8y_3 + a_9y_4)x_3 + (a_4y_1 + a_7y_2 + a_9y_3 + 2a_{10}y_4)x_4)f(z)$$

$$P^{(0)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) = Z_1^2x_1^2 + 2Z_1Z_2x_1x_2 + 2Z_1Z_3x_1x_3 + 2Z_1Z_4x_1x_4 + Z_2^2x_2^2$$

$$+2Z_2Z_3x_2x_3 + 2Z_2Z_4x_2x_4 + Z_3^2x_3^2 + 2Z_3Z_4x_3x_4 + Z_4^2x_4^2 - 2f(\mathbf{x})f(\mathbf{z})$$

Now we can write $B(f;\mathbf{x};\mathbf{y};\mathbf{z})$ using $f^{(i)}(\mathbf{x};\mathbf{y})$ and $P^(i)_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})$ and we can derive Brill's equation(s) by letting each coefficient of $B(f;\mathbf{x};\mathbf{y};\mathbf{z})$ to be zero.

$$B(f;\mathbf{x};\mathbf{y};\mathbf{z}) = \frac{1}{1+n}\sum_{i=0}^{n}(-1)^i i!(n-i)! f^{(i)}(\mathbf{x};\mathbf{y})P^{(i)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})$$

$$= \frac{1}{3}\left(2f^{(0)}(\mathbf{x};\mathbf{y})P^{(0)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) - f^{(1)}(\mathbf{x};\mathbf{y})P^{(1)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) + 2f^{(2)}(\mathbf{x};\mathbf{y})P^{(2)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})\right)$$

$$= \frac{1}{3}\left(2f(\mathbf{y})P^{(0)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) - f^{(1)}(\mathbf{x};\mathbf{y})P^{(1)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) + 2f(\mathbf{x})P^{(2)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})\right)$$

We need to compute the coefficients of $B(f;\mathbf{x};\mathbf{y};\mathbf{z})$ but there are too many terms to check. Therefore we will reduce the case by letting $\mathbf{z} = (1,0,0,0)$(see [3]). Here are $P^{(i)}_{f,\mathbf{z}}(\mathbf{y}+\mathbf{x})$'s using $\mathbf{z} = (1,0,0,0)$.

$$P_{f,\mathbf{z}}(\mathbf{y}+\mathbf{x}) = (2a_1(y_1+x_1) + a_2(y_2+x_2) + a_3(y_3+x_3) + a_4(y_4+x_4))^2 - 2f(\mathbf{y}+\mathbf{x})a_1$$

$$P^{(2)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x}) = 4a_1^2y_1^2 + 4a_1a_2y_1y_2 + 4a_1a_3y_1y_3 + 4a_1a_4y_1y_4 + a_2^2y_2^2$$

$$+2a_2a_3y_2y_3 + 2a_2a_4y_2y_4 + a_3^2y_3^2 + 2a_3a_4y_3y_4 + a_4^2y_4^2 - 2f(\mathbf{y})a_1$$

$$= (2a_1^2)y_1^2 + (2a_1a_2)y_1y_2 + (2a_1a_3)y_1y_3 + (2a_1a_4)y_1y_4 + (a_2^2 - 2a_1a_5)y_2^2$$

$$+(2a_2a_3 - 2a_1a_6)y_2y_3 + (2a_2a_4 - 2a_1a_7)y_2y_4 + (a_3^2 - 2a_1a_8)y_3^2$$

$$+(2a_3a_4 - 2a_1a_9)y_3y_4 + (a_4^2 - 2a_1a_{10})y_4^2$$

$$P^{(1)}_{f,\mathbf{z}}(\mathbf{y};\mathbf{x})$$

$$= 8a_1^2 y_1 x_1 + 4a_1 a_2 y_1 x_2 + 4a_1 a_3 y_1 x_3 + 4a_1 a_4 y_1 x_4$$

$$+4a_2 a_1 y_2 x_1 + 2a_2^2 y_2 x_2 + 2a_2 a_3 y_2 x_3 + 2a_2 a_4 y_2 x_4$$

$$+4a_3 a_1 y_3 x_1 + 2a_3 a_2 y_3 x_2 + 2a_3^2 y_3 x_3 + 2a_3 a_4 y_3 x_4$$

$$+4a_4 a_1 y_4 x_1 + 2a_4 a_2 y_4 x_2 + 2a_4 a_3 y_4 x_3 + 2a_4^2 y_4 x_4$$

$$-((4a_1^2 y_1 + 2a_1 a_2 y_2 + 2a_1 a_3 y_3 + 2a_1 a_4 y_4)x_1 + (2a_1 a_2 y_1 + 4a_1 a_5 y_2 + 2a_1 a_6 y_3 + 2a_1 a_7 y_4)x_2$$

$$+(2a_1 a_3 y_1 + 2a_1 a_6 y_2 + 4a_1 a_8 y_3 + 2a_1 a_9 y_4)x_3 + (2a_1 a_4 y_1 + 2a_1 a_7 y_2 + 2a_1 a_9 y_3 + 4a_1 a_{10} y_4)x_4)$$

$$= (4a_1^2) y_1 x_1 + (2a_1 a_2) y_1 x_2 + (2a_1 a_3) y_1 x_3 + (2a_1 a_4) y_1 x_4$$

$$+(2a_1 a_2) y_2 x_1 + (2a_2^2 - 4a_1 a_5) y_2 x_2 + (2a_2 a_3 - 2a_1 a_6) y_2 x_3 + (2a_2 a_4 - 2a_1 a_7) y_2 x_4$$

$$+(2a_1 a_3) y_3 x_1 + (2a_2 a_3 - 2a_1 a_6) y_3 x_2 + (2a_3^2 - 4a_1 a_8) y_3 x_3 + (2a_3 a_4 - 2a_1 a_9) y_3 x_4$$

$$+(2a_1 a_4) y_4 x_1 + (2a_2 a_4 - 2a_1 a_7) y_4 x_2 + (2a_3 a_4 - 2a_1 a_9) y_4 x_3 + (2a_4^2 - 4a_1 a_{10}) y_4 x_4$$

$$P_{f,\mathbf{z}}^{(0)}(\mathbf{y};\mathbf{x}) = 4a_1^2 x_1^2 + 4a_1 a_2 x_1 x_2 + 4a_1 a_3 x_1 x_3 + 4a_1 a_4 x_1 x_4 + a_2^2 x_2^2$$

$$+2a_2 a_3 x_2 x_3 + 2a_2 a_4 x_2 x_4 + a_3^2 x_3^2 + 2a_3 a_4 x_3 x_4 + a_4^2 x_4^2 - 2f(\mathbf{x}) a_1$$

$$= (2a_1^2) x_1^2 + (2a_1 a_2) x_1 x_2 + (2a_1 a_3) x_1 x_3 + (2a_1 a_4) x_1 x_4 + (a_2^2 - 2a_1 a_5) x_2^2$$

$$+(2a_2 a_3 - 2a_1 a_6) x_2 x_3 + (2a_2 a_4 - 2a_1 a_7) x_2 x_4 + (a_3^2 - 2a_1 a_8) x_3^2$$

$$+(2a_3 a_4 - 2a_1 a_9) x_3 x_4 + (a_4^2 - 2a_1 a_{10}) x_4^2$$

Using the above $P_{f,\mathbf{z}}^{(i)}(\mathbf{y}+\mathbf{x})$'s we have

$$B(f;\mathbf{x};\mathbf{y};\mathbf{z}) = \frac{1}{1+n} \sum_{i=0}^{n} (-1)^i i!(n-i)! f^{(i)}(\mathbf{x};\mathbf{y}) P_{f,\mathbf{z}}^{(i)}(\mathbf{y};\mathbf{x})$$

$$= \frac{1}{3}\left(2f^{(0)}(\mathbf{x};\mathbf{y}) P_{f,\mathbf{z}}^{(0)}(\mathbf{y};\mathbf{x}) - f^{(1)}(\mathbf{x};\mathbf{y}) P_{f,\mathbf{z}}^{(1)}(\mathbf{y};\mathbf{x}) + 2f^{(2)}(\mathbf{x};\mathbf{y}) P_{f,\mathbf{z}}^{(2)}(\mathbf{y};\mathbf{x})\right)$$

$$= \frac{1}{3}\left(2f(\mathbf{y}) P_{f,\mathbf{z}}^{(0)}(\mathbf{y};\mathbf{x}) - f^{(1)}(\mathbf{x};\mathbf{y}) P_{f,\mathbf{z}}^{(1)}(\mathbf{y};\mathbf{x}) + 2f(\mathbf{x}) P_{f,\mathbf{z}}^{(2)}(\mathbf{y};\mathbf{x})\right) = 0.$$

Here is $B(f;\mathbf{x};\mathbf{y};\mathbf{z})$ after distribution.

