ON SOLVING FINITELY REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

by

WILBER VENTURA

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To my church for their endless support, overwhelming kindness, and relentless love.

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ABSTRACT

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WILBER VENTURA, Ph.D.

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Supervising Professor: Andrzej Korzeniowski

Classical theory gives a closed form representation of the density $p(t, x)$, a solution to a linear parabolic PDE, via the Feynman-Kac Formula of the underlying diffusion process. In the non-linear PDE case there is no closed form representation for $p(t, x)$ and instead one solves a SDE running back in time whose initial (deterministic value) coincides with $p(t, x)$. This method of solving semi-linear parabolic PDEs is an effective alternative to known numerical schemes. Furthermore, the FBSDE approach allows for treatment of non-smooth coefficients in the PDE that cannot be handled by classical deterministic methods. One of the most important extensions of BSDEs is that of adding reflections. Roughly speaking, the solution of a Reflected BSDE (RBSDE) is forced to remain within some region by a so-called reflection process. We prove the existence and uniqueness of FR-FBSDE (Finitely Reflected Forward Backward SDE) along with a Donsker-type computational algorithm for effective approximate solution. Applications to option pricing in finance serve as an illustration of our results.

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CHAPTER 1

Introduction

The notion of the Backward Stochastic Differential Equations (BSDEs) and Forward Backward Stochastic Differential Equations (FBSDEs) were first introduced during the 1990s in a series of works by Pardoux, Peng and El-Karoui [detailed references to these articles will be provided in the course of development of our results in the subsequent chapters]. In a terminal value problem of an SDE the terminal distribution of the solution process is given. It seems natural to think that the solution could be obtained by simply progressing backwards in time. Unfortunately it is generally not so simple since solutions are required to be adapted. Thus a terminal value problem of an SDE must be reformulated into a BSDE problem where the solution is a pair of adapted processes.

Over the past two decades BSDEs became a subject of intense research and showed direct connections to Partial Differential Equations (PDEs), with numerous applications to Optimal Control and Finance among others. A continued interest in BSDEs culminated in the recent monographs on the subject by Touzi (2012) [29], Crépey (2013) [9], Delong (2013) [11], Pardoux and Răşcanu (2014) [25], which further underscore its growing relevance and generate an interest in the development of effective numerical solution algorithms.

1.1 Reflected Backward Stochastic Differential Equations

One of the most important extensions of BSDEs is that of adding reflections. Reflected Backward Stochastic Differential Equations (RBSDEs) were first introduced in [14]; here the authors considered an RBSDE with a given lower continuous obstacle so that solution must always be above that obstacle. The solution process is forced above the obstacle by a so called reflection process, hence called "Reflected" BSDEs. Additionally in [10] Cvitanic and Karatzas generalized the work in [14] by introducing the notion of BSDEs with two reflecting barriers. Roughly speaking, in [10] the authors look for a solution of a BSDE that is forced to remain between two given continuous processes.

Since [14] and [10] RBSDEs have received much attention and different extensions have been considered. For the purpose of our work we focus on the Finitely Reflected BSDEs (FR-BSDEs), first introduced in [22] . In the case of FR-BSDEs, the solution process undergoes projections at only finitely many points in time. For more details on FR-BSDE we refer the interested reader to [22] and [6].

1.2 Numerical Methods for Backward Stochastic Differential Equations

It is well known that second order semilinear PDEs can be interpreted in terms of decoupled Forward Backward SDEs, see [22]. In fact Ma, Protter, and Young proposed the four-step scheme as a method of approximating solutions of FBSDEs by way of standard numerical PDE methods, refer to [22].

As an alternative to the purely deterministic approached, probabilistic methods directly based on the resolution of the BSDEs have been introduce. Monte-Carlo methods based on solving conditional expectations have been proposed, refer to [3] and [4]. Techniques using random walks have been widely studied, see e.g. [8] and [20] , and our research focuses on developing such a method for FR-BSDEs.

1.3 Outline

In Chapter 2 we introduce basic notation, along with the necessary definitions and statements of key facts which constitute a basis for developing our results.

In Chapter 3 we prove the existence and uniqueness of the solution for the Finitely Reflected Forward Backward Stochastic Differential Equations (FR-FBSDEs) via a strict contraction arguments in a suitably defined norm. Unlike the typical approach, based on maximal inequality for semi-martingales, we employ a generalized Föllmer path-wise analysis and then combine with the weak convergence of Random Measures for right continuous process with left limits, referred to in the literature as cadlag processes [French "continue à droite, limite a gauche"].

In Chapter 4 we develop a random walk algorithm for effective approximate solution to the FR-FBSDEs and prove its convergence in Probability to the exact solution.

In Chapter 5 we present the standard financial mathematics problem of option valuations in complete markets. Furthermore in the case of Bermudan Options we demonstrate that the algorithm developed in Chapter 4 indeed yields the correct arbitrage-free price.

Chapter 6 is devoted to ideas for future work, among which are the extension to time continuous reflection boundaries, non-Lipschitz generating functions, and tractable rates of convergence.

CHAPTER 2

General Framework

2.1 Stochastic Analysis Overview

We begin with some important concepts from probability and stochastic analysis. All the information contained in this section is standard and is presented with further detail in books such as [2] and [19].

2.1.1 Basic Concepts

Definition 2.1.1. If Ω is a given set, then a σ-algebra $\mathbb{F}$ on Ω is a family of subsets of Ω with the following properties:

(i) $\emptyset \in \mathbb{F}$

(ii) If $F \in \mathbb{F}$, then $F^c \in \mathbb{F}$, where $F^c = \Omega \backslash F$ the complement of F in Ω

(iii) If $A_1, A_2, ... \in \mathbb{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathbb{F}$.

Given a set Ω the σ-algebra generated by a collection of sets $C \subset \Omega$ is the smallest σ-algebra containing C. The Borel σ-algebra is the σ-algebra generated by the collection of open sets of some topological space. Given a set Ω and a σ-algebra $\mathbb{F}$ on Ω the pair $(\Omega, \mathbb{F})$ is called a measurable space.

Definition 2.1.2. A probability measure P on a measurable space $(\Omega, \mathbb{F})$ is a set function $P : \mathbb{F} \rightarrow [0, 1]$ such that

(i) $P(\emptyset) = 0$ and$P(\Omega) = 1$

(ii) If $A_1, A_2, ... \in \mathbb{F}$ is a sequence of pairwise disjoint sets, then

$$P\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} P(A_i).$$

If P is a probability measure on the measurable space $(\Omega, \mathbb{F})$, then the triple $(\Omega, \mathbb{F}, P)$ is called a probability space. Given a probability space $(\Omega, \mathbb{F}, P)$, a random variable X is an $\mathbb{F}$-measurable function $X : \Omega \to \mathbb{R}^n$. The σ-algebra generated by a random variable X is the smallest σ-algebra containing all sets of the form $\{X^{-1}(B)|\ B \text{ is Borel set on } \mathbb{R}^n\}$.

Definition 2.1.3. The expectation or expected value of the random variable X is

$$E(X) := \int_\Omega X(\omega)dP(\omega).$$

A σ-algebra $\mathcal{H}$ is a sub-σ-algebra of $\mathbb{F}$ iff $\mathcal{H} \subset \mathbb{F}$.

Definition 2.1.4. Given a probability space $(\Omega, \mathbb{F}, P)$, $\mathcal{H}$ a sub-σ-algebra of $\mathbb{F}$, and random variable $X : \Omega \to \mathbb{R}^n$ such that

$$E(|X|) < \infty,$$

the conditional expectation of X given $\mathcal{H}$ denoted by $E(X|\mathcal{H})$ is the (a.s. unique) function from Ω to $\mathbb{R}^n$ satisfying:

(i) $E(X|\mathcal{H})$ is $\mathcal{H}$-measurable

(ii) $\int_H E(X|\mathcal{H})dP = \int_H X dP$, for all $H \in \mathcal{H}$.

2.1.2 Stochastic Processes

Definition 2.1.5. If I is a given totally order set, then a stochastic process is a parametrized collection of random variables

$$(X_t)_{t \in I}$$

defined on a probability space $(\Omega, \mathbb{F}, P)$ and assuming values in $\mathbb{R}^n$.

Usually I is taken to be $\mathbb{N}$, $\mathbb{N}_0$, $[0, T]$, or $[0, \infty)$. Note that for a fixed $t \in I$ we have the $\mathbb{F}$-measurable random variable

$$\omega \to X_t(\omega) \text{ where } \omega \in \Omega.$$

Furthermore for a fixed $\omega \in \Omega$ we have a function of I

$$t \to X_t(\omega) \text{ where } t \in I$$

often referred to as a path, realization, or trajectory of the stochastic process.

It is worth noting that a stochastic process can be viewed in many equivalent ways. For example one can recognize that a stochastic process is a map from the probability space $(\Omega, \mathbb{F}, P)$ into G, where G denotes the space of all $\mathbb{R}^n$ valued functions on I. Consider the case where we identify each $\omega \in \Omega$ with the function $t \to X_t(\omega)$ going from I into $\mathbb{R}^n$. Thus we may regard Ω as a subset of G. It follows that the σ-algebra $\mathcal{A}$ generated by the sets of the form

$$\{\omega | X_{t_1}(\omega) \in F_1, \cdot\cdot\cdot, X_{t_k}(\omega) \in F_k\}, \text{ for } F_i \subset \mathbb{R}^n \text{ Borel sets}$$

is a sub-σ-algebra of $\mathcal{F}$. In fact taking I to be $[0, \infty)$ gives that $\mathcal{A}$ is the Borel σ-algebra on G if G is endowed with the product topology. This implies that a stochastic process with index set $[0, \infty)$ can be viewed as some probability measure Q on the measurable space $(G, \mathcal{A})$.

Definition 2.1.6. The finite-dimensional distributions of the process $X = (X_t)_{t \in I}$ are the measures $\mu_{t_1,...,t_k}$ defined on $\mathbb{R}^{n \times k}$, $k = 1, 2, ...$, by

$$\mu_{t_1,...,t_k}(F_1 \times \cdot\cdot\cdot \times F_k) = P(X_{t_1} \in F_1, \cdot\cdot\cdot, X_{t_k} \in F_k) \text{ where } t_i \in I.$$

Here $F_1, ..., F_k$ are Borel sets in $\mathbb{R}^n$.

Given a family $\{\nu_{t_1,...,t_k} | k \in \mathbb{N}, t_i \in I\}$ of probability measures on $\mathbb{R}^{n \times k}$ it is important to construct a stochastic process $Y = (Y_t)_{t \in I}$ having $\nu_{t_1,...,t_k}$ as its finite-dimensional distributions. Kolmogorov's extension theorem states that this can be done given $\{\nu_{t_1,...,t_k}\}$ satisfies two natural consistency conditions.

Theorem 2.1.1. *For all* $t_1, ..., t_k \in I$, $k \in \mathbb{N}$ *let* $\nu_{t_1,...,t_k}$ *be probability measures on* $\mathbb{R}^{n\times k}$ *s.t.*

$$\nu_{t_{\sigma(1)},...,t_{\sigma(k)}}(F_1 \times \cdots \times F_k) = \nu_{t_1,...,t_k}(F_{\sigma(1)} \times \cdots \times F_{\sigma(k)}) \tag{2.1}$$

for all permutations σ *on* $\{1, ..., k\}$ *and*

$$\nu_{t_1,...,t_k}(F_1 \times \cdots \times F_k) = \nu_{t_1,...,t_k,t_{k+1},...,t_{k+m}}(F_1 \times \cdots \times F_k \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n) \tag{2.2}$$

for all $m \in \mathbb{N}$, *where the set on the right hand side has a total of* $k + m$ *factors.*

Then there exists a probability space $(\Omega, \mathbb{F}, P)$ *and a stochastic process* $(X_t)_{t\in I}$ *on* Ω, $X_t : \Omega \to \mathbb{R}^n$, *s.t.*

$$\nu_{t_1,...,t_k}(F_1 \times \cdots \times F_k) = P(X_{t_1} \in F_1, \cdots, X_{t_k} \in F_k)$$

for all $t_i \in I$, $k \in \mathbb{N}$ *and all Borel sets* F_i.

A very well known and widely used stochastic process is the Brownian Motion process also called the Wiener Process denoted by $W = (W_t)_{t\geq 0}$. This process describes the irregular motion of pollen grains suspended in water, here $W_t(\omega)$ gives the position at time t of the pollen grain ω. Using Kolmogorov's extension theorem we can construct and define the Brownian Motion process. In light of Kolmogorov's extension theorem, it suffices to specify a family $\{\nu_{t_1,...,t_k}\}$ of probability measures that agree with the observations of pollen grain behavior and satisfies (2.1) and (2.2).

Fix $x \in \mathbb{R}^n$ and define

$$p(t, x, y) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2t}} \text{ for } y \in \mathbb{R}^n, t > 0.$$

If $0 \leq t_1 \leq \cdots \leq t_k$ defined a measure $\nu_{t_1,...,t_k}$ on $\mathbb{R}^{n\times k}$ by

$$\begin{aligned} &\nu_{t_1,...,t_k}(F_1 \times \cdots \times F_k) \\ &= \int_{F_1\times\cdots\times F_k} p(t_1, x, x_1)p(t_2 - t_1, x_1, x_2) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k)dx_1 \cdots dx_k \end{aligned} \tag{2.3}$$

where the notation $dy = dy_1 \cdots dy_k$ represents the Lebesgue measure and $p(t, x, y)dy = 1_x(y)$, the unit point mass at x.

Using (2.1) we can extend the definition of $\nu_{t_1,...,t_k}$ to all sequences of the form $(t_1, ..., t_k)$. Since $\int_{\mathbb{R}^n} p(t, x, y)dy = 1$ for all $t \geq 0$, (2.1) holds. Thus by Kolmogorov's extension theorem there exists a probability space $(\Omega, \mathbb{F}, P^x)$ and stochastic process $W = (W_t)_{t\geq 0}$ on Ω such that the finite-dimensional distributions are defined by (2.3), that is

$$\begin{aligned} &P^x(W_{t_1} \in F_1, \cdots, W_{t_k} \in F_k) = \\ &\int_{F_1\times\cdots\times F_k} p(t_1, x, x_1)p(t_2 - t_1, x_1, x_2) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k)dx_1 \cdots dx_k. \end{aligned} \tag{2.4}$$

Definition 2.1.7. A process constructed from the finite-dimensional distributions defined by (2.3) is a Brownian motion starting at x.

Remark 1. Some properties of a Brownian motion are:

(i) W is a Gaussian process, that is for all $0 \leq t_1 \leq \cdots \leq t_k$ the random variable

$$Z = (W_{t_1}, \cdots, W_{t_k}) \in \mathbb{R}^{n\times k}$$

has a multi-normal distribution.

(ii) W has independent increments, that is for all $0 \leq t_1 \leq \cdots \leq t_k$

$$W_{t_1}, W_{t_2} - W_{t_1}, \cdots, W_{t_k} - W_{t_{k-1}}$$

are independent.

(iii) There exists a continuous version of W, this is a direct result from Kolmogorov's continuity theorem.

Definition 2.1.8. The family of σ-algebras $(\mathcal{F}_t)_{t\in I}$ is increasing if

$$t_1, t_2 \in I \text{ and } t_1 < t_2 \text{ implies } \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}.$$

Definition 2.1.9. A filtration of the σ-algebra $\mathbb{F}$ is an increasing family of σ-algebras $(\mathcal{F}_t)_{t\in I}$ such that

$$\mathcal{F}_t \subset \mathbb{F} \text{ for each } t \in I.$$

Definition 2.1.10. A filtered probability space, also referred to as a stochastic basis, $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$, is a probability space equipped with the filtration $(\mathcal{F}_t)_{t\in I}$ of its σ-algebra $\mathbb{F}$.

Definition 2.1.11. A stochastic process $(X_t)_{t\in I}$ is adapted to the filtration $(\mathcal{F}_t)_{t\in I}$ of $\mathbb{F}$ and has a stochastic basis $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ if

$$X_t : \Omega \to \mathbb{R}^n$$

is an $\mathcal{F}_t$-measurable function for each $t \in I$.

Definition 2.1.12. A stochastic process $(X_t)_{t\in I}$ is called a martingale with respect to the filtration $(\mathcal{F}_t)_{t\in I}$ (and with respect to P) if

(i) $(X_t)_{t\in I}$ is adapted to the filtration $(\mathcal{F}_t)_{t\in I}$

(ii) $E(|X_t|) < \infty$ for all $t \in I$

(iii) $s > t$ implies $E(X_s|\mathcal{F}_t) = X_t$ for all $s, t \in I$.

Remark 2. An example of a martingale process is the Brownian Motion.

2.1.3 Stochastic Differential Equations and The Itô Integral

We will now focus on presenting the development of a differential equation with the form

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot \text{“}noise\text{”}.$$

It is sensible to desire that the “*noise*” term be expressed by a stochastic process $(N_t)_{t\geq 0}$, so that

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)N_t, \tag{2.5}$$

for $(N_t)_{t\geq 0}$ satisfying these properties:

(i) $t_1 \neq t_2$ implies N_{t_1} and N_{t_2} are independent

(ii) N is stationary, that is the joint distribution of $(N_{t_1+t}, ..., N_{t_k+t})$ does not depend on t

(iii) $E(N_t) = 0$ for all t.

Unfortunately there does not exist any "reasonable" stochastic process satisfying properties (i) and (ii). Therefore we instead consider the discretized version of (2.5)

$$X_{k+1} - X_k = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k)N_k\Delta t_k, \tag{2.6}$$

where $0 = t_0 < \cdots < t_m = t$,

$$X_j = X(t_j),\ N_k = N_{t_k},\ \text{and}\ \Delta t_k = t_{k+1} - t_k.$$

Furthermore by replacing $N_k\Delta t_k$ by $\Delta V_k = V_{t_{k+1}} - V_{t_k}$, where $(V_t)_{t\geq 0}$ is some suitable stochastic process. Suitable in the sense that $(V_t)_{t\geq 0}$ has stationery independent increments with mean 0, as is suggested by properties (i), (ii), and (iii) for the process $(N_t)_{t\geq 0}$. Such a process with continuous paths is uniquely given by the Brownian motion $(W_t)_{t\geq 0}$, so setting $V_t = W_t$ (2.6) gives:

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta W_j. \tag{2.7}$$

If we could demonstrate that the limit of the right hand side exist in some sense, for $\Delta t_j \to 0$, then by using the usual integration notation we have that the limit can be expressed by

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s. \tag{2.8}$$

Moreover we could interpret X_t satisfying (2.5) to be the stochastic process $X_t(\omega)$ satisfying (2.8). Thus the existence, in a certain sense, of

$$\int_0^t \sigma(s, X_s) dW_s.$$

must be prove.

