INTERNAL STRUCTURE AND DYNAMIC DECISIONS

FOR COALITIONS ON GRAPHS

by

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Abstract

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The notions of collaboration, advantage, cost and impact are investigated in this dissertation, following are the details.

Contribution to Cooperation between Agents for Routing in an Unknown Graph Using Reinforcement Learning

First of all this work investigates the cooperation between agents to achieve a common goal. The problem of steering a swarm of autonomous agents out of an unknown maze to some goal located at an unknown location is discussed in this context. The routing algorithm given here provides a mechanism for storing data based on the experiences of previous agents visiting a node that results in routing decisions that improve with time.

Contribution to Coalitional Advantage in Agents and Structure in Coalitions on Graphs

Next a certain graphical coalitional game is introduced, where the internal topology of the coalition depends on a prescribed communication graph structure among the agents. Three measures of the contributions of agents to a coalition are introduced: marginal contribution, competitive contribution, and altruistic contribution. Results are established regarding the dependence of these three types of contributions on the graph topology, and changes in these contributions due to changes in graph topology. Based on these different contributions, three

online sequential decision games are defined on top of the graphical coalitional game. The stable graphs under each of these sequential decision games are studied.

Contribution to Coalitional Cost and Advantage in Agents and Structure in Coalitions on Graphs

Here a Positional Cost is also introduced. Based on the advantage and cost, a notion of Net Payoff or Allocation is defined; this notion is used to further define three measures of net advantages. Taking maximization of these measures of net advantages as the objective functions of agents, three online sequential decision games are defined. A threshold of cost is reached above which no agent is interested to stay in a coalition irrespective of their motives.

Contribution to Coalitional Advantage in Agents and Structure in Coalitions on Digraphs

Novel digraph structures, are defined. The marginal contribution made by an agent in a digraph is introduced. Results are established regarding the dependence of marginal contributions on the graph topology, and changes in these contributions due to changes in graph topology. Based on marginal contributions, and by varying the rules of the game, three online sequential decision games are defined on top of the graphical coalitional game. The stable graphs under each of these sequential decision games are studied, and give the structures of the coalitions that form in each sequential game.

Contribution to Coalitional Impact of Agents on Other Agents

Finally the notion of impact of one agent in the coalition upon the other agents is investigated, and a complete impact propagation framework is proposed. The analogy of impact propagation is use to present a comprehensive, and integrated framework to compute the impact made by a scientific work, a scientist, an institution, and a scientific journal.

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Chapter 1

Introduction

This chapter provides an outline of introductory and background material for this thesis. More detailed introductions are given at the beginning of each chapter.

1.1 Cooperation between Agents for Routing in an Unknown Graph Using Reinforcement Learning

In the literature there is a great deal of material about swarming of agents and their consensus [1], [2], [3], [4], [5], [6], [7], [8]. This material is about agents communicating with each other in a local manner in a Euclidian space to reach a global consensus or a goal. Swarm intelligence (SI) is another class of decentralized algorithms based on the cooperative behavior of the agents to achieve a common goal [13], [14]. In this work an SI algorithm based on reinforcement learning is proposed. Reinforcement learning is a type of real-time machine learning which refers to modifying one's actions or control policies based on learning from one's experience. It is inspired by learning mechanisms that occur in nature, where living beings modify their control actions based on feedback received from their surroundings [49], [52], [53]. In a distributed environment the scope for reinforcement learning is wide, the agents not only learn from their own experiences, but also from the experiences of their peers.

1.2 Cooperative Game Theory and Positional Advantage in coalitions and Structure in Coalitions on Graphs

Game theory was introduced as a formal discipline of mathematics by J. von Neumann in 1928 through his classic work [113] and its extension [114]; it is primarily divided into two areas: noncooperative game theory [84], and cooperative game theory [116], [130]. In noncooperative game theory, the fundamental unit of study is the individual agent, and it deals with its performance and strategies in interaction with other individual agents. By contrast, in cooperative game theory, the fundamental unit is the set of agents or coalition. Cooperative game theory deals with the value of the coalition, payoff allocations to individual players, and the stability of coalitions [116], [130]. Methods are sought to allocate the net value of the coalition to individual agents in such a way that agents are encouraged to join the coalition. A fair allocation [110] that often accomplishes this is the Shapley value [127]. In his classical work [110], Myerson used graph theoretic ideas to analyze cooperation in coalitional graph games. He proposed to restrict the interactions in coalitions based on the underlying communication graph structure. He showed that the unique fair allocation of the net value of the coalition to the agents is given by the Shapley value [127].

1.3 Positional Cost and Advantage in coalitions and structure in coalitions on graphs

Considerations of cost are instrumental in the formation of coalitions. Various definitions of cost are used in coalitions, including probability of false alarm, vulnerabilities from other agents, download delay, mean waiting times, and path delay [81], [82], [104], [119], [120]. In their paper [104], Jackson and Wolinsky analyzed the stability of networks when the individual agents choose to form and maintain the links between them. An agent gains value on connecting to an agent which is well-connected to other agents in the graph, and accrues a cost based on maintaining direct communication links with its neighbors. It is shown that different relations between the link cost and the propagation of value along a path result in stability of different structures, such as complete graph, star graph [87], etc. In [82] a constrained coalitional game, based on Jackson and Wolinsky model [104], for networks of autonomous agents is defined. In [81] a trust based game is proposed. In this game payoffs and costs are dependent upon the gain and loss in mutual trust value. In [119] a cooperative game with non-transferable utility is proposed; in this game advantages and costs are based on probability measure. In [120] a coalitional game is introduced; in this game the value function is based on the effective throughput of the agents and cost is based on the delay.

1.4 Graphical Coalitional Game and Marginal Contribution on digraphs

When there is a possibility of unidirectional flow of information in coalitions, then such coalitions can only be represented as digraphs. In the situation of asymmetric flow of information, agents join hands to make digraph structures to pursue a common goal. In case of digraphs, due to asymmetric flow of information, the study of contribution made by an agent is more involved than for the undirected graphs. The idea of a graphical coalition games (GCG) is introduced in [77]. Novel digraph structures required for the development of GCG on digraphs are defined and are used to state the Axioms of Value for GCG on digraphs, to assign values to digraphs. These novel structures include semi-strongly connected digraphs, and multi-chains. Notion of total or marginal contribution made by an agent [74] is defied for agents within a coalition modeled as a digraph, and the dependence of the marginal contribution made by the agents, on their position in coalition is studied. The GCG on digraphs and novel digraph structures, can be used to provide further insight to the cooperative control theory [1], [2], [3], [7], [8], [79], [135]. Algorithms are also devised to get these digraph structures.

1.5 Sequential Decision Games

Closely related to the coalitional graph games are online or sequential-in-time decision games. These are games where agents make moves through time sequentially to maximize their prescribed objective functions [130]. These games are defined by specifying the method of selection of the agent to make moves at each time, the allowed moves of the agents, and the objective function the agents seek to maximize. Agents might make moves according to some fixed round-robin procedure, or randomly according to some probability distribution function. These online sequential decision games best model real-life situations where the players are free to change their alliances as considered suitable by them to obtain their objectives.

This dissertation provides tools to study the internal structure of coalitions on graphs on the basis of different motives of the agents. A graphical coalitional game (GCG), with novel properties, based on a Value Function that is required to satisfy four formal axioms, is defined. Owing to these axioms imposed on the Value Function, it is possible to perform a rigorous study of the internal structure of coalitions on graph topologies. The allocation or payoff to individual agents is taken as their Shapley value with the Value Function satisfying the four Axioms, and is interpreted as the worth of an agent in a coalition, and is called the Positional Advantage (PA). PA formalizes the notion of well-connectedness in communication graphs. PA makes it possible to prove certain properties of the GCG, including convexity, fairness, full cooperativeness, and cohesiveness. Three types of contributions of agents within a coalition- the marginal, competitive, and altruistic contributions [74] are rigorously defined. The PA, which includes the formal Axioms of Value, allows the rigorous development of certain properties of these three contributions, including their dependence on graph topology and changes in topology.

In this dissertation, a graphical coalitional game (GCG) with cost is further defined. Cost of a coalition depends upon the connectivity of the agents and the number of agents involved in the coalition. The cost is initially allocated to the edges by using the Shapley value. The Shapley value with the Value Function satisfying the four Axioms of cost is interpreted as the cost of a communication link within a coalition. The cost is than allocated to individual agents by using the symmetric connection model of Jackson and Wolinsky [104]. The cost of an agent in a coalition is called the Positional Cost (PC). Based on cost and advantage, a Graphical Advantage and Cost Game (GACG) is defined. The advantage and cost are used to define Net Payoff or Allocation (NPA); it is further used to define three net advantages: Net Marginal Advantage (NMA), Net Competitive Advantage (NCA), and Net Altruistic Advantage (NAA). These net advantages are based on the components of cost and components of advantage and according to the concepts in [74]. A number of results about the dependence of these net advantages on coalition structure are presented.

Three online sequential decision games are also defined based on the marginal, competitive, and altruistic contributions, wherein agents make or break edges to maximize these respective contributions. It is shown these three sequential decision games have different stable coalition structures. Three online sequential decision games are also defined on top of the graphical coalitional advantage and cost game; these three online sequential decision games are: max-NMA, max-NCA, and max-NAA. The preferred graphs under each sequential decision game, under certain relations between the advantages and costs are studied. It is shown that the stable graphs in max-NMA are any connected graph, including a tree. The preferred graph in max-NCA is a connected graph or completely disconnected graph under certain other condition. The completely disconnected graph is stable in max-NAA under certain conditions. These preferences in the three sequential games yield thresholds of cost beyond which agents stay in completely disconnected or trivial coalition irrespective of sequential game.

The sequential decision games are also introduced on top of GCG on digraphs. The marginal contribution made by an agent within a coalition is taken as the objective function of these sequential decision games. The games are defined by varying the rules of the game, and agents maximize their objective functions while taking turns sequential in time under the allowed rules. It is established that stable structures under these games are, multi-chains, semi-strongly connected digraphs and chain of command.

1.6 Impact of an Agent on another Agent in a Coalition

Impact made by one agent in a coalition to another agent in the same coalition plays a fundamental role in coalitions when agents join hands to pursue a common goal. The agents not only impact the agents which directly communicate with them, rather they have seminal impact on the other agents. A comprehensive impact propagation framework is defined with reference to the scientific work produced and all the persons involved in the scientific activity.

1.7 Organization of this Dissertation

The rest of the thesis is organized as follows. Chapter 2 studies the cooperation in the coalitions by providing a rigorous theoretical framework for an intelligent distributed search of a maze by a swarm of agents. Two Theorems are proven, showing that minimal local information based on principles of RL at the maze nodes is sufficient to explore the maze. Based on these results a routing scheme is presented. Simulation results show the superiority of the proposed scheme over some of the existing schemes; these results also show that the swarm of agents achieves the goal as a swarm and not as separate agents.

In Chapter 3, a graphical coalition game is defined, with its formal axioms on the Value Function. The Positional Advantage (PA) of an agent within a graph topology is defined in terms of its connectedness properties. It is shown that the GCG is convex, fair, cohesive, and fully cooperative. Changes in PA are related to changes in graph topology. Motivated by [74], three types of contributions of agents in a coalition, the marginal, competitive, and altruistic contributions are defined. Formal results about these three contributions based on topological graph properties are derived. Further, three online sequential decision games are defined on top of the GCG. The stable graph structures under each of these three games are studied. Simulation results for several online sequential decision games are presented and shown to support the stable graph structures.

In Chapter 4, the framework of GCG with Positional Cost (PC) within a coalition is defined; some fundamental results about this framework are also presented. Based on the cost and advantage, a Graphical Advantage and Cost Game (GACG) is defined. Components of cost in a coalition are also defined and dependence of the components of cost on graph topology is elaborated. The notions of NMA, NCA, and NAA are also defined. Online Sequential Coalition Decision Games; stability of graph structures under these games and cost thresholds are also discussed here.

Chapter 5 introduces the structure of a Graphical Coalition Game on digraphs which assigns a value to each digraph based on its connectivity. Fundamental properties of the game are established in the form of technical lemmas. Novel graph theoretic structures including multi-chain, and semi-strongly connected digraphs, are defined; these structures are pivotal in defining the Graphical Coalition Game on digraphs, and algorithms are devised to compute these structures. Marginal contribution made by an agent within a coalition, modeled as a digraph, is defined; results are established about the dependence of marginal contribution made by an agent upon its position in the digraph. On top of marginal contribution, three sequential decision games are defined and stable coalitional structures under these games are established. Stable structures for these games are multi-chains, semi-connected digraphs and chains of command; these structures are useful in cooperative control theory.

Impact propagation framework is introduced in Chapter 6. Here a seamless, comprehensive, and integrated framework to compute the impact made by a scientific work, a scientist, an institution, a scientific journal, and a funding agency, is presented. The framework provides solutions to a number of problems about the existing indexing systems, under discussion, in the scientific community. These problems include but are not limited to the allocation of impact to scientists, excessive self-citation, gifted authorship, dependence of indices on disciplines, and excessive citations. The framework is evolved around a computation algorithm which is network-based, and distributed in nature. The framework thus includes the seminal effect of a work in the computation of impacts made by various entities and is readily implementable. Certain guidelines are provided for the implementation of the proposed framework. Work examples and simulation results show the working and usefulness of the proposed system in comparison to other existing systems. The proposed system has complete provision of peer input to cover delicate issues in impact calculations, yet the system can be employed fully automated during the earlier implementation stages. Chapter 7 discusses the future work to complete this dissertation. Some technical results are established in the Appendices at the end of this thesis.

Chapter 2

Cooperation between Agents for Routing in an Unknown Graph

This chapter investigates the cooperation between agents to achieve a common goal. The problem of steering a swarm of autonomous agents out of an unknown maze to some goal located at an unknown location is discussed in this context. Two theorems show the importance of certain minimal information in improving decision-making for routing. Based on these results, an ε–greedy reinforcement learning method using only local information exchanges is used to balance exploitation and exploration of the unknown maze. The routing algorithm given here provides a mechanism for storing data based on the experiences of previous agents visiting a node that results in routing decisions that improve with time. Simulation examples show that simple rules of learning from past experience significantly improve performance over random search and search based on Ant Colony Optimization, a metaheuristic algorithm.

2.1 Introduction

In the literature there is a great deal of material about swarming of agents and consensus [1], [2], [3], [4], [5], [6], [7], [8]. This material is about agents communicating with each other in a local manner in a Euclidian space to reach a global consensus or a goal. Some work also deals with obstacles and potential fields [9], [10], [11], [12].

Swarm intelligence (SI) is another class of decentralized algorithms based on the cooperative behavior of the agents to achieve a common goal. These algorithms are based on simple rules inspired from the biological systems in nature. There are great deals of swarm SI algorithms proposed in literature [13], [14]. These include Ant Colony Optimization (ACO) [15], Artificial Bee Colony (ABC) [16], Artificial Immune System (AIS) [17], Charged System Search (CSS) [18], Cuckoo Search (CS) [19], [20], Firefly algorithm (FA) [21], [22], Gravitational

Search Algorithm (GSA) [23], Intelligent Water Drops algorithm (IWD) [24], Particle swarm optimization (PSO) [25], Multi-Swarm Optimization (MSO) [26], River Formation Dynamics (RFD) [27], Self-propelled particles (SPP) [28], and Stochastic Diffusion Search (SDS) [29], [30]. These algorithms can be applied to flocking behavior in discrete surroundings. Discrete surroundings are of various kinds including grids, space filling curves [32], [33] space filling surfaces [34] and mazes [35]. All the above algorithms can be used to solve the maze; in [31], ACO is deployed to solve the maze. But these algorithms generally have some centralized element. Unlike most of the metaheuristic based algorithms this paper presents an SI algorithm with rigorous mathematical base and addresses the problem of steering a swarm of agents through an unknown maze to a goal at some unknown location, using only local information exchange.

Mazes are classified in several ways: according to their dimension, topology of their manifold, shape and orientation of their cells, nature of the route from the starting point to the goal and so on. Details of these classifications can be found at [36].

The fundamental maze exploration algorithm is the random search. Though random search does not guarantee finding the goal, in many cases this is the only available exploration method. The rest of the available maze exploration algorithms are generally deterministic in nature. The most basic may be the wall follower method [37], [38]. In this method, the searcher keeps one of his hands on one wall and keeps following. It is guaranteed that either he will reach the goal if it is located somewhere at the outer boundary of a 2D maze from an initial point also located at the outer boundary; otherwise it will return to the initial point. This method works well with 2D perfect mazes [36], [37], [38]. The perfect maze is the one which is a tree when represented as a graph [39]. But it does not work for imperfect mazes or to find a goal within the maze [36], [37]. An improvement to the wall follower algorithm is the Pledge Algorithm [37], [38], [40]. This algorithm works with an agent equipped with a perfect compass and assures the escape of an agent from within the maze to a goal at the outer boundary of the maze. The Pledge Algorithm does not work in the reverse direction, that is to say, it is not guaranteed to find a goal within a maze starting from the initial point [37], [38]. Trémaux's algorithm is a sure algorithm for solving a maze that works where passages are well-defined and there is a provision to draw lines on the floor [37]. Trémaux's algorithm is based on bidirectional double tracing, a form of depth first search of the graphs. There are some other algorithms to explore the maze given complete knowledge of the maze [37], [38].

The problem of robot learning to escape a maze is not new to the machine learning research community; it was originally posed many decades back by H. Abelson and A. A. diSessa in [40]. There is a great deal of research in robots learning to escape from a maze [41], [42], [43]. In [44] an architecture for autonomous mobile agents is proposed that explores and map a two-dimensional environment, and gives safe paths to unexplored regions. In [45] algorithms of meeting of two heterogeneous robots in an unknown environment while exploring the environment are proposed. The robots take starts from unknown starting points and cannot communicate with each other over long distances. In [46] an ultrasonic sensor localization system for autonomous mobile robot navigation in an indoor semi-structured environment is presented.

Several approaches are adopted to make the cyber machines learn to find their way out of mazes. Various approaches of automated maze search are practically implemented and several testing environments are proposed [41], [47]. A knowledge-guided route search is proposed based on obstacle adaptive spatial cells, routing knowledge and routing algorithms is proposed in

[42]. Neural network based approach is used in [43] for a robot to solve a maze while avoiding concave obstacles. Maze exploration is also used as a standard test benchmark to test artificial intelligence techniques [48]. Principles of machine learning have also being used to get a single robot out of mazes [49].

There is also some study showing that antibodies in an immune system use a mechanism of learning from their surroundings to efficiently fight against antigens. Behavioral study of antibodies has led scientists to use methods of machine learning for the development of artificial immune systems [49], [50], [51]. These future immune systems are tested on moving robots in mazes [49].

In this paper a randomized approach is presented to steer a swarm of agents out of any type of unknown maze. The approach uses distributed computations and principles of Reinforcement Learning (RL) [49], [52], [53] to achieve the goal, and improves on random search and metaheuristic based search. Reinforcement learning is a type of real-time machine learning which refers to modifying one's actions or control policies based on learning from one's experience. It is inspired by learning mechanisms that occur in nature, where living beings modify their control actions based on feedback received from their surroundings [49], [52], [53].

In a distributed environment the scope for reinforcement learning is wide. In this paper, the agents not only learn from their own experiences, but also from the experiences of their peers. The routing algorithm given here provides a mechanism for storing data based on the experiences of previous agents visiting a node that results in routing decisions that improve with time. Reinforcement learning is used to reward the agents in response to a good move and give them a setback for a wrong move. In this paper the structure of the maze is unknown to the agents and the goal is positioned at some anonymous location. A location is defined as an intersection point where a routing decision is needed. The maze is represented as a static graph [39] with locations in the maze taken as nodes and adjacency of two locations represented as an edge between the corresponding nodes in the graph. The swarm of agents is sent into the maze at a prescribed location and all of these agents are free to move from one node to the other in the graph along its edges, seeking for the goal. As these agents move they develop a database and make an expanding memory node network [54] by placing memory nodes at the explored nodes. The memory node network facilitates the updating of the database used by each agent. This database helps an agent to intelligently choose an edge from one node to another node by using principles of reinforcement learning [49], [52], [53].

The algorithm of this paper is compared to ACO in Section 2.3. Simulation results show that the algorithm in this paper outperforms ACO.

The contribution of this paper is to provide a rigorous theoretical framework for an intelligent distributed search of a maze by a swarm of agents. In Section 2.2, two Theorems are proven, showing that minimal local information based on principles of RL at the maze nodes is sufficient to explore the maze. Based on these results a routing scheme is presented in Section 2.3. It is proposed that each agent is equipped with Fuzzy Logic System (FLS) [55], [56], [57], [58], [59], [60], [61] to obtain the advantages of Reinforcement Learning (RL). Simulation results are presented in Section 2.4 show the superiority of the proposed scheme over random search and the search based on ACO and a combination of both of them. These results also show that the swarm of agents achieves the goal as a swarm and not as separate agents. Conclusions are presented in the Section 2.5.

2.2 Formulation, Routing Results on finite Graphs, and Data Structures

This section formally defines the problem of routing in an unknown maze by explaining the system architecture and information structure. A graph theoretic formulation is used [39]. Two theorems concerning routing in finite mazes provide the basis for the data structures maintained by the agents.

A swarm of agents is sent into an unknown finite maze from a single location, and the agents are required to reach a goal located at some unknown location in the maze by using only information available locally to each agent. To do so, on visiting a node, an agent is required to make a routing decision about which edge to follow on leaving that node. It is desired to use minimal information and data exchanges yet to significantly improve upon the results of using random exploration by each agent.

2.2.1 Representing a Maze as a Graph

To formalize the routing problem and objectives graph theory [39] is used. Represent the unknown maze as an undirected and unknown graph $G(V, E)$ with vertices V the set of locations within the maze and $|G|$ represents the number of locations in G. A location or node is defined to be an intersection in the maze where a route direction decision is needed. There is an edge $\{i, j\} \in E$ if and only if the locations $i \in V$ and $j \in V$ are adjacent in the maze. The degree d_i of node i is the number of edges incident on node i. A swarm of N agents is sent into the graph G at some starting node $1 \in V$, and the agents are required to reach a goal located at some unknown node $g \in V$. Each agent must make routing decisions using only information available locally. It is assumed that at least one path exists from the starting node $1 \in V$ to the goal $g \in V$. A path from the starting node $1 \in V$ to the goal $g \in V$ is defined as a finite sequence of distinct vertices

$\{1 = v_1, v_2, \ldots, v_n = g\}$ with an edge existing in *G* between each pair of consecutive vertices in the sequence. The goal is reachable from node *i* if there exists a path from node *i* to the goal.

Memory Nodes and Explored Subgraph

Any agent arriving at an unvisited node for the first time places a memory node with a memory at that node, which is then termed visited. This produces a growing network of visited nodes V^* that forms a graph $G^*(V^*, E^*)$. Since G^* grows by the agents starting from the same initial point $1 \in V$ and by traversing paths on *G*, $G^* \subseteq G$ is a connected subgraph of *G*. Each node in G^* refers to the corresponding node in *G*. The graph G^* is referred as the explored subgraph of *G*.

2.2.2 Routing Results for Finite Mazes

In this section two theorems are presented which show that certain specific local information stored at each node $i \in V^*$ is sufficient for routing in an unknown maze and yields performance far better than random exploration of maze by the agents. This information at node *i* include the number of agents $N_{ij}(t)$ that have gone along an edge $\{i, j\}$ prior to time *t*, and the number of agents $\eta_{ij}(t)$ of the agents $N_{ij}(t)$ who have returned to the node *i* prior to time *t*. In fact, as the number of agents *N* sent into the maze becomes large, this minimal information provides routing information that allows the agents to reach the unknown goal with probability tending to 1. This makes possible the distributed intelligent pursuit of the unknown goal by each agent using information related to $N_{ij}(t)$ and $\eta_{ij}(t)$, stored at the nodes and agents.

The first theorem shows that as $N_{ij}(t)$ becomes large, while the number of agents $\eta_{ij}(t)$ returning to node *i* is equal to zero, then the probability that the goal is reachable from node *i* through a path containing edge $\{i, j\}$ tends to 1. The result relies on the finite size of the maze.

Theorem 1: Let $N_{ij}(t)$ be the number of agents who have passed node *i* along an edge $\{i, j\}$ prior to time *t*. Assume that $\eta_{ij}(t)$, the number of agents who have returned to node *i*, is equal to 0. Then as $N_{ij}(t)$ and *t* increase, the probability that there exists a path containing $\{i, j\}$ from node *i* to the goal tends to 1.

Proof: The maze is represented as a connected graph *G*. Consider a node *i* from where a number of agents $N_{ij}(t)$ have gone along an edge $\{i, j\}$ prior to time *t*. Let us assume that there is no path leading to the goal from node *i* containing edge $\{i, j\}$. Split each edge of the graph into a pair of directed edges and consider the graph as a Markov chain. The chain does not have self-loops at any node other than the goal which is an absorbing node. Let the probability of going along any edge be uniformly distributed. Then there is a nonzero probability *p* that an agent will come back to the node *i* in finite time *t*. Thus there is some nonzero average rate of return λ of the agents $N_{ij}(t)$ back to node *i*. As the number of agents $N_{ij}(t)$ increases the probability that *k* agents will come back in time *t*, given that there is no path leading to goal from node *i* containing edge $\{i, j\}$ is given by Poisson distribution [64].

$$p(k) = \frac{(\lambda t)^k . e^{-\lambda t}}{k!}; k = 0, 1, 2, ... \tag{2.1}$$

From (2.1), for $k = 0$, $p(0) = e^{-\lambda t}$, which implies that $P(k = 0 \mid \neg S) = e^{-\lambda t}$, here

S=There exist a path from node *i* containing $\{i, j\}$ leading to the goal. This leads to

$$P(\neg S \mid k=0)\frac{P(k=0)}{P(\neg S)} = e^{-\lambda t}$$

or

$$P(S \mid k=0) = 1 - \frac{P(\neg S)}{P(k=0)}.e^{-\lambda t} \tag{2.2}$$

In (2.2), as time t increases, $P(k=0)$ reaches 1 and $P(\neg S)$ is fixed. Thus $P(S \mid k=0)$ reaches 1.■

The second result concerns the case $\eta_{ij}(t) \neq 0$ and shows that at a node i the steady state probability of finding the goal while following an edge $\{i, j\}$ is maximized if the ratio $\frac{\eta_{ij}(t)}{N_{ij}(t)}$ is minimum.

Theorem 2: Let $N_{ij}(t)$ be the number of agents who have passed an intermediate node i along edge $\{i, j\}$ prior to time t, and $\eta_{ij}(t)$ the number of agents out of $N_{ij}(t)$ who have returned to node i. Then as $N_{ij}(t)$ increases, the steady state probability that there exists a path containing $\{i, j\}$ from node i to the goal is maximum for the edge $\{i, j\}$ for which the ratio $\frac{\eta_{ij}(t)}{N_{ij}(t)}$ is minimum.

Proof: For an agent present at an intermediate node i at some time t, whole of the maze is viewed as d systems reachable from it. Here d is the degree of the node i. This visualization of the maze is shown in Figure 2.1.

Supposing that these systems are randomly selected by the agent at node i. In Figure 2.1 these systems are represented as states of a Markov chain numbered from 1 to d. Also the state g shows the goal node g. Let p_j be the transition probability of return of an agent from the system

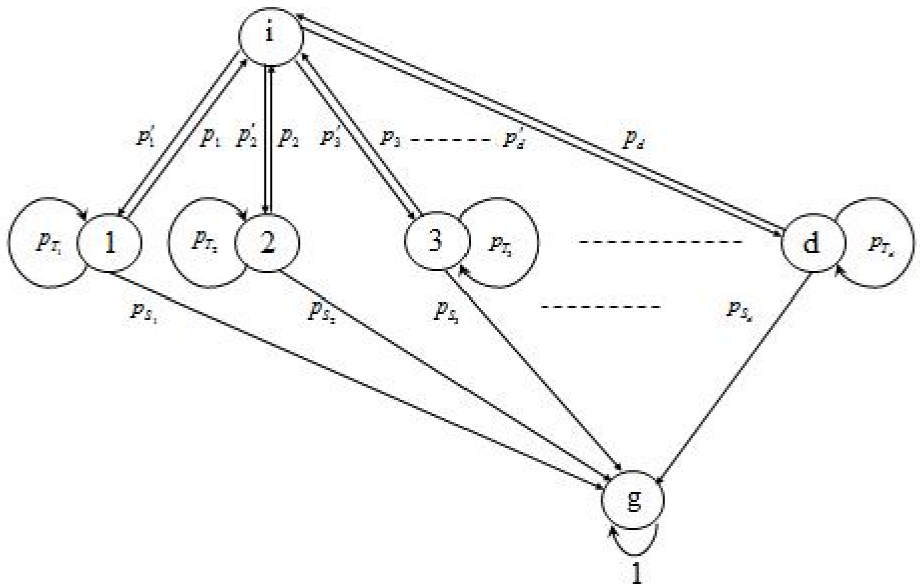


Figure 2.1: Visualization of maze as *d* systems

j to the node *i*. Also let p_{S_j} be the transition probability of going from the system *j* to the goal node *g* which is an absorbing node. The transition probability that an agent within the system *j* remains within the same system is p_{T_j}. Also p'_j is the probability to select a particular system from the node *i* so that the probability $p_S = P(s)$ of reaching the goal is maximized. Using the conditioning it is written as

$$p_s = \sum_{j=1}^{d} P(s \mid j) P_{ij} \tag{2.3}$$

Here

$$P(s \mid j) = p_{sj}, P_{ij} = p'_j \tag{2.4}$$

Substituting these values from (2.4) above into (2.3), yields

$$p_s = \sum_{j=1}^{d} p_{s_j} p'_j \tag{2.5}$$

For each of the state $j=1,2,...,d$, the following equation holds

$$p_{sj}+p_{T_j}+p_j=1 \tag{2.6}$$

Substituting the value of p_{sj} from (2.6) above into (2.5) above, yields

$$p_s=\sum_{j=1}^{d}(1-p_{T_j}-p_j)p'_j \tag{2.7}$$

or

$$p_S=1-\sum_{j=1}^{d}p_{T_j}p'_j-\sum_{j=1}^{d}p_jp'_j \tag{2.8}$$

The steady state transition probability p_{T_j} of keeping an agent within the same system for all the systems $j=1,2,...,d$ is 0. The problem of maximizing the probability p_s of an agent to reach the goal from the node *i* is transformed into the problem of minimizing the summation $\sum_{j=1}^{d}p_jp'_j$ on the right hand side of the (2.8), under the condition

$$\sum_{j=1}^{d}p'_j=1 \tag{2.9}$$

Thus the summation $\sum_{j=1}^{d}p_jp'_j$ is a convex sum under condition (2.9) and it has the minimum given in (2.10) below.

$$\min\sum_{j=1}^{d}p_jp'_j=p_{j_0} \tag{2.10}$$

Here $p_{j_0}=\min(p_j)$ when $p'_j=0\forall j\neq j_0\wedge p'_{j_0}=1$. This means that p_s is maximized if the edge with minimum probability of an agent coming back to node *i* is taken at time *t*. Under the given condition of equal complexity of the systems and assuming $N_{ij}(t)\forall j:\{i,j\}\in E$ is large enough, as the time passes, the ratio $\frac{\eta_{ij}(t)}{N_{ij}(t)}$ reaches to the steady state probability of return of an agent back to node *i*. It is thus established that the greater the value of $\frac{\eta_{ij}(t)}{N_{ij}(t)}$ the smaller are the chances of finding the goal while going along the edge $\{i, j\}$. ■

2.2.3 Data Structures of Agents and Nodes

Theorem 1 and Theorem 2 in the previous section showed the importance in routing of $N_{ij}(t)$, the number of agents who have passed an intermediate node *i* along edge $\{i, j\}$ prior to time *t*, and $\eta_{ij}(t)$, the number of agents out of $N_{ij}(t)$ who have returned to node *i*. This section capitalizes on these results to define minimal data structures to be maintained by both the memory nodes and the exploring agents.

The data structures presented here contain parameters used to make routing decisions by the agents for maze exploration. Upon its arrival at a memory node, an agent exchanges data with the memory node, and based on the results established in the previous section makes a decision about which edge to follow on leaving the node. Theorems 1 and 2 provide the basis for using minimal information storage and data exchanges to significantly improve upon the results of using random exploration by each agent.

Principles of reinforcement learning (RL) are used in the next section to update the data structures using local information.

Data Maintained by Visited Nodes

Each memory node $i \in V^*$ at the visited locations maintains a time-varying matrix $P_i(t)$ of size $d_i \times 2$, where d_i is the degree of the node *i*. This matrix is given by

$$P_i(t) = [\underline{N}_i(t) \;\; \underline{\gamma}_i(t)] \tag{2.11}$$

Here, $\underline{N}_i(t)$ and $\underline{\gamma}_i(t)$ are vectors of length d_i, each having one entry corresponding to each edge incident on node *i*.

The vector $\underline{N}_i(t)$ has the *j-th* entry equal to the number of agents $N_{ij}(t)$ who have travelled along the edge $\{i, j\}$ prior to the time *t*. The *j-th* entry in the vector $\underline{\gamma}_i(t)$ is the negative of the number of agents who have visited node *i* prior to time *t* and returned to it. That is to say, $\gamma_{ij}(t) = -\eta_{ij}(t)$. If none of the agents who have gone along edge $\{i, j\}$ have returned to node *i*, then the *j-th* entry in vector $\underline{\gamma}_i(t)$ is equal to zero.

According to the Theorems 1 and 2, routing decisions that select edge $\{i, j\}$ on leaving node *i* are beneficial if the *j-th* entry in the vector $\underline{\gamma}_i(t)$ is equal to zero and detrimental as the *j-th* entry in the vector $\underline{\gamma}_i(t)$ becomes larger negative. Thus the vector $\underline{\gamma}_i(t)$ can be considered as a reinforcement learning gain vector. Element *j* of $\underline{\gamma}_i(t)$ is called the RL gain of the edge $\{i, j\}$ at time *t*. This gain can be considered as a setback or a cost incurred by an agent on following edge $\{i, j\}$.

Data Maintained by Agents

Nodes $i \in V^*$ in the explored graph G^* are identified by *EUI-64 IDs* [65]. Every agent $m \in N$ keeps track of the edges it has visited in the form of a finite list L_m of the ordered pairs of *EUI-64 IDs* of nodes in V^* it has visited. The *l-th* entry of L_m is represented as $L_m(l)$ and it is equal to $(i(m,l), i(m,l+1))$, the edge of V^* traversed by the agent *m* at time *l*. Note that the first coordinate of $L_m(1)$, is 1 $\forall m \in N$ since all the agents start from the same node 1.

2.2.4 Node and Agent Data Updates

In this section the computational details for updating the system data parameters are discussed. The node parameters are given in (2.11) and the agent parameters are the list L_m of edges it has visited. These data parameters are updated based on only the local information passed between an agent and the node at which it currently resides. Based on the data updates, an agent makes a routing decision by selecting an edge along which to proceed on leaving the node.

The system parameter update mechanism works through update algorithms operating at the agents and at the nodes based on the communication between them. Suppose that at some time *t* there are *n* agents $m_1, m_2, ..., m_n$ at some node $i(t) \in G^* : i(t) \neq g$. All these agents receive the *EUI-64 ID* of the node $i(t)$, which is also referred as $i(t)$. All these agents also receive the parameter matrix $P_i(t)$ from the node $i(t)$.

The following data update algorithm is performed by an agent m_k who is present at the node $i(t)$ at time *t*. The algorithm has three steps: data update, routing decision, and communicate to node.

Algorithm 1- Agent Update and Routing Decision

1) *Data Update: Define,*

$\underline{r_{m_k}}(t) : r_{m_k j}(t) = 0 \forall j \in V^* : \{i(t), j\} \in E^*$

Search L_{m_n} *for* $i(t)$

$[r_{m_k}(t)]_{L_{m_k}(l+1)} = -1 : l < \mathrm{t}\, and :$

$L_{m_k}(l)(first\ coordinate) == i(t)$

The update specified here says that at the most there is only one nonzero component of $\underline{r_{m_k}}(t)$, which is -1. This nonzero component corresponds to the edge the agent went along when it visited the node i at the previous time. The agent updates its RL vector as specified in (2.12) *below.*

$$\underline{\gamma}_i(t) = \underline{\gamma}_i(t) + \underline{r_{m_k}}(t) \tag{2.12}$$

2) Decision and Further Update: Select edge $\{i, j\}$ to follow on leaving node i based on reinforcement gains $\gamma_{ij}(t)$ and number of agents $N_{ij}(t)$. This decision algorithm is given in the Section 2.3. Moreover, if the agent reaches an already visited node, one same as the first coordinate of an entry in L_{m_n} then it updates the second coordinate of the corresponding entry in L_{m_n} to the one the agent is going to visit the next time, else it will create a new entry in the list L_{m_n}.

3) Communicate: Form $\underline{e_{m_k}}(t)$ a vector of length $d_{i(t)}$ consisting of zero entries, with only one entry equal to 1 at the position corresponding to the edge $\{i, j\}$ selected.

Communicate $\underline{e_{m_k}}(t)$ and $\underline{r_{m_k}}(t)$ to $i(t)$. ■

In Section 2.3 an algorithm is given for the decision in Step 2. This routing algorithm is based on Theorems 1 and 2 and balances exploitation of the data stored in the matrix $P_i(t)$ as specified in (2.11) above, with exploration to ensure that the entire maze is eventually explored and every node visited.

In Algorithm 1, Step 1, an agent gives a setback of -1 to the edge incident at $i(t)$ which it followed if it visited the node at the previous time. This setback is communicated to the node $i(t)$

in the form of vector $\underline{r_{m_k}}(t)$. Also, the agent communicates the edge taken by it at time t in the form of vector $\underline{e_{m_k}}(t)$.

Upon receiving the above information from the agents visiting a node $i(t) \in V$ at time t, the node $i(t)$ adds all the setbacks to $\underline{\gamma_i}(t)$, to get the new RL setback vector $\underline{\gamma_i}(t+1)$. The vector $\underline{N_i}(t)$ is also updated according to the decisions of the agents to take specific edges to compute $\underline{N_i}(t+1)$. These actions are summarized in the following algorithm.

Algorithm 2: Node Update Algorithm

Update RL gain and number of agents visiting node i.

$$\underline{\gamma_i}(t+1) = \underline{\gamma_i}(t) + \sum_{k=1}^{n} \underline{r_{m_k}}(t) \tag{2.13}$$

$$\underline{N_i}(t+1) = \underline{N_i}(t) + \sum_{k=1}^{n} \underline{e_{m_k}}(t) \tag{2.14}$$ ■

Equations (2.13) and (2.14) above, elaborates the updates of the data structure $P_i(t)$ of node i on the basis of the RL feedbacks given by the visiting agents to the incident edges and the edges taken by agents at time t.

2.3 Routing in a Maze Using reinforcement Learning

This section describes the routing decision algorithm used by each agent in deciding which edge to follow when leaving a node i at time t. This corresponds to the decision Step 2 of Algorithm 1, the agent update and routing decision algorithm. This algorithm uses exploitation of the data about the explored graph in Theorems 1 and 2, to reach the goal. It also uses exploration to obtain data about the unexplored portion of the maze. The importance of balancing exploitation and exploration is well known in reinforcement learning [52]. The routing algorithm given here provides a mechanism for storing data based on the experiences of previous agents visiting a node that results in routing decisions that improve with time.

2.3.1 Exploitation of Information for Intelligent Routing

According to Theorem 1 and 2, the data contained in vectors $P_i(t) = [\underline{N}_i(t)\ \ \underline{\gamma}_i(t)]$, (2.11) above, contains information that can be used to make intelligent routing decisions based on minimal information. Specifically, as the *j*-th entry $N_{ij}(t)$ of $\underline{N}_i(t)$ becomes large while the absolute value of the *j*-th entry $\gamma_{ij}(t) = -\eta_{ij}(t)$ of $\underline{\gamma}_i(t)$ is small, the probability is high that the goal node is reachable by following edge $\{i, j\}$ when leaving node *i*.

Routing in an unknown maze requires a balance between using or exploiting available information to select the route most likely to reach the goal and exploring the unknown portion of the maze. This balance has been formalized as the 'exploitation vs. exploration' dilemma [52]. The data in $P_i(t) = [\underline{N}_i(t)\ \ \underline{\gamma}_i(t)]$ as specified in (2.11) above, can be used to make intelligent routing decisions at each node, which corresponds to exploiting the available information. On the other hand, even if two entries of $\underline{\gamma}_i(t)$ are zero, the agent should not necessarily take the edge with largest corresponding entry in $\underline{N}_i(t)$. This is because there should always be a finite probability that the agent will not proceed along the best routing path in order to maintain a finite probability of finding other better paths to the goal.

Taking the edge with the highest number of nodes not returning to node *i* is a greedy form of routing decision. Allowing a finite probability, say ε, of following a different edge is termed an ε-greedy action. It has been shown that ε-greedy actions preserve the balance between exploitation and exploration [52]. The RL based system gives the edge with maximum probability of getting the goal in the explored graph. It is proposed that an agent follows the edge suggested by the RL system with some probability $(1-\varepsilon)\approx 1$, according to the analogy of the $\varepsilon-$ greedy approach [52]. In a graph with multiple paths to the goal it may happen that a suboptimal path is found prior to the complete exploration of the graph. A balance between exploration and exploitation [52] is thus suggested to fully explore the unknown graph and to obtain the shortest path.

The data contained in vectors $\underline{N}_i(t)$, $\underline{\gamma}_i(t)$ are of different types or modalities, with the former containing numbers of agents who have taken edge $\{i,j\}$ prior to time t, and the latter containing a RL setback gain associated with edge $\{i,j\}$. Combining data of different modalities and allowing for a small probability of taking a non-greedy decision can be handled by soft decision schemes such as fuzzy logic. Therefore, in this paper fuzzy logic is used for the agents' routing decisions on leaving each node in Step 2 of Algorithm 1.

2.3.2 Fuzzy Logic Routing

A Fuzzy Logic System (FLS) consists of a Fuzzifier, an Inference Engine and a Defuzzifier. The Fuzzifier takes the two columns of $P_i(t)=[\underline{N}_i(t)\ \ \underline{\gamma}_i(t)]$ communicated to an agent by the node $i(t)$ as crisp antecedents and converts them into fuzzy antecedents. The Inference Engine generates the fuzzy output under certain rules laid down according to the results provided by Theorems 1 and 2. The defuzzifier converts the fuzzy output to the crisp output. The output membership functions are selected to balance the exploitation of the data in Theorems 1 and 2 and a desire to maintain exploration using ε-greedy methods. This crisp output is used to select the edge that the agent follows so that its probability of getting to the goal are increased.

Upon reaching at some node $i(t)$ an agent gets a matrix $P_i(t) = [\underline{N}_i(t)\ \underline{\gamma}_i(t)]$ of parameters. Motivated by Theorems 1 and 2, the information in the two columns $\underline{N}_i(t)$ and $\underline{\gamma}_i(t)$ is taken as antecedents or inputs to the fuzzy system for a particular edge $\{i, j\}$.

1) The number of agents $N_{ij}(t)$ who have gone along the edge $\{i, j\}$ from node *i* prior to time *t*. This antecedent, for a particular node and edge $\{i, j\}$ is expressed by X_1.
2) The reinforcement learning gain $\gamma_{ij}(t)$ at time *t*, of an edge $\{i, j\}$ incident at the node *i*. This antecedent, for a particular node *i* and edge $\{i, j\}$ is denoted as X_2.

The fuzzy sets under consideration are discrete in nature. The following structure for the fuzzy subsets is proposed.

1) For antecedent X_1 the following linguistic terms are defined.
    a) A Few (AF)
    b) Moderate (M)
    c) Too Many (TM)
    d) All of them (AT)

The corresponding fuzzy subset functions are defined as

$$\underline{\mu}_1(X_1) = [\mu_{AF}(X_1), \mu_M(X_1), \mu_{TM}(X_1), \mu_{AT}(X_1)]^T \tag{2.15}$$

Equation (2.16) below is obtained for all the membership functions in (2.15) above, from the same family

$$\underline{\mu}_1(X_1) = [f(X_1), f(X_1 - m_1), f(X_1 - m_2), f(X_1 - N_i)]^T \tag{2.16}$$

Where f is given by

$$f(X) = e^{-\frac{(X)^2}{2\sigma^2}} \tag{2.17}$$

In (2.16) above, $m_1 = \frac{2}{3}\frac{N_i}{D_i}$, where $N_i = \sum_j N_{ij}$ is the total number of agents who have visited the node *i* before time *t* and D_i is the degree of the node *i*. Similarly, $m_2 = \frac{3m_1 + N_i}{3}$. These values of m_1 and m_2 keep the fuzzy subsets evenly distributed from 0 to N_i. Also σ is the standard deviation of agents gone along the various edges at the node *i*.

2) For antecedent X_2 the following linguistic terms are defined.

   a) Zero (Z)

   b) A Little Negative (LN)

   c) Moderately Negative (MN)

   d) Negative (N)

The corresponding fuzzy subset functions are as defined below.

$$\underline{\mu}_2(X_2) = [\mu_Z(X_2), \mu_{LN}(X_2), \mu_{MN}(X_2), \mu_N(X_2)]^T \tag{2.18}$$

The following equation (2.19) is obtained for all the membership functions in (2.18) above, from the same family

$$\underline{\mu}_2(X_2) = [f^*(X_2), f^*(X_2 + \tfrac{1}{3}N_{ij}), f^*(X_2 + \tfrac{2}{3}N_{ij}), f^*(X_2 + N_{ij})]^T \tag{2.19}$$

Here f^* is given by

$$f^*(X) = e^{-\frac{(X)^2}{2\sigma_1^2}} \tag{2.20}$$

In (2.19) above, σ_1 is taken as $\frac{N_{ij}}{6}$. The values taken above keep the fuzzy subsets for the second antecedent evenly distributed.

2.3.3 Rule Consequents and ε-Greedy Exploration

Singleton membership functions are used for the outputs. The centroids of these membership functions are related to the probability to find a path to the goal by selecting the edge under consideration optimally as detailed by Theorems 1 and 2. The rule consequents must allow for a small probability of following non-optimal paths so that the unknown portion of the maze is explored. This corresponds to ε-greedy exploration of the maze.

Table 2.1: Rules' Centroids C_G^L with X_1 in the Vertical Direction and X_2 in the Horizontal Direction

	AF	M	TM	AT
N	0.3	0.2	0.1	0.05
MN	0.5	0.4	0.25	0.1
LN	0.7	0.6	0.5	0.3
Z	0.9	0.93	0.96	1

The centroids are given in Table 2.1, which was constructed using the following heuristics:

a. If at a node i all the agents have gone along an edge $\{i, j\}$ and none have returned back then the probability to find the goal while going along the edge $\{i, j\}$ is the best.

b. If at a node i a few agents have gone along an edge $\{i, j\}$ and none have returned back then the probability to find the goal while going along the edge $\{i, j\}$ is better.

c. If at a node i a few agents have gone along an edge $\{i, j\}$ and all have returned back then the probability to find the goal while going along the edge $\{i, j\}$ is good.

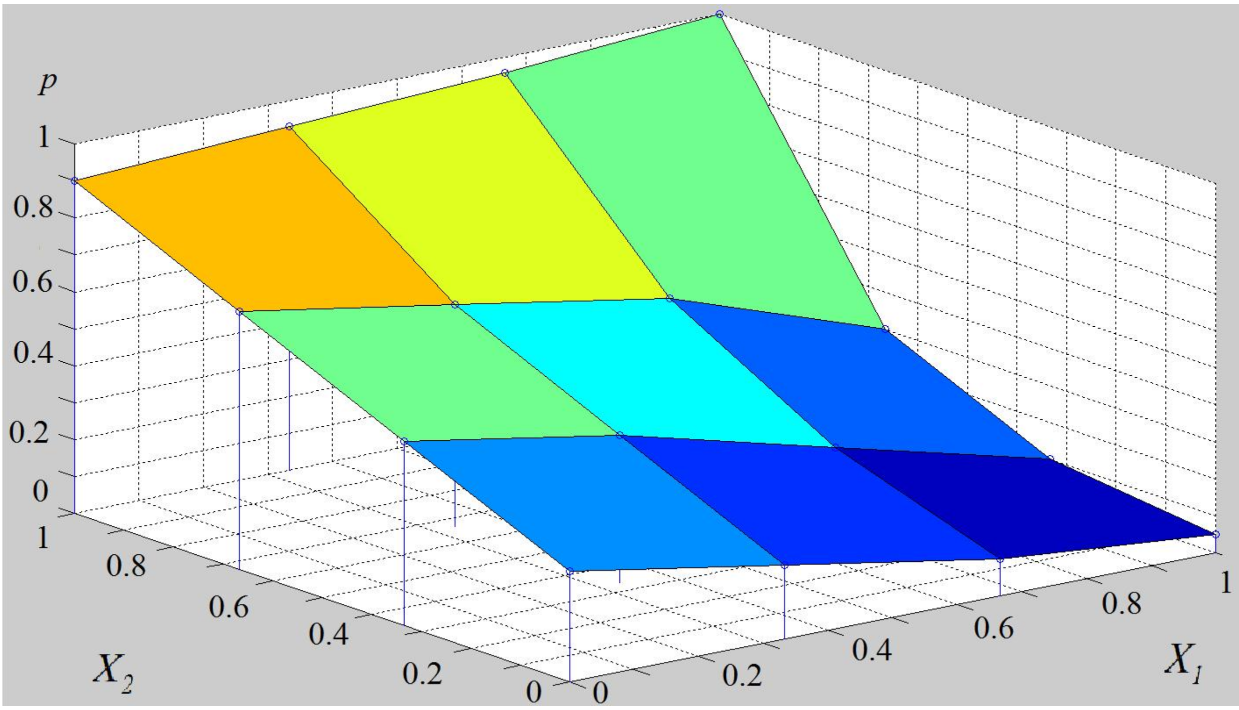


Figure 2.2: Visual illustration of the points in the Table I , probability (*p*) of finding the goal by following an edge is along the vertical axis.

d. If at a node *i* all the agents have gone along an edge $\{i, j\}$ and all have returned back then the probability to find the goal while going along the edge $\{i, j\}$ is none.

These four heuristics are used to select the four corner entries of Figure 2.2. The table lists the centroids of the rules used in the process of defuzzification. The rest of the entries in the table are interpolated from the corner entries so that there is no local extrema when viewed along the rows, columns and the diagonals of the table. The surface shown in Figure 2.2 is generated by the linear interpolation of the points in Table 2.1.

2.3.4 Inference Rules and Defuzzification

The above options for the antecedents require a set of 16 rules. The antecedent parts of the rules are the combination of the possibilities of the antecedents and the inference part is the centroids of the consequents as listed in the Table 2.1. If the *l-th* rule is denoted as R^l, then this rule is written as

R^l : if X_1 is F_1^l and X_2 is F_2^l then y is c_{G^l}

Here c_{G^l} are the centroids of various rules.

Center of Set (CoS) defuzzification [55], [56], [58] is used to determine the probability to find the goal upon selecting an edge. This uses product inferencing and centroid defuzzification. The general relation for CoS defuzzification is as given below.

$$y_{\cos} = \frac{\sum_{l=1}^{M} c_{G^l} T_{i=1}^{p} \mu_i^l (x_i^{'})}{\sum_{l=1}^{M} T_{i=1}^{p} \mu_i^l (x_i^{'})} \tag{2.21}$$

In (2.21) above, M is the number of rules and p is the number of antecedents. Also T represents the *t*-norm over all the antecedents. Here product *t*-norm and product Mamdani inferencing [55] are used.