$$B(f;\mathbf{x};\mathbf{y};\mathbf{z})$$

$$= (2a_1y_1^2 + 2a_2y_1y_2 + 2a_3y_1y_3 + 2a_4y_1y_4 + 2a_5y_2^2$$

$$+2a_6y_2y_3 + 2a_7y_2y_4 + 2a_8y_3^2 + 2a_9y_3y_4 + 2a_{10}y_4^2)$$

$$((2a_1^2)x_1^2 + (2a_1a_2)x_1x_2 + (2a_1a_3)x_1x_3 + (2a_1a_4)x_1x_4 + (a_2^2 - 2a_1a_5)x_2^2$$

$$+(2a_2a_3 - 2a_1a_6)x_2x_3 + (2a_2a_4 - 2a_1a_7)x_2x_4 + (a_3^2 - 2a_1a_8)x_3^2$$

$$+(2a_3a_4 - 2a_1a_9)x_3x_4 + (a_4^2 - 2a_1a_{10})x_4^2)$$

$$-((2a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4)x_1 + (a_2y_1 + 2a_5y_2 + a_6y_3 + a_7y_4)x_2$$

$$+(a_3y_1 + a_6y_2 + 2a_8y_3 + a_9y_4)x_3 + (a_4y_1 + a_7y_2 + a_9y_3 + 2a_{10}y_4)x_4)$$

$$((4a_1^2)y_1x_1 + (2a_1a_2)y_1x_2 + (2a_1a_3)y_1x_3 + (2a_1a_4)y_1x_4$$

$$+(2a_1a_2)y_2x_1 + (2a_2^2 - 4a_1a_5)y_2x_2 + (2a_2a_3 - 2a_1a_6)y_2x_3 + (2a_2a_4 - 2a_1a_7)y_2x_4$$

$$+(2a_1a_3)y_3x_1 + (2a_2a_3 - 2a_1a_6)y_3x_2 + (2a_3^2 - 4a_1a_8)y_3x_3 + (2a_3a_4 - 2a_1a_9)y_3x_4$$

$$+(2a_1a_4)y_4x_1 + (2a_2a_4 - 2a_1a_7)y_4x_2 + (2a_3a_4 - 2a_1a_9)y_4x_3 + (2a_4^2 - 4a_1a_{10})y_4x_4)$$

$$(2a_1x_1^2 + 2a_2x_1x_2 + 2a_3x_1x_3 + 2a_4x_1x_4 + 2a_5x_2^2$$

$$+2a_6x_2x_3 + 2a_7x_2x_4 + 2a_8x_3^2 + 2a_9x_3x_4 + 2a_{10}x_4^2)$$

$$((2a_1^2)y_1^2 + (2a_1a_2)y_1y_2 + (2a_1a_3)y_1y_3 + (2a_1a_4)y_1y_4 + (a_2^2 - 2a_1a_5)y_2^2$$

$$+(2a_2a_3 - 2a_1a_6)y_2y_3 + (2a_2a_4 - 2a_1a_7)y_2y_4 + (a_3^2 - 2a_1a_8)y_3^2$$

$$+(2a_3a_4 - 2a_1a_9)y_3y_4 + (a_4^2 - 2a_1a_{10})y_4^2)$$

After checking 100 coefficients, we have six coefficients that are different from others and not trivial.

The tables below show all equations obtained from making the coefficients of terms of Brill's covariant containing z_1^2 equal to zero.

Term	Equation	Term	Equation	Term	Equation	Term	Equation
$y_1^2x_1^2$	$0=0$	$y_1y_2x_2x_4$	$0=0$	$y_1y_4x_1x_3$	$0=0$	$x_1x_2y_2y_4$	$0=0$
$y_1^2x_1x_2$	$0=0$	$y_1y_2x_3^2$	$0=0$	$y_1y_4x_1x_4$	$0=0$	$x_1x_2y_3^2$	$0=0$
$y_1^2x_1x_3$	$0=0$	$y_1y_2x_3x_4$	$0=0$	$y_1y_4x_2^2$	$0=0$	$x_1x_2y_3y_4$	$0=0$
$y_1^2x_1x_4$	$0=0$	$y_1y_2x_4^2$	$0=0$	$y_1y_4x_2x_3$	$0=0$	$x_1x_2y_4^2$	$0=0$
$y_1^2x_2^2$	$0=0$	$y_1y_3x_1^2$	$0=0$	$y_1y_4x_2x_4$	$0=0$	$x_1x_3y_2^2$	$0=0$
$y_1^2x_2x_3$	$0=0$	$y_1y_3x_1x_2$	$0=0$	$y_1y_4x_3^2$	$0=0$	$x_1x_3y_2y_3$	$0=0$
$y_1^2x_2x_4$	$0=0$	$y_1y_3x_1x_3$	$0=0$	$y_1y_4x_3x_4$	$0=0$	$x_1x_3y_2y_4$	$0=0$
$y_1^2x_3^2$	$0=0$	$y_1y_3x_1x_4$	$0=0$	$y_1y_4x_4^2$	$0=0$	$x_1x_3y_3^2$	$0=0$
$y_1^2x_3x_4$	$0=0$	$y_1y_3x_2^2$	$0=0$	$x_1^2y_2^2$	$0=0$	$x_1x_3y_3y_4$	$0=0$
$y_1^2x_4^2$	$0=0$	$y_1y_3x_2x_3$	$0=0$	$x_1^2y_2y_3$	$0=0$	$x_1x_3y_4^2$	$0=0$
$y_1y_2x_1^2$	$0=0$	$y_1y_3x_2x_4$	$0=0$	$x_1^2y_2y_4$	$0=0$	$x_1x_4y_2^2$	$0=0$
$y_1y_2x_1x_2$	$0=0$	$y_1y_3x_3^2$	$0=0$	$x_1^2y_3^2$	$0=0$	$x_1x_4y_2y_3$	$0=0$
$y_1y_2x_1x_3$	$0=0$	$y_1y_3x_3x_4$	$0=0$	$x_1^2y_3y_4$	$0=0$	$x_1x_4y_2y_4$	$0=0$
$y_1y_2x_1x_4$	$0=0$	$y_1y_3x_4^2$	$0=0$	$x_1^2y_4^2$	$0=0$	$x_1x_4y_3^2$	$0=0$
$y_1y_2x_2^2$	$0=0$	$y_1y_4x_1^2$	$0=0$	$x_1x_2y_2^2$	$0=0$	$x_1x_4y_3y_4$	$0=0$
$y_1y_2x_2x_3$	$0=0$	$y_1y_4x_1x_2$	$0=0$	$x_1x_2y_2y_3$	$0=0$	$x_1x_4y_4^2$	$0=0$

Term	Equation
$x_2^2y_2^2$	$0 = 0$
$x_2^2y_2y_3$	$0 = 0$
$x_2^2y_2y_4$	$0 = 0$
$x_2^2y_3^2$	$a_3^2a_5 - 4a_1a_5a_8 - a_2a_3a_6 + a_1a_6^2 + a_2^2a_8 = 0$
$x_2^2y_3y_4$	$a_2a_4a_6 - 2a_1a_6a_7 - 2a_3a_4a_5 + 4a_1a_5a_9 + a_2a_3a_7 - a_2^2a_9 = 0$
$x_2^2y_4^2$	$a_4^2a_5 - 4a_1a_5a_{10} - a_2a_4a_7 + a_1a_7^2 + a_2^2a_{10} = 0$
$x_2x_3y_2^2$	$0 = 0$
$x_2x_3y_2y_3$	$a_3^2a_5 - 4a_1a_5a_8 - a_2a_3a_6 + a_1a_6^2 + a_2^2a_8 = 0$
$x_2x_3y_2y_4$	$a_2a_4a_6 - 2a_1a_6a_7 - 2a_3a_4a_5 + 4a_1a_5a_9 + a_2a_3a_7 - a_2^2a_9 = 0$
$x_2x_3y_3^2$	$0 = 0$
$x_2x_3y_3y_4$	$a_3^2a_7 - 4a_1a_7a_8 - a_3a_4a_6 + 2a_1a_6a_9 - a_2a_3a_9 + 2a_2a_4a_8 = 0$
$x_2x_3y_4^2$	$a_4^2a_6 - 4a_1a_6a_{10} - a_3a_4a_7 + 2a_1a_7a_9 - a_2a_4a_9 + 2a_2a_3a_{10} = 0$
$x_2x_4y_2^2$	$0 = 0$
$x_2x_4y_2y_3$	$a_2a_4a_6 - 2a_1a_6a_7 - 2a_3a_4a_5 + 4a_1a_5a_9 + a_2a_3a_7 - a_2^2a_9 = 0$
$x_2x_4y_2y_4$	$a_4^2a_5 - 4a_1a_5a_{10} - a_2a_4a_7 + a_1a_7^2 + a_2^2a_{10} = 0$
$x_2x_4y_3^2$	$a_3^2a_7 - 4a_1a_7a_8 - a_3a_4a_6 + 2a_1a_6a_9 - a_2a_3a_9 + 2a_2a_4a_8 = 0$
$x_2x_4y_3y_4$	$a_4^2a_6 - 4a_1a_6a_{10} - a_3a_4a_7 + 2a_1a_7a_9 - a_2a_4a_9 + 2a_2a_3a_{10} = 0$
$x_2x_4y_4^2$	$0 = 0$

