One of the most significant conceptual jumps involved in algebra is the use of letters (and other symbols) to represent numbers. One way to get students used to this notion is via puzzles. This paper describes a generalization of one such puzzle, based on a terse telegraph message from impoverished college student to parents — “SEND MORE MONEY” — and well-known in some mathematical circles (a December 2001 web search turned up over two dozen hits). In this type of puzzle, which has been termed alphametics, each word is associated to a number with as many digits as there are letters, and each letter is associated with a distinct digit, 0 through 9. A puzzle solution identifies which digit corresponds to each of the letters.

The “Send More Money” problem, credited to Henry E. Dudeney in 1924, reads as follows:

\[
\begin{array}{c}
S E N D \\
+ M O R E \\
\hline
M O N E Y
\end{array}
\]

How many solutions are there, and what are they?

It is generally recognized that this problem has a unique solution using our base ten number system (but see caveat in the last paragraph below). However, after working with prospective mathematics teachers on a unit involving arithmetic in other bases, it occurred to me to extend the problem. Can this problem be solved in other bases than ten? The solution to this question requires some persistence like any good problem, but no more complicated mathematics than “puzzle algebra”, that is, the notion of each letter representing a different digit (0 through 9 for base ten, 0 through \(b - 1\) for
This extension, like the unit in the aforementioned course, makes one take a second look at the conceptual underpinnings of the multidigit addition algorithm.

**Unique values for some letters** First, by listing out the letters involved — S, E, N, D, M, O, R, Y — we can see that the base b must be at least eight, since there are eight different letters with different values. We also need to decide whether 0 is an allowable value for S or M, the lead digits in the addends and sum. Most students will rule out such possibilities, since we normally write whole numbers without leading zeroes. The mathematicians in the audience may, however, object in favor of leading zeroes since, for example, it is still true that 8324 + 0913 = 09237. As always, the specification of what constitutes an acceptable solution must be defined by those considering the problem. In this case, the convention not to allow leading zeroes not only significantly reduces the complexity and number of solutions of the problem to a level manageable for high school and college students, but follows the decision such groups commonly make in solving the base ten version. Consequently the discussion below will follow this convention, but we will return to this point.

Next we need to decide on a plan of attack. After either some trial and error or some inspiration we find that it will be easier if we work from left to right, i.e., from greater place value to lesser. If we look at the far left places, we can see that the only nonzero possibility for M is 1, since the sum in the next lowest place, which is either S + M or S + M + 1, can be at most \((b - 1) + (b - 2) + 1 = 2b - 2 < 2b\). (The sum is \(S + M + 1\) if the E + O place generated an extra “1” — that is, a set of \(b^3\) to put in the \(b^3\)s’ place along with S and M.) In other words, no more than one set of \(b\) can be regrouped up to the next place, where M sits as the “answer”.

Since we have already seen that the issue of whether a 1 is “carried” from one place to the next higher place is likely to come into play frequently, it will simplify our discussion if we develop clear notation for such 1s. Therefore
we will write \(1_{01}\) for a regrouping from the \(b^0\)'s (rightmost) place to the \(b^1\)'s (second from the right) place, \(1_{12}\) for a regrouping from \(b^1\)s to \(b^2\)s, etc.

If we now consider the \(b^3\)'s place, we have either \(S + 1 = “1O” = b + O\) (if there is no \(1_{23}\)) or \(S + 1 + 1_{23} = “1O” = b + O\) (where “1O” is the two-digit number made of 1 followed by whatever digit is represented by the letter O).

In either case, \(S + 2 \geq b\), i.e., \(S \geq b - 2\). Since \(S < b\) as a single digit, \(S \) must be either \(b - 2\) or \(b - 1\). If \(S = b - 2\), then \(S + M = b - 1 < b\), so in order to generate the \(M = 1\) in the \(b^4\)'s place of the sum, there must be a \(1_{23}\), so that the addition in this column is \(S + M + 1_{23} = b\), making \(O = 0\). If instead \(S = b - 1\), then we have \(S + M = b + O\) if there is no \(1_{23}\), and \(S + M + 1_{23} = b + 1 = b + O\) if there is a \(1_{23}\). In the former case, \(O = 0\), and in the latter case \(O = 1\). However, since by hypothesis all letters are distinct digits and \(M = 1\), then \(O \) cannot be 1. Hence \(O = 0\) (which fortunate conclusion also relieves us of further typographical confusion).

If we now move to the \(b^2\)'s place, we have four possibilities: either \(E + 0 = N\) or \(E + 0 + 1_{12} = N\) (if there is no \(1_{23}\), and \(E + 0 = “1N” = b + N\) or \(E + 0 + 1_{12} = “1N” = b + N\) if there is to be a \(1_{23}\)). Since \(E \neq N\), the first possibility is ruled out; the fact that \(E < b\) as a digit rules out the third. (Thus we see that there must be a \(1_{12}\).) The fourth possibility, \(E + 1 = N + b\), can only be true if \(N = 0\) and \(E = b - 1\). But \(O = 0\) so \(N \neq 0\). Thus it must be that \(E + 1 = N < b\), and there is no \(1_{23}\). This makes the addition in the \(b^3\)'s place \(S + M = “MO”\), that is, \(S + 1 = “10” = b + 0\). Hence \(S = b - 1\).