2.3.5 Routing Decision

An agent arriving at node i performs Algorithm 1 and uses the FLS to make its routing decision in Step 2 of the algorithm. This uses the results of Theorems 1 and 2 to select the edge with highest probability of reaching the goal. This amounts to exploitation of the data stored at the nodes and the agents. However, a small probability ε exists that any other edge could be selected. This guarantees ε-greedy exploration of the maze so that the shortest path from the initial entry point node 1 to the goal node g is eventually found, since the maze is finite.

2.3.6 Comparison with ACO

Ant Colony Optimization is a graph search algorithm initially proposed by M. Dorigo [13]. This is a randomized swarm steering algorithm to get a path within a graph. The algorithm is inspired from the mechanism of optimization used by ants in their colonies to get the minimum path to their food [66], [67]. In an ant colony ants move around randomly in search of food. Whenever an ant finds some food it immediately returns to its colony while leaving a trail of pheromone along its way back. As early other ants find the trail of pheromone they stop their random search and start following the trail. All these ants following the trail continue to lay the pheromone along the path they follow. Pheromone is a chemical which evaporates as the time passes; thus if there are more than one paths available then the deposition of pheromone is more on the shorter path. Ants tend to follow the path with more pheromone deposition; this instinctive behavior of ants leads them to follow the optimal path to the food.

The ant colony algorithm proposed by Dorigo, explores an unknown graph by multiple agents in a distributed manner by using a centralized database [68]. The algorithm implements two local decision policies: Trail and attractiveness [13]. Based on these local decision policies the algorithm assign a probability p for an agent k to get the desired vertex or goal while following an edge incident at the present vertex x. Moreover, there are mechanisms of trail evaporation and daemon action in the ACO. Trail evaporation is analogous to the evaporation of pheromone from an ant trail and it reduces the probability p with the passage of time, while the daemon action biases the search process from a non-local perspective.

In literature certain improvements are also made in ACO. In [69], it is suggested that the earliest strategy is used with an appropriate number of elitist ants This is reported to allow ant system to work better. But if too many elitist ants are used, the search concentrates early around suboptimal solutions leading to a premature stagnation of the search. Search stagnation is defined in [69] as the situation where all ants follow the same path and construct the same solution over and over again, such that better solutions cannot be found anymore [68]. In [68] a max min strategy is proposed to exploit the best solutions found during iteration; after each iteration only one ant is allowed to add pheromone. This ant may be the one which found the best solution in the current iteration or the one which found the best solution from the beginning of the trial. To avoid sluggishness of the search the range of possible pheromone trails on each solution component is limited between maximum and minimum levels. Moreover, the pheromone trails is initialized to the maximum level to achieve a higher exploration of solutions at the beginning of the algorithm. Since this approach also make use of the best move made in iteration, it is also based on a global database.

The SI algorithm presented in this paper is completely based on the decisions made by the agents on the local information. The algorithm avoids the search stagnation by making use of ε-greedy RL approach and by using the analogy of giving setbacks. A mathematical foundation for the presented algorithm is also developed.

2.4 Simulation

This section presents the simulation setup and results for agents exploring the maze using the methods proposed in this paper. The RL-based search algorithm of Sections 2.2 and 2.3 is simulated. The results are compared to those for agents searching the maze with random search, with ACO and with hybrid ACO and random search systems. It is seen that the search directed by the results of Theorems 1 and 2 significantly speeds up the search.

To estimate the effectiveness of a search a performance measure is defined.

Definition (Performance Measure): The size of a 2D maze is defined as $M = LW$, where L is the length of the maze and W its width. The Performance Measure (PM) of a search is defined as the ratio of the mean time T taken by the first agent to reach the goal to the maze size M. That is to say $PM = \frac{T}{M}$.

It is desired for this performance measure to be small.

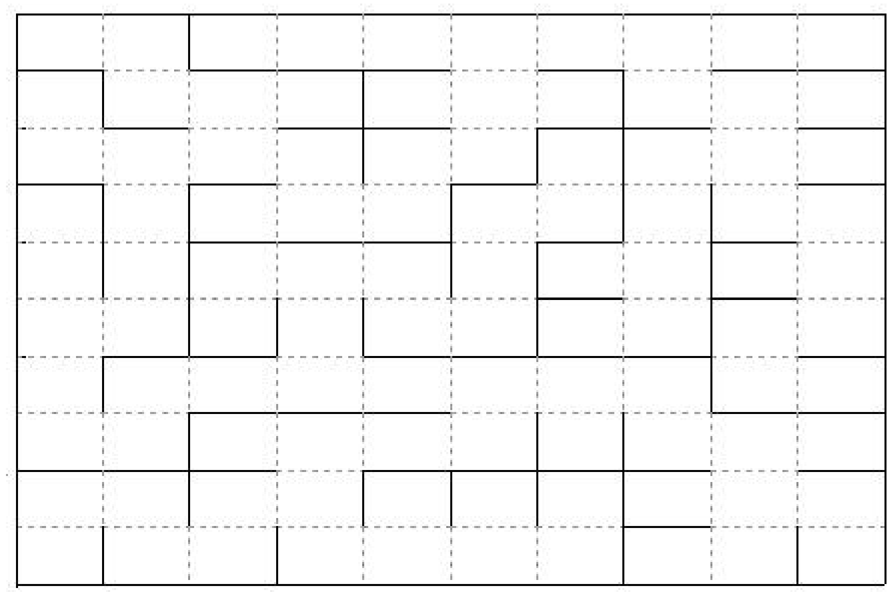


Figure 2.3: A Maze of size 10×10

2.4.1 Simulation Setup

For the purpose of this simulation a grid type maze like the one shown in Figure 2.3 is considered. The equivalent graph of the maze is represented by a unique adjacency matrix A of size $M \times 4$ [70]. The rows of this matrix are indexed by the nodes of the grid while listing them from top to bottom and then from left to right. The matrix consists of four columns with Boolean entries in each column represent the available directions of movement at the corresponding nodes. They are kept in order up, right, down and left. The node corresponding to the first row of A is taken as the initial node while the node corresponding to its last row is taken as the destination node or the goal. However it must be noted here that the location of goal is unknown to the agents.

2.4.2 Simulation Results

For the purpose of simulation several maze structures were generated using the Matlab Maze Toolbox [70]. An example maze structure is shown in Figure 2.3. The path to the goal is of the order $O(L+W)$.

Outcomes of four typical simulations are shown in Figure 2.4. These results are for maze of size 5×5 with 10 agents exploring it. All the agents start at the location $(1,1)$ and their unknown goal is present at the location $(5,5)$. These two locations are numbered as maze cell numbers 1 and 25 respectively along the bottom left axis of Figure 2.4.

Figure 2.4 (a) shows the case when the agents explore the maze through random search and without any intelligent decision-making. It is observed that the first agent reaches the goal after 45 time slots. This is noted by observing when the first point appears in the upper left plane of the 3D graph. At the end of the simulation spanning 100 time slots, only 3 agents have reached the goal using random search.

In [31], ACO is implemented to explore a penalty maze. In this paper ACO is implemented for the maze as mentioned above. Figure 2.4 (b) shows the case when the agents explore the maze through a local version of ACO without its global part of Daemon Action so that the comparison between two approaches can be made fair. It is observed that mostly the agents do not reach the goal. This happens since ACO takes in account the number of agents

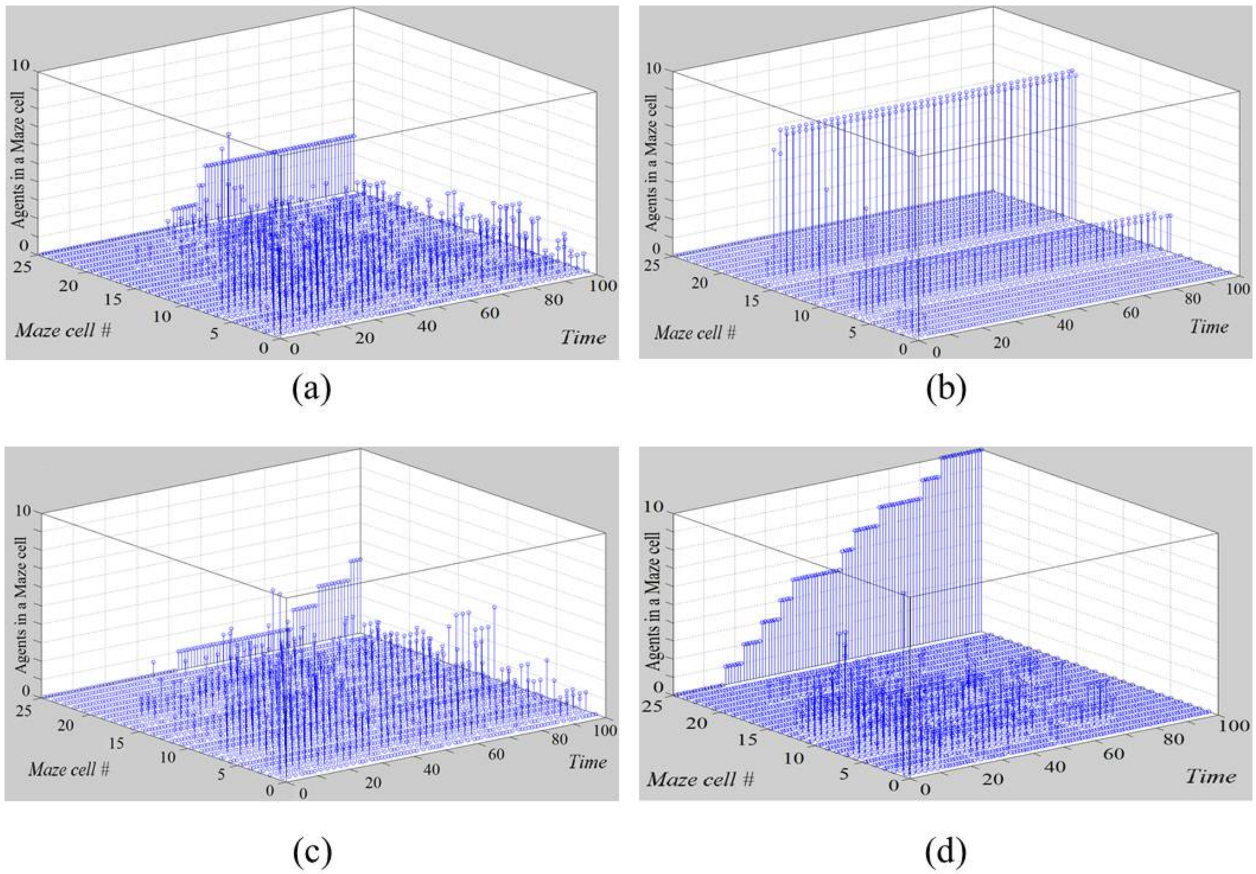


Figure 2.4: Simulation results of Maze Exploration, (a) Random Search (b) ACO (c) ACO and Random search (d) RL Based System

went through a path and does not take into account for the number of agents who visit the same node again. In this way the agents get into stagnation of the search by visiting the same set of nodes again and again. The ACO can also be implemented along with random search. In this hybrid system there are two types of agents, one following ACO and others following the random search. The agents following random search also deposit pheromones as the other agents do. Figure 2.4 (c) shows a typical case of maze search by the hybrid system. In case of the hybrid system there is some improvement as compared to simple random search and ACO, since now agents have pheromone deposition due to random agents this helps the agents to come out of the stagnation of the search and avoid visiting the same nodes again and again. This is noted by observing when the first point appears in the upper left plane of the 3D graph. At the end of the simulation spanning 100 time slots, only 4 agents have reached the goal using hybrid search.

Figure 2.4 (d) shows a typical case when the agents are equipped with the RL system based on Theorems 1 and 2 as developed in Sections 2.2 and 2.3. It is observed that one agent reaches the goal within 10 time slots, the minimum time for this size of maze, while the next two reach the goal within 16 time slots. At the end of the simulation, all agents have reached the goal. Note that, as time passes, more and more information in Theorems 1 and 2 is stored in the maze in the form of memory nodes dropped by the agents, thus routing successively improves.

In the second part of the simulation, results were obtained for five different cases with different maze sizes M. Each of these simulations was run M times, and the mean time required for the first agent to reach the goal, T is calculated in each case. The performance measures PM for these maze sizes and with various numbers of agents are calculated. The results are summarized in Figure 2.5. There, it is observed that the performance measure PM for the RL-equipped agents Figure 2.5 (d) is almost twice as small as that for agents using the random search Figure 2.5 (a) and hybrid search Figure 2.5 (c). It is observed from these figures that the best performance measure using random search and hybrid search is approximately 1, while the performance measure using the RL-based routing system is approximately 0.5. The purpose of the simulation is to show the benefits of the Algorithm given in Sections 2.2 and 2.3.

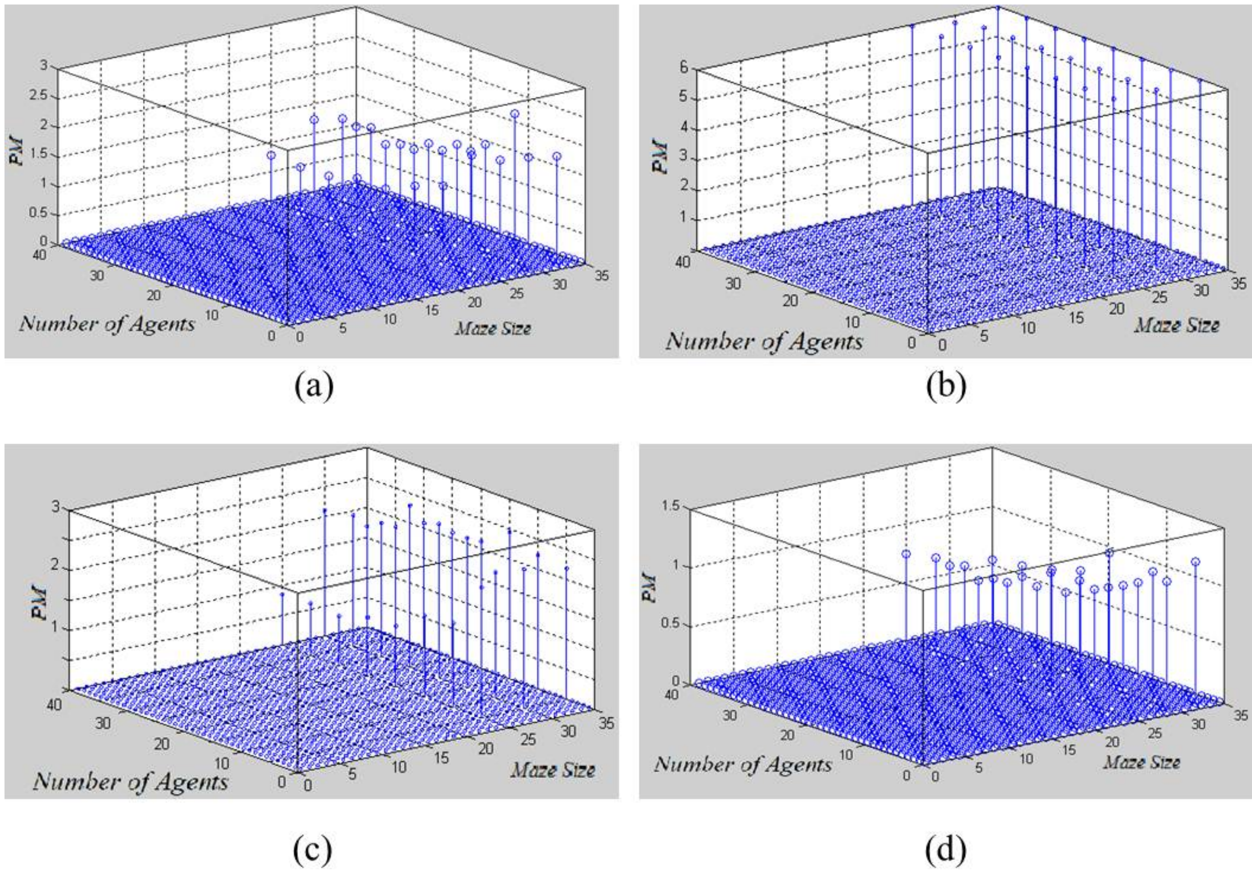


Figure 2.5: Performance Measure (PM) of, (a) Random Search (b) ACO (c) ACO and Random search (d) RL Based System

2.5 Conclusion

This paper establishes a strategy for steering a swarm of autonomous agents out of an unknown maze to some goal located at an unknown location. The strategy is based on principles of reinforcement learning. It is shown that simple rules of learning from past experience make the maze exploration significantly faster than standard random search. Two theorems in Section 2.2 show the importance of information about the number of agents who have previously gone along an edge, and how many have returned to the same node, in improving decision-making for routing. Based on these results, a ε–greedy reinforcement learning routing algorithm that uses only local information exchanges is developed in Section 2.3 to balance exploitation and exploration of the unknown maze.

Simulation results show that maze exploration using minimal information and based on RL is far superior to exploration using random search, search based on ACO and search based on hybrid ACO and random search.

Chapter 3

Positional Advantage in Coalitions and Structure in Coalitions on Graphs

This chapter introduces a certain graphical coalitional game where the internal topology of the coalition depends on a prescribed communication graph structure among the agents. The game Value Function is required to satisfy four axioms. These axioms make it possible to define a formal graphical game based on Shapley value and to assign a Positional Advantage to each agent in a coalition based on its connectivity properties within the graph. Under the Axioms of Value the graphical coalitional game satisfies basic properties of convexity, fairness, cohesiveness, and full cooperativeness. Three measures of the contributions of agents to a coalition are introduced: marginal contribution, competitive contribution, and altruistic contribution. Results are established regarding the dependence of these three types of contributions on the graph topology, and changes in these contributions due to changes in graph topology. Based on these different contributions, three online sequential decision games are defined on top of the graphical coalitional game. The stable graphs under each of these sequential decision games are studied, and give the structures of the coalitions that form in each sequential game. It is shown that the stable graphs under the objective of maximizing the marginal contribution are any connected graph. The stable graphs under the objective of maximizing the competitive contribution are the complete graph. The stable graphs under the objective of maximizing the altruistic contribution are any tree.

3.1 Introduction

The objective of this chapter is to provide a rigorous study of the internal structure of coalitions on communication graph topologies by defining a graphical coalitional game wherein the payoff to agents is taken as the Shapley value where the Value Function is required to satisfy four formal axioms.

Though game theory was introduced as a formal discipline of mathematics by J. von Neumann in 1928 through his classic work [113] and its extension [114], the diversity in the applications of game theory [132], shows that it has a rich and old history [95]. The oldest known written account of game theory is the work of the Chinese scholar Sun Tzu, *The Art of War* [136], which is nearly 2300 years old. It is an established fact that the principles of game theory among individuals and biological species have played a fundamental role in the evolution of life and ecological systems since the beginning of life on earth [103], [106], [133], [141]. Principles of game theory are thus extensively used to understand the behavior of living beings in ecological systems [73], [90], [92]. In general, game theory is a mathematical discipline that deals with issues and strategies involving competitions and cooperation between several entities [116]. In the scope of mathematical game theory these entities are called players or agents [84], [116], [119], [120], [121], [122], [123], [130].

Game theory is used in many walks of life involving situations of competition and cooperation. These areas include but are not limited to economics, finance, business [71], [93], [99], [100], [111], [112], law [80], [85], [118], political science [75], [124], strategic science [88], [107], [131], social science [83], [89], [126], and engineering [76], [81], [82], [91], [108], [109], [117], [125], [137], [138], [139], [140], [142]. Owing to such diverse applications of game theory, the subject is bound to break into various areas of study [132]. Though the boundaries between these areas of study are not very crisp [116], [132], game theory is primarily divided into two areas: noncooperative game theory [84], and cooperative game theory [116], [130]. In noncooperative game theory the fundamental unit of study is the individual agent, and the theory deals with its performance and strategies in interaction with other individual agents. By contrast, in cooperative game theory the fundamental unit is the set of agents or coalition. Cooperative game theory deals with the value of the coalition, payoff allocations to individual players, and the stability of coalitions [116], [130].

Cooperative games can be divided into three classes: Canonical Coalitional Games, Coalition Formation Games, and Coalitional Graph Games [84], [123]. Canonical coalitional games mainly deal with the stabilization of the grand coalition of all the agents. Methods are sought to allocate the net value of the coalition to individual agents in such a way that agents are encouraged to join the coalition. A fair allocation [110] that often accomplishes this is the Shapley value [127]. Coalition formation games mainly deal with coalitions based on gains and costs. Given prescribed gains and costs, the structures of the resulting coalitions are studied. Finally, the coalitional graph games deal with the formation and stability of coalitions given an underlying communication graph structure [82], [123]. In the work of Baras [81], [82] and of Başar [119], [120], [121], [122], [123] coalitional graph games are studied with applications to communication networks. Various definitions of value are used in [119], [120], [121], [122], [123], including probability of detection, gain of resources of other agents, effective throughput, and packet success rate. Various definitions of cost are used including probability of false alarm, vulnerabilities from other agents, download delay, mean waiting times, and path delay. Given the total value, algorithms are developed to form effective coalitions for communications.

Closely related to the coalitional graph games are online or sequential-in-time decision games. These are games where agents make moves through time sequentially to maximize their prescribed objective functions [130]. These games are defined by specifying the method of selection of the agent to make moves at each time, the allowed moves of the agents, and the objective function the agents seek to maximize. Agents might make moves according to some fixed round-robin procedure, or randomly according to some probability distribution function. These online sequential decision games best model real-life situations where the players are free to change their alliances as considered suitable by them to obtain their objectives.

In his classical work [110], Myerson used graph theoretic ideas to analyze cooperation in coalitional graph games. He proposed to restrict the interactions in coalitions based on the underlying communication graph structure. He showed that the unique fair (in his sense) allocation of the net value of the coalition to the agents is given by the Shapley value [127]. In their paper [104], Jackson and Wolinsky analyzed the stability of networks when the individual agents can choose to form and maintain the links between them. A node gains value on connecting to a node which is well-connected to other nodes in the graph, and accrues a cost based on maintaining direct communication edge links with its neighbors. It is shown that different relations between the link cost and the propagation of value along a path result in stability of different structures, such as complete graph, star graph, etc.

The objective of this chapter is to provide tools to study the internal structure of coalitions on graphs on the basis of different motives of the agents. This chapter defines a graphical coalitional game (GCG) with novel properties. The first point of impact of the chapter is based on a Value Function that is required to satisfy four formal axioms. Owing to these axioms imposed on the Value Function, it is possible to perform a rigorous study of the internal structure of coalitions on graph topologies. The allocation or payoff to individual agents is taken as their Shapley value. The Shapley value with the Value Function satisfying the four Axioms is interpreted as the worth of an agent in a coalition, and is called the Positional Advantage (PA). PA strengthens the definition of Shapley value and formalizes the notion of well-connectedness in communication graphs. PA makes it possible to prove certain properties of the GCG, including convexity, fairness, full cooperativeness, and cohesiveness.

The second point of impact of the chapter is to study three types of contributions of agents within a coalition- the marginal, competitive, and altruistic contributions [74]. The PA, which includes the formal Axioms of Value, allows the rigorous development of certain properties of these three contributions, including their dependence on graph topology and changes in topology. The third point of impact is the definition of three online sequential decision games based on the marginal, competitive, and altruistic contributions, wherein agents make or break edges to maximize these respective contributions. It is shown these three sequential decision games have different stable coalition structures. It is proven that the stable structures are respectively the connected graph, the complete graph, and the tree. These stable structures are inherent properties of the objective functions of the three games, not parameter dependent as in [104]. A preliminary conference paper [78] contains partial results.

The chapter is organized as follows. A graphical coalition game is defined, with its formal axioms on the Value Function, in Section 3.2. The Positional Advantage (PA) of an agent within a graph topology is defined in terms of its connectedness properties. In Section 3.3 it is shown that the GCG is convex, fair, cohesive, and fully cooperative. Changes in PA are related to changes in graph topology. Section 3.4 defines, motivated by [74], three types of contributions of agents in a coalition, the marginal, competitive, and altruistic contributions. The PA, where the Value Function is required to satisfy the four axioms, allows the derivation of formal results about these three contributions based on topological graph properties. In Section 3.5, three online sequential decision games are defined on top of the GCG. The stable graph structures under each of these three games are studied. Simulation results for several online sequential decision games are presented in Section 3.6 and shown to support the stable graph structures.

3.2 Positional Advantage in Graphical Coalitional Games

This section starts with a few essential notions of graph theory [87], [94]. Then, a graphical coalitional game (GCG) is defined where coalitions of agents joined by a graph topology are considered. The Value Function in the GCG is required to satisfy four axioms. A Positional Advantage (PA) is defined which captures the worth of an agent in a coalition where the Axioms of value hold. PA formalizes the notion of well-connectedness in communication graphs and makes it possible to provide a rigorous study of properties and structures of coalitions in the GCG in subsequent sections.

3.2.1 Graph Definitions

Consider a graph $G=(V,E)$ with V a finite nonempty set of agents and $E \subseteq [V]^2$ a set of edges. Here $[V]^2$ is the unordered set containing all the subsets of V with two elements. Two agents are interpreted to have an edge between them if and only if they directly communicate with each other. The elements of V are also called vertices or nodes. The number of elements in V is called the order or size of the graph and is denoted as $|G|$, also denoted as N. A simple graph does not contain self-loops and multiple edges. Moreover, all its edges are undirected, connecting two vertices, and do not have any weight associated with them. In this chapter simple graphs are considered.

Two vertices with an edge between them are called neighbors of each other. If all the vertices of G are neighbors of each other then G is called a complete graph. A complete graph with N vertices is denoted as K_N. A sequence of distinct vertices $i=i_0,i_1,...,i_M=j$ starting from a vertex i to another vertex j such that each pair of consecutive vertices are neighbors in G is called a path from i to j within G. Any two vertices having a path between them are called connected in G. A maximal set of connected vertices along with all the edges between them in G is called a component of G. A vertex that is not connected with any other vertex is called an isolated vertex. An isolated vertex is thus a graph component. A graph is called connected if every vertex of the graph is connected to every other vertex. A graph is called minimally connected if the number of its edges is the smallest necessary to connect all its vertices. A graph which is not connected is called a disconnected graph. A connected graph is said to have one component while a disconnected graph is the union of more than one component. The size of a component S of G is denoted as $|S|$. Since undirected graphs are considered in this paper, two components are mutually disjoint.

If there are at least two vertices i and j in G with at least two distinct paths existing from i to j then G is called a cyclic graph. A graph without a cycle is called an acyclic graph or a forest. A connected forest is called a tree. A tree is thus a minimally connected graph. A connected graph which is not a tree is thus non-minimally connected.

If $G=(V,E)$ and $S=(V',E')$ are two graphs such that $V'\subseteq V$ and $E'\subseteq E$, then S is called a subgraph of G. A subgraph is a graph in its own capacity. Moreover if E' contains all the edges $e=\{i,j\}\in E$ with $i,j\in V'$, then S is called an induced subgraph of G, in this paper denoted as $S\subseteq G$. An induced subgraph of G obtained by excluding a vertex i from V is denoted as $G\setminus\{i\}$; similarly, $G\setminus\{\{i\}\cup\{j\}\}$ is the induced subgraph of G obtained by excluding both the vertices i and j from V, the notation can be extended for more than two agents. If the number of components in $G\setminus\{i\}$ is more than the number of components in G, then i is called a cut vertex of *G*. Similarly if the deletion of edge increases the number of components of a graph then it is called a cut edge. If an edge *e* is deleted from a graph *G* then the new graph obtained is denoted as $G-e$, and if an edge *e* is added in *G* then the new graph is denoted as $G \cup e$.

3.2.2 Graphical Coalitional Game

In this section a graphical coalitional game $\Gamma = (G, v)$ is proposed. The game is based upon the communication structure of the agents within a coalition. Here the undirected graph *G* represents the communication topology of the coalition with agents as nodes, and edges between them if and only if the agents directly communicate with each other within the coalition. The allocation or payoff to individual agents is taken as their Shapley value. The Shapley value with the Value Function satisfying the four Axioms is interpreted as the worth of an agent in a coalition, and is called the Positional Advantage (PA). PA strengthens the definition of Shapley value and formalizes the notion of well-connectedness in communication graphs. PA makes it possible to prove certain properties of the GCG, including convexity, fairness, full cooperativeness, and cohesiveness.

Consider a graph *G*, with agents as nodes, where there exist edges between the nodes if and only if the corresponding agents directly communicate with each other. Define a Value Function *v* as the value of an agent for being in a coalition of size *N*. The Value Function *v* is formally defined as

$$v : 2^G \rightarrow \Re \quad with \quad v(\phi) = v_0 = 0 \qquad (3.1)$$

where 2^G is the collection of all the induced subgraphs of *G* and ϕ is the empty set. The Value Function satisfies the following Axioms of Value. In these axioms $S \in 2^G$ is an induced subgraph of *G*.

Axioms of Value:

1. If S is a connected component with $|S| = m$ then $v(S) = v_m \geq 0$
2. If S is having k connected components $S_i : i = 1, 2, \ldots k$ with $|S_i| = m_i$ then $v(S) = \sum_i v_{m_i} : v_{m_i} \geq 0$
3. If $N \geq m > n \geq 0$ then $n.v_m \geq m.v_n$
4. If $N - 1 \geq m > n \geq 0$ then $v_{m+1} - v_m \geq v_{n+1} - v_n$

Axiom 2 is according to the allocation rules in the graphical coalitional game defined by Myerson in Section 3 of [110], where the coalitions are restricted by the underlying communication graph. Axioms 3 and 4 are additional requirements on the Value Function that allow a rigorous study of the internal structure of coalitions in graph games.

Axiom 3 is essentially a super linearity requirement $v_m / m \geq v_n / n$ for $m > n$. This means that an agent does no worse by belonging to a large coalition. According to Axiom 3, one has $v_2 \geq 2v_1$. A condition on v_1 and v_2 stronger than Axiom 3 is

$$v_2 > 2v_1 \tag{3.2}$$

This is a strict super-linearity requirement. This condition is used to establish some refinements and strengthening of results in Sections 3.3, 3.4 and 3.5.

The next definition provides a fundamental notion used in this paper.

Definition 1: Graphical Coalitional Game. Given a graph G, the graphical coalitional game (GCG) is defined as the game $\Gamma = (G, v)$ where the Value Function v satisfies Axiom 1-4.■

Under the Axioms of Value, no agent is motivated to leave a coalition and join a smaller one. If condition (3.2) holds, agents are always motivated to join larger coalitions. The following remark provides further insight.

Remark 1:

1. If a coalition G has m components of sizes $n_1, n_2, ..., n_m$, then by using Axiom 2 the net value of the coalition G is given by $v(G) = v_{n_1} + v_{n_2} + ... + v_{n_m}$.
2. It can be seen by using Axiom 4 and (3.1) that for any $0 \le n \le N$ $v_n \ge nv_1$.
3. It can be seen by using Axiom 4 and the above remark that for $0 \le n < m \le N$ $nv_m \ge (m-n)v_n + nv_n$.
4. From the above remark it is clear that Axiom 3 is implied by the Axiom 4. Similarly it can be seen that Axiom 1 is implied by the Axiom 2. Nevertheless, Axioms 1 and 3 are retained as axioms because of the ease of their use in establishing the game properties.
5. Consider a possible game with $v_1 = 1$, $v_2 = 3$ and $v_3 = 4.51$. Clearly these values satisfy Axiom 3 since (1)(3)>(2)(1), (1)(4.5)>(3)(1) and (2)(4.51)>(3)(3). Yet they do not satisfy the Axiom 4 since 4.51-3 < 3-1. Thus Axiom 4 is not implied by Axiom 3.
6. Consider a possible game with $v_1 = 1$, $v_2 = 2.1$ and $v_3 = 3.9$. It can be easily seen that this list satisfies Axiom 4 and hence Axiom 3. ■

In the Axioms of Value it is assumed that all the agents are identical and similarly that all the edges are identical. Thus, the game can be used to study the advantage of one node over the other based on its position in the graph structure. It is established in Section 3.3 that the game is fair [110], both fully cooperative and cohesive [74], convex [128], and super-additive [115]. These properties are formally defined in Section 3.3.

The allocation of the net value of a coalition to its individual agents is a fundamental problem in coalitional games. Allocation is the share given to each agent of the net value of the coalition efforts. It is established in [110] that for a value function defined in games on graphs, the Shapley value function [127] is the only possible function that provides a fair allocation in Myerson's sense [110].

Definition 2: Shapley value of an Agent in the Graphical Coalitional Game. Given the graphical coalitional game $\Gamma = (G, v)$, define the Shapley value of agent *i* as the

$$\varphi_{G,v}(i) = \frac{1}{|G|} \sum_{S \subseteq G \setminus \{i\}} \frac{(v(S \cup \{i\}) - v(S))}{\binom{|G|-1}{|S|}} \tag{3.3}$$

In this equation $S \subseteq G \setminus \{i\}$ means *S* is an induced subgraph of the graph $G \setminus \{i\}$. Moreover, $S \cup \{i\}$ denotes an induced subgraph of *G*, containing all the agents in *S* and the agent *i*. ■

Therefore, the allocation of value to an agent *i* in the GCG $\Gamma = (G, v)$ is made here using the Shapley value function. The Shapley value with the Value Function satisfying the four Axioms is interpreted as the worth of an agent in a coalition, and is called the Positional Advantage; PA determines the importance of an agent within the coalition based on its location. Therefore, the next definition is made and provides a fundamental notion used in this paper.

Definition 3: Positional Advantage of an Agent in the Graphical Coalitional Game. Given the GCG $\Gamma = (G, v)$, define the Positional Advantage (PA) of agent *i* as the Shapley value (3.3), where the Value Function *v* satisfies the Axioms for the Value Function. ■

PA, where the Value Function satisfies the Axioms of Value, is a stronger concept than Shapley value and allows the rigorous study of GCG in this paper.

The subscripts *G* and *v* are dropped from the notation if these are clear from the context. Since this allocation is dependent only upon the position of the agent *i* within the coalition represented by the graph *G*, it is called the Positional Advantage of the node *i*. The PA is also dependent on the Value Function *v* in Definition 1. The Value Function can be represented as a real number list of size $|G|$. Therefore, the next definition is motivated.

Definition 4: Valid Game List. A list of non-negative real numbers $v_1, v_2, ..., v_{|G|}$ is called a valid game list of size $|G|$ if it satisfies the value Axioms 1- 4. ■

The following examples explain the procedure to compute the PA of agents within a coalition and show the rationale for Axioms 1-4 of the game as laid down. They reveal the importance of PA in comparing the relative importance of agents in contributing to the communication structure of a coalition as represented by the graph *G*.

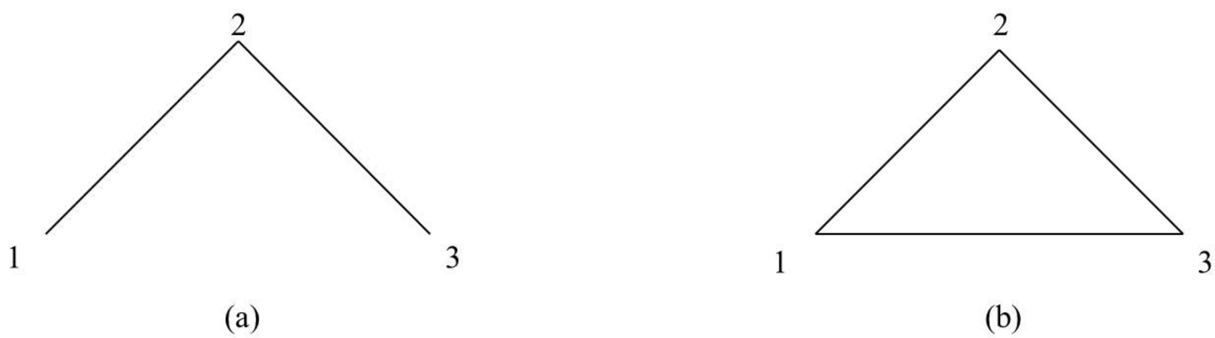


Figure 3.1. Two simple graphs (a) Example 1- Three agents in a chain (b). Example 2- Three agents in a complete graph

Example 1: Consider a chain of three agents $G = \{1,2,3\}$ as shown in Figure 3.1 (a). The PA of the agents is calculated by using Definition 3. For agent 1

$$\varphi(1) = \tfrac{1}{3}\left(\tfrac{v(\{1,2,3\}) - v(\{2,3\})}{1} + \tfrac{(v(\{1,2\}) - v(\{2\})) + (v(\{1,3\}) - v(\{3\}))}{2} + \tfrac{v(\{1\}) - v(\varphi)}{1}\right) \tag{3.4}$$

Using Axioms 1-4 of the game this is simplified to $\varphi(1) = \frac{1}{3}(v_3 - \frac{1}{2}v_2 + v_1)$. It can be easily seen that $\varphi(3) = \varphi(1)$. The PA of the agent 2 is given by

$$\varphi(2)=\tfrac{1}{3}\left(\tfrac{v(\{1,2,3\})-v(\{1,3\})}{1}+\tfrac{(v(\{1,2\})-v(\{1\}))+(v(\{2,3\})-v(\{3\}))}{2}+\tfrac{v(\{2\})-v(\varphi)}{1}\right) \qquad (3.5)$$

Using the Axioms of Value this is simplified to $\varphi(2)=\frac{1}{3}(v_3+v_2-2v_1)$.

In order to have further insight to the game, numerical values satisfying the Axioms of Value are assigned to v_1, v_2, and v_3 to form a valid game list $v_1=1$, $v_2=5$, and $v_3=10$. Substitution of these values in (3.4) and (3.5) give $\varphi(1)=2.83$ .and $\varphi(2)=4.33$. These numerical values show that $\varphi(2)\geq\varphi(1),\varphi(3)$; this also holds true in general and can be seen by using the Axioms of Value. This is according to the heuristics for the given communication structure, since 2 is in the middle of 1 and 3 and so logically contributes more to the communication structure of the coalition. ■

Example 2: Considering a complete graph of three agents $G=\{1,2,3\}$ as shown in Figure 3.1 (b), the PA of agent 1 is calculated by using the definition given in Definition 3.

$$\varphi(1)=\tfrac{1}{3}\left(\tfrac{v(\{1,2,3\})-v(\{2,3\})}{1}+\tfrac{(v(\{1,2\})-v(\{2\}))+(v(\{1,3\})-v(\{3\}))}{2}+\tfrac{v(\{1\})-v(\varphi)}{1}\right)$$

By Axioms of Value this is simplified as $\varphi(1)=\frac{1}{3}v_3$.

It can be easily seen that both $\varphi(2)$ and $\varphi(3)$ have the same PA; this again is according to intuition, since all the three nodes are symmetrically distributed in the graph and evenly contribute to the communications within the coalition. Considering the same Valid Game List used in the last example, in case of complete graph $\varphi(1)=\varphi(2)=\varphi(3)=3.33$. ■

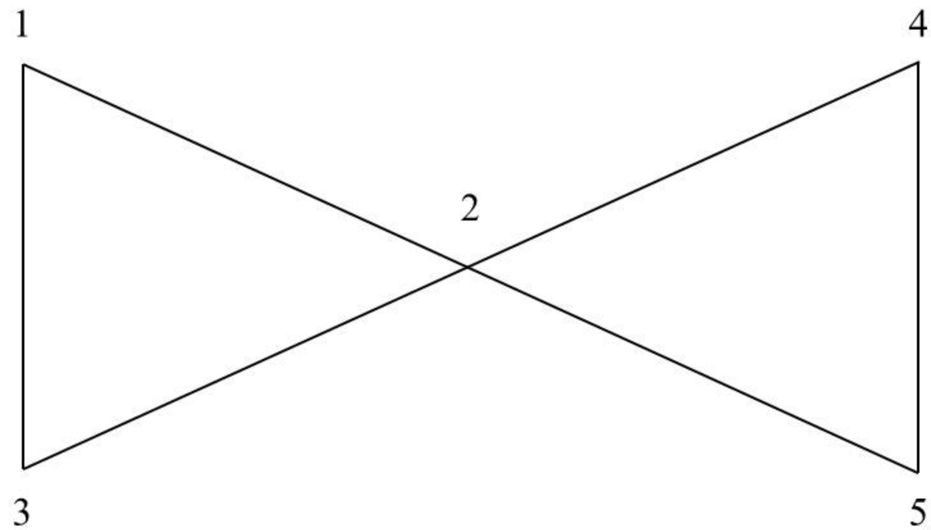


Figure 3.2. Communication graph for Example 3.

The next example further reveals the significance of PA in the game $\Gamma = (G, v)$.

Example 3: Consider the communication structure graph shown Figure 3.2, which is the same graph, stated in Fig 5 of [74]. The PAs of agents are calculated using Definition 3 and then simplified by using Axioms 1 and 2. For the cut vertex 2 one has

$$\varphi(2) = \tfrac{1}{5}(-\tfrac{7}{3}v_1 - \tfrac{7}{3}v_2 + v_3 + v_4 + v_5) \tag{3.6}$$

and for vertex 1

$$\varphi(1) = \tfrac{1}{5}(\tfrac{7}{12}v_1 + \tfrac{7}{12}v_2 - \tfrac{1}{4}v_3 - \tfrac{1}{4}v_4 + v_5) \tag{3.7}$$

By the symmetry of the graph and the relationship of the PA in Definition 3 it follows that the PAs of agents 3,4,5 are equal $\varphi(1)$.

Extending the numerical valid game list used in Examples 1 and 2, to have $v_1 = 1$, $v_2 = 5$, $v_3 = 10$, $v_4 = 20$, and $v_5 = 40$, so that .these numerical values also satisfy the Axioms of Value. Using these values in (3.6) and (3.7) give $\varphi(1) = 7.2$, and $\varphi(2) = 11.2$; this gives $\varphi(2) \geq \varphi(1)$.

Values of $\varphi(1)$ and $\varphi(2)$ in (3.6) and (3.7) can also be compared in general by using axioms to give $\varphi(2) \geq \varphi(1)$; this is according to the heuristics for the communication structure, since 2 is in the middle of the graph and is a cut vertex. Its importance in the coalition is therefore greater than the other vertices. The same graph is considered in Example 4. ■

In these three examples, small graphs were taken to demonstrate the utility of the game $\Gamma = (G, v)$ with respect to the communication structures. In these examples, the nodes that are placed more advantageously and so contribute more to the communications within a coalition have a greater PA as calculated through the payoff function, Definition 3, of the game. Moreover, the nodes which, according to the communication heuristics should have same relative importance actually do have the same PA as calculated through the payoff function of the game.

3.3 Fairness, Cooperation, and Cohesiveness in Graphical Coalitional Games

Positional Advantage, which is a strengthened version of Shapley value where the Value Function is required to satisfy the four Axioms of Value, makes it possible to rigorously establish certain properties of the graphical coalitional game. This is exploited in this section where some vital properties of GCG are introduced. These include convexity, super-additivity, fairness, full cooperativeness, and cohesiveness. These ideas require the Axioms of Value and provide the basis for defining online sequential decision games in Section 3.5.

3.3.1 Convexity and Super-Additivity

Convexity is an important property in canonical coalitional games. If the graphical coalition game is convex, all agents are motivated to form the so-called grand coalition. The intuitive idea of convexity was formally introduced by Shapley [128] in cooperative game theory.

Definition 5: Convex Cooperative Game. A cooperative game $\Gamma = (G, v)$ with transferable utility is convex if the Value Function v is super-modular, that is to say $v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \forall S, T \subseteq G$. ■

It is established by Driessen in [98] that condition of convexity is equivalent to

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S), \forall S \subseteq T \subseteq G \setminus \{i\} \forall i \in G \tag{3.8}$$

This definition of convex game is directly induced in the graphical coalition game in Definition 1 by considering G a graph on N agents and taking the notation of subset as representing an induced subgraph. The next lemma establishes that the GCG given in Definition 1 is convex.

Lemma 1: The GCG $\Gamma = (G, v)$ in Definition 1 is convex.

Proof: In the graphical coalition game $\Gamma = (G, v)$, the coalition of agents is the undirected graph G and v is the Value Function given by Definition 1. Consider any agent i in the coalition G. Moreover, consider S and T such that $S \subseteq T \subseteq G \setminus \{i\}$. Lemma A.4 directly implies that $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$. Thus the game under consideration is convex. ■

For a coalitional game $\Gamma = (G, v)$ an allocation is coalitional rational if for each S such as $S \subseteq G, S \neq \phi$, the sum of the allocations made to all the elements of S is at least equal to the allocation made to S. The core of a coalitional game $\Gamma = (G, v)$ is the collection of all such allocations which are coalition rational and completely allocate $v(G)$ [116]. For a convex coalitional game with transferable utility, the core [100], [102] of the game is nonempty [86], [129] and the allocation of each agent provided by the Shapley value is the centroid of the core.

The centroid of the core is the mean of all the allocations taken over the core. The Positional Advantage of an agent *i* given by Definition 3 is thus within the core and hence is coalitional rational.

Definition 6: Super-additive Cooperative Game. A coalitional game $\Gamma = (G, v)$, where *G* is a graph on the set of agents and *v* is the Value Function, is called super-additive if the value of the union of the disjoint coalitions is not less than the sum of the values of smaller coalitions. That is, for two disjoint induced subgraphs *S* and *T* of *G* $v(S \cup T) \geq v(S) + v(T)$. ■

Note that convexity implies super-additivity. Therefore, the GCG $\Gamma = (G, v)$ is super-additive. Under this condition no agent is motivated to leave a coalition and join a smaller one.

3.3.2 Fairness of Allocation and Changes in PA with Graph Topology Changes

In this section we first discuss fairness of the graphical games $\Gamma = (G, v)$. Then, changes in Positional Advantage on various changes in communication graph topology are detailed. This section provides essential ingredients for the introduction of online sequential coalition decision games in Section 3.5. The next definition is inspired by the definition of fairness in Myerson [110].

Definition 7: Fairness in Graphical Coalition Game. A graphical coalition game $\Gamma = (G, v)$ is fair if for all agents *i* and *j* not neighbors of each other in *G*, addition of the edge $\{i, j\}$ changes the PA of both the agents *i* and *j* by the same nonnegative value. ■

The following two results establish the fairness of the graphical coalitional game in Definition 1 for the PA in Definition 3. In the proofs of the following results in this section, G' represents a graph obtained from a graph *G* by the addition of an edge $\{i, j\}$ between two agents *i* and *j*, not neighbors of each other in *G*. Some basic graph concepts are defined in Section 3.2.1, and those being used in the following results are reiterated here. Recall that two agents with an edge between them in the graph G are neighbors of each other while two agents reachable from each other through a sequence of edges are called connected. An induced subgraph of G having all the agents connected is called a component of G. Two agents existing in different components are called disconnected.

Lemma 2: If an agent i is not a neighbor of another agent j in a graph G, then upon making the new edge $\{i,j\}$ the PA of agent i increases or stays the same.

Proof: Appendix B. ■

Theorem 1: The GCG in Definition 1 is fair.

Proof: Let G is a graph with the nodes i and j not neighbors of each other. By the definition of PA, Definition 3, and the underlying game Definition 1**,** this equation can be reached

$$\varphi_{G'}(i)-\varphi_G(i)=\frac{1}{|G|}\sum_{S'\subseteq G'\setminus\{\{i\}\cup\{j\}\}}\frac{v(S'\cup\{i\}\cup\{j\})}{\binom{|G|-1}{|S'|+1}}-\frac{1}{|G|}\sum_{S\subseteq G\setminus\{\{i\}\cup\{j\}\}}\frac{v(S\cup\{i\}\cup\{j\})}{\binom{|G|-1}{|S|+1}} \tag{3.9}$$

In this equation $S\subseteq G\setminus\{\{i\}\cup\{j\}\}$ and $S'\subseteq G'\setminus\{\{i\}\cup\{j\}\}$ mean S and S' are induced subgraph of $G\setminus\{\{i\}\cup\{j\}\}$.and $G'\setminus\{\{i\}\cup\{j\}\}$ respectively. Moreover, $S\cup\{i\}\cup\{j\}$ and $S'\cup\{i\}\cup\{j\}$ are induced subgraphs of G and G' respectively, containing i and j. By the symmetry of (3.9), $\varphi_{G'}(j)-\varphi_G(j)$ is the same as $\varphi_{G'}(i)-\varphi_G(i)$;this along with Lemma 2 establishes that the GCG in Definition 1 is fair. ■

Remark 2: The definition of Fairness and Theorem 1 imply that if an edge is broken then the PAs of both of its end agents decrease by the same nonnegative value. ■

The next sequence of lemmas shows how gains in PA on making an edge depend on the topological properties of the graph and provides essential ingredients for the introduction of online sequential coalition decision games in Section 3.5. The next result strengthens Lemma 2.

Lemma 3: If an agent *i* is not a neighbor of agent *j* in a graph *G* and a game list having $v_2 > 2v_1$, then upon making the edge $\{i, j\}$ the PA of agent *i* increases.

Proof: Appendix B. ■

Lemma 4: Let two agents *i* and *j* be connected but not neighbors in a graph *G*. Then upon making the edge $\{i, j\}$ the PA of at least one agent within the same component decreases or remains constant.

Proof: Appendix B. ■

The next lemma strengthens the above lemma; the proof is omitted.

Lemma 5: Let two agents *i* and *j* be connected but not neighbors in a graph *G* and a game list having $v_2 > 2v_1$. Then upon making the edge $\{i, j\}$ the PA of at least one agent within the same component decreases. ■

The following two lemmas extend the above two lemmas for agents in different graph components.

Lemma 6: Let two agents *i* and *j* be in different components of a disconnected graph *G*. Then upon making the edge $\{i, j\}$ the PA of all the nodes either remains constant or increases.

Proof: Appendix B. ■

The next lemma strengthens the above lemma; the proof is omitted.

Lemma 7: Let two agents i and j be in different components of a disconnected graph G and a game list having $v_2 > 2v_1$. Then upon making the edge $\{i, j\}$ the PAs of all agents connected with the agents i or j increases. ■

Remark 3:

1. If two agents i and j are in different components of a disconnected graph G and k is neither connected with i nor with j, then upon making the edge $\{i, j\}$ the PA of k remains unchanged; this fact follows from the Lemma A.1.
2. If an edge $\{i, j\}$ is a cut edge of a graph then, upon deletion of this edge, the PAs of all the agents connected with agents i and j either decrease or remain constant. Further, the PAs of the agents which are not connected with i or j do not change by the deletion of the edge.
3. If an edge $\{i, j\}$ is a cut edge of a graph and a game list having $v_2 > 2v_1$, then, upon deletion of this edge, the PAs of all the agents connected with agents i and j decrease. Further, the PAs of the agents which are not connected with i or j do not change by the deletion of the edge. ■

3.3.3 Fully Cooperative and Cohesive Games

The ideas in this section are important for elucidation of the cooperation among agents in the graphical coalitional game $\Gamma = (G, v)$ of Definition 1. These ideas are significant in the definition of contributions if agents in Section 3.4. The properties of being fully cooperative and cohesive are defined in [74] and formally explored here based on the definition of Positional Advantage. These properties elaborate the cooperation among the agents in the pursuit of a common cause. Informally, in a fully cooperative game, as more agents join the coalition the payoff of the already existing agents increases. On the other hand, in a cohesive game, allocation of a bigger body of agents within a coalition is bigger. These properties are formally defined next in the perspective of the GCG $\Gamma = (G, v)$. Let A, B, C, D be induced subgraphs of G such that $A \subseteq B \subseteq C \subseteq D \subseteq G$.

Definition 8: Allocation of a Set of Agents in an Induced Subgraph. The allocation or payoff of the agents in coalition A when only the coalition B is considered is denoted as $\mu_A(B)$ and it is defined as

$$\mu_A(B) = \sum\nolimits_{i \in A} \varphi_B(i) \tag{3.10} \blacksquare$$

It is to be noted that the value of $\mu_A(B)$ is dependent upon both the connectivity of the induced subgraph A within itself and its connectivity with the induced subgraph B. Moreover, relations hold.

$$\mu_{\{i\}}(G) = \varphi_G(i) \tag{3.11}$$

$$\mu_G(G) = v(G) \tag{3.12}$$

Equation (3.11) follows from (3.10) by substituting B by G and taking $A = \{i\}$, also (3.12) follows from (3.10) by substituting both A and B by G and by using Definition 3 which is based on Shapley value [127]. The next definitions are inspired by Arney [74]. However, the Axioms of Value allow a more rigorous treatment.