In what follows we will outline Itô's construction of the integral with respect to the Brownian motion, this construction yields the celebrated Itô-integral. We begin by giving a class of functions for which the Itô integrable is defined:

Definition 2.1.13. Let $\mathcal{V} = \mathcal{V}(S, T)$ be the class of functions

$$f(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}$$

s.t.

(i) $(t, \omega) \to f(t, \omega)$ is $\mathcal{B} \times \mathbb{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0, \infty)$

(ii) $f(t, \omega)$ is $\mathcal{F}_t$-adapted, where $\mathcal{F}_t$ is the σ-algebra generated by the random variables $(W_i(s))_{1 \leq i \leq n, 0 \leq s \leq t}$

(iii) $E\big(\int_S^T f(t, \omega)^2 dt\big)$

To prove that the Itô-integrale is defined for such a class we begin by considering W_t as the 1-dimensional Brownian motion and the simple class of functions having the form

$$\phi(t, \omega) = \sum_{j \geq 0} e_j(\omega) 1_{[j \cdot 2^{-n}, (j+1) \cdot 2^{-n})}(t), \tag{2.9}$$

where $n \in \mathbb{N}$. For such functions it is natural to define

$$\int_S^T \phi(t, \omega) dW_s = \sum_{j \geq 0} e_j(\omega)[W_{t_{j+1}} - W_{t_j}](\omega),$$

where

$$t_k = \begin{cases} k2^{-n} & \text{if} \quad S \leq k2^{-n} \leq T \\ S & \text{if} \quad k2^{-n} < S \\ T & \text{if} \quad k2^{-n} > T \end{cases} .$$

Thus it can be shown that for any function $f \in \mathcal{V}$ there exists a sequence ϕ_n of simple functions defined as in (2.9) so that

$$E\big(\int_S^T (f - \phi_n)^2 dt\big) \to 0 \text{ as } n \to \infty,$$

note the convergence is of L^2 on the product space $[0, \infty) \times \Omega$. The proof relies on the Itô-isometry property, which states

$$E\big(\int_S^T (\phi(t,\omega))^2 dt\big) = E\big((\int_S^T \phi(t,\omega) dW_t)^2\big).$$

This allows the existence of the limit which defines the Itô-integrale for $f \in \mathcal{V}$

$$\int_S^T f(t,\omega) dW_t := \lim_{n\to\infty} \int_S^T \phi_n(t,\omega) dW_t$$

in the sense of $L^2(P)$.

We now define the multi-dimensional Itô-integral as follows:

Definition 2.1.14. Let $W = (W_1, ..., W_n)$ denote an n-dimensional Brownian motion and denote by $\mathcal{V}^{m\times n}(S,T)$ the set of $m \times n$ matrices

$$\begin{pmatrix} v_{11}(t,\omega) & \dots & v_{1n}(t,\omega) \\ \vdots & & \vdots \\ v_{m1}(t,\omega) & \dots & v_{mn}(t,\omega) \end{pmatrix}$$

where each $v_{ij}(t,\omega) \in \mathcal{V}(S,T)$. If $v \in \mathcal{V}^{m\times n}(S,T)$ we define

$$\int_S^T v \; dW := \int_S^T \begin{pmatrix} v_{11}(t,\omega) & \dots & v_{1n}(t,\omega) \\ \vdots & & \vdots \\ v_{m1}(t,\omega) & \dots & v_{mn}(t,\omega) \end{pmatrix} \begin{pmatrix} dW_1 \\ \vdots \\ dW_n \end{pmatrix}$$

to be the $m \times 1$ column vector whose $i'th$ component is the following sum of 1-dimensional Itô-integrals:

$$\sum_{j=1}^{n} \int_{S}^{T} v_{ij}(t,\omega) dW_j(t,\omega).$$

Thus the Itô-integral gives the existence of X_t satisfying the stochastic integral equation (2.8) or its equivalent differential form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{2.10}$$

It can be said that X_t is an Itô process. It follows that the Itô-integral gives a proper interpretation of the stochastic differential equation (2.5). The ideas present above are the building blocks of Itô-calculus. Next we give the very useful 1-dimensional "change of variable formula-type" theorem for Itô-calculus called the Itô-formula.

Theorem 2.1.2. *Let X_t be an Itô process given by*

$$dX_t = u(t,\omega)dt + v(t,\omega)dW_t. \tag{2.11}$$

where W is the Brownian motion. Let $g(t,\omega) \in C^2([0,\infty) \times \mathbb{R})$. Then

$$dg(t, X_t) = g_t(t, X_t)dt + g_x(t, X_t)dX_t + \frac{1}{2} g_{xx}(t, X_t)(dX_t)^2, \tag{2.12}$$

where $(dX_t)^2 = dX_t \cdot dX_t$ in computed according to the rules

$$dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0, \;\; dW_t \cdot dW_t = dt. \tag{2.13}$$

2.2 Space $D[0,T]$

In this section we will give definitions central to the work done in chapters 3,4,5. We begin with D Space.

Definition 2.2.1. Let $D = D[0,T]$ be the space of real valued functions x on $[0,T]$ that are right-continuous and have left-hand limits:

(i) For $0 \leq t < T$, $x(t+) = \lim_{s \to t^+} x(s)$ exists and $x(t+) = x(t)$.

(ii) For $0 < t \leq T$, $x(t-) = \lim_{s \to t^-} x(s)$ exists.

In analysis it is desirable to have a Polish space (separable and complete). Unlike $C = C[0,T]$, the space of real valued continuous functions on $[0,T]$, $D[0,T]$ is not Polish when equipped with the Uniform Norm.

Therefore it is necessary to construct a reasonable metric for $D[0,T]$ so that the space is Polish. To this end, we will present the Skorohod Topology under which we consider functions x and y to be near one another if the graph of x can be carried onto the graph of y by crying out uniformly small perturbations of the ordinates and time scale. Mathematically this amounts to defining the topology on $D[0,T]$ by the metric

$$d(x,y) = \inf_{\lambda \in \Lambda} \{ ||\lambda - I|| \vee ||x - y(\lambda)|| \} \tag{2.14}$$

where Λ is the class of all strictly increasing, continuous mappings of $[0,T]$ onto itself, I is the identity map on $[0,T]$, and $|| \cdot || = \sup_{t \in [0,T]} | \cdot |$. This is the Skorohod Topology and a reason for which reasonable is that if relativized to $C[0,T]$ it coincides with the uniform topology there. Notwithstanding the d metric is not so reasonable since $D[0,T]$ is not complete under d. Thus it is desirable to find an equivalent metric, meaning the metric still gives the Skorohod Topology, under which $D[0,T]$ is complete. Such a metric is given by

$$d^o(x,y) = \inf_{\lambda \in \Lambda} \{ ||\lambda||^o \vee ||x - y(\lambda)|| \} \tag{2.15}$$

where

$$||\lambda||^o = \sup_{s<t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right|. \tag{2.16}$$

We now seek to present the idea of weak convergence. First note that the notion ∂ signifies boundary of a set

Definition 2.2.2. For a probability measure P on S, a set $A \in S$ whose boundary ∂A satisfies $P(\partial A) = 0$ is called a P-continuity set.

Definition 2.2.3. Given probability measures P_n and P on S. The sequence P_n converges weakly to P if the numeric sequence

$$P_n(A) \to P(A) \tag{2.17}$$

for every P-continuity set A, that is

$$P_n(A) \to P(A) \text{ if } P(\partial A) = 0. \tag{2.18}$$

Often weak convergence of P_n to P is denoted by $P_n \Rightarrow P$.

Theorem 2.2.1. *Given probability measures P_n and P on S. These five conditions are equivalent:*

(i) $P_n \Rightarrow P$

(ii) $\int_S f dP_n \to \int_S f dP$ *for all bounded, uniformly continuous* f.

(iii) $\limsup_n P_n(F) \leq P(F)$ *for all closed* F

(iv) $\liminf_n P_n(G) \leq P(G)$ *for all open* G

(v) $P_n(A) \to P(A)$ *for every P-continuity set* A

We finish this section with the definition of weak convergence of filtrations given in [8].

Definition 2.2.4. A sequence of filtrations $(\mathcal{F}_t^n)_{t<T}$ converges weakly to a filtration $(\mathcal{F}_t)_{t<T}$ iff, for all $B \in \mathcal{F}_T$, the sequence of processes $\big(E(1_B|\mathcal{F}_t^n)\big)_{t<T}$ converges in probability under the Skorokhod topology to the process $\big(E(1_B|\mathcal{F}_t)\big)_{t<T}$.

2.3 Change of Variable Formula for Functionals of a Cadlag Path

The following notation, definitions, and theorems can be found in [7] (unless otherwise mentioned) . The content of this section is necessary for the work done in chapter 3.

Definition 2.3.1. Let $\pi_n = (t_0^n, ..., t_{k(n)}^n)$, where $0 = t_0^n \leq \cdots \leq t_{k(n)}^n = T$ be a sequence of subdivisions of $[0, T]$ with steps decreasing to 0 as $n \to \infty$. $f \in D([0, T], \mathbb{R})$ is said to have finite quadratic variation along (π_n) if the sequence of discrete measures:

$$\xi^n = \sum_{i=0}^{k(n)-1} (f(t_{i+1}^n) - f(t_i^n))^2 1_{t_i^n} \tag{2.19}$$

converge weakly to a Radon measure ξ on $[0, T]$ such that

$$[f](t) = \xi([0, T]) = [f]^c(t) + \sum_{0<s\leq t} (\delta f(s))^2 \tag{2.20}$$

where $[f]^c$ is the continuous part of $[f]$. $[f]$ is called the quadratic variation of f along the sequence (π_n). $x \in D([0, T), \mathbb{R}^d)$ is said to be of quadratic variation along the sequence (π_n) if the functions x_i, $1 \leq i \leq d$ and $x_i + x_j$, $1 \leq i < j \leq d$ do.

In [15] Föllmer prove the following pathwise change of variable formula theorem:

Theorem 2.3.1. *Let x be of quadratic variation along (π_n) and F a function of class C^2 on $\mathbb{R}$. Then the Itô formula*

$$\begin{aligned} F(x_t) &= F(x_0) + \int_0^t F'(x_{s-})dx_s + \tfrac{1}{2}\int_0^t F''(x_{s-})d[x,x]_s \\ &\quad + \textstyle\sum_{s\leq t} \left(F(x_s) - F(x_{s-}) - F'(x_{s-})\Delta x_s - \tfrac{1}{2}F''(x_{s-})\Delta x_s^2\right), \end{aligned} \tag{2.21}$$

holds with the pathwise integral

$$\int_0^t F'(x_{s-})dx_s = \lim_{n\to\infty} \sum_{\pi_n \ni t_i \leq t} F'(x_{t_i})(x_{t_{i+1}} - x_{t_i}) \tag{2.22}$$

as a limit of Riemann sums along the subdivision (π_n).

In the case where X is a semimartingale, the Föllmer integral (2.22) applied to the paths of X coincides, with probability one, with the Itô-integral.

In [7] the authors extend Föllmer's pathwise change of variable formula to non-anticipative functionals on the space $D([0, T], \mathbb{R}^d)$. We will present the theorem later in the section but first we need some definitions.

Definition 2.3.2. Let $\mathcal{U}_t = D([0,t), U)$ and $\mathcal{S}_t = D([0,t), S)$, where $U \subset \mathbb{R}^d$ is an open subset of $\mathbb{R}^d$ and $S \subset \mathbb{R}^m$ is a Borel subset of $\mathbb{R}^m$. A non-anticipative functional on $\mathcal{U}_T$ is a family $F = (F_t)_{t\in[0,T]}$ of maps

$$F_t : \mathcal{U}_T \times \mathcal{S}_t \to \mathbb{R}. \tag{2.23}$$

Assume F is a non-anticipative functional from herein.

Definition 2.3.3. F is predictable in the second variable iff

$$\forall t \in [0,T],\ \forall (x,v) \in \mathcal{U}_t \times \mathcal{S}_t, \quad F_t(x_t, v_t) = F_t(x_t, v_{t-}). \tag{2.24}$$

Definition 2.3.4. A Left-continuous functional is a functional $F = (F_t, t \in [0,T])$ such that

$$\begin{gathered}\forall t \in [0,T], \forall \epsilon > 0, \forall (x,v) \in \mathcal{U}_t \times \mathcal{S}_t, \exists \eta > 0, \forall h \in [0,t],\\ \forall (x',v') \in \mathcal{U}_{t-h} \times \mathcal{S}_{t-h}, d_\infty\big((x,v),(x',v')\big) < \eta \Rightarrow |F_t(x,v) - F_{t-h}(x',v')| < \epsilon\end{gathered} \tag{2.25}$$

Denote by $\mathbb{F}_l^\infty$ the set of Left-continuous functionals.

Definition 2.3.5. F is locally bounded if

$$\begin{gathered}\forall (x,v) \in \mathcal{U}_t \times \mathcal{S}_t, \exists C > 0, \exists \eta > 0, \forall t \in [0,T], \forall (x',v') \in \mathcal{U}_t \times \mathcal{S}_t,\\ d_\infty\big((x_t,v_t),(x',v')\big) < \eta \Rightarrow \forall t \in [0,T], |F_t(x',v')| < C\end{gathered} \tag{2.26}$$

The following notation

$$y_{t,h}(u) = \begin{cases} y(u) & \text{if } u \in [0,t] \\ y(t) & \text{if } u \in (t, t+h] \end{cases}$$

is used below.

Definition 2.3.6. F is said to have the horizontal local Lipschitz property iff

$$\begin{aligned}&\forall (x,v) \in \mathcal{U}_t \times \mathcal{S}_t, \exists C > 0, \exists \eta > 0, \forall t_1 < t_2 \le T, \forall (x',v') \in \mathcal{U}_t \times \mathcal{S}_t,\\ &d_\infty\big((x_{t_1}, v_{t_1}),(x',v')\big) < \eta \Rightarrow \forall t \in [0,T],\\ &|F_{t_2}(x'_{t_1,t_2-t_1}, v'_{t_1,t_2-t_1}) - F_{t_1}(x'_{t_1}, v'_{t_1})| < C(t_2 - t_1)\end{aligned} \tag{2.27}$$

Definition 2.3.7. The horizontal derivative at $(x, v) \in \mathcal{U}_t \times \mathcal{S}_t$ of non-anticipative functional $F = (F_t)_{t \in [0,T)}$ is defined as

$$\mathcal{D}_t F(x, v) = \lim_{h \to 0^+} \frac{F_{t+h}(x_{t,h}, v_{t,h}) - F_t(x, v)}{h} \tag{2.28}$$

if the corresponding limit exists.

The following notation

$$y_t^h(u) = \begin{cases} y_t(u) & \text{if } u \in [0, t) \\ y(t) + h & \text{if } u = t \end{cases}$$

is used below.

Definition 2.3.8. Let $(e_i, i = 1, ..., d)$ be the canonical basis in $\mathbb{R}^d$. The vertical derivative at $(x, v) \in \mathcal{U}_t \times \mathcal{S}_t$ of non-anticipative functional $F = (F_t)_{t \in [0,T]}$ is defined as

$$\nabla_x F_t(x, v) = (\partial_i F_t(x, v), i = 1, ..., d) \text{ where } \partial_i F_t(x, v) = \lim_{h \to 0} \frac{F_t(x_t^{he_i}, v) - F_t(x, v)}{h} \tag{2.29}$$

if the corresponding limit exists.

Remark 3. If $F_t(x, v) = f(t, x(t))$ with $f \in C^{1,1}([0, T) \times \mathbb{R}^d)$ then we retrieve the usual partial derivatives

$$\mathcal{D}_t F(x, v) = \partial_t f(t, x(t)), \ \nabla_x F_t(x, v) = \nabla_x f(t, x(t)) \tag{2.30}$$

In chapter 3 we make use of the following theorem called the change of variable formula for non-anticipative functionals on the space $D([0, T], \mathbb{R}^d)$.

CHAPTER 3

Existence and Uniqueness

3.1 Prelininaries

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a stochastic basis supporting a d-dimensional Brownian Motion W, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ generated by W satisfies the usual assumptions.

For a finite terminal time $T > 0$, consider the following spaces:

- $\mathbb{L}_t^2(\mathbb{R}^d)$ denote the space of all $\mathcal{F}_t$-measurable random variables
  $l \equiv l_t(\omega) : \Omega \to \mathbb{R}^d$ s.t. $E[|l|^2] < \infty$.

- $\mathbb{H}^2(\mathbb{R}^{d \times d})$ denote the space of all predictable processes
  $Z \equiv Z_t(\omega) : \Omega \times [0, T] \to \mathbb{R}^{d \times d}$ s.t. $E[\int_0^T |Z_t|^2 dt] < \infty$.

- $\mathbb{H}^2(\mathbb{R}^d)$ denote the space of all predictable processes
  $Y \equiv Y_t(\omega) : \Omega \times [0, T] \to \mathbb{R}^d$ s.t. $E[\int_0^T |Y_t|^2 dt] < \infty$.

3.2 Formulation

In this chapter we are concerned with proving the existence and uniqueness of the adapted solution the to Finitely Reflected Forward Backward Stochastic Differential Equation (FR-FBSDE). Suppose we are given the reflection set

$$\mathcal{R} := \{0 < r_1 < \cdots < r_k := T\} \subset [0, T], \qquad k \geq 1$$

and the two random boundary processes

$$(\psi_{r_1}, ..., \psi_{r_k}) \in \mathbb{L}_{r_1}^2(\mathbb{R}^d) \times \cdots \times \mathbb{L}_{r_k}^2(\mathbb{R}^d)$$

$$(\Psi_{r_1}, ..., \Psi_{r_k}) \in \mathbb{L}^2_{r_1}(\mathbb{R}^d) \times \cdots \times \mathbb{L}^2_{r_k}(\mathbb{R}^d)$$

where $\psi^q_{r_i} < \Psi^q_{r_i}$ for all $i \in \{1, ..., k\}$ and for all $q \in \{1, ..., d\}$ (where q represents the q-th component of the d-dimensional vector).