Term	Equation
$x_3^2y_2^2$	$a_3^2a_5 - 4a_1a_5a_8 - a_2a_3a_6 + a_1a_6^2 + a_2^2a_8 = 0$
$x_3^2y_2y_3$	$0 = 0$
$x_3^2y_2y_4$	$a_3^2a_7 - 4a_1a_7a_8 - a_3a_4a_6 + 2a_1a_6a_9 - a_2a_3a_9 + 2a_2a_4a_8 = 0$
$x_3^2y_3^2$	$0 = 0$
$x_3^2y_3y_4$	$0 = 0$
$x_3^2y_4^2$	$a_4^2a_8 - 4a_1a_8a_{10} - a_3a_4a_9 + a_1a_9^2 + a_3^2a_{10} = 0$
$x_3x_4y_2^2$	$a_2a_4a_6 - 2a_1a_6a_7 - 2a_3a_4a_5 + 4a_1a_5a_9 + a_2a_3a_7 - a_2^2a_9 = 0$
$x_3x_4y_2y_3$	$a_3^2a_7 - 4a_1a_7a_8 - a_3a_4a_6 + 2a_1a_6a_9 - a_2a_3a_9 + 2a_2a_4a_8 = 0$
$x_3x_4y_2y_4$	$a_4^2a_6 - 4a_1a_6a_{10} - a_3a_4a_7 + 2a_1a_7a_9 - a_2a_4a_9 + 2a_2a_3a_{10} = 0$
$x_3x_4y_3^2$	$0 = 0$
$x_3x_4y_3y_4$	$a_4^2a_8 - 4a_1a_8a_{10} - a_3a_4a_9 + a_1a_9^2 + a_3^2a_{10} = 0$
$x_3x_4y_4^2$	$0 = 0$
$x_4^2y_2^2$	$a_4^2a_5 - 4a_1a_5a_{10} - a_2a_4a_7 + a_1a_7^2 + a_2^2a_{10} = 0$
$x_4^2y_2y_3$	$a_4^2a_6 - 4a_1a_6a_{10} - a_3a_4a_7 + 2a_1a_7a_9 - a_2a_4a_9 + 2a_2a_3a_{10} = 0$
$x_4^2y_2y_4$	$0 = 0$
$x_4^2y_3^2$	$a_4^2a_8 - 4a_1a_8a_{10} - a_3a_4a_9 + a_1a_9^2 + a_3^2a_{10} = 0$
$x_4^2y_3y_4$	$0 = 0$
$x_4^2y_4^2$	$0 = 0$

From the previous two tables, we have the following 6 non-trivial Brill's equations out of 21 distinct equations.

$$a_3^2a_5 - 4a_1a_5a_8 - a_2a_3a_6 + a_1a_6^2 + a_2^2a_8 = 0$$

$$a_4^2a_5 - 4a_1a_5a_{10} - a_2a_4a_7 + a_1a_7^2 + a_2^2a_{10} = 0$$

$$a_4^2a_8 - 4a_1a_8a_{10} - a_3a_4a_9 + a_1a_9^2 + a_3^2a_{10} = 0$$

$$a_2^2a_9 - 4a_1a_5a_9 + 2a_1a_6a_7 + 2a_3a_4a_5 - a_2a_4a_6 - a_2a_3a_7 = 0$$

$$a_4^2a_6 - 4a_1a_6a_{10} + 2a_1a_7a_9 + 2a_2a_3a_{10} - a_3a_4a_7 - a_2a_4a_9 = 0$$

$$a_3^2a_7 - 4a_1a_7a_8 + 2a_1a_6a_9 + 2a_2a_4a_8 - a_3a_4a_6 - a_2a_3a_9 = 0$$

By using the symmetry of Brill Covariant and Veronese map, we have 4 more Brill's equations from previous 6 Brill's equations.

$$a_6^2a_{10} - 4a_5a_8a_{10} - a_6a_7a_9 + a_7^2a_8 + a_9^2a_5 = 0$$

$$a_6^2a_4 - 4a_4a_5a_8 + 2a_5a_9a_3 + 2a_8a_7a_2 - a_6a_9a_2 - a_6a_3a_7 = 0$$

$$a_7^2a_3 - 4a_3a_5a_10 + 2a_5a_9a_4 + 2a_10a_2a_6 - a_7a_9a_2 - a_7a_4a_6 = 0$$

$$a_9^2a_2 - 4a_2a_8a_10 + 2a_8a_7a_4 + 2a_10a_3a_6 - a_9a_7a_3 - a_9a_4a_6 = 0$$

One can finish finding all the Brill's equations using the recursive equations (2.11).

CHAPTER 3

EXAMPLES

3.1 Noise-free Data

Example 3.1.1. Consider two circles

$$(x-3)^2 + (y-4)^2 = 1$$

and

$$(x-5)^2 + (y-12)^2 = 4.$$

For sample points, consider the following 9 tangent lines to these two circles:

$$l_1(\pi, -4),$$

$$l_2(\frac{3\pi}{2}, -5),$$

$$l_3(\frac{4\pi}{3}, -(1 + 5\cos(\frac{\pi}{3} - \cos^{-1}(\frac{3}{5})))),$$

$$l_4(\frac{5\pi}{4}, -(1 + 5\cos(\cos^{-1}(\frac{3}{5}) - \frac{\pi}{4}))),$$

$$l_5(\frac{6\pi}{5}, -(1 + 5\cos(\cos^{-1}(\frac{3}{5}) - \frac{\pi}{5})))$$

tangent to the circle $(x-3)^2 + (y-4)^2 = 1$ and

$$l_6(\frac{7\pi}{6}, -(2+13\cos(\cos^{-1}(\frac{5}{13})-\frac{\pi}{6}))),$$

$$l_7(\frac{8\pi}{7}, -(2+13\cos(\cos^{-1}(\frac{5}{13})-\frac{\pi}{7}))),$$

$$l_8(\frac{9\pi}{8}, -(2+13\cos(\cos^{-1}(\frac{5}{13})-\frac{\pi}{8}))),$$

$$l_9(\frac{10\pi}{9}, -(2+13\cos(\cos^{-1}(\frac{5}{13})-\frac{\pi}{9})))$$

tangent to the circle $(x-5)^2+(y-12)^2=4$.

There are two hyperplanes corresponding to each circle and intersecting the cylinder. Here we choose one hyperplane for each circle where the hyperplanes are in the following form

$$a\tilde{x}+b\tilde{y}-\tilde{z}=r.$$

The two hyperplanes in $\mathbb{R}^3$ are

$$3\tilde{x}+4\tilde{y}-\tilde{z}=1$$

and

$$5\tilde{x}+12\tilde{y}-\tilde{z}=2.$$

The sample points in $\mathbb{R}^3$ will be

$$\mathbf{x}^1(-1,0,-4),$$

$$\mathbf{x}^2(0, -1, -5, 1)$$

$$\mathbf{x}^3(-0.5, -0.866025403784438, -5.964101615137755),$$

$$\mathbf{x}^4(-0.707106781186548, -0.707106781186547, -5.949747468305833),$$

$$\mathbf{x}^5(-0.809016994374947, -0.587785252292473, -5.778191992294735),$$

$$\mathbf{x}^6(-0.866025403784439, -0.5, -12.330127018922193),$$

$$\mathbf{x}^7(-0.900968867902419, -0.433883739117558, -11.711449208922794),$$

$$\mathbf{x}^8(-0.923879532511287, -0.382683432365090, -11.211598850937513),$$

$$\mathbf{x}^9(-0.939692620785908, -0.342020143325669, -10.802704823837567).$$

Notice, that the two hyperplanes $3\tilde{x} + 4\tilde{y} - \tilde{z} = 1$ and $5\tilde{x} + 12\tilde{y} - \tilde{z} = 2$ are not passing through the origin, i.e. these are affine spaces. Since GPCA works for sample points corresponding to subspaces (not affine spaces), we need to modify our setup to use GPCA. Therefore we consider new sample points in $\mathbb{R}^4$ by adding 1 to all points as a 4-th component. It is easy to notice that the new sample points will belong to hyperplanes in $\mathbb{R}^4$ passing through the origin, i.e. to subspaces.