Definition 9: Fully Cooperative Game. A coalitional game $\Gamma = (G, v)$ is fully cooperative if $\mu_A(C) \le \mu_A(D)$ for any A, C and D such as $A \subseteq C \subseteq D \subseteq G$ ■

Definition 10: Cohesive Game. A coalitional game $\Gamma = (G, v)$ is cohesive if $\mu_A(C) \le \mu_B(C)$ for any A, B and C such as $A \subseteq B \subseteq C \subseteq G$. ■

Theorem 2: The GCG is fully cooperative.

Proof: For the GCG $\Gamma = (G, v)$ let A, C, D be induced subgraphs of G such that $A \subseteq C \subseteq D \subseteq G$. Then by using Definition 8 one has

$$\mu_A(D) - \mu_A(C) = \sum_{i \in A} \varphi_D(i) - \sum_{i \in A} \varphi_C(i) \tag{3.13}$$

This can also be written as

$$\mu_A(D) - \mu_A(C) = \sum_{i \in A} (\varphi_D(i) - \varphi_C(i)) \tag{3.14}$$

Suppose that D' is another subgraph of G obtained from D by removing all edges in G between C and $D \setminus C$. Then by using Lemma A.1, PAs of all the agents in A are the same whether they are evaluated for C or D', since by the construction of D' there is no edge between agents in C and $D' \setminus C$ That is to say $\varphi_C(i) = \varphi_{D'}(i) \forall i \in A$, and (3.14) can be rewritten as

$$\mu_A(D) - \mu_A(C) = \sum_{i \in A} (\varphi_D(i) - \varphi_{D'}(i)) \tag{3.15}$$

Using the definition of the PA, it modifies into

$$\mu_A(D) - \mu_A(C) = \frac{1}{|D|} \sum_{i \in A} \Big(\sum_{\substack{S \subseteq D \setminus \{i\} \\ S' \subseteq D' \setminus \{i\}}} \frac{(v(S \cup \{i\}) - v(S)) - (v(S' \cup \{i\}) - v(S'))}{\binom{|D|-1}{|S|}} \Big) \tag{3.16}$$

In this equation the terms are arranged such that S and S' are having the same vertex sets. Using Lemma A.4 all the terms $(v(S \cup \{i\}) - v(S)) - (v(S' \cup \{i\}) - v(S'))$ within the inner summation of equation (3.16) are non-negative. This establishes that $\mu_A(D) - \mu_A(C) \geq 0$ thus the game is fully cooperative. ■

Theorem 3: The GCG is cohesive.

Proof: For the GCG $\Gamma = (G, v)$ let A, B, C be induced subgraphs of G such that $A \subseteq B \subseteq C \subseteq G$. Then by using Definition 8 one can write

$$\mu_B(C) - \mu_A(C) = \sum\nolimits_{i \in B} \varphi_C(i) - \sum\nolimits_{i \in A} \varphi_C(i)$$

This equation can also be written as

$$\mu_B(C) - \mu_A(C) = \sum\nolimits_{i \in A} \varphi_C(i) + \sum\nolimits_{i \in B \setminus A} \varphi_C(i) - \sum\nolimits_{i \in A} \varphi_C(i)$$

or

$$\mu_B(C) - \mu_A(C) = \sum\nolimits_{i \in B \setminus A} \varphi_C(i)$$

The right hand side of this equation is clearly nonnegative; this completes the proof. ■

3.4 Contribution of Agents within a Coalition

This section studies three types of contributions of agents within a coalition- the marginal, competitive, and altruistic contributions as introduced in [74]. The Positional Advantage, which includes the formal Axioms of Value, allows the rigorous development of various properties of these three contributions, including their dependence on graph topology and changes in topology. The notions presented in this section provide the basis for defining online sequential decision games in Section 3.5.

3.4.1 Definitions of Contributions of Agents in a Coalition

When agents participate with other agents to make a coalition, they contribute towards the overall coalition cause. The total contribution of a set $A \subseteq G$ of agents in a coalition G is called the marginal contribution of A in G and written as $m_G(A)$ [74]. The marginal contribution of a set of agents can be divided in two parts: one part is the contribution of the agents in the subset A for the sake of themselves, and the second part is the contribution of agents in A for the sake of the other agents in $G \setminus A$. These two parts are termed the competitive contribution and the altruistic contribution respectively. These contributions are represented as $c_G(A)$ and $a_G(A)$ respectively [74]. These contributions are formally defined next.

Definition 11: Marginal Contribution of a Set of Agents. The marginal contribution $m_G(A)$ is defined as

$$m_G(A) = \mu_G(G) - \mu_{G\setminus A}(G \setminus A) \tag{3.17}$$

where $\mu_G(G)$ and $\mu_{G\setminus A}(G \setminus A)$ are the allocations or payoffs specified in Definition 8. ■

Definition 12: Competitive Contribution of a Set of Agents. The competitive contribution $c_G(A)$ is defined as

$$c_G(A) = \mu_G(G) - \mu_{G\setminus A}(G) \tag{3.18}$$ ■

Definition 13: Altruistic Contribution of a Set of Agents. The altruistic contribution $a_G(A)$ is defined as

$$a_G(A) = \mu_{G\setminus A}(G) - \mu_{G\setminus A}(G \setminus A) \tag{3.19}$$ ■

From (3.17), (3.18) and (3.19) it follows that these definitions are according to their rationales established at the beginning of this section: $m_G(A)$ is the total contribution of the agents in A, $c_G(A)$ is the contribution of agents in A for their own sake and $a_G(A)$ is the contribution of A for the sake of the rest of the coalition $G \setminus A$. Moreover, according to these definitions

$$m_G(A) = c_G(A) + a_G(A) \tag{3.20}$$

Remark 4: It is shown in Section 3.3.3, that the graphical coalition game $\Gamma = (G, v)$ in Definition 1 is cohesive. Therefore, from (3.18) the competitive contribution of a set of agents A

is non-negative. Moreover, the game is fully cooperative, so that from (3.19) the altruistic contribution of a set A of agents is non-negative. ■

For a singleton set A consisting of one agent i, the marginal contribution, competitive contribution and altruistic contribution are represented as $m_G(i)$, $c_G(i)$, and $a_G(i)$ respectively. From (3.17), (3.18), and (3.19), these contributions can be written as

$$m_G(i) = \mu_G(G) - \mu_{G\backslash i}(G \backslash i) \tag{3.21}$$

$$c_G(i) = \mu_G(G) - \mu_{G\backslash i}(G) \tag{3.22}$$

$$a_G(i) = \mu_{G\backslash i}(G) - \mu_{G\backslash i}(G \backslash i) \tag{3.23}$$

Using (3.10) in these three equations gives

$$m_G(i) = \sum\nolimits_{j \in G} \varphi_G(j) - \sum\nolimits_{j \in G\backslash i} \varphi_{G\backslash i}(j) \tag{3.24}$$

$$c_G(i) = \sum\nolimits_{j \in G} \varphi_G(j) - \sum\nolimits_{j \in G\backslash i} \varphi_G(j) \tag{3.25}$$

$$a_G(i) = \sum\nolimits_{j \in G\backslash i} \varphi_G(j) - \sum\nolimits_{j \in G\backslash i} \varphi_{G\backslash i}(j) \tag{3.26}$$

If G is connected, using Axioms 1 and 2 of the graphical coalition game, (3.10) and (3.12), equation (3.24) can be written as

$$m_G(i) = v_{|G|} - \sum_{j=1}^{p} v_{k_j} : \sum_{j=1}^{p} k_j = |G| - 1 \tag{3.27}$$

Here, p is the number of disconnected components of G obtained by the deletion of the agent i and $k_j : j = 1, 2, .., p$ are the sizes of these components.

Lemma 8: The competitive contribution of an agent in a graphical coalitional game is the same as its Positional Advantage. That is

$$c_G(i) = \varphi_G(i) \tag{3.28}$$

Proof: Appendix B. ■

Remark 5: From (3.25), the competitive contribution of an induced subgraph A of G is given by

$$c_G(A) = \sum_{j \in A} \varphi_G(j) \tag{3.29}$$ ■

The next example elaborates these concepts.

Example 4: Continuing with Example 3, consider the graph shown in Figure 3.2. Here, the marginal contributions of the individual agents are calculated and compared. From (3.17), the marginal contribution of the agent 2 is given by $m_G(2) = \mu_G(G) - \mu_{G\setminus\{2\}}(G \setminus \{2\})$, which by using Axiom 2 becomes

$$m_G(2) = v_5 - 2v_2 \tag{3.30}$$

By the symmetry of the graph and (3.17), the marginal contributions of the rest of the agents are equal and given by

$$m_G(1) = v_5 - v_4 \tag{3.31}$$

Using the valid game list introduced in Example 3, the marginal contribution made by the agents 1 and 2 is $m_G(2) = 40 - 20 = 20$ .and $m_G(2) = 40 - 10 = 30$.

Comparison of (3.30) and (3.31) by using Lemma A.2, gives $m_G(2) \geq m_G(1)$. This result is according to the heuristics of the communication structure, since agent 2 is a cut vertex and makes more contribution to the communication in the coalition. ■

3.4.2 Dependence of Contribution of Agents on Graph Topology

Some lemmas about the three types of contributions of the agents are presented next. They demonstrate the dependence of the marginal, competitive, and altruistic contributions on the topology of the graph. These results are established on the basis of the Axioms of Value. As defined in the Section 3.2.1, a cut vertex is one whose removal increases the number of disconnected components. The first results concern the marginal contribution.

Lemma 9: Given the GCG $\Gamma = (G, v)$ in Definition 1, in any connected graph *G* all the agents which are not cut vertices of *G* have the same marginal contribution. Moreover, their marginal contribution is the minimum possible marginal contribution within the connected graph. This minimum marginal contribution only depends upon $|G|$ of the connected graph *G*.

Proof: Appendix B. ■

Remark 6: In a connected graph *G*, if there is no cut vertex then the marginal contributions of all the agents are identical and independent of the graph structure. ■

Lemma 10: In a connected graph *G* of size *N*, the maximum possible marginal contribution an agent may have is of the center point of a star.

Proof: Appendix B. ■

The next results concern the altruistic contribution.

Lemma 11: In a GCG, if an agent is isolated then its altruistic contribution is 0.

Proof: Appendix B. ■

The next result shows that under condition (3.2), this result is also sufficient.

Lemma 12: In a GCG with a game list having $v_2 > 2v_1$, if the altruistic contribution of an agent is 0 then it is isolated.

Proof: Appendix B. ■

3.4.3 Change in Agent Contributions due to Changes in Graph Topology

Changes in marginal, competitive, and altruistic contributions are important in the online sequential decision games detailed in Section 3.5. The competitive contribution of an agent is its

Positional Advantage, as shown in Lemma 8. Changes in PA of an agent, when edges are made or broken are detailed in Section 3.3.2. The next result describes changes in marginal and altruistic contributions when an edge is formed.

Theorem 4: In a graphical coalition game, if a new edge is formed between two connected agents *i* and *j* in a graph *G*, then the marginal contribution of its end vertices remains constant. Moreover the altruistic contributions of its end vertices change by equal non-positive values which are the negatives of the changes in the competitive contributions of its end vertices.

Proof: Let G' be the graph obtained from *G* by adding the edge $\{i, j\}$. With the help of (3.24) The marginal contribution of the agent *i* in the graph G' is

$$m_{G'}(i) = \sum_{j \in G'} \varphi_{G'}(j) - \sum_{j \in G' \backslash i} \varphi_{G' \backslash i}(j) \tag{3.32}$$

Since the new edge is formed within the same component of G, thus the first term in the right hand side of the above equation is same as the first term in the right hand side of (3.24). Moreover the value of the second term is independent of any edge incident at agent *i*. Thus from (3.24) and (3.32) it is implied that $m_{G'}(i) = m_G(i)$. Rest of the result follows from (3.20), Lemma 8 and Lemma 2. ■

Remark 7:

1. In a non-minimally connected graph there always exists an edge whose deletion does not change the marginal contribution of the end vertices.
2. From the above remark it implies that for a non-minimally connected graph there always exist an edge whose deletion changes the altruistic contribution of the end vertices by the same non-negative value. Moreover, for a game with the game list having $v_2 > 2v_1$, the altruistic contribution increases upon deletion of such edge.

3. In a graphical coalition game, if a new edge is formed between two connected agents i and j in a graph G, with the game list having $v_2 > 2v_1$, then the altruistic contributions of its end vertices change by equal negative values which are the negatives of the changes in the competitive contributions of its end vertices. ■

3.5 Online Sequential Coalition Decision Games

This section defines three online sequential decision games based on the marginal, competitive, and altruistic contributions introduced in Section 3.4. These online decision games are defined on top of the graphical coalitional game $\Gamma = (G, v)$ of Definition 1. A background on sequential decision games can be found in Chapter 5 of [130].

In a sequential decision game, agents take turns sequentially in time to make valid or allowed moves (e.g. make or break an edge) to maximize their yield in terms of a prescribed objective function. Here, three online decision games are defined in terms of the three objective functions taken as the marginal, competitive, and altruistic contributions. It is shown these three sequential decision games have different stable coalition structures. It is proven that the stable structures are respectively the connected graph, the complete graph, and the tree. The machinery for establishing the results in this section rests on the formal Axioms of Value.

The properties of sequential decision games depend on the allowed moves and the prescribed objective function. An important concept in sequential coalition decision games is stability of graph topologies [104], [110]. Stability is important in studying the steady-state graph topologies of sequential decision games. Stable topologies show the structure of the coalitions that form under various allowed moves and decision objective functions. These stable structures are inherent properties of the objective functions of the three games, not parameter dependent as in [104].

3.5.1 Sequential Decision Games

In the online decision games defined here, agents are free to make coalitions by making or breaking edges with other agents. In contrast to [104], there is no edge cost involved. Based on marginal, competitive and altruistic contributions, three online sequential decision games can be defined. These online decision games are defined on top of the graphical coalitional game $\Gamma = (G, v)$ of Definition 1. The agents make allowed moves sequentially through time; the moves are made to maximize the prescribed objective function.

Allowed Moves. In the sequential decision games, at each move, an agent is selected at random; this agent is free to unilaterally break any edge incident at it or to bilaterally make an edge, provided the other agent incident on the edge agrees to make it, as detailed below. In a single step, an agent is allowed either to make or break several edges.

Objective Functions. An objective function $f_G(i)$, for each agent i in a coalition represented by graph G is a real, nonnegative function. Edges are made or broken by a selected agent in order to maximize $f_G(i)$.

Based on the Allowed Moves and the Objective Function the sequential decision game is defined as follows.

Definition 14: Sequential Decision Game.

In a sequential decision game a selected agent makes or breaks edges according to the rules:

a) An agent i forms an edge $e = \{i, j\}$ if $f_{G \cup e}(i) > f_G(i)$ *and* $f_{G \cup e}(j) \geq f_G(j)$

b) An agent i breaks an edge $e = \{i, j\}$ if $f_{G-e}(i) > f_G(i)$

Based on the marginal, competitive and altruistic contributions in Section 3.4 the motives of agents for forming and breaking the edges are different. Taking these contributions as objective functions, three sequential decision games can be defined.

i. Game of Maximal Marginal Contribution (MMC)

In this online game the objective function $f_G(i) = m_G(i)$.

ii. Game of Maximal Competitive Contribution (MCC)

In this online game the objective function $f_G(i) = c_G(i)$.

iii. Game of Maximal Altruistic Contribution (MAC)

In this online game the objective function $f_G(i) = a_G(i)$. ■

In these three sequential decision games an agent i is said to have a motive to make an edge if the condition *(a)* in Definition 14 is satisfied and it is said to have a motive to break an edge if the condition *(b)* in Definition 14 is satisfied.

3.5.2 Stability of Graph Topologies Under Sequential Decision Games

For a set of N agents there are $2^{N(N-1)/2}$ possible simple graphs. When agents are allowed to make valid moves, as they proceed, they may reach a graph where no agent has a motive to make any further moves. Such graphs are called stable graphs. The structure of stable graphs is thus dependent on the allowed moves and the objective function of the sequential decision game.

Definition 15: Stable Graph. In any online sequential decision game, a graph is called stable when no agent has a motive either to make an edge or to break an edge. ■

In [110], Myerson used the Shapley value as the objective function and allowed only the breakage of an edge as a valid move. Since breakage of an edge does not increase the competitive advantage, Lemma 2, Lemma 8, under such allowance, for the game in Definition 1 every graph is stable. In [104] the rules of making and breaking edges are nearly the same as those in the sequential decision games of Definition 14. However, in [104] there are costs associated with making edges. There, the balance between the value of being connected and the cost of maintaining edges has a pivotal role in determining the stable graph structures.

The next development specifies the stable graphs for the online *Game of Maximal Marginal Contribution (MMC)* , *Game of Maximal Competitive Contribution (MCC)*, and *Game of Maximal Altruistic Contribution (MAC)*.

Remark 8: It was established in Section 3.3.2 that the underlying game $\Gamma = (G, v)$ in Definition 1 is fair. Therefore, according to Remark 2 and Lemma 8, in the online Game of MCC no agent has a motive to break an edge. Similarly by Remark 7, in the online Game of MMC no agent has a motive to break an edge. Only in the online Game of MAC, may an agent have a motive to break an edge. ■

Theorem 5: In an online *Game of Maximal Marginal Contribution* any connected graph G is stable. Moreover, with a game list having the strengthened condition $v_2 > 2v_1$, any stable graph is connected.

Proof: In an online *Game of Maximal Marginal Contribution*, if the graph G is connected an agent i which is not a cut vertex, never becomes a cut vertex no matter how many edges it makes. Thus according to Lemma 9 it will continue to have the same minimal marginal contribution, thus it has no motive to make an edge. For an agent i which is a cut vertex, its marginal contribution is given by (3.21). In the right hand side of (3.21), the first term is constant for a connected graph also no matter how many new edges agent *i* makes the second term will also remain unchanged. Thus agent *i* has no motive to make any edge.

The second term in the right hand side of (3.21) is independent of any edge incident at the agent *i*, while it follows from Lemma A.3, under the given condition $v_2 > 2v_1$, the first term is maximum only when the graph *G* is connected. Thus in an online *Game of Maximal Marginal Contribution* a disconnected graph cannot be stable. ■

Remark 9: Let an online Game of Maximal Marginal Contribution be started from a completely disconnected graph, and let every agent be allowed to make as many edges as it desires. Then the agent who gets the first move to make edges will make edges with all the other agents. This move will make a star, which is also a tree, with the first agent at the center. Then there is no motive for any other player to make or break an edge. ■

Theorem 6: In an online *Game of Maximal Competitive Contribution* any complete graph is stable. Moreover, with a game list having $v_2 > 2v_1$, any stable graph is complete.

Proof: In an online *Game of Maximal Competitive Contribution* a complete graph is stable since there is no more edge to make and there is no motive to break an edge according to Theorem 1 and condition *(b)* of *Game of Maximal Competitive Contribution*.

It is established in Lemma 8 that the competitive contribution of an agent is the same as the Positional Advantage of an agent. It has already seen in Lemma 3 that whenever an edge is added in a graph, the PAs of both of its end vertices increase under the given condition. Thus there is always a motive to make an edge whenever it is possible. ■

Theorem 7: In an online Game of Maximal Altruistic Contribution any tree is a stable graph. Moreover, with a game list having $v_2 > 2v_1$, if a graph *G* is stable then it is a tree.

Proof: The altruistic contribution of an agent *i* is given by (3.26)

$$a_G(i) = \sum_{j \in G \setminus i} \varphi_G(j) - \sum_{j \in G \setminus i} \varphi_{G \setminus i}(j) \tag{3.33}$$

or

$$a_G(i) = \sum_{j \in G} \varphi_G(j) - \varphi_G(i) - \sum_{j \in G \setminus i} \varphi_{G \setminus i}(j) \tag{3.34}$$

By the Axioms of Value in Definition 1 and using the known facts about the PA in Definition 3, inherited from the Shapley value [127], all the connected graphs having the same number of agents have the same value of $\sum_{j \in G} \varphi_G(j) = v_{|G|}$. Since tree is a connected graph, formation of a new edge by any agent *i* does not change the value of the first term at the right hand side of (3.34). Moreover, from Theorem 1 it is clear that for each new edge agent *i* makes, its PA increases or remains constant. Moreover the last term at the right hand side of (3.34) is independent of agent *i*. Thus the agent *i* have no motive to make an edge.

Since a tree is minimally connected graph, all of its edges are cut edges and all the agents are reachable from any other vertex in the tree. Thus, by Lemma A.2, breakage of any edge by the agent *i*, incident at it will not increase the first term in the right hand side of (3.33). Moreover the second term at the right hand side of (3.33) is independent of any edge incident at the agent *i*. The altruistic contribution $a_G(i)$ of the agent *i* thus reduces or remains constant upon the breakage of any edge incident at it. Thus agents have no motive to break any of the edge incidents at them.

The set of simple graphs can be partitioned into three classes: disconnected graphs, minimally connected graphs and non-minimally connected graphs. It is to be established that under given condition $v_2 > 2v_1$, *G* is neither disconnected nor it is non-minimally connected.

If G is disconnected then there always exist at least two agents i and j which are not reachable from each other. From Definition 3, (3.33), and Theorem 1, under the given condition, making of the edge fulfills the condition *(a)* of Game of Maximal Altruistic Contribution. A disconnected graph is thus unstable. If G is non-minimally connected then there must exist an edge $e = \{i, j\}$ such that G remains connected even after its removal. Removal of edge $e = \{i, j\}$ by the agent i thus does not change the first term in the right hand side of (3.34), moreover the last term in the right hand side of (3.34) is independent of any edge incident at i, and according to Theorem 1 and under given condition, $\varphi_G(i)$ decreases upon removal of edge e. A non-minimal connected graph is also unstable. ■

Remark 10: It follows from the results established in this section that if $v_2 > 2v_1$, then in any of the three online sequential decision games defined in this section a stable graph is always connected. ■

3.5.3 Applications of GCG

The graphical coalitional game and the sequential decision games proposed in this paper can be used in a variety of ways in problems involving situations of simultaneous competition and collaboration among anonymous agents. The GCG with Positional Advantage can be used to determine the social standing of various kinds of agents purely on the basis of the communication structure. GCG also distinguishes between the events of making a communication link for self-interest and for the coalition's sake [71], [75], [124], [126]. The development in Section 3.4 can be used to determine the strategic importance of graph points. The development in this paper can also be used to understand the notions of competition and cooperation in groups of biological species [90], [92], [103], [106], [133], [141]. The sequential decision games in the Section 3.5 can be used to understand the internal structure of a coalition based on the notions of competition and altruism. Situations in economics, communication, and swarm control are very complex; here a lot of agents interact in situations of simultaneous competition and cooperation. The theory developed in this paper can be used to understand complex situations of joint competition and cooperation [74], [81], [82], [111], [117], [123], [125], [131], [140].

3.6 Simulation Examples of Online Sequential Decision Games

Simulation results for the three sequential decision games in Section 3.5 are presented here. In these simulations the games are started from an initial graph and the agents are free to make allowed moves as in Definition 14. One agent is randomly selected to makes moves at each time, and it can make or break as many edges as it desires to improve its contribution objective function. The method established in [96], [97] for fast computation of Shapley value is used in the simulation.

The simulations were run until one of the stable graphs is reached. The simulation results are shown and explained in Figures 3-5. These simulation results support the theory developed in Section 3.5. These results show that any connected graph is stable in *Game of Maximal Marginal Contribution (MMC)*, only a complete graph is stable in *Game of Maximal Competitive Contribution (MCC)*, and any tree is stable in *Game of Maximal Altruistic Contribution (MAC)*.

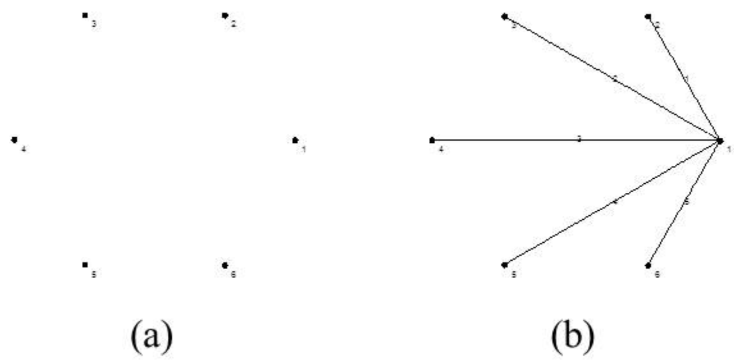


Figure 3.3. Evolution of graph in MMC when agents are allowed to make or break as many edges as desired. (a) Initial, completely disconnected graph. (b) Stable Connected Graph (a tree) results after one move.

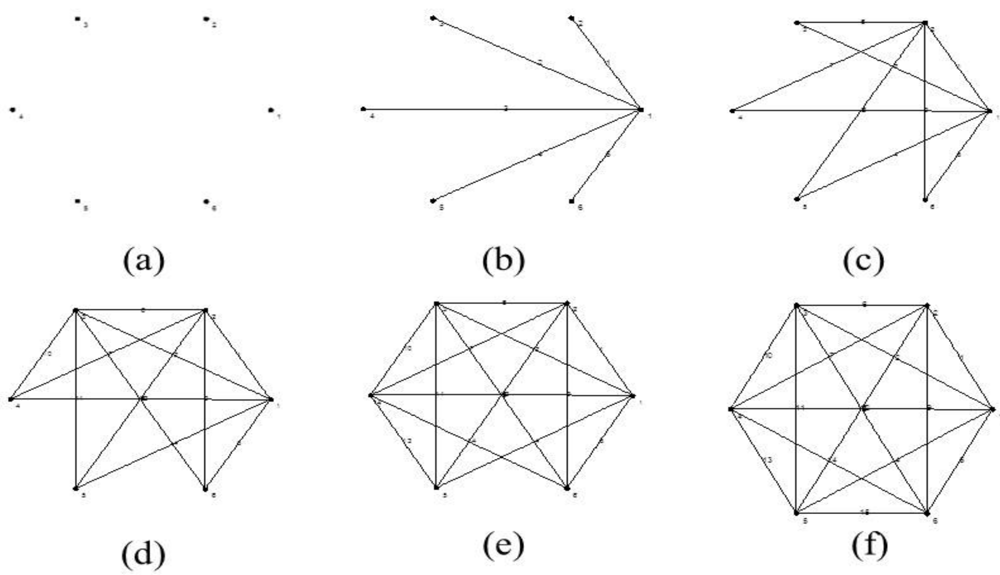


Figure 3.4. Evolution of graph in MCC when agents are allowed to make or break as many edges as desired. (a) Initial, completely disconnected graph. (b)-(e) Transition states on sequential moves of randomly selected agents. (f) Stable, Complete Graph.

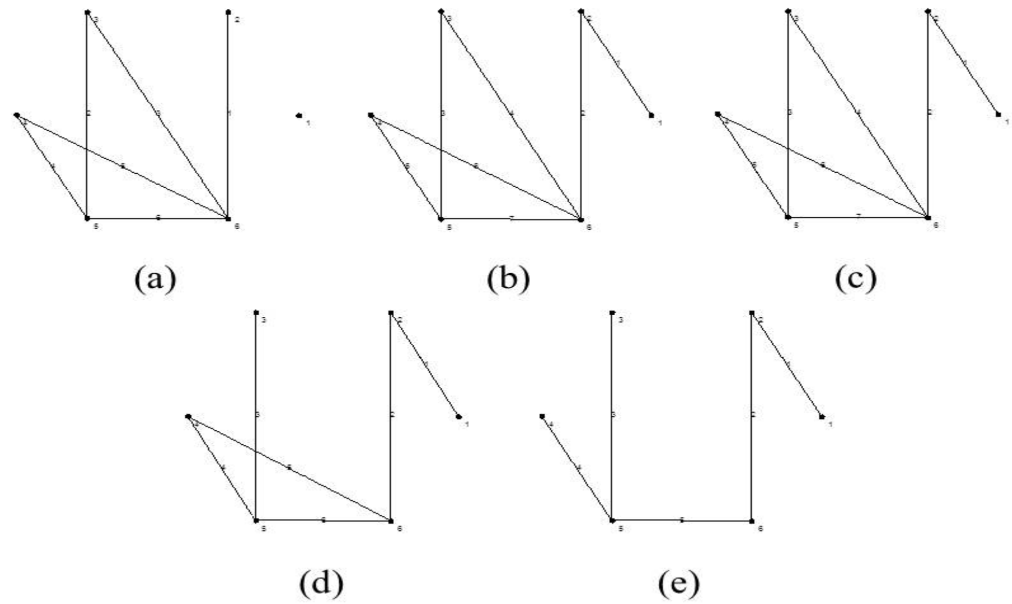


Figure 3.5. Evolution of graph in MAC when agents are allowed to make or break as many edges as desired. (a) Initial, random graph. (b)-(d) Transition states on sequential moves of randomly selected agents. (e) Stable graph, a Tree.

3.7 Conclusions

A Graphical Coalition Game is presented in this paper. Shapley value strengthened by the Axioms of Value is used to define the notion of Positional Advantage of agents in a coalition from a graph theoretic view point based on their connectivity. Vital properties including convexity, fairness, cohesiveness, and full cooperativeness are verified for the graphical coalitional game. The marginal contribution of an agent and its components competitive and altruistic contributions are defined in the framework of GCGs. A number of results are established regarding the dependence of these contributions on the graph topology, and changes in these contributions due to changes in graph topology. Further, on top of the GCG, and based on these three contributions, three online sequential decision games are defined. The concept of stability is defined in these sequential decision games, and the stable graph topologies under the three games are detailed. Certain elementary properties of the game are established in the Appendix for this framework.

Chapter 4

Positional Cost and Advantage in Coalitions on Graphs

This chapter introduces a graphical coalitional game where the internal topology of the coalition depends on a prescribed communication graph structure among the agents. The Value Function is required to satisfy four axioms. Here a Positional Cost is also introduced; the cost is assigned to each agent based on Shapley value and connectivity of the agent within the communication graph. A graphical coalitional game with Positional Advantage is also outlined. Based on the advantage and cost, a notion of Net Payoff or Allocation is defined; this notion is used to further define three measures of net advantages. Taking maximization of these measures of net advantages as the objective functions of agents, three online sequential decision games are defined on top of the coalitional graph game. Stable graphs under each sequential decision game are studied by varying the cost, and certain results about the coalition structure are established. A threshold of cost is reached above which no agent is interested to stay in a coalition irrespective of their motives.

4.1 Introduction

Game theory was introduced as a formal discipline of mathematics by J. von Neumann in 1928 through his classic work [113] and its extension [114]. Yet the diversity in the applications of game theory [132] is enough to know that it has a rich, old history [95]. The oldest, known, written account of game theory is the work of the Chinese scholar Sun Tzu, *The Art of War* [136], which is nearly 2300 years old. It is an established fact that since the beginning of life on earth the principles of game theory among individuals and biological species have played a fundamental role in the evolution of life and ecological systems [71], [103], [106], [133], [141]. Thus principles of game theory are extensively used to understand the behavior of living beings in ecological systems [73], [90], [92]. In general, game theory is a mathematical discipline that deals with issues and strategies involving competitions and cooperation between several entities [116]. In the scope of mathematical game theory these entities are called players or agents [84], [116], [119], [120], [121], [122], [123], [130].

Game theory is used in many walks of life involving situations of competition and cooperation. These areas include but are not limited to economics, finance, business [93], [99], [111], [112], law [80], [118], political science [75], [124], strategic science [88], [107], [131], social science [83], [89], [100], [126], and engineering [76], [81], [82], [91], [108], [109], [117], [125], [137], [138], [139], [140], [142]. Owing to such diverse applications of game theory, the subject is bound to break into various areas of study [132]. Though the boundaries between these areas of study are not very crisp [116], [132], game theory is primarily divided into two areas: noncooperative game theory [84], and cooperative game theory [116], [130]. In noncooperative game theory the fundamental unit of study is the individual agent, and the theory deals with its performance and strategies in interaction with other individual agents. By contrast, in cooperative game theory the fundamental unit is the group or coalition. Cooperative game theory deals with the value of the coalition, payoff allocations to individual players, and the stability of coalitions [116], [130].

Cooperative games can be divided into three classes: Canonical coalitional games, coalition formation games, and coalitional graph games [84], [123]. Canonical coalitional games mainly deal with the stabilization of the grand coalition of all the agents. Methods are sought to allocate the net value of the coalition to individual agents in such a way that agents are encouraged to join the coalition. A fair allocation that often accomplishes this is the Shapley value [127]. Coalition formation games mainly deal with coalitions based on gains and costs.

Given prescribed gains and costs, the structures of the resulting coalitions are studied. Finally, the coalitional graph games deal with the formation and stability of coalitions given an underlying communication graph structure [82], [123]. In the work of Baras [81], [82] and of Başar [119], [120], [121], [122], [123] coalitional graph games are studied with applications to communication networks. Various definitions of value are used in [119], [120], [121], [122], [123], including probability of detection, gain of resources of other agents, effective throughput, and packet success rate. In his classical work [110], Myerson used graph theoretic ideas to analyze cooperation in graphical coalitional games. He proposed to restrict the coalitions based on the underlying communication graph structure. He showed that the unique fair allocation of the net value of the coalition to the agents is given by the Shapley value [127].

Closely related to the coalitional graph games are online or sequential-in-time decision games. These are games where agents make moves through time sequentially to maximize their prescribed objective functions [130]. These games are defined by specifying the method of selection of the agent to make moves at each time, the allowed moves of the agents, and the objective function the agents seek to maximize. Agents might make moves according to some fixed round-robin procedure, or randomly according to some probability mass function. These online sequential decision games best model real-life situations where the players are free to change their alliances as considered suitable by them to obtain their objectives.

Considerations of cost are instrumental in the formation of coalitions. Various definitions of cost are used in coalitions, including probability of false alarm, vulnerabilities from other agents, download delay, mean waiting times, and path delay [81], [82], [104], [119], [120]. In their paper [104], Jackson and Wolinsky analyzed the stability of networks when the individual agents choose to form and maintain the links between them. An agent gains value on connecting to an agent which is well-connected to other agents in the graph, and accrues a cost based on maintaining direct communication links with its neighbors. It is shown that different relations between the link cost and the propagation of value along a path result in stability of different structures, such as complete graph, star graph [87], etc. In [82] a constrained coalitional game, based on Jackson and Wolinsky model [104], for networks of autonomous agents is defined. In [81] a trust based game is proposed. In this game payoffs and costs are dependent upon the gain and loss in mutual trust value. In [119] a cooperative game with non-transferable utility is proposed; in this game advantages and costs are based on probability measure. In [120] a coalitional game is introduced; in this game the value function is based on the effective throughput of the agents and cost is based on the delay.

The objective of this paper is to provide tools to study the internal structure of coalitions on graphs on the basis of different motives of the agents. This paper defines a graphical coalitional game (GCG) with transferable utility having novel properties. In this game, the total cost of a coalition depends upon the connectivity of the agents and the number of agents involved in the coalition. The first point of impact of the paper is based on a Value Function of cost that is required to satisfy four formal axioms. Owing to these axioms imposed on the Value Function, it is possible to perform a rigorous study of the internal structure of coalitions on graph topologies. The cost is initially allocated to the edges by using the Shapley value. The Shapley value with the Value Function satisfying the four Axioms is interpreted as the cost of a communication link within a coalition. The cost is than allocated to individual agents by using the symmetric connection model of Jackson and Wolinsky [104]. The cost of an agent in a coalition is called the Positional Cost (PC). Allocation rules based on Shapley values strengthened by the Axioms of Value assign the cost to the agents or vertices [104], [105]. In this paper allocation rules of advantage are the same as those introduced in [77]. The second point of impact of this paper is the use of advantage and cost to define Net Payoff or Allocation (NPA); it is further used to define three net advantages: Net Marginal Advantage (NMA), Net Competitive Advantage (NCA), and Net Altruistic Advantage (NAA). These net advantages are based on the components of cost defined in this paper and components of advantage defined in [77] and according to the concepts in [74]. A number of results about the dependence of these net advantages on coalition structure are presented. The third point of impact is the definition of three online sequential decision games on top of the graphical coalitional advantage and cost game; these three online sequential decision games are: max-NMA, max-NCA, and max-NAA. The preferred graphs under each sequential decision game, under certain relations between the advantages and costs are studied. It is shown that the stable graphs in max-NMA are any connected graph, including a tree. The preferred graph in max-NCA is a connected graph or completely disconnected graph under certain other condition. The completely disconnected graph is stable in max-NAA under certain conditions. These preferences in the three sequential games yield thresholds of cost beyond which agents stay in completely disconnected or trivial coalition irrespective of sequential game.

The paper is organized as follows. The concepts of Graph Theory used in this paper are elaborated in Section 4.2. A Graphical Coalition Game (GCG) is outlined in Section 4.3. The framework of GCG with Positional Cost (PC) within a coalition is defined in Section 4.4; some fundamental results about this framework are also presented in this section. Components of advantage and cost in a coalition are defined in Section 4.5; dependence of the components of cost on graph topology is also elaborated in this section. Section 4.6 deals with three Graphical Advantage and Cost Game (GACG). The notions of NMA, NCA, and NAA are also defined in this section. Section 4.7 presents Online Sequential Coalition Decision Games; stability of graph structures under these games and cost thresholds are also discussed here. Some technical lemmas required for the development of this framework are presented in the Appendix C at the end.

4.2 Graph Theory Background

Consider a graph $G=(V,E)$ with V a finite nonempty set of agents and $E \subseteq [V]^2$ a set of edges. Here $[V]^2$ is the unordered set containing all the subsets of V with two elements. Two agents are interpreted to have an edge between them if and only if they directly communicate with each other. The elements of V are also called vertices. The number of elements in V is called the order or size of the graph and is denoted as $|G|$ or N. In this paper simple graphs are considered. A simple graph does not contain self-loops and multiple edges; moreover, all its edges are undirected, connecting two vertices, and do not have any weight associated with them.

Two vertices with an edge between them are called neighbors of each other. If all the vertices of G are neighbors of each other than G is called a complete graph. A complete graph with N vertices is denoted as K_N A sequence of distinct vertices $i=i_0,i_1,...,i_M=j$ starting from a vertex i to another vertex j such that each pair of consecutive vertices are neighbors in G is called a path from i to j within G. Any two vertices having a path between them are called connected in G. A maximal set of connected vertices along with all the edges between them in G is called a component of G. A vertex that is not connected with any other vertex is called an isolated vertex. An isolated vertex is thus a graph component. If all the vertices of a graph are isolated then the graph is called completely disconnected. A graph is called connected if every vertex of the graph is connected to every other vertex. A graph is called minimally connected if the number of its edges is the smallest necessary to connect all its vertices. A graph which is not connected is called a disconnected graph. A connected graph is said to have one component while a disconnected graph is the union of more than one component. The size of a component S of G is denoted as $|S|$. Since undirected graphs are considered in this paper, two components are mutually disjoint.

If there are at least two vertices i and j in G with at least two distinct paths existing from i to j then G is called a cyclic graph. A graph without a cycle is called an acyclic graph or a forest. A connected forest is called a tree. A tree is thus a minimally connected graph. A connected graph which is not a tree is thus non-minimally connected.

If $G=(V,E)$ and $S=(V',E')$ are two graphs such that $V' \subseteq V$ and $E' \subseteq E$, then S is called a subgraph of G. A subgraph is a graph in its own capacity. Moreover if E' contains all the edges $e=\{i,j\} \in E$ with $i,j \in V'$, then S is called an induced subgraph of G. An induced subgraph of G obtained by excluding a vertex i from V is denoted as $G \setminus \{i\}$. If the number of components in $G \setminus \{i\}$ is more than the number of components in G, then i is called a cut vertex of G. If an edge e is deleted from a graph G then the new graph obtained is denoted as $G \setminus e$. Similarly if the deletion of edge increases the number of components of a graph then it is called a cut edge of G. If an edge e is added in G then the new graph is denoted as $G \cup e$. Detailed study on Graph Theory can be had in [87], [94].

4.3 Graphical Coalitional Game with Positional Advantage (PA)

In [77] a graphical coalitional game $\Gamma=(G,v)$ with transferable utility in the form of advantage is proposed. The game is based on the communication structure of the agents within a coalition. Here the undirected graph G represents the communication topology of the coalition with agents as vertices, and edges between them if and only if the agents directly communicate with each other within the coalition. In this section the Graphical Coalition Game and the related concepts are briefly outlined.

Consider a graph *G*, with agents as nodes, where there exist edges between the nodes if and only if the corresponding agents directly communicate with each other. The Value Function *v* is formally defined as

$$v : 2^G \to \Re \;\; with \;\; v(\phi) = v_0 = 0 \tag{4.1}$$

where 2^G is the collection of all the subgraphs of *G* and ϕ is the empty set. The Value Function satisfies the following axioms. In these axioms $S \in 2^G$ is a subgraph of *G*.

Axiom I: Axioms of Value in GCG with PA

1. If *S* is a connected component with $|S| = m$ then $v(S) = v_m \geq 0$
2. If *S* is having *k* connected components $S_i : i = 1, 2, ...k$ with $|S_i| = m_i$ then $v(S) = \sum_i v_{m_i} : v_{m_i} \geq 0$
3. If $N \geq m > n \geq 0$ then $n.v_m \geq m.v_n$
4. If $N - 1 \geq m > n \geq 0$ then $v_{m+1} - v_m \geq v_{n+1} - v_n$ ■

It is to be mentioned here that the Axiom I.2 is according to allocation rules in the coalitional graph game defined by Myerson in Section 3 of [110], where the coalitions are restricted by the underlying communication graph.

According to Axiom I.3, one has $v_2 \geq 2v_1$. A condition stronger than Axiom I.3 is

$$v_2 > 2v_1 \tag{4.2}$$

This condition is used to strengthen many results in [77]; it is also used in this paper to establish more results.

Definition 1: Graphical Coalitional Game with Positional Advantage. Given a graph *G*, the graphical coalitional game is defined as the game $\Gamma = (G, v)$ where the value function *v* satisfies Axiom I. ■

Remark 1:

1. If a coalition *G* has *m* components of sizes $n_1, n_2, ..., n_m$, then by using Axiom I.2 the value of the coalition *G* is given by $v(G) = v_{n_1} + v_{n_2} + ... + v_{n_m}$.
2. It can be seen by using Axiom I.4 and (3.1) that for any $0 \le n \le N$ $v_n \ge nv_1$.
3. It can be seen by using Axiom I.4 and the above remark that for $0 \le n < m \le N$ $nv_m \ge (m-n)v_n + nv_n$.
4. From the above remark it is clear that Axiom I.3 is implied by the Axiom I.4. Similarly it can be seen that Axiom I.1 is implied by the Axiom I.2. Nevertheless, Axioms I.1 and I.3 are retained as axioms because of the ease of their use in establishing the game properties.
5. Consider a possible game with $v_1 = 1$, $v_2 = 3$ and $v_3 = 4.51$. Clearly these values satisfy Axiom I.3 since (1)(3)>(2)(1), (1)(4.5)>(3)(1) and (2)(4.51)>(3)(3). Yet they do not satisfy the Axiom I.4 since 4.51-3 < 3-1. Thus Axiom I.4 is not implied by Axiom I.3.
6. Consider a possible game with $v_1 = 1$, $v_2 = 2.1$ and $v_3 = 3.9$. It can be easily seen that this list satisfies Axiom I.4 and hence Axiom I.3. ■

In the axioms of the game $\Gamma = (G, v)$, it is assumed that all the agents are identical and similarly that all the edges are identical. Thus, the game can be used to study the advantage of one vertex over the other based on its position in the graph structure. It is established in Section 3 of [77] that the game is fair, both fully cooperative and cohesive [74], convex [128], and super-additive [116].

The allocation of the total value of a coalition to its individual agents is a fundamental problem in coalitional games. Allocation is the share given to each agent of the total value of the coalition efforts. It is established in [110] that for a value function defined in games on graphs, the Shapley value function [127] is the only possible function that provides a fair allocation. Therefore, the allocation of PA to an agent *i* in the graphical coalitional game $\Gamma = (G, v)$ is made here using the Shapley value function. Therefore, the next definition is made. It provides a fundamental notion used in this paper.

Definition 2: Positional Advantage (PA) of an Agent in the Graphical Coalitional Game. Given the Graphical Coalitional Game with PA, $\Gamma = (G, v)$, define the PA of agent *i* as the Shapley value

$$\varphi_{G,v}(i) = \frac{1}{|G|} \sum_{S \subseteq G \setminus \{i\}} \frac{(v(S \cup \{i\}) - v(S))}{\binom{|G|-1}{|S|}} \tag{4.3}$$

In this equation the Value Function *v* satisfies the Axioms for the Value for PA. Moreover, $S \subseteq G \setminus \{i\}$ means *S* is an induced subgraph of the graph $G \setminus \{i\}$. Moreover, $S \cup \{i\}$ denotes an induced subgraph of *G*, containing all the agents in *S* and the agent *i*. ■

The subscripts *G* and *v* can be dropped from the notation if these are clear from the context. Since this allocation is dependent only upon the position of the agent *i* within the coalition represented by the graph *G*, it is called the PA of the vertex *i*. The PA is also dependent on the value function *v* in Definition 1. The value function can be represented as a real number list of size $|G|$. Therefore, the next definition is motivated.

Definition 3: Valid Game List. A list of non-negative real numbers $v_1, v_2, ..., v_{|G|}$ is called a valid game list of size $|G|$ if it satisfies the value Axiom I. ■

4.4 Graphical Coalitional Game with Positional Cost

In this section a Graphical Coalitional Game with Positional Cost is defined. The game is based on axioms, similar to those of the Graphical Coalitional Game with Positional Advantage, mentioned in the last section. Valid Game List for this game is also defined in this section. Some elementary concepts about the game list and the game are also established in the form of lemmas, remarks and examples. These concepts provide the background for Sections 4.5, 4.6, and 4.7. Some technical lemmas needed in the proofs are given in the Appendix.

4.4.1 Graphical Coalitional Game with Positional Cost

If there is a possibility of some nonzero cost associated with joining the coalition *G* a Graphical Coalitional Game with Positional Cost $\Upsilon = (G, u)$ coexists with the GCG with PA. Here *G* is the same graph representing the coalition in the last section.

The cost function *u* is formally defined as

$$u : 2^G \to \Re \ \ with \ \ u(\phi) = u_0 = 0 \tag{4.4}$$

where 2^G is the collection of all the subgraphs of *G* and ϕ is the empty set. The Value Function satisfies the following axioms. In these axioms $S \in 2^G$ is a subgraph of *G*.

Axiom II: Axioms of Cost in GCG with PC

1. If *S* is a connected component with $|S| = m$ then $u(S) = u_m \geq 0, u_1 = 0$
2. If S is having k connected components $S_i : i = 1, 2, ...k$ with $|S_i| = m_i$ then $u(S) = \sum_i u_{m_i} : u_{m_i} \geq 0$

3. If $N \geq m > n \geq 0$ then $n.u_m \geq m.u_n$
4. If $N-1 \geq m > n \geq 0$ then $u_{m+1} - u_m \geq u_{n+1} - u_n$ ■

The Axiom II is similar to the Axioms I of the GCG in Section 4.3. The only difference is the equality $u_1 = 0$ in Axiom II.1 for the PC case. Since the GCG with PC in this section is introduced to analyze the costs involved in making a coalition and allocation of cost to the agents involved in the coalition; the cost is incurred only when two or more agents come in contact with each other to make a coalition. This provides the rational to set u_1 to 0 in Axiom II.1.

Axiom II.3 is essentially a super linearity requirement $u_m / m \geq u_n / n$ for $m > n$. According to Axiom II.3, one has $u_2 \geq 2u_1 = 0$. A condition on u_2 stronger than Axiom 3 is

$$u_2 > 0 \tag{4.5}$$

It is to be noted that in the Axiom II, $u_1 = 0$, so that condition (4.5) is equivalent to $u_2 > 2u_1 = 0$ which is the same as condition (4.2) for the Graphical Coalition Game in Definition 1. This condition strengthens several results in the following development. It is established in Lemma C.4 in the Appendix that under the condition (4.5) in general, for $N \geq m > n \geq 0$ then $n.u_m > m.u_n$ which is a stronger condition than Axiom II.3.

The next definition of GCG with PC provides a fundamental notion used in this paper.

Definition 4: Graphical Coalitional Game with Positional Cost. Given a graph *G*, the graphical cost game is defined as the game $\Upsilon = (G, u)$ where the cost function *u* satisfies Axiom II. ■

Similar to the allocation of advantage of a coalition to the agents, the allocation of the cost of a coalition to its individual agents is a fundamental problem in coalitional games. Since coalition cost is associated to the communication of the agents within the coalition, thus cost is assigned to the edges which represent the communication among the agents. The cost associated to any edge depends upon the location of the edge within the communication graph *G*. Let $S=(V,E')$ be a randomly selected subgraph of $G=(V,E)$ having the same vertex set as that of *G* but having a randomly selected edge set E' out of *E*, not containing an edge *e*. Then the additional cost incurred by the edge *e* to the subgraph $S\cup e$ is given by $u(S\cup e)-u(S)$. The expected value, taken over all such values of *S*, of the cost incurred by the edge *e* is given by the Shapley value of that edge within *G* [96], [105] and in this paper it is taken as the cost of the edge *e*. The following definition is thus made.

Definition 5: Positional Cost of an Edge in GCG with PC. Given the GCG with PC $\Upsilon=(G,u)$, define the Positional Cost of edge *e* as the Shapley value

$$\psi_{G,u}(e)=\frac{1}{\|G\|}\sum_{S\subseteq G\setminus e}\frac{(u(S\cup e)-u(S))}{\binom{\|G\|-1}{\|S\|}} \tag{4.6}$$

In this equation the cost function *u* satisfies the Axioms for the cost. Moreover, $S\subseteq G\setminus e$ means *S* is a subgraph of the graph $G\setminus e$.with the same vertex set as that of *G*. Moreover, $S\cup e$ denotes a subgraph of *G*, containing all the edges in *S* and the edge *e*. ■

Remark 2: In equation (4.6) the value of the expression $(u(S\cup e)-u(S))$ is 0 when the subgraphs *S* and $S\cup e$ have the same number of components. ■

In their *Symmetric Connections Model* [104] Jackson and Wolinsky has associated a cost to each edge and equally divided it between the end vertices of that edge which best suits the communication models where the cost of each edge is equally born up by its end vertices. Therefore, the next definition is made. It splits the cost of an edge equally between its end vertices or agents.

Definition 6: Positional Cost of an Agent in the Graphical Coalitional Game with Positional Cost. Given the GCG with PC $\Upsilon = (G,u)$, define the Positional Cost of an agent *i* as

$$\psi_{G,u}(i) = \tfrac{1}{2}\sum\nolimits_{e\ :i\in e}\psi_{G,u}(e) \qquad (4.7)\blacksquare$$

The PC of an agent as defined above is different than the symmetric connection model of Jackson and Wolinsky in [104], in the sense that here in this paper the total cost of the coalition is first efficiently allocated to edges by using Shapley value [127] and then cost of an edge is equally divided into its end vertices according to symmetric connection model, whereas in [104] an arbitrary cost is allocated to each edge, which is generally taken as a constant for each edge. Allocation of a constant cost to all the communication links limits its use in communication structures where cost is dependent upon the location of the link in the communication structure.