Definition 3.2.1. The pair of cadlag processes $(Y^{\mathcal{R}}, Z^{\mathcal{R}}) \in \mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$ is called a solution of the Finitely Reflected Forward Backward Stochastic Differential Equation (FR-FBSDE) if it satisfies the following equation

$$Y^{\mathcal{R}}_t = g(X_T) + \int_t^T f(X_s, Y^{\mathcal{R}}_s, Z^{\mathcal{R}}_s)ds - \int_t^T (Z^{\mathcal{R}}_s)^\top dW_s + \sum_{s\in\mathcal{R}^t} [(\psi_s - Y^{\mathcal{R}}_s)^+ - (Y^{\mathcal{R}}_s - \Psi_s)^+] \tag{3.1}$$

where $\mathcal{R}^t = \{s \in \mathcal{R} | s > t\}$ if $0 \le t < T$ and $\mathcal{R}^T = \{T\}$, $\psi_T \le g(X_T) \le \Psi_T$ *a.s.*, and the X process solves the forward SDE

$$X_t = X_0 + \int_t^T b(X_s)ds - \int_t^T \sigma(X_s)dW_s.$$

Finitely reflected BSDE were introduced [22] and our formulation follows [6].

Remark 4. Due to multidimensional setting, where $Y^{\mathcal{R}}_t(\omega) \in \mathbb{R}^d$ and $Z^{\mathcal{R}}_t(\omega) \in \mathbb{R}^{d\times d}$ for all $\omega \in \Omega$, equation (3.1) stands for $q \in \{1, ..., d\}$ scalar equations

$$\begin{aligned} Y^{q,\mathcal{R}}_t &= g^q(X_T) + \int_t^T f^q(X_s, Y^{\mathcal{R}}_s, Z^{\mathcal{R}}_s)ds - \int_t^T (Z^{q,\mathcal{R}}_s)^\top dW_s \\ &\quad + \sum_{s\in\mathcal{R}^t} [(\psi^q_s - Y^{q,\mathcal{R}}_s)^+ - (Y^{q,\mathcal{R}}_s - \Psi^q_s)^+]. \end{aligned} \tag{3.2}$$

Our objective in this section is to prove the following

Theorem 3.2.1. *If*

(i) $g(X_T) \in \mathbb{L}^2_T(\mathbb{R}^d)$

(ii) $f(\cdot, u, v) \in \mathbb{H}^2(\mathbb{R}^d)$ *for all* $(u, v) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$

(iii) $\exists K$ *s.t.* $t \in [0, T]$, $u, u' \in R^d$, $v, v' \in R^{d\times d}$

$|f(t, u, v) - f(t, u', v')| < K(|u - u'| + |v - v'|)$

(iv) $E(\int_0^T f^2(t, 0, 0)dt) < \infty$

then FR-FBSDE (3.1) has a unique solution in $\mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$.

3.3 Simple Case

Instead of FR-FBSDE (3.1) we first consider a much simpler BSDE. As before, we have the reflection set

$$\mathcal{R} := \{0 < r_1 < \cdots < r_k := T\} \subset [0,T], \qquad k \geq 1$$

the sequence of random variables

$$(l_{r_1}, ..., l_{r_k}) \in \mathbb{L}^2_{r_1}(\mathbb{R}^d) \times \cdots \times \mathbb{L}^2_{r_k}(\mathbb{R}^d)$$

the process

$$L^{\mathcal{R}}_t := \sum_{s \in \mathcal{R}^t} l_s \tag{3.3}$$

and the terminal random variable ξ.

Definition 3.3.1. The pair of processes $(Y^{\mathcal{R}}, Z^{\mathcal{R}}) \in \mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$ is called a solution of the BSDE if it satisfies the following equation

$$Y^{\mathcal{R}}_t = \xi + \int_t^T \tilde{f}(s)ds - \int_t^T (Z^{\mathcal{R}}_s)^\top dW_s + L^{\mathcal{R}}_t. \tag{3.4}$$

Remark 5. Note that $\tilde{f}$ and $L^{\mathcal{R}}$ do not depend on $Y^{\mathcal{R}}$ or $Z^{\mathcal{R}}$ and that $L^{\mathcal{R}} \in \mathbb{H}^2(\mathbb{R}^d)$.

Theorem 3.3.1. *If*

(i) $\xi \in \mathbb{L}^2_T(\mathbb{R}^d)$

(ii) $\tilde{f} \in \mathbb{H}^2(\mathbb{R}^d)$

(iii) $\exists K$ *s.t.* $t \in [0,T]$, $u, u' \in R^d$, $v, v' \in R^{d\times d}$

$|f(t,u,v) - f(t,u',v')| < K(|u-u'| + |v-v'|)$

(iv) $E(\int_0^T f^2(t,0,0)dt) < \infty$

then BSDE (3.4) has a unique solution in $\mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$.

Proof. BSDE (3.4) can be decomposed into the following backward stochastic equations

$$S^{\mathcal{R}}_t = \xi + \int_t^T \tilde{f}(s)ds - \int_t^T (Z^{\mathcal{R}}_s)^\top dW_s \tag{3.5a}$$

$$Y_t^{\mathcal{R}} = S_t^{\mathcal{R}} + L_t^{\mathcal{R}}. \tag{3.5b}$$

By Lemma 2.1 from [24] it follows that the pair of processes $(S^{\mathcal{R}}, Z^{\mathcal{R}}) \in \mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$ is the unique solution to $(3.5a)$ and clearly $Y^{\mathcal{R}} \in \mathbb{H}^2(\mathbb{R}^d)$ is uniquely determined by $(3.5b)$. Therefore the pair of processes $(Y^{\mathcal{R}}, Z^{\mathcal{R}}) \in \mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$ satisfying equations $(3.5a)$ and $(3.5b)$, must also satisfy equation (3.4) and consequently (3.4) has a unique pair of adapted solutions. □

3.4 The Contraction Map

Consider a special case of the BSDE of Definition 1.2 with the reflection set

$$\mathcal{R} := \{0 < r_1 < \cdots < r_k := T\} \subset [0, T], \qquad k \geq 1,$$

a given pair of sequences of boundary random variables

$$(\psi_{r_1}, ..., \psi_{r_k}) \in \mathbb{L}^2_{r_1}(\mathbb{R}^d) \times \cdots \times \mathbb{L}^2_{r_k}(\mathbb{R}^d)$$

$$(\Psi_{r_1}, ..., \Psi_{r_k}) \in \mathbb{L}^2_{r_1}(\mathbb{R}^d) \times \cdots \times \mathbb{L}^2_{r_k}(\mathbb{R}^d),$$

where $\psi^q_{r_i} < \Psi^q_{r_i}$ for all $i \in \{1, ..., k\}$ and for $q \in \{1, ..., d\}$ (where q represents the q-th component of the d-dimensional vector), the process

$$L_t^{\mathcal{R}} := \sum_{s\in\mathcal{R}^t} \rho(y_s) := \sum_{s\in\mathcal{R}^t} [(\psi_s - y_s)^+ - (y_s - \Psi_s)^+],$$

with $y \in \mathbb{H}^2(\mathbb{R}^d)$ and the terminal random variable ξ s.t.

$$\psi_T \leq \xi \leq \Psi_T \qquad a.s.$$

Definition 3.4.1. The pair of processes $(Y^{\mathcal{R}}, Z^{\mathcal{R}}) \in \mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$ is called a solution of the BSDE if it satisfies the following equation

$$Y_t^{\mathcal{R}} = \xi + \int_t^T f(s, y_s, z_s) ds - \int_t^T (Z_s^{\mathcal{R}})^\top dW_s + L_t^{\mathcal{R}}. \tag{3.6}$$

for a given pair of processes $(y, z) \in \mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$.

Proposition 3.4.1. *If*

(i) $(y,z) \in \mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$

(ii) $\xi \in \mathbb{L}^2_T(\mathbb{R}^d)$

(iii) $f(\cdot,u,v) \in \mathbb{H}^2(\mathbb{R}^d)$ *for all* $(u,v) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$

(iv) $\exists K$ *s.t.* $t \in [0,T]$, $u,u' \in R^d$, $v,v' \in R^{d\times d}$

$|f(t,u,v) - f(t,u',v')| < K(|u-u'| + |v-v'|)$

then BSDE (3.6) has a unique solution in $\mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$.

Proof. It is clear that BSDE (3.6) is a special case of BSDE (3.4) with $\tilde{f}(s) = f(s,y_s,z_s)$, therefore by Theorem 3.3.1 the result follows. □

Define the following mapping $\phi : \mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d}) \to \mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$ where

$$(Y^{\mathcal{R}}, Z^{\mathcal{R}}) = \phi(y^{\mathcal{R}}, z^{\mathcal{R}}) \tag{3.7}$$

is the unique solution to (3.6). The main goal of this section is to show ϕ is a contraction on $\mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$ with a suitable chosen norm $||\cdot||$, to be defined later. This will guarantee the existence of a unique fix point for the mapping ϕ, i.e., $(Y^{\mathcal{R}}, Z^{\mathcal{R}})$ such that $(Y^{\mathcal{R}}, Z^{\mathcal{R}}) = \phi(Y^{\mathcal{R}}, Z^{\mathcal{R}})$. Equivalently, the fixed point $(Y^{\mathcal{R}}, Z^{\mathcal{R}})$ satisfies the FR-BSDE

$$Y^{\mathcal{R}}_t = \xi + \int_t^T f(s, Y^{\mathcal{R}}_s, Z^{\mathcal{R}}_s)ds - \int_t^T (Z^{\mathcal{R}}_s)^\top dW_s + L^{\mathcal{R}}_t. \qquad (3.6)'$$

We need to show that for the norm $||\cdot||$ there exist $0 < \eta < 1$ such that

$$||(Y^{\mathcal{R}}_1 - Y^{\mathcal{R}}_2, Z^{\mathcal{R}}_1 - Z^{\mathcal{R}}_2)|| < \eta \; ||(y^{\mathcal{R}}_1 - y^{\mathcal{R}}_2, z^{\mathcal{R}}_1 - z^{\mathcal{R}}_2)||$$

where $(Y^{\mathcal{R}}_1, Z^{\mathcal{R}}_1) = \phi(y^{\mathcal{R}}_1, z^{\mathcal{R}}_1)$ and $(Y^{\mathcal{R}}_2, Z^{\mathcal{R}}_2) = \phi(y^{\mathcal{R}}_2, z^{\mathcal{R}}_2)$.

We utilize a generalization of the celebrated Itô-formula for non-anticipating functionals with discontinuous paths ([7],Th.4) based on the Föllmer integral [15], which is a path integral coinciding with the Itô-integral with probability 1.

Consider the following non-anticipating functional

$$F_t(y) = F_t(y_t) = e^{\beta t}(y_t)^2, \qquad \beta > 1 \tag{3.8}$$

and set

$$\hat{Y}_t^{\mathcal{R}} = Y_1^{\mathcal{R}} - Y_2^{\mathcal{R}}. \tag{3.9}$$

Remark 6. In what follows, F should be interpreted as the family of functionals $(F^q)_{q\in\{1,...,d\}}$, where for all $q \in \{1, ..., d\}$, $F^q : D([0,T], \mathbb{R}^d) \to \mathbb{R}$ reads

$$F_t^q(y) = e^{\beta t}(y_t^q)^2 \tag{3.10}$$

with y^q denoting the q-th component of the d-dimensional vector. Consequently, all of the following analyses regarding F and $\hat{Y}^{\mathcal{R}}$ are done component-wise.

In order to apply the change of variables formula to the functional F and the process $\hat{Y}^{\mathcal{R}}$ we must verify that the following conditions are satisfied:

(i) F is predictable in the second variable

(ii) $\bigtriangledown_y^2 F$ and $\mathcal{D}F$ have the local boundedness property

(iii) $F, \bigtriangledown_y F, \bigtriangledown_y^2 F \in \mathbb{F}_l^\infty$, where $\mathbb{F}_l^\infty$

(iv) $\bigtriangledown_y F$ has the horizontal local Lipschitz property

(v) $\sup_{t\in[0,T]-\pi_n} |\hat{Y}^{\mathcal{R}}(t) - \hat{Y}^{\mathcal{R}}(t-)| \to 0$ as $n \to \infty$, for some $\pi_n = (0 = t_0^n, ..., t_{k(n)}^n = T)$ sequence of subdivisions of $[0,T]$ such that $||\pi_n|| \to 0$ as $n \to \infty$

(vi) $\hat{Y}^{\mathcal{R}}$ has finite quadratic variation along (π_n).

Again, for definitions and notation regarding (i)-(vi) we refer the reader to Section 2.3.

Proposition 3.4.2. *F is predictable in the second variable.*

Proof. F is independent of the second argument, i.e., $F(y,v) = F(y)$ for any $v \in D([0,T], \mathbb{R})$ so the result clearly follows. □

Proposition 3.4.3. *$\bigtriangledown_y^2 F$ and $\mathcal{D}F$ have the local boundedness property.*

Proof. Part 1($\bigtriangledown_y^2 F$ is locally bounded). Since $\bigtriangledown_y^2 F_t(y) = 2e^{\beta t}$, hence

$$|\bigtriangledown_y^2 F_t(y)| \leq 2e^{\beta T}$$

gives local boundedness.

Part 2($\mathcal{D}F$ is locally bounded). Let $y \in D([0,T], \mathbb{R}^d)$, $t \in [0,T]$, and $\eta > 0$ and assume

$$d_\infty(y_t, y') = \sup_{u \in [0,t]} |y(t) - y'(u)| < \eta, \text{ where } y' \in D([0,t], \mathbb{R}^d).$$

Since $\mathcal{D}F_t(y) = \beta e^{\beta t} y^2$, therefore

$$\begin{aligned} |\mathcal{D}F_t(y')| = \beta e^{\beta t}(y'_t)^2 = \beta e^{\beta t}|y'_t - y_t + y_t|^2 &\leq \beta e^{\beta t}(|y'_t - y_t| + |y_t|)^2 \\ &< \beta e^{\beta t}(\eta + |y_t|)^2 \\ &\leq \beta e^{\beta t}(\eta + \sup_{u \in [0,T]} |y_t|)^2 \end{aligned}$$

hence $\mathcal{D}F$ is locally bounded. □

Proposition 3.4.4. *$F, \bigtriangledown_y F, \bigtriangledown_y^2 F \in \mathbb{F}_l^\infty$ (Left-continous functionals).*

Let us introduce the notation

$$y_{t,h}(u) = \begin{cases} y(u) & \text{if } u \in [0,t] \\ y(t) & \text{if } u \in (t, t+h] \end{cases}.$$

Proof. Let $t \in [0,T]$, $y \in D([0,t], \mathbb{R}^d)$, and $\eta > 0$. Assume

$$d_\infty(y, y') = \sup_{u \in [0,t]} |y(u) - y'_{t-h,h}(u)| + h < \eta, \text{ where } h > 0 \text{ and } y' \in D([0,t-h], \mathbb{R}^d).$$

Part 1($F \in \mathbb{F}_l^\infty$). Note that

$$\begin{aligned} |F_t(y) - F_{t-h}(y')| &= e^{\beta t}|y_t^2 - \tfrac{(y'_{t-h})^2}{e^{\beta h}}| \\ &= e^{\beta t}|y_t^2 - \tfrac{(y'_{t-h,h}(t) - y_t + y_t)^2}{e^{\beta h}}| \\ &= e^{\beta t}|(1 - \tfrac{1}{e^{\beta h}})y_t^2 - \tfrac{(y'_{t-h,h}(t) - y_t)}{e^{\beta h}}(y'_{t-h,h}(t) + y_t)|. \end{aligned}$$

Thus it follows

$$\begin{aligned}|F_t(y) - F_{t-h}(y')| &< e^{\beta t}((1 - \tfrac{1}{e^{\beta\eta}})y_t^2 + \eta(y'_{t-h,h}(t) + y_t)) \\ &\leq e^{\beta t}((1 - \tfrac{1}{e^{\beta\eta}})y_t^2 + \eta(2y_t + \eta)).\end{aligned}$$

Finally, by the above inequality,

$$\lim_{\eta\to 0} |F_t(y) - F_{t-h}(y')| \leq \lim_{\eta\to 0} e^{\beta t}((1 - \frac{1}{e^{\beta\eta}})y_t^2 + \eta(2y_t + \eta)) = 0$$

implies $F \in \mathbb{F}_l^\infty$.

Part 2 ($\bigtriangledown_y F \in \mathbb{F}_l^\infty$). Since $\bigtriangledown_y F(y_t) = 2e^{\beta t}y$, hence

$$| \bigtriangledown_y F(y_t) - \bigtriangledown_y F(y'_{t-h})| = 2e^{\beta t}|y_t - \frac{y'_{t-h}}{e^{\beta h}}| = 2e^{\beta t}|(1 - \frac{1}{e^{\beta h}})y_t - \frac{(y'_{t-h,h}(t) - y_t)}{e^{\beta h}}|$$

which gives

$$|F_t(y) - F_{t-h}(y')| < e^{\beta t}((1 - \frac{1}{e^{\beta\eta}})y_t + \eta).$$

Finally, by the above inequality, we have

$$\lim_{\eta\to 0} |F_t(y) - F_{t-h}(y')| \leq \lim_{\eta\to 0} e^{\beta t}((1 - \frac{1}{e^{\beta\eta}})y_t + \eta) = 0$$

and $\bigtriangledown_y F \in \mathbb{F}_l^\infty$.