The two hyperplanes in $\mathbb{R}^4$ are $3\tilde{x} + 4\tilde{y} - \tilde{z} - \tilde{w} = 0$ and $5\tilde{x} + 12\tilde{y} - \tilde{z} - 2\tilde{w} = 0$ and the 9 sample points are

$$\mathbf{x}^1(-1, 0, -4, 1),$$

$$\mathbf{x}^2(0, -1, -5, 1),$$

$$\mathbf{x}^3(-0.5, -0.866025403784438, -5.964101615137755, 1),$$

$$\mathbf{x}^4(-0.707106781186548, -0.707106781186547, -5.949747468305833, 1),$$

$$\mathbf{x}^5(-0.809016994374947, -0.587785252292473, -5.778191992294735, 1),$$

$$\mathbf{x}^6(-0.866025403784439, -0.5, -12.330127018922193, 1),$$

$$\mathbf{x}^7(-0.900968867902419, -0.433883739117558, -11.711449208922794, 1),$$

$$\mathbf{x}^8(-0.923879532511287, -0.382683432365090, -11.211598850937513, 1),$$

$$\mathbf{x}^9(-0.939692620785908, -0.342020143325669, -10.802704823837567, 1).$$

As a first step of our modification of GPCA let us find the number of hyperplanes using the 9 sample points from above. For $i = 1$, we have

$$L_1 = \begin{bmatrix} \mathcal{V}_1(-1, 0, -4, 1) \\ \mathcal{V}_1(0, -1, -5, 1) \\ \mathcal{V}_1(-0.5, -0.866025403784438, -5.964101615137755, 1) \\ \mathcal{V}_1(-0.707106781186548, -0.707106781186547, -5.949747468305833, 1) \\ \mathcal{V}_1(-0.809016994374947, -0.587785252292473, -5.778191992294735, 1) \\ \mathcal{V}_1(-0.866025403784439, -0.5, -12.330127018922193, 1) \\ \mathcal{V}_1(-0.900968867902419, -0.433883739117558, -11.711449208922794, 1) \\ \mathcal{V}_1(-0.923879532511287, -0.382683432365090, -11.211598850937513, 1) \\ \mathcal{V}_1(-0.939692620785908, -0.342020143325669, -10.802704823837567, 1) \end{bmatrix}$$

$$L_1 = \begin{bmatrix} -1.000000000000000 & 0 & -4.000000000000000 & 1 \\ 0 & -1.000000000000000 & -5.000000000000000 & 1 \\ -0.500000000000000 & -0.866025403784438 & -5.964101615137755 & 1 \\ -0.707106781186548 & -0.707106781186547 & -5.949747468305833 & 1 \\ -0.809016994374947 & -0.587785252292473 & -5.778191992294735 & 1 \\ -0.866025403784439 & -0.500000000000000 & -12.330127018922193 & 1 \\ -0.900968867902419 & -0.433883739117558 & -11.711449208922794 & 1 \\ -0.923879532511287 & -0.382683432365090 & -11.211598850937513 & 1 \\ -0.939692620785908 & -0.342020143325669 & -10.802704823837567 & 1 \end{bmatrix}$$

where $\mathcal{V}_1(x) = (x_1, x_2, x_3, x_4)$.

We have $\text{rank}(L_1) = 4 > 3 = M_1 - 1 = \binom{4}{1} - 1$. Therefore, we conclude that the number of hyperplanes n is greater than 1.

For $i = 2$, we have the matrix in (3.1).

$$L_2 = \begin{bmatrix} \mathcal{V}_2(-1, 0, -4, 1) \\ \mathcal{V}_2(0, -1, -5, 1) \\ \mathcal{V}_2(-0.5, -0.866025403784438, -5.964101615137755, 1) \\ \mathcal{V}_2(-0.707106781186548, -0.707106781186547, -5.949747468305833, 1) \\ \mathcal{V}_2(-0.809016994374947, -0.587785252292473, -5.778191992294735, 1) \\ \mathcal{V}_2(-0.866025403784439, -0.5, -12.330127018922193, 1) \\ \mathcal{V}_2(-0.900968867902419, -0.433883739117558, -11.711449208922794, 1) \\ \mathcal{V}_2(-0.923879532511287, -0.382683432365090, -11.211598850937513, 1) \\ \mathcal{V}_2(-0.939692620785908, -0.342020143325669, -10.802704823837567, 1) \end{bmatrix}$$

where $\mathcal{V}_2(x) = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2)$.

We have $\text{rank}(L_2) = 8 < 9 = \binom{5}{2} - 1 = M_2 - 1$. Therefore the number of hyperplanes maybe two. To confirm that the number of hyperplanes is indeed two, we need to make sure that there is a vector in $\text{Ker}(L_2)$ satisfying Brill's equations.

Consider the following $\mathbf{C}_1$ and $\mathbf{C}_2$ which make a basis of $\text{Ker}(L_2)$

$$\mathbf{C}_1 = (-0.146266519878173, 0.764434487275566, -0.109204926753680, -0.150156774285714, 0.304203802979989, -0.218409853507136, -0.273012316883171, 0.013650615844191, 0.040951847532400, 0.378326989229750)$$

$$= (a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}, a_{1,5}, a_{1,6}, a_{1,7}, a_{1,8}, a_{1,9}, a_{1,10})$$

Figure 3.1. .

$$
L_2 = \begin{bmatrix}
1.0000000000000 & 0 & 4.0000000000000 & -1.0000000000000 & 0 \\
0 & 0 & 0 & 0 & 1.0000000000000 \\
0.2500000000000 & 0.4330127018922 & 2.9820508075689 & -0.5000000000000 & 0.7500000000000 \\
0.5000000000000 & 0.5000000000000 & 4.2071067811866 & -0.7071067811865 & 0.5000000000000 \\
0.6545084971875 & 0.4755282581476 & 4.6746555185277 & -0.8090169943749 & 0.3454915028125 \\
0.7500000000000 & 0.4330127018922 & 0.6782032302755 & -0.8660254037844 & 0.2500000000000 \\
0.8117449009294 & 0.3909157412340 & 10.5516511352599 & -0.9009688679024 & 0.1882550990706 \\
0.8535533905933 & 0.3535533905933 & 10.3581667051082 & -0.9238795325113 & 0.1464466094067 \\
0.8830222215595 & 0.3213938048433 & 10.1512220074885 & -0.9396926207859 & 0.1169777784405
\end{matrix} \cdots
$$

$$
\cdots \begin{matrix}
0 & 0 & 16.0000000000000 & -4.0000000000000 & 1.0000000000000 \\
5.0000000000000 & -1.0000000000000 & 25.0000000000000 & -5.0000000000000 & 1.0000000000000 \\
5.1650635094611 & -0.8660254037844 & 35.5705080756888 & -5.9641016151378 & 1.0000000000000 \\
4.2071067811865 & -0.7071067811865 & 35.3994949366117 & -5.9497474683058 & 1.0000000000000 \\
3.3963360379853 & -0.5877852522925 & 33.3875026998190 & -5.7781919922947 & 1.0000000000000 \\
6.1650635094611 & -0.5000000000000 & 152.0320323027551 & -12.3301270189222 & 1.0000000000000 \\
5.0814073732528 & -0.4338837391176 & 137.1580425731783 & -11.7114492089228 & 1.0000000000000 \\
4.2904931305773 & -0.3826834323651 & 125.6999487943434 & -11.2115988509375 & 1.0000000000000 \\
3.6947426521538 & -0.3420201433257 & 116.6984315109634 & -10.8027048238376 & 1.0000000000000
\end{bmatrix}
$$

and

$$
\begin{aligned}
\mathbf{C}_2 = (&-0.563317835689615, -0.098065320807445, 0.014009331543937, \\
&0.019262830872911, -0.621106328308222, 0.028018663087836, \\
&0.035023328859776, -0.001751166442989, -0.005253499328940, \\
&0.533548006158840)
\end{aligned}
$$

$$
= (a_{2,1}, a_{2,2}, a_{2,3}, a_{2,4}, a_{2,5}, a_{2,6}, a_{2,7}, a_{2,8}, a_{2,9}, a_{2,10}).
$$

Let us recall the Brill's equations for the case n=2, K=4.

$$a_3^2a_5 - 4a_1a_5a_8 - a_2a_3a_6 + a_1a_6^2 + a_2^2a_8 = 0$$

$$a_4^2a_5 - 4a_1a_5a_{10} - a_2a_4a_7 + a_1a_7^2 + a_2^2a_{10} = 0$$

$$a_4^2a_8 - 4a_1a_8a_{10} - a_3a_4a_9 + a_1a_9^2 + a_3^2a_{10} = 0$$

$$a_6^2a_{10} - 4a_5a_8a_{10} - a_6a_7a_9 + a_7^2a_8 + a_9^2a_5 = 0$$

$$a_2^2a_9 - 4a_1a_5a_9 + 2a_1a_6a_7 + 2a_3a_4a_5 - a_2a_4a_6 - a_2a_3a_7 = 0$$

$$a_4^2a_6 - 4a_1a_6a_{10} + 2a_1a_7a_9 + 2a_2a_3a_{10} - a_3a_4a_7 - a_2a_4a_9 = 0$$

$$a_3^2a_7 - 4a_1a_7a_8 + 2a_1a_6a_9 + 2a_2a_4a_8 - a_3a_4a_6 - a_2a_3a_9 = 0$$

$$a_6^2a_4 - 4a_4a_5a_8 + 2a_5a_9a_3 + 2a_8a_7a_2 - a_6a_9a_2 - a_6a_3a_7 = 0$$

$$a_7^2a_3 - 4a_3a_5a_10 + 2a_5a_9a_4 + 2a_10a_2a_6 - a_7a_9a_2 - a_7a_4a_6 = 0$$

$$a_9^2a_2 - 4a_2a_8a_10 + 2a_8a_7a_4 + 2a_10a_3a_6 - a_9a_7a_3 - a_9a_4a_6 = 0$$

