ADVANCEMENTS AND APPLICATIONS OF NONSTANDARD FINITE DIFFERENCE METHODS

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To my aunt Wanda, for her unending love and support.

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ABSTRACT

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A class of dynamically consistent numerical methods are analyzed for general n-dimensional productive-destructive systems (PDS). Using this analysis, a methodology for constructing positive and elementary stable nonstandard numerical methods is established. The nonstandard approach results in qualitatively superior numerical methods when compared to the standard ones. PDS model a wide range of dynamical systems, including ones with biological, chemical and physical interactions. Building upon this, a nonstandard finite difference method for solving autonomous dynamical systems with positive solutions is constructed. The proposed numerical methods are computationally efficient and easy to implement. Several examples are given which show that the numerical results agree with the theoretical results.

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CHAPTER 1

Introduction

In the early 1980s, Potts considered "best" difference equation approximations to linear ordinary differential equations (ODEs) [1] and nonlinear ODEs such as the simple Duffing equation [2]. Based on Potts' observations, Mickens proved that given any ODE, there exists an exact difference equation which has zero local truncation error [3, 4]. However, in order to construct an exact difference equation, in general, one needs the exact solution to the differential equation. Since the standard approach to constructing finite difference methods for solving differential equations can lead to incorrect behavior in the solutions (e.g. "ghost solutions", numerical instabilities and chaotic behavior [5]), Mickens, using the exact difference equations as a guide, proposed the modeling rules which would come to define what it means for a finite difference method to be nonstandard [4]. These rules were later expanded by Mickens in his book [6]:

Rule 1 *The orders of the discrete derivatives should be equal to the orders of the corresponding derivatives of the differential equation.*

Rule 2 *Denominator functions for the discrete derivatives, must, in general, be expressed in terms of more complicated functions of the step-sizes than those conventionally used.*

Rule 3 *Nonlinear terms should, in general, be replaced by nonlocal discrete representations.*

Rule 4 *Special conditions that hold for the solutions of the differential equations should also hold for the solutions of the finite difference scheme.*

Rule 5 *The scheme should not introduce extraneous or spurious solutions.*

Rule 6 *For differential equations having $N(\geq 3)$ terms, it is generally useful to construct finite difference schemes for various sub-equations composed of M terms, where $M < N$, and then combine all the schemes together in an overall consistent finite difference model.*

In 1994 [7], Mickens introduced the concept of *elementary stability*, the property which brings correspondence between the local stability at equilibria of the differential equation and the numerical method. Since the onset, several NSFD methods have been constructed for specific ODEs and PDEs (e.g. [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]).

In 2001, Anguelov and Lubuma [25] formalized some of the foundations which motivate Mickens' rules, including convergence properties of nonstandard schemes. They defined *qualitative stability* with respect to a given property, such as positivity of solutions, conservation laws, and equillibria, to mean that a numerical method satisfies such a property for any time-step. They went further to prove by construction, how to construct elementary stable finite difference schemes, including one based on the second order Runge-Kutta method. In 2003, Lubuma and Roux [26] constructed the nonstandard $\theta-$method which is elementary stable method for any value $\theta \in [0, 1]$. In 2005, Mickens coined the term *dynamic consistency* [27], to mean that a numerical method is qualitatively stable with respect to all desired properties of the solutions to the differential equation. Between 2005 and 2008 Roeger constructed NSFD methods for Lotka-Volterra (LV) models [28, 29, 30, 31]. These NSFD methods preserve the periodicity of the LV models and several other qualitative properties. In 2011, Anguelov, Lubuma and Shillor [32], in an effort to expand the theory connecting continuous dynamical systems to discrete dynamical systems of numerical

methods, introduced the concept of *topological dynamical consistency*, which expands more carefully upon the idea of dynamic consistency. This concept describes the size of the domain on which a numerical method is topologically equivalent (conjugate) with the continuous dynamical system it solves. They went on to prove the theory which connects $1-$dimensional continuous and discrete dynamical systems and provided examples of how the nonstandard approach can create topological dyanmically consistent methods. In 2014, Anguelov, Dumont, Lubuma and Shillor [33] used the NSFD methodology to construct a numerical method which is dynamically consistent for a large class of dynamical systems used in epidemiology. The NSFD method is elementary stable, and preserves several properties, including positivity of solutions, dissipativity, and global asymptotic stability of the disease free equilibrium.

Some other notable NSFD methods include methods for delay differential equations [17, 34, 35, 36], including the most recent one by Garba, Gumel, Hassan, and Lubuma [37], methods for fractional differential equations [38, 39], and group-preserving methods [40]. It should also be noted that many similar discretization ideas in the literature have been inspired by other work (e.g. Kahan's unpublished lecture notes [41]). There are also other publications which are in a similar vein to the NSFD methods which do not reference the NSFD method literature (e.g. [42]).

From 2005 to 2011, Dimitrov and Kojouharov designed NSFD methods that preserve positivity and elementary stability for a variety of special classes of autonomous dynamical systems [43, 44, 45, 46, 47] culminating with positive and elementary stable nonstandard (PESN) methods for general $2-$ and $3-$dimensional productive-destructive systems (PDS). One of the main subjects of this dissertation is to extend these results.

In Chapter 2, definitions and preliminary work needed for the rest of the dissertation are given. In Chapter 3, we extend a NSFD method for $2-$ and $3-$dimensional

PDS to work for general $n-$dimensional PDS. In Chapter 4, a new NSFD method for solving general autonomous systems with positive solutions is constructed. In Chapter 5, several examples and applications are given which demonstrate the effectiveness of the proposed NSFD methods developed in the previous chapters. Finally, in Chapter 6, concluding remarks are given.

CHAPTER 2

Definitions and Preliminaries

The general framework which will be used to understand and prove the results of the next few chapters will be presented here. Results and definitions needed for specific proofs will be given in their respective chapters.

A system of autonomous ordinary differential equations can be written as

$$\frac{dx}{dt} = f(x(t)); \;\; x(t_0) = x_0, \tag{2.1}$$

where $x = [x^1, x^2, \ldots, x^n]^T : [t_0, T) \to \mathbb{R}^n$, $f = [f^1, f^2, \ldots, f^n]^T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is differentiable and $x_0 \in \mathbb{R}^n$.

A finite difference method to approximate System (2.1) can be written as

$$D_h(x_k) = F_h(f; x_k), \tag{2.2}$$

where $D_h(x_k) \approx \left.\frac{dx}{dt}\right|_{t=t_k}$, $x_k \approx x(t_k)$, $F_h(f; x_k)$ approximates $f(x)$ in System (2.1) and $t_k = t_0 + kh$, where $h > 0$.

The qualitative properties that we focus on preserving are quantified with the following definitions.

Definition 2.0.1. The finite difference method (2.2) is called *positive*, if, for any value of the step size h, and $x_0 \in \mathbb{R}^n_+$ its solution remains positive, i.e., $x_k \in \mathbb{R}^n_+$ for all $k \in \mathbb{N}$.

Positivity of the solution is an important property in applications to biological systems and various physical systems where negative values are generally not meaningful.

Definition 2.0.2. The finite difference method (2.2) is called *elementary stable* if, for any value of the step size h, its only fixed points $\bar{x}$ are the same as the equilibria of the differential system (2.1) and the local stability properties of each $\bar{x}$ are the same for both the differential system and the discrete method.

Elementary stability [25] resolves the issue of the numerical method introducing spurious fixed points or the wrong stability behavior. For the purposes of this dissertation only positivity and elementary stability are considered.

Definition 2.0.3. The finite difference method (2.2) is *dynamically consistent* with the differential System (2.1) with respect to the positivity of solutions, equilibria and their local stability if it is both, positive and elementary stable.

The numerical methods proposed in this dissertation are also NSFD methods according to the following definition introduced by Anguelov and Lubuma in [25].

Definition 2.0.4. The one-step finite-difference scheme (2.2) for solving System (2.1) is a NSFD method if at least one of the following conditions is satisfied:

- $D_h(x_k) = \dfrac{x_{k+1} - x_k}{\varphi(h)}$, where $\varphi(h) = h + \mathcal{O}(h^2)$ is a non-negative function;

- $F_h(f; x_k) = g(x_k, x_{k+1}, h)$, where $g(x_k, x_{k+1}, h)$ is a non-local approximation of the right-hand side of System (2.1).

It should be noted that a more robust and technical definition is given by Lubuma and Patidar in [48].

An equilibrium of the system (2.1) is defined as any constant $\bar{x}$ such that $f(\bar{x}) = 0$ and its local stability can be analyzed by the following theorem.

Theorem 2.0.1. *Let $\bar{x}$ be an equilibrium of System (2.1), $J_f(\bar{x})$ be the Jacobian of System (2.1) at $\bar{x}$ and $\sigma(J_f(\bar{x}))$ denotes the spectrum of $J_f(\bar{x})$. An equilibrium $\bar{x}$ of System (2.1) is locally stable if $Re(\lambda) < 0$ for all $\lambda \in \sigma(J_f(\bar{x}))$ and locally unstable if $Re(\lambda) > 0$ for at least one $\lambda \in \sigma(J_f(\bar{x}))$.*

If the numerical method (2.2) is explicit, the local stability of its fixed points can be determined by the following theorem.

Theorem 2.0.2. *Assume that System* (2.2) *has the explicit form:*

$$x_{k+1} = G(x_k), \tag{2.3}$$

where the function $G = [G^1, G^2, \ldots, G^n]^T : \mathbb{R}^n \mapsto \mathbb{R}^n$ *is differentiable. A fixed point* $\bar{x}$ *of System* (2.3) *is locally stable if and only if all eigenvalues of* $J_G(\bar{x})$ *are less than one in absolute values, where* $J_G(\bar{x})$ *denotes the Jacobian* $\left(\frac{\partial G^i(\bar{x})}{\partial x^j}\right)_{1\leq i,j\leq n}$.

To prove dynamical consistency between a system of differential equations and a NSFD method, as stated in Definition 2.0.3, one should establish correspondence between the stability conditions described in Theorem 2.0.1 and Theorem 2.0.2.

CHAPTER 3

A Nonstandard Finite Difference Method for Productive-Destructive Systems

3.1 Introduction

Productive-destructive systems (PDS) find applications in biology, chemistry and engineering [49]. The correspondence between these systems and their numerical approximations have been studied from various perspectives [50, 51]. Nonstandard finite-difference methods aim to secure consistency with PDS with respect to different dynamical characteristics of the systems. Among the most studied properties which are preserved to achieve dynamical consistency are positivity of solutions and the existence and stability of the equilibrium points.