Part 3 ($\bigtriangledown_y^2 F \in \mathbb{F}_l^\infty$). Since $\bigtriangledown_y^2 F_t(y) = 2e^{\beta t}$ and

$$| \bigtriangledown_y^2 F(y_t) - \bigtriangledown_y^2 F(y'_{t-h})| = 2e^{\beta t}(1 - \frac{1}{e^{\beta h}}) < 2e^{\beta t}(1 - \frac{1}{e^{\beta\eta}})$$

we get

$$\lim_{\eta\to 0} | \bigtriangledown_y^2 F(y_t) - \bigtriangledown_y^2 F(y'_{t-h})| \leq \lim_{\eta\to 0} 2e^{\beta t}(1 - \frac{1}{e^{\beta\eta}}) = 0$$

and $\bigtriangledown_y^2 F \in \mathbb{F}_l^\infty$. □

Proposition 3.4.5. *$\bigtriangledown_y F$ has the horizontal local Lipschitz property.*

Proof. Let $y \in D([0,T],\mathbb{R}^d)$, and $\eta > 0$ and assume

$$d_\infty(y_{t_1}, y') = \sup_{u\in[0,t]} |y(t_1) - y'(u)| < \eta, \text{ where } 0 \leq t_1 < t_2 \leq T \text{ and } y' \in D([0,t_1],\mathbb{R}^d).$$

Thus it follows

$$\begin{aligned} |\bigtriangledown_y F_{t_2}(y'_{t_1,t_2-t_1}) - \bigtriangledown_y F_{t_1}(y'_{t_1})| &= |2e^{\beta t_2}(y'_{t_1,t_2-t_1}(t_2)) - 2e^{\beta t_1} y'_{t_1}| \\ &= 2|y'_{t_1}|(e^{\beta t_2} - 2e^{\beta t_1}) \\ &\leq 2|y_{t_1} + \eta|\beta e^{\beta T}(t_2 - t_1) \end{aligned}$$

and $\bigtriangledown_y F$ has the horizontal local Lipschitz property. □

Proposition 3.4.6. *If $\pi_n \supset \mathcal{R}$ for all $n \in \mathbb{N}$, then*

$$\sup_{t\in[0,T]-\pi_n} |\hat{Y}^{\mathcal{R}}(t) - \hat{Y}^{\mathcal{R}}(t-)| \to 0 \text{ as } n \to \infty.$$

Proof. Let us define

$$\triangle y(t) = y(t) - y(t-).$$

We have

$$\begin{aligned} \hat{Y}^{\mathcal{R}}_t &= (Y^{\mathcal{R}}_1 - Y^{\mathcal{R}}_2)_t = \xi + \int_t^T f(s, y^{\mathcal{R}}_{1,s}, z^{\mathcal{R}}_{1,s})ds - \int_t^T (Z^{\mathcal{R}}_{1,s})^\top dW_s + \textstyle\sum_{s\in\mathcal{R}^t} \rho(y^{\mathcal{R}}_{1,s}) \\ &\quad -[\xi + \int_t^T f(s, y^{\mathcal{R}}_{2,s}, z^{\mathcal{R}}_{2,s})ds - \int_t^T (Z^{\mathcal{R}}_{2,s})^\top dW_s + \textstyle\sum_{s\in\mathcal{R}^t} \rho(y^{\mathcal{R}}_{2,s})]. \end{aligned}$$

Upon setting

$$\hat{f}(s, \bar{y}^{\mathcal{R}}_s, \bar{z}^{\mathcal{R}}_s) = f(s, y^{\mathcal{R}}_{1,s}, z^{\mathcal{R}}_{1,s}) - f(t, y^{\mathcal{R}}_{2,s}, z^{\mathcal{R}}_{2,s})$$

and

$$\hat{\rho}(s, \bar{y}^{\mathcal{R}}_s) = \rho(s, y^{\mathcal{R}}_{1,s}) - \rho(t, y^{\mathcal{R}}_{2,s})$$

where $\bar{y}^{\mathcal{R}} = (y^{\mathcal{R}}_1, y^{\mathcal{R}}_2)$ and $\bar{z}^{\mathcal{R}} = (z^{\mathcal{R}}_1, z^{\mathcal{R}}_2)$ and

$$\hat{Z}^{\mathcal{R}} = Z^{\mathcal{R}}_1 - Z^{\mathcal{R}}_2.$$

we have

$$\hat{Y}_t^{\mathcal{R}} = \int_t^T \hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds - \int_t^T (\hat{Z}^{\mathcal{R}})^\top dW_s + \sum_{s\in\mathcal{R}^t} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}}). \tag{3.11}$$

Let $\pi_n \supset \mathcal{R}$ for all $n \in \mathbb{N}$, and observe that the support of $\triangle \sum_{s\in\mathcal{R}^\cdot} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}})$ is a subset of π_n for all $n \in \mathbb{N}$. Therefore $\triangle \sum_{s\in\mathcal{R}^t} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}}) = 0$ for all $t \in [0, T] - \pi_n$. Furthermore, note that by continuity of the Lebesgue and Itô integrals $\triangle \hat{Y}_t^{\mathcal{R}} = \triangle \sum_{s\in\mathcal{R}^t} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}})$ for all $t \in [0, T]$. Thus it follows that $| \triangle \hat{Y}_t^{\mathcal{R}}| = |\hat{Y}_t^{\mathcal{R}} - \hat{Y}_{t-}^{\mathcal{R}}| = 0$ for all $t \in [0, T] - \pi_n$ which concludes the proof. □

To verify assumption (vi) we will first need to state some lemmas.

Lemma 3.4.7. *The process* $\sum_{s\in\mathcal{R}_t} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}})$ *is of finite variation, where* $\mathcal{R}_t = \{s \in \mathcal{R} | 0 < s \leq t\}$ *and* $\mathcal{R}_0 \equiv$.

Proof. Let $\delta_n = (0 = t_0^n, ..., t_{k(n)}^n = t)$ be any sequence of partitions of [0,t] such that $||\delta_n|| \to 0$ as $n \to \infty$. Thus there exists N large enough such that for any n greater than N, $||\delta_n|| < \frac{||\mathcal{R}||}{2}$. Note if $t \neq T$, then there exists $m < k$ such that $r_m \leq t < r_{m+1}$. Furthermore, if $||\delta_n|| < \frac{||\mathcal{R}||}{2}$, then for all $j \in \{1, ..., m\}$ (if $t = T$ let $m = k$) there exists a unique $i \in \{1, ..., k(n)\}$ such that $\{r_i\} = \mathcal{R}_{t_i} - \mathcal{R}_{t_{i-1}}$. This implies that for all n greater than N

$$\sum_{i=1}^{k(n)} | \sum_{s\in\mathcal{R}_{t_i}} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}}) - \sum_{s\in\mathcal{R}_{t_{i-1}}} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}})| = \sum_{i=1}^{k(n)} | \sum_{s\in\mathcal{R}_{t_i} - \mathcal{R}_{t_{i-1}}} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}})| = \sum_{i=1}^{m} |\hat{\rho}(r_i, \bar{y}_{r_i}^{\mathcal{R}})|. \tag{3.12}$$

Therefore the variation of the process $\sum_{s\in\mathcal{R}_t} \hat{g}(s, \bar{y}_s^{\mathcal{R}})$ is

$$\lim_{\delta_n\to 0} \sum_{i=1}^{k(n)} | \sum_{s\in\mathcal{R}_{t_i}} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}}) - \sum_{s\in\mathcal{R}_{t_{i-1}}} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}})| = \sum_{i=1}^{m} |\hat{\rho}(r_i, \bar{y}_{r_i}^{\mathcal{R}})| < \infty. \tag{3.13}$$

□

Lemma 3.4.8. *The process* $\hat{Y}_t^{\mathcal{R}}$ *is of finite quadratic variation.*

Proof. From (3.11) we obtain

$$\hat{Y}_t^{\mathcal{R}} = \hat{Y}_0^{\mathcal{R}} - \int_0^t \hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds + \int_0^t (\hat{Z}^{\mathcal{R}})^\top dW_s - \sum_{s\in\mathcal{R}_t} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}}).$$

Using the polarization identity and the linearity properties, the quadratic variation of $\hat{Y}_t^{\mathcal{R}}$ gives

$$\begin{aligned}
[\hat{Y}^{\mathcal{R}}](t) &= [\hat{Y}_0^{\mathcal{R}} - \textstyle\int_0^\cdot \hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds + \int_0^\cdot (\hat{Z}_s^{\mathcal{R}})^\top dW_s - \sum_{s\in\mathcal{R}_\cdot} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}})](t) \\
&= [\hat{Y}_0^{\mathcal{R}} - \textstyle\int_0^\cdot \hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds](t) + [\int_0^\cdot (\hat{Z}_s^{\mathcal{R}})^\top dW_s](t) + [\sum_{s\in\mathcal{R}_\cdot} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}})](t) \\
&+ 2([\hat{Y}_0^{\mathcal{R}} - \textstyle\int_0^\cdot \hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds, \int_0^\cdot (\hat{Z}_s^{\mathcal{R}})^\top dW_s](t) \\
&- [\hat{Y}_0^{\mathcal{R}} - \textstyle\int_0^\cdot \hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds, \sum_{s\in\mathcal{R}_\cdot} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}})](t) \\
&- [\textstyle\int_0^\cdot (\hat{Z}_s^{\mathcal{R}})^\top dW_s, \sum_{s\in\mathcal{R}_\cdot} \hat{\rho}(s, \bar{y}_s^{\mathcal{R}})](t)).
\end{aligned} \tag{3.14}$$

By properties of integrals the process $\int_0^t \hat{f}(s, \bar{y}_{1,s}^{\mathcal{R}}, \bar{z}_{1,s}^{\mathcal{R}})ds$ is continuous and of finite variation, which implies that $\hat{Y}_0^{\mathcal{R}} - \int_0^t \hat{f}(s, \bar{y}_{1,s}^{\mathcal{R}}, \bar{z}_{1,s}^{\mathcal{R}})ds$ is also continuous and of finite variation. Furthermore, note that the Itô integral $\int_0^t (\hat{Z}_s^{\mathcal{R}})^\top dW_s$ is continuous, and by Lemma 3.4.7 the process $\sum_{s\in\mathcal{R}_t} \hat{\rho}(s, \bar{y}_{1,s}^{\mathcal{R}})$ is of finite variation. Hence covariation terms are zero and $[\hat{Y}_0^{\mathcal{R}} - \int_0^\cdot \hat{f}(s, \bar{y}_{1,s}^{\mathcal{R}}, \bar{z}_{1,s}^{\mathcal{R}})ds](t) = 0$. Thus the equation for the quadratic variation of $\hat{Y}_t^{\mathcal{R}}$ can be simplified as follows

$$[\hat{Y}^{\mathcal{R}}](t) = [\int_0^\cdot (\hat{Z}_s^{\mathcal{R}})^\top dW_s](t) + [\sum_{s\in\mathcal{R}_\cdot} \hat{\rho}(s, \bar{y}_{1,s}^{\mathcal{R}})](t). \tag{3.15}$$

The first term is the quadratic variation of an Itô integral is given by

$$[\int_0^\cdot (\hat{Z}_s^{\mathcal{R}})^\top dW_s](t) = \int_0^t |(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds. \tag{3.16}$$

The second term is the quadratic variation of the process $\sum_{s\in\mathcal{R}_t}\hat{\rho}(s,\bar{y}_s^{\mathcal{R}})$. Let $\delta_n = (0 = t_0^n, ..., t_{k(n)}^n = t)$ be any sequence of partitions of [0,t] such that $||\delta_n|| \to 0$ as $n \to \infty$. Then there exists N large enough such that for any n greater than N, $||\delta_n|| < \frac{||\mathcal{R}||}{2}$. Furthermore, if $||\delta_n|| < \frac{||\mathcal{R}||}{2}$, then for all $j \in \{1, ..., m\}$ (if $t \neq T$, then there exist $m < k$ such that $r_m \leq t < r_{m+1}$; if $t = T$ let $m = k$) there exist a unique $i \in \{1, ..., k(n)\}$ such that $\{r_i\} = \mathcal{R}_{t_i} - \mathcal{R}_{t_{i-1}}$. This implies that for all n greater than N

$$\sum_{i=1}^{k(n)}(\sum_{s\in\mathcal{R}_{t_i}}\hat{\rho}(s,\bar{y}_s^{\mathcal{R}}) - \sum_{s\in\mathcal{R}_{t_{i-1}}}\hat{\rho}(s,\bar{y}_s^{\mathcal{R}}))^2 = \sum_{i=1}^{k(n)}(\sum_{s\in\mathcal{R}_{t_i}-\mathcal{R}_{t_{i-1}}}\hat{\rho}(s,\bar{y}_s^{\mathcal{R}}))^2 = \sum_{j=1}^{m}(\hat{\rho}(r_j,\bar{y}_{r_j}^{\mathcal{R}}))^2. \tag{3.17}$$

Therefore the quadratic variation of the process $\sum_{s\in\mathcal{R}_t}\hat{\rho}(s,\bar{y}_s^{\mathcal{R}})$ is

$$[\sum_{s\in\mathcal{R}.}\hat{\rho}(s,\bar{y}_s^{\mathcal{R}})](t) = \lim_{\delta_n\to 0}\sum_{i=1}^{k(n)}(\sum_{s\in\mathcal{R}_{t_i}}\hat{\rho}(s,\bar{y}_s^{\mathcal{R}}) - \sum_{s\in\mathcal{R}_{t_{i-1}}}\hat{\rho}(s,\bar{y}_s^{\mathcal{R}}))^2 = \sum_{j=1}^{m}(\hat{\rho}(s,\bar{y}_{r_j}^{\mathcal{R}}))^2. \tag{3.18}$$

Furthermore, this implies that the quadratic variation of $\hat{Y}_t^{\mathcal{R}}$ is

$$\begin{aligned} [\hat{Y}^{\mathcal{R}}](t) &= [\int_0^{\cdot}(\hat{Z}_s^{\mathcal{R}})^{\top}dW_s](t) + [\textstyle\sum_{s\in\mathcal{R}.}\hat{\rho}(s,\bar{y}_{1,s}^{\mathcal{R}})](t) \\ &= \int_0^t|(\hat{Z}_s^{\mathcal{R}})^{\top}|^2ds + \textstyle\sum_{j=1}^{m}(\hat{\rho}(r_j,\bar{y}_{r_j}^{\mathcal{R}}))^2 < \infty. \end{aligned} \tag{3.19}$$

Note that the quadratic variation is defined as limit of convergence in probability.

□

Next we will show that the quadratic variation of $\hat{Y}^{\mathcal{R}}$ along a sequence $\pi_n = (0 = t_0^n, ..., t_{k(n)}^n = T)$ of partitions of $[0, T]$, such that $||\pi_n|| \to 0$ as $n \to \infty$, is in fact equal to quadratic variation of $\hat{Y}^{\mathcal{R}}$ as defined above. In other words we will show that the random measures on $[0, T]$

$$\nu^{\pi_n}([0,t]) = \sum_{i=0}^{k(n)-1}(\hat{Y}_{t_{i+1}^n}^{\mathcal{R}} - \hat{Y}_{t_i^n}^{\mathcal{R}})^2 1_{t_i^n\in[0,t]} \tag{3.20}$$

converge weakly to the random measure μ on $[0,T]$ such that

$$\mu([0,t]) = [\hat{Y}^{\mathcal{R}}](t) = \int_0^t |(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds + \sum_{j=1}^{m} (\hat{\rho}(s, \bar{y}_{r_j}^{\mathcal{R}}))^2 = [\hat{Y}^{\mathcal{R}}]^c(t) + \sum_{0<s\leq t} (\Delta \hat{Y}_s^{\mathcal{R}})^2 \tag{3.21}$$

where $[\hat{Y}^{\mathcal{R}}]^c$ is the continuous part of $[\hat{Y}^{\mathcal{R}}]$. Therefore, by ([17],Th.1.1), it is sufficient to show that there exists $\mathcal{I} \subset \mathcal{B}([0,T])$ = Borel σ-algebra on [0,T] such that

(i) $\mathcal{I}$ is a fundamental system of dissecting sets in the space $([0,T], \mathcal{B}([0,T]))$, defined in [1]

(ii) $\mu(\partial I) = 0$, a.s., for all $I \in \mathcal{I}$

(iii) $\nu^{\pi_n}(I) \xrightarrow{\omega} \mu(I)$, as $n \to \infty$ for all $I \in \mathcal{I}$.

Lemma 3.4.9. *If* $\mathcal{A} = \{(s,t] | s,t \in [0,T] \backslash (\mathcal{R} \backslash \{T\})$ *and* $s < t\}$,
$\mathcal{I}_n = \{(\frac{j-1}{2^n}T, \frac{j}{2^n}T] | j = 1, ..., 2^n\}$, *and*

$$\mathcal{I} = (\cup_{n=1}^{\infty} \mathcal{I}_n \cap \mathcal{A}) \cup \{\} \cup \{\{0\}\} \tag{3.22}$$

then $\mathcal{I}$ *satisfies conditions* (i) *and* (ii) *above.*

Proof. Let $\mathcal{I}$ be as defined by (3.22).

Part 1($\mathcal{I}$ *is a fundamental system of dissecting sets in the space* $([0,T], \mathcal{B}([0,T]))$. From the construction it is clear that $\mathcal{I}$ is a fundamental system of dissecting sets in the space $([0,T], \mathcal{B}([0,T]))$.