If we check the Brill's equations for $\mathbf{C}_1$ and $\mathbf{C}_2$, then both fail and we cannot continue the algorithm. For example, using $\mathbf{C}_1$ in $a_3^2a_5 - 4a_1a_5a_8 - a_2a_3a_6 + a_1a_6^2 + a_2^2a_8$ we have

$$a_3^2a_5 - 4a_1a_5a_8 - a_2a_3a_6 + a_1a_6^2 + a_2^2a_8 = -0.011175938533824 \neq 0.$$

and if we apply $\mathbf{C}_2$ in $a_4^2a_5 - 4a_1a_5a_{10} - a_2a_4a_7 + a_1a_7^2 + a_2^2a_{10}$ we have

$$a_4^2a_5 - 4a_1a_5a_{10} - a_2a_4a_7 + a_1a_7^2 + a_2^2a_{10} = -0.189097766457781 \neq 0.$$

Let us check if we can find scalars k_1 and k_2 such that $k_1\mathbf{C}_1 + k_2\mathbf{C}_2$ satisfies the Brill's equations.

Let

$$d_1 = k_1a_{1,1} + k_2a_{2,1}$$

$$d_2 = k_1a_{1,2} + k_2a_{2,2}$$

$$d_3 = k_1a_{1,3} + k_2a_{2,3}$$

$$d_4 = k_1a_{1,4} + k_2a_{2,4}$$

$$d_5 = k_1a_{1,5} + k_2a_{2,5}$$

$$d_6 = k_1a_{1,6} + k_2a_{2,6}$$

$$d_7 = k_1a_{1,7} + k_2a_{2,7}$$

$$d_8 = k_1a_{1,8} + k_2a_{2,8}$$

$$d_9 = k_1a_{1,9} + k_2a_{2,9}$$

$$d_{10} = k_1 a_{1,10} + k_2 a_{2,10}$$

After we plug in $(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10})$ into above Brill's equations we get six equations in terms of k_1 and k_2. Since Brill's equations are homogeneous of degree 3, we can reorder with k_1^3, $k_1^2 k_2$, $k_1 k_2^2$, k_2^3 terms.

From here we get six Brill's equation. After reordering and using $\frac{k_1}{k_2} = x$ we have the following equations.

$$E_1 = x^3 + 2.271653148339396x^2$$

$$+1.023165363024656x - 0.170752308808336 = 0$$

and three roots are

$$-1.5299456735200716, -0.8699922689250938,$$

and

$$0.1282847941057694$$

.

$$E_2 = x^3 + 1.6953341565236044x^2 - 1.6647781622525613x - 2.9341505530393284 = 0$$

and three roots are

$$-1.5299456735201447, -1.4700124796328748,$$

and

$$1.3046239966294175$$

.

$$E_3 = x^3 + 2.884613457839037x^2 + 1.882327849483978x - 0.291057272189508 = 0$$

and three roots are

$$-1.5299456735202328, -1.4829525784245736,$$

and

$$0.12828479410576987$$

.

$$E_4 = x^3 + 2.301747105931861x^2 + 1.065346886088750x - 0.176658812720654 = 0$$

and three roots are

$$-1.5299456735200976, -0.9000862265175333,$$

and

$$0.12828479410576998$$

.

$$E_5 = x^3 + 2.661668445993830x^2 + 1.569834548120250x - 0.247300129890861 = 0$$

and three roots are

$$-1.529945673519751, -1.2600075665797306,$$

and

$$0.1282847941056522$$

.

$$E_6 = x^3 + 2.990018426001852x^2 + 2.030069869853704x - 0.311744975191658 = 0$$

and three roots are

$$-1.5883575465874809, -1.5299456735201429,$$

and

$$0.12828479410577767$$

.

$$E_7 = x^3 + 1.27337608531049x^2 - 0.3760805430393185x + 0.025178298199436247 = 0$$

and three roots are

$$-1.5299456735199217, 0.12828478920302763,$$

and

$$0.1282847991591374$$

.

$$E_8 = x^3 + 1.273376085317707x^2 - 0.37608054303341537x + 0.025178298198560056 = 0$$

and three roots are

$$-1.529945673522462, 0.12828478374340213,$$

and

$$0.12828480465948586$$

.

$$E_9 = x^3 + 2.5404855534804424x^2 - 1.399977228429608x - 0.223515713151144 = 0$$

and three roots are

$$-1.5299456735201225, -1.1388246740661232,$$

and

$$0.12828479410582$$

.

$$E_10 = x^3+1.2733760853076657x^2-0.3760805430436313x+0.025178298200036035 = 0$$

and three roots are

$$-1.529945673520135, -0.9000862265175333,$$

and

$$0.12828479410576998$$

.

From six equations we have one common root $x = -1.5299456735$.

Now that we have verified that there are exactly two hyperplanes, we will find the normals to these hyperplanes.

Let $k_1 = -1.5299456735$ then

$$\mathbf{A} = k_1\mathbf{C}_1 + \mathbf{C}_2$$

$$= (-0.339538006421189, -1.267608557304117, 0.181086936757790,$$

$$0.248994538041047, -1.086521620545763, 0.362173873515199,$$

$$0.452717341892833, -0.022635867094692, -0.067907601283783,$$

$$-0.045271734189034)$$

$$= (c_{2,0,0,0}, c_{1,1,0,0}, c_{1,0,1,0}, c_{1,0,0,1}, c_{0,2,0,0}, c_{0,1,1,0}, c_{0,1,0,1}, c_{0,0,2,0}, c_{0,0,1,1}, c_{0,0,0,2}).$$

Let the two normal vectors be $(b_{11}, b_{12}, b_{13}, b_{14})$ and $(b_{21}, b_{22}, b_{23}, b_{24})$. Then we have the following equation.

$$(b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4)(b_{21}x_1 + b_{22}x_2 + b_{23}x_3 + b_{24}x_4)$$

$$= c_{2,0,0,0}x_1^2 + c_{1,1,0,0}x_1x_2 + c_{1,0,1,0}x_1x_3 + c_{1,0,0,1}x_1x_4 + c_{0,2,0,0}x_2^2$$

$$+c_{0,1,1,0}x_2x_3 + c_{0,1,0,1}x_2x_4 + c_{0,0,2,0}x_3^2 + c_{0,0,1,1}x_3x_4 + c_{0,0,0,2}x_4^2 = p_2(\mathbf{x}) \quad (3.1)$$

From the equation we can see that last two elements of normals correspond to only the last three terms of $p_2(\mathbf{x})$.

$$(b_{13}x_3 + b_{14}x_4)(b_{23}x_3 + b_{24}x_4) = c_{0,0,2,0}x_3^2 + c_{0,0,1,1}x_3x_4 + c_{0,0,0,2}x_4^2$$

Now, if we factor our x_4, we get a quadratic equation in terms of $t = \frac{x_3}{x_4}$, namely:$q_2(t)$ Therefore using $q_2(t)$ we will derive (b_{13}, b_{14}) and (b_{23}, b_{24}).

$$\frac{1}{{x_4}^2}(b_{13}x_3 + b_{14}x_4)(b_{23}x_3 + b_{24}x_4)$$
$$= \frac{1}{{x_4}^2}(b_{13}b_{23}{x_3}^2 + (b_{13}b_{24} + b_{14}b_{23})x_3x_4 + b_{14}b_{24}x_4^2)$$
$$= \frac{1}{{x_4}^2}(c_{0,0,2,0}{x_3}^2 + c_{0,0,1,1}x_3x_4 + c_{0,0,0,2}x_4^2)$$
$$= (c_{0,0,2,0}t^2 + c_{0,0,1,1}t + c_{0,0,0,2})$$

$$= q_2(t) = t^2 + 2.999999999987050t + 1.999999999984536$$

Here $q_2(t)$ has two real roots $t_1 = -2$ and $t_2 = -1$.

$$\therefore (b_{13}, b_{14}) = (1, 2), (b_{23}, b_{24}) = (1, 1)$$

Now we will try to solve for the first 2 entries of each b_i. Using the above equation (3.1) for $p_2(\mathbf{x})$ we get

$$(b_{12}x_2 + b_{13}x_3 + b_{14}x_4)(b_{22}x_2 + b_{23}x_3 + b_{24}x_4)$$

$$= c_{0,2,0,0}x_2^2 c_{0,1,1,0}x_2x_3 + c_{0,1,0,1}x_2x_4 + c_{0,0,2,0}x_3^2 + c_{0,0,1,1}x_3x_4 + c_{0,0,0,2}x_4^2$$