Definition 7: Valid Game List for GCG with PC. A list of non-negative real numbers $u_1, u_2, ..., u_{|G|}$ is called a valid game list of size $|G|$ for the GCG with PC if it satisfies the cost Axiom II. ■

4.4.2 Valid Game List of Graphical Coalitional Game with Positional Cost

In this subsection validates certain game lists for the graphical coalitional game with Positional Cost. Instrumental in this discussion is the cost under the condition

$$u_i = k_i(v_i - iv_1), \forall i = 1,2,3..., N, k_i > 0 \qquad (4.8)$$

This condition captures a relationship between the cost function u_i and the value function v_i.

In the following lemmas it is established that if $v = (v_1, v_2, ...v_N)$ is a Valid Game List for the GCG with Positional Advantage in Definition 1 then under the condition (4.8) $u = (u_1, u_2, ..., u_N)$ is a Valid Game List for some values of k_i, for the GCG with PC in Definition

4. These game lists are motivating for their relationship with the Valid Game List for the corresponding GCG with PA. These Valid Game Lists are used as benchmark for the PC in the results in Sections 4.6 and 4.7.

Lemma 1: (Validity of Game List) If $v = (v_1, v_2, ...v_N)$ is a Valid Game List for a GCG with PA then u_i in (4.8) with $k_i = 1 \forall i = 1, 2, ..., N$, that is, $u = (u_1, u_2, ...u_N : u_i = v_i - iv_1, \forall i = 1, 2, 3..., N)$ is a Valid Game List for a GCG with PC.

Proof: The proof follows by establishing that u satisfies Axioms II.1 and II.4.From the given value of u,

$$u_1 = v_1 - 1v_1 = 0 \tag{4.9}$$

This is according to Axiom II.1. For any arbitrary integers n and m, such that $N - 1 \geq m > n \geq 0$

$$u_{m+1} - u_m = (v_{m+1} - (m+1)v_1) - (v_m - (m)v_1) \tag{4.10}$$

or

$$u_{m+1} - u_m = v_{m+1} - v_m - v_1 \tag{4.11}$$

Similarly,

$$u_{n+1} - u_n = v_{n+1} - v_n - v_1 \tag{4.12}$$

Since v is a Valid Game List, thus by Axiom I.4 for the given values of m and n, satisfying $N - 1 \geq m > n \geq 0$, $v_{m+1} - v_m \geq v_{n+1} - v_n$. Using (4.11) and (4.12) it follows that for $N - 1 \geq m > n \geq 0$, $u_{m+1} - u_m \geq u_{n+1} - u_n$. This proves the desired result. ■

Remark 3: If $v = (v_1, v_2, ...v_N)$ is a Valid Game List for a GCG with PA then u_i in (4.8) with $k_i = k \forall i = 1, 2, ..., N$, that is, $u = (u_1, u_2, ...u_N : u_i = k(v_i - iv_1) \forall i = 1, 2, 3...N : k \geq 0)$ is a Valid Game List for a GCG with PC. In the above list the constant k plays a basic role in varying the values of the list. It is established in the results in Sections 4.6 and 4.7 that by tuning the value of k various graph topologies are achieved. ■

The following lemma provides a greatest lower bound on the last entry u_N of a Valid Game List for a GCG with PC u, given the preceding entries in the list.

Lemma 2: (Validity of Game List) If $v=(v_1,v_2,....v_N)$ is a Valid Game List with $v_2>2v_1$ for a GCG with PA then u_i in (4.8) with $k_i=k\forall i=1,2,...,N-1,k_N=k'$, that is, $u=(u_1,u_2,....u_N:u_i=k(v_i-iv_1)\forall i=1,2,3,...,(N-1):k>0,u_N=k'(v_N-Nv_1))$ is a Valid Game List for a GCG with PC if and only if

$$k'>\frac{k(2v_{N-1}-v_{N-2}-Nv_1)}{(v_N-Nv_1)} \tag{4.13}$$

Proof: The proof follows by establishing that u satisfies Axioms II.1 and II.4.From the given value of u,

$$u_1=v_1-1v_1=0 \tag{4.14}$$

This is according to Axiom II.1. For any integers n and m, such that $N-2\geq m>n\geq 0$, then by Remark 3

$$u_{m+1}-u_m\geq u_{n+1}-u_n \tag{4.15}$$

For $m=N$

$$u_N-u_{N-1}=k'(v_N-Nv_1)-k(v_{N-1}-(N-1)v_1) \tag{4.16}$$

Using the given condition on k', it can be written as

$$u_N-u_{N-1}\geq k(2v_N-v_{N-1}-Nv_1)-k(v_{N-1}-(N-1)v_1) \tag{4.17}$$

On simplification and using the value for u_i the above equation becomes

$$u_N - u_{N-1} \geq u_{N-1} - u_{N-2} \tag{4.18}$$

Combining (4.15) and (4.18), it is established that for values of *m* and *n*, satisfying $N-1 \geq m > n \geq 0$, $v_{m+1} - v_m \geq v_{n+1} - v_n$. This proves that *u* is a valid game list.

Conversely suppose that the given value of *u* is a valid game list. Thus *u* satisfies Axiom II.4. In particular for $m = N-1$ and $n = N-2$ it can be written as

$$u_N - u_{N-1} \geq u_{N-1} - u_{N-2} \tag{4.19}$$

Substituting the values of u_i for $i = N, N-1, N-2$ and simplifying under the given condition that $v_2 > 2v_1$ the desired result $k' > \dfrac{k(2v_{N-1} - v_{N-2} - Nv_1)}{(v_N - Nv_1)}$ is obtained. ■

Remark 4:

1. The condition $v_2 > 2v_1$ in the above lemma implies that $v_N > Nv_1$ Lemma 5 of [77], thus the denominator in (4.13) is positive.
2. In subsequent results it is desired to have $k' < 1$ when $k > 1$, then by using order axioms of real numbers [72] this is possible only if $k < \dfrac{(v_N - Nv_1)}{(2v_{N-1} - v_{N-2} - Nv_1)}$. ■

A generalization of the above two lemmas is established in Appendix Lemma C.**5**. This lemma is reported to support further research in the area of Graphical Coalition Games. Moreover the result established in Lemma C.4 also assures that $u_i = k(v_i - iv_1), \forall i = 2,3..., N, k > 0$ given in (4.8) under condition (4.5) are positive.

The following two examples illustrate the calculation of PA of agents in a GCG with PA $\Gamma = (G, v)$ and Position Cost of agents in a GCG with PC $\Upsilon = (G, u)$ in a chain and in a complete graph of sizes 3.

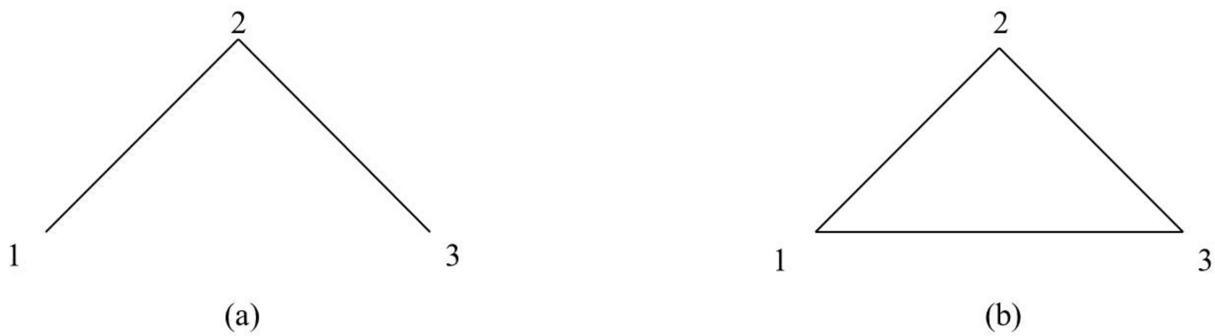


Figure 4.1. Two simple graphs . (a) Example 1- Three agents in a chain (b). Example 2- Three agents in a complete graph

Example 1: Consider a chain of three agents $G = \{1,2,3\}$ as shown in Figure 4.1 (a). The PA of the agents is calculated by using (4.3). For agent 1

$$\varphi_{G,v}(1) = \tfrac{1}{3}\left(\tfrac{v(\{1,2,3\})-v(\{2,3\})}{1} + \tfrac{(v(\{1,2\})-v(\{2\}))+(v(\{1,3\})-v(\{3\}))}{2} + \tfrac{v(\{1\})-v(\varphi)}{1}\right)$$

Using Axiom I of the game this is simplified to $\varphi_{G,v}(1) = \frac{1}{3}(v_3 - \frac{1}{2}v_2 + v_1)$. By the symmetry of the graph *G* and of (4.3), it can be seen that $\varphi_{G,v}(3) = \varphi_{G,v}(1)$. The PA of the agent 2 is given by

$$\varphi_{G,v}(2) = \tfrac{1}{3}\left(\tfrac{v(\{1,2,3\})-v(\{1,3\})}{1} + \tfrac{(v(\{1,2\})-v(\{1\}))+(v(\{2,3\})-v(\{3\}))}{2} + \tfrac{v(\{2\})-v(\varphi)}{1}\right)$$

Using the axioms of the game this is simplified as $\varphi_{G,v}(2) = \frac{1}{3}(v_3 + v_2 - 2v_1)$.

Now for the PC, using (4.6) $\psi_{G,u}(\{1,2\}) = \frac{1}{2}\left(\frac{u(\{1,2\})-u(\{\phi\})}{1} + \frac{u(\{1,2\}\cup\{2,3\})-u(\{2,3\})}{1}\right)$. Using Axiom II of the game this is simplified to $\psi_{G,u}(\{1,2\}) = \frac{1}{2}(u_3 - 3u_1) = \frac{1}{2}u_3$. Similarly, $\psi_{G,u}(\{2,3\}) = \frac{1}{2}u_3$, thus by (4.7) $\psi_{G,u}(1) = \frac{1}{4}u_3 = \psi_{G,u}(3)$ and $\psi_{G,u}(2) = \frac{1}{2}u_3$

It can be readily seen that $\psi_{G,u}(2) \geq \psi_{G,u}(1), \psi_{G,u}(3)$ This is according to the heuristics for the given communication structure, since 2 is in the middle of 1 and 3 and so logically contributes more to the communication structure of the coalition and hence bears more cost. ■

Example 2: Considering a complete graph of three agents $G = \{1,2,3\}$ as shown in Figure 4.1 (b), the PA of agent 1 is calculated by using the definition given in (4.3)

$$\varphi_{G,v}(1) = \tfrac{1}{3}\left(\frac{v(\{1,2,3\}) - v(\{2,3\})}{1} + \frac{(v(\{1,2\}) - v(\{2\})) + (v(\{1,3\}) - v(\{3\}))}{2} + \frac{v(\{1\}) - v(\varphi)}{1}\right)$$

Using the axioms of the game this equation is simplified as $\varphi(1) = \frac{1}{3}v_3$.

Now for the PC, using (4.6)

$$\psi_{G,u}(\{1,2\}) = \tfrac{1}{3}\left(\frac{u(\{1,2\}) - u(\{\phi\})}{1} + \frac{(u(\{1,2\} \cup \{2,3\}) - u(\{2,3\})) + (u(\{1,2\} \cup \{1,3\}) - u(\{1,3\}))}{2} + \frac{(u(\{1,2\} \cup \{2,3\} \cup \{1,3\}) - u(\{2,3\} \cup \{1,3\}))}{1}\right)$$

Using Axiom II of the game this is simplified to $\psi_{G,u}(\{1,2\}) = \frac{1}{3}u_3 = \psi_{G,u}(\{3,2\}) = \psi_{G,u}(\{1,3\})$.

Thus by (4.7) $\psi_{G,u}(1) = \frac{1}{3}u_3 = \psi_{G,u}(3) = \psi_{G,u}(2)$

It can be easily seen that all the vertices have the same PC. This again is according to intuition, since all the three vertices are symmetrically distributed in the graph and evenly contribute to the communications within the coalition. ■

4.4.3 Dependence of Positional Cost on Graph Topology

In this subsection some lemmas about the dependence of Positional Cost on the graph topology are presented. These lemmas establish the usual properties of PC; like dependence of cost on number of connected agents and increase in cost due to increase in component sizes. Moreover, these lemmas are important to establish the distribution of cost among agents and are also needed to prove the upcoming results in Sections 4.6 and 4.7.

The following lemma provides a fundamental result in the game presented in this paper; it shows that unlike the direct dependence of coalitional cost on edges in [104], the cost associated with a coalition is independent of addition of a new edge *e* as long the addition does not reduce the number of graph components.

Lemma 3: (Addition of a New Edge in a Connected Graph) In a GCG with PC, addition of a new edge in an already connected graph does not change the overall cost of the coalition.

Proof: Consider a GCG with PC $\Upsilon = (G,u)$ in Definition 4. Since the graph is connected, according to Axiom II.2 $u(G) = u_{|G|}$. Using the same axiom this value of $u(G)$ does not change upon making of a new edge. ■

Remark 5: For a subgraph S of a graph G and an edge e in G, if the number of components in S and $S \cup e$ are the same then $u(S \cup e) - u(S) = 0$. ■

The following two lemmas illustrate the change in PC on a coalition when a cut edge is added to a disconnected graph representing it. In the following two lemmas G' presents a graph obtained from G by adding a new edge in it.

Lemma 4: (Addition of New Cut Edge in a Disconnected Graph) In a Graphical GCG with PC, if the addition of an edge reduces the number of components of the graph then the overall PC of the coalition increases or remains constant.

Proof: Consider a GCG with PC $\Upsilon = (G,u)$ in Definition 4. Since the graph is disconnected, according to Axiom II.2 the PC of the Coalition G is

$$u(G) = \sum_{i=1}^{k} u_{m_i} : \sum_{i=1}^{k} m_i = |G|, k > 1 \tag{4.20}$$

Making of a new edge gives a new graph G', its PC is given by

$$u(G') = \sum_{i=1}^{k-1} u_{m_i'} : \sum_{i=1}^{k} m_i' = |G|, k > 1 \tag{4.21}$$

Without loss of generality it can be assumed that

$$m_{k-1}' = m_{k-1} + m_k, m_i' = m_i \forall i = 1,2,...,k-2 \tag{4.22}$$

Under the above condition, By Lemma 4 of [77]

$$u_{m'_{k-1}} \geq u_{m_{k-1}} + u_{m_k}, u_{m'_i} = u_{m_i} \forall i = 1, 2, ..., k-2 \tag{4.23}$$

Comparison of (4.20) and (4.21), under the above condition gives

$$u(G') \geq u(G) \tag{4.24}$$

This completes the desired proof. ■

The next lemma strengthens the above lemma under condition (4.5).

Lemma 5: (Addition of New Cut Edge in a Disconnected Graph) In a GCG with PC, if the addition of an edge reduces the number of components of a graph *G* and there is a cost game with $u_2 > 0$, then the overall PC of the coalition increases.

Proof: The proof goes on the same lines as the above proof; however, under hypothesis $u_2 > 0$, by Remark 4 in [77], (4.23) turns into a strict inequality

$$u_{m'_{k-1}} > u_{m_{k-1}} + u_{m_k}, u_{m'_i} = u_{m_i} \forall i = 1, 2, ..., k-2 \tag{4.25}$$

Comparison of (4.20) and (4.21), under the above condition gives

$$u(G') > u(G) \tag{4.26}$$

This completes the desired proof. ■

The following lemma supports an observation that the maximum cost an agent may bear may be half of the total cost of the coalition.

Lemma 6: (Maximum PC of an Agent) In a GCG with PC, the maximum possible PC of an agent in a connected graph *G* is $\frac{1}{2}u_{|G|}$.

Proof: The PC of a coalition is primarily allocated to the edges (4.6), and then equally distributed between their end vertices (4.7). Since this paper deals with simple graphs with no self-loops allowed; Positional Coast of an agent *i* can be maximum when every edge is incident at it. It is hypothesized that *G* is connected, thus by Axiom II.1

$$u(G) = u_{|G|} \tag{4.27}$$

By the distribution of cost to agents (4.7), $\psi_{G,u}(i) = \frac{1}{2}u_{|G|}$, where i is a point with every edge incident at it. ■

Remark 6: The maximum possible PC a vertex may have in any graph G is that of the star point in a Star Graph [94]. ■

Making a New Edge in a Graph. In the following results G' represents a graph obtained from a graph G by adding a new edge e' in it.

Lemma 7: (Increase in PC of a Cut Edge) In a GCG with PC, if a graph G' is obtained from a graph G by adding a new edge e' in it then PC of an edge $e \in G$ increases by a non-negative value if e is a cut-edge in G'.

Proof: Change in the PC of an edge e in a graph G for a GCG with PC $\Upsilon = (G,u)$ is given by Lemma C.9

$$\psi_{G',u}(e) - \psi_{G,u}(e) = \frac{1}{\| G \| + 1} \sum_{S \subseteq G \setminus e} \frac{(u(S \cup e' \cup e) - u(S \cup e')) - (u(S \cup e) - u(S))}{\binom{\|G\|}{\|S\|+1}} \tag{4.28}$$

Since e' is a cut edge $(u(S \cup e' \cup e) - u(S \cup e')) \geq (u(S \cup e) - u(S))$, and the above equation yields that $\psi_{G',u}(e) \geq \psi_{G,u}(e)$. This shows that the PC of a cut edge in G' increases by a nonnegative value. ■

Remark 7: For the GCG with PC $\Upsilon = (G,u)$ if u satisfies $u_2 > 0$, mentioned in (4.5) then $(u(S \cup e' \cup e) - u(S \cup e')) > (u(S \cup e) - u(S))$ and the PC of a cut edge in G' increases upon making a new edge in G. ■

4.5 Components of Advantage and Cost in Coalitions

Whenever agents join hands to make a coalition there is some advantage associated with the coalition and there is some cost involved in making the coalition. Both the advantage and the cost are transferred to the member agents of the coalition. Both the marginal or total advantage and cost contributed by an agent to a coalition can be divided into selfish and selfless components. This section discusses the components of advantage in Graphical Coalitional Game with Positional Advantage, in Definition 1 and contribution in cost of an agent within the setting of GCG with Positional Cost, in Definition 4. The definitions and results about the components of marginal advantage are presented in [77]. Definitions of marginal cost and its components along with results about the dependence of these components on graph topology are presented in this section. The notions presented in this section provide the basis for defining net advantages in Section 4.6 and online sequential decision games in Section 4.7.

4.5.1 Components of Advantage of Agents in Coalitions

When agents participate with other agents to make a coalition, they contribute towards the overall coalition cause. The total contribution of a group A of agents in a GCG $\Gamma(G,v)$ is called the marginal contribution of A in G and written as $m_{G,v}(A)$ [74]. The marginal contribution of a group of agents is divided in two components: one component is the contribution of the agents in the subset A for the sake of themselves, and the second component is the contribution of agents in A for the sake of the other agents in $G \setminus A$. These two components are termed the competitive contribution and the altruistic contribution respectively. These contributions are represented as $c_{G,v}(A)$ and $a_{G,v}(A)$ respectively [74]. These contributions are formally defined in [77] and are summarized here.

The Marginal Contribution and its components are defined on the basis of the definition of allocations or payoffs in Definition 8 below. In the following definition $A \subseteq B \subseteq G$, A is an induced subgraph of B which is an induced subgraph of the coalition G.

Definition 8: Allocation of a Set of Agents in an Induced Subgraph. The allocation or payoff of the agents in coalition A when only the coalition B is considered is denoted as $\mu_A(B)$ and it is defined as

$$\mu_A(B,v) = \sum_{i \in A} \varphi_{B,v}(i) \tag{4.29}$$ ■

Based on this definition, the following components of advantage in a coalition are defined.

Definition 9: Marginal Contribution of a Set of Agents. The marginal contribution $m_{G.v}(A)$ is defined as

$$m_{G,v}(A) = \mu_G(G,v) - \mu_{G \setminus A}(G \setminus A, v) \tag{4.30}$$ ■

Definition 10: Competitive Contribution of a Set of Agents. The competitive contribution $c_{G,v}(A)$ is defined as

$$c_{G,v}(A) = \mu_G(G,v) - \mu_{G \setminus A}(G,v) \tag{4.31}$$ ■

Definition 11: Altruistic Contribution of a Set of Agents. The altruistic contribution $a_{G,v}(A)$ is defined as

$$a_{G,v}(A) = \mu_{G \setminus A}(G,v) - \mu_{G \setminus A}(G \setminus A, v) \tag{4.32}$$ ■

From (4.30), (4.31) and (4.32) it follows that these definitions are according to their rationales established at the beginning of this section: $m_{G,v}(A)$ is the total contribution of the agents in A, $c_{G,v}(A)$ is the contribution of agents in A for their own sake and $a_{G,v}(A)$ is the contribution of A for the sake of the rest of the coalition $G \setminus A$. Moreover, according to these definitions

$$m_{G,v}(A) = c_{G,v}(A) + a_{G,v}(A) \tag{4.33}$$

For a singleton set A consisting of one agent i, the marginal contribution, competitive contribution and altruistic contribution are represented as $m_{G,v}(i)$, $c_{G,v}(i)$, and $a_{G,v}(i)$ respectively. From (4.31), (4.32), and (4.33), these contributions can be written as

$$m_{G,v}(i) = \mu_G(G,v) - \mu_{G\setminus i}(G \setminus i, v) \tag{4.34}$$

$$c_{G,v}(i) = \mu_G(G,v) - \mu_{G\setminus i}(G,v) \tag{4.35}$$

$$a_{G,v}(i) = \mu_{G\setminus i}(G,v) - \mu_{G\setminus i}(G \setminus i, v) \tag{4.36}$$

Using (4.29) in these three equations gives

$$m_{G,v}(i) = \sum\nolimits_{j\in G} \varphi_{G,v}(j) - \sum\nolimits_{j\in G\setminus i} \varphi_{G\setminus i,v}(j) \tag{4.37}$$

$$c_{G,v}(i) = \sum\nolimits_{j\in G} \varphi_{G,v}(j) - \sum\nolimits_{j\in G\setminus i} \varphi_{G,v}(j) \tag{4.38}$$

$$a_{G,v}(i) = \sum\nolimits_{j\in G\setminus i} \varphi_{G,v}(j) - \sum\nolimits_{j\in G\setminus i} \varphi_{G\setminus i,v}(j) \tag{4.39}$$

If G is connected, using Axioms I.1 and I.2 of the graphical coalition game and (4.29), equation (4.37) can be written as

$$m_{G,v}(i) = v_{|G|} - \sum_{j=1}^{p} v_{k_j} : \sum_{j=1}^{p} k_j = |G| - 1 \tag{4.40}$$

Here, p is the number of disconnected components of G obtained by the deletion of the agent i and $k_j : j = 1, 2, .., p$ are the sizes of these components.

4.5.2 Components of Cost of Agents in Coalitions

When agents participate with other agents to make a coalition, and there is some cost involved in making the coalition then they contribute towards the overall coalition cost. The total cost of a group *A* of agents in a coalition *G* is called the Marginal Cost of *A* in *G*, the names of the total cost and its components are adopted from [74] and [77]. The Marginal Cost of *A* in a Coalition *G* is written as $\omega_{G,u}(A)$. The Marginal Cost of a group of agents *A* is divided in two parts: one part is the cost incurred by the agents in the subset *A* and born by them, and the second part is the cost incurred by the agents in *A* but born by the agents in $G \setminus A$. These two parts are termed the Competitive Cost and the Divisive Cost respectively. These components are represented as $\varsigma_{G,u}(A)$ and $\alpha_{G,u}(A)$ respectively. Moreover, the Competitive Cost is divided further into two components: Fair Cost, and Surplus Cost.

The Marginal Cost and its components are defined on the basis of the following definition of Cost Allocation in Definition 8; in this definition $A \subseteq B \subseteq G$, *A* is an induced subgraph of *B* which is an induced subgraph of the coalition *G*.

Definition 12: Cost Allocation of a Set of Agents in an Induced Subgraph. The cost allocation of the agents in coalition *A* when only the coalition *B* is considered is denoted as $\eta_A(B)$ and it is defined as

$$\eta_A(B,u) = \sum_{i \in A} \psi_{B,u}(i) \qquad (4.41) \blacksquare$$

Based on this definition, the following components of cost in a coalition are defined.

Definition 13: Marginal Cost of Set of Agents. The marginal cost $m_{G,u}(A)$ of a set *A* of agents is defined as

$$\omega_{G,u}(A) = \eta_G(G,u) - \eta_{G \setminus A}(G \setminus A, u) \qquad (4.42)$$

where $\eta_G(G,u)$ and $\eta_{G\setminus A}(G\setminus A,u)$ are the Cost Allocations specified in Definition 8. ■

As mentioned earlier, the Marginal Cost can be split into two components: Competitive, and Divisive. These components are defined below.

Definition 14: Competitive Cost of a Set of Agents. The competitive cost $\varsigma_{G,u}(A)$ of a set A of agents is defined as

$$\varsigma_{G,u}(A)=\eta_G(G,u)-\eta_{G\setminus A}(G,u) \tag{4.43}$$ ■

Definition 15: Divisive Cost of a Set of Agents. The divisive cost $\alpha_G(A)$ of a set A of agents is defined as

$$\alpha_{G,u}(A)=\eta_{G\setminus A}(G,u)-\eta_{G\setminus A}(G\setminus A,u) \tag{4.44}$$ ■

It follows from (4.42), (4.43), and (4.44) that

$$\omega_{G,u}(A)=\varsigma_{G,u}(A)+\alpha_{G,u}(A) \tag{4.45}$$

For a singleton set A consisting of an agent i these costs are given by

$$\omega_{G,u}(i)=\eta_G(G,u)-\eta_{G\setminus i}(G\setminus i,u) \tag{4.46}$$

$$\varsigma_{G,u}(i)=\eta_G(G,u)-\eta_{G\setminus i}(G,u) \tag{4.47}$$

$$\alpha_{G,u}(i)=\eta_{G\setminus i}(G,u)-\eta_{G\setminus i}(G\setminus i,u) \tag{4.48}$$

In this paper singleton sets are referred by the elements instead of set notation. If G is connected, using Axioms II.1 and II.2 of the GCG with Positional Cost and (4.41), the Marginal Cost (4.46) can be written as

$$\omega_{G,u}(i)=u_{|G|}-\sum_{j=1}^{p}u_{k_j}:\sum_{j=1}^{p}k_j=|G|-1 \tag{4.49}$$

Here, p is the number of disconnected components of G obtained by the deletion of the agent i and $k_j : j = 1, 2, .., p$ are the sizes of these components.

Remark 8: It follows from the Axiom II and (4.42), (4.43), and (4.44) that Marginal, Competitive and Divisive costs are all nonnegative. ■

Lemma 8: (Equality of Cost Allocation and PC) The Cost Allocation of an agent in a Graphical Coalitional Game with PC is the same as its PC. That is

$$\varsigma_{G,u}(i) = \eta_G(i,u) = \psi_{G,u}(i) \tag{4.50}$$

Proof: For $A = \{i\}$ and $B = G$, (4.41) can be written as $\eta_G(i,u) = \psi_G(i,u)$. This establishes one of the desired equalities. Now for the second equality, substitution of $A = \{i\}$ in (4.43) gives

$$\varsigma_{G,u}(i) = \eta_G(G,u) - \eta_{G\setminus i}(G,u) \tag{4.51}$$

or, by using (4.41)

$$\varsigma_{G,u}(i) = \sum\nolimits_{j\in G} \psi_{G,u}(j) - \sum\nolimits_{i\in G\setminus i} \psi_{G,u}(j) \tag{4.52}$$

This gives $\varsigma_{G,u}(i) = \psi_{G,u}(i)$, the second desired equality. ■

The following example illustrates the computation of PC and its components in a given graph.

Example 3: PC of agents in a GCG with PC $\Upsilon = (G,u)$ in a chain of three agents $G = \{1,2,3\}$ as shown in Figure 4.1 (a).

Now for the PC, using (4.6) $\psi_{G,u}(\{1,2\}) = \frac{1}{2}\left(\frac{u(\{1,2\}) - u(\{\phi\})}{1} + \frac{u(\{1,2\}\cup\{2,3\}) - u(\{2,3\})}{1}\right)$. Using Axiom II of the game this is simplified to $\psi_{G,u}(\{1,2\}) = \frac{1}{2}(u_3 - 3u_1) = \frac{1}{2}u_3$. Similarly, $\psi_{G,u}(\{2,3\}) = \frac{1}{2}u_3$, thus by (4.7) and (4.50) $\eta_{G,u}(1) = \psi_{G,u}(1) = \frac{1}{4}u_3 = \psi_{G,u}(3) = \eta_{G,u}(3)$ and $\eta_{G,u}(2) = \psi_{G,u}(2) = \frac{1}{2}u_3$.

Using (4.46) for the computation of the Marginal Cost of agent 1 $\omega_{G,u}(1) = \eta_G(G,u) - \eta_{G\setminus 1}(G \setminus 1, u)$. By using (4.41) and symmetry of the graph structure $\omega_{G,u}(1) = u_3 - u_2 = \omega_{G,u}(3)$

Similarly $\omega_{G,u}(2) = u_3$. And using (4.48) for the computation of the Divisive Cost of agent 1is $\alpha_{G,u}(1) = \eta_{G\setminus 1}(G,u) - \eta_{G\setminus 1}(G \setminus 1, u)$. By using (4.41) and symmetry of the graph structure $\alpha_{G,u}(1) = \frac{1}{4}u_3 - u_2 = \alpha_{G,u}(1)$. Similarly $\alpha_{G,u}(2) = \frac{1}{2}u_3$ ■

4.5.3 Fair Advantage and Fair Cost

It is established by Myerson [110] that in graphical coalition games Shapley value [127] is the only fair allocation in the sense that every edge equally contributes to each of its end vertices. Since Shapley value (4.3) is used in (4.29), the allocation of PA (4.29) is fair.

In the computation of the allocation of cost to agents in (4.41) is not made directly by the Shapley value (4.6), but is made by (4.7), which is based on the *Symmetric Connections Model* of Jackson Wolinsky [104]. Therefore, the allocation of cost to the agents made in (4.41) is generally not fair in Myerson's sense [110]. Thus the Cost Allocation of a Set of Agents (4.41) can be further divided into two parts: Fair Cost of a Set of Agents, and Surplus Cost of a Set of Agents. The first one is based on the Shapley value and the second one is the remaining part of the Cost Allocation.

Definition 16: Fair Cost of a Set of Agents. The fair cost $f_{G,u}(A)$ of a set A of agents is defined as

$$f_{G,u}(A) = \sum_{i \in A} \varphi_{G,u}(i) \qquad (4.53)$$ ■

It is to be noted here that in this equation the individual cost function is based on the GCG with PC $\Upsilon = (G,u)$ in Definition 4 and not on Graphical Coalition Game $\Gamma = (G,v)$ in Definition 1, thus $\varphi_{G,u}(i)$ is not same as $\varphi_{G,v}(i)$.

Definition 17: Surplus Cost of a Set of Agents. The surplus cost $s_{G,u}(A)$ of a set A of agents is defined as

$$s_{G,u}(A) = \sum_{i\in A}(\psi_{G,u}(i) - \varphi_{G,u}(i)) \tag{4.54}$$ ■

It can be noted here that

$$\varsigma_{G,u}(A) = f_{G,u}(A) + s_{G,u}(A) \tag{4.55}$$

These notions of Fair and Surplus costs are mentioned here in this paper to support further research in Graphical Coalitions Games with PC.

Remark 9: It implies from the above two definitions that if the Surplus Cost of an agent is positive then that agent is paying more than the fair share of the coalitional cost in Myerson's sense [110]. Similarly negative value of Surplus cast of an agent mean that the agent is paying less than the fair share of coalitional cost in the same sense. For a graph with all the agents having 0 surplus cost implies all the agents are paying their fair share of cost, this generally happens in regular graphs. ■

4.5.4 Dependence of Cost Components of Agents on Graph Topology

Some lemmas about the contributions in cost of the agents are presented next. They demonstrate the dependence of the Marginal, Competitive, and Divisive costs on the topology of the graph. As defined in the Section 4.2, a cut vertex is one whose removal increases the number of disconnected components. The first results concern the marginal contribution.

Lemma 9: (Marginal Cost of Vertices) Given the graphical cost game $\Upsilon = (G,u)$ in Definition 4, in any connected graph G all the agents which are not cut vertices of G have the same marginal cost. Moreover their marginal cost is the minimum possible marginal cost within the connected graph. This minimum marginal cost is independent of the connected graph G and only depends upon $|G|$.

Proof: The Marginal Cost $\omega_{G,u}(i)$ for an agent i within a connected graph G is given by using (4.49). For a connected graph G the first term in the right hand side of this equation is constant, thus $\omega_{G,u}(i)$ is minimum when the second term $\sum_{j=1}^{p} u_{k_j}$ in the right hand side of this equation is maximum, which under the given condition in (4.49) and according to Remark 3 in [77] is $v_{|G|-1}$, when the agent i is not a cut vertex. This minimum marginal contribution is given by

$$\omega_{G,u}(i) = u_{|G|} - u_{|G|-1} \tag{4.56}$$

This minimum value is independent of the structure of G and only depends upon $|G|$. ■

Remark 10: In a connected graph G, if there is no cut vertexes then the Marginal Costs of all the agents are identical and independent of the graph structure. ■

Lemma 10: (Maximum Marginal Cost of an Agent) In a connected graph G of size N, the maximum possible Marginal Cost an agent may have is of the center point of a star.

Proof: For an agent i within a connected graph G the Marginal Cost $\omega_{G,u}(i)$ is given by (4.49). For a connected graph G the first term in the right hand side of this equation is constant, thus $\omega_G(i)$ is maximum when $\sum_{j=1}^{p} u_{k_j}$ is minimum, which according to Remark 3 of [77], under the given condition in (4.49) is $(|G|-1)u_1 = 0$, which is possible only when removal of agent *i* from *G* leaves rest of the agents isolated. ■

Remark 11: The Marginal Cost of the star point in a Star of size *N* is the total cost u_N. Moreover from (4.7) and Lemma 8 it follows that the maximum Competitive Cost of an agent is also possible only when agent is the star point and it is equal to $\frac{1}{2}u_N$. ■

The next two lemmas provide the conditions for an agent to have zero Marginal Cost.

Lemma 11: (Marginal Cost of an Isolated Agent) In a GCG with PC, if an agent is isolated then its Marginal Cost is 0.

Proof: For an agent *i* within a graph *G* the Marginal Cost $\omega_G(i)$ is given by using (4.46)

$$\omega_{G,u}(i) = \eta_G(G,u) - \eta_{G\setminus i}(G \setminus i, u)$$

Since the agent *i* is isolated, thus $\eta_G(G,u) = \eta_{G\setminus i}(G,u) + u_1 = \eta_{G\setminus i}(G \setminus i, u) + u_1$, using the Axiom II, the above equation gives $\omega_{G,u}(i) = 0$. ■

The next result shows that under condition (4.5), the above result is also sufficient.

Lemma 12: (Agents with 0 Marginal Cost) In a GCG with PC with a game list having $u_2 > 0$, if the Marginal Cost of an agent is 0 then it is isolated.

Proof: The Marginal Cost of an agent *i* in a GCG with PC is given by (4.46)

$$\omega_{G,u}(i) = \eta_G(G,u) - \eta_{G\setminus i}(G \setminus i, u)$$

Under the given condition $\omega_{G,u}(i) = 0$, the above equation becomes $\eta_G(G,u) = \eta_{G\setminus i}(G \setminus i, u)$. If G'' is the connected component in *G* containing the agent *i* then by using the Axiom II, the above equation can be written as

$$\eta_{G''}(G'',u) = \eta_{G''\setminus i}(G'' \setminus i, u)$$

or

$$u_{|G''|} = \sum_{i=1}^{p} u_{k_i} : \sum_{i=1}^{p} k_i = |G''| - 1$$

Under the given condition $u_2 > 0$, by using Axioms II and Lemma 4 of [77], the above equation holds only if $|G''| = 1$. This implies that the agent i is isolated. ■

Remark 12: It is already mentioned in Remark 8 all of Marginal, Competitive and Divisive costs are nonnegative. The zero value of Marginal Cost thus implies that both the Competitive and Divisive costs are zero. ■

4.6 Graphical Coalitional Games of Simultaneous Positional Advantage and Positional Cost

In this section a Graphical Advantage and Cost Game (GACG) is defined. The game is based on the analogy that when agents join to make a coalition, both advantages and costs are involved. In such real-life situations, agents distribute the advantage among each other and share the cost. On the basis of GACG, three net advantages are defined. Results giving the bounds on these net advantages are also presented in this section.

4.6.1 Components of Net Advantages in Games of Simultaneous Positional Advantage and Positional Cost

In this subsection, under the game of Graphical Advantage and Cost, three kinds of net advantages are defined. These net advantages are Net Marginal Advantage (NMA), Net Competitive Advantage (NCA) and Net Altruistic Advantage (NAA). These net advantages are further used to study the situations where the objectives of the agents within the coalition are to optimize their respective advantages.

Definition 18: Graphical Advantage and Cost Game. Given a graph *G*, the graphical advantage and cost game is defined as the game $\Sigma = (G, v, u)$ where the advantage function *v* satisfies Axiom I and cost function *u* satisfies Axiom II. ■

The Net Marginal Contribution and its components are defined on the basis of the definition of Net Allocation or Net Payoffs in Definition 19 below. In the following definition $A \subseteq B \subseteq G$, *A* is an induced subgraph of *B* which is an induced subgraph of the coalition *G*.

Definition 19: Net Payoff or Allocation (NPA) of a Set of Agents in an Induced Subgraph. The allocation or payoff of the agents in coalition *A* when only the coalition *B* is considered is denoted as $\lambda_A(B, v, u)$ and it is defined as

$$\lambda_A(B,v,u) = \mu_A(B,v) - \eta_A(B,u) = \sum\nolimits_{i\in A} \varphi_{B,v}(i) - \sum\nolimits_{i\in A} \psi_{B,u}(i) \tag{4.57}$$

In this equation $\mu_A(B,v)$ and $\eta_A(B,u)$ are from (4.29) and (3.10) respectively. ■

Remark 13: From this definition the Net Payoff or Allocation (NPA) to a single agent *i* under GACG $\Sigma = (G, v, u)$ is

$$\lambda_i(G,v,u) = \varphi_{G,v}(i) - \psi_{G,u}(i) \tag{4.58}$$

In this equation $\varphi_{B,v}(i)$ and $\psi_{B,u}(i)$ are from (4.3) and (4.7) respectively. ■

Definition 20: Net Marginal Advantage (NMA) of an Agent in the GACG. Given the GACG $\Sigma = (G, v, u)$, define the Net Marginal Advantage of agent or vertex *i* as the difference of its Marginal Contribution $m_{G,v}(i)$ [77] and Marginal Cost $\omega_{G,u}(i)$ and given by

$$\varpi_{G,v,u}(i) = m_{G,v}(i) - \omega_{G,u}(i) \tag{4.59}$$

where $m_{G,v}(i)$ is given by (4.34) and $\omega_{G,u}(i)$ is given by (4.46). ■

Definition 21: Net Competitive Advantage (NCA) of an Agent in the GACG. Given the GACG $\Sigma = (G, v, u)$, define the Net Competitive Advantage of agent or vertex i as

$$\sigma_{G,v,u}(i) = c_{G,v}(i) - \varsigma_{G,u}(i) \tag{4.60}$$ ■

Remark 14: Using Lemma 8 and Lemma 17 of [77], the above equation can also be written as

$$\sigma_{G,v,u}(i) = \varphi_{G,v}(i) - \psi_{G,u}(i) \tag{4.61}$$

which is the same as (4.58). ■

Definition 22: Net Altruistic Advantage (NAA) of an Agent in the GACG. Given the GACG $\Sigma = (G, v, u)$, define the Net Altruistic Advantage (NAA) of agent or vertex i as the difference of its Altruistic Contribution $a_{G,v}(i)$ [77] and Divisive Cost $\alpha_{G,u}(i)$ and given by

$$\chi_{G,v,u}(i) = a_{G,v}(i) - \alpha_{G,u}(i) \tag{4.62}$$

where $a_{G,v}(i)$ is given by (4.36) and $\alpha_{G,u}(i)$ is given by (4.48). ■

In Section 4.4.2 were given conditions on $u_i : i = 1, 2, ..., N$ for cost u to be a valid game list given that v is a valid game list. These results depend on condition (4.8) and some variants. The next results and those in Section 4.7 are obtained under the following conditions on Game Lists $v = (v_1, v_2, ...v_N)$ and $u = (u_1, u_2, ...u_N)$.

Relative Cost and Advantage Conditions on Game Lists for Coalitions:

$$u_i = k(v_i - iv_1) : i = 1, 2, ..., N-1, k \in \Re \tag{4.63}$$

$$u_N = k'(v_N - Nv_1), k' \in \Re \tag{4.64}$$

The following two lemmas give results about NMA and NAA bounds under certain condition on cost.

Lemma 13: (NMA in Connected Components) In a GACG $\Sigma = (G, v, u)$ if conditions (4.63) and (4.64) hold for $0 < k = k' < 1$ then in the graph G the NMA of each agent i in a connected component of size n is greater than or equal to v_1.

Proof: In a GACG $\Sigma = (G, v, u)$, the NMA is given by (4.59) $\varpi_{G,v,u}(i) = m_{G,v}(i) - \omega_{G,u}(i)$, where $m_{G,v}(i)$ is given by (4.34) and $\omega_{G,u}(i)$ is given by (4.46). For a graph G with the agent i in a connected component of size n, (4.59) can be written as

$$\varpi_{G,v,u}(i) = (v_n - \sum_{i=1}^{p} v_{k_i}) - (u_n - \sum_{i=1}^{p} u_{k_i}) : \sum_{i=1}^{p} k_i = n-1 \tag{4.65}$$

Using the given value of the Cost List u, the above equation can be written as

$$\varpi_{G,v,u}(i) = (v_n - \sum_{i=1}^{p} v_{k_i}) - (k(v_n - nv_1) - k\sum_{i=1}^{p}(v_{k_i} - k_i v_1)) : \sum_{i=1}^{p} k_i = n-1 \tag{4.66}$$

Rearrangement of the above equation yields

$$\varpi_{G,v,u}(i) = (1-k)(v_n - \sum_{i=1}^{p} v_{k_i}) + kv_1 : \sum_{i=1}^{p} k_i = n-1 \tag{4.67}$$

Using Lemma 3 and 4 of [77] $v_n - \sum_{i=1}^{p} v_{k_i} \geq v_1 : \sum_{i=1}^{p} k_i = n-1$, since $0 < k = k' < 1$ the above equation yields $\varpi_{G,v,u}(i) \geq v_1$. ■

The following lemma strengthens the above lemma under condition $v_2 > 2v_1$.

Lemma 14: (NMA in Connected Components) In a GACG $\Sigma = (G, v, u)$ if conditions (4.63) and (4.64) hold for $0 < k = k' < 1$ and $v_2 > 2v_1$ then in the graph G the NMA of each agent i in a connected component of size $n(>1)$ is greater than v_1.

Proof: The proof follows that same ways as the proof of the above lemma to get the same equation (4.67) for the NMA of an agent i

$$\varpi_{G,v,u}(i) = (1-k)(v_n - \sum_{i=1}^{p} v_{k_i}) + kv_1 : \sum_{i=1}^{p} k_i = n-1$$

Using Remark 4 of [77] $v_n - \sum_{i=1}^{p} v_{k_i} > v_1 : \sum_{i=1}^{p} k_i = n-1, n > 1$, since $0 < k = k' < 1$ the above equation yields $\varpi_{G,v,u}(i) > v_1$. ■

The following lemma gives a condition when all the agents have NMA less than v_1 the value they have as standalone agent and without any coalition.

Lemma 15: (NMA in Connected Components) In a GACG $\Sigma = (G, v, u)$ if conditions (4.63) and (4.64) hold for $k = k' > 1$ and $v_2 > 2v_1$ then in the graph G the NMA of each agent i in a connected component of size n $(>1)$is less than v_1.

Proof: The proof follows that same ways as the proof of the Lemma 13 to get the same equation (4.67) for the NMA of an agent i

$$\varpi_{G,v,u}(i) = (1-k)(v_n - \sum_{i=1}^{p} v_{k_i}) + kv_1 : \sum_{i=1}^{p} k_i = n-1$$

Using Remark 4 of [77] $v_n - \sum_{i=1}^{p} v_{k_i} > v_1 : \sum_{i=1}^{p} k_i = n-1, n > 1$, since $k = k' > 1$ the above equation yields $\varpi_{G,v,u}(i) < v_1$. ■

Remark 15: For $k = k' = 1$, (4.67) gives $\varpi_{G,v,u}(i) = v_1$ ■

Lemma 16: (Bounds on NAA) In a GACG $\Sigma = (G, v, u)$ if conditions (4.63) and (4.64) hold for $k = k' = 1$ then the Net Altruistic Advantage (NAA) of all the agents is either zero or there is at least one agent with negative value NAA.

Proof: The Net Altruistic Advantage (NAA) of an agent *i* is given by (4.62)

$$\chi_{G,v,u}(i) = a_{G,v}(i) - \alpha_{G,u}(i) \tag{4.68}$$

where $a_{G,v}(i)$ is given by (4.39) and $\alpha_{G,u}(i)$ is given by (4.48); using these equations, the above equation can be written as.

$$\chi_{G,v,u}(i) = (\mu_{G\backslash i}(G,v) - \eta_{G\backslash i}(G,u)) - (\mu_{G\backslash i}(G\setminus i,v) - \eta_{G\backslash i}(G\setminus i,u)) \tag{4.69}$$

Using the Game Lists, the above equation can be written as

$$\chi_{G,v,u}(i) = \sum_{i=1}^{q}(v_{k_i} - u_{k_i}) - \sum_{i=1}^{p}(v_{j_i} - u_{j_i}) - (\varphi_{G,v}(i) - \psi_{G,u}(i)) : \sum_{i=1}^{q} k_i = |G|, \sum_{i=1}^{p} j_i = |G|-1 \tag{4.70}$$

For the given value of the Game List

$$\chi_{G,v,u}(i) = v_1 - (\varphi_{G,v}(i) - \psi_{G,u}(i)) \tag{4.71}$$

If all the agents are having same value of $(\varphi_{G,v}(i) - \psi_{G,u}(i))$, then by using the efficiency of the Shapley value [127] that $(\varphi_{G,v}(i) - \psi_{G,u}(i)) = v_1$, otherwise at least one of the agent has $(\varphi_{G,v}(i) - \psi_{G,u}(i)) > v_1$. This establishes the desired result. ■

The following lemma strengthens the above lemma under condition (4.2) and conditions (4.63) and (4.64) for $k = k' > 1$.

Lemma 17: (Bounds on NAA) In a GACG $\Sigma = (G, v, u)$ if conditions (4.63) and (4.64) hold for $k = k' > 1$ and $v_2 > 2v_1$ then in a coalition having at least one edge, the Net Altruistic Advantage (NAA) of at least one agent with negative value NAA.

Proof: Using (4.70) the Net Altruistic Advantage of an agent in a graph component of size *n* is

$$\chi_{G,v,u}(i) = (v_n - u_n) - \sum_{i=1}^{p}(v_{j_i} - u_{j_i}) - (\varphi_{G,v}(i) - \psi_{G,u}(i)) : \sum_{i=1}^{p} j_i = n-1 \tag{4.72}$$

Using (4.63) and (4.64) for the value of valid game list for cost, the above equation can be written as

$$\chi_{G,v,u}(i) = (1-k)v_n + knv_1 - (1-k)\sum_{i=1}^{p} v_{j_i} - k(n-1)v_1 - (\varphi_{G,v}(i) - \psi_{G,u}(i)) : \sum_{i=1}^{p} j_i = n-1 \quad (4.73)$$

By using the efficiency of the Shapley value [127], a connected component of size *n* there must exist an agent *i* with $(\varphi_{G,v}(i) - \psi_{G,u}(i)) \geq \frac{v_n - u_n}{n}$. For this agent i the above equation become

$$\chi_{G,v,u}(i) \leq (1-k)v_n + knv_1 - (1-k)\sum_{i=1}^{p} v_{j_i} - k(n-1)v_1 - \frac{(1-k)v_n + knv_1}{n} : \sum_{i=1}^{p} j_i = n-1 \quad (4.74)$$

Simplification of the above inequality gives

$$\chi_{G,v,u}(i) \leq (1-k)(v_n - \sum_{i=1}^{p} v_{j_i} - \frac{v_n}{n}) : \sum_{i=1}^{p} j_i = n-1 \quad (4.75)$$

By Lemma C.3, under the given conditions $\chi_{G,v,u}(i) \leq 0$. ■

4.6.2 Dependence of Net Payoff or Allocation on the Cost of Coalition

In the next results all agents are motivated to stay in a coalition if each agent receives a Net Payoff or Allocation (4.58) greater than v_1, its payoff in a coalition consisting of itself. In the next results such NCA is called sufficient.

Theorem 1: (Insufficient Net Payoff to a Single Agent) In a Graphical Advantage and Cost Game $\Sigma = (G, v, u)$ if conditions (4.63) and (4.64) hold for $k = k' > 1$ and $v_2 > 2v_1, N > 1$ then in a connected graph *G* there is at least one agent with NCA less than v_1.

Proof: The Net Competitive Advantage of an agent *i*, in a graph *G* is given by (4.60), and sum of all the NCAs taken over all the agents is given by

$$\sum_{i=1}^{N}\sigma_{G,v,u}(i)=\sum_{i=1}^{N}(\varphi_{G,v}(i)-\psi_{G,u}(i))=\sum_{i=1}^{N}\varphi_{G,v}(i)-\sum_{i=1}^{N}\psi_{G,u}(i) \tag{4.76}$$

The Positional Advantage $\varphi_{G,v}(i)$ and the Positional Cost $\psi_{G,u}(i)$ are given by (4.3) and (4.7) respectively, since both of them are based on the Shapley value thus these are efficient [127]. If the graph G is connected then using Axioms I.1 and II.1, (4.76) can be written as

$$\sum_{i=1}^{N}\sigma_{G,v,u}(i)=v_N-u_N \tag{4.77}$$

This implies that there exists at least one agent j such that

$$\sigma_{G,v,u}(j)\le\frac{v_N-u_N}{N} \tag{4.78}$$

Using the given condition for u_i, for $i = N$ the above inequality becomes

$$\sigma_{G,v,u}(j)\le\frac{v_N-k(v_N-Nv_1)}{N}$$

or

$$\sigma_{G,v,u}(j)\le(1-k)\frac{v_N}{N}+kv_1 \tag{4.79}$$

The given condition $v_2>2v_1$ implies that $v_N>Nv_1$, Lemma 5 [77]. Moreover, it is given that $k>1$, thus (4.79) becomes

$$\sigma_{G,v,u}(j)<(1-k)v_1+kv_1$$

or

$$\sigma_{G,v,u}(j)<v_1 \tag{4.80}$$

This proves the desired result. ■

Remark 16: This theorem means that under the hypotheses, there always exists at least one agent with NCA less than the NCA of the standalone agent. Thereby, this agent has a motive to break away from the coalition. ■

The following lemmas provide the foundation for upcoming results. They provide a condition when there is no agent in the coalition with NCA less than v_1.

Lemma 18: (Insufficient Net Payoff to the Center Agent in a Star Graph) In a GACG $\Sigma = (G, v, u)$ if conditions (4.63) and (4.64) hold for $1 < k = k' < 2\left(\frac{v_N - \frac{N-1}{2}v_1 - \frac{1}{N}\sum_{i=1}^{N} v_i}{v_N - Nv_1}\right)$, and $v_2 > 2v_1, N > 1$, and the connected graph G is a star then the star point is the only agent with NCA less than v_1.