Part 2($\mu(\partial I) = 0$*, a.s., for all* $I \in \mathcal{I}$*).* If $I =$ or $I = \{0\}$, then we are done. Suppose $I = (s,t] \in \mathcal{I}$, this implies that $\partial I = \{s,t\}$. Thus $\mu(\partial I) = \mu(\{s\}) + \mu(\{t\})$ it follows that

$$\begin{aligned} \mu(\partial I) &= (\mu([0,s]) - \mu([0,s))) + (\mu([0,t]) - \mu([0,t))) \\ &= ([\hat{Y}^{\mathcal{R}}](s) - [\hat{Y}^{\mathcal{R}}](s-)) + ([\hat{Y}^{\mathcal{R}}](t) - [\hat{Y}^{\mathcal{R}}](t-)). \end{aligned} \tag{3.23}$$

Since $s,t \notin (\mathcal{R} \backslash \{T\})$ and $\xi \in \mathbb{L}_T^2(\mathbb{R}^d)$ it follows that

$$\mu(\partial I) = ([\hat{Y}^{\mathcal{R}}]^c(s) - [\hat{Y}^{\mathcal{R}}]^c(s-)) + ([\hat{Y}^{\mathcal{R}}]^c(t) - [\hat{Y}^{\mathcal{R}}]^c(t-)) = 0. \tag{3.24}$$

$\square$

Lemma 3.4.10. *Let $\pi_n \supset \mathcal{R}$ be such that $||\pi_n|| \to 0$ as $n \to \infty$, the random measures ν^{π_n} are defined by (3.20) for all $n \in \mathbb{N}$, the random measure μ is defined by (3.21), and $\mathcal{I}$ is defined by (3.22). Then*

$$\nu^{\pi_n}(I) \xrightarrow{\omega} \mu(I), \ \ as\ n \to \infty, \ \ for\ all\ I \in \mathcal{I}.$$

Let us introduce the notation

$$j_t^n = max\{i | t_i^n \leq t\}, \tag{3.25}$$

and

$$\mu^{\pi_n}([0,t]) = \sum_{i=0}^{j_t^n - 1} (\hat{Y}^{\mathcal{R}}_{t^n_{i+1}} - \hat{Y}^{\mathcal{R}}_{t^n_i})^2 1_{t^n_i \in [0,t]} + (\hat{Y}^{\mathcal{R}}_t - \hat{Y}^{\mathcal{R}}_{t_{j^n_t}})^2. \tag{3.26}$$

Proof. Let $\pi_n \supset \mathcal{R}$, the random measures ν^{π_n} be defined by (3.20) for all $n \in \mathbb{N}$, the random measure μ be defined by (3.21), and $\mathcal{I}$ be defined by (3.22). Again, if $I =$ or $I = \{0\}$, then we are done. So suppose $I = (s,t] \in \mathcal{I}$ and observe

$$\nu^{\pi_n}(I) = \nu^{\pi_n}((s,t]) = \nu^{\pi_n}([0,t]) - \nu^{\pi_n}([0,s]) \text{ for all } n \in \mathbb{N} \tag{3.27}$$

and

$$\mu^{\pi_n}(I) = \mu^{\pi_n}((s,t]) = \mu^{\pi_n}([0,t]) - \mu^{\pi_n}([0,s]). \tag{3.28}$$

Therefore, it follows that for all $n \in \mathbb{N}$

$$\begin{aligned}
|\nu^{\pi_n}(I) - \mu(I)| &= |(\nu^{\pi_n}([0,t]) - \nu^{\pi_n}([0,s])) - (\mu([0,t]) - \mu([0,s]))| \\
&= |(\nu^{\pi_n}([0,t]) - \mu^{\pi_n}([0,t]) + \mu^{\pi_n}([0,t]) - \mu([0,t])) \\
&\quad -(\nu^{\pi_n}([0,s]) - \mu^{\pi_n}([0,s]) + \mu^{\pi_n}([0,s]) - \mu([0,s]))| \\
&\leq |\nu^{\pi_n}([0,t]) - \mu^{\pi_n}([0,t])| + |\mu^{\pi_n}([0,t]) - \mu([0,t])| \\
&\quad +|\nu^{\pi_n}([0,s]) - \mu^{\pi_n}([0,s])| + |\mu^{\pi_n}([0,s]) - \mu([0,s])| \\
&= |(\hat{Y}^{\mathcal{R}}_{t_{j_t^n+1}} - \hat{Y}^{\mathcal{R}}_{t_{j_t^n}})^2 - (\hat{Y}^{\mathcal{R}}_t - \hat{Y}^{\mathcal{R}}_{t_{j_t^n}})^2| + |\mu^{\pi_n}([0,t]) - \mu([0,t])| \\
&\quad +|(\hat{Y}^{\mathcal{R}}_{t_{j_s^n+1}} - \hat{Y}^{\mathcal{R}}_{t_{j_s^n}})^2 - (\hat{Y}^{\mathcal{R}}_s - \hat{Y}^{\mathcal{R}}_{t_{j_s^n}})^2| + |\mu^{\pi_n}([0,s]) - \mu([0,s])| \\
&\leq |(\hat{Y}^{\mathcal{R}}_{t_{j_t^n+1}} - \hat{Y}^{\mathcal{R}}_{t_{j_t^n}})^2| + |(\hat{Y}^{\mathcal{R}}_t - \hat{Y}^{\mathcal{R}}_{t_{j_t^n}})^2| + |\mu^{\pi_n}([0,t]) - \mu([0,t])| \\
&\quad +|(\hat{Y}^{\mathcal{R}}_{t_{j_s^n+1}} - \hat{Y}^{\mathcal{R}}_{t_{j_s^n}})^2| + |(\hat{Y}^{\mathcal{R}}_s - \hat{Y}^{\mathcal{R}}_{t_{j_s^n}})^2| + |\mu^{\pi_n}([0,s]) - \mu([0,s])|.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
t_{j_t^n+1} &\to t \quad \text{as } n \to \infty \\
t_{j_t^n} &\to t \quad \text{as } n \to \infty \\
t_{j_s^n+1} &\to s \quad \text{as } n \to \infty \\
t_{j_s^n} &\to s \quad \text{as } n \to \infty.
\end{aligned}$$

Since $s, t \notin (\mathcal{R} \backslash \{T\})$ and $\xi \in \mathbb{L}^2_T(\mathbb{R}^d)$ it follows that

$$\begin{aligned}
|(\hat{Y}^{\mathcal{R}}_{t_{j_t^n+1}} - \hat{Y}^{\mathcal{R}}_{t_{j_t^n}})^2| &\xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty \\
|(\hat{Y}^{\mathcal{R}}_t - \hat{Y}^{\mathcal{R}}_{t_{j_t^n}})^2| &\xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty \\
|(\hat{Y}^{\mathcal{R}}_{t_{j_s^n+1}} - \hat{Y}^{\mathcal{R}}_{t_{j_s^n}})^2| &\xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty \\
|(\hat{Y}^{\mathcal{R}}_s - \hat{Y}^{\mathcal{R}}_{t_{j_s^n}})^2| &\xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Furthermore, since $[\hat{Y}^{\mathcal{R}}] < \infty$ it follows that

$$\begin{aligned}
|\mu^{\pi_n}([0,t]) - \mu([0,t])| &\xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty \\
|\mu^{\pi_n}([0,s]) - \mu([0,s])| &\xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore

$$|\nu^{\pi_n}(I) - \mu(I)| \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty \tag{3.29}$$

which concludes the proof. □

Proposition 3.4.11. *If $\pi_n \supset \mathcal{R}$ such that $||\pi_n|| \to 0$ as $n \to \infty$, then the process $\hat{Y}^{\mathcal{R}}$ is of finite quadratic variation along (π_n).*

Proof. Let $\pi_n \supset \mathcal{R}$ be such that $||\pi_n|| \to 0$ as $n \to \infty$, then, by ([17],Th.1.1), the result follows from Lemmas 3.4.9 and 3.4.10. □

Given the above propositions we can now apply Theorem 4 in [7] to obtain the following

Proposition 3.4.12. *If ϕ satisfies (3.7), $(Y_1^{\mathcal{R}}, Z_1^{\mathcal{R}}) = \phi(y_1^{\mathcal{R}}, z_1^{\mathcal{R}})$, and $(Y_2^{\mathcal{R}}, Z_2^{\mathcal{R}}) = \phi(y_2^{\mathcal{R}}, z_2^{\mathcal{R}})$, then there exist $0 < \eta < 1$ and a norm $||\cdot||$ on $\mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$ such that*

$$||(Y_1^{\mathcal{R}} - Y_2^{\mathcal{R}}, Z_1^{\mathcal{R}} - Z_2^{\mathcal{R}})|| < \eta\ (||(y_1^{\mathcal{R}} - y_2^{\mathcal{R}}, z_1^{\mathcal{R}} - z_2^{\mathcal{R}})||. \tag{3.30}$$

Proof. Let ϕ satisfies (3.7), $(Y_1^{\mathcal{R}}, Z_1^{\mathcal{R}}) = \phi(y_1^{\mathcal{R}}, z_1^{\mathcal{R}})$, $(Y_2^{\mathcal{R}}, Z_2^{\mathcal{R}}) = \phi(y_2^{\mathcal{R}}, z_2^{\mathcal{R}})$, and $F_t(y) = e^{\beta t}(y_t)^2$ for some $\beta \geq 1$, to be determined later. Furthermore let $i, j \in \{0, ..., k\}$ where $i < j$ and $r_0 := 0$. From $\mathcal{D}_t F(y) = \beta e^{\beta t}(y)^2$ we have

$$\int_{(r_i, r_j]} \mathcal{D}_t F_t(\hat{Y}_{s-}^{\mathcal{R}}) ds = \int_{(r_i, r_j] - \mathcal{R}} \beta e^{\beta s} (\hat{Y}_s^{\mathcal{R}})^2 ds. \tag{3.31}$$

In addition, we have shown that $[\hat{Y}^{\mathcal{R}}]^c(t) = \int_0^t |(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds$, thus $d[\hat{Y}^{\mathcal{R}}]^c(t) = |(\hat{Z}_t^{\mathcal{R}})^\top|^2 dt$. Then $\bigtriangledown_y^2 F_t(y) = 2e^{\beta t}$ gives

$$\int_{(r_i, r_j]} \frac{1}{2} tr(\bigtriangledown_y^2 F_s(\hat{Y}_{s-}^{\mathcal{R}}) d[\hat{Y}^{\mathcal{R}}]^c(s)) = \int_{(r_i, r_j] - \mathcal{R}} e^{\beta s} |(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds. \tag{3.32}$$

The above propositions gives the existence of the Föllmer integral defined by the following limit: $\int_{(0,T]} \bigtriangledown_y F_t(y_{t-}) d^\pi y := \lim_{n\to\infty} \sum_{i=0}^{k(n)-1} \bigtriangledown_y F_{t_i^n}(y_{t_i^n -}^{n, \bigtriangledown y(t_i^n)})(y(t_{i+1}^n) - y(t_i^n))$, where $y^n(t) = \sum_{i=0}^{k(n)-1} y(t_i-)1_{[t_i,t_{i+1})}(t) + y(T)1_T(t)$. Therefore, since $\bigtriangledown_y F_t(y) = 2e^{\beta t}(y)$, the Föllmer integral reads

$$\begin{aligned}\int_{(r_i,r_j]} \bigtriangledown_y F_s(\hat{Y}^{\mathcal{R}}_{s-}) d^\pi \hat{Y}^{\mathcal{R}}_s &= 2\int_{(r_i,r_j]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s d\hat{Y}^{\mathcal{R}}_s + 2\sum_{s\in(r_i,r_j]} e^{\beta s}\hat{Y}^{\mathcal{R}}_{s-} \triangle \hat{Y}^{\mathcal{R}}_s \\ &= 2\int_{(r_i,r_j]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s d\hat{Y}^{\mathcal{R}}_s + 2\sum_{m=i+1}^{j} e^{\beta r_m}\hat{Y}^{\mathcal{R}}_{r_m-} \triangle \hat{Y}^{\mathcal{R}}_{r_m}.\end{aligned} \tag{3.33}$$

Furthermore,

$$\begin{aligned}&\sum_{s\in(r_i,r_j]}(F_s(\hat{Y}^{\mathcal{R}}_s) - F_s(\hat{Y}^{\mathcal{R}}_{s-}) - \bigtriangledown_y F_s(\hat{Y}^{\mathcal{R}}_{s-}) \triangle \hat{Y}^{\mathcal{R}}_s) \\ &= \sum_{s\in(r_i,r_j]} e^{\beta s}((\hat{Y}^{\mathcal{R}}_s)^2 - (\hat{Y}^{\mathcal{R}}_{s-})^2) - 2\sum_{s\in(r_i,r_j]} e^{\beta s}(\hat{Y}^{\mathcal{R}}_{s-}) \triangle \hat{Y}^{\mathcal{R}}_s \\ &= \sum_{m=i+1}^{j} e^{\beta r_m}((\hat{Y}^{\mathcal{R}}_{r_m})^2 - (\hat{Y}^{\mathcal{R}}_{r_m-})^2) - 2\sum_{m=i+1}^{j} e^{\beta r_m}(\hat{Y}^{\mathcal{R}}_{r_m-}) \triangle \hat{Y}^{\mathcal{R}}_{r_m}.\end{aligned} \tag{3.34}$$

Thus from the equations above and Theorem 4 in [7] we get

$$\begin{aligned}e^{\beta r_j}(\hat{Y}^{\mathcal{R}}_{r_j})^2 - e^{\beta r_i}(\hat{Y}^{\mathcal{R}}_{r_i})^2 &= F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j}) - F_{r_i}(\hat{Y}^{\mathcal{R}}_{r_i}) \\ &= \int_{(r_i,r_j]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}^{\mathcal{R}}_s)^2 ds + \int_{(r_i,r_j]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds \\ &\quad + 2\int_{(r_i,r_j]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s d\hat{Y}^{\mathcal{R}}_s + 2\sum_{m=i+1}^{j} e^{\beta r_m}\hat{Y}^{\mathcal{R}}_{r_m-} \triangle \hat{Y}^{\mathcal{R}}_{r_m} \\ &\quad + \sum_{m=i+1}^{j} e^{\beta r_m}((\hat{Y}^{\mathcal{R}}_{r_m})^2 - (\hat{Y}^{\mathcal{R}}_{r_m-})^2) \\ &\quad - 2\sum_{m=i+1}^{j} e^{\beta r_m}(\hat{Y}^{\mathcal{R}}_{r_m-}) \triangle \hat{Y}^{\mathcal{R}}_{r_m}. \\ &= \int_{(r_i,r_j]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}^{\mathcal{R}}_s)^2 ds + \int_{(r_i,r_j]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds \\ &\quad + 2\int_{(r_i,r_j]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s d\hat{Y}^{\mathcal{R}}_s \\ &\quad + \sum_{m=i+1}^{j} e^{\beta r_m}((\hat{Y}^{\mathcal{R}}_{r_m})^2 - (\hat{Y}^{\mathcal{R}}_{r_m-})^2) \\ &= \int_{(r_i,r_j]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}^{\mathcal{R}}_s)^2 ds + \int_{(r_i,r_j]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds \\ &\quad + 2\int_{(r_i,r_j]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s \hat{f}(s,\bar{y}^{\mathcal{R}}_s,\bar{z}^{\mathcal{R}}_s) ds \\ &\quad + 2\int_{(r_i,r_j]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s (\hat{Z}^{\mathcal{R}}_s)^\top dW_s \\ &\quad + \sum_{m=i+1}^{j} F_{r_m}(\hat{Y}^{\mathcal{R}}_{r_m}) - F_{r_m}(\hat{Y}^{\mathcal{R}}_{r_m-}).\end{aligned} \tag{3.35}$$

Observe that $\hat{Y}_T^{\mathcal{R}} = \xi - \xi = 0$, thus $F_T(\hat{Y}_T^{\mathcal{R}}) = 0$. Moreover, $Y_T^{\mathcal{R}} = \xi \in \mathbb{L}_T^2$, thus $\hat{Y}_{T-}^{\mathcal{R}} = \hat{Y}_T^{\mathcal{R}} = 0$. It is clear that $\mathrm{E}(\hat{Y}_T^{\mathcal{R}}) = 0$ and $\mathrm{E}(\hat{Y}_{T-}^{\mathcal{R}}) = 0$. Recall that $r_k = T$, thus

$$\begin{aligned}
\mathrm{E}(F_{r_{k-1}}(\hat{Y}_{r_{k-1}}^{\mathcal{R}})) &= -\mathrm{E}(0 - F_{r_{k-1}}(\hat{Y}_{r_{k-1}}^{\mathcal{R}})) \\
&= -\mathrm{E}(F_{r_k}(\hat{Y}_T^{\mathcal{R}}) - F_{r_{k-1}}(\hat{Y}_{r_{k-1}}^{\mathcal{R}})) \\
&= -\mathrm{E}(\textstyle\int_{(r_{k-1},T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_{k-1},T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds \\
&+ 2\textstyle\int_{(r_{k-1},T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}}\hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds \\
&+ 2\textstyle\int_{(r_{k-1},T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}}(\hat{Z}_s^{\mathcal{R}})^\top dW_s \\
&+ \textstyle\sum_{s=T}^{T} F_s(\hat{Y}_s^{\mathcal{R}}) - F_s(\hat{Y}_{s-}^{\mathcal{R}})) \\
&= -\mathrm{E}(\textstyle\int_{(r_{k-1},T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_{k-1},T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top)|^2 ds) \\
&+ 2\mathrm{E}(-\textstyle\int_{(r_{k-1},T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}}\hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds).
\end{aligned} \tag{3.36}$$

It follows that

$$\begin{aligned}
\mathrm{E}(F_{r_{k-2}}(\hat{Y}_{r_{k-2}}^{\mathcal{R}})) &= -\mathrm{E}(F_{r_{k-1}}(\hat{Y}_{r_{k-1}}^{\mathcal{R}}) - F_{r_{k-2}}(\hat{Y}_{r_{k-2}}^{\mathcal{R}})) + \mathrm{E}(F_{r_{k-1}}(\hat{Y}_{r_{k-1}}^{\mathcal{R}})) \\
&= -\mathrm{E}(\textstyle\int_{(r_{k-2},r_{k-1}]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_{k-2},r_{k-1}]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\
&+ 2\mathrm{E}(-\textstyle\int_{(r_{k-2},r_{k-1}]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}}\hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds) \\
&+ \mathrm{E}(-\textstyle\sum_{s=r_{k-1}}^{r_{k-1}} F_s(\hat{Y}_s^{\mathcal{R}}) - F_s(\hat{Y}_{s-}^{\mathcal{R}})) \\
&+ -\mathrm{E}(\textstyle\int_{(r_{k-1},T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_{k-1},T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\
&+ 2\mathrm{E}(-\textstyle\int_{(r_{k-1},T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}}\hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds) \\
&= -\mathrm{E}(\textstyle\int_{(r_{k-2},T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_{k-2},T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\
&+ 2\mathrm{E}(-\textstyle\int_{(r_{k-2},T]-\mathcal{R}} e^{\beta t}\hat{Y}_t^{\mathcal{R}}\hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds) \\
&+ \mathrm{E}(F_{r_{k-1}}(\hat{Y}_{r_{k-1}-}^{\mathcal{R}}) - F_{r_{k-1}}(\hat{Y}_{r_{k-1}}^{\mathcal{R}})).
\end{aligned} \tag{3.37}$$