Hence

$$\begin{bmatrix} b_{23} & b_{13} \\ b_{24} & b_{14} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} \frac{c_{0,1,1,0}}{c_{0,0,2,0}} \\ \frac{c_{0,1,0,1}}{c_{0,0,2,0}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} -16.000000000005596 \\ -19.999999999955527 \end{bmatrix}$$

$$\therefore b_{12} = -12.000000000055664, b_{22} = -3.999999999949932$$

If we check the nonlinear equation,

$$b_{12}b_{22} = 47.999999999621842 \approx 48.000000000024166 = \frac{c_{0,2,0,0}}{c_{0,0,2,0}}$$

Continue the previous step using

$$(b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4)(b_{21}x_1 + b_{22}x_2 + b_{23}x_3 + b_{24}x_4)$$

$$= c_{2,0,0,0}x_1^2 + c_{1,1,0,0}x_1x_2 + c_{1,0,1,0}x_1x_3 + c_{1,0,0,1}x_1x_4 + c_{0,2,0,0}x_2^2$$

$$+ c_{0,1,1,0}x_2x_3 + c_{0,1,0,1}x_2x_4 + c_{0,0,2,0}x_3^2 + c_{0,0,1,1}x_3x_4 + c_{0,0,0,2}x_4^2$$

$$\begin{bmatrix} b_{22} & b_{12} \\ b_{23} & b_{13} \\ b_{24} & b_{14} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} \frac{c_{1,1,0,0}}{c_{0,0,2,0}} \\ \frac{c_{1,0,1,0}}{c_{0,0,2,0}} \\ \frac{c_{1,0,0,1}}{c_{0,0,2,0}} \end{bmatrix}$$

$$\begin{bmatrix} -3.999999999949932 & -12.000000000055664 \\ 1 & 1 \\ 1.0002299290618177 & 1.9997696752754424 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 56.000000000060290 \\ -8.000000000011205 \\ -10.999999999975021 \end{bmatrix}$$

$$\therefore b_{11} = -4.999733698582360, b_{21} = -3.000084995146104$$

If we check the nonlinear equation,

$$b_{11}b_{21} = 14.999626048843272 \approx 15.000000000035739 = \frac{c_{2,0,0,0}}{c_{0,0,2,0}}$$

The computed two normal vectors of hyperplanes are

$$(-4.999733698582360, -12.000000000055664, 1, 1.9997696752754424)$$

and

$$(-3.000084995146104, -3.999999999949932, 1, 1.0002299290618177)$$

which are very close to the true values $(-5, -12, 1, 2)$ and $(-3, -4, 1, 1)$.

3.2 Noisy Data

For the previous example, we were able to recover the two normal vectors using the GPCA. However, if we have noise in the sample points we might not be able to compute the correct rank(L_2). Next we demonstrate how to recover the two normal vectors when there is noise in the sample points. Here we have the sample points from the previous example.

$$\mathbf{x}^1(-1, 0, -4, 1),$$

$$\mathbf{x}^2(0, -1, -5, 1),$$

$$\mathbf{x}^3(-0.5, -0.866025403784438, -5.964101615137755, 1),$$

$$\mathbf{x}^4(-0.707106781186548, -0.707106781186547, -5.949747468305833, 1),$$

$$\mathbf{x}^5(-0.809016994374947, -0.587785252292473, -5.778191992294735, 1),$$

$$\mathbf{x}^6(-0.866025403784439, -0.5, -12.330127018922193, 1),$$

$$\mathbf{x}^7(-0.900968867902419, -0.433883739117558, -11.711449208922794, 1),$$

$$\mathbf{x}^8(-0.923879532511287, -0.382683432365090, -11.211598850937513, 1),$$

$$\mathbf{x}^{9}(-0.939692620785908, -0.342020143325669, -10.802704823837567, 1).$$

Now we will add small random numbers (of the order 10^{-8}) to the values of sample points. Then the new 9 sample points are

$$\mathbf{x}'^{1}(-1, -1.576130813606815 \times 10^{-8}, -3.999999960777298, 1),$$

$$\mathbf{x}'^{2}(9.571669483070567 \times 10^{-8}, -0.999999999999995, -4.999999934452211, 1),$$

$$\mathbf{x}'^{3}(-0.499999930693677, -0.866025443798459, -5.964101598019086, 1),$$

$$\mathbf{x}'^{4}(-0.707106771153668, -0.707106791219427, -5.949747397701224, 1),$$

$$\mathbf{x}'^{5}(-0.809016969584441, -0.587785286413677, -5.778191989111450, 1),$$

$$\mathbf{x}'^{6}(-0.866025357997659, -0.500000079305021, -12.330126991229895, 1),$$

$$\mathbf{x}'^{7}(-0.900968833529828, -0.433883810492971, -11.711449204305655, 1),$$

$$\mathbf{x}'^{8}(-0.923879495793097, -0.382683521010629, -11.211598841224335, 1),$$

$$\mathbf{x}'^9(-0.939692598358254, -0.342020204945137, -10.802704741491784, 1).$$

We have emphasized with the red font the randomly changed digits. We will now check the number of hyperplanes using these new 9 sample points.

For $i = 1$, we have

$$L'_1 = \begin{bmatrix} \mathcal{V}_1(-1.000000000000000, -1.576130813606815 \times 10^{-8}, -3.999999960777298, 1) \\ \mathcal{V}_1(9.571669483070567 \times 10^{-8}, -0.999999999999995, -4.999999934452211, 1) \\ \mathcal{V}_1(-0.499999930693677, -0.866025443798459, -5.964101598019086, 1) \\ \mathcal{V}_1(-0.707106771153668, -0.707106791219427, -5.949747397701224, 1) \\ \mathcal{V}_1(-0.809016969584441, -0.587785286413677, -5.778191989111450, 1) \\ \mathcal{V}_1(-0.866025357997659, -0.500000079305021, -12.330126991229895, 1) \\ \mathcal{V}_1(-0.900968833529828, -0.433883810492971, -11.711449204305655, 1) \\ \mathcal{V}_1(-0.923879495793097, -0.382683521010629, -11.211598841224335, 1) \\ \mathcal{V}_1(-0.939692598358254, -0.342020204945137, -10.802704741491784, 1) \end{bmatrix}$$

$$L'_1 = \begin{bmatrix} -1.000000000000000 & -0.000000015761308 & -3.999999960777298 & 1 \\ 0.000000095716695 & -0.999999999999995 & -4.999999934452211 & 1 \\ -0.499999930693677 & -0.866025443798459 & -5.964101598019086 & 1 \\ -0.707106771153668 & -0.707106791219427 & -5.949747397701224 & 1 \\ -0.809016969584441 & -0.587785286413677 & -5.778191989111450 & 1 \\ -0.866025357997659 & -0.500000079305021 & -12.330126991229895 & 1 \\ -0.900968833529828 & -0.433883810492971 & -11.711449204305655 & 1 \\ -0.923879495793097 & -0.382683521010629 & -11.211598841224335 & 1 \\ -0.939692598358254 & -0.342020204945137 & -10.802704741491784 & 1 \end{bmatrix}$$

where $\mathcal{V}_1(x) = (x_1, x_2, x_3, x_4)$. We have $\text{rank}(L'_1) = 4 > 3 = M_1 - 1 = \binom{4}{1} - 1$. Therefore, we conclude that the number of hyperplanes n is greater than 1.

For $i = 2$, we have the matrix in (3.2)

$$L'_1 =$$
$$\begin{bmatrix} \mathcal{V}_2(-1.000000000000000, -1.576130813606815 \times 10^{-8}, -3.999999960777298, 1) \\ \mathcal{V}_2(9.571669483070567 \times 10^{-8}, -0.999999999999995, -4.999999934452211, 1) \\ \mathcal{V}_2(-0.499999930693677, -0.866025443798459, -5.964101598019086, 1) \\ \mathcal{V}_2(-0.707106771153668, -0.707106791219427, -5.949747397701224, 1) \\ \mathcal{V}_2(-0.809016969584441, -0.587785286413677, -5.778191989111450, 1) \\ \mathcal{V}_2(-0.866025357997659, -0.500000079305021, -12.330126991229895, 1) \\ \mathcal{V}_2(-0.900968833529828, -0.433883810492971, -11.711449204305655, 1) \\ \mathcal{V}_2(-0.923879495793097, -0.382683521010629, -11.211598841224335, 1) \\ \mathcal{V}_2(-0.939692598358254, -0.342020204945137, -10.802704741491784, 1) \end{bmatrix}$$

where $\mathcal{V}_2(x) = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2)$. Using these sample points, now the $\text{rank}(L'_2) = 9 = M_2 - 1 = \binom{5}{2} - 1$.