Following a sequence of studies designing NSFD methods for specific PDS [43, 45, 44, 46], Dimitrov and Kojouharov proposed a numerical method which is dynamically consistent with general 2− and 3− dimensional PDS and suggested that the result is generalizable to higher-dimensional PDS [47]. In this chapter, their results are extended [52] and a dynamically consistent numerical method for general $n-$dimensional PDS is designed. A novel approach based on bilinear transformations in the complex plane is applied in order to establish the corresponding stability between PDS and a NSFD scheme.

The chapter is organized as follows. In Section 3.2, a dynamically consistent numerical method is constructed for general multi-dimensional PDS. In Section 3.3, proofs of the main results are given.

3.2 Main Results

A general PDS is a special case of (2.1) that can be written as

$$\frac{dx}{dt} = P(x(t)) - D(x(t)); \quad x(0) \in \mathbb{R}^n_+,$$

where the set $\mathbb{R}^n_+$ is defined as

$$\mathbb{R}^n_+ := \{(x^1, x^2, \ldots, x^n) \in \mathbb{R}^n : x^1 \geq 0, x^2 \geq 0, \ldots, x^n \geq 0\},$$

$x = [x^1, x^2, \ldots, x^n]^T : \mathbb{R}_+ \to \mathbb{R}^n_+$, and $P, D \in C^1(\mathbb{R}^n_+, \mathbb{R}^n_+)$. Many PDS which are used to model biological interactions, such as population dynamics, are of the form

$$\begin{aligned}
\frac{dx^1}{dt} &= P^1(x^1(t), x^2(t), \cdots, x^n(t)) - N^1(x^1(t), x^2(t), \cdots, x^n(t))x^1(t); \qquad x^1(0) \geq 0, \\
\frac{dx^2}{dt} &= P^2(x^1(t), x^2(t), \cdots, x^n(t)) - N^2(x^1(t), x^2(t), \cdots, x^n(t))x^2(t); \qquad x^2(0) \geq 0, \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
\frac{dx^n}{dt} &= P^n(x^1(t), x^2(t), \cdots, x^n(t)) - N^n(x^1(t), x^2(t), \cdots, x^n(t))x^n(t); \qquad x^n(0) \geq 0,
\end{aligned} \tag{3.1}$$

where the positive functions $P^i, N^i \in C^1(\mathbb{R}^n_+, \mathbb{R}_+)$ model the inflows to and outflows from the component x^i, respectively. Differential equations of this form are guaranteed to exhibit non-negative solutions [53, 54]. This can be seen since System (3.1) satisfies

$$\frac{dx^i}{dt} \geq 0 \text{ whenever } x^i = 0, \quad i = 1, \ldots, n. \tag{3.2}$$

In what follows, the system (3.1) is assumed to have a finite number of hyperbolic equilibria. This condition is needed in order to obtain elementary stability of the NSFD method.

To prove dynamical consistency between PDS and NSFD method, as stated in Definition 2.0.3, one should establish correspondence between the stability conditions described in Theorem 2.0.1 and Theorem 2.0.2. Next, definitions for when a polynomial has roots which satisfy Theorems 2.0.1 and 2.0.2 are presented.

Definition 3.2.1. A polynomial $p(z)$ is said to be *Hurwitz stable* if all of its roots, z_i, satisfy $Re(z_i) < 0$. If all of its roots, z_i, satisfy $|z_i| < 1$, $p(z)$ is said to be *Schur stable*.

The traditional approach, employed in previous studies [43, 45, 44, 46, 47], required a set of explicit conditions to be listed for the characteristic polynomials of the Jacobians, J_f, of the continuous system and, J_G, of the discrete system from Theorems 2.0.1 and 2.0.2 to be Hurwitz and Schur stable, respectively and equivalency of those sets to be proved by direct comparison. Unfortunately, the complexity and computational intensity of this process grows with the system size and becomes inapplicable in the general case. In this analysis, an alternative approach based on bilinear transformations in the complex plane is proposed. This makes it possible to work with the characteristic polynomial of J_G without any need to evaluate J_f.

A few definitions and lemmas regarding properties of matrices and polynomials that are useful for the analysis are stated below.

Definition 3.2.2. Given an $n \times n$ matrix, $A = (a_{ij})$,

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$

where S_n is the set of all permutations of $\{1, \ldots, n\}$.

Definition 3.2.3. Given an $n \times n$ matrix $A = (a_{ij})$, $I \subseteq \{1, 2, \ldots, n\}$, where $|I| = k$ is the cardinality of I, a *principal submatrix* of A is the $k \times k$ matrix, A_I, formed by deleting the rows and columns of A with indices in $\{1, 2, \ldots, n\} \backslash I$. The determinant of A_I is called a *principal minor* of A. If $I = \{1, 2, \ldots, k\}$, the determinant, denoted Δ_k, is called a *leading principal minor* of A.

If A is a square matrix, the coefficients of the *characteristic polynomial*, $p_A(\lambda) = det(A - \lambda I)$, can be evaluated using the following lemma from [55].

Lemma 3.2.1. *Given an $n \times n$ matrix, A, the coefficients c_k of the characteristic polynomial $p_A(\lambda) = (-1)^n(\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n)$ can be written as $c_k = (-1)^k \sum_{|I|=k} det(A_I)$.*

Lemma 3.2.2. *If $z_i \neq 0$ is a root of the polynomial $p(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n$, then $\frac{1}{z_i}$ is a root of the polynomial $p^*(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0$.*

Lemma 3.2.3. *Assume that $p(h) = a_0 + a_1h + \ldots + a_nh^n$ is a polynomial of degree n and the following conditions are satisfied for some $k < n$:*

- *$a_0 = a_1 = \ldots = a_{k-1} = 0$ and*
- *$a_k \neq 0$.*

Then there exists $q > 0$ such that $a_kp(h) > 0$ for $h \in (0, q]$, i.e., $p(h)$ has the sign of a_k for sufficiently small positive values of h.

The following theorem [56] provides conditions for Hurwitz stability of a polynomial.

Theorem 3.2.4. *Given the polynomial*

$$p(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n \tag{3.3}$$

construct the following Hurwitz matrix:

$$\begin{pmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \cdots & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 & 0 \\ 0 & a_0 & a_2 & a_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_{n-2} & a_n \end{pmatrix}. \tag{3.4}$$

The polynomial (3.3) is Hurwitz stable if and only if $a_0 > 0$ and all the leading principal minors of (3.4) satisfy $\Delta_1 > 0, \Delta_2 > 0, \ldots, \Delta_n > 0$.

The next Theorem 3.2.4 provides a connection between Hurwitz and Schur stability using a bilinear transform as follows [57].

Theorem 3.2.5. *A polynomial $p(z)$ of degree n, is Schur stable if and only if the polynomial $r(z) = (z-1)^n p\left(\dfrac{z+1}{z-1}\right)$ is Hurwitz stable, provided $z = 1$ is not a root of $p(z)$.*

The following NSFD method is constructed based on the procedure described in [47]:

$$\begin{aligned}
\frac{x_{k+1}^1 - x_k^1}{\varphi(h)} &= P^1(x_k^1, x_k^2, \cdots, x_k^n) - N^1(x_k^1, x_k^2, \cdots, x_k^n)x_{k+1}^1; \\
\frac{x_{k+1}^2 - x_k^2}{\varphi(h)} &= P^2(x_k^1, x_k^2, \cdots, x_k^n) - N^2(x_k^1, x_k^2, \cdots, x_k^n)x_{k+1}^2; \\
&\vdots \\
\frac{x_{k+1}^n - x_k^n}{\varphi(h)} &= P^n(x_k^1, x_k^2, \cdots, x_k^n) - N^n(x_k^1, x_k^2, \cdots, x_k^n)x_{k+1}^n.
\end{aligned} \tag{3.5}$$

It will be shown that the NSFD method (3.5) is dynamically consistent with the PDS (3.1), for appropriate choice of the denominator function $\varphi(h)$. The function $\varphi(h)$ can be selected as $\varphi(h) = \phi(hq)/q$ using any function $\phi(h)$ that satisfies the property:

$$\phi(h) = h + O(h^2) \quad \textit{and} \quad 0 < \phi(h) < 1 \quad \textit{for all} \quad h > 0. \tag{3.6}$$

Remark 1. The function $\phi(h) = 1 - e^{-h}$ is one such function that satisfies the property (3.6). More information on constructing denominator functions for NSFD methods can be found in [58].

The proof of the dynamical consistency between the NSFD method (3.5) and the PDS (3.1) is divided into two parts. The next theorem establishes a correspondence between the systems for small step-sizes using the traditional denominator function $\varphi(h) = h$.

Theorem 3.2.6. *The NSFD scheme* (3.5) *is positive and has the same set of equilibrium points as the PDS (3.1). If $\varphi(h) = h$, the local stability of each equilibrium point*

with regard to the PDS (3.1) and the NSFD scheme (3.5) is the same for sufficiently small step-size h.

Next, it is demonstrated that the denominator function $\varphi(h)$ can be selected to guarantee dynamical consistency for an arbitrary step size.

Theorem 3.2.7. *Let ϕ be a real-valued function on $\mathbb{R}$ that satisfies the property* (3.6)*. Let Q be a constant such that for all $h \in (0, Q]$, the PDS* (3.1) *and the NSFD scheme* (3.5) *have the same local stability at each equilibrium point. Let the denominator function in the NSFD scheme* (3.5) *be $\varphi(h) = \phi(hq)/q$ for some $q > \frac{1}{Q}$, then the NSFD scheme* (3.5) *is dynamically consistent with the PDS (3.1).*

3.3 Proofs of Main Results from Section 3.2

Proof. (Theorem 3.2.6)

Suppose $\varphi(h) = h$. To show that the NSFD scheme is positive, write it in an explicit form as follows:

$$\begin{aligned}
x_{k+1}^1 &= \frac{hP^1(x_k^1, x_k^2, \cdots, x_k^n) + x_k^1}{1 + hN^1(x_k^1, x_k^2, \cdots, x_k^n)}, \\
x_{k+1}^2 &= \frac{hP^2(x_k^1, x_k^2, \cdots, x_k^n) + x_k^2}{1 + hN^2(x_k^1, x_k^2, \cdots, x_k^n)}, \\
&\vdots \\
x_{k+1}^n &= \frac{hP^n(x_k^1, x_k^2, \cdots, x_k^n) + x_k^n}{1 + hN^n(x_k^1, x_k^2, \cdots, x_k^n)}.
\end{aligned} \tag{3.7}$$

Since $h > 0$, $P^i \geq 0$, and $N^i \geq 0$ for $i = 1, 2, \ldots, n$, the method is positive.