Proof: Let the vertex 1 be at the star point. Using (4.3), (4.7) and (4.60) NCA of the star point can be written as

$$\sigma_{G,v,u}(1) = \frac{1}{N}\sum_{i=1}^{N} v_i - \frac{N-1}{2}v_1 - \frac{k(v_N - Nv_1)}{2} \tag{4.81}$$

Since by Lemma C.2 $\frac{1}{N}\sum_{i=1}^{N} v_i \le \frac{v_N + v_1}{2}$, the above equation can be written as

$$\sigma_{G,v,u}(1) \le \frac{v_N + v_1}{2} - \frac{N-1}{2}v_1 - \frac{k(v_N - Nv_1)}{2} \tag{4.82}$$

or

$$\sigma_{G,v,u}(1) \le (1-k)\frac{v_N - Nv_1}{2} + v_1 \tag{4.83}$$

Since $k > 1$ and $v_2 > 2v_1$, thus the above equation can be written as

$$\sigma_{G,v,u}(1) < v_1 \tag{4.84}$$

It is thus established that NCA of the star point is lesser than v_1 .Consider any other agent $j(\neq 1)$, then owing to the symmetry in the star and using (4.3), (4.7) and (4.60) NCA of the point j can be written as

$$\sigma_{G,v,u}(j) = \frac{1}{N-1}(v_N - (\frac{1}{N}\sum_{i=1}^{N} v_i - \frac{N-1}{2}v_1) - \frac{k(v_N - Nv_1)}{2}) \tag{4.85}$$

It follows form the given conditions that

$$\sigma_{G,v,u}(j) > v_1 \tag{4.86}$$

This proves the desired result. ■

Remark 17:

1. It follows from the above proof that the partial condition $1 < k = k'$ establishes that the star point has NCA less than v_1.
2. To assure a valid range for k for in the above lemma, it is required that $2(\frac{v_N - \frac{N-1}{2}v_1 - \frac{1}{N}\sum_{i=1}^{N} v_i}{v_N - Nv_1}) > 1$; it follows from Lemma C.2 this inequality holds.
3. Lemma 18 shows that under the given conditions only the star point has a motive to break the coalition. This result motivates that if the last member u_n, of a Valid Game List for Graphical Coalitional Game with PC, is lowered such that NCA of the star point is greater than v_1 then no agent has a motive to disassociate from the coalition. This idea is capitalized upon in the next lemma. ■

Remark 18: In Lemma 2 and Remark 4 it is established that for

$$u = (u_1, u_2, \ldots u_N : u_i = k(v_i - iv_1), \forall i = 1,2,3\ldots, N-1, k > 1, u_N = k'(v_N - Nv_1), k' < 1, v_2 > 2v_1)$$

to be a Valid Game List for GCG with PC the following two inequalities hold $1 < k < \frac{v_N - Nv_1}{2v_{N-1} - v_{n-2} - Nv_1}$ and $\frac{k(2v_{N-1} - v_{n-2} - Nv_1)}{v_N - Nv_1} < k' < 1$. ■

The following Lemma gives a sufficient condition assuring that there is no agent in a star with NCA less than v_1.

Lemma 19: (Sufficient Net Payoff for every Agent in a Star) In a GACG $\Sigma = (G, v, u)$ if conditions (4.63) and (4.64) hold for $k' < \frac{2(\frac{1}{N}\sum_{i=1}^{N} v_i - \frac{N+1}{2} v_1)}{v_N - Nv_1} < 1 < k$ and u is a valid game list with $v_2 > 2v_1$ then there is no agent in a star with NCA less than v_1.

Proof: It is established in Lemma 18 that for

$u_i = k(v_i - iv_1) : i = 1, 2, ..., N, N > 1, 1 < k < 2(\frac{v_N - \frac{N-1}{2} v_1 - \frac{1}{N}\sum_{i=1}^{N} v_i}{v_N - Nv_1})$, $v_2 > 2v_1$ only the star point is the one with NCA less than v_1. It is given that $k' < 1$ thus it further reduces the cost of all the agents and guarantees that all agents other than the star point would have NCA less than v_1. Let 1 is the star point, thus from (4.3), (4.7) and (4.60) it can be written as

$$\sigma_{G,v,u}(1) = \frac{1}{N}\sum_{i=1}^{N} v_i - \frac{N-1}{2} v_1 - \frac{k'(v_N - Nv_1)}{2} \tag{4.87}$$

Under the given condition on k' the above equation can be written as

$$\sigma_{G,v,u}(1) > \frac{1}{N}\sum_{i=1}^{N} v_i - \frac{N-1}{2} v_1 - (\frac{1}{N}\sum_{i=1}^{N} v_i - \frac{N+1}{2} v_1) \tag{4.88}$$

This equation yields $\sigma_{G,v,u}(1) > v_1$ and the desired result is established. ■

Remark 19: Combining the result in the above lemma with Remark 18, it is established that for $u = (u_1, u_2, \ldots u_N : u_i = k(v_i - iv_1), \forall i = 1, 2, 3\ldots, N-1, k > 1, u_N = k'(v_N - Nv_1), k' < 1, v_2 > 2v_1)$ to be a Valid Game List for GCG with PC and for all the agents to have NCA greater than v_1 a sufficiency is provided by the following two inequalities $1 < k < \frac{v_N - Nv_1}{2v_{N-1} - v_{n-2} - Nv_1}$ and $\frac{k(2v_{N-1} - v_{n-2} - Nv_1)}{v_N - Nv_1} < k' < \frac{2(\frac{1}{N}\sum_{i=1}^{N} v_i - \frac{N+1}{2}v_1)}{v_N - Nv_1}$. Moreover from Lemma C.2 it follows that $\frac{2(\frac{1}{N}\sum_{i=1}^{N} v_i - \frac{N+1}{2}v_1)}{v_N - Nv_1} < 1$, which assures that condition that $k' < 1$ of the above lemma is not violated in the above inequality. ■

4.7 Online Sequential Decision Coalition Games and Stable Graphs

Based on the Graphical Advantage and Cost Game and Net Marginal Advantage, Net Competitive Advantage, and Net Altruistic Advantage in (4.59), (4.60) and (4.62) of an agent, three online coalition sequential decision games are defined in this section. Online or real-time refers to games that develop through time. In a sequential decision game, agents take turns sequentially in time to make valid or allowed moves (e.g. make or break an edge) to maximize their advantage in terms of a prescribed objective function. The agents can take turns either in a fixed order or randomly according to some probability distribution. A background on sequential decision games can be found in Chapter 5 of [130].

The properties of sequential decision games depend on the allowed moves and the prescribed objective function. An important concept in sequential coalition decision games is stability of graph topologies [104], [110]. Stability is important in studying the steady-state graph topologies of sequential decision games. Stable topologies show the structure of the coalitions that form under various allowed moves and decision objective functions.

In this section, first we define the sequential decision games. Next, the stable coalition graph topologies under these games are presented.

4.7.1 Sequential Decision Games

In the online game defined here, agents are free to make coalitions by making or breaking edges with other agents. This online decision game is defined on top of the GACG and Net Marginal Advantage, Net Competitive Advantage, and Net Altruistic Advantage in (4.59), (4.60) and (4.62) of an agent. The agents make allowed moves sequentially through time; the moves are made to maximize a prescribed objective function.

Allowed Moves. In this online game, at each move, an agent is selected at random; this agent is free to unilaterally break any edge incident at it or to bilaterally make an edge provided the other agent incident on the edge agrees to make it, as detailed below. In a single step, an agent is allowed either to make or break several edges.

Objective Functions. An objective function $f_G(i)$, for each agent i in a coalition represented by graph G is a real, nonnegative function. Edges are made or broken by a selected agent in order to maximize $f_G(i)$.

Based on the Allowed Moves and the Objective Function the sequential decision game is defined as follows.

Definition 23: Sequential Decision Games. In a sequential decision game a selected agent makes or breaks edges according to the rules:

a) An agent i forms an edge $e = \{i, j\}$ if $f_{G\cup e}(i) > f_G(i)$ *and* $f_{G\cup e}(j) \geq f_G(j)$

b) An agent i breaks an edge $e = \{i, j\}$ if $f_{G \setminus e}(i) > f_G(i)$

Based on the net marginal, competitive and altruistic contributions in Section 4.6 the motives of agents for forming and breaking the edges are different. Taking these contributions as objective functions, three sequential decision games can be defined.

i. *Game of Maximal Net Marginal Advantage (max-NMA)*

In this online game the objective function $f_G(i) = \varpi_{G,v,u}(i)$.

ii. *Game of Maximal Net Competitive Advantage (max-NCA)*

In this online game the objective function $f_G(i) = \sigma_{G,v,u}(i)$.

iii. *Game of Maximal Net Altruistic Advantage (max-NAA)*

In this online game the objective function $f_G(i) = \chi_{G,v,u}(i)$. ■

In these three sequential decision games an agent i is said to have a motive to make an edge if the condition *(a)* in Definition 23 is satisfied and it is said to have a motive to break an edge if the condition *(b)* in Definition 23 is satisfied.

4.7.2 Stability of Graph Topologies Under Sequential Decision Games

For a group of N agents there are $2^{N(N-1)/2}$ possible simple graphs. When agents are allowed to make valid moves, as they proceed, they may reach a graph where no agent has a motive to make any further moves. Such graphs are called stable graphs. The Structure of stable graphs is thus dependent on the allowed moves and the objective function of the sequential decision game.

In [110] Myerson allowed only the breakage of an edge as a valid move. Under such allowance, for the game in Definition 1 every graph is stable. In [104] the rules of making and breaking edges are nearly the same as those in the sequential decision games of Definition 23. However, in [104] there is a fixed cost associated with making edges.

The next development specifies the stable graphs for the Sequential Decision Games of Maximal Net Marginal Advantage (max-NMA), Maximal Net Competitive Advantage (max-NCA), and Maximal Net Altruistic Advantage (max-NAA). The balance between the value of being connected and the cost of maintaining edges has a pivotal role in determining the stable graph structures. The balance is captured in the relative cost and advantage conditions (4.63) and (4.64).

Definition 24: Stable Graph. In any online sequential decision game, a graph is called stable when no agent has a motive either to make an edge or to break an edge. ■

The following theorem is based on Theorem 1, and gives a condition on the cost so that no agent is willing to make a coalition in the game of max-NMA.

Theorem 2: (Stability of a Connected Graph in max-NMA) In a max-NMA under conditions (4.63) and (4.64) for $0 < k = k' < 1$ and under condition $v_2 > 2v_1$, a graph G is stable if and only if it is connected.

Proof: The proof of this theorem is divided into three parts: In the first part it is established that if G is connected then no agent has a motive to make an edge, the second part proves that in a connected graph G no agent has a motive to break an edge and in the third and final part the proof is concluded by proving that if the graph G is disconnected then there is always a motive to make an edge. First two parts prove the sufficiency of the theorem while the last part proves the necessity of the theorem.

In the first part it is established that if G is connected then no agent has a motive to make an edge. The Net Marginal Advantage of an agent i is given by (4.67), for a connected graph G, $n = N = |G|$ and it can be written as

$$\varpi_{G,v,u}(i) = (1-k)(v_N - \sum_{i=1}^{p} v_{k_i}) + kv_1 : \sum_{i=1}^{p} k_i = N-1 \tag{4.89}$$

It follows from this equation that making of an edge by the agent i incident at it does not change any of the term it. Thus, under Definition 23 of max-NMA, in a connected graph G no agent has motive to make an edge.

In the second part of this proof it is established that no agent has a motive to break an edge. The NMA of an agent i is given by (4.67)

$$\varpi_{G,v,u}(i) = (1-k)(v_n - \sum_{i=1}^{p} v_{k_i}) + kv_1 : \sum_{i=1}^{p} k_i = n-1 \tag{4.90}$$

By Lemma C.12, breakage of an edge by the agent i incident at it results into a possible decrease in the value of term $v_n - \sum_{i=1}^{p} v_{k_i}$ in the above equation; under condition $0 < k < 1$ results in possible decrease in the value of NMA of the agent i. Thus, under Definition 23 of max-NMA, in a graph G no agent has motive to break an edge.

In the third part of the proof it is established that in a disconnected graph G, an agent i has a motive to make an edge. It implies from Lemma C.12, that making of an edge by the agent i incident at it with another agent j, which is disconnected form it in G, results into a sure increase in the value of term $v_n - \sum_{i=1}^{p} v_{k_i}$ in the above equation; under conditions $0 < k \leq 1$ and $v_2 > 2v_1$, it results in sure increase in the value of NMA of the agent i. Thus, under Definition 23 of max-NMA, in a graph G agent i has motive to make an edge. Moreover, it implies from the first part of the proof that this motive of the agent i to make an edge remains there till it gets connected with all the agents. This establishes the desired result. ■

Remark 20: It follows from the above theorem that under the game max-NMA and under the given conditions of the theorem, all the connected graphs are Nash Points of the space of all the graphs on N points. Moreover, a tree being connected, is also stable in the game max-NMA under the given conditions. ■

Theorem 3: (Stability of a Completely Disconnected Graph in max-NMA) In a max-NMA under conditions (4.63) and (4.64) for $k = k' > 1$ and under condition $v_2 > 2v_1$, a graph G is stable if and only if it is completely disconnected.

Proof: Let the graph is completely disconnected. In a completely disconnected graph each agent has NMA v_1. According to Lemma 15 under the given conditions all the agents in connected components of size n (>1) have NMA less than v_1. Thus no agent has a motive to make an edge. It implies that a completely disconnected graph is stable.

Conversely suppose that the graph G is not completely disconnected. It implies that there is at least one component of size greater than 1. By Lemma 15 under the given conditions all the agents in connected components of size n (>1) have NMA less than v_1. Thus all these agents have motive to break all their edges, since making of any number of edges by these agents still maintains the same situation. Thus a graph G which is not completely disconnected is not stable.■

The above two theorems establish that (4.63) and (4.64) for $k = k' = 1$ act as a thresholds of costs for max-NMA given by

$$u_i = (v_i - iv_1) : i = 1, 2, ..., N \tag{4.91}$$

Every agent is willing to stay in a coalition below the threshold and all break up as early cost goes above the threshold.

The following two theorems are based on Theorem 1, and give conditions on the cost so that either no agent is willing to make a coalition in the game of max-NMA or all agents are willing to make coalition. Furthermore, the following theorem gives an upper bound on cost for the existence of a non-trivial coalition in max-NCA sequential decision game.

Theorem 4: (Stability of a Completely Disconnected Graph in max-NCA) In a Sequential Decision Game of Maximal Net Competitive Advantage (max-NCA) under conditions (4.63) and (4.64) for $k = k' > 1$ and under condition $v_2 > 2v_1$, a graph G is stable if and only if it is completely disconnected.

Proof: It is established in Theorem 1 that for a connected graph of size N there exists at least one agent with NCA lesser than v_1. That is to say in a connected graph of size N there always exists an agent j such that

$$\sigma_{G,v,u}(j) < v_1 \tag{4.92}$$

This implies that in a connected graph G there always exists at least one agent j with motive to break the entire edges incident at it. The desired sufficiency is a domino effect of the above result for the connected components of sizes between 2 and $N-1$.

Conversely suppose that the graph G is completely disconnected. Then by Remark 17 no agent has a motive to make any edge. This establishes the necessity and the desired result. ■

The above theorem for max-NCA is analogous to Theorem 3 for max-NMA, since it establishes the fact that above the threshold of cost given by (4.91) no agent is willing to remain in a coalition.

Remark 21: In a max-NCA if $u_i > (v_i - iv_1) : i = 1, 2, ..., N$ and $v_2 > 2v_1$ then only the completely disconnected graph is stable. ■

The following theorem is based on Lemma 19 and gives a condition on the cost so that no agent is willing to leave a connected coalition. It is to be noted that the following theorem does not establish the stability of a connected coalition.

Theorem 5: (Stability of a Connected Graph in max-NCA) In a max-NCA if conditions (4.63) and (4.64) hold for $k' < \frac{2(\frac{1}{N}\sum_{i=1}^{N} v_i - \frac{N+1}{2} v_1)}{v_N - Nv_1} < 1 < k$, $v_2 > 2v_1$,with u is a Valid Game List then every agent benefits by remaining in a connected coalition of size *N*. That is a stable graph must be connected.

Proof: Without the loss of generality suppose that the game starts with a connected initial graph G_o . If all agents *j* are having $\sigma_{G,v,u}(j) \geq v_1$ then by Theorem 1 it is not possible to make a coalition of smaller component size with all the agents having $\sigma_{G,v,u}(j) \geq v_1$. Suppose that there exists an agent *j* in G_o with $\sigma(j) < v_1$ and it is unable to find a configuration with $\sigma_{G,v,u}(j) \geq v_1$ then again by Theorem 1 all the agents will disassociate under the domino effect to make the completely disconnected graph. By Lemma 18 the agent who gets the next move makes a star, which is a connected graph. ■

The above theorem gives thresholds of cost in the game max-NCA below which each agent wants to remain in a connected coalition. Though these thresholds are different than those established by Theorem 2 for max-NMA yet the above theorem for max-NCA is analogous to it in the sense that both the theorems give thresholds for the costs co that agents remain connected.

Remark 22: Under conditions hypothesized in the above theorem, if an agent is made to leave the coalition then rest of the agents get out of the coalition and the coalition fragments into isolated agents; such coalition is called a fragile coalition. ■

This gives rise to the following definition.

Definition 25: Fragility of a Graph. In any online sequential decision game, a graph representing a coalition is called fragile when no agent has a motive to get itself disconnected from the graph but if one agent gets disconnected then rest of the coalition disintegrates into a completely disconnected graph. ■

Needless to mention that under the conditions mentioned in Theorem 5 a connected graph is fragile. The fragility of coalitions is to be discussed in next research.

The following theorem gives a stable graph in a Sequential Decision Game of Maximal Net Altruistic Advantage (max-NAA).

Theorem 6: (Stability of a Completely Disconnected Graph in max-NAA) In a max-NAA for conditions (4.63) and (4.64) hold for $k = k' > 1$ then a graph is stable if and only if it is completely disconnected.

Proof: In a completely isolated graph the only move an agent i may have is to make some edges. No matter how many edges an agent makes, according to Lemma 16, under the given conditions, either all the agents have zero value of NAA or there is at least one agent with negative value of NAA. This decimates any motive to make an edge according to Definition 23 of the Sequential Decision Game max-NAA. Moreover, the result follows for $k = k' > 1$ since the more the values of k and k' the lesser the value of NAA. Thus the completely disconnected graph is stable.

Conversely suppose that there exists a stable graph G which is not completely disconnected. Thus there exists at least one connected component S of size greater than 1 in G. According to Lemma 17, under the hypothesized conditions there must exists one agent with negative value of NAA; such agent has at least one motive to break all of its edges and get itself disconnected from the rest of the graph. This implies that G is not stable. ■

The above theorem establishes that beyond the same threshold given by (4.91) the agents prefer to stay disconnected. With this theorem it is proven that above the thresholds of cost given by (4.91) the agents prefer to stay in a completely disconnected or trivial coalition irrespective of their motive or objective function. Thus the threshold given by (4.91) is instrumental in examining the feasibility of making a coalition when both the advantages and costs are involved.

4.8 Conclusions

A graphical coalitional game with Positional Cost is defined in this paper. In this game, the total cost of a coalition depends upon the connectivity of the agents and the number of agents involved in the coalition. Allocation rules based on Shapley values assign the cost primarily to the edges and then equally divided to the agents or vertices [104], [105]. The game defines the notion of PC of agents in a coalition from a graph theoretic view point. Certain elementary properties of the game are established for this framework. In this paper allocation rules of advantage are the same as those introduced in [77]. The advantage and cost are used to define Net Payoff or Allocation; it is further used to define three net advantages: Net Marginal Advantage, Net Competitive Advantage, and Net Altruistic Advantage. These net advantages are based on the components of cost defined in this paper and components of advantage defined [77] and according to the concepts in [74]. A number of results about the dependence of these net advantages on coalition structure are presented. This provides the constructions needed to define, on top of the graphical coalitional advantage and cost game, three online sequential decision games: max-NMA, max-NCA, and max-NAA. The preferred graphs under each sequential decision game, under certain relations between the advantages and costs are studied. It is shown that the stable graphs in max-NMA are any connected graph, including a tree. The preferred graph in max-NCA is a connected graph or completely disconnected graph under certain other condition. The completely disconnected graph is stable in max-NAA under certain conditions. These preferences in the three sequential games yield thresholds of cost beyond which agents stay in completely disconnected or trivial coalition irrespective of sequential game. It is anticipated that under the conditions in Theorem 5 if the graph representing the coalition is a tree then it is stable; the result will be reported in the next research.

The complete setup established in this paper is suitable to understand the significance of a graph vertex when only the graph topology is known. The setup provides a guideline for the formation of coalitions, and serves to examine the competitiveness and altruism aspects of the coalition. It is also established that above certain cost there is no motive left for any agent to remain connected in a coalition.

Chapter 5

Internal Structure of Coalitions in Digraphs

This chapter introduces the structure of a Graphical Coalition Game on digraphs which assigns a value to each digraph based on its connectivity. Fundamental properties of the game are established in the form of technical lemmas. Novel graph theoretic structures including multi-chain, and semi-strongly connected digraphs, are defined; these structures are pivotal in defining the Graphical Coalition Game on digraphs, and algorithms are devised to compute these structures. Marginal contribution made by an agent within a coalition, modeled as a digraph, is defined; results are established about the dependence of marginal contribution made by an agent upon its position in the digraph. On top of marginal contribution, three sequential decision games are defined and stable coalitional structures under these games are established. Stable structures for these games are multi-chains, semi-connected digraphs and chains of command; these structures are useful in cooperative control theory.

5.1 Introduction

This Chapter extends the idea of a graphical coalition games (GCG) introduced in [77], on digraphs. The game is used to provide a rigorous study of the internal structure of coalitions where the flow of information is asymmetric and modeled as a digraph. Novel digraph structures required for the development in this paper are defined and are used to state the Axioms of Value for GCG on digraphs, to assign values to digraphs. Notion of total or marginal contribution made by an agent [74] is defied for agents within a coalition modeled as a digraph, and the dependence of the marginal contribution made by the agents, on their position in coalition is studied.

Game theory is a mathematical discipline [71], [114], [132], [136], that deals with issues and strategies involving competitions and cooperation between several entities [73], [90], [92],

[103], [106], [133], [116], [141]. In the scope of mathematical game theory these entities are called players or agents [84], [116], [119], [120], [121], [122], [123], [130]. With no exceptions, in engineering systems, game theory is extensively used in situations involving competition and cooperation [76], [81], [82], [91], [108], [109], [117], [125], [137], [138], [139], [140], [142]. Game theory is primarily divided into two areas [116], [132]: noncooperative game theory [84], and cooperative game theory [116], [130]. In noncooperative game theory the fundamental unit of study is the individual agent, and the theory deals with its performance and strategies in interaction with other individual agents. By contrast, in cooperative game theory, the fundamental unit is the set of agents or coalition. Cooperative game theory deals with the value of the coalition, payoff allocations to individual players, and the stability of coalitions [116], [130].

Cooperative games can be divided into three classes: Canonical Coalitional Games, Coalition Formation Games, and Coalitional Graph Games [84], [123]. Canonical coalitional games mainly deal with the stabilization of the grand coalition of all the agents. Methods are sought to allocate the net value of the coalition to individual agents in such a way that agents are encouraged to join the coalition. A fair allocation [110] that often accomplishes this is the Shapley value [127]. Coalition formation games mainly deal with coalitions based on gains and costs. Given prescribed gains and costs, the structures of the resulting coalitions are studied. Finally, the coalitional graph games deal with the formation and stability of coalitions given an underlying communication graph structure [82], [123]. In the work of Baras [81], [82] and of Başar [119], [120], [121], [122], [123] coalitional graph games are studied with applications to communication networks. Various definitions of value are used in [119], [120], [121], [122], [123], including probability of detection, gain of resources of other agents, effective throughput, and packet success rate. Various definitions of cost are used including probability of false alarm, vulnerabilities from other agents, download delay, mean waiting times, and path delay. Given the total value, algorithms are developed to form effective coalitions for communications.

Closely related to the coalitional graph games are online or sequential-in-time decision games. These are games where agents make moves through time sequentially to maximize their prescribed objective functions [130]. These games are defined by specifying the method of selection of the agent to make moves at each time, the allowed moves of the agents, and the objective function the agents seek to maximize. Agents might make moves according to some fixed round-robin procedure, or randomly according to some probability distribution function. These online sequential decision games best model real-life situations where the players are free to change their alliances as considered suitable by them to obtain their objectives.

In his classical work [110], Myerson used graph theoretic ideas to analyze cooperation in coalitional graph games. He proposed to restrict the interactions in coalitions based on the underlying communication graph structure. He showed that the unique fair (in his sense) allocation of the net value of the coalition to the agents is given by the Shapley value [127]. In their paper [104], Jackson and Wolinsky analyzed the stability of networks when the individual agents can choose to form and maintain the links between them. An agent gains value on connecting to an agent which is well-connected to other agents in the graph, and accrues a cost based on maintaining direct communication edge links with its neighbors. It is shown that different relations between the link cost and the propagation of value along a path result in stability of different structures, such as complete graph, star graph, etc.

The objective of this paper is to provide tools to study the internal structure of coalitions on digraphs on the basis of different motives of the agents. This paper extends the idea of graphical coalitional game (GCG) introduced in [77], on digraphs. The first contribution of this paper is to introduce novel digraph structures. These digraph structures are not only pivotal for the development in this paper, but they can also provide further insight to the cooperative control theory [1], [2], [3], [7], [8], [79], [135]. Algorithms are also devised to get these digraph structures.

The second contribution of the paper is definition of GCG on digraphs. The game is based on a Value Function that is required to satisfy five formal axioms. Owing to these axioms imposed on the Value Function, it is possible to perform a rigorous study of the internal structure of coalitions on digraphs. A detailed machinery of the game is developed in the form of technical lemmas in the Appendix at the end. It is established that the game is convex, thereby guarantees a fair allocation to the agents.

The third contribution of the paper is to study the marginal contribution [74] of agents within a coalition represented as a digraph; the dependence of marginal contribution on graph topology is also studied. The fourth contribution is the definition of three online sequential decision games based on the marginal contribution. These games are defined by varying the rules for agents to make or break arcs to maximize their marginal contributions. It is shown that the three sequential decision games have different stable coalition structures; these are multi-chain, semi-strongly connected digraphs, and chain of command; these structures are formally defined in Section 5.2. These stable structures are inherent properties of the objective function and rules of the three games, not parameter dependent as in [104].

The GCG on digraphs, and the sequential decision games proposed in this paper can be used in a variety of ways in problems involving situations of simultaneous competition and collaboration among anonymous agents. The GCG on digraphs can be used to determine the social standing of various kinds of agents purely on the basis of the communication structure, where there are asymmetric relations between agents [71], [75], [124], [132]. The notion of marginal contribution made by the agent developed in Section 5.4 can be used to determine the comprehensive contribution of an agent to digraph structure [74]. The sequential decision games in the Section 5.5 can be used to understand the internal structure of a coalition based on the notions of marginal contribution. The novel digraph structures and theory of sequential decision games developed in this paper can be used in cooperative control theory [1], [2], [3], [7], [8], [79], [135]. Situations in economics, communication, and swarm control are very complex; here a lot of agents interact in situations of simultaneous competition and cooperation. The theory developed in this paper can be used to understand complex situations of joint competition and cooperation [74], [81], [82], [111], [117], [123], [125], [131], [140].

The paper is organized as follows. Novel digraph structures are defined Section 5.2, algorithms to get these structures in polynomial time are also elaborated in this section. A graphical coalition game on digraphs is defined by formal Axioms of Value Function, in Section 5.3, it is also established that the game is convex. Motivated by [74], Section 5.4 defines the marginal contribution made by an agent towards a coalition; results about the dependence of the marginal contribution made by an agent upon its position within a digraph are also established in this section. In Section 5.5, three online sequential decision games are defined on top of the GCG on digraphs. The stable digraph structures under each of these three games are studied. These structures also provide insight to the cooperation of agents within a coalition, where flow of information is asymmetric, and modeled as a digraph. Simulation results for several online sequential decision games are presented in Section 5.6 and shown to support the stable graph structures.

5.2 Digraphs Definitions and Algorithms

In this paper a graphical coalitional game GCG on digraphs is defined. The game provides an extension to the graphical coalitional game defined in [77]. The GCG defined in [77] is applicable to the coalitions which can be modeled as simple graphs; this extension defined a GCG for the coalitions which are modeled as digraphs. This section provides essential background knowledge about digraphs. Moreover, some novel digraph structures are introduced; these structures are essential to define GCG on digraphs. Algorithms are also devised to get these structures in digraphs.

5.2.1 Digraph Definitions

A digraph is an ordered pair $D=(V,E)$, such that V is a finite set and $E \subseteq V \times V$ where $V \times V$ is the set of all the ordered pairs of the elements of V. The cardinality of $V \times V$ is represented as $|V \times V|$ and equal to $|V|^2$ where $|V|$ is the cardinality of V called the order of the graph and, in our discussion, represented by N or $|D|$. In this paper, digraph do not contain self-loops and multiple arcs, moreover, the arcs do not have any weight associated to them, thus $E \subseteq \{(a,b): a,b \in V, a \neq b\}$ and its maximum cardinality is $N^2 - N$. In this paper V is called the set of vertices, or agents and E is called the set of arcs. Consequently the elements of V are called vertices of the digraph and usually denoted by v and the elements of E are called arcs of the digraph and usually denoted by e. If $e=(v_1,v_2)$ with $v_1, v_2 \in V$ is an arc, then e is said to connect v_1 to v_2. It is also said that v_1 dominates v_2 or in other words v_1 is at the tail of e and v_2 is at the head of e. The number of arcs with heads at a vertex v is the in-degree of v and denoted as $d^-(v)$. Similarly, the number of arcs with tails at the vertex v is the out-degree of v and denoted as

$d^+(v)$. The sum of in-degree and out-degree of a vertex v is called degree of v and denoted as $d(v)$.

Digraphs and simple graphs [77], [94], [87], sharing the same set of vertices are linked in several ways. A simple graph *G* can be taken as a bidirectional graph by replacing each of its undirected edge by a pair of opposite arcs connecting the same pair of vertices. This bidirectional graph *D* thus obtained is called associated bidirectional graph of *G*. Another way to get a digraph *D* from a simple graph *G* is to replace each of its undirected edge with one of the two possible arcs connecting the same two ends. In this case *D* is called an orientation of *G*. It can be seen that there are $2^{|E|}$ possible orientations of a labeled graph $G=(V,E)$. If all the arcs of a digraph *D* are replaced by edges, the resulting simple graph *G* is called underlying undirected graph.

A sequence of distinct vertices $u=v_0,v_1,...,v_M=v;M\geq 1$ starting from a vertex u to another vertex v such that there exist an arc from vertex v_i to the vertex v_{i+1} for all $i=0,1,2,...,M-1$ is called a directed path from u to v within a digraph *D*, it is written as $P=\{u=v_0,v_1,...,v_M=v\}$ and the existence of the directed path is denoted as uPv; if there exists an arc e from the last vertex of *P* to the first vertex then $P\cup\{v_0\}$ is called a directed cycle or a circuit. A vertex u is said to be connected to a vertex v if there is a directed path from u to v. The minimum hop count over the paths from u to v is called distance from u to v. A digraph is said to have a root $r\in V$ if r is connected to every vertex v. A digraph is called strongly connected if every vertex of the digraph is connected to every other vertex that is to say in case of strongly connected digraph every vertex is a root vertex. It is possible for a digraph which is not strongly connected to have a connected underlying undirected graph; such digraph is called weakly connected. There is still another notion of connectivity known as quasi strong connectivity; a digraph is said to quasi strongly connected if for each pair of vertices $u, v \in V$ there always exists a vertex w such that both u and v are reachable from w through directed paths. Thus a digraph having at least three root vertices is at least quasi strongly connected. A digraph whose underlying undirected graph is disconnected is called a disconnected digraph. A digraph having tree as underlying undirected graph, and having exactly one root vertex is called a chain of command. Another notion of connectivity in digraphs is introduced in the next subsection.

If $D = (V, E)$ and $S = (V', E')$ are two graphs such that $V' \subseteq V$ and $E' \subseteq E$ then S is called a subgraph of D; a subgraph of a digraph is a digraph in its own capacity. Moreover if E' contains all the arcs $e = (u, v) \in E$ with $u, v \in V'$ then S is called an induced subgraph of D, in this paper it is denoted as $S \subseteq D$. An induced subgraph of D obtained by excluding a vertex u from V is denoted as $D \setminus \{u\}$ or simply $D \setminus u$. A maximal induced subgraph of a digraph D with all its vertices connected to each other is called a strong component in D. Algorithms to find strong components of a digraph include Kosaraju's Algorithm, Tarjan's Algorithm [134], and Gabow's Algorithm [101]. A strongly connected digraph is said to have one strong component. It can be proved that a strict digraph is acyclic if and only if it has no non-trivial strongly connected component. If a component is not strongly connected then it is called a weakly connected component. Another notion of components in digraphs is detailed in the next subsection.

Structural contraction of the digraph $D = (V, E)$ is obtained by taking its strong components as a single vertex to get a new digraph $\underline{D} = (\underline{V}, \underline{E})$, such that each vertex $\underline{v} \in \underline{V}$ is a strong component of D and there is an arc from $\underline{v}_i \in \underline{V}$ to $\underline{v}_j \in \underline{V}$ if and only if there is an arc from any vertex in the strong component $\underline{v}_i \in \underline{V}$ to a vertex in the other strong component $\underline{v}_j \in \underline{V}$, in *D*. Further details of these graph theoretic concepts are available in [94], [87].

5.2.2 Pivot Vertex and Key Concepts for Graphical Coalition Games on Digraphs

This subsection introduces novel concepts in digraphs required for defining the Axioms of Value for the Graphical Coalition Game on digraphs, introduced in this paper. These concepts play a pivotal role in Section 5.3 in formulating axioms to assign value to the digraph structures. These digraph concepts also provide further insight to the cooperative control [1], [2], [3], [7], [8], [79], [135].

Definition 1: Pivot Vertex. Given a digraph $D = (V, E)$ a vertex $v \in V$ is called a pivot vertex in *D* if there does not exist a vertex $u \in V$ such that *v* is reachable from *u* but *u* is not reachable from *v*. ■

Remark 1: A root vertex is by definition a pivot vertex, converse is not true. Thus, if a graph is strongly connected then all of its vertices are pivot vertices. Moreover, an isolated vertex is trivially a pivot vertex. ■

The following algorithm provides a procedure to get all the pivot vertices in a digraph.

Algorithm 1: Search for all pivot vertices.

1. Input = Digraph $D = (V, E)$
2. Output = Set of pivot vertices = $S = \phi$
3. Get $\underline{D} = (\underline{V}, \underline{E})$, the structural contraction of $D = (V, E)$
4. $\underline{S}$ = Set of pivot vertices in $\underline{D} = (\underline{V}, \underline{E}) = \phi$
5. $\underline{S} \longleftarrow$ All the vertices $\underline{v} \in \underline{V}$ with zero in-degree

6. $S \longleftarrow \{v : v \in V, v \in \underline{v} \in \underline{S}\}$

Proof of Correctness: Proof follows from the definition of structural contraction and pivot vertex. ■

It follows from the definition of pivot vertex and strong component of a digraph *D*, that if one vertex of a strong component of *D* is a pivot vertex then all the vertices of this strong component are also pivot vertices of *D*. It motivates the following definition.

Definition 2: Set of Independent Pivot Vertices. Pivot vertices existing in different strong components of the digraph $D = (V, E)$ are called independent pivot vertices. A maximal collection of pivot vertices by taking one pivot vertex from each of the strong component having pivot vertices is called a set of independent pivot vertices. In this paper the set of independent pivot vertices is denoted as P^D. ■

All the sets of independent pivot vertices have the same cardinality. If a set of independent pivot vertices P^D of a digraph $D = (V, E)$ is singleton then pivot vertices are also root vertices, in this paper such digraphs are named as semi-strongly connected digraphs.

Definition 3: Semi-Strongly Connected Digraph. If the sets of independent pivot vertices P^D of a digraph $D = (V, E)$ are singleton then the digraph is called semi-strongly connected digraph. ■

Remark 2: All the concepts of connectivity in digraphs are mutually related. As the name implies, strong connectivity implies all the concepts of connectivity discussed in this paper, while converse in not true in general; whereas, weak connectivity does not imply any other connectivity discussed in this paper and all the other concepts of connectivity discussed, imply weak connectivity. Quasi strong connectivity implies semi-strong connectivity while converse is not true in general; for example, a chain of command is semi-strongly connected digraph with its root vertex as the unique pivot vertex, and it is not quasi strongly connected. ■

Remark 3: In a coalition represented by a semi-strongly connected digraph, when agents want to reach a consensus by local averaging, all the agents reach a unique consensus value.■

It follows from the definition of a pivot vertex that generally there are certain vertices which are reachable from a pivot vertex while some vertices are not reachable from the pivot vertex. This motivates the following definition.

Definition 4: Semi-Strongly Connected Component. For a digraph $D=(V,E)$ and P^D as a set of independent pivot vertices, let $v \in P^D$ be a pivot vertex of the digraph. Then the subgraph of $D=(V,E)$ induced by the vertices reachable from *v* and *v* is called a semi-strongly connected component of *D* induced by *v* and represented by D^v. All the vertices included in the component D^v are part of the component. The collection of all the semi-strongly connected components of *D* is represented by D^P. ■

Remark 4: Semi-strongly connected components are vital for the development of the graphical coalition game on digraphs in this paper. Moreover, this notion gives further insight to the consensus of agents within a coalition where the flow of information is modeled as digraphs; a pinning agent affects the consensus value [1], [2], [3], [7], [8], [79], [135] of only those agents who are within its semi-strongly connected component. ■

Definition 5: Multi-Chain. For a weakly connected digraph $D=(V,E)$, if each semi-strongly connected component is a chain of command then the digraph is called multi-chain.■

Remark 5: Multi-chains are important structures in cooperative control where there is more than 1 pinning agents [1], [2], [3], [7], [8], [79], [135]. In such situations, coalitions modeled as multi-chains, have minimal Fidler Eigen Values and maximal rate of convergence to the consensus [1], [2], [3], [7], [8], [79], [135]. Multi-chains have no circuits but the underlying undirected graph may be cyclic. Moreover, the set of independent pivot vertices P^D of a Multi-chain is unique. ■

The following algorithm finds all the semi-strongly connected components of a digraph $D=(V,E)$; the working of this algorithm is similar to that of Algorithm 1.

Algorithm 2: Search for all Semi-Strongly Connected Components.

1. Input = Digraph $D=(V,E)$.
2. Output = Set of semi-strongly connected components.
3. Get $\underline{D}=(\underline{V},\underline{E})$, the structural contraction of $D=(V,E)$.
4. $\underline{S}=$Set of pivot vertices in $\underline{D}=(\underline{V},\underline{E})=\phi$.
5. $\underline{S}\longleftarrow$All the vertices $\underline{v}\in\underline{V}$ with zero in-degree.
6. For each element $\underline{v}\in\underline{S}$
   a. Find the set vertices $\underline{D}^{\underline{v}}$ of $\underline{D}=(\underline{V},\underline{E})$ reachable from $\underline{v}$.
   b. Corresponding semi-strongly connected component is the induced sub-graph of $D=(V,E)$ with vertices $v\in\underline{v}\in\underline{D}^{\underline{v}}$.

Proof of Correctness: Proof follows from the definitions of structural contraction, pivot vertex, and semi-strongly connected component. ■

Remark 6: This algorithm also implies that any finite digraph can be represented as union of semi-strongly connected components. ■

It follows from the definitions of pivot vertex and semi-strongly connected component that the number of semi-strongly connected components in a digraph $D = (V, E)$ is equal to the cardinality of its set of independent pivot vertices P^D. These semi-strongly connected components are always different from each other but might be mutually intersecting. Using the principles of counting it can be shown that the order of a digraph $D = (V, E)$ is

$$|D| = \sum_{k=1}^{|P^D|} (-1)^{k+1} \sum_{\substack{i_1 < i_2 < \ldots < i_k \\ v_{i_j} \in P^D}} |\bigcap_{j=1}^{k} D^{v_{i_j}}| \tag{5.1}$$

The concept of cut vertex is defined next on the basis of semi-strongly connected components in a digraph.

Definition 6: Cut Vertex. In a digraph $D = (V, E)$ a vertex v is called cut vertex if the number of semi-strongly connected components in $D \setminus \{i\}$ are more than the number of semi-strongly connected components in D. ■

Example 1: Semi-strongly Connected Components in a Digraph. Some digraphs $D = (V, E)$ with their semi-strongly connected components D^P as induced sub-graphs are:

a) $V = \{1,2,3\}, E = \phi, D^P = \{\{1\},\{2\},\{3\}\}$

b) $V = \{1,2,3\}, E = \{(2,1),(2,3)\}, D^P = \{\{1,2,3\}\}$

c) $V = \{1,2,3\}, E = \{(2,1),(2,3),(1,3)\}, D^P = \{\{1,2,3\}\}$

d) $V = \{1,2,3\}, E = \{(1,2),(3,2)\}, D^P = \{\{1,2\},\{1,3\}\}$

e) $V = \{1,2,3,4,5\}$, $E = \{(1,5),(5,2),(3,5),(5,4)\}$, $D^P = \{\{1,5,2,4\},\{3,5,2,4\}\}$ ■

Remark 7: If a weekly connected component of a digraph $D = (V, E)$ has at least two independent pivot vertices then it has at least two non-distinct semi-strongly connected components. ■

<u>5.3 Graphical Coalitional Game on Digraphs</u>

A graphical coalitional game GCG on digraphs is introduced in this section; this game is an extension of GCG introduced in [77]. The GCG introduced in [77] is defined for the coalitions which can be represented as simple graphs and was used to study the internal structures of such coalitions. In this extension a GCG on digraphs is introduced. This extension is used in Sections 5.4 and 5.5 to study the internal structures of the coalitions with asymmetric communication structure and can be modeled as digraphs. The game is based on certain axioms. These axioms are developed on the assumption that all the agents under consideration are identical and supportive to each other. Construction of these axioms makes it possible to assign a value to any of the digraph structure, and to establish the notion of marginal contribution of agents within a coalition, purely, on the basis of their position within the digraph. The GCG on digraphs is represented as $\Gamma = (D, v)$, here the digraph *D* represents the communication topology of the coalition with agents as vertices, and arcs between them in the direction of communication between agents within the coalition. The value function *v* is formally defined as

$$v : 2^D \to \mathbb{R} \;\; with \;\; v(\varphi) = v_0 = 0 \tag{5.2}$$

where 2^D is the collection of all the induced subgraphs of *D*. The game has the following axioms. In these axioms $S \in 2^D$ is an induced subgraph of *D* and P^S is a set of independent pivot vertices of *S*.

Axioms of Value in the GCG on Digraphs:

1. If S is a single semi-strongly connected component with $|S|=m$ then $v(S)=v_m \geq 0$

2. If $|P^S|=k$ and S is having k nonintersecting semi-strongly connected components $S^{v_i}: i=1,2,\ldots k, v_i \in P^S$ with $|S^{v_i}|=m_i$ then $v(S)=\sum_i v_{m_i} : v_{m_i} \geq 0$.

3. If $|P^S|=k$ that is S is having k semi-strongly connected components $S^{v_i}: i=1,2,\ldots k=|P^S|, v_i \in P^S$ then $v(S)=\sum_{l=1}^{|P^S|}(-1)^{l+1}\sum_{\substack{i_1<i_2<\ldots<i_l \\ v_{i_j}\in P^S}} v(\bigcap_{j=1}^{l} S^{v_{i_j}})$

4. If $N \geq m > n \geq 0$ then $n.v_m \geq m.v_n$

5. If $N-1 \geq m \geq 0$ then $v_{m+1}-v_m \geq v_m-(m-1)v_1$

Definition 7: GCG on Digraphs. Given a digraph D, the GCG on it is defined as the game $\Gamma=(D,v)$, where the value function v satisfies the Axioms of Value. ■

Some technical lemmas establishing the bases of GCG on digraphs are presented in the Appendix A1.

Remark 8:

1. Axioms 1 to 3 are related to the structure of the digraph D while Axioms 4 and 5 are related to the value function v.

2. It is established in technical Lemma D 1 that Axiom 5 implies Axiom 4. Similarly, Axioms 1 and 2 are implied from Axiom 3. The Axioms 1, 2 and 4 are retained as axiom because of the ease of their use in establishing the game properties.

3. It is established in technical Lemma D 2 that Axiom 5 implies that for $N-1 \geq m > n \geq 0$, $v_{m+1}-v_m \geq v_{n+1}-v_n$, which is Axiom 4 of GCG on simple graphs in [77]. The Axiom 5 is

thus strengthened version of Axiom 4 in [77]. This strengthening is required due to additional connectivity structures in digraphs which do not exist in simple graphs.

4. If the digraph D consists of k disconnected components $D_i : i = 1, 2, ...k$ then by using Axiom 3 $v(D) = \sum_i v(D_i)$.

5. It is established in the technical Lemma D 3 that for a GCG on digraph, $\Gamma = (D, v)$, if S is a subgraph of D then $|S| v_1 \le v(S) \le v_{|S|}$. One incidence of the least value is completely disconnected digraph, and an incidence of the maximum value is a semi-strongly connected digraph.

6. Axioms of Value for the GCG on digraphs imply Axioms of Value in [77] for bidirectional digraphs. ■

In the Axioms of Value, it is assumed that all the agents are identical and similarly all the arcs are identical. Thus the game can be used to study the advantage of one vertex over the other based on its position in the digraph structure. It is established later in this paper that the game is convex [128], thereby it is possible to have a fair allocation for each agent.

Example 2: Values of Some Digraphs. Some digraphs and their values are given for further insight to the value function.

a) $V = \{1, 2, 3\}, E = \phi, D^P = \{\{1\}, \{2\}, \{3\}\}$. The value of this digraph is $v(D) = 3v_1$.

b) $V = \{1, 2, 3\}, E = \{(2,1), (2,3)\}, D^P = \{\{1, 2, 3\}\}$. The value of this digraph is $v(D) = v_3$.

c) $V = \{1, 2, 3\}, E = \{(2,1), (2,3), (1,3)\}, D^P = \{\{1, 2, 3\}\}$. The value of this digraph is $v(D) = v_3$.

d) $V=\{1,2,3\}, E=\{(1,2),(3,2)\}, D^P=\{\{1,2\},\{1,3\}\}$. The value of this digraph is $v(D)=2v_2-v_1$.

e) $V=\{1,2,3,4,5\}, E=\{(1,5),(5,2),(3,5),(5,4)\}, D^P=\{\{1,5,2,4\},\{3,5,2,4\}\}$. The value of this digraph is $v(D)=2v_4-v_3$.■

The value of the digraphs D can be represented as a real numbers list of size $|D|$. Therefore the next definition is motivated.

Definition 8: Valid Game List. A list of non-negative real numbers $v_1, v_2, ..., v_{|D|}$ is called a valid game list of size $|D|$ if it satisfies the Axiom 5 of the game. ■

Convexity is an important property in canonical coalitional games. If the graphical coalition game is convex, all agents are motivated to form the so-called grand coalition. In cooperative game theory, the intuitive idea of convexity was formally introduced by Shapley [128].

Definition 9: Convex Cooperative Game. A cooperative game $\Gamma=(D,v)$ with transferable utility is convex if the Value Function v is super-modular, that is to say $v(A\cup B)+v(A\cap B)\geq v(A)+v(B), \forall A,B\subseteq D$. ■

This definition of convex game is directly induced in the GCG on digraphs in Definition 1 by considering D a digraph on N agents and taking the notation of subset as representing an induced subgraph. The next lemma establishes that the GCG given in Definition 1 is convex.

Lemma 1: The GCG on digraphs in Definition 1 is convex.

Proof: In the graphical coalition game on digraphs $\Gamma=(D,v)$, in Definition 1, the coalition of agents is the digraph D and v is the Value Function. Let A and B are two induced subgraphs of D and let $A \cup B$ is the subgraph of D induced by the vertices in A and B. By technical Lemma D 16 in the Appendix A1, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B), \forall A, B \subseteq D$. ■

A stronger version of Axiom 5 is the strict inequality

$$v_{m+1} - v_m > v_m - (m-1)v_1 : N-1 \geq m \geq 1 \tag{5.3}$$

This stronger version is used to establish some refinements in the proofs of some results in the Sections 5.4 and 5.5.

5.4 Marginal Contribution in Graphical Coalitional Game on Digraphs

Agents join hands to form a coalition to pursue a common goal. The value of a coalition formed by cooperative agents is generally more than simple algebraic sum of the values of the individual agents. The contribution made by the agent towards the coalition is thus both competitive and altruistic [74]. Contributions made by agents in a coalition introduced in [74], are further elaborated in [77] for graphical the graphical coalitional games on simple graphs. In this section marginal contribution made by agents in a GCG on digraphs is defied and results are established elaborating the dependence of marginal contribution upon the position of the agents on digraphs and their dependence on graph topology and changes in topology. The notions presented in this section provide the basis for defining online sequential decision games in Section 5.5.

Definition 10: Marginal Contribution of an Agent. In a GCG on digraphs $\Gamma = (D, v)$, the marginal contribution made by an agent j is denoted as $m_D(j)$ and defined as

$$m_D(j) = v(D) - v(D \setminus j) \tag{5.4}$$

where $v(D)$ and $v(D \setminus j)$ are the values of the digraphs D and $D \setminus j$ respectively as elaborated in Definition 1. ■

Some lemmas about the marginal contributions made by the agents are presented next. They demonstrate the dependence of the marginal contributions on the topology of the graph. These results are established on the basis of the Axioms of Value, provide insight to the marginal contribution made by agents, and form the basis for the sequential decision games defined in the next section. The first results concern the marginal contribution of vertices which are not cut vertices. As elaborated in Definition 6, a cut vertex is the one whose removal increases the number of semi-strongly connected components.

Lemma 2: Given the GCG on digraph $\Gamma = (D, v)$ in Definition 1, in any semi-strongly connected graph *D* all the agents which are not cut vertices of *D* have the same marginal contribution. Moreover, their marginal contribution is the minimum possible marginal contribution within the connected graph. This minimum marginal contribution only depends upon the size $|D|$ of the semi-strongly connected graph *D*.

Proof: The marginal contribution made by an agent *j* for the GCG on digraph $\Gamma = (D, v)$ is given by (5.4)

$$m_D(j) = v(D) - v(D \setminus j) \tag{5.5}$$

Since the digraph *D* is semi-strongly connected, by Axiom 1, $v(D) = v_{|D|}$. Moreover, for vertex *j*, which is not a cut vertex, Definition 6, $v(D \setminus j) = v_{|D|-1}$. The marginal contribution, thus becomes

$$m_D(j) = v_{|D|} - v_{|D|-1} \tag{5.6}$$

Since the digraph is semi-strongly connected, $v(D) = v_{|D|}$ is constant and by technical Lemma D 3, $v(D \setminus j) = v_{|D|-1}$ is the maximum possible value of $v(D \setminus j)$. This completes the proof. ■

Remark 9: In a semi-strongly connected digraph *D*, if there is no cut vertex then the marginal contributions of all the agents are identical and independent of the digraph structure. ■

The next lemma establishes the maximum possible value of the marginal contribution made by an agent in a digraph of fixed size. This along with the last lemma marks the bounds of the marginal contribution made by an agent.

Lemma 3: Given the GCG on digraph $\Gamma = (D, v)$ in Definition 1. The maximum possible marginal contribution an agent may have is of the center point of a star which is also dominating to the rest of all the vertices of *D*.