Consider $\mathbf{C}'_1$ which is a basis of $\text{Ker}(L'_2)$

$$\mathbf{C}'_1 = (-0.577350244951306, -0.000000254494387, 0.000000036356339, \\ 0.000000049991779, -0.577350394922908, 0.000000072713188, \\ 0.000000090893635, -0.000000004544585, -0.000000013634128, \\ 0.577350167694568).$$

If we check the Brill's equations for $\mathbf{C}'_1$, it does not satisfy Brill's equations and we cannot continue the algorithm. For example, using $\mathbf{C}'_1$ in $a_4^2a_5 - 4a_1a_5a_{10} - a_2a_4a_7 + a_1a_7^2 + a_2^2a_{10}$ we have

Figure 3.2. .

$$L_2' = \begin{bmatrix} 0.010000000000000 & 0.000000000157613 & 0.039999999607773 & -0.010000000000000 & 0.000000000000000 \\ 0.000000000000000 & -0.000000000957167 & -0.000000004785835 & 0.000000000957167 & 0.010000000000000 \\ 0.002499999306937 & 0.004330126618782 & 0.029820503856596 & -0.004999999306937 & 0.007500000693063 \\ 0.004999999858114 & 0.005000000000000 & 0.042071066715685 & -0.007071067711537 & 0.005000000141886 \\ 0.006545084570756 & 0.004755282711807 & 0.046746553727080 & -0.008090169695844 & 0.003454915429244 \\ 0.007499999206950 & 0.004330127476790 & 0.106782026417365 & -0.008660253579977 & 0.002500000793050 \\ 0.008117448389921 & 0.003909157906273 & 0.105516507285471 & -0.009009688335298 & 0.001882551610079 \\ 0.008535533227469 & 0.003535534584396 & 0.103581662844648 & -0.009238794957931 & 0.001464466772531 \\ 0.008830221794093 & 0.003213938550759 & 0.101512216878294 & -0.009396925983583 & 0.001169778205907 \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} 0.000000000630452 & -0.000000000157613 & 0.159999996862184 & -0.039999999607773 & 0.010000000000000 \\ 0.049999999344522 & -0.010000000000000 & 0.249999993445221 & -0.049999999344522 & 0.010000000000000 \\ 0.051650637332836 & -0.008660254437985 & 0.355705078714938 & -0.059641015980191 & 0.010000000000000 \\ 0.042071067909546 & -0.007071067912194 & 0.353994940964525 & -0.059497473977012 & 0.010000000000000 \\ 0.033963362332731 & -0.005877852864137 & 0.333875026630317 & -0.057781919891115 & 0.010000000000000 \\ 0.061650644734559 & -0.005000000793050 & 1.520320316198560 & -0.123301269912299 & 0.010000000000000 \\ 0.050814082071590 & -0.004338838104930 & 1.371580424650316 & -0.117114492043057 & 0.010000000000000 \\ 0.042904941207184 & -0.003826835210106 & 1.256999485765429 & -0.112115988412243 & 0.010000000000000 \\ 0.036947432896468 & -0.003420202049451 & 1.166984297318491 & -0.108027047414918 & 0.010000000000000 \end{bmatrix}$$

$$a_4^2 a_5 - 4a_1 a_5 a_{10} - a_2 a_4 a_7 + a_1 a_7^2 + a_2^2 a_{10} = 0.769800358919500 \neq 0.$$

Because of the noise, $\mathrm{rank}(L_2')$ maybe larger than $\mathrm{rank}(L_2)$.

Let L_2'' be a new matrix obtained by adding one 0 row at the bottom of L_2'. If we check the eigenvalues of L_2', we will have one eigenvalue that is exactly equal to 0, and another one very close to zero, namely equal to 0.0000001350072. Here are all eigenvalues in the decreasing order of magnitude

$$123.1812669555284$$

$$0.5733274537547 + 0.026931470031760i$$

$$0.5733274537547 - 0.026931470031760i$$

$$-0.5437894833168$$

$$0.4362298613775$$

$$0.0497281593562$$

$$-0.0192161331569$$

$$-0.0020145967624$$

$$-0.0000001108077$$

$$0$$

One can introduce a threshold value of accuracy, below which all numbers are considered to be 0. For example, in this case let us choose the accuracy threshold to be 10^{-6}.

We will interpret this as the rank(L_2') being "approximately" 8.

This means number of hyperplanes may be equal to 2, depending on whether any vector of the kernel satisfies Brill's equations. Therefore, we continue with GPCA using basis of "approximate" Ker(L_2), i.e. the span of eigenvectors corresponding to the smallest two eigenvalues.

Consider the following $\mathbf{C}_1$ and $\mathbf{C}_2$ eigenvectors of -0.0000001108077 and 0 where

$$\mathbf{C}_1 = (-0.206525641293860, -0.680300981723725, 0.097184598891111,$$

$$0.133622263511375, -0.607416783149077, 0.194370177246331,$$

$$0.242955244820273, -0.012148145384769, -0.036444965502194,$$

$$0.000000000000000)$$

$$= (a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}, a_{1,5}, a_{1,6}, a_{1,7}, a_{1,8}, a_{1,9}, a_{1,10})$$

and

$$\mathbf{C}_2 = (-0.577350269017577, -0.000000001805945, 0.000000000257993,$$

$$0.000000000354870, -0.577350270081916, 0.000000000516019,$$

$$0.000000000645168, -0.000000000032252, -0.000000000096780,$$

$$0.577350268469384)$$

$$= (a_{2,1}, a_{2,2}, a_{2,3}, a_{2,4}, a_{2,5}, a_{2,6}, a_{2,7}, a_{2,8}, a_{2,9}, a_{2,10}).$$

If we check the Brill's equations for $\mathbf{C}_1$ and $\mathbf{C}_2$, then both fail and we cannot continue the algorithm. For example, using $\mathbf{C}_1$ in $a_3^2a_5 - 4a_1a_5a_8 - a_2a_3a_6 + a_1a_6^2 + a_2^2a_8$ we have

$$a_3^2a_5 - 4a_1a_5a_8 - a_2a_3a_6 + a_1a_6^2 + a_2^2a_8 = -0.0002151831461315815 \neq 0.$$

using $\mathbf{C}_2$ in $a_4^2a_5 - 4a_1a_5a_{10} - a_2a_4a_7 + a_1a_7^2 + a_2^2a_{10}$ we have

$$a_4^2a_5 - 4a_1a_5a_{10} - a_2a_4a_7 + a_1a_7^2 + a_2^2a_{10} = -0.769800358919501 \neq 0.$$

Let us check if we can find scalars k_1 and k_2 such that $k_1\mathbf{C}_1 + k_2\mathbf{C}_2$ satisfies the Brill's equations.

Let

$$d_1 = k_1a_{1,1} + k_2a_{2,1}$$

$$d_2 = k_1 a_{1,2} + k_2 a_{2,2}$$

$$d_3 = k_1 a_{1,3} + k_2 a_{2,3}$$

$$d_4 = k_1 a_{1,4} + k_2 a_{2,4}$$

$$d_5 = k_1 a_{1,5} + k_2 a_{2,5}$$

$$d_6 = k_1 a_{1,6} + k_2 a_{2,6}$$

$$d_7 = k_1 a_{1,7} + k_2 a_{2,7}$$

$$d_8 = k_1 a_{1,8} + k_2 a_{2,8}$$

$$d_9 = k_1 a_{1,9} + k_2 a_{2,9}$$

$$d_{10} = k_1 a_{1,10} + k_2 a_{2,10}$$

After we plug in $(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10})$ into above Brill's equations we get six equations in terms of k_1 and k_2. Since Brill's equations are homogeneous of degree 3, we can reorder with k_1^3, $k_1^2 k_2$, $k_1 k_2^2$, k_2^3 terms.