To show that the system (3.1) and the numerical scheme (3.5) share the same equilibria, notice that the condition for some $\bar{x} \in \mathbb{R}^n_+$ to be an equilibrium is the same for both. Suppose $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$ is an equilibrium of the system (3.1). Then the following conditions hold:

$$\begin{aligned} P^1(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) &= N^1(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)\bar{x}^1, \\ P^2(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) &= N^2(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)\bar{x}^2, \\ &\vdots \\ P^n(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) &= N^n(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)\bar{x}^n. \end{aligned} \tag{3.8}$$

It is clear that the conditions are the same for the numerical method.

It will now be shown that for a sufficiently small step-size h, the PDS and the NSFD scheme have the same local stability at each equilibrium point. To simplify notation, define $P^i := P^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$, $N^i := N^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$, $P^i_j := \dfrac{\partial P^i}{\partial x^j}(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$ and $N^i_j := \dfrac{\partial N^i}{\partial x^j}(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$, for $i, j = 1, 2, \ldots, n$.

With this notation, the Jacobian of the continuous system (3.1) evaluated at the equilibrium point $(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)$ is:

$$A = \begin{pmatrix} -N^1 - \bar{x}^1 N^1_1 + P^1_1 & -\bar{x}^1 N^1_2 + P^1_2 & \cdots & -\bar{x}^1 N^1_n + P^1_n \\ -\bar{x}^2 N^2_1 + P^2_1 & -N^2 - \bar{x}^2 N^2_2 + P^2_2 & \cdots & -\bar{x}^2 N^2_n + P^2_n \\ \vdots & \vdots & & \vdots \\ -\bar{x}^n N^n_1 + P^n_1 & -\bar{x}^n N^n_2 + P^n_2 & \cdots & -N^n - \bar{x}^n N^n_n + P^n_n \end{pmatrix}$$

and the Jacobian of the NSFD method (3.5) evaluated at the equilibrium point $(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)$ is:

$$B = \begin{pmatrix} \dfrac{1 - h\bar{x}^1 N^1_1 + hP^1_1}{1 + hN^1} & \dfrac{-h\bar{x}^1 N^1_2 + hP^1_2}{1 + hN^1} & \cdots & \dfrac{-h\bar{x}^1 N^1_n + hP^1_n}{1 + hN^1} \\ \dfrac{-h\bar{x}^2 N^2_1 + hP^2_1}{1 + hN^2} & \dfrac{1 - h\bar{x}^2 N^2_2 + hP^2_2}{1 + hN^2} & \cdots & \dfrac{-h\bar{x}^2 N^2_n + hP^2_n}{1 + hN^2} \\ \vdots & \vdots & & \vdots \\ \dfrac{-h\bar{x}^n N^n_1 + hP^n_1}{1 + hN^n} & \dfrac{-h\bar{x}^n N^n_2 + hP^n_2}{1 + hN^n} & \cdots & \dfrac{1 - h\bar{x}^n N^n_n + hP^n_n}{1 + hN^n} \end{pmatrix}.$$

Let

$$p_A(\lambda) = det(A - \lambda I) = (-1)^n(\lambda^n + \alpha_1\lambda^{n-1} + \alpha_2\lambda^{n-2} + \cdots + \alpha_{n-1}\lambda + \alpha_n)$$

be the characteristic polynomial of $A = (a_{ij})$, and

$$p_B(\lambda) = det(B - \lambda I) = (-1)^n(\lambda^n + \beta_1\lambda^{n-1} + \beta_2\lambda^{n-2} + \cdots + \beta_{n-1}\lambda + \beta_n)$$

be the characteristic polynomial of $B = (b_{ij})$. Instead of testing the Schur stability of $p_B(\lambda)$, one can transform the polynomial and test its Hurwitz stability following Theorem 3.2.5. Note that B can be written in terms of A in the following manner:

$$B - I = \left(\frac{ha_{ij}}{1 + hN^i}\right) = diag\left(\frac{h}{1 + hN^1}, \frac{h}{1 + hN^2}, \cdots, \frac{h}{1 + hN^n}\right) A.$$

Then taking the determinant of both sides yields

$$det(B - I) = \frac{h^n}{(1 + hN^1)\cdots(1 + hN^n)} det(A).$$

Since the system (3.1) has only hyperbolic equilibria, $\lambda = 0$ is not an eigenvalue of A. This implies that $det(A) \neq 0$. Since $h > 0$ and $N^i \geq 0$ for $i = 1, 2, \ldots, n$, it must be true that $det(B - I) \neq 0$. Therefore $\lambda = 1$ is not an eigenvalue of B, and thus not a root of $p_B(\lambda)$. Therefore, the application of Theorem 3.2.5 is valid.

Let $C = B - I$. Consider

$$p_B\left(\frac{\lambda + 1}{\lambda - 1}\right) = det\left(B - \frac{\lambda + 1}{\lambda - 1} I\right) = det\left(C - \frac{2}{\lambda - 1} I\right) = p_C\left(\frac{2}{\lambda - 1}\right),$$

where $p_C(\omega)$ is the characteristic polynomial of C:

$$\begin{aligned} p_C(\omega) &:= det(C - \omega I) \\ &= (-1)^n(\omega^n + \gamma_1\omega^{n-1} + \gamma_2\omega^{n-2} + \cdots + \gamma_{n-1}\omega + \gamma_n). \end{aligned}$$

Now use Lemma 3.2.1 to find γ_i. It is clear that if

$$A_I = \begin{pmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_k} \\ a_{i_2 i_1} & a_{i_2 i_2} & \cdots & a_{i_2 i_k} \\ \vdots & \vdots & & \vdots \\ a_{i_k i_1} & a_{i_k i_2} & \cdots & a_{i_k i_k} \end{pmatrix}$$

where $I = \{i_1, i_2, \ldots, i_k\}$, is any $k \times k$ principal submatrix of A, then one can write the corresponding principal submatrix for C as

$$C_I = \begin{pmatrix} \frac{h}{1+hN^{i_1}} & 0 & \cdots & 0 \\ 0 & \frac{h}{1+hN^{i_2}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{h}{1+hN^{i_k}} \end{pmatrix} \begin{pmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_k} \\ a_{i_2 i_1} & a_{i_2 i_2} & \cdots & a_{i_2 i_k} \\ \vdots & \vdots & & \vdots \\ a_{i_k i_1} & a_{i_k i_2} & \cdots & a_{i_k i_k} \end{pmatrix}.$$

Let $l_r \in \{1, \ldots, n\} \backslash I$. Then the principal minor

$$\begin{aligned} det(C_I) &= \frac{h^k}{(1+hN^{i_1}) \cdots (1+hN^{i_k})} det(A_I) \\ &= \frac{h^k}{(1+hN^1) \cdots (1+hN^n)} (1+hN^{l_1}) \ldots (1+hN^{l_{n-k}}) det(A_I) \\ &= \frac{h^k}{(1+hN^1) \cdots (1+hN^n)} (det(A_I) + hd_1 + \cdots + h^{n-k} d_{n-k}), \end{aligned}$$

for some constants $d_1, \ldots, d_{n-k}$ dependent on C_I.

Next, by Lemma 3.2.1:

$$\begin{aligned} \gamma_k &= (-1)^k \sum_{|I|=k} det(C_I) \\ &= \frac{(-1)^k h^k}{(1+hN^1) \cdots (1+hN^n)} \sum_{|I|=k} (det(A_I) + hd_1 + \cdots + h^{n-k} d_{n-k}) \\ &= \frac{1}{(1+hN^1) \cdots (1+hN^n)} (h^k \alpha_k + h^{k+1} \bar{d}_1 + \cdots + h^n \bar{d}_{n-k}), \end{aligned}$$

for $k = 1 \ldots, n$ and for some constants $\bar{d}_1, \ldots, \bar{d}_{n-k}$ dependent on γ_k.

Let $\gamma_0 = 1$ and find the coefficients to the polynomial

$$\begin{aligned} r(\lambda) &= (\lambda-1)^n p_C(\omega) \\ &= (-1)^n(\lambda-1)^n(\gamma_0\omega^n + \gamma_1\omega^{n-1} + \cdots + \gamma_{n-1}\omega + \gamma_n) \\ &= (-1)^n(\lambda-1)^n(\gamma_0\left(\frac{2}{\lambda-1}\right)^n + \gamma_1\left(\frac{2}{\lambda-1}\right)^{n-1} + \cdots + \gamma_{n-1}\left(\frac{2}{\lambda-1}\right) + \gamma_n) \\ &= (-1)^n(\gamma_0 2^n + \gamma_1 2^{n-1}(\lambda-1) + \cdots + \gamma_{n-1}2(\lambda-1)^{n-1} + (\lambda-1)^n\gamma_n). \end{aligned}$$

Using the binomial theorem

$$(\lambda-1)^m = \lambda^m + \cdots + \binom{m}{j}\lambda^{m-j}(-1)^j + \cdots + (-1)^m,$$

for $m = 1, \ldots, n$.

Let $\alpha_0 = 1$ and thus $r(\lambda) = (-1)^n(\bar{\alpha}_n\lambda^n + \bar{\alpha}_{n-1}\lambda^{n-1} + \cdots + \bar{\alpha_1}\lambda + \bar{\alpha}_0)$, where

$$\begin{aligned} \bar{\alpha}_k &= \gamma_n\binom{n}{k}(-1)^{n-k} + 2\gamma_{n-1}\binom{n-1}{k}(-1)^{n-k-1} + \cdots + 2^{n-k}\gamma_k \\ &= \frac{1}{(1+hN^1)\cdots(1+hN^n)}(2^{n-k}h^k\alpha_k + h^{k+1}\hat{d}_1 + \cdots + h^n\hat{d}_{n-k}) \end{aligned}$$

for $k = 0, \ldots, n$ and for some constants $\hat{d}_1, \ldots, \hat{d}_{n-k}$ dependent on $\bar{\alpha_k}$.