Proof: Star is a digraph *D* with $|D|-1$ arcs, having $|D|-1$ vertices with degree 1 and one vertex with degree $|D|-1$; this vertex is called star point. If all the arcs are having their tails at it then it is the only dominating point in this digraph. The marginal contribution made by a star point *j* is given by (5.4)

$$m_D(j) = v(D) - v(D \setminus j) \tag{5.7}$$

Since the star *D* with the star point *j* dominating, is semi-strongly connected, by Axiom 1, $v(D) = v_{|D|}$. Moreover, for the star point *j*, $v(D \setminus j) = (|D|-1)v_1$. The marginal contribution, thus becomes

$$m_D(j) = v_{|D|} - (|D|-1)v_1 \tag{5.8}$$

By technical Lemma D 3, the maximum possible value of a digraph of size $|D|$ is $v(D) = v_{|D|}$ and the minimum possible value of a digraph of size $|D|-1$ is $v(D \setminus j) = (|D|-1)v_1$. This completes the proof. ■

It is established in [77] that the contribution made by an agent is the contribution made by it within its connected component. The connectivity in digraphs is a more involved concept and it is discussed in Section 5.2. The following lemma establishes that the marginal contribution made by an agent is only due to the marginal contribution made by it within the semi-strongly connected components it is a part.

Lemma 4: Given the GCG on digraph $\Gamma = (D, v)$ in Definition 1. The marginal contribution made by an agent *j* is the marginal contribution due to semi-strongly connected components of *D* which the agent is a part.

Proof: Without the loss of generality, assume that the digraph *D* has $k+l$, semi-strongly connected components $S^{v_i} : i = 1, 2, ..., k, k+1, ..., k+l$, such that agent *j* is a part of $S^{v_i} : i = k+1, ..., k+l$ then

$$v(D) = v(\bigcup_{i=1}^{k+l} S^{v_i}) \tag{5.9}$$

or

$$v(D) = v(\bigcup_{i=1}^{k} S^{v_i} \cup \bigcup_{i=k+1}^{k+l} S^{v_i}) \tag{5.10}$$

Considering the digraph $D \setminus j$, since *j* is a part of $S^{v_i} : i = k+1, ..., k+l$ in *D*, it implies that the semi-strongly connected components in $D \setminus j$, are $S^{v_i} : i = 1, 2, ..., k, S'^{v_i} k+1, ..., k+l'$, then

$$v(D \setminus j) = v(\bigcup_{i=1}^{k} S^{v_i} \cup \bigcup_{i=k+1}^{k+l'} S'^{v_i}) \tag{5.11}$$

By using (5.10) and (5.11) in (5.4)

$$m_D(j) = v(\bigcup_{i=1}^{k} S^{v_i} \cup \bigcup_{i=k+1}^{k+l} S^{v_i}) - v(\bigcup_{i=1}^{k} S^{v_i} \cup \bigcup_{i=k+1}^{k+l'} S'^{v_i}) \tag{5.12}$$

Rearrangement of this gives

$$
\begin{aligned}
m_D(j) = {} & v(\bigcup_{i=1}^{k} S^{v_i}) + v(\bigcup_{i=k+1}^{k+l} S^{v_i}) - v(\bigcup_{i=1}^{k} S^{v_i} \cap \bigcup_{i=k+1}^{k+l} S^{v_i}) \\
& - v(\bigcup_{i=1}^{k} S^{v_i}) + v(\bigcup_{i=k+1}^{k+l'} S'^{v_i}) + v(\bigcup_{i=1}^{k} S^{v_i} \cap \bigcup_{i=k+1}^{k+l'} S'^{v_i})
\end{aligned}
\tag{5.13}
$$

Since $v(\bigcup_{i=1}^{k} S^{v_i} \cap \bigcup_{i=k+1}^{k+l} S^{v_i}) = v(\bigcup_{i=1}^{k} S^{v_i} \cap \bigcup_{i=k+1}^{k+l'} S'^{v_i})$, this becomes

$$
m_D(j) = v(\bigcup_{i=k+1}^{k+l} S^{v_i}) - v(\bigcup_{i=k+1}^{k+l'} S'^{v_i}) \tag{5.14}
$$

This establishes the desired result. ■

The following lemma elaborates the effect of making an arc incident on an agent *j* on its marginal contribution. This result is used to establish results about the stable digraph structures in the sequential decision games discussed in Section 5.5.

Lemma 5: Given the GCG on digraph $\Gamma = (D, v)$ in Definition 1, and an arc incident at an agent *j* is constructed to make a new digraph $D \cup e$. Then

i. If both the end of the newly constructed arc are within the same semi-strongly connected component then $m_{D\cup e}(j) \le m_D(j)$.

ii. If both the end of the newly constructed arc are in different semi-strongly connected component then $m_{D\cup e}(j) \ge m_D(j)$. $m_{D\cup e}(j) = v(D \cup e) - v(D \cup e \setminus j)$

Proof: The marginal contribution made by the agent *j* within the digraph *D* is given by (5.4)

$$
m_D(j) = v(D) - v(D \setminus j) \tag{5.15}
$$

Similarly, the marginal contribution made by the agent *j* within the digraph $D \cup e$.

$$
m_{D\cup e}(j) = v(D \cup e) - v(D \cup e \setminus j) \tag{5.16}
$$

Since the newly constructed arc is incident at the agent *j*, $v(D\setminus j)=v(D\cup e\setminus j)$, (5.15) and (5.16) imply

$$m_{D\cup e}(j)-m_D(j)=v(D\cup e)-v(D) \tag{5.17}$$

By the technical Lemma D 12, making of an arc such that both the ends of the newly constructed arc are within the same semi-strongly connected components, then $v(D\cup e)\le v(D)$; thus (5.17) establishes the first part of the result.

Similarly, by the technical Lemma D 10, making of an arc such that both the ends of the newly constructed arc are in the different semi-strongly connected components, then $v(D\cup e)\ge v(D)$; thus (5.17) establishes the desired result. ■

Remark 10: It follows from this lemma that if a digraph is semi-strongly connected then making of an arc by an agent *j* does not change the marginal contribution made by the agent. ■

The following result strengthens the above result under the strong version (3.2) of Axiom 5. This lemma assures a definite increase or decrease in the marginal contribution made by an agent upon making an arc, thereby encourages the agents to form certain coalitions in the sequential decision games in the next section.

Lemma 6: Given the GCG on digraph $\Gamma=(D,v)$ in Definition 1, with stronger version (3.2) of Axiom 5, and an arc incident at an agent *j* is constructed to make a new graph $D\cup e$. Then

i. If both the ends of the newly constructed arc are within the intersection of two or more semi-strongly connected components then $m_{D\cup e}(j)<m_D(j)$.
ii. If both the end of the newly constructed arc are in different semi-strongly connected component then $m_{D\cup e}(j)>m_D(j)$.

Proof: The marginal contribution made by the agent j within the digraph D is given by (5.4)

$$m_D(j) = v(D) - v(D \setminus j) \tag{5.18}$$

Similarly, the marginal contribution made by the agent j within the digraph $D \cup e$

$$m_{D \cup e}(j) = v(D \cup e) - v(D \cup e \setminus j) \tag{5.19}$$

Since the newly constructed arc is incident at the agent j, $v(D \setminus j) = v(D \cup e \setminus j)$, (5.18) and (5.19) imply

$$m_{D \cup e}(j) - m_D(j) = v(D \cup e) - v(D) \tag{5.20}$$

By the technical Lemma D 13 , making of an arc such that both the ends of the newly constructed arc are within two or more semi-strongly connected components of D, then $v(D \cup e) < v(D)$; thus (5.20) establishes the first part of the result.

Similarly, by the technical Lemma D 11, making of an arc, such that both the ends of the newly constructed arc are in the different semi-strongly connected components, then $v(D \cup e) > v(D)$; thus (5.20) establishes the result. ■

The following lemma establishes the effect of the deletion of an arc within a digraph; it supplements the result obtained in Lemma 6.

Lemma 7: Given the GCG on digraph $\Gamma = (D, v)$ in Definition 1, with stronger version (3.2) of Axiom 5, if $D - e$ is a digraph produced by deleting an arc e incident at an agent j, such that it does not change the vertices in any of the semi-strongly connected components in D, then $m_{D-e}(j) \geq m_D(j)$. Moreover if the deletion of an arc e changes the vertices in a semi-strongly connected component then $m_{D-e}(j) < m_D(j)$.

Proof: The marginal contribution made by the agent j within the digraph D is given by (5.4)

$$m_D(j) = v(D) - v(D \setminus j) \tag{5.21}$$

Similarly, the marginal contribution made by the agent j within the digraph $D-e$

$$m_{D-e}(j) = v(D-e) - v((D-e) \setminus j) \tag{5.22}$$

Since the newly constructed arc is incident at the agent j, $v(D \setminus j) = v((D-e) \setminus j)$, (5.21) and (5.22) imply

$$m_{D-e}(j) - m_D(j) = v(D-e) - v(D) \tag{5.23}$$

By the technical Lemma D 14 , breaking of an arc e such that it does not change the vertices in any of the semi-strongly connected components in D, then $v(D-e) \geq v(D)$; thus (5.23) establishes the first part of the result.

Again by the technical Lemma D 14, breaking of an arc e such that deletion of the arc changes the vertices in a semi-strongly connected component then $v(D-e) < v(D)$; thus (5.23) establishes the result. ■

Remark 11: It follows from this lemma that if a digraph D is semi-strongly connected and deletion of an arc e, incident at an agent j, keeps it semi-strongly connected then $m_{D-e}(j) = m_D(j)$ and if the semi-strongly connected digraph breaks into two semi-strong components then $m_{D-e}(j) < m_D(j)$. ■

5.5 Online Sequential Coalition Decision Games

This section defines three online sequential decision games. These games are based on the marginal contribution made by the agents, and the results about its dependence on the graph topology established in the last section. These online decision games are defined on top of the graphical coalitional game $\Gamma = (D, v)$ of Definition 1. A background on sequential decision games can be found in Chapter 5 of [130].

In a sequential decision game, agents take turns sequentially in time to make valid or allowed moves (e.g. make or break an arc) to maximize their yield in terms of a prescribed objective function. Here, three online decision games are defined in terms of the objective function taken as the marginal contributions made by the agents towards the coalition, and varying the rules of the games. It is shown these three sequential decision games have different stable coalition structures. It is established that the stable structures are respectively multi-chains, semi-strongly connected digraphs, and chains of command. The machinery for establishing the results in this section rests on the formal Axioms of Value.

The properties of sequential decision games depend on the allowed moves and the prescribed objective function. An important concept in sequential coalition decision games is stability of graph topologies [104], [110]. Stability is important in studying the steady-state graph topologies of sequential decision games. Stable topologies show the structure of the coalitions that form under various allowed moves and decision objective functions. These stable structures are inherent properties of the objective functions of the three games, not parameter dependent as in [104].

5.5.1 Sequential Decision Games

In the online decision games defined here, agents are free to make coalitions by making or breaking arcs with other agents according to the rules of the game. In contrast to [104], there is no cost associated with arcs involved. Based on marginal contributions, three online sequential decision games are defined. These online decision games are defined on top of the graphical coalitional game $\Gamma = (D, v)$ of Definition 1. The agents make allowed moves sequentially through time; the moves are made to maximize the prescribed objective function.

In the sequential decision games, at each move, an agent is selected at random; this agent is free to unilaterally break any arc incident at it or to bilaterally make an arc, provided that the other agent incident on the arc agrees to make it, as detailed below. In a single step, an agent is allowed either to make or break several arcs.

Definition 11: Allowed Moves. In a sequential decision game the move which a selected agent is allowed to make is called an allowed move. ■

The objective function is the marginal contribution $m_D(j)$ made by an agent, for each agent j in a coalition represented by graph D. Arcs are made or broken by a selected agent in order to maximize $m_D(j)$ and according to the rules of sequential decision games defined next.

Definition 12: Objective Function. In a sequential decision game the selected agents make moves according to the rules of the game to maximize this real valued function. In this paper, the objective function of an agent j is the marginal contribution $m_D(j)$ made by the agent. ■

Based on the Allowed Moves and the Objective Function the sequential decision games are defined as follows.

Definition 13: No Make but Break Sequential Decision Game. In this sequential decision game a selected agent is not allowed to makes any arc and allowed to break arcs incident at it according to the rule

a) An agent j breaks an arc $e = (i, j)$ or (j, i) if $m_{D-e}(j) \geq m_D(j)$. ■

Definition 14: Make and Strong Break Sequential Decision Game. In this sequential decision game a selected agent is allowed to makes an arc and to break an arc incident at it according to the rules

a) An agent *j* makes an arc $e = (i, j)$ or (j, i) if $m_{D\cup e}(j) > m_D(j)$ and $m_{D\cup e}(i) \geq m_D(i)$.

b) An agent *j* breaks an arc $e = (i, j)$ or (j, i) if $m_{D-e}(j) > m_D(j)$. ■

Definition 15: Make and Break Sequential Decision Game. In this sequential decision game a selected agent is allowed to makes an arc and to break an arc incident at it according to the rules

a) An agent *j* makes an arc $e = (i, j)$ or (j, i) if $m_{D\cup e}(j) > m_D(j)$ and $m_{D\cup e}(i) \geq m_D(i)$.

b) An agent *j* breaks an arc $e = (i, j)$ or (j, i) if $m_{D-e}(j) \geq m_D(j)$. ■

In these three sequential decision games an agent *j* is said to have a motive to make an arc if the condition of making an arc in the above sequential decision games is satisfied, and it is said to have a motive to break an arc if the condition of breaking an in the above sequential decision games is satisfied.

5.5.2 Stability of Graph Topologies Under Sequential Decision Games

For a set of *N* agents there are $2^{N(N-1)}$ possible strict digraphs. When agents are allowed to make valid moves, as they proceed, they may reach a digraph where no agent has a motive to make any further moves. Such graphs are called stable graphs. The structure of stable graphs is thus dependent on the allowed moves and the objective function of the sequential decision game.

Definition 16: Stable Graph. In any online sequential decision game, a graph is called stable when no agent has a motive either to make an arc or to break an arc. ■

In [110], Myerson used the Shapley value as the objective function and allowed only the breakage of an edge of a simple graph as a valid move. Since breakage of an arc does not increase the competitive advantage, Lemmas 2 and 8 of [77], under such allowance, for the game in Definition 1 of [77], every graph is stable. In [104] the rules of making and breaking arcs are nearly the same as those in the sequential decision games defined in the last subsection. However, in [104], simple graphs are involved and there are costs associated with making edges. There, the balance between the value of being connected and the cost of maintaining arcs has a pivotal role in determining the stable graph structures. Here in this section, the stable structures are only dependent upon the marginal contribution made by the agents. The next development specifies the stable digraphs for the online sequential decision games. The next result establishes that if the initial digraph is weakly connected then under the No Make but Break Sequential Decision Game the stable structure is a multi-chain.

Theorem 1: In an online No Make but Break Sequential Decision Game, with a game list obeying strong version (5.3) of Axiom 5, if the initial digraph is weakly connected, then the stable digraph is a multi-chain. Conversely, any multi-chain is stable under No Make but Break Sequential Decision Game.

Proof: In an online, No Make but Break Sequential Decision Game, no agent is allowed to make a new arc. The agents are allowed to break an arc *e* within the digraph *D*, under condition $m_{D-e}(j) \geq m_D(j)$. By Lemma 7 , if breaking of an arc *e* does not change the vertices in any of the semi-strongly connected components in *D*, then $m_{D-e}(j) \geq m_D(j)$, and if the deletion of the arc *e* changes the vertices in a semi-strongly connected component then $m_{D-e}(j) < m_D(j)$;

this implies that agents have the motive to break only all arcs which do not change the semi-strong connectivity of the agents. Breakage of all these arcs makes the digraph a multi-chain.

Conversely suppose that the initial digraph in a No Make but Break Sequential Decision Game, is multi-chain. It implies that every semi-strongly connected component in this digraph is a chain of command. This implies that breaking of an arc definitely breaks at least one semi-strongly connected component; it implies from Lemma 7, under such condition, $m_{D-e}(j) < m_D(j)$, which leaves no agent with a motive to break an arc. ■

Remark 12: It follows from the technical Lemma D 14, and Lemma 7 that in the process of breaking the arcs in Theorem 5, no agent moves out of any semi-strongly connected component it already is a part. Breaking of arcs without changing the agents' semi-strong components guarantees that in the coalitions with pinning agents [1], [2] , [3], [7], [8], [79], [135], the agents reachable by various pinning agents remain unchanged. ■

The following result establishes that the stable digraph structure for Make and Strong Break Sequential Decision Game, is a semi-strongly connected digraph. Semi-strongly connected digraphs assure a unique consensus value [1], [2] , [3], [7], [8], [79], [135].

Theorem 2: In an online Make and Strong Break Sequential Decision Game, with a game list obeying strong version (5.3) of Axiom 5, the stable digraph is a semi-strongly connected digraph. Conversely, any semi-strongly connected digraph is stable under Make and Strong Break Sequential Decision Game.

Proof: In an online, Make and Strong Break Sequential Decision Game, an agent j is allowed to make a new arc $e=(i,j)$ or (j,i) in the digraph D if $m_{D\cup e}(j) > m_D(j)$ and $m_{D\cup e}(i) \geq m_D(i)$. If the digraph is not semi-strongly connected there always exists an agent j and agent i which do not share the same semi-strong component, and according to Lemma 6, $m_{D\cup e}(j) > m_D(j)$ and $m_{D\cup e}(i) > m_D(i)$; which implies that according to the rule *(a)* of Make and Strong Break Sequential Decision Game, both the agents j and i have motives to make the arcs e. If the agent j or i is pivot vertex then making of this arc merge the two semi-strong components into one semi-strong component, technical Lemma D 5. Under rule *(a)* of the game, agents keep on making the arcs until a semi-connected digraph is reached. Within a semi-strongly connected digraph, according to Remark 11 and rule *(b)* of Make and Strong Break Sequential Decision Game, no agent has a motive to break an arc.

Conversely suppose that the initial digraph in a Make and Strong Break Sequential Decision Game, is semi-strongly connected. Then according to Remark 11 and rules of the game no agent has any motive to make or break any arc. It implies that every semi-strongly connected digraph is stable. ■

The following result establishes the stable digraph for Make and Break Sequential Decision Game. It is established that chain of command is the stable structure in this case. Chain of command is an important digraph structure where there is only one pinning agent who controls the whole coalition [1], [2], [3], [7], [8], [79], [135].

Theorem 3: In an online Make and Break Sequential Decision Game, with a game list obeying strong version (5.3) of Axiom 5, the stable digraph is a chain of command. Conversely, any chain of command is stable under Make and Break Sequential Decision Game.

Proof: In an online, Make and Break Sequential Decision Game, an agent j is allowed to make a new arc $e=(i,j)$ or (j,i) in the digraph D if $m_{D\cup e}(j) > m_D(j)$ and $m_{D\cup e}(i) \geq m_D(i)$. If the digraph is not semi-strongly connected there always exists an agent j and agent i which do not share the same semi-strong component, and according to Lemma 6, $m_{D\cup e}(j) > m_D(j)$ and $m_{D\cup e}(i) > m_D(i)$; which implies that according to the rule *(a)* of Make and Break Sequential Decision Game, both the agents *j* and *i* have motives to make the arcs *e*. If the agent *j* or *i* is pivot vertex then making of this arc merge the two semi-strong components into one semi-strong component, technical Lemma D 5. Under rule *(a)* of the game, agents keep on making the arcs until a semi-connected digraph is reached. For any arc *e*, whose deletion does not change the semi-strong connectivity of the digraph, $m_{D-e}(j) \geq m_D(j)$, Lemma 7, and its deletion is motivated according to rule *(b)* of Make and Break Sequential Decision Game. Deletion of arc makes every semi-strong component a chain of command, while making of arcs leaves only one of them.

Conversely suppose that the initial digraph in a Make and Break Sequential Decision Game, is a chain of command. According to rule *(a)* of the game, an agent *j* has the motive to make an arc $e = (i, j)$ or (j,i) if $m_{D\cup e}(j) > m_D(j)$ and $m_{D\cup e}(i) \geq m_D(i)$, since agent is in a chain of command, it is in a unique semi-connected component; according to Remark 10, making of an arc does not increase the marginal contribution made by the agent. Similarly breaking of an arc will break the only semi-connected component, thereby reducing the marginal contribution made by the agent Remark 11, and thus according to rule *(b)* of the game no agent has any motive to make or break any arc. It implies that every chain of command is stable. ■

5.6 Simulation Examples of Online Sequential Decision Games

Simulation results for the three sequential decision games in Section 5.5 are presented here. In these simulations the games are started from an initial digraph and the agents are free to make allowed moves as in Definition 14, Definition 14, and Definition 15. One agent is randomly selected to make moves at each time, and it can make or break some arcs to improve its marginal contribution.

The simulations were run until one of the stable digraph structures is reached. The simulation results are shown and explained in Figure 5.1. These simulation results support the theory developed in Section 5.5. It is established that stable digraph under No Make but Break Sequential Decision Game is a multi-chain, the stable digraph under Make and Strong Break Sequential Decision Game is a semi-strongly connected digraph, and the stable digraph under Make and Break Sequential Decision Game is a chain of command.

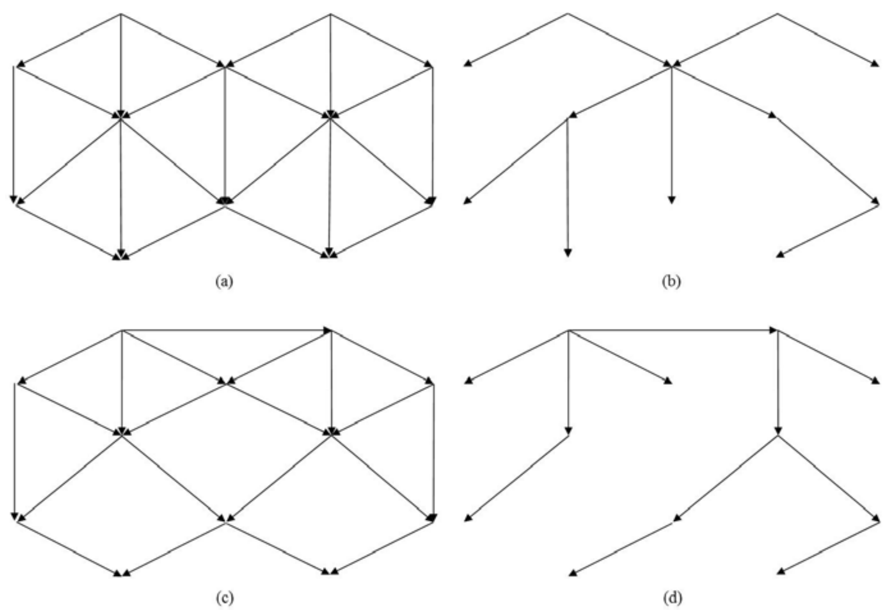


Figure 5.1: Evolution of digraphs in online sequential decision games. (a) The initial digraph for all the sequential decision games. (b) Stable graph is a multi-chain in No Make but Break Sequential Decision Game when randomly selected agents are allowed to break arcs according to the rules of the game. (c) Stable graph is a semi-strongly connected digraph in Make and Strong Break Sequential Decision Game when randomly selected agents are allowed to make/break arcs according to the rules of the game. (d) Stable graph is a chain of command in Make and Break Sequential Decision Game

Sequential Decision Game when randomly selected agents are allowed to make/break arcs according to the rules of the game.

5.7 Conclusions

A Graphical Coalition Game on digraphs is presented in this paper. Axioms of Value assign value to each digraph and are used to define the notion of marginal contribution made by an agent in a coalition from a graph theoretic view point, based on its connectivity. It is established that the game is convex; thereby, the core of the game is non-empty. The machinery of GCG on digraph is developed through a number of technical lemmas in the Appendix. These results elaborate the bunds of the value of digraph and about the dependence of value on the graph topology, and changes in the value due to changes in digraph topology. Further, on top of the GCG on digraphs, and based on the marginal contribution made by the agents, three online sequential decision games are defined by varying the rules of the game. The concept of stability is defined in these sequential decision games, and the stable graph topologies under the three games are detailed. It is established that stable graph structures are multi-chain, semi-strongly connected digraph, and chain of command; these digraph structures are essential in distributed control and consensus problems.

Chapter 6

Impact Propagation Framework, the End of the Number Game

This chapter presents a seamless, comprehensive, and integrated framework to compute the impact made by a scientific work, a scientist, an institution, a scientific journal, and a funding agency. The framework provides solutions to a number of problems about the existing indexing systems, under discussion, in the scientific community. These problems include but are not limited to the allocation of impact to scientists, excessive self-citation, gifted authorship, dependence of indices on disciplines, and excessive citations. The framework is evolved around a computation algorithm which is network-based, and distributed in nature. The framework thus includes the seminal effect of a work in the computation of impacts made by various entities and is readily implementable. Certain guidelines are provided for the implementation of the proposed framework. Work examples and simulation results show the working and usefulness of the proposed system in comparison to other existing systems. The proposed system has complete provision of peer input to cover delicate issues in impact calculations, yet the system can be employed fully automated during the earlier implementation stages.

6.1 Introduction

Citation of the work of earlier scholars has occurred throughout written history. The history of scientific citation can be tracked back at least to 428BC-322BC, the era of Plato and Aristotle who cited the work of earlier phosphors [143]. In the modern age of scientific work and publication, citations to earlier work can even be found in the first 1869 issue of Nature [144], the oldest continuing scientific journal. Yet the earliest account of popularizing the idea of systematic ranking of scientists by performance [145] is made by James McKeen Cattell through his 1906 work, American Men of Science [146]. In 1960, Dr. Eugene Garfield, based on his

1955 work [147], began to publish the Science Citation Index through the newly founded Institute of Scientific Information (ISI). Since then Garfield and ISI are best known for a citation based metric, the Impact Factor [145]. The paramount contribution made by them is to take the initiative in the compilation of a citation database, which, in spite of certain imperfections [145], [148] is a great asset to the scientific community. Though many other mechanisms are devised [145], [149], [150], [151], [152], [153], [154], [155] to measure the impact made by a scientific article or a journal or a scientist, all these mechanisms generally work around the citation database.

The Institute of Scientific Information, being the pioneer in the area of bibliographical metrics, enjoyed a monopoly over the field for over four decades. Their pioneer product is the Journal Impact Factor (JIF) [156]. JIF is a ratio of the number of citations in the current year to any items published in the journal during the previous two years divided by the number of substantive articles published in the same two years [156]. Another metric to measure the impact made by a journal is the five years Journal Impact Factor [154]; this metric is very much similar to JIF except that two years duration in the JIF is replaced by five years [154]. Immediacy Index is another related index to judge the quality of a journal [154]; it is used to determine the current yearly rate at which a journal is cited. It is obtained by dividing the number of citations to articles published in a year by the number of articles published in that year. This index is reported by Thomson Reuters in Journal Citation Reports [157]. There are two other metrics to measure the performance of a journal, these are cited half-life and citing half-life [154]; details about them are omitted.

There are certain bibliographical metrics which are based on the Google Page Rank algorithm; it is thus worthwhile to briefly explain the algorithm [149], [155], [158], [159], [160].

In this algorithm, the importance of each page is defined as the sum of the ratios of the importance of pages where this page is cited to the number of citations at the citing page. Under the condition that each page has at least one citation, this is analogous to visualizing all the Web pages as nodes of a Markov Chain with no self-loops and transition probabilities equal to the reciprocal of the number of cited links at a page and to the number of cited pages [161]. The definition of the importance of each page is thus equal to the steady-state probability of a web surfer to be found at that web page, when the web surfer is following the hyperlinks at its current page and switching to a random page when there is no hyperlink at the current page.

Eigenfactor is a web based [162] metric to calculate the impact of a scientific journal; it is based on a modified Google Page Rank algorithm implemented on the ISI database [163]. The modifications made to the Google page rank are the dilution of the effect of a dangling node [163] and accommodation of teleportation by means of weighted random search of scientific journals [163]. Thus Eigenfactor nicely automates a surfer who is going from one journal to another and sometime teleporting his course with a probability proportional to the number of articles produced by the journal [163], while ignoring the possibility of the surfer to stay at the same place, at least for some time slots. In this way, the Eigenfactor value is sensitive to the interconnectivity of journals. The Eigenfactor Score of a journal is defined as the percentage of the total weighted citations that the journal receives from their 7611 source items [163]. Another metric directly related to Eigenfactor Score is the Article Influence Score for each journal, which is a measure of the per-article citation influence of the journal [163]. Another very similar service is SCImago Journal Rank (SJR) [164]. SJR is on per article basis so it is comparable to Article Influence Score for each journal with some differences. First difference is SJR uses Scopus [165] as the database [166]. The second difference is that instead of dropping self-citations altogether these are capped at one third of total citations. Finally there are some minor differences in the mathematical equations the two methods use. SCImago also offers a service to rank countries [164].

Another journal ranking created in 2002 is F1000. The organization with a faculty of 10, 000 experts is aiming to filter the research articles by selecting the prominent articles and rate them on a scale up to 10, with 6 as recommended and 10 exceptional. Then they produce normalized sum of these scores over the journals [153], [167].

A number of methods are proposed to rank scientists; some of them are discussed here. The first one is total number of papers published by a scientist [150]. The second one is total number of citations [150]. The third one is citations per paper [150]. The fourth one is number of significant papers [150]. Other than all these conventional metrics there are some newly emerging metrics. The most prominent of them all is H-Index [150]. H-Index of a scientist is defined to be h, if h of the publications has at least h citations and the rest of them have less than or equal to h citations [150]. Many variants of H-Index are introduced, a few of them are discussed here, yet most scientists still prefer the H-Index [145]. One is m-parameter which is obtained by dividing h by the scientific age of a scientist [151]. Another variant is g-index which is defined to be an integer g less than or equal to the number of highly cited articles such that each of them has on average g citation [168]. Since both h and g indices are integers by definition, their rational number variants are also proposed. Another index is Jin's index which is defined as the average number of citations got by the articles counted towards H-Index [169].

The combined current system has served over a long time and gives some good outputs, yet there are certain issues with the existing system of bibliographical metrics which have caused a continuing unrest in the scientific community [145], [148], [150], [151], [156], [167], [170],

[171], [172], [173], [174], [175], [176], [177], [178]. Apart from the fact that there does not exist a single integrated system to give impact made by various entities like articles, journals, and authors there are also certain issues with the individual metrics. Taking the cases of JIF or its variant 5 years JIF; the first major issue which the scientific community is pointing to is the inconsistency in the nominator and denominator of the ratio JIF [156]. The nominator contains citations to all kinds of articles including editorials, letters, meeting abstracts, primary research papers, reviews and even the retracted papers while the denominator counts only the primary research papers, notes, and the reviews; this results in an inflated value of impact factor [148], [154], [178]. The second major issue is that this method to calculate the impact of a journal incites the editorial staff to persuade and promote self-citation of the journal, which causes the shadowing of the significant citations and results in the complete annihilation of its significance; this also promotes the editorial staff to publish more and more review articles instead of actual scientific articles since review articles attract more citations [170], [148], [174], [178]. The third major issue is that this method to calculate the impact of a journal disfavors those areas of study like mathematics and physics which have a natural tendency of lower numbers of citations compared to those having higher number of citations like clinical sciences and engineering [148]. Though measure such as normalization of the impact factor across disciplines is proposed to overcome this issue [156], [179], inter discipline boundaries are so soft that the scientific community is still concerned over it [145]; in this reference these concerns are expressed at the level of an individual scientist yet these concerns are valid at the level of a journal. In case of Immediacy Index, besides the problems discussed above there is another issue of disadvantage for those journals who publish in the later part of the year [154].

One problem with Eigenfactor based approaches is that they favor journals publishing a large number of papers [154]. Moreover, Eigenfactor completely ignores self-citations [163] which is not completely correct, since self-citations of journals are significant. Further, the method to calculate Eigenfactor takes care of the dangling node yet no such special measures are taken to take care of small inter-citing groups [162], [163]; the situation of dangling node and small inter-citing groups may be tackled on a case by case basis. In SJR the nominator contains all the citations of the journal but the denominator contains only the main articles, reviews and conference papers [166]. Though SJR makes a somewhat different importance wise ordering of journals [152], it is shown in [166] that a strong positive correlation exists between SJR and JIF which indicates that in spite of the entire background mathematical modeling, web based approaches are unable to make a big difference from the simple technique of JIF. In these approaches more importance is given to the citations coming from more important journals [149], [155], [158], [159], [160], whereas the importance of a citation is based on its use in the research [151], [170], [178]. In Google Page Rank based approaches a scientist is modeled as a random surfer from one journal to another, this model is by itself very divergent from the working of a scientist which is more deterministic and objective. Moreover, by virtue of using this model one may capture the frequency by which a reader comes across a journal, but how does it calculate the impact made by the journal is still under question [145].

There are certain questions posed about the methodology devised by F1000. The first one is based on the human nature which may cause to bias the scores according to the interests of the member faculty. The second one is based on the limited capacity of the system when compared to the huge size of the scientific literature produced every year. The third question is about an inherent system problem that a few over enthusiastic members can enhance the score of a journal; another aspect of this question is related to the integrity of the reviewers. Even at the initial implementation stage of the project such problems are already pointed out [167].

Certain disadvantages of metrics evaluating the scientists are also under discussion in the scientific community. The fundamental problem is that all these metrics encourage scientists not only to aggressively join the number game but also that there is a rampant increase in the self-citations. Scientists prefer to present half-baked ideas rather than striving for significant results. Scientists prefer to join overly large groups with the name of every member of the group on the papers produced by the group, publishing pacts to include each other's name are also common, similarly clique building is also observed, repetition of the same idea is also a significant issue, irrelevant self or gifted citations are also there [170], [171], [173], [175], [178], [180]. All these metrics are unable to measure the stature of scientists across disciplines [148], [171], though certain normalizing measures are proposed; yet the scientific community has some valid issues with the normalization [145]. None of these metrics are designed to measure the seminal effects of a research work. Some other problems with the classical individual level metrics are detailed in [150]. Other than the problems discussed above, problems specifically related to H-Index are under discussion in the scientific community [151], [180]. Since H-Index results in loss of information about a person, some even consider average citation per paper a better metric then H-Index [180].

Owing to the existence of a vacuum of having a satisfactory system to evaluate a scientist, many times scientists are judged by the impact of the journals they are publishing in; many times jobs, promotions and grants get associated with this criterion. There is almost a consensus in the scientific community that this is an improper practice while some say that it is a sin [145], [148], [171], [177], [181].

All the analysis made above indicates that there are certain issues with the existing system of measuring scientific impact of various entities, including authors, scientists, articles, presentations, journals etc. The various components of the system are related but they lack in complete harmony. There is generally a want of a new generation system which is capable of giving some meaningful numbers related to the impact actually made, decimate the problems associated with the current system, and take care of delicate issues of the peer review and human interaction [145], [170], [172], [174], [176], [177], [178], [182]. This paper provides an integrated system which takes care of all these needs. The proposed system is built on the citation database but additionally takes input about an article like its nature, its authors, and the nature of the articles it is citing. Based on this input this system primarily quantifies the impact made by an article. Moreover, impact made by an article on the human life is also calculated. These impacts are then translated into the impacts made by scientists, authors, agencies etc. Universities and institutes are integral part of the research activity. Many popular university rankings exist [183], [184], [185], [186]. These approaches make use of various inputs like citation data base, number of alumni winning Nobel Prizes, Field Medals etc. and many other useful entities. The system elaborated in this paper proposes an approach integrated with the ranking of scientific work, articles, scientists and all the other affiliated entities, to rank universities as well. Since implementation of such a system needs a lot of comments and suggestions from the scientific community many proposed parameters are left open for discussion.

The next section provides all the formal details of the proposed framework. These details include required definitions, notations, graph theory background, computation algorithms, and mathematical results. Some examples and numbers of remarks are also provided in this section for better insight of the framework, and to invite feedback from the scientific community. Section 6.3 provides the simulation results to further elaborate the usefulness of the proposed system, and presents conclusions. Proofs of the results in Section 6.2 are provided in the 0 at the end.

6.2 Impact Calculation Framework

In this section, a completely unified and integrated system to calculate the impact of a scientific article, a scientific journal and an author upon research activity is introduced. The proposed system provides a single framework to calculate impact of every single entity involved in the research activity including articles, journals, periodicals, conference proceedings, scientific articles, surveys, thesis, book chapters, webpages, posters, authors, mentors, coauthors, funding agencies, institutes and universities, etc. In this paper any kind of single scientific work is referred to as *work*, any journal or conference is referred to as *object*, any author, coauthor, presenter is referred to as *person*, any other human involved is referred to as *mentor*, any funding agency involved as *agency* and any institute, laboratory or university as *institute*. A *person* and *mentor* is jointly referred to as *human*. If a *work* i is presented/published in an *object* k and a *person* p is author/coauthor/presenter of this *work* then in this paper it is said that *person* p is a part of *work* i and the *work* i is a part of the *object* k, and denoted as

$$p \lhd i, i \lhd k \tag{6.1}$$

It is proposed that only those journals and conferences are included in the proposed systems having some standard procedure to accept a work; similarly, only those works which are part of such journals or conferences are considered. The standard procedure is discussed in this work.

Similarly, if a funding *agency g* has supported a *work i* then it is said that *agency g* is a of the *work i* and it is written as

$$g \prec i \tag{6.2}$$

Similarly, if an *institution y* is in the list of affiliations of a *work i* then it is said that the *institution y* has some affiliation in *work i* and it is written as

$$y \rightarrow i \tag{6.3}$$

And if a *human p* is working with an *institution y* then it is said that *human p* is a part of *institution y* and it is written as

$$p \rightarrow y \tag{6.4}$$

In this paper the scientific impact upon research activity at some time *t* of a *work i* is denoted as $TW_i(t)$, scientific impact of an *object k* is denoted as $TO_k(t)$, and the scientific impact of a *person p* is denoted as $TP_p(t)$. Some other notations are defined later in this section. This framework can be dynamically tuned in such a way that it is immune to the problems in the current indexing systems [148], [150], [151], [170], [174], [175], [177], [178]. Impact of a *work i* is not only upon a *work* where it is cited rather any *work* has seminal effect on the further *works*; the framework proposed in this paper also captures this part of impact made by a *work*. The proposed framework models the complete bibliographic system as a graph; this section, thus takes the start from a brief account of graph theory.

6.2.1 Graph Theory Background

Graphs are mathematical objects extensively used to study entities and their interactions. Entities can be people, robots, organisms, scholarly articles or a group of any other things in some kind of interaction with each other. Entities are represented as vertices in a graph. In this paper these entities are generally scientific *works* and scientists involved in these *works*. Interactions between the entities can be of a variety of type; these can be communication, chain of command or dependence of one on others. These interactions thus can be non-directional or directional; these are respectively represented by edges or arcs between the vertices. On the bases of the existence of edges or arcs between vertices, graphs are generally classified as simple graphs or digraphs. Based on the nature of interactions, weights can also be associated with the edges or arcs. Since the interactions under study are generally asymmetric and directional in nature; weighted directional graphs or digraphs are used in this paper. In this paper the weights can assume any real value between 0 and 1.

In this paper a weighted digraphs are generally represented as $G=(V,E)$ with V a finite nonempty set of entities and set $E \subseteq V \times V \times I$ a set of arcs. Here $V \times V$ is the set of all the ordered pairs of the elements of V and I is the closed interval of real numbers between 0 and 1. The triple product $V \times V \times I$ is used here to represent a situation of an ordered pair associated with a weight within the interval I. Cardinality of the elements in V is taken as N.

If an arc of nonzero weight exists between a vertex i and a vertex j then they are called neighbors of each other and the collection of all the neighbors of a vertex i is called its neighborhood set denoted as $N_i = \{j : (i,j,t_{ij}) \in E, t_{ij} \neq 0 \vee t_{ji} \neq 0\}$. A sequence of distinct vertices $i = i_0, i_1, ..., i_M = j; M \geq 1$ starting from a vertex i to another vertex j such as there exist an arc of nonzero weight from vertex i_k to the vertex i_{k+1} for all $k = 0,1,2,...,M-1$ is called a directed path of hop count M from i to j for a weighted digraph G, it is written as $p = \{i = i_0, i_1, i_2, ..., i_k = j : \forall l \neq m \Rightarrow i_l \neq i_m, \forall l = 0,1,2,...,k-1 \ (i_l, i_{l+1}, t_{i_l i_{l+1}}) \in E, t_{i_l i_{l+1}} \neq 0\}$. If there is a path from vertex i to vertex j then the later is said to be reachable from the earlier otherwise vertex *j* is said to be disconnected or unreachable from vertex *i*. The number of arcs in a path is known as path length and the minimum path length from a vertex *i* to a vertex *j* is called directed distance from *i* to *j*. A detailed account on graphs can be found at [187].

6.2.2 Definitions and Preliminaries

This section defines the basic elements involved in the framework of impact and their mutual relationships.

Definition 1: Direct Impact Factor of a *Work* on another *Work*. In a situation when *work i* is cited in another *work j*, the first one has certain factor of impact on the second one; this is named Direct Impact Factor of *work i* on *work j* and denoted as t_{ij} and it may assume any value within the interval $(0,1)$. ■

A null or zero value of the direct impact factor t_{ij} means that *work i* has no direct impact on *work j*. A mechanism elaborating the assignment of the Direct Impact Factor value is outlined in the next subsection.

In this paper all the *works* are represented as vertices of a weighted digraph $G(V,E)$ with matching labels, and weighted arcs from vertex *i* to vertex *j* if and only if *work i* is cited in *work j*. The weight of the arcs (i,j,t_{ij}) is the corresponding values of direct impact factor t_{ij}.

Definition 2: System of Works. The weighted digraph $G(V,E)$ as elaborated in the above paragraph is defined as System of Works. ■

Definition 3: Path Impact Factor of a *Work* on another *Work*. For a *work j* reachable from a *work i* through a path

$$p = \{i = i_0, i_1, i_2, ..., i_k = j : \forall l \neq m \Rightarrow i_l \neq i_m, \forall l = 0,1,2,...,k-1 \ (i_l, i_{l+1}, t_{i_l i_{l+1}}) \in E, t_{i_l i_{l+1}} \neq 0\}$$ in

the System of Works $G(V,E)$, the Path Impact Factor of the *work i* upon the *work j* is denoted as $T_{\substack{p \\ i\sim j}}$ and it is defined as the product of the Direct Impact Factors taken on the arcs along the path *p* and given by the equation

$$T_{\substack{p \\ i\sim j}} = \prod_{l=0}^{k}(t_{i_l i_{l+1}}) \tag{6.5}$$ ■

Definition 4: Impact Factor of a *Work* upon another *Work*. The Impact Factor of a *work i* upon another *work j* where the later is reachable from the earlier in the System of Works $G(V,E)$, is denoted as T_{ij} and given by

$$T_{ij} = \max_{p}(T_{\substack{p \\ i\sim j}}) \tag{6.6}$$ ■

Remark 1: The Impact Factor T_{ij} defined in this equation gives the maximum lower bound of the impact factor of *work i* on *work j* over all the directed paths from *i* to *j* in the digraphs; further, it is shown in the later sections that it makes it possible to devise a framework of trust propagations. Impact Factor of a *Work* upon another *Work* of the work *i* upon itself is defined as 1, that is to say $T_{ii} = 1$. Moreover, if there is no path from the *work i* to the *work j* then the Impact Factor T_{ij} of the *work i* upon the *work j* is defined as 0. Furthermore, if there is a unique path *p* from the *work i* to the *work j* then $T_{ij} = T_{\substack{p \\ i\sim j}}$. ■

6.2.3 The Framework of Impacts

This subsection details the proposed framework of the impacts involving *works*, *objects* and *persons*. The proposed framework addresses various problems involved in the existing indexing systems [148], [150], [151], [170], [174], [175], [177], [178] by quantifying the impacts made and by using a competitive approach.

6.2.3.1 Direct Impact Factor Assignment and Impact Factor Propagation

When a *work i* is a part of an *object k*; this *work i* cites some other *works j*. The citation of *works* in another *work* implies that these cited *works* have certain impacts upon the new work. In the current system of indexing, no mechanism of measuring the impact of a cited *work* upon the citing *work* exists; each cited *work* is given the same credit [150], [156]. Since some *works* are cited just to make a connection with the existing literature [170] while others are cited since the results established or surveys made in them are used in the citing *work*, it is unfair to assign equal credit to each cited *work*. Moreover, currently there is no defined finite amount of impact which is to be distributed to the cited *works*. This results in excessively high numbers of citations with motives other than to indicate research impacts [148], [151], [170], [174]; this flaw is also highlighted by an editor by citing all articles published during the last two years in an editorial [188], as a protest to the current system. In this subsection some rules are outlined to assign the Direct Impact Factor (Definition 1) of the cited *works* on the citing *work*.

1. *Proportional Assignment*: Sum of all the Direct Impact Factors taken over all the cited *woks* at a *work i* is 1, that is to say

$$\sum\nolimits_{\forall j} t_{ij} = 1 \tag{6.7}$$

This rule is proposed for the proportional assignment of the impact and to contain the citations to those *works* having real impact on the citing *work* by offering a competitive distribution.

2. *Trivial Assignment*: For a *work i* citing total *J* number of *works j*, each of the t_{ij} is assigned as reciprocal of *J*, that is to say

$$t_{ij} = \frac{1}{J} \forall j = 1, 2, \ldots J \tag{6.8}$$

Though this trivial assignment does not resolve the issue of uneven impacts made by cited *works* yet it is according to the rule of proportional assignment and up to quite a good extent contains the number of citations to the *works* with real impact on the citing *work*. Moreover, this assignment is easy to apply at the initial stages of the implementation of the proposed framework. Further, trivial assignment works well with *works* like surveys, reviews, news articles and editorials.

3. *Weighted Assignment*: A number of weighted assignments can be proposed. These weighted assignments can be discussed at the level of scientific community and at the level of *objects'* editorial boards. Following are general guidelines for the weighted assignments:
    a. *Persons* and referees involved agree upon the background citations and citations which are used in a *work*. This classification can be generally made by separating those citations which are not involved but in the introduction part of the *work*. This mechanism offers an interface to provide delicate human input to the system while minimizing the issues related to human interaction in ranking the research [167].
    b. If there is at least one citation which is involved in the development of the *work* then the Direct Impact Factor of a background citation is always less than the Direct Impact Factor of other citations.

c. If there is at least one citation which is involved in the development of the *work* then the sum of the Direct Impact Factors of a background citation is always less than the sum of Direct Impact Factor of other citations.

d. An upper limit on the sum of Direct Impact Factors of self-citations is recommended if possible while obeying b and c.

4. *Communication of Assignment*: If a new *work j* is a part of *object k* and it has a citation of *work i* which is a part of *object l*, then *object k* communicates the Direct Impact Factor t_{ij} of *work i* upon *work j* to *object l*.

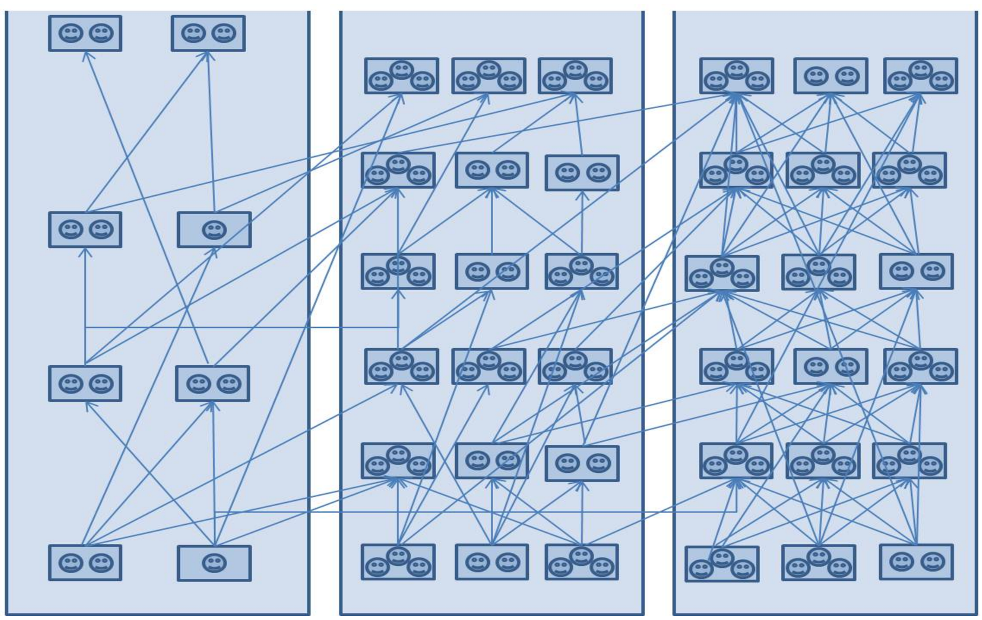


Figure 6.1: System of Works showing three objects in the form of three big top open rectangles. Time is increasing from bottom towards top. The small rectangles represent the *works* and happy faces in the *works* are the *persons* involved. Arcs showing the citations of the *works* at the tail ends of the arcs in the *works* at the head ends of the arcs. Weights on the arcs are not shown to avoid rush in the figure.

Example 1: System of Works. Consider a small system of works shown in Figure 6.1. In this figure three big rectangles open from the top represent three *objects*. The small rectangles in these *objects* are the *works*, parts of these *objects*. Moreover, the happy faces within the *works* show the *persons* involved in these *works*. Time is increasing from bottom towards the top. *Object* at the left represents a pure mathematical *object* with lower number of *works* in each of its issue while the *object* in the center represents an applied mathematics and natural science based *object* while the one at the right represents an applied *object*. The arcs between the *works* show the citation of a *works* at the tail ends in the *works* at the head ends. Weights on the arcs are dropped in the figure to avoid rush in the figure. ■

Example 2: Direct Impact Factor . Consider a small segment of system of *works* shown in Figure 6.2. Here, weights are also shown on the arcs as detailed in this section. ■

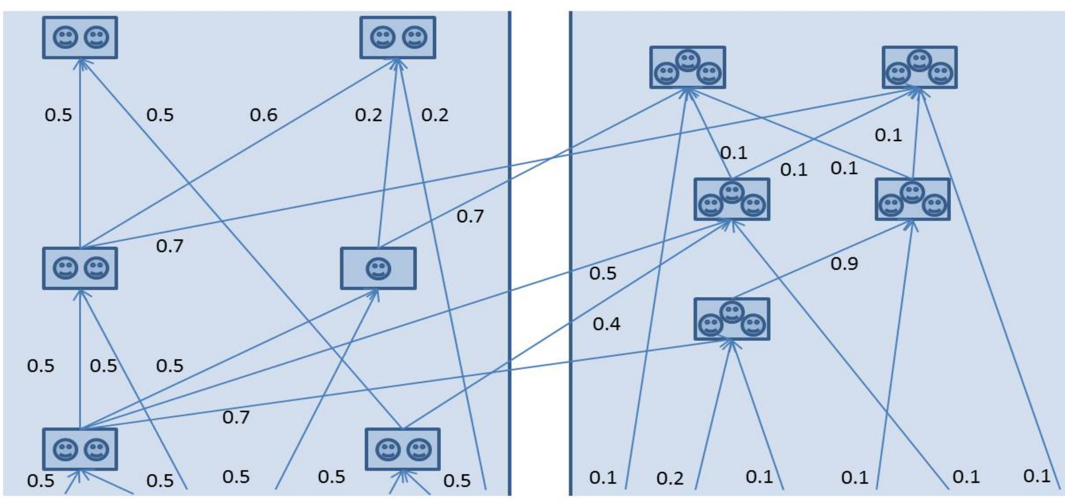


Figure 6.2: A segment of System of Works with Direct Impact Factors shown.

6.2.3.2 Properties of Impact Factor

Certain important properties of the Impact Factor T_{ij} which the *work i* has on the *work j* within the System of Works are mentioned in this subsection. These properties are important to establish the Impact Factor propagation results in the next subsection.

6.2.3.2.1 Range of Impact Factor

For any two *works* i and j, T_{ij} lies between 0 and 1. The Impact Factor value $T_{ij} = 0$ means i has absolutely no impact on j, while the impact factor value 1 means i has full impact on j.