From here we get six Brill's equation. After reordering and using $\frac{k_1}{k_2} = x$ we have the following equations.

$$E_1 = x^3 + 20.58728416313517x^2 - 75.27321450097105x$$

$$-1.9984216840966619 \times 10^{-7} = 0$$

and have three roots

$$-23.755896045743782, -2.6548908478191776 \times 10^{-9}$$

and

$$3.1686118852635046.$$

$$E_2 = x^3 + 70.37135394189214x^2 + 1141.6962355517692x$$

$$+809.8344680934741 = 0$$

and have three roots

$$-45.862452029778744, -23.765909201191526$$

$$-0.7429927109218613.$$

$$E_3 = x^3 + 61.74603821492648x^2 + 902.6984922715698x$$

$$+2.396565965036748 * 10^-6 = 0$$

and have three roots

$$-37.975470557584444, -23.770567654687138$$

and

$$-2.654890847819178 \times 10^{-9}.$$

$$E_4 = x^3 + 20.307048099855663x^2 - 82.14352009679232x$$

$$+2.1808207985575776 \times 10^{-7} = 0$$

and have three roots

$$-23.763725030752187, 2.65489085130413 \times 10^{-9}$$

and

$$3.456676928241635.$$

$$E_5 = x^3 + 7.8830252291184735x^2 - 376.51993331444237x$$

$$-9.998527545870223 \times 10^{-7} = 0$$

and have three roots

$$-23.741902887557863, -2.6555108138097653 \times 10^{-9}$$

and

$$15.858877661094898.$$

$$E_6 = x^3 + 40.292941118755664x^2 + 392.7683353484685x$$

$$+1.0427315880158164 \times 10^{-6} = 0$$

and have three roots

$$-23.76751194208549, -16.52542917401535$$

and

$$-2.654825999083357 \times 10^{-9}.$$

$$E_7 = x^3 + 23.749340648505118x^2 + 1.261104134416643 \times 10^{-7}x$$

$$+1.6741344557652653 \times 10^{-16} = 0$$

and have three roots

$$-23.749340643195058, -2.6555401470890325 \times 10^{-9}$$

and

$$-2.654519417479263 \times 10^{-9}.$$

$$E_8 = x^3 + 23.861236221690845x^2 + 1.2658146096349668 \times 10^{-7}x$$

$$+1.6787530197063586 \times 10^{-16} = 0$$

and have three roots

$$-23.861236216385947, -2.655207254844815 \times 10^{-9}$$

and

$$-2.6496922821511506 \times 10^{-9}.$$

$$E_9 = x^3 + 15.75992266009988x^2 - 190.22445627684752x$$

$$-5.049830808481658 * 10^-16 = 0$$

and have three roots

$$-23.764489866960627, -2.654669597862996\times^{-18}$$

and

$$8.004567206860747.$$

$$E_{10} = x^3 + 23.76524420403986x^2 + 1.2617784396817842 \times 10^{-7}x$$

$$+1.6748037937687166 * 10^-16 = 0$$

and have three roots

$$-23.765244198730517, -2.6546717320891496 \times 10^{-9} - 2.584218023965645 \times 10^{-13}i$$

and

$$-2.6546717320891496 \times 10^{-9} + 2.584218023965645 \times 10^{-13}.$$

From six equations, the closest common root is $x = -23.7$.

Now that we have verified that there are exactly two hyperplanes, we will find the normals to these hyperplanes.

Let $k_1 = -23.7$ then

$$\mathbf{A} = k_1\mathbf{C}_1 + \mathbf{C}_2$$

$$= (4.317307429646905, 16.123133265046338, -2.303274993461338,$$

$$-3.166847644864717, 13.818427490551208, -4.606573200222025,$$

$$-5.758039301595302, 0.287911045586773, 0.863745682305218,$$

$$0.577350268469384)$$

$$= (c_{2,0,0,0}, c_{1,1,0,0}, c_{1,0,1,0}, c_{1,0,0,1}, c_{0,2,0,0}, c_{0,1,1,0}, c_{0,1,0,1}, c_{0,0,2,0}, c_{0,0,1,1}, c_{0,0,0,2}).$$

Let the two normal vectors be $(b_{11}, b_{12}, b_{13}, b_{14})$ and $(b_{21}, b_{22}, b_{23}, b_{24})$. Then we have the following (3.1) equation.

$$(b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4)(b_{21}x_1 + b_{22}x_2 + b_{23}x_3 + b_{24}x_4)$$

$$= c_{2,0,0,0}x_1^2 + c_{1,1,0,0}x_1x_2 + c_{1,0,1,0}x_1x_3 + c_{1,0,0,1}x_1x_4 + c_{0,2,0,0}x_2^2$$

$$+c_{0,1,1,0}x_2x_3 + c_{0,1,0,1}x_2x_4 + c_{0,0,2,0}x_3^2 + c_{0,0,1,1}x_3x_4 + c_{0,0,0,2}x_4^2 = p_2(\mathbf{x})$$

From the equation we can see that last two elements of normals correspond to only the last three terms of $p_2(\mathbf{x})$.

$$(b_{13}x_3 + b_{14}x_4)(b_{23}x_3 + b_{24}x_4) = c_{0,0,2,0}x_3^2 + c_{0,0,1,1}x_3x_4 + c_{0,0,0,2}x_4^2$$

Now, if we factor our x_4, we get a quadratic equation in terms of $t = \frac{x_3}{x_4}$, namely: $q_2(t)$. Therefore using $q_2(t)$ we will derive (b_{13}, b_{14}) and (b_{23}, b_{24}).

$$\frac{1}{{x_4}^2}(b_{13}x_3 + b_{14}x_4)(b_{23}x_3 + b_{24}x_4)$$

$$= \frac{1}{{x_4}^2}(b_{13}b_{23}{x_3}^2 + (b_{13}b_{24} + b_{14}b_{23})x_3x_4 + b_{14}b_{24}x_4^2)$$

$$= \frac{1}{{x_4}^2}(c_{0,0,2,0}{x_3}^2 + c_{0,0,1,1}x_3x_4 + c_{0,0,0,2}x_4^2)$$

$$= (c_{0,0,2,0}t^2 + c_{0,0,1,1}t + c_{0,0,0,2})$$

$$= q_2(t) = t^2 + 3.000043574378584t + 2.005307810586855$$

Here $q_2(t)$ has two real roots

$$t_1 = -1.9947515633860582$$

and

$$t_2 = -1.0052920109925259.$$

$$\therefore (b_{13}, b_{14}) = (1, 1.9947515633860582), (b_{23}, b_{24}) = (1, 1.0052920109925259)$$

Now we will try to solve for the first 2 entries of each b_i. Using the above equation (3.1) for $p_2(\mathbf{x})$ we get

$$(b_{12}x_2 + b_{13}x_3 + b_{14}x_4)(b_{22}x_2 + b_{23}x_3 + b_{24}x_4)$$

$$= c_{0,2,0,0}x_2^2 c_{0,1,1,0}x_2x_3 + c_{0,1,0,1}x_2x_4 + c_{0,0,2,0}x_3^2 + c_{0,0,1,1}x_3x_4 + c_{0,0,0,2}x_4^2$$

Hence

$$\begin{bmatrix} b_{23} & b_{13} \\ b_{24} & b_{14} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} \frac{c_{0,1,1,0}}{c_{0,0,2,0}} \\ \frac{c_{0,1,0,1}}{c_{0,0,2,0}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1.005292010992525 & 1.9947515633860582 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} -15.999987742164127 \\ -19.999369214405135 \end{bmatrix}$$

$$\therefore b_{12} = -12.043576030579342, b_{22} = -3.956411711584783$$

If we check the nonlinear equation,

$$b_{12}b_{22} = 47.661810617308099 \approx 47.992693106428113 = \frac{c_{0,2,0,0}}{c_{0,0,2,0}}$$

Continue the previous step using

$$(b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4)(b_{21}x_1 + b_{22}x_2 + b_{23}x_3 + b_{24}x_4)$$

$$= c_{2,0,0,0}x_1^2 + c_{1,1,0,0}x_1x_2 + c_{1,0,1,0}x_1x_3 + c_{1,0,0,1}x_1x_4 + c_{0,2,0,0}x_2^2$$

$$+c_{0,1,1,0}x_2x_3 + c_{0,1,0,1}x_2x_4 + c_{0,0,2,0}x_3^2 + c_{0,0,1,1}x_3x_4 + c_{0,0,0,2}x_4^2$$

$$\begin{bmatrix} b_{22} & b_{12} \\ b_{23} & b_{13} \\ b_{24} & b_{14} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} \frac{c_{1,1,0,0}}{c_{0,0,2,0}} \\ \frac{c_{1,0,1,0}}{c_{0,0,2,0}} \\ \frac{c_{1,0,0,1}}{c_{0,0,2,0}} \end{bmatrix}$$

$$\begin{bmatrix} -3.956411711584783 & -12.043576030579342 \\ 1 & 1 \\ 1.005292010992525 & 1.9947515633860582 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 56.000398429267683 \\ -7.999953557763584 \\ -10.999396144773009 \end{bmatrix}$$

$$\therefore b_{11} = -4.976075610989117, b_{21} = -3.014960108149451$$

If we check the nonlinear equation,

$$b_{11}b_{21} = 15.004571503207352 \approx 14.993152939820282 = \frac{c_{2,0,0,0}}{c_{0,0,2,0}}$$

The computed two normal vectors of hyperplanes are

$$(-4.976075610989117, -12.043576030579342, 1, 1.9947515633860582)$$

and

$$(-3.014960108149451, -3.956411711584783, 1, 1.005292010992525).$$

which are very close to the true values $(-5, -12, 1, 2)$ and $(-3, -4, 1, 1)$.

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BIOGRAPHICAL STATEMENT

Choi, Sl-ghi was born in Virginia, USA, in 1989. He received his B.S. in Mathematics from Chungnam National University in 2012 and Ph.D. in Mathematics from The University of Texas at Arlington in 2017 in Mathematics. His research interest include inverse problems, Radon transform and Generalized Principal Component Analysis (GPCA). Apart from the research, he is interested in math education.