Since $\alpha_0 = 1$ it follows that $\bar{\alpha_0} \neq 0$. Therefore $r(0) = (-1)^n\bar{\alpha_0} \neq 0$. Thus by Lemma 3.2.2, λ_i is a root of $r(\lambda)$ if and only if $\dfrac{1}{\lambda_i}$ is a root of the polynomial

$$r^*(\lambda) := (-1)^n(\bar{\alpha}_0\lambda^n + \bar{\alpha}_1\lambda^{n-1} + \cdots + \bar{\alpha}_{n-1}\lambda + \bar{\alpha}_n).$$

Since $Re(\lambda) < 0$ if and only if $Re\left(\dfrac{1}{\lambda}\right) < 0$ and $Re(\lambda) > 0$ if and only if $Re\left(\dfrac{1}{\lambda}\right) > 0$, then it is true that $r(\lambda)$ is Hurwitz stable if and only if $r^*(\lambda)$ is Hurwitz stable. Thus the Hurwitz stability of $p_A(\lambda)$ and $r^*(\lambda)$ can be compared using Theorem 3.2.4 (the Routh-Hurwitz Criterion).

Construct Hurwitz matrices for $(-1)^n p_A(\lambda)$ and $(-1)^n r^*(\lambda)$ as:

$$H_{p_A} = \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_5 & \alpha_7 & \cdots & 0 & 0 \\ \alpha_0 & \alpha_2 & \alpha_4 & \alpha_6 & \cdots & 0 & 0 \\ 0 & \alpha_1 & \alpha_3 & \alpha_5 & \cdots & 0 & 0 \\ 0 & \alpha_0 & \alpha_2 & \alpha_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \alpha_{n-2} & \alpha_n \end{pmatrix} \tag{3.9}$$

and

$$H_{r^*} = \begin{pmatrix} \bar{\alpha}_1 & \bar{\alpha}_3 & \bar{\alpha}_5 & \bar{\alpha}_7 & \cdots & 0 & 0 \\ \bar{\alpha}_0 & \bar{\alpha}_2 & \bar{\alpha}_4 & \bar{\alpha}_6 & \cdots & 0 & 0 \\ 0 & \bar{\alpha}_1 & \bar{\alpha}_3 & \bar{\alpha}_5 & \cdots & 0 & 0 \\ 0 & \bar{\alpha}_0 & \bar{\alpha}_2 & \bar{\alpha}_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \bar{\alpha}_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \bar{\alpha}_{n-2} & \bar{\alpha}_n \end{pmatrix}. \tag{3.10}$$

One can write $H_{p_A} = (\eta_{ij})$ and $H_{r^*} = (\bar{\eta}_{ij})$, where

$$\eta_{ij} = \begin{cases} \alpha_{2j-i}, & \text{if} \quad 0 \leq 2j - i \leq n \\ 0, & \text{otherwise} \end{cases}$$

and

$$\bar{\eta}_{ij} = \begin{cases} \bar{\alpha}_{2j-i}, & \text{if} \quad 0 \leq 2j - i \leq n \\ 0, & \text{otherwise} \end{cases}$$

Using the notation from Definition 3.2.2 and defining,

$$S := \{\sigma \in S_k \mid 0 \leq 2i - \sigma(i) \leq n,\ i = 1, \ldots, k\}$$

one can write the kth leading principal minor of (3.9) as:

$$\begin{aligned}\Delta_k^{p_A} &= \sum_{\sigma\in S_k} sgn(\sigma)\prod_{i=1}^{k}\eta_{\sigma(i),i}\\ &= \sum_{\sigma\in S} sgn(\sigma)\prod_{i=1}^{k}\alpha_{2i-\sigma(i)}\end{aligned}$$

and the kth leading principal minor of (3.10) as:

$$\begin{aligned}\Delta_k^{r^*} &= \sum_{\sigma\in S_k} sgn(\sigma)\prod_{i=1}^{k}\bar{\eta}_{\sigma(i),i}\\ &= \sum_{\sigma\in S} sgn(\sigma)\prod_{i=1}^{k}\bar{\alpha}_{2i-\sigma(i)}\\ &= \sum_{\sigma\in S} sgn(\sigma)\prod_{i=1}^{k}\left(\frac{\left((2^{n-(2i-\sigma(i))})(h^{2i-\sigma(i)})\alpha_{2i-\sigma(i)}+h^{2i-\sigma(i)+1}\hat{d}_1+\cdots+h^n\hat{d}_{n-(2i-\sigma(i))}\right)}{(1+hN^1)\cdots(1+hN^n)}\right)\\ &= \frac{\left(2^{\sum_{i=1}^{k}(n-i)}\right)\left(h^{\sum_{i=1}^{k}i}\right)\left(\sum_{\sigma\in S} sgn(\sigma)\left(\prod_{i=1}^{k}(\alpha_{2i-\sigma(i)}+h\tilde{d}_1+\cdots+h^{n-(2i-\sigma(i))}\tilde{d}_{n-(2i-\sigma(i))})\right)\right)}{(1+hN^1)^k\cdots(1+hN^n)^k}\\ &= \frac{\left(2^{kn-\frac{k(k+1)}{2}}\right)\left(h^{\frac{k(k+1)}{2}}\right)\left(\sum_{\sigma\in S} sgn(\sigma)\prod_{i=1}^{k}\alpha_{2i-\sigma(i)}\right)+h^{(\frac{k(k+1)}{2}+1)}\breve{d}_{(\frac{k(k+1)}{2}+1)}+\cdots+h^{nk}\breve{d}_{nk}}{(1+hN^1)^k\cdots(1+hN^n)^k}\\ &= \frac{\left(2^{kn-\frac{k(k+1)}{2}}\right)\left(h^{\frac{k(k+1)}{2}}\right)\Delta_k^{p_A}+h^{(\frac{k(k+1)}{2}+1)}\breve{d}_{(\frac{k(k+1)}{2}+1)}+\cdots+h^{nk}\breve{d}_{nk}}{(1+hN^1)^k\cdots(1+hN^n)^k}\end{aligned}$$

for $k=1,\ldots,n$, where for each i, $\tilde{d}_i$ and $\breve{d}_i$ are some constants dependent on $\Delta_k^{r^*}$.

Since the numerator of $\Delta_k^{r^*}$ is a polynomial in h, then Lemma 3.2.3 implies that there exists $q_k>0$ such that for $h\in(0,q_k]$, $\Delta_k^{r^*}$ has the same sign as $\Delta_k^{p_A}$ for $k=1,\ldots,n$. Also, there exists $q_0>0$ such that for $h\in(0,q_0]$, the coefficient $\bar{\alpha}_0$ has the same sign as α_0.

Let

$$q_{\bar{x}} = \min_{0\le k\le n}(q_k)$$

be the restriction on h for the equillibrium $\bar{x}=(\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_n)$. Let Γ be the finite set of all equilibria of the PDS (3.1), and let

$$Q = \min_{\bar{x}\in\Gamma} q_{\bar{x}}. \tag{3.11}$$

Then by Theorem 3.2.4 and the application of the bilinear transform Theorem 3.2.5 it is found that $p_B(\lambda)$ is Schur stable if and only if $p_A(\lambda)$ is Hurwitz stable for $h \in (0, Q]$.

Finally one can conclude by Theorem 2.0.1 and its discrete counterpart Theorem 2.0.2 that the equilibria of the PDS (3.1) and the fixed points of the NSFD scheme (3.5) have the same local stability for $h \in (0, Q]$. □

Proof. (Theorem 3.2.7)

Suppose that the PDS (3.1) and the NSFD scheme (3.5) have the same local stability at each equilibrium for $h \in (0, Q]$. Let $\varphi(h) = \phi(hq)/q$ for some $q > \frac{1}{Q}$, then $\varphi(h) < Q$. Thus, for all $h > 0$, the PDS (3.1) and the NSFD scheme (3.5) have the same local stability at each equilibrium and the NSFD scheme (3.5) is dynamically consistent with the PDS (3.1). □

CHAPTER 4

A Nonstardard Finite Difference Method for Autonomous Dynamical Systems With Positive Solutions

4.1 Introduction

Consider an $n-$dimensional dynamical system of the form

$$\frac{dx}{dt} = f(x(t)); \quad x(0) \in \mathbb{R}^n_+, \tag{4.1}$$

whose solutions are of the form $x = [x^1, x^2, \ldots, x^n]^T : \mathbb{R}_+ \to \mathbb{R}^n_+$ and f is a $C^1(\mathbb{R}^n_+, \mathbb{R}^n_+)$ function that satisfies conditions which guarantee that $\mathbb{R}^n_+$ is positively invariant (e.g. see Appendix B of [54]). It is assumed that the system (4.1) has a finite number of equilibria, each of which is hyperbolic. This condition is needed in order to obtain elementary stability of the NSFD method.

There have been several successful attempts at designing numerical methods which preserve positivity of solutions and the local behavior, but only for special classes of systems of ordinary differential equations which exhibit certain structure (e.g. [33, 44, 47, 59, 60, 61, 62, 63]). In Section 4.2 we present a new NSFD method [64] which preserves the aforementioned qualitative properties for the general $n-$dimensional system (4.1). The method is analyzed in Section 4.3.

4.2 Construction of the Numerical Method

Using the NSFD framework introduced in Definition 2.0.4, the new numerical method which approximates the system (4.1) is constructed as follows:

$$\frac{x_{k+1}^i - x_k^i}{\varphi(h)} = f^i(x_k)\omega_k^i, \tag{4.2}$$

where

$$\omega_k^i = \begin{cases} 1, & \text{if } f^i(x_k) \geq 0 \\ \dfrac{x_{k+1}^i}{x_k^i}, & \text{if } f^i(x_k) < 0 \end{cases} \tag{4.3}$$

for $i = 1, 2, \ldots, n$ and $\varphi(h) = h + \mathcal{O}(h^2)$ is a denominator function which is to be determined in the next section.

The numerical treatment of the right-hand side function in the Method (4.2) is motivated by some of the discretization ideas of J. Benz et al. [65]. In [65], the authors propose a numerical method that works for both conservative and non-conservative systems by using a term similar to equation (4.3) which switches based on the conservativity-property of the system.