6.2.3.2.2 Bounds on Impact Factor

It follows from the definitions in this section that for any three *works* i, j and k the Impact Factor T_{ik} always lies between $T_{ij}\,T_{jk}$ and 1 that is to say $T_{ij}\,T_{jk} \le T_{ik} \le 1$.

6.2.3.2.3 Semi Ring Property of Impact Factor

In this subsection, the semi-ring property of the Impact Factor is established. In a framework with this property it is possible to propagate Impact Factor over a digraph [189]. Let T be the set containing $T_{\overset{p}{i \sim j}}$ for all the possible vertex pairs, for all the possible paths in all possible digraphs and for all the possible direct Impact Factor values, that is to say $T = \{T_{\overset{p}{i \sim j}} : i, j \in V, \forall t_{ij} \in [0,1], \forall p \in P\}$, here P is the set of all paths between i and j of a weighted digraph $G(V, E)$ with $|V| > 2$ and having every possible arc. It is to establish that the triplet $(T, Max, *)$ forms a semi-ring. In this formulation the product operation is used to combine Impact Factor values along a path. The *Max* operation is used to combine Impact Factor options received about a *work* from more than one intermediate *work* by taking the maximum over all the received values.

Lemma 1: The triplet $(T, Max, *)$ forms a semi-ring.

Proof: Proof is in the Appendix. ■

With the setting up of $(T, Max, *)$ being a semi-ring, it can be said that the Impact Factor as defined above can be treated algebraically in all operation which does not involve cancellation [189], as done in proving result about its propagation in the next section.

6.2.4 Impact Factor Propagation

Once the Direct Impact Factor between the neighboring *works* in the System of Works is known, this subsection shows how to determine the Impact Factor T_{ij} for the two *works* $i, j \in V$ who are not necessarily neighbors in the System of Works $G(V, E)$. The constructions of the previous subsection are used to propose a mechanism for the Impact Factor T_{ij} propagation for all pairs of *works* $i, j \in V$ within the System of Works. The computational complexity of the mechanism is $O(N^2)$ for a *work* $i \in V$ computing its Impact Factor T_{ij} for another *work* $j \in V$.

In this algorithm a *work i* maintains an *N* vector T^i; at the end of the following algorithm its *j-th* entry $T^i{}_j$, represents T_{ij} the Impact factor *work i* has on the *work j*.

Algorithm 1: Computation of Impact Factor of *works i*. Following are the steps involved in the algorithm.

1. Preliminaries

   Input: The System of Works $G(V, E)$ and *N* vector $T^i (\forall i = 1, 2, \ldots N)$

   Output: Vector T^i

   Initialization: $T^i{}_k \leftarrow 0 \forall i, k = 1, 2, \ldots, N$, set $T^i{}_i \leftarrow 1$

2. Relaxation

for $(N-1)$ time do

for each work i do

for each arc $(i, j, t_{ij}) \in E$ do

for each $k \in V$ do

if $T^{i}{}_{k} < t_{ij} T^{j}{}_{k}$ then

$T^{i}{}_{k} \leftarrow t_{ij} T^{j}{}_{k}$

Return T^{i}

The above algorithm can also be expressed as follows.

Mathematical formulation of Algorithm 1: Computation of Impact Factor of *works i*.

$$
\begin{aligned}
&T^{i} = 0 \\
&T^{i}{}_{i} = 1 \\
&for\ \ count = |V| - 1 \\
&(\forall i \in V\, \forall (i, j, t_{ij}) \in E\, \forall k \in V (T^{i}{}_{k} = \max_{j \in N_i \cup \{i\}} [t_{ij} T^{j}{}_{k}])) \\
&end
\end{aligned}
\tag{6.9}
$$

return T^{i} ■

Theorem 1: The Algorithm 1 converges to give $T^{i}{}_{k} = T_{ik}$.

Proof: Proof is in Appendix. ■

Since the vector T^{i} can be updated by corresponding *work i* by using only the local information, the above algorithm can be readily implemented distributed on each of the *works i* [190]. The following example depicts the distributed working of Algorithm 1.

Example 3: Impact Factor Propagation. Consider a System of Works as shown in Figure 6.3 (a), four tables show the vectors $T^{i} : i = 1, 2, 3, 4$ and the weighted arcs between then show the direct impact factors of the *works* upon each other. Figure 6.3 (b)-(c) show the working of the Algorithm 1, and Figure 6.3 (d) shows the steady state value of all the impact factors of *works* on each other. ■

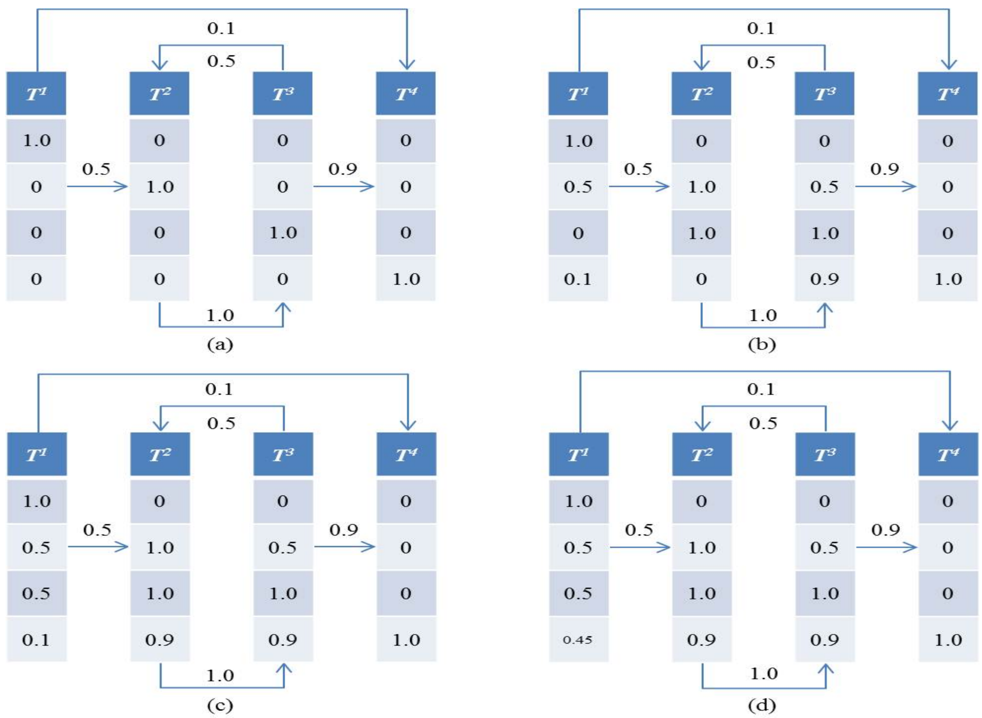


Figure 6.3: Impact Factor Propagation. (a) Initial setting. (b)-(c) Propagation. (d) Steady state.

Remark 2: The Impact Factor T_{ij} computed by the Algorithm 1 is the highest lower bound of the contribution made by work i to work j. ■

6.2.4.1 Impact of a *Work* on another *Work*

By using the analogy developed so far in this section, all the *works* form a System of Works $G(V,E)$ with the *works* are represented as vertices with matching labels, and weighted arcs from vertex i to vertex j if and only if *work* i is cited in *work* j. The weight of the arcs (i,j,t_{ij}) is the corresponding value of direct impact factor t_{ij}. Currently, only those *works* j citing a *work* i are generally counted while evaluating the impact of a work [150]. This is equivalent to counting only those *works* at 1 hop form the *work i* in the System of Works $G(V, E)$. Generally, in the existing system there is no way in use to directly calculate the impact made by a *work* on another *work*; impacts made by *works* are projected from either the journal or from the citations they got when calculated for persons involved. This system to calculate the impact of a *work* has many shortfalls [150], [151], these include:

1. *Not a Measure of Real Impact*. The simple count is not a measure of a *work*'s impact, since a citation could merely be for the introduction purposes. Moreover, *persons* involved are driven to give more and more self-citations [151], [170], [178].
2. *Consideration of Citing Object*. Owing to the absence of a reliable system to calculate the impact of a *work* many times *works* are evaluated by using the impact of the *objects* which they are part of. There is a consensus in the scientific community that this is not fair due to many reasons [145], [148], [171], [177], [181].
3. *No Consideration for Further Referencing.* As mentioned earlier, the current system counts the citations at one hop of the cited *work* [150], [151]. This is completely against the way cause and effect work in nature [191], [192], [193].

The mechanism to calculate the impact of a *work*, step by step developed in the later part of this section, addresses these problems. The Impact Factor of one *work* upon the other work thus given by Algorithm 1 takes care of the seminal effect of a *work*. This leads to the following definition.

Definition 5: Direct and Indirect Citation. In the System of Works $G(V, E)$ if there exists a path of length 1 from a *work i* to a *work j* then the first one is called a direct citation of the second one and if a path of length greater than 1 exists then it is called an indirect citation. ■

Remark 2: It is to be noted that if no more arcs between the existing vertices of the System of Works are added after T_{ik} is calculated then it remains fixed for the *works i* and the *k* which were in the System of Works at that time. That is to say if no omission is made in the System of Works then for the given *works i* and the *k* calculation of T_{ik} is a one-time process.■

6.2.4.2 Computation of Scientific Impact of a Work $TW_i(t)$

In spite of the fact that *work* is the fundamental entity in the whole of the citation scenario, little effort is made to elaborate the impact of a *work*. This section defines the impact made by a *work* which is further used in the next sections to define the impacts made by other entities involved. The impact made by each *work* depends upon the type of the *work* and its participation in scientific activity. For example, while a regular scientific *work* has full participation, a survey may have some partial participation and an editorial or correspondence may have minimal participation in scientific activity [148], [154], [178]. The following definition is thus made.

Definition 6: Participation Factor. Participation Factor of a *work j* in an *object k* is denoted as η_{jk} and is a number between 0 and 1, that is to say $0 < \eta_{jk} \leq 1$. Participation Factor signifies the type of *work j* in *object k*. ■

Remark 3: The unit value may be assigned to all η_{jk} at the time of the implementation of the proposed system; the scientific community is invited to give input to tune the assignment of η_{jk} for various types of *works*, including regular papers, survey papers, short papers, correspondences, reviews, news articles, editorials etc. etc. ■

Once the System of Works is set and Impact Factors are calculated, the framework is ready to define the Impacts of individual *works* on the research activity.

Definition 7: Current Scientific Impact of a *Work*. The Current Scientific Impact of *work* i at some time t is written as $TW_i(t)$ and defied as

$$TW_i(t) = \sum_{j:j \lhd k} T_{ij}.\eta_{jk} \tag{6.10}$$ ■

Since more and more *works* are produced at every moment, the System of Works is a changing entity. Suppose that at some time t the System of Works is $G(V(t), E(t))$ and the Scientific Impact of a *work i* is $TW_i(t)$. Then every *work* can asynchronously update its impact in the event of getting a new direct or indirect citation (Definition 5). Representing the instance of updating as $t+1$, the System of Works at this instance is represented as $G(V(t+1), E(t+1))$. The updated value of the Scientific Impact of *Work* is written as $TW_i(t+1)$ and given by the following algorithm.

Algorithm 2: Updating the Impact of a Work.

$$TW_i(t+1) = TW_i(t) + \sum_{j:j \lhd k} (T'_{ij} - T_{ij}).\eta_{jk} \tag{6.11}$$

Here T'_{ij} is the Impact Factor (Definition 4) of *work i* on *work j* in the System of Works $G(V(t+1), E(t+1))$. ■

Remark 4: Computationally more efficient formulation can be made for the implementation of Algorithm 2, the algorithm in (6.11) is presented in its present form for its better readability. ■

Remark 5: The updating of the Scientific Impact of Work in Algorithm 2 is a one-time process, and happens only when a new direct or indirect citation is reported to the *work* by Algorithm 1 or there is some change in the existing values of T_{ij}. Generally there is an increase in T_{ij} except when some *work* is withdrawn, in this way the system has the flexibility to take care of such changes. ■

6.2.4.3 Computation of Scientific Impact of an *Object* $TO_k(t)$

Currently the impact of an *object* is computed by calculating the average number of citations made by the *works* which are parts of the *object* [147], [154]. This current method to compute the impact of an *object* has certain problems, these include:

1. *Not a Measure of Real Impact*. The simple average is not a measure of an *object*'s impact, since a citation could merely be for introduction purposes. Moreover, the editorial board involved is driven to coercive citation, the act of giving more and more self-citations [174].
2. *No Consideration of Citing Work*. The current method only counts the number of citations without any consideration of the impact of citing *work*.
3. *No Consideration of Past History*. The current method to calculate the Impact of an *object*, considers only the articles of the *object* cited during the past one year with absolutely no consideration of past history. Moreover, citations of older *works* are not considered [147], [163], [166].

In Google Page Rank based approaches, Eigenfactor and SRJ, a scientist is modeled as a random surfer from one journal to another, this model is by itself very divergent from the working of a scientist which is more deterministic and objective. Moreover, by virtue of using this model one may get the frequency by which a reader comes across a journal but how does it calculates the impact made by the journal is still under question [145]. In this subsection a mechanism to compute the Scientific Impact of *objects* is defined. This method takes care of the problems mentioned above and gives a dynamic and stable system. This section also proposes a transition of this proposed system without disturbing the existing systems. For the purpose of this transition, the following definition is made. This definition gives the initial value of the Scientific Impact of an *object*.

Definition 8: Scientific Impact of an *Object*. The Scientific Impact of an *object* k is written as TO_k and the initial value assigned to it is equal to the current 5 years impact factor of the *object* in the present system. A mechanism to update this assignment is presented in this subsection. The value of TO_k is published. ■

The Scientific Impact of an *object* for a past one year is generally calculated as the ratio of the number of citations made to the past one year *works* which are parts of the *object* and the number of *works* added in the *object* during that year [147], [163], [166]; in contrast, it is an established fact that scientific works generally have long term effects [194], thus it is not rational to deprive an *object* of the credit of an older *work* being cited currently. Following the same lines, but adopting the proposed framework, in this paper the Scientific Impact of an *object* for a year is proposed to be dependent upon the ratio of the sum of changes in the Scientific Impact of the *works* which are parts of the *object* to the number of *works* increased during the year. Thus the following definitions are made.

Definition 9: Yearly Change in the Current Scientific Impact of a Work. For a *work* i who's Scientific Impact was $TW_i(t)$ at the start of a year or just after its becoming a part of an *object* k, whichever occurrence is latest, and $TW_i(t')$ is its Scientific Impact at the end of the year then Yearly Change in the Scientific Impact of the Work i is denoted as ΔTW_i and defined as

$$\Delta TW_i = TW_i(t') - TW_i(t) \qquad (6.12)■$$

Remark 6: Any event of the citation of a *work* is considered to occur after the *work* has become part of some *object*. ■

Definition 10: Yearly Change in Scientific Impact of an Object. Yearly change in the Scientific Impact of an *object k* for a year is denoted ΔTO_k and is defined as the ratio of the sum of that's yearly changes in the impacts of works which are parts of the *object* to the sum of participation factors (Definition 3) of all the *works i* added in the *object k* during that year. That is to say

$$\Delta TO_k = \frac{\sum_{i \lhd k} \Delta TW_i}{\sum_{i \lhd k} \eta_{ik}} \tag{6.13}$$ ■

Remark 7:

1. As the time passes impacts of the existing *works* generally increase. This results in nonnegative value of the Yearly Change in the Scientific Impact of an Object in the above definition.
2. The ratio in (6.13) neither favors nor disfavor an *object* as attributed to the current system [147], [166]. ■

The Scientific Impact of an *object* is proposed to be updated any time, preferably every year, which is in line with the current practice [147], [163], [166]. The following algorithm is proposed to update the Scientific Impact of an *object* TO_k every year.

Algorithm 3: Updating the Scientific Impact of an *Object k*.

$$TO_k \leftarrow (1-\alpha)TO_k + \alpha \Delta TO_k \tag{6.14}$$

The constant α can be typically taken 0.2, if 5 years Impact is required. ■

Remark 8: The above algorithm gives a running average of the impact made by the *object* which is a reasonably true measure of the current value [195]. ■

6.2.4.4 Scientific Impact of a Person

The current ways to measure the research impact of a person have certain issues [145], [148], [150], [151], [170], [171], [173], [175], [177], [178], [180], these include.

1. *Self-Citation*. The current system drives the authors to make more and more self-citations.
2. *Excessive Works*. The current system drives *persons* to play the number game, and to act on the analogy of publish or perish. This makes it difficult to find real *works*.
3. *Credit of Work*. Currently there is almost no mechanism to distribute credit of work among authors. Generally some intangible credit is given to the first author but credit allocation to the second author is much more mystical, who could have put a lot more effort in the work.
4. *Gifted Authorship*. The current system to measure the research impact of a *person* does not give any penalty to gift authorship to some *persons* who are not involved in the research. Many times names of all the persons working in a lab are placed in the list of *persons* without considering their real efforts involved in producing the *work*.

The mechanism proposed in this subsection addresses these issues by offering a competitive environment. In this competitive environment every *person p*, who is a part of *work i*, is assigned with a fraction of the Impact made by the *work*. This is reached by the following definition.

Definition 11: Participation Fraction of a Person in a Work. For a *person p* who is a part of a *work i*, the Participation Fraction is a number f_{pi} existing within the interval $(0,1]$. ■

The following recommendations are made for the assignment of values to f_{pi}, and the scientific community is requested to comment to reach a consensus over it.

1. *Proportional Assignment*: Sum of all the Participation Fractions taken over all the *persons* involved in a *work i* is 1, that is to say

$$\sum_{p \lhd i} f_{pi} = 1 \tag{6.15}$$

This rule is proposed for the proportional assignment of the Impact of Work and to contain the *persons* involved to those who put real efforts, by offering a competitive distribution.

2. *Trivial Assignment*: For a *work i* having total *P* number of *persons* as its parts, each of the f_{pi} is assigned as reciprocal of *P*, that is to say

$$f_{pi} = \frac{1}{P} \forall p = 1, 2, ..., P \tag{6.16}$$

Though this trivial assignment does not resolve the issue of uneven contributions made by contributing *persons* yet it is according to the rule of proportional assignment and to quite a good extent contains the number of *persons* to real contributors. Moreover, this assignment is easy to apply at the initial stages of the implementation of the proposed framework. Further, trivial assignment works well with all the *persons* having equal contribution in a *work*.

3. *Weighted Assignment*: A number of weighted assignments can be proposed. These weighted assignments can be discussed at the level of scientific community. Following are general guidelines for the weighted assignments:
    a. *Persons* involved may reach a consensus over Participation Fraction of all.
    b. Participation Fraction can be based upon the quantum of the impact of *persons* in the topic *work*.

c. If there are more than two *persons* the sum of Participation Fractions of first and second *persons* may be greater than or equal to 0.5.

d. Participation Fraction of any other *person* involved is always less than the Participation Fraction of the first and second *persons*.

e. Special recommendations can be made for *works* produced in the partial fulfillment of an academic degree.

f. Some probability distribution can be devised by keeping in view the opinions of established *persons*. For example assuming that in an ordered list of *persons* who are the part of a *work*, each subsequent *person* is picked by the preceding *person* for any reason; and accordingly the preceding person shares half of its Participation Fraction. The distribution of Participation Fraction f_{pi} of a *person* at index *p* of an ordered list of *P persons* who are part of a *work i* is given by

$$f_{pi} = \begin{cases} 1/2^{P-1} & p = P \\ 1/2^{p} & p = 1,2,\ldots,P-1 \quad for \quad P > 1 \end{cases} \tag{6.17}$$

Based on the Participation Fraction of *persons* the Scientific Impact of a Person is defined next.

Definition 12: Scientific Impact of a Person. Scientific Impact of a *person p* is denoted as $TP_p(t)$ and defined as

$$TP_p(t) = \sum_{p \lhd i} f_{pi} TW_i(t) \tag{6.18}$$ ■

Remark 9: The Scientific Impact of a Person changes upon participation of a new *work* or upon change in the Impacts of Works that the person has involved. It follows from the above definition that Scientific Impact of a Person directly depends upon the quantum of participation made by the *person*, the Direct and Indirect Impacts upon works. Moreover, it quantifies the seminal effect of a *person* on research activity. ■

6.2.5 Freshness of Entities

In whole of the above development the Current Scientific Impact of a Work (Definition 7) and given by (6.10) plays the pivotal role in whole of the framework defined in this paper. All the impacts defied in this paper are directly or indirectly depend upon this entity. The Current Scientific Impact of a Work adds up all the impact made by a *work* at any time; thus, it does not count for the freshness of the impact made by the *work*. The following definition is made to take care of the freshness in the impact made.

Definition 33: Fresh Scientific Impact of a Work. The fresh scientific Impact of a work at some time *t* (in years) is written as $FTW_i(t)$ and defined as

$$FTW_i(t) = \sum_{j:j \lhd k} \frac{T_{ij} . \eta_{jk}}{t_j - t + 1} \tag{6.19}$$

in this equation t_j is the time (in year) when the *work j* was made part of the *object k*.■

Involving $FTW_i(t)$ in all the above definitions gives the freshness versions of all these impacts defined earlier. These impacts are also part of the framework; further details are omitted to contain the length of this paper.

6.2.6 Implementation and Scalability of the Proposed Framework

Implementation of the proposed system is possible with the help of the existing data base. It is proposed that at the first stage of the implementation trivial assignments are made to the Direct Impact Factors (Definition 1) and Participation Fractions (Definition 11). With the availability of the citation database these trivial assignments can be automated, thereby incurring little cost. This first step of the implementation will setup the entities of *works*, *objects*, *persons* and institutes. In the next step of implementations the governing bodies of the established *objects* will be requested to make the nontrivial assignments to the Direct Impact Factors (Definition 1) and Participation Fractions (Definition 11) during their review processes.

6.3 Results and Conclusion

A comprehensive and seamless framework for the impact factor propagation and computation of impacts made by scientific works produced is elaborated in the last section. The proposed framework takes a work as the fundamental entity and then assigns the value of impacts made by other entities including journals, scientists, and institutions. This section establishes the superiority of the proposed system by presenting examples, simulation results and mathematical modeling.

6.3.1 Comparison of the Proposed System of Impacts with the Existing Systems

In this subsection comparison of the existing systems of indexing, impacts and factors is made with the proposed system. The most distinctive feature of the proposed framework is that it provides an integrated and seamless system to measure the impact on research activity of every entity involved including journals, conference proceedings, articles, papers, thesis, books, authors, coauthors, presenters, mentors, universities, funding agencies and others.

The effectiveness of the proposed framework in addressing the issues like measuring the impact of a scientist, excessive self-citations, and gifted authorship, is evident from the structure of the framework by itself. To further emphasize the significance of the propose system and its superiority over the existing systems, consider a segment of the system of works in Figure 6.4; here all the nodes are *works* with triangle, oval, rectangle, and hexagon represent *works* significant in this example. It is evident that in the existing system, both triangle and over will contribute exactly same in, ISI index, H-Index, number of citations etc. From the Figure 6.4 it is clear that oval has much more impact to the scientific activity, since it has seminal effect on it, and the proposed system takes care of this seminal effect assigns more worth to the oval *work* than triangle *work*. Hexagon is a typical case of a survey *work*, in the current system it is given much more worth than any other *work* in the Figure 6.4, but it is evident that the oval one has more significance, which is duly assigned by the proposed system. Similarly the *work* represented as a rectangle is superior to the oval one but its superiority cannot be judged by the present system.

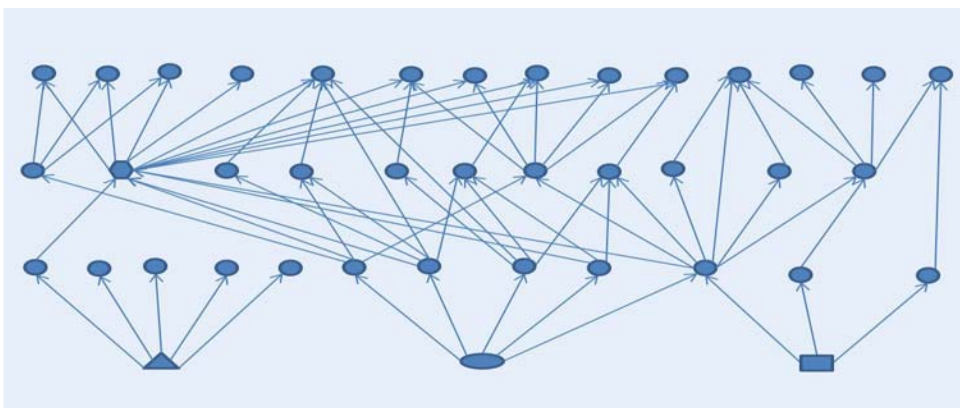
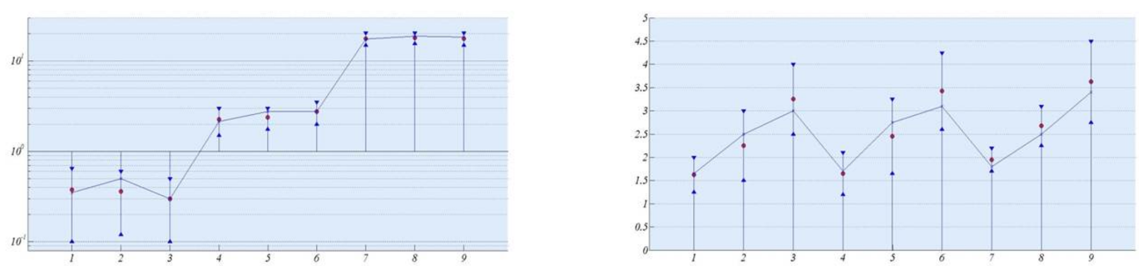


Figure 6.4: A section of the systems of works.

It is also worth noting that equation (6.13), which computes the impact made by an *object* corrects the discrepancy of denominator which favors the publishing of reviews, survey articles, and other non-scientific works. The numerical values of the entities involved in the proposed framework and their significance is also apparent from the dynamics of the underlying algorithm and illustrated examples, but may need further insight about the extent of the proposed system in dealing with the cross discipline distribution of impact. To provide this insight, the citation system is simulated for 27 *objects*. The citations of the *works* in these are simulated according to the power law distribution, as proposed in [202]. In all 27 *objects* are simulated in three categories, the first nine simulated *objects* are pure theory *objects* having typical trend of having 3-4 citations in a *work*, the next nine simulated *objects* are applied theory *objects* having typical trend of having 15-25 citations in a *work*, and the last nine simulated *objects* are application science *objects* having typical trend of having 30-60 citations in a *work*; the power law distributions of these different types of *objects* are tuned to give typical Impact Factor values in the present system; the Impact Factors of the nine representative simulated *objects*, three from each category, in the current citation system are shown in Figure 6.5 (a); first three of them are pure theory *objects*, the next three are applied theory *objects*, and the last three are application science *objects*. The impact factors of these *objects* are calculated by using the current method being used by ISI as explained in the Section 6.1. The proposed framework is also simulated for the same *objects* and the seminal impacts made by the *objects* are calculated by using (6.13); the simulation results are shown in Figure 6.5 (b).



(a) (b)

Figure 6.5: Scientific Impacts of various simulated *objects* in the (a) current system and in the (b) proposed system. The first three simulated *objects* are pure theory *objects* having typical trend of having 3-4 citations in a *work* but are cited in the applied *work*, the next three simulated *objects* are Applied Theory *objects* having typical trend of having 15-25 citations in a *work* but are cited in pure application science *works*, and the last three simulated *objects* are application science *objects* having typical trend of having 30-60 citations in a *work*. The figure shows

minimum, maximum and mean of the scientific impacts made by various *objects*; some instances of the impacts made by *objects* are shown in line graphs. Graph in (a) shows that in the current system of calculating scientific impact of an *object*, those *objects* having a trend of fewer number of citations in a *work* get a severe penalty in the form of their lower scientific impacts; while graph is (b) shows that the *objects* containing *works* with minimal value of direct impact factor has minimal value of the scientific impact made, *objects* containing *works* with moderate value of direct impact factor has moderate value of the scientific impact made, and *objects* containing *works* with maximal value of direct impact factor has maximal value of the scientific impact made.

The simulation results in Figure 6.5 (b) show that there is a significant effect of indirect impact made by the *works* in an *object* on the impact made by it. The first object in each category is the one containing *works* with small value of direct impact factors, the second object in each category is the one containing *works* with moderate direct impact factors and the last one containing *works* with high value of direct impact factors. The simulation results show that the first *object* in each category generally has minimal value of the scientific impact made, the second one has moderate vale of the scientific impact made and the third one has the maximal value of the scientific impact made. Moreover, a trend of having the excessive number of citations, generally do not result in excessive value of scientific impact made.

The example and simulation results presented in this section so far provide an insight to the superiority of the proposed framework over the existing system. The following theorem establishes the robustness of the proposed framework from game theoretic view point [203], when the persons and the anonymous referees are rational [203]. Moreover the referees are rationally picked, rationally picked referees mean that if a *work* is places in the system of works, the *work* has some nonzero impact on the referee.

Theorem 2: If all the persons and rationally picked anonymous referees are rational then the proposed system is immune against gifted authorship and against irrelevant citations.

Proof: Proof is in Appendix. ■

6.3.2 Conclusion

This paper provides a comprehensive system to measure the impact made by a *work* not only over the research activity but also to the human life. The seamless system presented in this paper takes care about a number of problems with the current system, under discussion in the scientific community; these include but not limited to: excessive self-citations, excessive publishing of non-scientific material, publishing of half cooked ideas, cross discipline normalization, seminal effect made by a *work*, and gifted authorship. The proposed system takes a research *work* as a basic unit of contribution and on its basis, calculates the impact made by the rest of the entities like scientists, institutes, universities, funding agencies, mentors etc. The system does not calculates the impact made by the *work* on the basis of its citations rather it computes the impact made by a *work* on the basis of its seminal effect on the research activity. The network based approach used in the proposed system does not change the representative graph of the system by adding teleporting arcs as done in present network based approaches [162], [164], in contrast, the proposed system does not add additional arcs to the network; furthermore, instead of using a random surf model as in [162], [164], the proposed system assigns the impact made by a *work i* on another *work j* on the basis of proportional contribution made by *work i* on *work j*. Moreover, the system assign's the impact made by the scientists on the basis of proportional contribution made and bars the nonscientific practices inculcated in the scientific activity. Though the system can be implemented in an autonomous way but there is lot of room in the system to take delicate inputs from the authors and the reviewers, making it more realistic and useful. By making *work* as the fundamental unit of contribution, and by allocating the impacts made by the rest of the entities involved on the basis of their contribution in the *works*, the proposed system eliminates many non-scientific practices in evaluation of scientists, and journals. The proposed system ends the number game, and promotes the true scientific activity by discouraging the gifted authorship and excessive self-citations, and by providing the advantage of the quantification of seminal impact made by a *work*.

Chapter 7

Future Work

This work studies various aspects of internal structure and dynamics decision involved in coalitions on graphs. Moreover, the work provides a machinery to explore the contributions made by individual agents within the coalition and cost associated with the agents joining the coalition. Furthermore, impact made by one agent over the other is also studied. There are five aspects of the contribution made in this work. These aspects are summarized in the next subsection and the futuristic expansion of this work is discussed in the further subsection.

7.1 Summary of this Work

First of all this work investigates the cooperation between agents to achieve a common goal. The problem of steering a swarm of autonomous agents out of an unknown maze to some goal located at an unknown location is discussed in this context. The routing algorithm given here provides a mechanism for storing data based on the experiences of previous agents visiting a node that results in routing decisions that improve with time. Algorithms are based on an underlying network of communicating nodes.

Next a certain graphical coalitional game is introduced, where the internal topology of the coalition depends on a prescribed communication graph structure among the agents. The graphical coalitional game satisfies basic properties of convexity, fairness, cohesiveness, and full cooperativeness. Three measures of the contributions of agents to a coalition are introduced: marginal contribution, competitive contribution, and altruistic contribution. Based on these different contributions, three online sequential decision games are defined on top of the graphical coalitional game. The stable graphs under each of these sequential decision games are studied, and give the structures of the coalitions that form in each sequential game.

A Positional Cost is also introduced next; the cost is assigned to each agent based on Shapley value and connectivity of the agent within the communication graph. Based on the advantage and cost, a notion of Net Payoff or Allocation is defined; this notion is used to further define three measures of net advantages. Taking maximization of these measures of net advantages as the objective functions of agents, three online sequential decision games are defined on top of the coalitional graph game. Stable graphs under each sequential decision game are studied by varying the cost, and certain results about the coalition structure are established. A threshold of cost is reached above which no agent is interested to stay in a coalition irrespective of their motives.

Next a certain graphical coalitional game on digraphs is introduced, where the internal topology of the coalition depends on a prescribed communication graph structure among the agents, when the flow of information can be unidirectional. Novel digraph structures, including semi-strongly connected digraphs and multi-chain, are defined. The marginal contribution made by an agent in a digraph is introduced. Based on marginal contributions, and by varying the rules of the game, three online sequential decision games are defined on top of the graphical coalitional game. The stable graphs under each of these sequential decision games are studied, and give the structures of the coalitions that form in each sequential game. It is shown that the stable graphs under these games are semi-strongly connected digraphs, multi-chain, and chain of command.

Finally the notion of impact of one agent in the coalition upon the other agents is investigated, and a complete impact propagation framework is proposed. The analogy of impact propagation is use to present a seamless, comprehensive, and integrated framework to compute the impact made by a scientific work, a scientist, an institution, a scientific journal, and a funding agency. The framework provides solutions to a number of problems about the existing indexing systems, under discussion, in the scientific community. These problems include but are not limited to the allocation of impact to scientists, excessive self-citation, gifted authorship, dependence of indices on disciplines, and excessive citations. The proposed system has complete provision of peer input to cover delicate issues in impact calculations, yet the system can be employed fully automated during the earlier implementation stages.

7.2 Applications and Future Expansions of the Work

The work can be applied to various areas of study and there is a lot of room to expand all the five aspects of this work in multiple ways. Owing to immense diversity in the fields of study, these applications and expansions cannot be fully comprehended at this stage, yet some very immediate ones of them are mentioned here.

7.2.1 Routing Algorithm without Communication Network

The routing algorithm devised in Chapter 1, computes the shortest path in an unknown graph to an unknown goal. The working of the algorithm is based on an underlying communication network. In future this part of the work can be expanded to devise a routing algorithm independent of the communication network.

The algorithm in this work, and the proposed expansion can be used in various engineering applications, including cyber cooperation, and information hunting over the Internet [1], [2], [3], [4], [5], [6], [7], [8]. [9], [10], [11], [12].

7.2.2 Graphical Coalition Game with Positional Advantage

The graphical coalition game proposed in this work can be expanded in variety of ways. First of all the machinery developed in this work can be further developed mathematically.

Certain mathematical results can be obtained connecting the graph theoretic parameters with GCG and contribution of an agent within a coalition.

It seems to be tractable to work out a distributed approach to compute the positional advantage of an agent. Advent of such an algorithm has a potential to revolutionaries the techniques involving the assignment of value to the network nodes [78], [149], [155].

The graphical coalitional game and the sequential decision games proposed in this paper can be used in a variety of ways in problems involving situations of simultaneous competition and collaboration among anonymous agents. The GCG with Positional Advantage can be used to determine the social standing of various kinds of agents purely on the basis of the communication structure. GCG also distinguishes between the events of making a communication link for self-interest and for the coalition's sake [71], [75], [124], [126]. The development in Section 3.4 can be used to determine the strategic importance of graph points. The development in this paper can also be used to understand the notions of competition and cooperation in groups of biological species [90], [92], [103], [106], [133], [141]. The sequential decision games in the Section 3.5 can be used to understand the internal structure of a coalition based on the notions of competition and altruism. Situations in economics, communication, and swarm control are very complex; here a lot of agents interact in situations of simultaneous competition and cooperation. The theory developed in this paper can be used to understand complex situations of joint competition and cooperation [74], [81], [82], [111], [117], [123], [125], [131], [140].

7.2.3 GCG on Digraphs and Cooperative Control theory

Novel digraph structures and GCG on digraphs is introduced in Chapter 5. This works can also be expanded mathematically to show the connection between digraph structures and other digraph parameters like, circuits, in-degree, out-degree etc. etc.

The GCG on digraphs, and the sequential decision games proposed in this paper can be used in a variety of ways in problems involving situations of simultaneous competition and collaboration among anonymous agents. The GCG on digraphs can be used to determine the social standing of various kinds of agents purely on the basis of the communication structure, where there are asymmetric relations between agents [71], [75], [124], [132]. The notion of marginal contribution made by the agent developed in Section 5.4 can be used to determine the comprehensive contribution of an agent to digraph structure [74]. The sequential decision games in the Section 5.5 can be used to understand the internal structure of a coalition based on the notions of marginal contribution. The novel digraph structures and theory of sequential decision games developed in this paper can be used in cooperative control theory [1], [2], [3], [7], [8], [79], [135].

7.2.4 Graphical Coalition Game and Fidler Eigen Value

It is known that Eigen Values of the graph Laplasian matrix plays a fundamental role in determining the rate of convergence to the consensus value of the coalition. In situations where agents converge to coalitions through local voting protocols, the graph structures like multi-chains, semi-strongly connected digraphs, chain of commands and trees are the key structures having minimal Fidler Eigen Values and maximal rate of convergence to the consensus [1], [2], [3], [7], [8], [79], [135]. The relation between positional advantage and Fidler Eigen values can be explored.

7.2.5 Graphical Coalition Game and Economics

Since the economic activity is also determined by how feasible it is for the agents to make coalitions, the graphical coalitional games introduced in this paper can be used to study the coalition formation in economic setup. In economic setup these games can be used to study the state of recession and study of measure to avoid it.

7.2.6 Impact Propagation Framework

The impact propagation framework introduced in Chapter 6 is ready to use to make a next generation tool to compute impact made by any entity in the research activity. The proposed framework is immune to many problems which are attributed to the current system.

Appendix A

Technical Lemmas for Chapter 3

In the following results symbols *k*, *n*, and *m* are integers.

Lemma A.1: The Positional Advantage of a node within a graph is the PA of the node within its connected component.

Proof : Let the vertex *i* is contained in the component G_n of the graph *G*. Further suppose that $G_m = G \setminus G_n$ is the rest of the graph not connected with *i*. Now every $S' \subseteq G \setminus \{i\}$ can be partitioned into S and S_m such that $S \subseteq G_n \setminus \{i\}$ and $S_m \subseteq G_m$. That is to say

$$\forall S' \subseteq G \setminus \{i\}, S' = S \cup S_m : S \subseteq G_n \setminus \{i\} \wedge S_m \subseteq G_m \wedge S \cap S_m = \varphi \tag{8.1}$$

Since S and S_m are disconnected, thus by using Axioms 1 and 2, this holds

$$v(S') = v(S \cup S_m) = v(S) + v(S_m) \tag{8.2}$$

Similarly this also holds

$$v(S' \cup \{i\}) = \quad v(S \cup \{i\} \cup S_m) = v(S \cup \{i\}) + v(S_m) \tag{8.3}$$

Using (8.2) and (8.3)

$$v(S' \cup \{i\}) - v(S') = v(S \cup \{i\}) - v(S) \tag{8.4}$$

Using Definition 3 the PA of node *i* within the graph *G* is given by

$$\varphi_G(i) = \frac{1}{|G|} \sum_{S' \subseteq G \setminus \{i\}} \frac{(v(S' \cup \{i\}) - v(S'))}{\binom{|G|-1}{|S'|}} \tag{8.5}$$

And the positional advantage of node *i* within its component G_n is given by

$$\varphi_{G_n}(i) = \frac{1}{|G_n|} \sum_{S \subseteq G_n \setminus \{i\}} \frac{(v(S \cup \{i\}) - v(S))}{\binom{|G_n|-1}{|S|}} \tag{8.6}$$

Using (8.4), (8.5) can be written as

$$\varphi_G(i) = \frac{1}{|G|} \sum_{\substack{S' \subseteq G \setminus \{i\} \\ S \subseteq G_n \setminus \{i\}}} \frac{(v(S \cup \{i\}) - v(S))}{\binom{|G|-1}{|S'|}} \tag{8.7}$$

Further, rearranging this by collecting all the terms involving $(v(S\cup\{i\})-v(S))$ for the same value of *S*, using (8.1) and the fact that $G=|G_n|+|G_m|$ gives

$$\varphi_G(i)=\frac{1}{(|G_n|+|G_m|)}\sum_{S\subseteq G_n\setminus\{i\}}\left(\sum_{k=0}^{|G_m|}\frac{\binom{|G_m|}{k}}{\binom{|G_m|+|G_n|-1}{|S|+k}}\right)\frac{(v(S\cup\{i\})-v(S))}{\binom{|G|-1}{|S|}} \tag{8.8}$$

Using mathematical induction on $|G_m|$ it can be established that

$$\frac{1}{(|G_n|+|G_m|)}\left(\sum_{k=0}^{|G_m|}\frac{\binom{|G_m|}{k}}{\binom{|G_m|+|G_n|-1}{|S|+k}}\right)=\frac{1}{|G_n|}\frac{1}{\binom{|G_n|-1}{|S|}} \tag{8.9}$$

Using this result in (8.8), the desired result that $\varphi_{G_n}(i)=\varphi_G(i)$ is obtained. ■

Lemma A.2: If a positive integer *n*, such that $0<n\le N$, is written as a sum of positive integers $n=n_1+n_2+...+n_m$ then $v_n\ge v_{n_1}+v_{n_2}+...+v_{n_m}$

Proof: Using the Axiom 3 these inequalities hold $n_i v_n\ge nv_{n_1}\forall i=1,2,...,m$. Summation of these inequalities for all values of *i* gives $\sum_{i=1}^{m}n_i v_n\ge\sum_{i=1}^{m}nv_{n_1}$ or $v_n\sum_{i=1}^{m}n_i\ge n\sum_{i=1}^{m}v_{n_1}$. Using the given condition that $n=n_1+n_2+...+n_m$, the desired result is obtained. ■

Lemma A.3: If $v_2>2v_1$ then $mv_{m+1}>(m+1)v_m$ with $0<m\le N-1$.

Proof: The result is true for $m=1$. Making the induction hypothesis that $kv_{k+1}>(k+1)v_k$. Using the Axiom 4 $v_{k+2}-v_{k+1}\ge v_{k+1}-v_k$. Using the induction hypothesis in this inequality proves the desired result. ■

Lemma A.4: If *m*, *n* and *k* satisfy $0\le n<m\le N-k, k>0$ then $v_{m+k}-v_m\ge v_{n+k}-v_n$.

Proof: Using the Axiom 4 these inequalities hold $v_{m+k} - v_{m+k-1} \geq v_{n+k} - v_{n+k-1}$, $v_{m+k-1} - v_{m+k-2} \geq v_{n+k-1} - v_{n+k-2}, \ldots,$ $v_{m+1} - v_m \geq v_{n+1} - v_n$. Addition of these inequalities gives the desired result. ■

Appendix B

Proofs of Lemmas in Chapter 3

Proof of Lemma 2: Let G is a graph with the nodes i and j not neighbors of each other. By the definition of Positional Advantage, Definition 3, and the underlying game Definition 1, this can be written

$$\varphi_{G'}(i)-\varphi_G(i)=\frac{1}{|G'|}\sum_{S'\subseteq G'\setminus\{i\}}\frac{(v(S'\cup\{i\})-v(S'))}{\binom{|G'|-1}{|S'|}}-\frac{1}{|G|}\sum_{S\subseteq G\setminus\{i\}}\frac{(v(S\cup\{i\})-v(S))}{\binom{|G|-1}{|S|}}$$

For each $S'\subseteq G'\setminus\{i\}$, not containing j, an identical $S\subseteq G\setminus\{i\}$ exists. Thus above equation be written as

$$\varphi_{G'}(i)-\varphi_G(i)=\frac{1}{|G'|}\sum_{\substack{S'\subseteq G'\setminus\{i\}\\ j\in S'}}\frac{(v(S'\cup\{i\})-v(S'))}{\binom{|G'|-1}{|S'|}}-\frac{1}{|G|}\sum_{\substack{S\subseteq G\setminus\{i\}\\ j\in S}}\frac{(v(S\cup\{i\})-v(S))}{\binom{|G|-1}{|S|}}$$

Since $i\notin S,S'$, thus for each $S'\subseteq G'\setminus\{i\}$, an identical $S\subseteq G\setminus\{i\}$ exists. Above equation can be written

$$\varphi_{G'}(i)-\varphi_G(i)=\frac{1}{|G'|}\sum_{\substack{S\subseteq G'\setminus\{i\}\\ j\in S'}}\frac{v(S'\cup\{i\})}{\binom{|G'|-1}{|S'|}}-\frac{1}{|G|}\sum_{\substack{S\subseteq G\setminus\{i\}\\ j\in S}}\frac{v(S\cup\{i\})}{\binom{|G|-1}{|S|}}$$

Since $|G'|=|G|$, above equation can be written as

$$\varphi_{G'}(i)-\varphi_G(i)=\frac{1}{|G|}\sum_{S'\subseteq G'\setminus\{\{i\}\cup\{j\}\}}\frac{v(S'\cup\{i\}\cup\{j\})}{\binom{|G|-1}{|S'|+1}}-\frac{1}{|G|}\sum_{S\subseteq G\setminus\{\{i\}\cup\{j\}\}}\frac{v(S\cup\{i\}\cup\{j\})}{\binom{|G|-1}{|S|+1}}$$

In this equation $S'\cup\{i\}\cup\{j\}\subseteq G'$, $S\cup\{i\}\cup\{j\}\subseteq G$. For each S' an S can be found with identical vertex set, and for such pair of S and S', according to Lemma A.2, $v(S'\cup\{i\}\cup\{j\})\geq v(S\cup\{i\}\cup\{j\})$. Thus $\varphi_{G'}(i)-\varphi_G(i)$ is nonnegative. ■

Proof of Lemma 3: It is established in the proof of Lemma 2 that $\varphi_{G'}(i)-\varphi_G(i)$ in non-negative and is

$$\varphi_{G'}(i)-\varphi_G(i)=\frac{1}{|G|}\sum_{\substack{S'\subseteq G'\setminus\{\{i\}\cup\{j\}\}\\ S\subseteq G\setminus\{\{i\}\cup\{j\}\}}}\frac{v(S'\cup\{i\}\cup\{j\})-v(S\cup\{i\}\cup\{j\})}{\binom{|G|-1}{|S'|+1}}$$

In this equation both S and S' have the identical vertex set but S is an induced subgraph of G and S' is an induced subgraph of G'. According to Lemma A.2, $v(S' \cup \{i\} \cup \{j\}) \geq v(S \cup \{i\} \cup \{j\})$, for all S and S' which implies that $\varphi_{G'}(i) - \varphi_G(i)$ is nonnegative in general. Now for $S' = \phi = S$, $v(S' \cup \{i\} \cup \{j\}) = v_2$ and $v(S \cup \{i\} \cup \{j\}) = 2v_1$, thus under the given condition $v_2 > 2v_1$, $v(S' \cup \{i\} \cup \{j\}) > v(S \cup \{i\} \cup \{j\})$ which implies that $\varphi_{G'}(i) - \varphi_G(i)$ is positive. ■

Proof of Lemma 4: Since the vertices i and j are already connected in G then by Axiom 2 sum of the PAs of the coalition $v(G)$ does not change with the making of the new edge $\{i, j\}$. By Theorem 1, making the edge $\{i, j\}$ changes the PAs of both the end vertices by the same non-negative value. ■

Proof of Lemma 6: Consider a node k in a disconnected graph G with agents i and j in distinct components. Let G' be the graph obtained by G by making the edge $\{i, j\}$. Thus

$$\varphi_{G'}(k) - \varphi_G(k) = \frac{1}{|G'|} \sum_{S' \subseteq G' \setminus \{k\}} \frac{(v(S' \cup \{k\}) - v(S'))}{\binom{|G'|-1}{|S'|}} - \frac{1}{|G|} \sum_{S \subseteq G \setminus \{k\}} \frac{(v(S \cup \{k\}) - v(S))}{\binom{|G|-1}{|S|}} \tag{9.1}$$

For each $S' \subseteq G' \setminus \{k\}$, not containing i or j, an identical $S \subseteq G \setminus \{k\}$ exists. Thus above equation can be written as

$$\begin{aligned} &\varphi_{G'}(k) - \varphi_G(k) = \\ &\frac{1}{|G'|} \sum_{S' \subseteq G' \setminus \{k\} \cup \{i\} \cup \{j\}} \frac{(v(S' \cup \{k\} \cup \{i\} \cup \{j\}) - v(S') \cup \{i\} \cup \{j\})}{\binom{|G'|-1}{|S'|+2}} \\ &- \frac{1}{|G|} \sum_{S \subseteq G \setminus \{k\} \cup \{i\} \cup \{j\}} \frac{(v(S \cup \{k\} \cup \{i\} \cup \{j\}) - v(S \cup \{i\} \cup \{j\}))}{\binom{|G|-1}{|S|+2}} \end{aligned} \tag{9.2}$$

If k is connected to i in the original graph G then by using the Axiom 2 and the fact that $|G'| = |G|$, above equation can be written as

$$\varphi_{G'}(k) - \varphi_G(k) = \frac{1}{|G|} \sum_{S' \subseteq G' \setminus \{\{k\} \cup \{i\} \cup \{j\}\}} \frac{(v(S' \cup \{k\} \cup \{i\} \cup \{j\}) - v(S') \cup \{i\} \cup \{j\})}{\binom{|G|-1}{|S'|+2}} - \frac{1}{|G|} \sum_{S \subseteq G \setminus \{\{k\} \cup \{i\} \cup \{j\}\}} \frac{(v(S \cup \{k\} \cup \{i\}) - v(S \cup \{i\}))}{\binom{|G|-1}{|S|+2}}$$

Using Axiom 4 it follows that the right hand side of this equation is always nonnegative, that is to say $\varphi_{G'}(k)-\varphi_{G}(k)\geq 0$. It can be seen that (9.2) is symmetric with respect to *i* and *j*, thus the result also holds true for those agents *k* connected with agent *j* in the original graph *G*. ■

Proof of Lemma 8: From (3.25), the competitive contribution of the agent *i* is given by

$$c_G(i)=\sum\nolimits_{j\in G}\varphi_G(j)-\sum\nolimits_{j\in G\setminus i}\varphi_G(j)$$

or $c_G(i)=\varphi_G(i)$. This is the desired result. ■

Proof of Lemma 9: The marginal contribution $m_G(i)$ for an agent *i* within a connected graph *G* is given by using (3.27). For a connected graph *G* the first term in the right hand side of (3.27) is constant, thus $m_G(i)$ is minimum when the second term $\sum_{i=1}^{p} v_{k_i}$ in the right hand side of this equation is maximum, which under the given condition in (3.27) and Lemma A.2 is $v_{|G|-1}$ when the agent *i* is not a cut vertex. This minimum marginal contribution is given by

$$m_G(i)=v_{|G|}-v_{|G|-1} \tag{9.3}$$

This minimum value is independent of the structure of *G* and only depends upon $|G|$. ■

Proof of Lemma 10: For an agent *i* within a connected graph *G* the marginal contribution $m_G(i)$ is given by (3.27). For a connected graph *G* the first term in the right hand side of this equation is constant, thus $m_G(i)$ is maximum when $\sum_{j=1}^{p} v_{k_j}$ is minimum, which by Lemma A.2, under the given condition in (3.27) is $(|G|-1)v_1$, which is possible only when removal of agent *i* from *G* leaves rest of the agents isolated. ■

Proof of Lemma 11: For an agent *i* within a graph *G* the altruistic contribution $a_G(i)$ by using (3.26) is

$$a_G(i) = \sum_{j \in G \setminus i} \varphi_G(j) - \sum_{j \in G \setminus i} \varphi_{G \setminus i}(j)$$

Since the agent *i* is isolated, using Lemma A.1, $\varphi_G(j) = \varphi_{G \setminus i}(j) \forall j \neq i$. ■

Proof of Lemma 12: The altruistic contribution of an agent *i* is given by (3.26). Let the agent *i* is not isolated and G'' is a graph obtained from *G* by removing the entire edges incident at the agent *i*. By using (3.26) the altruistic contribution of the agent *i* can be written as

$$a_G(i) = \sum_{j \in G \setminus i} (\varphi_G(j) - \varphi_{G''}(j)) \tag{9.4}$$

Let the altruistic contribution of the agent *i* is 0. By Lemma 6, $(\varphi_G(j) - \varphi_{G''}(j))$ is nonnegative, thus

$$\varphi_G(j) - \varphi_{G''}(j) = 0 \forall j \neq i \tag{9.5}$$

Using Definition 3 above equation can be written as

$$\frac{1}{|G|} \sum_{S \subseteq G \setminus \{j\}} \frac{(v(S \cup \{j\}) - v(S))}{\binom{|G|-1}{|S|}} - \frac{1}{|G''|} \sum_{S' \subseteq G'' \setminus \{j\}} \frac{(v(S' \cup \{j\}) - v(S'))}{\binom{|G''|-1}{|S'|}} = 0 \forall j \neq i \tag{9.6}$$

Since both *G* and G'' share the same vertex set, thus this can be written as

$$\frac{1}{|G|} \sum_{\substack{S \subseteq G \setminus \{\{j\} \cup \{i\}\} \\ S' \subseteq G'' \setminus \{\{j\} \cup \{i\}\}}} \frac{(v(S \cup \{j\} \cup \{i\}) - v(S \cup \{i\})) - (v(S' \cup \{j\}) - v(S'))}{\binom{|G|-1}{|S|+1}} = 0 \forall j \neq i \tag{9.7}$$

Here *S* and S' have the same vertex set. Using Axiom 2 all the terms within the summation are non-negative with, under the given condition, at least one term is positive for $|S| = 0 = |S'|$ and *j* some neighbor of *i*. This proves the desired result. ■

Appendix C

Proofs of Lemmas in Chapter 4

Lemma C.1: For a valid game list $v = (v_1, v_2, ...v_N)$, $k(v_{k+1} - v_k) \geq v_{k+1} - v_1 \forall 1 \leq k < N$.