For the remaining two places, there are two possibilities:

(a) \(D + E = Y < b\) and \(N + R = “1E” = b + E\), or

(b) \(D + E = “1Y” = b + Y\) and \(N + R + 1_{01} = “1E” = b + E\).

If we substitute \(N = E + 1\) from above, the second part of (a) becomes \(E + 1 + R = b + E\), which implies \(R = b - 1\). But \(S = b - 1\), so \(R \neq b - 1\). Thus instead we must have (b) \(E + 1 + R + 1_{01} = b + E\), making \(R = b - 2\).

**Man Y ENDings** We have now found unique values for four of the eight letters in the problem. The values of the remaining four letters may or
may not be unique, depending on \( b \). We need distinct digits \( Y, E, D, N \) such that \( D + E = b + Y \) (since there is a 1\(_{001}\)) and \( N = E + 1 \). With two equations and four unknowns we can choose values for two of the four (within constraints), and then use the two equations to give us the remaining two values. Here there are many approaches possible, but as an example let us consider \( Y \) first. Since \( O = 0 \) and \( M = 1 \), we know \( Y \geq 2 \). To determine an upper bound on \( Y \), we can recall that \( D + E = b + Y \) and find an upper bound for \( D + E \). Since \( S = b - 1, R = b - 2, \) all remaining digits are bounded by \( b - 3 \). Since \( N = E + 1, E \leq b - 4 \) and \( D \neq E + 1 \). Therefore \( D + E \leq (b - 3) + (b - 5) = 2b - 8, \) so \( Y \leq (2b - 8) - b = b - 8. \) Hence \( 2 \leq Y \leq b - 8 \). Note that this requires \( 2 \leq b - 8, \) i.e., \( b \geq 10 \), so there are no solutions for \( b = 8, 9 \).

If, having chosen \( Y \), we next choose \( E \), then \( D \) and \( N \) will follow from the two equations. Recall \( E \leq b - 4 \). We also have that \( b + Y - E = D \leq b - 3, \) so that \( E \geq Y + 3 \). Thus \( Y + 3 \leq E \leq b - 4 \). However, we must make sure that \( D \neq E, N: \)

\[
\begin{align*}
  b + Y - E &\neq E, E + 1 \\
  b + Y &\neq 2E, 2E + 1 \\
  E &\neq \frac{b + Y}{2}, \frac{b + Y - 1}{2}
\end{align*}
\]

Only one of these latter two will be a whole number, so we can rewrite this condition as \( E \neq \lfloor \frac{b + Y}{2} \rfloor, \) where \( \lfloor x \rfloor \) is the greatest integer in \( x \).

To verify that \( \lfloor \frac{b + Y}{2} \rfloor \) falls within the range \( [Y + 3, b - 4] \), note first that \( b + Y \leq 2b - 8 \) (from above), so \( \lfloor \frac{b + Y}{2} \rfloor \leq \frac{b + Y}{2} \leq b - 4 \). Also \( Y < b - 7, \) so \( Y + 6 < b - 1, 2Y + 6 < b + Y - 1, \) and \( Y + 3 < \frac{b + Y - 1}{2} \leq \lfloor \frac{b + Y}{2} \rfloor, \) Thus, once \( Y \) is chosen, we may choose any \( E: Y + 3 \leq E \leq b - 4 \) except \( \lfloor \frac{b + Y}{2} \rfloor \). Then distinct \( D = Y + b - E, N = E + 1 \) will follow. (Note \( D \neq Y \) since \( b - E > 3 \).

For \( b = 10, \) the only solution is \( Y = 2, E = 5, \) which leads to \( D = 7, N = 6 \) and the following addition:
For $b=$eleven, there are three solutions: $(Y,E) = (2,5), (2,7)$ or $(3,6)$. More
generally there are $b−9$ choices for $Y$ (namely, the numbers $2, ..., b−8$),
and given $Y$ there are $b−Y−7$ choices for $E$ (after excluding $\lfloor \frac{b+Y}{2} \rfloor$).

For $Y = 2$, this is $b−9$ choices for $E$;
for $Y = 3$, this is $b−ten$ choices for $E$;
...
for $Y = b−8$, this is 1 choice for $E$.
Thus the total number of solutions is $(b−9)+(b−ten)+...+2+1 = \frac{(b−9)(b−8)}{2}$.
(This formula also correctly gives 0 for $b = 8, 9$.)

As a postscript, we reconsider allowing leading zeroes, in which case it
is possible that $M = 0$. This line of reasoning does not allow us to solve
uniquely for as many letters as was done above, but does lead to solutions in
all bases eight and above; in fact, there are more solutions with $M = 0$ than
there are with $M = 1$. However, the number is more difficult to count: for $b = 8$, there are three solutions (SENDMORY=26170345, 53140627, 54270613),
for $b = 9$ there are eleven, and in base ten there are twenty-four.

References