4.3 Analysis of the Numerical Method

It can be easily shown that the numerical method (4.2) is a first-order method [25]. The zero-stability of the method follows from the fact that $\varphi(h)f^i(x_k)\omega_k^i \to 0$ as $h \to 0$, which implies that Method (4.2) is convergent.

4.3.1 Positivity

The numerical method (4.2) can be written in an explicit form as follows:

$$x_{k+1}^i = \begin{cases} x_k^i + \varphi(h)f^i(x_k), & \text{if } f^i(x_k) \geq 0 \\ \dfrac{(x_k^i)^2}{x_k^i - \varphi(h)f^i(x_k)}, & \text{if } f^i(x_k) < 0 \end{cases} \tag{4.4}$$

for $i = 1, 2, \ldots, n$.

To show that the method is positive, observe that when $x_k^i \geq 0$ and $f^i(x_k) \geq 0$, then $x_{k+1}^i = x_k^i + \varphi(h) f^i(x_k) \geq 0$. Also, when $x_k^i \geq 0$ and $f^i(x_k) < 0$, then $(x_k^i)^2 \geq 0$ and $-\varphi(h) f^i(x_k) > 0$, thus $x_{k+1}^i = \dfrac{(x_k^i)^2}{x_k^i - \varphi(h) f^i(x_k)} \geq 0$.

4.3.2 Elementary Stability

In order to ensure that the numerical method (4.2) is elementary stable, a special denominator function, $\varphi(h)$, must be selected. Let $\phi(h)$ be a real-valued function which satisfies $0 < \phi(h) < 1$ for all $h > 0$ and $\phi(h) = h + \mathcal{O}(h^2)$. A relationship between the eigenvalues of the Jacobians of the differential system (4.1) and the numerical method (4.2) at the equilibrium points can be used to guarantee that both (4.1) and (4.2) share the same local stability.

Denote by A the Jacobian of the differential system (4.1) evaluated at an equilibrium, $\bar{x}$. Then the Jacobian, B, of the numerical method (4.2) evaluated at $\bar{x}$ can be written as

$$B = I + \phi(h) A.$$

The following relationship holds:

$$\det(A - \lambda I) = \det\left(\frac{B - I}{\phi(h)} - \lambda I\right) = \left(\frac{1}{\phi(h)}\right)^n \det\left(B - [1 + \phi(h)\lambda] I\right). \tag{4.5}$$

Therefore λ is an eigenvalue of A if and only if $1 + \phi(h)\lambda$ is an eigenvalue of B.

Now, take

$$Q \geq \max\left\{\frac{|\lambda|^2}{2|\mathrm{Re}\lambda|}\right\} \tag{4.6}$$

across all equilibria, $\bar{x}$. Then, by choosing the denominator function

$$\varphi(h) = \frac{\phi(qh)}{q}, \tag{4.7}$$

where $q > Q$, the numerical method (4.2) can be shown to be elementary stable as follows:

Equations (4.6) and (4.7), along with the fact that $0 < \phi(qh) < 1$, imply that for any λ,

$$\varphi(h) < \frac{2|\mathrm{Re}\lambda|}{|\lambda|^2}$$

which yields

$$-2\varphi(h)|Re\lambda| + \varphi(h)^2|\lambda|^2 < 0.$$

If the system (4.1) is locally stable at $\bar{x}$, then $Re(\lambda) < 0$. Using (4.5), which gives a relation between the eigenvalues of differential system (4.1) and the numerical method (4.2), the eigenvalues, $1 + \varphi(h)\lambda$, of B satisfy

$$|1 + \varphi(h)\lambda| = \sqrt{1 - 2\varphi(h)|Re\lambda| + \varphi(h)^2|\lambda|^2} < 1.$$

Next, if the system (4.1) is unstable at $\bar{x}$, then there exists an eigenvalue λ_0 such that $Re(\lambda_0) > 0$. This implies that

$$|1 + \varphi(h)\lambda_0| = \sqrt{1 + 2\varphi(h)Re\lambda_0 + \varphi(h)^2|\lambda_0|^2} > 1.$$

This result demonstrates that the numerical method (4.2) shares the same local stability at each equilibrium as the original system (4.1) and is, thus, elementary stable.

Remark 2. The inequality (4.6) used to find Q has been previously employed in other papers (e.g. most recently in [33]). Also, instead of finding the eigenvalues directly, one can make use of the Routh-Hurwitz conditions to more easily find a suitable constant Q (see also [63]).

Remark 3. Method (4.2) can easily be modified to work for dynamical systems with negative solutions, by switching the inequalities in Equation (4.3).

CHAPTER 5

Numerical Simulations

In this chapter the NSFD methods (3.5) and (4.2) are applied to some specific examples which show the usefulness of the proposed new methods, but also some potential hazards.

5.1 Example 1: HIV Intervention Model

In the last decade the HIV research focus moved toward development of prevention strategies and interventions. The field celebrated the effective solutions of almost eliminating mother-to-child transmission [66], proven reduction of male risk through circumcision [67, 68, 69], as well as the advances in the treatment for couples in which only one of the partners is HIV positive [70]. Recently, significant attention and hope is associated with the growing number of promising options for pre-exposure prophylaxis (PrEP) which when applied topically, in the form of microbicide gels, or taken as a daily pill (oral PrEP) substantially reduces the risk of HIV acquisition [71, 72, 73, 74, 75].

Here, we investigate the asymptotic dynamics of a new model [76] that includes PrEP interventions in populations with HIV. Consider the following system of differential equations:

$$\begin{aligned} \frac{dS^p}{dt} &= k\Lambda - (1-\alpha_s)\beta\frac{S^pI}{N} - \mu S^p \\ \frac{dS}{dt} &= (1-k)\Lambda - \beta\frac{SI}{N} - \mu S \\ \frac{dI}{dt} &= \beta\frac{SI}{N} + (1-\alpha_s)\beta\frac{S^pI}{N} - (\mu+d)I, \end{aligned} \tag{5.1}$$

where the total population, $N(t) = S^p(t) + S(t) + I(t)$, is divided into three major classes, according to their HIV and PrEP status: susceptible PrEP users (S^p), susceptible individuals who do not use PrEP (S) and infected individuals (I). The initial conditions are chosen to be as follows:

$$\begin{aligned} S^p(0) &= k(1-P)N(0) \\ S(0) &= (1-k)(1-P)N(0) \\ I(0) &= PN(0). \end{aligned}$$

Here P is the initial HIV prevalence, $N(0)$ is the initial population size and k is the initial coverage of PrEP among susceptible individuals. All parameters are described in Table 5.1. Frequency-dependent transmission is assumed and the cumulative HIV acquisition risk per year β is calculated based on the HIV risk per act (β_a) with a HIV-positive partner and the average number of sex acts per year (n):

$$\beta = 1 - (1-\beta_a)^n.$$

A constant proportion k of the new recruits are assume to start using PrEP. The same proportion of the susceptible individual are assume to start on PrEP initially. Since PrEP provides imperfect protection against HIV some of the PrEP users become infected. The risk of drug-resistance emergence among infected PrEP users has been discussed in the HIV prevention community [77, 78, 79] and wide-scale PrEP interventions will likely include periodic HIV screening of all prescribed users. Therefore, we assume that PrEP users stop using the product after acquiring HIV and all infected individuals accumulate in the compartment (I).

The basic reproduction number of model (5.1) is given by $R_0 = (1-k)R_0(S) + kR_0(S^p) \triangleq (1-k)\frac{\beta}{\mu+d} + k\frac{(1-\alpha_s)\beta}{\mu+d} = \frac{(1-\alpha_s k)\beta}{\mu+d}$. The following proposition characterizes the asymptotic dynamics.

Table 5.1. Model parameters

Par.	Description	Baseline Value	Ref.
Λ	Constant recruitment: Fixed number of individuals who become sexually active annually		
β_a	HIV acquisition risk per act	0.2%-0.7%	[80]
n	Average number of sexual acts per year	60-100	[81, 82]
β	Annual HIV acquisition risk in partnership with infected individual	11%-50%	calculated from β_a and n
$\frac{1}{\mu}$	Time (in years) to remain sexually active	30-40	[83]
d	Annual rate of progression to AIDS	8.3%-12.5%	[84, 85]
k	Proportion of the new recruits using PrEP		assumed
α_s	Efficacy of PrEP in reducing susceptibility	50%-90%	[71, 72, 73, 74]

Proposition 5.1.1. *With nonnegative initial conditions, solutions for* (5.1) *are nonnegative and bounded. If* $R_0 < 1$ *then the model has an unique disease-free equilibrium* $E_0 = (\frac{k}{\mu}\Lambda, \frac{1-k}{\mu}\Lambda, 0)$ *which is locally stable. Further, if* $R_0 < \frac{\mu}{\mu+d}$ *then* E_0 *is globally stable. If* $R_0 > 1$ *then* E_0 *is unstable and the model possesses an unique endemic equilibria* E^* *which is locally stable and satisfies:*

$$E^* = \left(\frac{\Lambda(\Lambda - dI^*)}{\mu} \frac{k}{\Lambda + ((1-\alpha_s)\beta - d)I^*}, \frac{\Lambda(\Lambda - dI^*)}{\mu} \frac{1-k}{\Lambda + (\beta - d)I^*}, I^* \right)$$

where I^* *is a solution of*

$$F(I) \triangleq \beta - d - \mu - \frac{\beta\mu I}{\Lambda - dI} - \frac{\alpha_s \beta k \Lambda}{(1-\alpha_s)\beta I + \Lambda - dI} = 0$$

in the interval $(0, \frac{\Lambda}{d})$.

For a proof of Proposition 5.1.1 see Appendix B.

For our numerical simulations, the following parameter set $\Lambda = 1 \times 10^6$, $\beta_a = 0.0038$, $n = 80$, $\mu = 1/35$, $d = 1/10$, $k = 1/5$, and $\alpha_s = 1/2$ is used. The initial conditions are calculated from (5.2), where $N(0) = 27,172,431$ and $P = 0.1766$. This gives us a basic reproductive number of $R_0 = 1.6337$, so that the endemic equilibrium E^* is stable, and the disease-free equilibrium E_0 is unstable.