Proof: Proof follows from Axiom I.4. ■

Lemma C.2: For a valid game list $v = (v_1, v_2, ...v_N)$, $\frac{1}{N}\sum_{i=1}^{N} v_i \leq \frac{v_N + v_1}{2}$

Proof: Proof follows by induction on *N* and by using Lemma C.1. ■

Lemma C.3: For a valid game list $v = (v_1, v_2, ...v_N)$ with $v_2 > 2v_1$, $v_n - \sum_{i=1}^{p} v_{j_i} > \frac{v_n}{n} : \sum_{i=1}^{p} j_i = n-1$.

Proof: From Remark 4 of [77] $\min(v_n - \sum_{i=1}^{p} v_{j_i}) = v_n - v_{n-1}$ and it follows from Axiom I.4 that under condition (4.2) $v_n - v_{n-1} > \frac{v_n}{n}$, the desired result follows. ■

Lemma C.4: For a GCG with Positional Cost $\Upsilon = (G, u)$ if *u* satisfies (4.5), $u_2 > 0$, then for $N \geq m > n \geq 0$ then $n.u_m > m.u_n$.

Proof: The proof of this Lemma follows from Lemma 5 of [77]. ■

Lemma C.5: If $v = (v_1, v_2, ...v_N)$ is a Valid Game List for a GCG with PA then for u_i in (4.8) such that $u = (u_1, u_2, ...u_N : u_i = k_i(v_i - iv_1), k_{m+1} - k_m \geq k_{n+1} - k_n : N \geq m > n \geq 0)$ is a Valid Game List for a GCG with PC.

Proof: The proof follows by establishing that *u* satisfies Axiom II.1 and II.4. From the given value of *u*,

$$u_1 = k_1(v_1 - 1v_1) = 0 \tag{10.1}$$

This is according to Axiom II.1. For any integers *n* and *m*, such that $N - 1 \geq m > n \geq 0$, then

$$u_{m+1} - u_m = k_{m+1}(u'_{m+1}) - k_m(u'_m) \tag{10.2}$$

where $u'_i = v_i - iv_1, i = m, m+1$, and it can be written as

$$u_{m+1} - u_m = k_{m+1}(u'_{m+1} - u'_m) + (k_{m+1} - k_m)u'_m \tag{10.3}$$

Similarly

$$u_{n+1} - u_n = k_{m+1}(u'_{n+1} - u'_n) + (k_{n+1} - k_n)u'_n \tag{10.4}$$

It is established in Lemma 1 that $u' = (u'_1, u'_2, \ldots u'_N : u'_i = v_i - iv_1, \forall i = 1,2,3\ldots,N)$ is a Valid Game List for a GCG with PC. Moreover under the given condition that $k_{m+1} - k_m \geq k_{n+1} - k_n : N \geq m > n \geq 0$, from (10.3) and (10.4)

$$u_{m+1} - u_m \geq u_{n+1} - u_n \tag{10.5}$$

Thus *u* being satisfying both the Axioms II.1 and II.4, is a Valid Game List. ■

Lemma C.6: The PC of an edge given by Definition 5 is always non-negative.

Proof: The PC of an edge *e* in a graph *G* for a GCG with PC $\Upsilon = (G,u)$ is given by (4.6)

$$\psi_{G,u}(e) = \frac{1}{\|G\|} \sum_{S \subseteq G \setminus e} \frac{(u(S \cup e) - u(S))}{\binom{\|G\|-1}{\|S\|}} \tag{10.6}$$

By Axiom II it follows that $u(S \cup e) - u(S) \geq 0$, this proves the desired result. ■

Lemma C.7: The PC of an edge *e* in a graph *G* is its PC in its connected component.

Proof: The Proof is similar to the proof of Lemma 1 of [77]. ■

Lemma C.8: The PC of an edge of and agent *i* in a graph *G* is its PC in its connected component.

Proof: The proof follows from the above lemma and (4.7). ■

Lemma C.9: If a graph G' is obtained from a graph *G* by adding a new edge e' in it then change in the PC of some edge $e \in G$ is given by

$$\psi_{G',u}(e)-\psi_{G,u}(e)=\frac{1}{\|G\|+1}.\sum_{S\subseteq G\setminus e}\frac{(u(S\cup e'\cup e)-u(S\cup e'))-(u(S\cup e)-u(S))}{\binom{\|G\|}{\|S\|+1}} \tag{10.7}$$

Proof: The PC of an edge e in a graph G for a GCG with PC $\Upsilon=(G,u)$ is given by (4.6)

$$\psi_{G,u}(e)=\frac{1}{\|G\|}\sum_{S\subseteq G\setminus e}\frac{(u(S\cup e)-u(S))}{\binom{\|G\|-1}{\|S\|}} \tag{10.8}$$

Similarly the PC of e in the graph G' for the GCG with PC Cost $\Upsilon=(G',u)$ is given by

$$\psi_{G',u}(e)=\frac{1}{\|G'\|}\sum_{S\subseteq G'\setminus e}\frac{(u(S\cup e)-u(S))}{\binom{\|G'\|-1}{\|S\|}} \tag{10.9}$$

Since $\|G'\|=\|G\|+1$, the above equation can be written as

$$\psi_{G',u}(e)=\frac{1}{\|G\|+1}\sum_{S\subseteq G'\setminus e}\frac{(u(S\cup e)-u(S))}{\binom{\|G\|}{\|S\|}} \tag{10.10}$$

or

$$\psi_{G',u}(e)=\frac{1}{\|G\|+1}\sum_{\substack{S\subseteq G'\setminus e\\ e'\in S}}\frac{(u(S\cup e)-u(S))}{\binom{\|G\|}{\|S\|}}+\frac{1}{\|G\|+1}\sum_{\substack{S\subseteq G'\setminus e\\ e'\notin S}}\frac{(u(S\cup e)-u(S))}{\binom{\|G\|}{\|S\|}} \tag{10.11}$$

or

$$\psi_{G',u}(e)=\frac{1}{\|G\|+1}(\sum_{S\subseteq G\setminus e}\frac{(u(S\cup e'\cup e)-u(S\cup e'))}{\binom{\|G\|}{\|S\|+1}}+\sum_{S\subseteq G\setminus e}\frac{(u(S\cup e)-u(S))}{\binom{\|G\|}{\|S\|}}) \tag{10.12}$$

From equations (10.8), (10.12) and by using the identity

$$\frac{1}{(\|G\|)\binom{\|G\|-1}{\|S\|}}-\frac{1}{(|G\|+1)\binom{\|G\|}{\|S\|}}=\frac{1}{(|G\|+1)\binom{\|G\|}{\|S\|+1}} \tag{10.13}$$

the following equation can be written

$$\psi_{G',u}(e)-\psi_{G,u}(e)=\frac{1}{\|G\|+1}\sum_{S\subseteq G\setminus e}\frac{(u(S\cup e'\cup e)-u(S\cup e'))-(u(S\cup e)-u(S))}{\binom{\|G\|}{\|S\|+1}} \tag{10.14}$$

This is the desired result. ■

Lemma C.10: In a GCG with PC $\Upsilon = (G,u)$, where G is a connected graph, if a graph G' is obtained from G by adding a new edge e' in it then sum of the changes in the PC of all edges $e \in G$ is equal to the PC of the new edge e'.

$$\psi_{G',u}(e') = \sum\nolimits_{e\in G}(\psi_{G',u}(e) - \psi_{G,u}(e)) \tag{10.15}$$

Proof: Since G is a connected graph thus by the Axiom II.1

$$u(G') = u(G) \tag{10.16}$$

Moreover Shapley value is efficient, thus using (4.6) the above equation can be written as

$$\sum\nolimits_{e\in G'}\psi_{G',u}(e) = \sum\nolimits_{e\in G}\psi_{G,u}(e) \tag{10.17}$$

Rearrangement of the above equation gives

$$\psi_{G',u}(e') + \sum\nolimits_{e\in G}\psi_{G',u}(e) = \sum\nolimits_{e\in G}\psi_{G,u}(e) \tag{10.18}$$

The desired result follows from the above equation. ■

Lemma C.11: In Graphical Advantage and Cost Games $\Sigma = (G,v,u)$ and $\Sigma = (G''',v,u)$, where G''' is obtained from G by disconnecting a part of the connected component of a cut vertex i, if $u_i = k(v_i - iv_1) : i = 1,2,...,N, 0 < k \le 1$ then in the graph G the Net Marginal Advantage of the agent i is at least the NMA of the agent i in G'''.

Proof: Let the agent i exists in a connected component of size n within the graph G; NMA of the agent i in this graph G is given by (4.67)

$$\varpi_{G,v,u}(i) = (1-k)(v_n - \sum_{i=1}^{p} v_{k_i}) + kv_1 : \sum_{i=1}^{p} k_i = n-1 \tag{10.19}$$

Supposing that the agent i exists in a connected component of size n''' within the graph G'''; NMA of the agent i in this graph G''' is similarly given by

$$\varpi_{G,v,u}(i) = (1-k)(v_{n'''} - \sum_{i=1}^{p} v_{k_i}) + kv_1 : \sum_{i=1}^{p} k_i = n''' - 1 \tag{10.20}$$

Moreover, by hypothesis $n''' < n$ and by the construction of G''' the above equation can be written as

$$\varpi_{G,v,u}(i) = (1-k)(v_{n'''} + v_{n_1} - \sum_{i=1}^{p} v_{k_i}) + kv_1 : \sum_{i=1}^{p} k_i = n-1, n''' + n_1 = n \tag{10.21}$$

where n_1 is the size of the component disconnected from *i* in the hypothesis. By Lemma 4 of [77] $v_n \geq v_{n'''} + v_{n_1}$, thus comparison of (10.19) and (10.21) gives the desired result. ■

Lemma C.12: In GACGs $\Sigma = (G, v, u)$ and $\Sigma = (G''', v, u)$, where G''' is obtained from *G* by disconnecting a part of the connected component of a cut vertex *i*, if $u_i = k(v_i - iv_1) : i = 1, 2, ..., N, 0 < k \leq 1$ and $v_2 > 2v_1$ then in the graph *G* the NMA of the agent *i* is more than the NMA of the agent *i* in G'''.

Proof: The proof is similar to the proof of the above lemma, except that by Remark 4 of [77], the given condition $v_2 > 2v_1$ implies that $v_n > v_{n'''} + v_{n_1}$. This establishes the desired strict inequality. ■

Appendix D

Proofs of Lemmas in Chapter 5

In Section A1 are given some technical lemmas required for the proofs of results in the paper. In Section A2 are given the proofs of some of the results in the paper. In the following results symbols *k*, *l*, *n*, and *m* are integers.

A1 Technical Lemmas

Lemma D 1: For a GCG on digraph, $\Gamma = (D, v)$, Axiom 5 implies Axiom 4.

Proof: Axiom 5 says that for $|D|-1 \geq m \geq 0$

$$v_{m+1} \geq 2v_m - (m-1)v_1 \tag{11.1}$$

For positive integers *n* and *k* such that $m = n + k$, *k* times recursive substitution of (11.1) yields

$$v_m \geq 2^k v_{m-k} - (2^{k-1}(m-k-1) + 2^{k-2}(m-k) + ... + 2^1(m-3) + (m-2))v_1 \tag{11.2}$$

or

$$v_m \geq 2^{m-n} v_n - (n2^{m-n} - m)v_1 \tag{11.3}$$

Substituting $n = 0$ and using (3.1), yields

$$v_m \geq mv_1 \tag{11.4}$$

From (11.3), this can be written

$$nv_m \geq n2^{m-n} v_n - n(n2^{m-n} - m)v_1 \tag{11.5}$$

By using the inequality $(n2^{m-n} - m) \geq 0 \forall |D| \geq m > n \geq 1$, and (11.4), this can be written as.

$$nv_m \geq n2^{m-n} v_n - (n2^{m-n} - m)v_n \tag{11.6}$$

or

$$nv_m \geq mv_n \tag{11.7}$$

For $n = 0$ this inequality trivially holds; this completes the proof. ■

Lemma D 2: For a GCG on digraph, $\Gamma = (D, v)$, Axiom 5 implies Axiom 4 of [77].

Proof: Axiom 5 says that for $|D| - 1 \geq m \geq 0$

$$v_{m+1} - v_m \geq v_m - (m-1)v_1 \tag{11.8}$$

Using Lemma D 1, $v_{m-1} \geq (m-1)v_1$, this gives

$$v_{m+1} - v_m \geq v_m - v_{m-1} \tag{11.9}$$

For positive integers *n* and *k* such that $m = n + k$, *k* times recursive substitution of (11.9) yields

$$v_{m+1} - v_m \geq v_n - v_{n-1} \tag{11.10}$$

This completes the proof. ■

Lemma D 3: For a GCG on digraph, $\Gamma = (D, v)$, if *S* is a subgraph of *D* then $|S| v_1 \leq v(S) \leq v_{|S|}$.

Proof: The result is established by using induction on $|S|$. By using the Axioms of value it is trivial to establish that the desired result holds for some small values $|S| = 2, 3$. Let the result is true for all digraphs with $|S| \leq m$.

Consider a subgraph S' with $|S'| = m + 1$. Clearly, by Axiom 4, the result holds true if S' is completely disconnected, or it is consisting of a single semi-strongly connected component. Similarly, by Remark 8.4, Axioms of Value, and Lemma A2 of [77], the result holds true. Suppose that S' consists of $k+1$ semi-strongly connected components, $S^{v_i} : i = 1, 2, \ldots, (k+1) = |P^S|, v_i \in P^S$, by using Axiom 3, it can be written as

$$v(S') = \sum_{l=1}^{k+1} (-1)^{l+1} \sum_{\substack{i_1 < i_2 < \ldots < i_l \\ v_{i_j} \in P^S}} v(\bigcap_{j=1}^{l} S^{v_{i_j}}) \tag{11.11}$$

It can be written as

$$v(S') = \sum_{l=1}^{k} (-1)^{l+1} \sum_{\substack{i_1 < i_2 < \ldots < i_l \\ v_{i_j} \in P^S - S^{v_{k+1}}}} v(\bigcap_{j=1}^{l} S^{v_{i_j}}) + v(S^{v_{k+1}}) - v((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}}) \tag{11.12}$$

or

$$v(S') = v(\bigcup_{l=1}^{k} S^{v_l}) + v(S^{v_{k+1}}) - v((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}}) \tag{11.13}$$

All the sets $\bigcup_{l=1}^{k} S^{v_l}, S^{v_{k+1}}, (\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}}$ are of the size less than or equal to *m*, thus by using the induction hypothesis, (11.13) can be written as

$$v(S') \geq v(\bigcup_{l=1}^{k} S^{v_l}) + v(S^{v_{k+1}}) - v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|} \tag{11.14}$$

By using Axioms of Value, and Lemma A2 of [77]

$$v(S^{v_{k+1}}) = v_{|S^{v_{k+1}}|} \geq v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|} + v_{(|S^{v_{k+1}}| - |(\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}}|)} \tag{11.15}$$

or

$$v(S^{v_{k+1}}) - v_{((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})} \geq v_{(|S^{v_{k+1}}| - |(\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}}|)} \tag{11.16}$$

By using Axiom 5, this can be written as

$$v(S^{v_{k+1}}) - v_{((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})} \geq (| S^{v_{k+1}} | - | (\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}} |) v_1 \tag{11.17}$$

and (11.14) becomes

$$v(S') \geq v(\bigcup_{l=1}^{k} S^{v_l}) + (| S^{v_{k+1}} | - | (\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}} |) v_1 \tag{11.18}$$

By induction hypothesis $v(\bigcup_{l=1}^{k} S^{v_l}) \geq |\bigcup_{l=1}^{k} S^{v_l} | v_1$ and (11.18) becomes

$$v(S') \geq (|\bigcup_{l=1}^{k} S^{v_l} | + (| S^{v_{k+1}} | - | (\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}} |) v_1 \tag{11.19}$$

By using the principal of counting this can be written as

$$v(S') \geq |\bigcup_{l=1}^{k+1} S^{v_l} | v_1 \tag{11.20}$$

or

$$v(S') \geq | S' | v_1 \tag{11.21}$$

This establishes one part of the desired inequality; for the other part of the desired inequality, consider (11.13)

$$v(S') = v(\bigcup_{l=1}^{k} S^{v_l}) + v(S^{v_{k+1}}) - v((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}}) \tag{11.22}$$

All the sets $\bigcup_{l=1}^{k} S^{v_l}, S^{v_{k+1}}, (\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}}$ are of the size less than or equal to *m*, thus by using the induction hypothesis, (11.22) can be written as

$$v(S') \leq v(\bigcup_{l=1}^{k} S^{v_l}) + v(S^{v_{k+1}}) - | ((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}}) | v_1 \tag{11.23}$$

or

$$v(S') \leq v_{|\bigcup_{l=1}^{k} S^{v_l}|} + v_{|S^{v_{k+1}}|} - | ((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}}) | v_1 \tag{11.24}$$

By Axiom 5

$$v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+2} - 2v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+1} \ge -|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|\, v_1 \tag{11.25}$$

Making this substitution in (11.24) yields

$$v(S') \le (v_{|\bigcup_{l=1}^{k} S^{v_l}|} - v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+1}) + (v_{|S^{v_{k+1}}|} - v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+1}) + v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+2} \tag{11.26}$$

Since $|S^{v_{k+1}}| \ge |((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})| + 1$, by using Lemma A4 of [77]

$$v_{|\bigcup_{l=1}^{k} S^{v_l}|+|S^{v_{k+1}}|-|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|-1} - v_{|S^{v_{k+1}}|} \ge v_{|\bigcup_{l=1}^{k} S^{v_l}|} - v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+1} \tag{11.27}$$

Using this in (11.26)

$$v(S') \le (v_{|\bigcup_{l=1}^{k} S^{v_l}|+|S^{v_{k+1}}|-|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|-1} - v_{|S^{v_{k+1}}|}) + (v_{|S^{v_{k+1}}|} - v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+1}) + v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+2} \tag{11.28}$$

or

$$v(S') \le (v_{|\bigcup_{l=1}^{k} S^{v_l}|+|S^{v_{k+1}}|-|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|-1} - v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+1}) + v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+2} \tag{11.29}$$

Again, by using Lemma A4 of [77]

$$v_{|\bigcup_{l=1}^{k} S^{v_l}|+|S^{v_{k+1}}|-|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|} - v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+2} \ge v_{|\bigcup_{l=1}^{k} S^{v_l}|+|S^{v_{k+1}}|-|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|-1} - v_{|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|+1} \tag{11.30}$$

and (11.29) becomes

$$v(S') \le v_{|\bigcup_{l=1}^{k} S^{v_l}|+|S^{v_{k+1}}|-|((\bigcup_{l=1}^{k} S^{v_l}) \cap S^{v_{k+1}})|} \tag{11.31}$$

or

$$v(S') \le v_{|S'|} \tag{11.32}$$

This and (11.21) complete the proof. ■

Lemma D 4: For a GCG on digraphs $\Gamma = (D, v)$ if *S* a semi-strongly connected component of *D* and *A* is a subgraph of *S* then $v(A) \le v(S)$.

Proof: Since S is a semi-strongly connected component, $v(S) = v_{|S|}$, it is already established in the Lemma D 3 that for any subgraph *A*, $v(A) \le v_{|A|}$. Since *A* is a subgraph of *S*, $|A| \le |S|$. Thus by the Axioms of Value $v(A) \le v_{|A|} \le v_{|S|} = v(S)$. This completes the proof. ■

Lemma D 5: For a GCG on digraphs $\Gamma = (D, v)$ if S^{v_1} and S^{v_2} are two semi-strongly connected components of the digraph *D* and *S* is a digraph produced by making an arc from v_1 to v_2, then *S* is semi-strongly connected and $v(S) \ge v(S^{v_1} \cup S^{v_2})$.

Proof: The semi-strong connectivity of the digraph *S* follows from the Definition 4, since with the making of the new arc from v_1 to v_2, v_1 becomes a pivot vertex, Definition 1, of *S*. The value of *S* is thus given by Axiom 1, and by (5.1)

$$v(S) = v_{|S|} = v_{|S_1|+|S_2|-|S_1 \cap S_2|} \tag{11.33}$$

The value of $S^{v_1} \cup S^{v_2}$, by Axiom 3 is

$$v(S^{v_1} \cup S^{v_2}) = v(S^{v_1}) + v(S^{v_2}) - v(S^{v_1} \cap S^{v_2}) \tag{11.34}$$

by using Axiom 1

$$v(S^{v_1} \cup S^{v_2}) = v_{|S^{v_1}|} + v_{|S^{v_2}|} - v(S^{v_1} \cap S^{v_2}) \tag{11.35}$$

Using Lemma D 3, it can be written as

$$v(S^{v_1} \cup S^{v_2}) \le v_{|S^{v_1}|} + v_{|S^{v_2}|} - |S^{v_1} \cap S^{v_2}| v_1 \tag{11.36}$$

Using Axiom 3, it can be written

$$v_{|S^{v_1} \cap S^{v_2}|+2} - v_{|S^{v_1} \cap S^{v_2}|+1} \ge v_{|S^{v_1} \cap S^{v_2}|+1} - |S^{v_1} \cap S^{v_2}| v_1 \tag{11.37}$$

For S^{v_1} and S^{v_2}, two different semi-strongly connected components of the digraph D $|S^{v_1}|,|S^{v_2}| \geq |S^{v_1} \cap S^{v_2}|+1$, by using Lemma A4 of [77]

$$v_{|S^{v_1}|+1} - v_{|S^{v_1}|} \geq v_{|S^{v_1} \cap S^{v_2}|+2} - v_{|S^{v_1} \cap S^{v_2}|+1} \tag{11.38}$$

using this in (11.37) gives

$$v_{|S^{v_1}|+1} - v_{|S^{v_1}|} \geq v_{|S^{v_1} \cap S^{v_2}|+1} - |S^{v_1} \cap S^{v_2}|\, v_1 \tag{11.39}$$

or

$$v_{|S^{v_1}|+1} - v_{|S^{v_1} \cap S^{v_2}|+1} \geq v_{|S^{v_1}|} - |S^{v_1} \cap S^{v_2}|\, v_1 \tag{11.40}$$

again

$$v_{|S^{v_1}|+|S^{v_2}|-|S^{v_1} \cap S^{v_2}|} - v_{|S^{v_1} \cap S^{v_2}|+|S^{v_2}|-|S^{v_1} \cap S^{v_2}|} \geq v_{|S^{v_1}|+1} - v_{|S^{v_1} \cap S^{v_2}|+1} \tag{11.41}$$

or

$$v_{|S^{v_1}|+|S^{v_2}|-|S^{v_1} \cap S^{v_2}|} - v_{|S^{v_2}|} \geq v_{|S^{v_1}|+1} - v_{|S^{v_1} \cap S^{v_2}|+1} \tag{11.42}$$

Using this in (11.40) and rearranging gives

$$v_{|S^{v_1}|+|S^{v_2}|-|S^{v_1} \cap S^{v_2}|} \geq v_{|S^{v_1}|} + v_{|S^{v_2}|} - |S^{v_1} \cap S^{v_2}|\, v_1 \tag{11.43}$$

Comparison of this and (11.36) gives

$$v(S^{v_1} \cup S^{v_2}) \leq v_{|S^{v_1}|+|S^{v_2}|-|S^{v_1} \cap S^{v_2}|} \tag{11.44}$$

using (11.33)

$$v(S^{v_1} \cup S^{v_2}) \leq v(S) \tag{11.45}$$

This completes the proof. ■

Lemma D 6: For a GCG on digraphs $\Gamma = (D, v)$ if $S^{v_i} : i = 1, 2, ..., k$ are semi-strongly connected components of the digraph *D* and *S* is a digraph produced by making arcs from v_1 to $v_i \forall i = 2, ..., k$, then *S* is semi-strongly connected and $v(S) \geq v(\bigcup_{i=1}^{k} S^{v_i})$.

Proof: The proof follows by extending Lemma D 5, by mathematical induction on the number of semi-strongly connected components. ■

Lemma D 7: For a GCG on digraphs $\Gamma = (D, v)$ if $S^{v_i} : i = 1, 2, ..., k$ are semi-strongly connected components of the digraph *D* and *S* is a digraph produced by making arcs from v_1 to v_i for some $i = 2, ..., k$, then $v(S) \geq v(\bigcup_{i=1}^{k} S^{v_i})$.

Proof: Making of a new arc from v_1 to v_2 leaves components $S'^{v_1}, S^{v_i} : i = 3, 4, ..., k$ such that $S'^{v_1} \cap S^{v_i} = (S^{v_1} \cup S^{v_2}) \cap S^{v_i} \forall i = 3, 4, ..., k$. This implies that

$$v(S'^{v_1} \cup \bigcup_{i=3}^{k} S^{v_i}) = v(S'^{v_1}) - (v(S^{v_1}) + v(S^{v_2}) - v(S^{v_1} \cap S^{v_2})) + v(\bigcup_{i=1}^{k} S^{v_i}) \tag{11.46}$$

Using Lemma D 5

$$v(S'^{v_1} \cup \bigcup_{i=3}^{k} S^{v_i}) \geq v(\bigcup_{i=1}^{k} S^{v_i}) \tag{11.47}$$

The desired result follows by making any number of arcs from v_1 to v_i for $i = 2, ..., k$. ■

Lemma D 8: For a GCG on digraphs $\Gamma = (D, v)$ if $S^{v_i} : i = 1, 2, ..., k$ are semi-strongly connected components of the digraph *D* and *S* is a digraph produced by making arcs from another vertex v_0 to v_i for some $i = 1, 2, ..., k$, then $v(S) \geq v(\bigcup_{i=1}^{k} S^{v_i}) + v_1$.

Proof: Using Lemma D 6,

$$v(\bigcup_{i=1}^{k} S^{v_i}) \leq v_{|\bigcup_{i=1}^{k} S^{v_i}|} \tag{11.48}$$

Whereas by the construction of *S*, the Axioms of Value, and the definition of semi-strongly connected components

$$v(S) = v_{|\bigcup_{i=1}^{k} S^{v_i}|+1} \tag{11.49}$$

by using Lemma A4 of [77]

$$v_{|\bigcup_{i=1}^{k} S^{v_i}|} + v_1 \leq v_{|\bigcup_{i=1}^{k} S^{v_i}|+1} \tag{11.50}$$

This along with (11.48) and (11.49) gives the desired result. ■

Lemma D 9: For a GCG on digraphs $\Gamma = (D, v)$ if S^{v_1} and S^{v_2} are two semi-strongly connected components of the digraph *D* and *S* is a digraph produced by making an arc from $v \in S^{v_1}$ to $v' \in S^{v_2}$, then $v(S) \geq v(S^{v_1} \cup S^{v_2})$.

Proof: Since S^{v_1} and S^{v_2} are two semi-strongly connected components of the digraph *D*, the value of $S^{v_1} \cup S^{v_2}$, by Axiom 3 is

$$v(S^{v_1} \cup S^{v_2}) = v(S^{v_1}) + v(S^{v_2}) - v(S^{v_1} \cap S^{v_2}) \tag{11.51}$$

by using Axiom 1

$$v(S^{v_1} \cup S^{v_2}) = v_{|S^{v_1}|} + v_{|S^{v_2}|} - v(S^{v_1} \cap S^{v_2}) \tag{11.52}$$

Using Lemma D 3, it can be written as

$$v(S^{v_1} \cup S^{v_2}) \leq v_{|S^{v_1}|} + v_{|S^{v_2}|} - |S^{v_1} \cap S^{v_2}|\, v_1 \tag{11.53}$$

Making of the new arc increases the size of the component S^{v_1} by $k : 1 \leq k \leq |S^{v_2}|$, to make a new component $S'^{v_1} : |S'^{v_1}| = |S^{v_1}| + k$; using Axiom 3 the value of the new digraph $S'^{v_1} \cup S^{v_2}$ is

$$v(S'^{v_1} \cup S^{v_2}) = v(S'^{v_1}) + v(S^{v_2}) - v(S'^{v_1} \cap S^{v_2}) \quad (11.54)$$

by using Axiom 1

$$v(S'^{v_1} \cup S^{v_2}) = v_{|S^{v_1}|+k} + v_{|S^{v_2}|} - v(S'^{v_1} \cap S^{v_2}) \quad (11.55)$$

Using Lemma D 3, it can be written as

$$v(S'^{v_1} \cup S^{v_2}) \geq v_{|S^{v_1}|+k} + v_{|S^{v_2}|} - v_{|S'^{v_1} \cap S^{v_2}|} \quad (11.56)$$

Making of the new arc increases the size of the component S^{v_1} by $k : 1 \leq k \leq |S^{v_2}|$, which may cause an increase in the size of $S'^{v_1} \cap S^{v_2}$ by at the most k, that is to say $|S'^{v_1} \cap S^{v_2}| = |S^{v_1} \cap S^{v_2}| + k$; and (11.56) becomes

$$v(S'^{v_1} \cup S^{v_2}) \geq v_{|S^{v_1}|+k} + v_{|S^{v_2}|} - v_{|S^{v_1} \cap S^{v_2}|+k} \quad (11.57)$$

For $|S^{v_1} \cap S^{v_2}| + k \geq |S^{v_1}|$, using (11.3) in the proof of Lemma D 1

$$\begin{aligned} v_{|S^{v_1}|+k} - v_{|S^{v_1} \cap S^{v_2}|+k} \geq\ & 2^{|S^{v_1}|-|S^{v_1} \cap S^{v_2}|} v_{|S^{v_1}|} - ((|S^{v_1} \cap S^{v_2}| + k) 2^{|S^{v_1}|-|S^{v_1} \cap S^{v_2}|} \\ & -(|S^{v_1}| + k)) v_1 - v_{|S^{v_1} \cap S^{v_2}|+k} \end{aligned} . \quad (11.58)$$

or

$$\begin{aligned} & v_{|S^{v_1}|+k} - v_{|S^{v_1} \cap S^{v_2}|+k} \geq (2^{|S^{v_1}|-|S^{v_1} \cap S^{v_2}|} - 2) v_{|S^{v_1}|} - ((|S^{v_1} \cap S^{v_2}| + k) 2^{|S^{v_1}|-|S^{v_1} \cap S^{v_2}|} \\ & -2(|S^{v_1}| + k)) v_1 + v_{|S^{v_1} \cap S^{v_2}|+k} - (2|S^{v_1} \cap S^{v_2}| + k - |S^{v_1}|) v_1 \end{aligned} \quad (11.59)$$

or

By Lemma A4 of [77]

$$v_{|S^{v_1} \cap S^{v_2}|+k} - (2|S^{v_1} \cap S^{v_2}| + k - |S^{v_1}|) v_1 \geq v_{|S^{v_2}|} - |S^{v_1} \cap S^{v_2}| v_1 \quad (11.60)$$

Using this to compare (11.53) and (11.59) for $|S^{v_1} \cap S^{v_2}| + k \geq |S^{v_1}|$ gives

$$v(S'^{v_1} \cup S^{v_2}) \geq v(S^{v_1} \cup S^{v_2}) \quad (11.61)$$

Similarly using (11.3) in the proof of Lemma D 1, for $|S^{v_1} \cap S^{v_2}|+k \leq |S^{v_1}|$,

$$v_{|S^{v_1}|+k} - v_{|S^{v_1} \cap S^{v_2}|+k} \geq 2^k v_{|S^{v_1}|} - (|S^{v_1}|2^k - (|S^{v_1}|+k))v_1 - v_{|S^{v_1} \cap S^{v_2}|+k} \tag{11.62}$$

or

$$\begin{aligned} v_{|S^{v_1}|+k} - v_{|S^{v_1} \cap S^{v_2}|+k} \geq (v_{|S^{v_1}|} - |S^{v_1} \cap S^{v_2}|v_1) + \\ (2^k - 1)v_{|S^{v_1}|} - (|S^{v_1}|2^k - (|S^{v_1}|+k+|S^{v_1} \cap S^{v_2}|))v_1 - v_{|S^{v_1} \cap S^{v_2}|+k} \end{aligned} \tag{11.63}$$

By Lemma A4 of [77]

$$(2^k - 1)v_{|S^{v_1}|} - (|S^{v_1}|2^k - (|S^{v_1}|+k+|S^{v_1} \cap S^{v_2}|))v_1 - v_{|S^{v_1} \cap S^{v_2}|+k} \geq 0 \tag{11.64}$$

or

$$v_{|S^{v_1} \cap S^{v_2}|+k} - (2|S^{v_1} \cap S^{v_2}|+k-|S^{v_1}|)v_1 \geq v_{|S^{v_2}|} - |S^{v_1} \cap S^{v_2}|v_1 \tag{11.65}$$

Using this to compare (11.53) and (11.59) for $|S^{v_1} \cap S^{v_2}|+k \geq |S^{v_1}|$ gives

$$v(S'^{v_1} \cup S^{v_2}) \geq v(S^{v_1} \cup S^{v_2}) \tag{11.66}$$

Using this to compare (11.53) and (11.59) for $|S^{v_1} \cap S^{v_2}|+k \leq |S^{v_1}|$ gives

$$v(S'^{v_1} \cup S^{v_2}) \geq v(S^{v_1} \cup S^{v_2}) \tag{11.67}$$

This and (11.61) gives the desired result. ■

Lemma D 10: For a GCG on digraphs $\Gamma = (D, v)$ if $S^{v_i} : i = 1, 2, ..., k$ are semi-strongly connected components of the digraph *D* and *S* is a digraph produced by making an arcs from $v \in S^{v_i}$ to $v' \in S^{v_j} : j \neq i$, then $v(S) \geq v(\bigcup_{i=1}^{k} S^{v_i})$.

Proof: The proof follows by extending Lemma D 9 by using mathematical induction and by iterating making the arcs. ■

Lemma D 11: For a GCG on digraphs $\Gamma = (D, v)$ with strict inequality in Axiom 5, if $S^{v_i} : i = 1, 2, ..., k$ are semi-strongly connected components of the digraph D and S is a digraph produced by making an arcs from $v \in S^{v_i}$ to $v' \in S^{v_j} : j \neq i$, then $v(S) > v(\bigcup_{i=1}^{k} S^{v_1})$.

Proof: A strict inequality in Axiom 5 leads to a strict inequality in (11.3), which leads to strict inequalities in (11.58) and (11.62), and consequently results in strict inequality in Lemma D 9. The proof follows by extending this result by using mathematical induction and by iterating making the arcs. ■

Lemma D 12: For a GCG on digraphs $\Gamma = (D, v)$ if $S^{v_i} : i = 1, 2, ..., k$ are semi-strongly connected components of the digraph D and S is a digraph produced by making an arcs from $v \in S^{v_i}$ to $v' \in S^{v_i}$, then $v(S) \leq v(\bigcup_{i=1}^{k} S^{v_1})$.

Proof: The proof follows by Lemma D 3, Lemma D 10 by using the fact that making of such an arc does not change the value of the individual semi-strongly connected component but it may cause to increase the value of the intersection of two or more semi-strongly connected components, according to Lemma D 10. ■

Lemma D 13: For a GCG on digraphs $\Gamma = (D, v)$ with strict inequality in Axiom 5, if $S^{v_i} : i = 1, 2, ..., k$ are semi-strongly connected components of the digraph D and S is a digraph produced by making an arc from $v \in S^{v_i}$ to $v' \in S^{v_i}$, such that both v and v' are in more than 1 semi-strongly connected components in D, then $v(S) < v(\bigcup_{i=1}^{k} S^{v_1})$.

Proof: The proof follows by Lemma D 3, Lemma D 11 by using the fact that making of such an arc does not change the value of the individual semi-strongly connected component but it causes to increase the value of the intersection of two or more semi-strongly connected components, according to Lemma D 11. ■

Lemma D 14: For a GCG on digraphs $\Gamma = (D, v)$ with strict inequality in Axiom 5, if $S^{v_i} : i = 1, 2, \ldots, k$ are semi-strongly connected components of the digraph D and $D - e$ is a digraph produced by deleting an arc e, such that it does not change the vertices in any of the semi-strongly connected components in D, then $v(D-e) \geq v(\bigcup_{i=1}^{k} S^{v_1})$. Moreover if the deletion of an arc e changes the vertices in a semi-strongly connected component then $v(D-e) < v(\bigcup_{i=1}^{k} S^{v_1})$.

Proof: The proof follows by Lemma D 3, Lemma D 13 by using the fact that deletion of such an arc does not change the value of the individual semi-strongly connected component but it causes to increase the value of the intersection of two or more semi-strongly connected components, according to Lemma D 13. ■

Lemma D 15: For a GCG on digraphs $\Gamma = (D, v)$ if A and B are two induced subgraphs of D then $v(A \cup B) = v(A) + v(B) - v(A \cap B)$, where $A \cup B$ is the union of induced subgraphs A and B.

Proof: Since A and B are two subgraphs they can be represented as unions of semi-strongly connected components, Remark 6, that is to say $S^{v_i} : i = 1, 2, \ldots k = |P^A|, v_i \in P^A$ and $S^{v_i} : i = k+1, k+2, \ldots k+m, m = |P^B|, v_i \in P^B$ are semi-strongly connected components of A and B. By using Axiom 3, the value of $A \cup B$ is

$$v(A \cup B) = \sum_{l=1}^{|P^A|+|P^B|} (-1)^{l+1} \sum_{\substack{i_1 < i_2 < \ldots < i_l \\ v_{i_j} \in P^A \cup P^B}} v(\bigcap_{j=1}^{l} S^{v_{i_j}}) \tag{11.68}$$

or

$$v(A \cup B) = \sum_{l=1}^{|P^A|} (-1)^{l+1} \sum_{\substack{i_1 < i_2 < \ldots < i_l \\ v_{i_j} \in P^A}} v(\bigcap_{j=1}^{l} S^{v_{i_j}}) +$$

$$\sum_{l=|P^A|+1}^{|P^A|+|P^B|} (-1)^{l+1} \sum_{\substack{i_1 < i_2 < \ldots < i_l \\ v_{i_j} \in P^B}} v(\bigcap_{j=1}^{l} S^{v_{i_j}}) - \sum_{(l,t)=(1,1)}^{(|P^A|,|P^B|)} (-1)^{l+t+1} \sum_{\substack{i_1 < i_2 < \ldots < i_l \\ i_1 < i_2 < \ldots < i_l \\ v_{i_j} \in P^A, v_{i_k} \in P^B}} v((\bigcap_{k=1}^{t} S^{v_{i_k}}) \cap (\bigcap_{j=1}^{l} S^{v_{i_j}})) \tag{11.69}$$

or

$$v(A \cup B) = v(A) + v(B) - v(A \cap B) \tag{11.70}$$

This completes the proof. ■

Lemma D 16: For a GCG on digraphs $\Gamma = (D, v)$ if A and B are two induced subgraphs of D then $v(A \cup B) \geq v(A) + v(B) - v(A \cap B)$, where $A \cup B$ is the subgraph of D induced by vertices in A and B.

Proof: The proof follows from Lemma D 10 and Lemma D 15. ■

Lemma D 17: For a GCG on digraphs $\Gamma = (D, v)$ if A and B are two induced subgraphs of D such that $A \subseteq B$ then $v(A) \leq v(B)$.

Proof: Proof follows from Lemma D 10 and Lemma D 15. ■

Lemma D 18: For a GCG on digraphs $\Gamma = (D, v)$ if S an induced subgraph of D then

$v(S) = a_1 v_1 + a_2 v_2 + \ldots + a_{|s|} v_{|S|}$ such that $\sum_{i=1}^{|S|} a_i i = |S|$ where $a_i \forall i = 1, 2, \ldots, |S|$ are integers.

Proof: The proof follows from the Axioms, the counting principle mentioned in (5.1), Axiom 3, and by using mathematical induction on the number of components in S. ■

Appendix E

Proofs of Results in Chapter 6

Lemma 1: The triplet $(T, Max, *)$ forms a semi-ring.

Proof: For this purpose the following three properties of semi-ring are to be satisfied.

Property 1: $(T,*)$ is a Commutative Monoid with Identity Element 1

1. $(T,*)$ is closed

Let $T_1, T_2 \in T$

$\Rightarrow \exists i, j, k \in V : T_{ij} = T_1, T_{jk} = T_2$

$\Rightarrow T_{i\sim j\sim k} = T_{ij}\, T_{jk}$

$\Rightarrow \exists i, k : T_{i\sim k} = T_1\, T_2$

Thus T is closed under *. In this framework it means that if there are two adjacent paths with given Impact Factor values then the combined path will have the Impact Factor value which is the product of the Impact Factor values of the two path's Impact Factor values and is present in T.

2. $(T,*)$ is associative

Let $T_1, T_2, T_3 \in T \subseteq R$ then associative property under * is followed from the real numbers. In this framework it signifies that if there are three adjacent paths with Impact Factor values T_1, T_2 *and* T_3 in whatever order they are combined the net Impact Factor value is the same.

3. Existence of identity in $(T,*)$

According to the definition of T in Section 6.2, $1 \in T$ also let $t \in T$

$\Rightarrow \exists i, j, k \in V : T_{ij} = 1, T_{jk} = t$

$\Rightarrow T_{i\sim k} = T_{ij}\, T_{ik} = 1.t = t = T_{jk}$

$\Rightarrow T_{ij} = 1$ is identity element in T under *

In this framework it means that if there is a path of Impact Factor value 1 and it is concatenated with another path with the Impact Factor value *t* then the Impact Factor value of the concatenated path is also *t*.

4. $(T,*)$ is commutative

Let $T_1, T_2 \in T \subseteq R$ then commutative property under * is followed from the real numbers. In this framework it means that whatever the order of concatenation of two paths is the Impact Factor value of the concatenated path remains the same.

Property 2: (T, Max) is a Commutative Monoid with Identity Element 0

1. (T, Max) is closed

Let $T_1, T_2 \in T$

$\Rightarrow \exists i, j \in V : T_{\underset{i\sim j}{P}} = T_1, T_{\underset{j\sim k}{Q}} = T_2$

$\Rightarrow T_{\underset{i\sim j}{R}} = Max(T_{\underset{i\sim j}{P}}, T_{\underset{j\sim k}{Q}})$ where R is the union of the paths P and Q. In this paper act of selecting a path of maximum Impact Factor out of two available paths is called the union of parallel paths.

Thus *T* is closed under *M*. In this framework it means that if there are two parallel paths with given Impact Factor values then the combined path will have the Impact Factor value which is the max of the Impact Factor values of the two path's Impact Factor values and is present in *T*.

2. (T, Max) is associative

Let $T_1, T_2, T_3 \in T \subseteq R$ then associative property under *Max* is followed from the real numbers. In this framework it signifies that if there are three parallel paths with Impact Factor values T_1, T_2 *and* T_3 in whatever order they are combined their Impact Factor value is the same.

3. Existence of identity under (T, Max)

According to the definition of *T* in Section 6.2, $0 \in T$ also let $t \in T$

$\Rightarrow \exists i, j \in V : T_{\overset{P}{i \sim j}} = 0, T_{\overset{Q}{i \sim j}} = t$

$\Rightarrow T_{\overset{R}{i \sim j}} = Max(T_{\overset{P}{i \sim j}}, T_{\overset{Q}{i \sim k}}) = Max(0, t) = t$

$\Rightarrow T_{ij} = 0$ is identity element in *T* under *Max*

In this framework it means that if there is a path of Impact Factor value 0 and it is parallel with another path with the Impact Factor value *T* then the Impact Factor value of their union is also *T*.

4. (T, Max) is commutative

Let $T_1, T_2 \in T \subseteq R$ then commutative property under *M* is followed from the real numbers. In this framework it means that whatever the order of two parallel paths is the Impact Factor value of their union remains the same.

Property 3: In $(T, Max, *)$ * is distributive over *Max*

Let $T_1, T_2, T_3 \in T \subseteq R$ then distribution of * over *Max* is followed from the real numbers. In this framework it means that if there are two paths between two *works* *i* and *j* such as parts of the paths, say from *i* to a vertex *k* are common with $T_{ik} = T_1, T_{\overset{P}{k \sim j}} = T_2, T_{\overset{Q}{k \sim j}} = T_3$. Then whether the combination of the two parallel paths is made first by taking the maximum of T_2 and T_3 and then concatenate the result with T_1 to get the Impact Factor value T_{ij} or the path $i \sim k$ is concatenated with $\overset{P}{i \sim k}$ and path $i \sim k$ is concatenated with $\overset{Q}{i \sim k}$ and then take the maximum over the two paths the same result is obtained. It can also be written as $T_1 \, Max(T_2, T_3) = Max(T_1 T_2, T_1 T_3)$. ■

Theorem 1: The Algorithm 1 converges to give $T^{i}{}_{k} = T_{ik}$.

Proof: It is sufficient to prove the correctness of the algorithm for one source vertex *i* to a destination vertex *k*. Since the numbers of *works* are finite and there is at the most one arc between two vertices so the numbers of arcs are also finite. Now for every arc $(i, j, t_{ij}) \in E$, the direct Impact Factor t_{ij} lies between 0 and 1, and T_{ik} is the maximum of the product of direct Impact Factor values taken along the paths between source and destination. Clearly each loop has a product lesser than or equal to 1 and the number of paths containing each loop at the most one are finite. Thus, a maximum T_{ik} can always be found out of these finite many paths between *i* and *k*.

Let $p = \{i = i_0, i_1, i_2, ..., i_s = k : \forall l \neq m \Rightarrow i_l \neq i_m, \forall l = 0,1,2,...,s-1 \ (i_l, i_{l+1}, t_{i_l i_{l+1}}) \in E, t_{i_l i_{l+1}} \neq 0\}$ be the path from *i* to *k* along which the maximum Impact Factor value T_{ik} exists. Also suppose that it is also the one with the least number of the vertices among such paths. Now such a path has at the most *N* vertices and hence has at the most *N-1* directed arcs. Also if *j* is some intermediate vertex in the path then the part of this path from *i* to *j* is also the path for maximum Impact Factor value or otherwise *p* could not be the path of maximum Impact Factor value from *i* to *k*.

The base case of strong induction for step number 0 i.e. at the time of initialization the self-Impact Factor value of *i* is 1 and its Impact Factor value for any other vertex is 0 which is true as there is no path of length 0 from *i* to *j* for all vertices in the path. Now suppose that at the *j-th* step of the outer most loop, *i* attains the maximum Impact Factor value of all the vertices along this path up to vertex *j* which is at a distance of *j* hops from *i*. Let *l* be the next vertex along the path. Path from *i* to *j* must be the path of maximum Impact Factor from *i* to *j*. In the next iteration of the outer most loop the algorithm makes the comparison $\forall (i, j) \in E, T^i{}_l < t_{ij} T^j{}_l$ and make the following substitutions

$$T^i{}_l = Max\{t_{jl}\, T^i{}_l \forall j \in V : (i, j, t_{ij}) \in E \vee j = i\}$$

Thus at the $(j+1)$ *-th* iteration $T^i{}_l$ is equal to the maximum Impact Factor value for the vertex *l*. This completes the proof. ■

Theorem 2: If all the persons and rationally picked anonymous referees are rational then the proposed system is immune against gifted authorship and against irrelevant citations.

Proof: The proof of the first part of the result follows from the rational behavior of persons involved. Involving a new author in the list of authors decreases the share of at least one author (Section 6.2.4.4) unless it changes the type of the work (Section 6.2.4.1). It implies that if involving a person does not add a value to the work there is always one author who suffers from the involvement of a new such author and thus will not allow to have a gifted authorship. Similarly, if an irrelevant citation does not belong to all the authors definitely reduce the impact made by any citation which an author is a part (Section 6.2.3.1). Moreover, if all the authors are part of an irrelevant citation then a rationally picked anonymous referees suffers from such citation (Section 6.2.3.1), and thus such citation cannot be in a work. ■

Appendix F

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Biographical Information

Muhammad Aurangzeb received B.Sc. Electrical Engineering EE, Communication, with honors, in 1993, from the University of Engineering and Technology Lahore Pakistan. Immediately after the graduation, he joined telecommunication industry and worked both in mobile telecommunication and land line telecommunication sector for the next nine years. During this time he got many base stations commissioned, installed and commissioned more than 1000km of Optical Fiber Links, and managed huge divisions of telecommunication sector in Pakistan. During this course of time he also had the opportunity to get a one year post graduate diploma from the Telecommunication Staff College Haripur Hazara Pakistan, and obtained a two months training on Optical Fiber SDH links and DWDM systems from Huawei Training Center Shenzhen China. Teaching is Aurangzeb's passion; during this course of nine years he was also actively involved in part time university teaching as visiting instructor. He taught BS and MS classes at reputed institutions including but not limited to, The University of the Punjab, Lahore Pakistan, and The National University of Computer and Emerging Technologies (NUCES) Lahore Pakistan. In 2003 he decided to continue with his studies; he completed MSCS from NUCES in 2005, while securing second position in a very capable batch of students. Looking at the qualification and teaching skills of Aurangzeb, NUCES offered him with an Assistant Professorship before the completion of the degree; which he accepted. In 2007 he completed MSEE from The University of Engineering and Technology Lahore, Pakistan. During this time, he also worked as a research consultant to Versonic Pte Ltd.; there he had the opportunity to work on a research based software project for two years. In 2009 he received invitation from Dr. F. L Lewis to join doctoral research program at The University of Texas at Arlington (UTA), USA. In Spring 2013, he successfully defended PhD EE under the supervision of Dr. Lewis, his area of research is coalitions and cooperative games. Moreover, by looking at his qualifications and teaching talent, UTA has also appointed him as a part time instructor for a graduate EE course, as a special case. In short, Aurangzeb has twenty years of teaching and eleven years of industry experience. Aurangzeb has passion to continue with university level teaching and research activity; he is looking forward to join a challenging position where he can express his talent; he will also be in touch with the industry to motivate his research.