Continuing in this manner, it can be shown that for any $j \in \{0, ..., k-1\}$

$$\begin{aligned}
\mathrm{E}(F_{r_j}(\hat{Y}_{r_j}^{\mathcal{R}})) &= -\mathrm{E}(\textstyle\int_{(r_j,T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\
&+ 2\mathrm{E}(-\textstyle\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}}\hat{f}(s, \bar{y}_s^{\mathcal{R}}, \bar{z}_s^{\mathcal{R}})ds) \\
&+ \mathrm{E}(\textstyle\sum_{m=j+1}^{k} F_{r_m}(\hat{Y}_{r_m-}^{\mathcal{R}}) - F_{r_m}(\hat{Y}_{r_m}^{\mathcal{R}})).
\end{aligned} \tag{3.38}$$

It follows that

$$\begin{aligned}
&\textstyle\sum_{j=0}^{k-1} \mathrm{E}(F_{r_j}(\hat{Y}_{r_j}^{\mathcal{R}}))\\
&+\textstyle\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s{}^{\mathcal{R}})^\top|^2 ds)\\
&= 2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1} \int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}} \hat{f}(s,\bar{y}_s^{\mathcal{R}},\bar{z}_s^{\mathcal{R}}) ds)\\
&+\textstyle\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}(F_{r_m}(\hat{Y}_{r_m-}^{\mathcal{R}}) - F_{r_m}(\hat{Y}_{r_m}^{\mathcal{R}})).
\end{aligned} \tag{3.39}$$

In fact, because $E(F_{r_k}(\hat{Y}_{r_k}^{\mathcal{R}}))=0$, we get

$$\begin{aligned}
&F_{r_0}(\hat{Y}_{r_0}^{\mathcal{R}}) + \textstyle\sum_{j=1}^{k} \mathrm{E}(F_{r_j}(\hat{Y}_{r_j}^{\mathcal{R}}))\\
&+\textstyle\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s{}^{\mathcal{R}})^\top|^2 ds)\\
&= 2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1} \int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}} \hat{f}(s,\bar{y}_s^{\mathcal{R}},\bar{z}_s^{\mathcal{R}}) ds)\\
&+\textstyle\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}(F_{r_m}(\hat{Y}_{r_m-}^{\mathcal{R}}) - F_{r_m}(\hat{Y}_{r_m}^{\mathcal{R}})).
\end{aligned} \tag{3.40}$$

Moreover, for $\delta \in (0,1)$

$$\begin{aligned}
&F_{r_0}(\hat{Y}_{r_0}^{\mathcal{R}}) + \textstyle\sum_{j=1}^{k} \mathrm{E}(F_{r_j}(\hat{Y}_{r_j}^{\mathcal{R}}))\\
&+\textstyle\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} \beta (e^{\beta s}\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s{}^{\mathcal{R}})^\top|^2 ds)\\
=\;& 2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1} \int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}} \hat{f}(s,\bar{y}_s^{\mathcal{R}},\bar{z}_s^{\mathcal{R}}) ds)\\
&+\textstyle\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}(F_{r_m}(\hat{Y}_{r_m-}^{\mathcal{R}}) - F_{r_m}(\hat{Y}_{r_m}^{\mathcal{R}}))\\
=\;& 2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1} \int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}} \hat{f}(s,\bar{y}_s^{\mathcal{R}},\bar{z}_s^{\mathcal{R}}) ds)\\
&+(1-\delta)\textstyle\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}(F_{r_m}(\hat{Y}_{r_m-}^{\mathcal{R}}) - F_{r_m}(\hat{Y}_{r_m}^{\mathcal{R}}))\\
&-\delta\textstyle\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}(F_{r_m}(\hat{Y}_{r_m}^{\mathcal{R}})) + \delta\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}(F_{r_m}(\hat{Y}_{r_m-}^{\mathcal{R}})).
\end{aligned} \tag{3.41}$$

Note that $\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} a_m = \sum_{j=1}^{k} j a_j$, so it follows from the above equation that

$$\begin{aligned}
&F_{r_0}(\hat{Y}_{r_0}^{\mathcal{R}}) + \textstyle\sum_{j=1}^{k}(1+\delta j) \mathrm{E}(F_{r_j}(\hat{Y}_{r_j}^{\mathcal{R}}))\\
&+\textstyle\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s{}^{\mathcal{R}})^\top|^2 ds)\\
=\;& 2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1} \int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}} \hat{f}(s,\bar{y}_s^{\mathcal{R}},\bar{z}_s^{\mathcal{R}}) ds)\\
&+(1-\delta)\textstyle\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}(F_{r_m}(\hat{Y}_{r_m-}^{\mathcal{R}}) - F_{r_m}(\hat{Y}_{r_m}^{\mathcal{R}}))\\
&+\delta\textstyle\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}(F_{r_m}(\hat{Y}_{r_m}^{\mathcal{R}})).
\end{aligned} \tag{3.42}$$

Let us now focus on the term

$$F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j-}) - F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j}) = e^{\beta r_j}(\hat{Y}^{\mathcal{R}}_{r_j-})^2 - e^{\beta r_j}(\hat{Y}^{\mathcal{R}}_{r_j})^2 = e^{\beta r_j}((\hat{Y}^{\mathcal{R}}_{r_j-})^2 - (\hat{Y}^{\mathcal{R}}_{r_j})^2). \tag{3.43}$$

Using $b^2 - a^2 = -(a-b)^2 + 2(b-a)b$ we get

$$\begin{aligned} F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j-}) - F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j}) &= e^{\beta r_j}((\hat{Y}^{\mathcal{R}}_{r_j-})^2 - (\hat{Y}^{\mathcal{R}}_{r_j})^2) \\ &= e^{\beta r_j}(-(\hat{Y}^{\mathcal{R}}_{r_j} - \hat{Y}^{\mathcal{R}}_{r_j-})^2 + 2(\hat{Y}^{\mathcal{R}}_{r_j-} - \hat{Y}^{\mathcal{R}}_{r_j})\hat{Y}^{\mathcal{R}}_{r_j-}) \end{aligned} \tag{3.44}$$

Note that for any real numbers a, b and $\gamma > 0$, $2ab \leq \gamma a^2 + \frac{1}{\gamma}b^2$, thus

$$\begin{aligned} F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j-}) - F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j}) &= e^{\beta r_j}(-(\hat{Y}^{\mathcal{R}}_{r_j} - \hat{Y}^{\mathcal{R}}_{r_j-})^2 + 2(\hat{Y}^{\mathcal{R}}_{r_j-} - \hat{Y}^{\mathcal{R}}_{r_j})\hat{Y}^{\mathcal{R}}_{r_j-}) \\ &\leq e^{\beta r_j}(-(\hat{Y}^{\mathcal{R}}_{r_j} - \hat{Y}^{\mathcal{R}}_{r_j-})^2 + \gamma(\hat{Y}^{\mathcal{R}}_{r_j-} - \hat{Y}^{\mathcal{R}}_{r_j})^2 + \tfrac{1}{\gamma}(\hat{Y}^{\mathcal{R}}_{r_j-})^2) \\ &= (\gamma-1)e^{\beta r_j}(\hat{Y}^{\mathcal{R}}_{r_j} - \hat{Y}^{\mathcal{R}}_{r_j-})^2 + \tfrac{e^{\beta r_j}}{\gamma}(\hat{Y}^{\mathcal{R}}_{r_j-})^2. \end{aligned} \tag{3.45}$$

It follows that

$$\begin{aligned} & F_{r_0}(\hat{Y}^{\mathcal{R}}_{r_0}) + \textstyle\sum_{j=1}^{k}(1+\delta j)\mathrm{E}(F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j})) \\ + \; & \textstyle\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}^{\mathcal{R}}_s)^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\ \leq \; & 2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1}\int_{(r_j,T]-\mathcal{R}} e^{\beta t}\hat{Y}^{\mathcal{R}}_t \hat{f}(s, \bar{y}^{\mathcal{R}}_s, \bar{z}^{\mathcal{R}}_s)ds) \\ + \; & (1-\delta)\textstyle\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}((\gamma-1)e^{\beta r_m}(\hat{Y}^{\mathcal{R}}_{r_m} - \hat{Y}^{\mathcal{R}}_{r_m-})^2 + \tfrac{e^{\beta r_m}}{\gamma}(\hat{Y}^{\mathcal{R}}_{r_m-})^2) \\ + \; & \delta\textstyle\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}(F_{r_m}(\hat{Y}^{\mathcal{R}}_{r_m-})) \\ \leq \; & 2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1}\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s \hat{f}(s, \bar{y}^{\mathcal{R}}_s, \bar{z}^{\mathcal{R}}_s)ds) \\ + \; & (1-\delta)(\gamma-1)\textstyle\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}(e^{\beta r_m}(\hat{Y}^{\mathcal{R}}_{r_m} - \hat{Y}^{\mathcal{R}}_{r_m-})^2) \\ + \; & (\delta + \tfrac{(1-\delta)}{\gamma})\textstyle\sum_{j=0}^{k-1}\sum_{m=j+1}^{k} \mathrm{E}(F_{r_m}(\hat{Y}^{\mathcal{R}}_{r_m-})). \end{aligned} \tag{3.46}$$

Observe that from (3.38) it follows that for any $j \in \{0, ..., k-1\}$

$$\begin{aligned} \textstyle\sum_{m=j+1}^{k} \mathrm{E}(F_{r_m}(\hat{Y}^{\mathcal{R}}_{r_m-})) &= \mathrm{E}(\textstyle\int_{(r_j,T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}^{\mathcal{R}}_s)^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\ &+ 2\mathrm{E}(\textstyle\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s \hat{f}(s, \bar{y}^{\mathcal{R}}_s, \bar{z}^{\mathcal{R}}_s)ds) \\ &+ \textstyle\sum_{m=j}^{k} \mathrm{E}(F_{r_m}(\hat{Y}^{\mathcal{R}}_{r_m})). \end{aligned} \tag{3.47}$$

It follows that

$$
\begin{aligned}
& F_{r_0}(\hat{Y}_{r_0}^{\mathcal{R}}) + \textstyle\sum_{j=1}^{k}(1+\delta j)\mathrm{E}(F_{r_j}(\hat{Y}_{r_j}^{\mathcal{R}})) \\
+ \ & \textstyle\sum_{j=0}^{k-1}\mathrm{E}(\int_{(r_j,T]-\mathcal{R}}\beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\
\leq \ & 2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1}\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}}\hat{f}(s,\bar{y}_s^{\mathcal{R}},\bar{z}_s^{\mathcal{R}})ds) \\
+ \ & (1-\delta)(\gamma-1)\textstyle\sum_{j=1}^{k} j\mathrm{E}(e^{\beta r_j}(\hat{Y}_{r_j}^{\mathcal{R}} - \hat{Y}_{r_j-}^{\mathcal{R}})^2) \\
+ \ & (\delta + \tfrac{(1-\delta)}{\gamma})\textstyle\sum_{j=0}^{k-1}\mathrm{E}(\int_{(r_j,T]-\mathcal{R}}\beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\
+ \ & (\delta + \tfrac{(1-\delta)}{\gamma})\textstyle\sum_{j=0}^{k-1} 2\mathrm{E}(\int_{(r_j,T]-\mathcal{R}} e^{\beta t}\hat{Y}_t^{\mathcal{R}}\hat{f}(s,\bar{y}_s^{\mathcal{R}},\bar{z}_s^{\mathcal{R}})ds) \\
+ \ & (\delta + \tfrac{(1-\delta)}{\gamma})\textstyle\sum_{j=1}^{k} j\mathrm{E}(F_{r_j}(\hat{Y}_{r_j}^{\mathcal{R}})).
\end{aligned}
\tag{3.48}
$$

Moreover

$$
\begin{aligned}
& F_{r_0}(\hat{Y}_{r_0}^{\mathcal{R}}) + \textstyle\sum_{j=1}^{k}(1+\delta j) - (\delta + \tfrac{(1-\delta)}{\gamma})\mathrm{E}(F_{r_j}(\hat{Y}_{r_j}^{\mathcal{R}})) \\
+ \ & \textstyle\sum_{j=0}^{k-1}\mathrm{E}(\int_{(r_j,T]-\mathcal{R}}\beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\
\leq \ & 2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1}\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}}\hat{f}(s,\bar{y}_s^{\mathcal{R}},\bar{z}_s^{\mathcal{R}})ds) \\
+ \ & (1-\delta)(\gamma-1)\textstyle\sum_{j=1}^{k} j\mathrm{E}(e^{\beta r_j}(\hat{Y}_{r_j}^{\mathcal{R}} - \hat{Y}_{r_j-}^{\mathcal{R}})^2) \\
+ \ & (\delta + \tfrac{(1-\delta)}{\gamma})\textstyle\sum_{j=0}^{k-1}\mathrm{E}(\int_{(r_j,T]-\mathcal{R}}\beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\
+ \ & (\delta + \tfrac{(1-\delta)}{\gamma})\textstyle\sum_{j=0}^{k-1} 2\mathrm{E}(\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}}\hat{f}(s,\bar{y}_s^{\mathcal{R}},\bar{z}_s^{\mathcal{R}})ds).
\end{aligned}
\tag{3.49}
$$

Since $F_{r_0}(\hat{Y}_{r_0}^{\mathcal{R}}) \geq 0$ and $(1+\delta j) - (\delta + \frac{(1-\delta)}{\gamma}) \geq (1 - j\frac{(1-\delta)}{\gamma})$, for $j \in \{1, ..., k\}$, we obtain

$$
\begin{aligned}
& \textstyle\sum_{j=1}^{k}(1 - j\tfrac{(1-\delta)}{\gamma})\mathrm{E}(F_{r_j}(\hat{Y}_{r_j}^{\mathcal{R}})) \\
+ \ & (1-\delta-\tfrac{(1-\delta)}{\gamma})\textstyle\sum_{j=0}^{k-1}\mathrm{E}(\int_{(r_j,T]-\mathcal{R}}\beta e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\
\leq \ & (1-\delta-\tfrac{(1-\delta)}{\gamma})2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1}\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}_s^{\mathcal{R}}\hat{f}(s,\bar{y}_s^{\mathcal{R}},\bar{z}_s^{\mathcal{R}})ds) \\
+ \ & (1-\delta)(\gamma-1)\textstyle\sum_{j=1}^{k} j\mathrm{E}(e^{\beta r_j}(\hat{Y}_{r_j}^{\mathcal{R}} - \hat{Y}_{r_j-}^{\mathcal{R}})^2).
\end{aligned}
\tag{3.50}
$$

Furthermore, observe that

$$
\begin{aligned}
|\hat{Y}^{\mathcal{R}}_{r_j} - \hat{Y}^{\mathcal{R}}_{r_j-}| &= |(Y^{\mathcal{R}}_{1,r_j} - Y^{\mathcal{R}}_{2,r_j}) - (Y^{\mathcal{R}}_{1,r_j-} - Y^{\mathcal{R}}_{2,r_j-})| \\
&= |(Y^{\mathcal{R}}_{1,r_j} - Y^{\mathcal{R}}_{1,r_j-}) - (Y^{\mathcal{R}}_{2,r_j} - Y^{\mathcal{R}}_{2,r_j-})| \\
&= |\rho(y^{\mathcal{R}}_{1,r_j}) - \rho(y^{\mathcal{R}}_{2,r_j})| \\
&\leq |y^{\mathcal{R}}_{1,r_j} - y^{\mathcal{R}}_{2,r_j}| \\
&= |\hat{y}^{\mathcal{R}}_{r_j}|.
\end{aligned}
\tag{3.51}
$$

Therefore

$$
\begin{aligned}
&\textstyle\sum_{j=1}^{k}(1 - j\frac{(1-\delta)}{\gamma})\mathrm{E}(F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j})) \\
+ \quad & (1-\delta-\tfrac{(1-\delta)}{\gamma})\textstyle\sum_{j=0}^{k-1}\mathrm{E}(\int_{(r_j,T]-\mathcal{R}}\beta e^{\beta s}(\hat{Y}^{\mathcal{R}}_s)^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s{}^{\mathcal{R}})^\top|^2 ds) \\
\leq \quad & (1-\delta-\tfrac{(1-\delta)}{\gamma})2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1}\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s \hat{f}(s,\bar{y}^{\mathcal{R}}_s,\bar{z}^{\mathcal{R}}_s)ds) \\
+ \quad & (1-\delta)(\gamma-1)\textstyle\sum_{j=1}^{k} j\mathrm{E}(e^{\beta r_j}(\hat{y}^{\mathcal{R}}_{r_j})^2) \\
= \quad & (1-\delta-\tfrac{(1-\delta)}{\gamma})2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1}\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s \hat{f}(s,\bar{y}^{\mathcal{R}}_s,\bar{z}^{\mathcal{R}}_s)ds) \\
+ \quad & (1-\delta)(\gamma-1)\textstyle\sum_{j=1}^{k} j\mathrm{E}(F_{r_j}(\hat{y}^{\mathcal{R}}_{r_j})).
\end{aligned}
\tag{3.52}
$$

Note that because $\delta < 1$, $k(\frac{1}{k} - \frac{1-\delta}{\gamma}) \leq (1 - j\frac{1-\delta}{\gamma})$ for all $j \in \{1, ..., k\}$, it follows that

$$
\begin{aligned}
&k(\tfrac{1}{k} - \tfrac{1-\delta}{\gamma})\textstyle\sum_{j=1}^{k}\mathrm{E}(F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j})) \\
+ \quad & (1-\delta-\tfrac{(1-\delta)}{\gamma})\textstyle\sum_{j=0}^{k-1}\mathrm{E}(\int_{(r_j,T]-\mathcal{R}}\beta e^{\beta s}(\hat{Y}^{\mathcal{R}}_s)^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s{}^{\mathcal{R}})^\top|^2 ds) \\
\leq \quad & (1-\delta-\tfrac{(1-\delta)}{\gamma})2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1}\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s \hat{f}(s,\bar{y}^{\mathcal{R}}_s,\bar{z}^{\mathcal{R}}_s)ds) \\
+ \quad & (1-\delta)(\gamma-1)k\textstyle\sum_{j=1}^{k}\mathrm{E}(F_{r_j}(\hat{y}^{\mathcal{R}}_{r_j})).
\end{aligned}
\tag{3.53}
$$