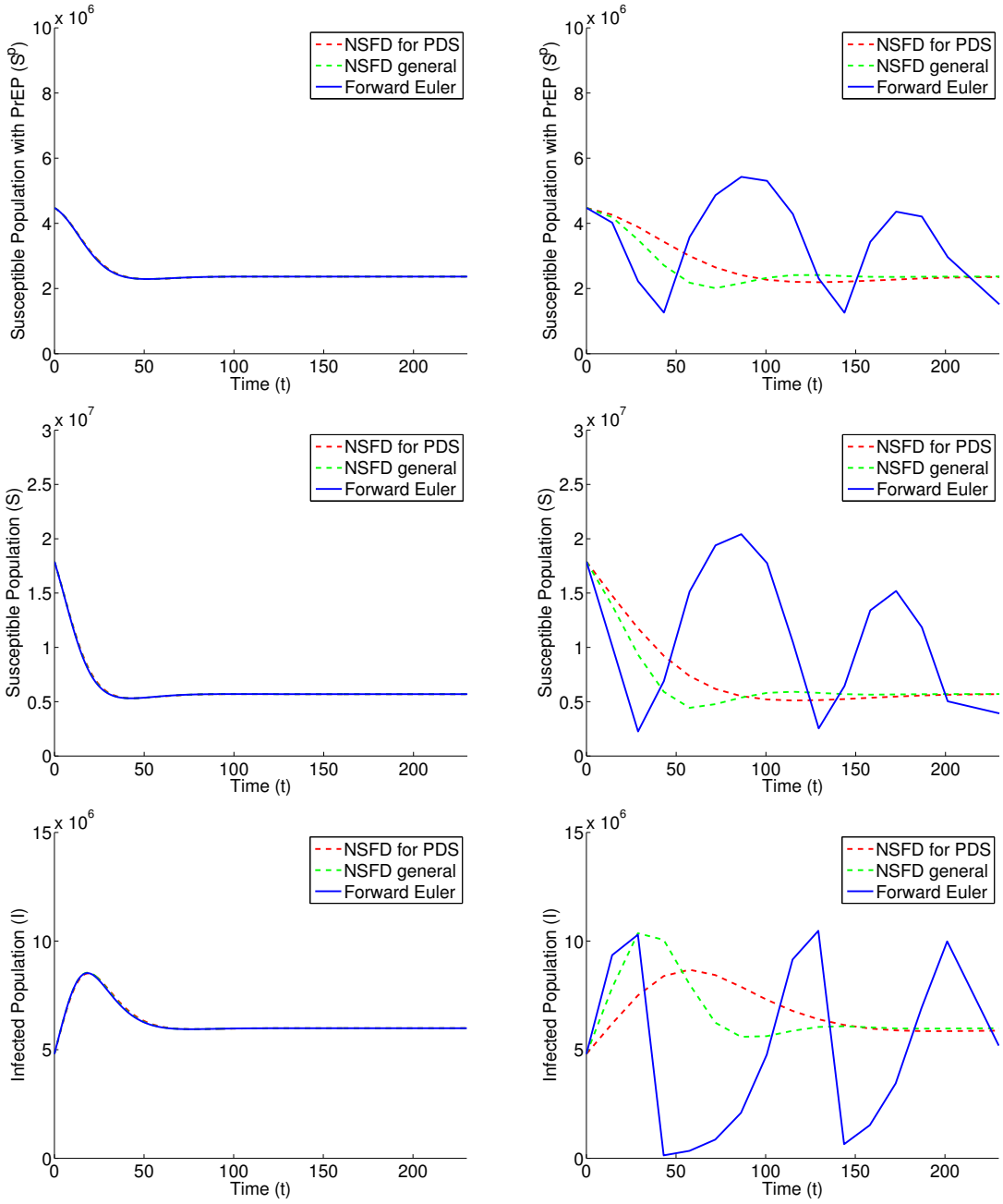

Figure 5.1. Numerical approximations of the system (5.4) using the NSFD method for PDS (5.2), the NSFD method for general systems (5.3) and the forward Euler method with timestep $h = 0.2556$ (left) and $h = 14.375$ (right).

To simulate population dynamics we use the NSFD method for PDS (3.5):

$$\begin{aligned}
\frac{S^p_{k+1}-S^p_k}{\varphi_1(h)} &= k\Lambda - ((1-\alpha_s)\beta\tfrac{I_k}{N_k}+\mu)S^p_{k+1},\\
\frac{S_{k+1}-S_k}{\varphi_1(h)} &= (1-k)\Lambda - (\beta\tfrac{I_k}{N_k}+\mu)S_{k+1},\\
\frac{I_{k+1}-I_k}{\varphi_1(h)} &= \beta\tfrac{S_kI_k}{N_k} + (1-\alpha_s)\beta\tfrac{S^p_kI_k}{N_k} - (\mu+d)I_{k+1},
\end{aligned} \tag{5.2}$$

and the NSFD method for general systems (4.2):

$$\begin{aligned}
\frac{S^p_{k+1}-S^p_k}{\varphi_2(h)} &= \left(k\Lambda - (1-\alpha_s)\beta\tfrac{S^p_kI_k}{N_k} - \mu S^p_k\right)\omega^1_k,\\
\frac{S_{k+1}-S_k}{\varphi_2(h)} &= \left((1-k)\Lambda - \beta\tfrac{S_kI_k}{N_k} - \mu S_k\right)\omega^2_k,\\
\frac{I_{k+1}-I_k}{\varphi_2(h)} &= \left(\beta\tfrac{S_kI_k}{N_k} + (1-\alpha_s)\beta\tfrac{S^p_kI_k}{N_k} - (\mu+d)I_k\right)\omega^3_k,
\end{aligned} \tag{5.3}$$

where ω^i_k is given by Equation (4.3). The denominator functions are given by $\varphi_1(h) = (1-\exp(-q_1h))/q_1$ and $\varphi_2(h) = (1-\exp(-q_2h))/q_2$, where $q_1 = 0.05$ which satisfies $q_1 > Q_1 = 0.0464016$, and $q_2 = 0.06$ which satisfies $q_2 > Q_2 = 0.0547568$. The value of the parameter Q_1 was calculated using methodology in Appendix A. The value Q_2 is calculated following the procedure described in Subsection 4.3.2. The forward Euler method [86] is used for comparison.

The first set of simulations are shown in Figure 5.1. One can see that for relatively small values of h, all of the numerical methods have the same behavior, but for larger values of h, the forward Euler method oscillates wildly, while the NSFD methods (albiet with different transient behaviour) still converge to the correct equilibria.

It is important to note that the NSFD methods are first order methods, and while they preserve the required properties for large values of h, one pays the price in accuracy for such an h. This is readily seen in the simulations, where the transient behavior is visibly skewed.

5.2 Example 2: Competition in a Chemostat

A chemostat is a system with a continuous inflow of medium and a balancing outflow of nutrients and organisms which keeps the volume of culture constant. The study of chemostat models serves as a basis for many biological models, including models of aquatic ecosystems. Mathematical theory concerning populations in a chemostat has been widely developed [54] and provides a firm foundation for analysis. Here, we consider the following $4-$dimensional dynamical system presented by Martines et al. [87], which models competition between two organisms in a chemostat with limited resources and allelopathic interactions:

$$\begin{aligned}
\frac{dN_1}{dt} &= (\mu_1(R) - m_1 - D)N_1, \\
\frac{dN_2}{dt} &= (\mu_2(R) - m_2(P) - D)N_2, \\
\frac{dR}{dt} &= D(R_{in} - R) - (\mu_1(R) - m_1)q_1N_1 - (\mu_2(R) - m_2(P))\, q_2N_2, \\
\frac{dP}{dt} &= \alpha(\mu_1^{max} - \mu_1(R))N_1 - (D + K)P.
\end{aligned} \tag{5.4}$$

In this model, $N_1(t)$ and $N_2(t)$ represent the densities of the competing populations, where the first species produces a toxin which negatively affects the second species. The nutrient concentration is represented by $R(t)$, and the poison concentration is represented by $P(t)$. Here D is the dilution rate of the chemostat, and R_{in} is the concentration of the limiting nutrient supplied in the inflowing medium. The positive constants α and K affect the poison production and dilution, respectively. The growth functions of the two species are given by $\mu_i(R) = \mu_i^{max}R/(k_i + R)$, where μ_i^{max} is the maximal growth rate as R approaches infinity, and k_i is the half-saturation constant. The per capita mortality rates of the species are m_i, $i = 1, 2$, where m_1 is a positive constant and $m_2(P) = \gamma P + m_2^{min}$, with $\gamma = \mu_2^{max}/(2k_2)$.

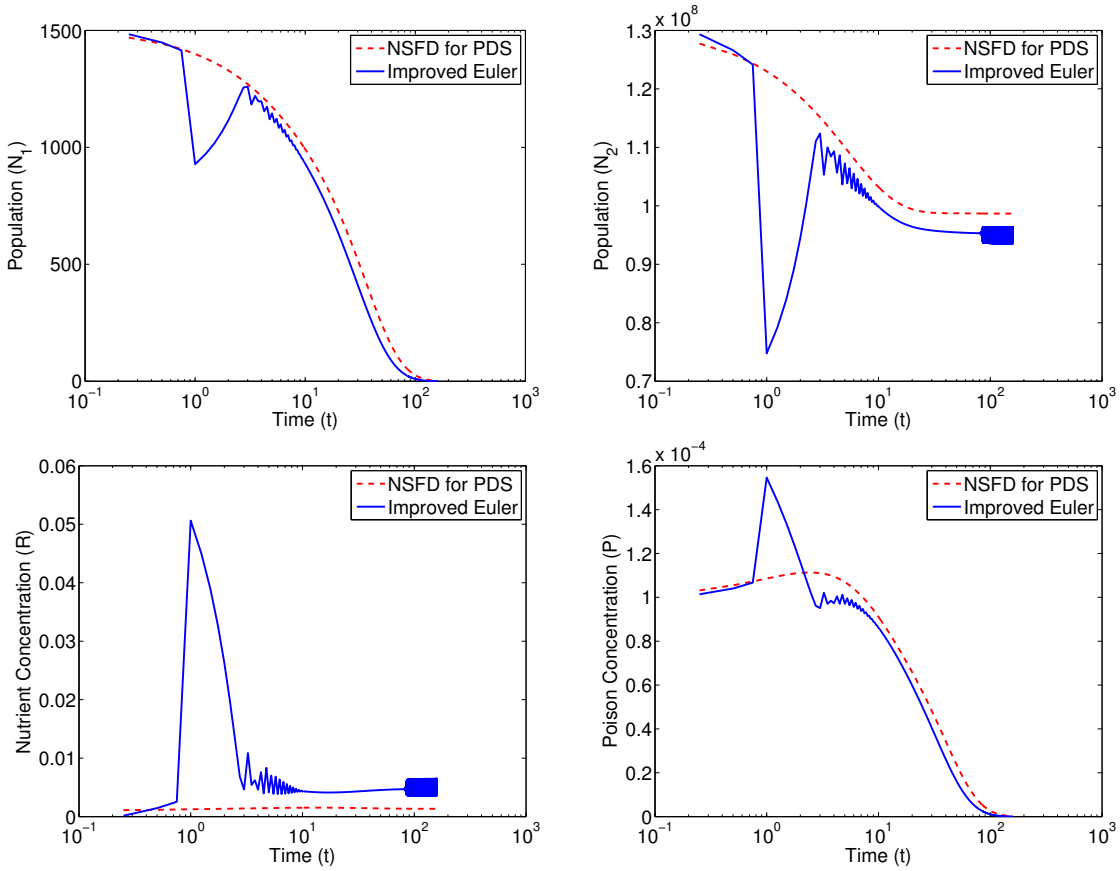

Figure 5.2. Numerical approximations of the system (5.4) using the NSFD method for PDS (5.5) and the improved Euler's method with timestep $h = 0.25$.