Furthermore, by dividing both sides by k, we get

$$
\begin{aligned}
&(\tfrac{1}{k} - \tfrac{1-\delta}{\gamma})\textstyle\sum_{j=1}^{k}\mathrm{E}(F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j})) \\
+ \quad & (1-\delta-\tfrac{(1-\delta)}{\gamma})\tfrac{1}{k}\textstyle\sum_{j=0}^{k-1}\mathrm{E}(\int_{(r_j,T]-\mathcal{R}}\beta e^{\beta s}(\hat{Y}^{\mathcal{R}}_s)^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s{}^{\mathcal{R}})^\top|^2 ds) \\
\leq \quad & (1-\delta-\tfrac{(1-\delta)}{\gamma})\tfrac{1}{k}2\mathrm{E}(-\textstyle\sum_{j=0}^{k-1}\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s \hat{f}(s,\bar{y}^{\mathcal{R}}_s,\bar{z}^{\mathcal{R}}_s)ds) \\
+ \quad & (1-\delta)(\gamma-1)\textstyle\sum_{j=1}^{k}\mathrm{E}(F_{r_j}(\hat{y}^{\mathcal{R}}_{r_j})).
\end{aligned}
\tag{3.54}
$$

Recall that the above inequality holds for $\delta \in (0,1)$ and $\gamma > 0$, so choose $\gamma = 2k+1$ and $\delta = 1 - \frac{1}{(2k+1)^2}$. This leads to the following equalities:

$$\begin{array}{rcl} (\frac{1}{k} - \frac{1-\delta}{\gamma}) & = & \frac{(2k+1)^3-k}{k(2k+1)^3} \\ (1-\delta-\frac{1-\delta}{\gamma})\frac{1}{k} & = & \frac{2}{(2k+1)^3} \\ (1-\delta)(\gamma-1) & = & \frac{2}{(2k+1)^2} \end{array} \tag{3.55}$$

We get

$$\begin{array}{rl} & \frac{(2k+1)^3-k}{k(2k+1)^3}\sum_{j=1}^{k} \mathrm{E}(F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j})) \\ + & \frac{2}{(2k+1)^3}\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}^{\mathcal{R}}_s)^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s{}^{\mathcal{R}})^\top|^2 ds) \\ \leq & \frac{2}{(2k+1)^3} 2\mathrm{E}(-\sum_{j=0}^{k-1}\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s \hat{f}(s,\bar{y}^{\mathcal{R}}_s,\bar{z}^{\mathcal{R}}_s)ds) \\ + & \frac{2}{(2k+1)^2}\sum_{j=1}^{k} \mathrm{E}(F_{r_j}(\hat{y}^{\mathcal{R}}_{r_j})). \end{array} \tag{3.56}$$

It follows that

$$\begin{array}{rl} & \sum_{j=1}^{k} \mathrm{E}(F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j})) \\ + & \frac{2k}{(2k+1)^3-k}\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}^{\mathcal{R}}_s)^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s{}^{\mathcal{R}})^\top|^2 ds) \\ \leq & \frac{2k}{(2k+1)^3-k} 2\mathrm{E}(-\sum_{j=0}^{k-1}\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\hat{Y}^{\mathcal{R}}_s \hat{f}(s,\bar{y}^{\mathcal{R}}_s,\bar{z}^{\mathcal{R}}_s)ds) \\ + & \frac{2k(2k+1)}{(2k+1)^3-k}\sum_{j=1}^{k} \mathrm{E}(F_{r_j}(\hat{y}^{\mathcal{R}}_{r_j})). \end{array} \tag{3.57}$$

Furthermore, for $\alpha > 0$,

$$\begin{array}{rl} & \sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} -2e^{\beta s}\hat{Y}^{\mathcal{R}}_s \hat{f}(s,\bar{y}^{\mathcal{R}}_s,\bar{z}^{\mathcal{R}}_s)ds) \\ \leq & \sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} e^{\beta s}(\frac{2K^2}{\alpha}\hat{Y}^{\mathcal{R}}_s)^2 + \alpha(\hat{y}^{\mathcal{R}}_s)^2 + \alpha|\hat{z}^{\mathcal{R}}_s|^2)ds) \end{array} \tag{3.58}$$

gives

$$\begin{array}{rl} & \sum_{j=1}^{k} \mathrm{E}(F_{r_j}(\hat{Y}^{\mathcal{R}}_{r_j})) \\ + & \frac{2k}{(2k+1)^3-k}\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} \beta e^{\beta s}(\hat{Y}^{\mathcal{R}}_s)^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s{}^{\mathcal{R}})^\top|^2 ds) \\ \leq & \frac{2k}{(2k+1)^3-k}\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} e^{\beta s}(\frac{2K^2}{\alpha}(\hat{Y}^{\mathcal{R}}_s)^2 + \alpha(\hat{y}^{\mathcal{R}}_s)^2 + \alpha|\hat{z}^{\mathcal{R}}_s|^2)ds) \\ + & \frac{2k(2k+1)}{(2k+1)^3-k}\sum_{j=1}^{k} \mathrm{E}(F_{r_j}(\hat{y}^{\mathcal{R}}_{r_j})) \end{array} \tag{3.59}$$

and

$$
\begin{aligned}
& \textstyle\sum_{j=1}^{k} \mathrm{E}(F_{r_j}(\hat{Y}_{r_j}^{\mathcal{R}})) \\
+ \quad & \tfrac{2k}{(2k+1)^3-k} \textstyle\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} (\beta - (\tfrac{2K^2}{\alpha}))e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\
\leq \quad & \tfrac{2k}{(2k+1)^3-k} \textstyle\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} e^{\beta s}\alpha(\hat{y}_s^{\mathcal{R}})^2 + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}\alpha|\hat{z}_s^{\mathcal{R}}|^2 ds) \\
+ \quad & \tfrac{2k(2k+1)}{(2k+1)^3-k} \textstyle\sum_{j=1}^{k} \mathrm{E}(F_{r_j}(\hat{y}_{r_j}^{\mathcal{R}})) \\
= \quad & \alpha \tfrac{2k}{(2k+1)^3-k} \textstyle\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} e^{\beta s}(\hat{y}_s^{\mathcal{R}})^2 + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|\hat{z}_s^{\mathcal{R}}|^2 ds) \\
+ \quad & \tfrac{2k(2k+1)}{(2k+1)^3-k} \textstyle\sum_{j=1}^{k} \mathrm{E}(F_{r_j}(\hat{y}_{r_j}^{\mathcal{R}})).
\end{aligned}
\tag{3.60}
$$

That means

$$
\begin{aligned}
& \textstyle\sum_{j=1}^{k} \mathrm{E}(e^{\beta r_j}(\hat{Y}_{r_j}^{\mathcal{R}})^2) \\
+ \quad & \tfrac{2k}{(2k+1)^3-k} \textstyle\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} (\beta - (\tfrac{2K^2}{\alpha}))e^{\beta s}(\hat{Y}_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(\hat{Z}_s^{\mathcal{R}})^\top|^2 ds) \\
\leq \quad & \alpha \tfrac{2k}{(2k+1)^3-k} \textstyle\sum_{j=0}^{k-1} \mathrm{E}(\int_{(r_j,T]-\mathcal{R}} e^{\beta s}(\hat{y}_s^{\mathcal{R}})^2 + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|\hat{z}_s^{\mathcal{R}}|^2 ds) \\
+ \quad & \tfrac{2k(2k+1)}{(2k+1)^3-k} \textstyle\sum_{j=1}^{k} \mathrm{E}(e^{\beta r_j}(\hat{y}_{r_j}^{\mathcal{R}})^2).
\end{aligned}
\tag{3.61}
$$

Let $\eta^2 = \alpha = \frac{2k(2k+1)}{(2k+1)^3-k}$ and $\beta = 1 + \frac{2K^2}{\alpha}$, then for the norm $||\cdot||$

$$
\begin{aligned}
||(Y,Z)|| \quad = \quad & \mathrm{E}\Bigg(\tfrac{2k}{(2k+1)^3-k} \textstyle\sum_{j=0}^{k-1} \int_{(r_j,T]-\mathcal{R}} e^{\beta s}(Y_s^{\mathcal{R}})^2 ds + \int_{(r_j,T]-\mathcal{R}} e^{\beta s}|(Z_s^{\mathcal{R}})^\top|^2 ds \\
& + \textstyle\sum_{j=1}^{k} e^{\beta r_j}(Y_{r_j}^{\mathcal{R}})^2 \Bigg)^{\frac{1}{2}}
\end{aligned}
\tag{3.62}
$$

on $\mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$ and we arrive at

$$
||(Y_1^{\mathcal{R}} - Y_2^{\mathcal{R}}, Z_1^{\mathcal{R}} - Z_2^{\mathcal{R}})|| < \eta\ (||(y_1^{\mathcal{R}} - y_2^{\mathcal{R}}, z_1^{\mathcal{R}} - z_2^{\mathcal{R}})||. \tag{3.63}
$$

as claimed . □

Corollary 1. *FR-BSDE (3.6)′ has a unique solution in* $\mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$.

Remark 7. The mapping ϕ is a contraction in the space $\mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^2(\mathbb{R}^{d\times d})$ equipped with the norm $||\cdot||$. Furthermore, Theorem 3.2.1 follows.

CHAPTER 4

Donsker-Type Theorem for FR-BSDEs

4.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a Brownian motion $(W_t)_{0\le t\le T}$ and an i.i.d. Bernoulli sequence $\{\epsilon_j^n\}_{j=1}^n$, $n \in \mathbb{N}$. For a finite terminal time $T > 0$, now define the 1-dimensional random walk process for a fixed $n \in \mathbb{N}$,

$$W_t^n := \sqrt{\delta_n} \sum_{j=1}^{[t/\delta_n]} \epsilon_j^n, \text{ for all } 0 \le t \le T, \delta_n = \frac{T}{n}. \tag{4.1}$$

In addition define $\mathcal{G}_j^n := \sigma\{\epsilon_1^n, \dots, \epsilon_j^n\}$, $t_j^n = j\delta_n$, and $t_0^n = 0$. Furthermore define the filtration $(\mathcal{F}_t^n)_{0\le t\le T}$ as the right continuous filtration generated by W^n. Also consider the space $\mathbb{S}^2(\mathbb{R})$ denote the space of all predictable processes
$Y \equiv Y_t(\omega) : \Omega \times [0,T] \to \mathbb{R}$ s.t. $||Y||^2_{\mathbb{S}^2} = E[\sup_{0\le t\le T} |Y_t|^2] < \infty$.

4.2 Formulation

The main goal of this chapter is to prove the convergence of a random walk type discretization of the FR-BSDE $(1.6)'$. In [5] the convergence is shown for the BSDE. Since the proof does not depend on the dimensions of W or Y, we present our analysis is done for real-valued processes. Given $T > 0$, consider the reflection set

$$\mathcal{R} := \{0 < r_1 < \cdots < r_k := T\}, \qquad k \ge 1$$

and a given pair of sequences of boundary random variables

$$(\psi_{r_1}, ..., \psi_{r_k}) \in \mathbb{L}^2_{r_1}(\mathbb{R}) \times \cdots \times \mathbb{L}^2_{r_k}(\mathbb{R})$$

$$(\Psi_{r_1}, ..., \Psi_{r_k}) \in \mathbb{L}^2_{r_1}(\mathbb{R}) \times \cdots \times \mathbb{L}^2_{r_k}(\mathbb{R}),$$

where $\psi_{r_q} < \Psi_{r_q}$ for $q = 1, ..., k$ and the terminal random variable ξ s.t.

$$\psi_T \leq \xi \leq \Psi_T \qquad a.s.$$

Due to the numerical applications, without loss of generality, we assume that all elements of the given reflection set $\mathcal{R} = \{0 < r_1 < \cdots < r_k := T\}$ are rational numbers. Select the smallest $\lambda \in \mathbb{N}$ s.t. we have

$$r_q = \frac{\lambda_q}{\lambda} \tag{4.2}$$

where $\lambda_q \in \mathbb{N}$ for $q = 1, ..., k$, and set $n_i = i\lambda$, $i = 1, 2, ...$ thus

$$\mathcal{R} \subset \{j\delta_{n_i} | j = 0, ..., n_i\} \qquad i = 1, 2, ... \tag{4.3}$$

Definition 4.2.1. For a give reflection set

$$\mathcal{R} := \{0 < r_1 < \cdots < r_k := T\} \subset \mathbb{Q}, \qquad k \geq 1$$

the pair of sequences of boundary random variables $(\psi^{n_i}_{r_1}, ..., \psi^{n_i}_{r_k})$, $(\Psi^{n_i}_{r_1}, ..., \Psi^{n_i}_{r_k})$ such that

(i) $\psi^{n_i}_{r_q}$ and $\Psi^{n_i}_{r_q}$ are $\mathcal{G}^{n_i}_{r_q(\frac{n_i}{T})}$ measurable

(ii) $\sup_i E((\psi^{n_i}_{r_q})^2) + \sup_i E((\Psi^{n_i}_{r_q})^2) < \infty$, $q = 1, ..., k$

(iii) $\psi^{n_i}_{r_q} < \Psi^{n_i}_{r_q}$ for all $q \in \{1, ..., k\}$

and the $\mathcal{G}^{n_i}_{n_i}$ measurable terminal random variable ξ^{n_i} such that

(iv) $\psi^{n_i}_T \leq \xi^{n_i} \leq \Psi^{n_i}_T$

the discrete processes $(y^{n_i,\mathcal{R}}, z^{n_i,\mathcal{R}})$, adapted with respect to $\mathcal{G}^n$, is called a solution of the state-time discretization of the FR-BSDE if it satisfies the following equation

$$\begin{aligned} y^{n_i,\mathcal{R}}_j &= y^{n_i,\mathcal{R}}_{j+1} 1_{\{t^{n_i}_{j+1} \notin \mathcal{R}\}} + [(y^{n_i,\mathcal{R}}_{j+1} \vee \psi^{n_i}_{t^{n_i}_{j+1}}) \wedge \Psi^{n_i}_{t^{n_i}_{j+1}}] 1_{\{t^{n_i}_{j+1} \in \mathcal{R}\}} \\ &\quad + f(t^{n_i}_j, y^{n_i,\mathcal{R}}_j, z^{n_i,\mathcal{R}}_j)\delta_{n_i} - z^{n_i,\mathcal{R}}_j \epsilon^{n_i}_{j+1} \sqrt{\delta_{n_i}} \\ &\quad j = n_i - 1, ..., 0, \\ y^{n_i,\mathcal{R}}_{n_i} &= \xi^{n_i}. \end{aligned} \tag{4.4}$$

In order to define the discrete process (4.4) on $[0,T]$ we set

$$Y_t^{n_i,\mathcal{R}} = y^{n_i,\mathcal{R}}_{\lfloor \frac{t}{\delta_{n_i}} \rfloor} \text{ and } Z_t^{n_i,\mathcal{R}} = z^{n_i,\mathcal{R}}_{\lfloor \frac{t}{\delta_{n_i}} \rfloor} \text{ for } 0 \le t \le T \tag{4.5}$$

so that $Y^{n_i,\mathcal{R}}$ and $Z^{n_i,\mathcal{R}}$ are cadlag processes.

Our objective is to prove the following

Theorem 4.2.1. *If*

(i) $\mathcal{R} \subset \mathbb{Q}$ *is a finite reflection set*

(ii) f *is continuous in the first component*

(iii) $f(\cdot, u, v) \in \mathbb{S}^2(\mathbb{R})$ *for all* $(u,v) \in \mathbb{R} \times \mathbb{R}$

(iv) $\exists K$ *s.t.* $t \in [0,T]$, $u, u', v, v' \in \mathbb{R}$ $|f(t,u,v) - f(t,u',v')| < K(|u-u'| + |v-v'|)$

(v) ξ *is* $\mathcal{F}_T$*-measurable and* ξ^{n_i} *is* $\mathcal{G}^{n_i}_{n_i}$*-measurable*

(vi) $\xi^{n_i} \to \xi$ *as* $i \to \infty$ *in* L^1

(vii) $(\psi^{n_i}_{r_1}, ..., \psi^{n_i}_{r_k}) \to (\psi_{r_1}, ..., \psi_{r_k})$ *as* $i \to \infty$ *in probability*

(viii) $(\Psi^{n_i}_{r_1}, ..., \Psi^{n_i}_{r_k}) \to (\Psi_{r_1}, ..., \Psi_{r_k})$ *as* $i \to \infty$ *in probability*

and $W^{n_i} \to W$ *as* $i \to \infty$ *in the sense that*

$$\sup_{0 \le t \le T} |W_t^{n_i} - W_t| \to 0 \qquad \textit{in probability,} \tag{4.6}$$

then we have $(Y^{n_i,\mathcal{R}}, Z^{n_i,\mathcal{R}}) \to (Y^{\mathcal{R}}, Z^{\mathcal{R}})$, *in the sense that*

$$\sup_{0 \le t \le T} |Y_t^{n_i,\mathcal{R}} - Y_t^{\mathcal{R}}|^2 + \int_0^T |Z_s^{n_i,\mathcal{R}} - Z_s^{\mathcal{R}}|^2 ds \to 0 \textit{ as } i \to \infty \textit{ in probability.} \tag{4.7}$$

Following [5] consider decompositions

$$\begin{aligned} Y_t^{n_i,\mathcal{R}} - Y_t^{\mathcal{R}} &= (Y_t^{n_i,\mathcal{R}} - Y_t^{n_i,p,\mathcal{R}}) + (Y_t^{n_i,p,\mathcal{R}} - Y_t^{\infty,p,\mathcal{R}}) + (Y_t^{\infty,p,\mathcal{R}} - Y_t^{\mathcal{R}}), \\ Z_t^{n_i,\mathcal{R}} - Z_t^{\mathcal{R}} &= (Z_t^{n_i,\mathcal{R}} - Z_t^{n_i,p,\mathcal{R}}) + (Z_t^{n_i,p,\mathcal{R}} - Z_t^{\infty,p,\mathcal{R}}) + (Z_t^{\infty,p,\mathcal{R}} - Z_t^{\mathcal{R}}). \end{aligned} \tag{4.8}$$

Here p represents the p-th Picard iteration approximating the solution of the FR-BSDE. In particular, we set $Y^{\infty,0,\mathcal{R}} = 0, Z^{\infty,0,\mathcal{R}} = 0, y^{n_i,0,\mathcal{R}} = 0, z^{n_i,0,\mathcal{R}} = 0$ and recursively define $(Y^{\infty,p+1,\mathcal{R}}, Z^{\infty,p+1,\mathcal{R}})$ by

$$Y_t^{\infty,p+1,\mathcal{R}} = \xi + \int_t^T f(s, Y_s^{\infty,p,\mathcal{R}}, Z_s^{\infty,p,\mathcal{R}})ds - \int_t^T (Z_s^{\infty,p+1,\mathcal{R}})^\top dW_s + \sum_{s\in\mathcal{R}^t} \rho(Y_s^{\infty,p,\mathcal{R}}) \tag{4.9}$$

and $(y^{n_i,p+1,\mathcal{R}}, z^{n_i,p+1,\mathcal{R}})$ by

$$\begin{aligned} y_j^{n_i,p+1,\mathcal{R}} &= y_{j+1}^{n_i,p,\mathcal{R}} 1_{\{t_{j+1}^{n_i}\notin\mathcal{R}\}} + [(y_{j+1}^{n_i,p,\mathcal{R}} \vee \psi_{t_{j+1}^{n_i}}^{n_i}) \wedge \Psi_{t_{j+1}^{n_i}}^{n_i}] 1_{\{t_{j+1}^{n_i}\in\mathcal{R}\}} \\ &\quad + f(t_j^{n_i}, y_j^{n_i,p,\mathcal{R}}, z_j^{n_i,p,\mathcal{R}})\delta_{n_i} - z_j^{n_i,p+1,\mathcal{R}} \epsilon_{j+1}^{n_i} \sqrt{\delta_{n_i}}, \\ &\quad j = n_i - 1, ..., 0 \\ y_{n_i}^{n_i,p+1,\mathcal{R}} &= \xi^{n_i}. \end{aligned} \tag{4.10}$$

Remark 8. $(Y_t^{\infty,p+1,\mathcal{R}}, Z_t^{\infty,p+1,\mathcal{R}})$ corresponds to the Picard iteration for the contraction given by (1.7).