All equilibria of system (5.4) are found and the local stability analysis is done in [87]. The parameter set $q_1 = q_2 = 10^{-9}, k_1 = k_2 = 0.001, K = 0.1, D = 0.2, R_{in} = 0.1, \alpha = 10^{-7}, \mu_1^{max} = 0.6, m_1 = 0.2, \mu_2^{max} = 0.7, m_2^{min} = 0.2$, is considered which leads to a case of bistability. For this parameter set there are four equilibria, $E_0 = (0, 0, 1/10, 0)$, $E_1 = (9.8 \times 10^7, 0, 1/500, 98/15)$, $E_2 = (0, 296000000/3, 1/750, 0)$, and $E_3 = (20000/7, 685980000/7, 1/500, 1/5250)$, where E_0 and E_3 are unstable and E_1 and E_2 are stable.

In our numerical simulations, the NSFD method (3.5) is used:

$$\begin{aligned}
\frac{N_{1,k+1}-N_{1,k}}{\varphi(h)} &= \mu_1(R_k)N_{1,k} - (m_1+D)N_{1,k+1},\\
\frac{N_{2,k+1}-N_{2,k}}{\varphi(h)} &= \mu_2(R_k)N_{1,k} - (m_2(P_k)+D)N_{2,k+1},\\
\frac{P_{k+1}-P_k}{\varphi(h)} &= DR_{in} + m_1q_1N_{1,k} + m_2(P_k)q_2N_{2,k}\\
&\quad - \left(D + \frac{\mu_1(R_k)}{R_k} + \frac{\mu_2(R_k)}{R_k}q_2N_{2,k}\right)R_{k+1},\\
\frac{P_{k+1}-P_k}{\varphi(h)} &= \alpha(\mu_1^{max} - \mu_1(R_k))N_{1,k} - (D+K)P_{k+1}.
\end{aligned} \tag{5.5}$$

with denominator function $\varphi(h) = (1-\exp(-qh))/q$, where $q = 0.9$ which satisfies $q > Q = 0.859574$. The value of the parameter Q was calculated using methodology in Appendix A. The NSFD method (5.5) is compared with the improved Euler's method (also known as Heun's method [86]) in order to show some of the useful properties of the nonstandard construction. The initial conditions used are $N_1(0) = 1.5\times10^3$, $N_2(0) = 1.3\times10^8$, $R(0) = 10^{-3}$, and $P(0) = 10^{-4}$. As it can be seen in Figure 5.2, the NSFD method (5.5) converges to the correct equilibrium E_2 even when large time-step h is used, while the improved Euler's method results in erratic short-term behavior and, in population N_2, oscillatory long-term behavior where none should exist.

5.3 Example 3: 2−Dimensional Autonomous System

In order to apply the method (3.5) from Chapter 3 to a system of differential equations, it is necessary to be able to write the system as a PDS. If this cannot be done, then the method cannot be applied to preserve the positivity of solutions and the local behavior of the dynamical system near equilibria. In this case, however, one

can still use the NSFD method (4.2). For illustrative purposes, the NSFD method (4.2) for general systems is applied to the following 2−dimensional system:

$$\begin{aligned} \frac{dx}{dt} &= y - x; \\ \frac{dy}{dt} &= \sin(xy) - y; \end{aligned} \tag{5.6}$$

with initial conditions $x(0) = 2$, $y(0) = 0.5$. It is easy to see that the system (5.6) has one trivial equilibrium at $x^* = 0$, $y^* = 0$, which is locally asymptotically stable.

The NSFD method (4.2) for System (5.6) is:

$$\begin{aligned} \frac{x_{k+1} - x_k}{\varphi(h)} &= (y_k - x_k)\,\omega_k^1, \\ \frac{y_{k+1} - y_k}{\varphi(h)} &= (\sin(x_k y_k) - y_k)\,\omega_k^2, \end{aligned} \tag{5.7}$$

where ω_k^i is given by Equation (4.3). Here, we have selected the denominator function for the NSFD method (4.2) as $\phi(h) = (1 - \exp(-qh))/q$, where $q = 0.6$ satisfies $q > Q = 0.5$. The value Q is calculated following the procedure described in Subsection 4.3.2.

In Figure 5.3, the first order NSFD method (5.7) is compared with the second order Heun's method. One can see that they both converge for the small time step $h = 0.01$. However, the Heun's method does not preserve positivity and does not approach the equilibrium for the larger time step $h = 2$, while the NSFD method (5.7) still preserves all the desired properties.

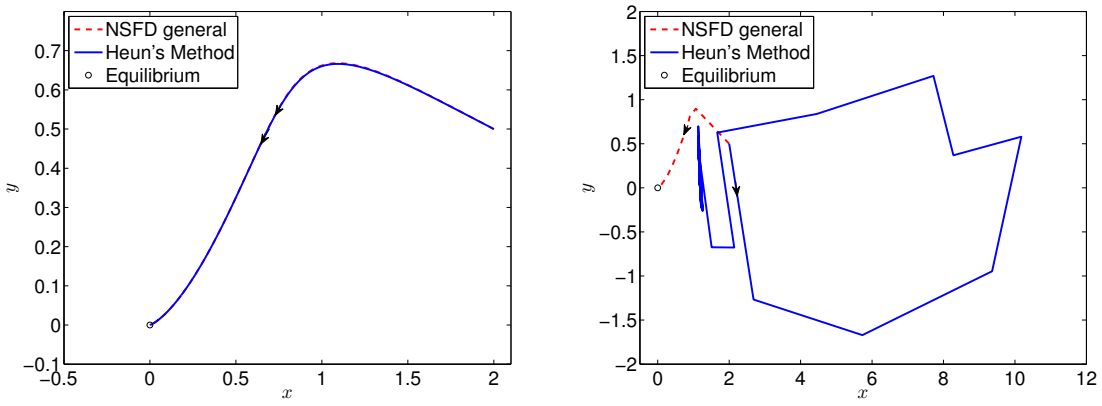

Figure 5.3. Numerical approximations of the system (5.6) using the NSFD method for general systems (5.7) and the Heun's method with time-step $h = 0.01$ (left) and $h = 2$ (right).

CHAPTER 6

Conclusions

In this study a nonstandard finite difference (NSFD) method for productive-destructive systems (PDS) of the form (3.1) is constructed according to the rules proposed in [47]. It is demonstrated that the numerical method is dynamically consistent with general $n-$dimensional PDS, for all $n \in \mathbb{N}$. A novel approach is used to establish the correspondence between the stability of the equilibrium points of the continuous and discrete systems. It is based on a bilinear transformation in the complex plane which allows one to convert the conditions for an equilibrium of the NSFD (Schur conditions) into conditions comparable with those for an equilibrium of the PDS (Hurwitz conditions). Using similar ideas, a new NSFD method is constructed for solving general dynamical systems with positive solutions. It preserves two important qualitative dynamical properties, namely, elementary stability and positivity of solutions. The method is both computationally efficient and easy to implement, and it can be used to solve a broad range of problems in science and engineering. Three examples were given which show both the usefulness of the numerical methods, but also some of the potential hazards which may be encountered. Numerical simulations were presented to illustrate the practical application of the methods and support the theoretical results.

APPENDIX A

Construction of Denominator Functions for Productive-Destructive Systems

Here, the methodology for constructing a denominator function which guarantees dynamical consistency for productive-destructive systems is given. The construction is based off the proof of Theorem 3.2.6. First, let $\phi(h) = h + O(h^2)$, where $0 < \phi(h) < 1$ for all $h > 0$, and Γ be the finite set of all equilibria of the PDS (3.1). For each equilibrium $\bar{x} \in \Gamma$, do the following:

1. Construct the Jacobian of the NSFD method (3.5) evaluated at equilibrium point $\bar{x} = (\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)$:

$$B = \begin{pmatrix} \frac{1 - h\bar{x}^1 N_1^1 + hP_1^1}{1 + hN^1} & \frac{-h\bar{x}^1 N_2^1 + hP_2^1}{1 + hN^1} & \cdots & \frac{-h\bar{x}^1 N_n^1 + hP_n^1}{1 + hN^1} \\ \frac{-h\bar{x}^2 N_1^2 + hP_1^2}{1 + hN^2} & \frac{1 - h\bar{x}^2 N_2^2 + hP_2^2}{1 + hN^2} & \cdots & \frac{-h\bar{x}^2 N_n^2 + hP_n^2}{1 + hN^2} \\ \vdots & \vdots & & \vdots \\ \frac{-h\bar{x}^n N_1^n + hP_1^n}{1 + hN^n} & \frac{-h\bar{x}^n N_2^n + hP_2^n}{1 + hN^n} & \cdots & \frac{1 - h\bar{x}^n N_n^n + hP_n^n}{1 + hN^n} \end{pmatrix}.$$

2. Find the characteristic polynomial of B:

$$p_B(\lambda) = det(B - \lambda I) = (-1)^n(\lambda^n + \beta_1\lambda^{n-1} + \beta_2\lambda^{n-2} + \cdots + \beta_{n-1}\lambda + \beta_n).$$

3. Form the polynomial $r(\lambda) = (\lambda - 1)^n p_B\left(\frac{\lambda + 1}{\lambda - 1}\right)$
$= (-1)^n(c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n).$

4. Construct the Hurwitz matrix

$$H = \begin{pmatrix} c_1 & c_3 & c_5 & c_7 & \cdots & 0 & 0 \\ c_0 & c_2 & c_4 & c_6 & \cdots & 0 & 0 \\ 0 & c_1 & c_3 & c_5 & \cdots & 0 & 0 \\ 0 & c_0 & c_2 & c_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & c_{n-2} & c_n \end{pmatrix}.$$

5. Evaluate the n leading principal minors of H: $\Delta_1, \ldots, \Delta_n$.

6. Find the least positive root, h_k, of the numerator of $\Delta_k(h)$, for $k = 1, \ldots, n$ and the least positive root, h_0, of c_0.

7. Take $q_{\bar{x}} = \min\limits_{0 \le k \le n} (h_k)$.

Let $Q = \min\limits_{\bar{x} \in \Gamma} q_{\bar{x}}$, so that if $q > \dfrac{1}{Q}$, then the function $\varphi(h) = \phi(hq)/q$ guarantees dynamical consistency. If Q is unbounded, i.e., there are no positive roots of $\Delta_k(h)$, for $k = 1, \ldots, n$, then one may choose $\varphi(h) = h$.

APPENDIX B

Proof of Proposition 5.1.1

Positivity and boundedness of solutions can be easily proved. Then by a Gronwall inequality [51] we have that $\frac{dS^p}{dt} \leq k\Lambda - \mu S^p$ implies $\limsup_{t\to\infty} S^p \leq \frac{k\Lambda}{\mu}$ and $\frac{dS}{dt} \leq (1-k)\Lambda - \mu S$ implies $\limsup_{t\to\infty} S \leq \frac{(1-k)\Lambda}{\mu}$. Therefore $\frac{dN}{dt} = \frac{dS^p}{dt} + \frac{dS}{dt} + \frac{dI}{dt} = \Lambda - \mu N - dI \geq \Lambda - (\mu + d)N$ implies $\liminf_{t\to\infty} N \geq \frac{\Lambda}{\mu+d}$. Then in the long term we have $\frac{S}{N} \leq \frac{(1-k)(\mu+d)}{\mu}$, and $\frac{S^p}{N} \leq \frac{k(\mu+d)}{\mu}$. This implies that the following inequality holds

$$\begin{aligned}\frac{dI}{dt} &\leq I\left[\beta\frac{(1-k)(\mu+d)}{\mu} + (1-\alpha_s)\beta\frac{k(\mu+d)}{\mu} - (\mu+d)\right] \\ &= I\frac{(\mu+d)^2}{\mu}\left[\beta\frac{1-k}{\mu+d} + (1-\alpha_s)\beta\frac{k}{\mu+d} - \frac{\mu}{\mu+d}\right] \\ &= I\frac{(\mu+d)^2}{\mu}\left(R_0 - \frac{\mu}{\mu+d}\right).\end{aligned}$$

Now if $R_0 < \frac{\mu}{\mu+d}$, then because of the positivity of the solution, we know $\lim_{t\to\infty} I = 0$. Then combining this result with the equations in Model (5.1), implies that $\lim_{t\to\infty} S^p = \frac{k\Lambda}{\mu}$ and $\lim_{t\to\infty} S = \frac{(1-k)\Lambda}{\mu}$. Thus, global stability of the infection-free steady state $E_0 = (\frac{k\Lambda}{\mu}, \frac{(1-k)\Lambda}{\mu}, 0)$ under condition $R_0 < \frac{\mu}{\mu+d}$ is proved. For local stability of E_0, we consider the corresponding eigenvalues $\lambda_1 = -\mu < 0$, $\lambda_2 = -\mu < 0$ and $\lambda_3 = (\mu+d)(R_0-1)$. Therefore, E_0 is stable when $R_0 < 1$ and unstable when $R_0 > 1$.

Now we consider E^*. By setting $F(I) = 0$ and dividing by $\Lambda^2(\mu+d)$, we obtain that I^* is a root of:

$$\begin{aligned}\hat{F}(I) \triangleq\ & (R_0-1) + \left((d-(1-\alpha_s)\beta) - (\mu+d-(1-\alpha_s)\beta)\frac{\beta}{\mu+d} - d(R_0-1)\right)\frac{I}{\Lambda} \\ & -(d-\beta)(d-(1-\alpha_s)\beta)\frac{I^2}{\Lambda^2}.\end{aligned}$$

Substituting in iN where $i := \frac{I}{N}$, and $N = \frac{\Lambda}{\mu+di}$ into $\hat{F}(I) = 0$ we notice that I^* is also a root of:

$$\bar{F}(iN) \triangleq \mu(\mu+d)(1-R_0) + \beta\left(\mu + d\alpha_s k - (\beta-\mu-d)(1-\alpha_s)\right)i + (1-\alpha_s)\beta^2 i^2.$$

Now substituting back in for I and $N = \dfrac{\Lambda - dI}{\mu}$, we notice that I^* is a root of:

$$\bar{F}(I) = \mu(\mu+d)(1-R_0) + \beta\left(\mu + d\alpha_s k - (\beta-\mu-d)(1-\alpha_s)\right)\frac{I}{N} + (1-\alpha_s)\beta^2\frac{I^2}{N^2}.$$

Now we have $\hat{F}(0) = (R_0 - 1)$ and $\hat{F}(\frac{\Lambda}{d}) = -\dfrac{(1-\alpha_s)\beta^2\mu}{d^2(\mu+d)} < 0$.

Assume $R_0 > 1$, then $\hat{F}(0) > 0$ and also $\beta > d$. Since $\hat{F}(0) > 0$, $\hat{F}(\dfrac{\Lambda}{d}) < 0$ and $\hat{F}$ is a second order polynomial, we have existence of a unique endemic equilibrium.

Assume $R_0 < 1$. Since

$$\mu + d\alpha_s k - (\beta - \mu - d)(1-\alpha_s) \geq \mu + d\alpha_s k - \alpha_s k(\mu+d)\left(\frac{1-\alpha_s}{1-\alpha_s k}\right)$$
$$\geq \alpha_s k(\mu+d)\left(1 - \frac{1-\alpha_s}{1-\alpha_s k}\right) > 0,$$

we have that $\bar{F}(I) > 0$ for all $I \in (0, \dfrac{\Lambda}{d})$ and therefore we have no solution I^*.

Therefore when $R_0 < 1$, there is only the disease-free equilibrium, E_0, and whenever $R_0 > 1$ there is both the disease-free equilibrium, E_0, and the endemic equilibrium, E^*.

Now we analyze the stability of the endemic equilibrium, E^*. Because of the complexity of the expressions, we will not express the positive steady state explicitly. Now assume that $R_0 = \frac{(1-\alpha_s k)\beta}{\mu+d} > 1$.

The Jacobian of the system is:

$$J = \begin{pmatrix} -\dfrac{(I+S)(1-\alpha_s)}{N^2}\beta I - \mu & \dfrac{(1-\alpha_s)}{N^2}\beta S^p I & -\dfrac{(S^p+S)(1-\alpha_s)}{N^2}\beta S^p \\ \dfrac{1}{N^2}\beta SI & -\dfrac{S^p+I}{N^2}\beta I - \mu & -\dfrac{S^p+S}{N^2}\beta S \\ \dfrac{I-(I+S)\alpha_s}{N^2}\beta I & \dfrac{I+S^p\alpha_s}{N^2}\beta I & \dfrac{(S^p+S)(S+S^p(1-\alpha_s))}{N^2}\beta - (\mu+d) \end{pmatrix}.$$

Using

$$P = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we see that the Jacobian is similar to

$$H = P^{-1}JP = \begin{pmatrix} -\mu & 0 & -d \\ \frac{1}{N^2}\beta SI & -\frac{1}{N}\beta I - \mu & -\frac{1}{N}\beta S \\ \frac{I-(I+S)\alpha_s}{N^2}\beta I & \frac{\alpha_s}{N}\beta I & \frac{(S^p-I)(1-\alpha_s)+S}{N}\beta - (\mu+d) \end{pmatrix}.$$

Rewriting H evaluated at the endemic equilibrium using $p^* := \frac{S^{p*}}{N^*}$, $s^* := \frac{S^*}{N^*}$, $i^* := \frac{I^*}{N^*}$, $N^* = \frac{\Lambda}{\mu+di^*}$ and $i^* + p^* + s^* = 1$, we obtain:

$$H = \begin{pmatrix} -\mu & 0 & -d \\ \beta s^* i^* & -\beta i^* - \mu & -\beta s^* \\ ((s^*+i^*)(1-\alpha_s) - s^*)\beta i^* & \alpha_s \beta i^* & -(1-\alpha_s)\beta i^* \end{pmatrix}$$

where $s^* = \dfrac{1}{\alpha_s \beta}(\mu + d - (1-i^*)(1-\alpha_s)\beta)$.

We have the characteristic polynomial:

$$f(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C$$

where

$$A = (2-\alpha_s)\beta i + 2\mu,$$

$$B = (1-\alpha_s)(\beta i^* + di^* + 2\mu)\beta i^* + \mu(\beta i^* + \mu) + \alpha_s(\beta - d)\beta s^* i^*$$

$$C = ((1-\alpha_s)(d\beta i^{*2} + d\mu i^* + \mu(\beta i^* + \mu)) + \alpha_s \mu(\beta - d)s^*)\beta i^*.$$

Clearly if $R_0 > 1$, then $\beta > d$, which implies that $A > 0, B > 0$ and $C > 0$. Also we have that

$$\begin{aligned} AB - C = \; & 3\mu^2\beta i^* + 2\mu^3 + \alpha_s^2 \mu \beta^2 i^{*2} + \alpha_s \mu(\beta - d)\beta i^* s^* + \alpha_s(2-\alpha_s)(\beta - d)\beta^2 i^{*2} s^* \\ & + (1-\alpha_s)(\mu d i^* + \beta^2 i^{*2} + 6\mu\beta i^* + 4\mu^2)\beta i^* \\ & + (1-\alpha_s)^2(\mu + (\beta + d)i^*)\beta^2 i^{*2} > 0. \end{aligned}$$

Therefore by the Routh-Hurwitz Criteria, the endemic equilibria, E^* is locally stable.

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BIOGRAPHICAL STATEMENT

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