Again, we define the real-valued cadlag processes

$$Y_t^{n_i,p,\mathcal{R}} = y_{\lfloor \frac{t}{\delta_{n_i}} \rfloor}^{n_i,p,\mathcal{R}} \text{ and } Z_t^{n_i,p,\mathcal{R}} = z_{\lfloor \frac{t}{\delta_{n_i}} \rfloor}^{n_i,p,\mathcal{R}} \text{ for } 0 \le t \le T. \tag{4.11}$$

4.3 Proof of Discrete Picard Convergence

We turn our attention to proving the following

Lemma 4.3.1. *Let $\mathcal{R} \subset \mathbb{Q}$ be a finite reflection set, $p \in \mathbb{N}$, K be the Lipschitz constant of f, and $0 < \eta < 1$ a fixed number. Take $\gamma > 1$ and $C > 0$ such that*

$$0 < \frac{8TK^2}{\gamma(1 - \frac{\gamma}{C})} \le \eta. \tag{4.12}$$

Furthermore, take $I \in \mathbb{N}$ large enough such that for all $i > I$, $\delta_{n_i} < \frac{1}{C}$ and

$$\frac{1}{2} e^{-t\gamma} < ((1 - \frac{T\gamma}{n_i})^{-n_i})^{\frac{t}{T}}, \text{ for all } t \in [0, T]. \tag{4.13}$$

Then for all $i > I$ and $p > 2$

$$\begin{aligned} &||(Y^{n_i,p+1,\mathcal{R}}_{t^{n_i}_j} - Y^{n_i,p,\mathcal{R}}_{t^{n_i}_j}, Z^{n_i,p+1,\mathcal{R}}_{t^{n_i}_j} - Z^{n_i,p,\mathcal{R}}_{t^{n_i}_j})||_\gamma \\ &< \eta\ ||(Y^{n_i,p,\mathcal{R}}_{t^{n_i}_j} - Y^{n_i,p-1,\mathcal{R}}_{t^{n_i}_j}, Z^{n_i,p,\mathcal{R}}_{t^{n_i}_j} - Z^{n_i,p-1,\mathcal{R}}_{t^{n_i}_j})||_\gamma \end{aligned} \tag{4.14}$$

where

$$||(Y,Z)||_\gamma = \left(\frac{1}{n_i} \sum_{j=0}^{n_i-1} e^{-t^{n_i}_j \gamma} E(|Y_{t^{n_i}_j}|^2 + |Z_{t^{n_i}_j}|^2) \right)^{\frac{1}{2}}. \tag{4.15}$$

Remark 9. Our proof follows ideas from [28], which however where used in a different setting of a constant lower boundary with continuous time reflections.

Proof. Assume $\mathcal{R} \subset \mathbb{Q}$ is a finite reflection set, $2 < p \in \mathbb{N}$, K is the Lipschitz constant of f, and $0 < \eta < 1$ is a fixed number. Take $\gamma > 1$ and $C > 0$ such that

$$0 < \frac{8TK^2}{\gamma(1 - \frac{\gamma}{C})} \leq \eta. \tag{4.16}$$

Furthermore, take $I \in \mathbb{N}$ large enough such that for all $i > I$, $\delta_{n_i} < \frac{1}{C}$ and

$$\frac{1}{2} e^{-t\gamma} < ((1 - \frac{T\gamma}{n_i})^{-n_i})^{\frac{t}{T}}, \text{ for all } t \in [0,T]. \tag{4.17}$$

Let $i > I$ and define

$$\begin{aligned} \hat{Y}^{n_i,p+1,\mathcal{R}}_j &:= Y^{n_i,p+1,\mathcal{R}}_{t^{n_i}_j} - Y^{n_i,p,\mathcal{R}}_{t^{n_i}_j} \\ \hat{Z}^{n_i,p+1,\mathcal{R}}_j &:= Z^{n_i,p+1,\mathcal{R}}_{t^{n_i}_j} - Z^{n_i,p,\mathcal{R}}_{t^{n_i}_j} \\ \hat{S}^{n_i,p+1,\mathcal{R}}_{j+1} &:= Y^{n_i,p+1,\mathcal{R}}_{t^{n_i}_{j+1}} 1_{\{t^{n_i}_{j+1} \notin \mathcal{R}\}} + [(Y^{n_i,p+1,\mathcal{R}}_{t^{n_i}_{j+1}} \vee \psi^{n_i}_{t^{n_i}_{j+1}}) \wedge \Psi^{n_i}_{t^{n_i}_{j+1}}] 1_{\{t^{n_i}_{j+1} \in \mathcal{R}\}} \\ &\quad - Y^{n_i,p,\mathcal{R}}_{t^{n_i}_{j+1}} 1_{\{t^{n_i}_{j+1} \notin \mathcal{R}\}} + [(Y^{n_i,p,\mathcal{R}}_{t^{n_i}_{j+1}} \vee \psi^{n_i}_{t^{n_i}_{j+1}}) \wedge \Psi^{n_i}_{t^{n_i}_{j+1}}] 1_{\{t^{n_i}_{j+1} \in \mathcal{R}\}} \\ \hat{f}^{n_i,p,\mathcal{R}}_j &:= f(t^{n_i}_j, Y^{n_i,p,\mathcal{R}}_{t^{n_i}_j}, Z^{n_i,p,\mathcal{R}}_{t^{n_i}_j}) - f(t^{n_i}_j, Y^{n_i,p-1,\mathcal{R}}_{t^{n_i}_j}, Z^{n_i,p-1,\mathcal{R}}_{t^{n_i}_j}). \end{aligned} \tag{4.18}$$

From (4.10) it follows that

$$\hat{S}^{n_i,p+1,\mathcal{R}}_{j+1} = \hat{Y}^{n_i,p+1,\mathcal{R}}_j - \delta_{n_i} \hat{f}^{n_i,p,\mathcal{R}}_j + \hat{Z}^{n_i,p+1,\mathcal{R}}_j \epsilon^{n_i}_{j+1} \sqrt{\delta_{n_i}}. \tag{4.19}$$

Thus

$$\begin{aligned}
E(|\hat{S}_{j+1}^{n_i,p+1,\mathcal{R}}|^2) &= E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2 + \delta_{n_i}^2|\hat{f}_j^{n_i,p,\mathcal{R}}|^2 + \delta_{n_i}|\hat{Z}_j^{n_i,p+1,\mathcal{R}}|^2 \\
&\quad -2\hat{f}_j^{n_i,p,\mathcal{R}}\delta_{n_i}\hat{Y}_j^{n_i,p+1,\mathcal{R}} + 2\hat{Z}_j^{n_i,p+1,\mathcal{R}}\epsilon_{j+1}^{n_i}\sqrt{\delta_{n_i}}(\hat{Y}_j^{n_i,p+1,\mathcal{R}} - \hat{f}_j^{n_i,p,\mathcal{R}})) \\
&= E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2 + \delta_{n_i}^2|\hat{f}_j^{n_i,p,\mathcal{R}}|^2 + \delta_{n_i}|\hat{Z}_j^{n_i,p+1,\mathcal{R}}|^2 \\
&\quad -2\hat{f}_j^{n_i,p,\mathcal{R}}\delta_{n_i}\hat{Y}_j^{n_i,p+1,\mathcal{R}})
\end{aligned} \tag{4.20}$$

giving

$$\begin{aligned}
&E(|\hat{S}_{j+1}^{n_i,p+1,\mathcal{R}}|^2) - E(\delta_{n_i}^2|\hat{f}_j^{n_i,p,\mathcal{R}}|^2) + E(2\hat{f}_j^{n_i,p,\mathcal{R}}\delta_{n_i}\hat{Y}_j^{n_i,p+1,\mathcal{R}}) \\
&= E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2) + E(\delta_{n_i}|\hat{Z}_j^{n_i,p+1,\mathcal{R}}|^2).
\end{aligned} \tag{4.21}$$

Furthermore, by $|\hat{S}_{j+1}^{n_i,p+1,\mathcal{R}}|^2 \le |\hat{Y}_{j+1}^{n_i,p+1,\mathcal{R}}|^2$

$$E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2) + E(\delta_{n_i}|\hat{Z}_j^{n_i,p+1,\mathcal{R}}|^2) \le E(|\hat{Y}_{j+1}^{n_i,p+1,\mathcal{R}}|^2) + E(2\hat{f}_j^{n_i,p,\mathcal{R}}\delta_{n_i}\hat{Y}_j^{n_i,p+1,\mathcal{R}}) \tag{4.22}$$

and $\gamma > 0$ implies

$$\begin{aligned}
&E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2) + E(\delta_{n_i}|\hat{Z}_j^{n_i,p+1,\mathcal{R}}|^2) \\
&\le E(|\hat{Y}_{j+1}^{n_i,p+1,\mathcal{R}}|^2) + \delta_{n_i}\gamma E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2) + \tfrac{\delta_{n_i}}{\gamma}E(|\hat{f}_j^{n_i,p,\mathcal{R}}|^2).
\end{aligned} \tag{4.23}$$

In particular

$$(1-\delta_{n_i}\gamma)E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2) \le E(|\hat{Y}_{j+1}^{n_i,p+1,\mathcal{R}}|^2) + \tfrac{\delta_{n_i}}{\gamma}E(|\hat{f}_j^{n_i,p,\mathcal{R}}|^2). \tag{4.24}$$

Since $\hat{Y}_T^{n_i,p+1,\mathcal{R}} = 0$ by iterating the above inequality we have

$$(1-\delta_{n_i}\gamma)^{n_i-j}E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2) \le \tfrac{\delta_{n_i}}{\gamma}\textstyle\sum_{m=j}^{n_i-1}(1-\delta_{n_i}\gamma)^{n_i-m-1}E(|\hat{f}_m^{n_i,p,\mathcal{R}}|^2). \tag{4.25}$$

Furthermore

$$\begin{aligned}
&\textstyle\sum_{j=0}^{n_i-1}(1-\delta_{n_i}\gamma)^{-j}E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2) \\
\le\ &\textstyle\sum_{j=0}^{n_i-1}\frac{\delta_{n_i}}{\gamma}\sum_{m=j}^{n_i-1}(1-\delta_{n_i}\gamma)^{-j-1}E(|\hat{f}_m^{n_i,p,\mathcal{R}}|^2) \\
=\ &\tfrac{\delta_{n_i}}{\gamma(1-\delta_{n_i}\gamma)}\textstyle\sum_{j=0}^{n_i-1}\sum_{m=j}^{n_i-1}(1-\delta_{n_i}\gamma)^{-j}E(|\hat{f}_m^{n_i,p,\mathcal{R}}|^2) \\
=\ &\tfrac{1}{\gamma(1-\delta_{n_i}\gamma)}\textstyle\sum_{j=0}^{n_i-1}\delta_{n_i}(j+1)(1-\delta_{n_i}\gamma)^{-j}E(|\hat{f}_m^{n_i,p,\mathcal{R}}|^2) \\
\le\ &\tfrac{T}{\gamma(1-\delta_{n_i}\gamma)}\textstyle\sum_{j=0}^{n_i-1}(1-\delta_{n_i}\gamma)^{-j}E(|\hat{f}_m^{n_i,p,\mathcal{R}}|^2).
\end{aligned} \tag{4.26}$$

Since f is Lipschitz with constant K therefore

$$\begin{aligned}&\textstyle\sum_{j=0}^{n_i-1}(1-\delta_{n_i}\gamma)^{-j}E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2)\\ &\le \tfrac{2TK^2}{\gamma(1-\delta_{n_i}\gamma)}\textstyle\sum_{j=0}^{n_i-1}(1-\delta_{n_i}\gamma)^{-j}E(|\hat{Y}_j^{n_i,p,\mathcal{R}}|^2+|\hat{Z}_j^{n_i,p,\mathcal{R}}|^2).\end{aligned} \tag{4.27}$$

By $0<(1-\frac{\gamma}{C})$, $\delta_{n_i}<\frac{1}{C}$, and (4.23) we get

$$E(|\hat{Z}_j^{n_i,p+1,\mathcal{R}}|^2) \;\le\; E(|\hat{Y}_{j+1}^{n_i,p+1,\mathcal{R}}|^2)+\tfrac{\delta_{n_i}}{\gamma}E(|\hat{f}_j^{n_i,p,\mathcal{R}}|^2) \tag{4.28}$$

hence

$$\begin{aligned}&(1-\delta_{n_i}\gamma)^{-j}E(|\hat{Z}_j^{n_i,p+1,\mathcal{R}}|^2)\\ &\le(1-\delta_{n_i}\gamma)^{-j}E(|\hat{Y}_{j+1}^{n_i,p+1,\mathcal{R}}|^2)+(1-\delta_{n_i}\gamma)^{-j}\tfrac{\delta_{n_i}}{\gamma}E(|\hat{f}_j^{n_i,p,\mathcal{R}}|^2).\end{aligned} \tag{4.29}$$

By (4.25), it follows that

$$(1-\delta_{n_i}\gamma)^{-j}E(|\hat{Z}_j^{n_i,p+1,\mathcal{R}}|^2) \;\le\; \tfrac{\delta_{n_i}}{\gamma}\textstyle\sum_{m=j}^{n_i-1}(1-\delta_{n_i}\gamma)^{-m}E(|\hat{f}_m^{n_i,p,\mathcal{R}}|^2). \tag{4.30}$$

By similar arguments to those above

$$\begin{aligned}&\textstyle\sum_{j=0}^{n_i-1}(1-\delta_{n_i}\gamma)^{-j}E(|\hat{Z}_j^{n_i,p+1,\mathcal{R}}|^2)\\ &\le \tfrac{2TK^2}{\gamma(1-\delta_{n_i}\gamma)}\textstyle\sum_{j=0}^{n_i-1}(1-\delta_{n_i}\gamma)^{-j}E(|\hat{Y}_j^{n_i,p,\mathcal{R}}|^2+|\hat{Z}_j^{n_i,p,\mathcal{R}}|^2)\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}&\textstyle\sum_{j=0}^{n_i-1}(1-\delta_{n_i}\gamma)^{-j}E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2+|\hat{Z}_j^{n_i,p+1,\mathcal{R}}|^2)\\ &\le \tfrac{4TK^2}{\gamma(1-\delta_{n_i}\gamma)}\textstyle\sum_{j=0}^{n_i-1}(1-\delta_{n_i}\gamma)^{-j}E(|\hat{Y}_j^{n_i,p,\mathcal{R}}|^2+|\hat{Z}_j^{n_i,p,\mathcal{R}}|^2).\end{aligned} \tag{4.32}$$

Recall that $\delta_{n_i}=\frac{T}{n_i}$ and $t_j^{n_i}=j\delta_{n_i}$, which implies $(1-\delta_{n_i}\gamma)^{-j}=((1-\frac{T\gamma}{n_i})^{-n_i})^{\frac{t_j^{n_i}}{T}}$. Since $i>I$ it follows that for all $j\in\{1,...,n_i\}$

$$\frac{1}{2}e^{-t_j^{n_i}\gamma}<((1-\frac{T\gamma}{n_i})^{-n_i})^{\frac{t_j^{n_i}}{T}}<e^{-t_j^{n_i}\gamma}. \tag{4.33}$$

Now

$$\begin{aligned}&\textstyle\sum_{j=0}^{n_i-1}e^{-t_j^{n_i}\gamma}E(|\hat{Y}_j^{n_i,p+1,\mathcal{R}}|^2+|\hat{Z}_j^{n_i,p+1,\mathcal{R}}|^2)\\ &\le \tfrac{8TK^2}{\gamma(1-\delta_{n_i}\gamma)}\textstyle\sum_{j=0}^{n-1}e^{-t_j^{n_i}\gamma}E(|\hat{Y}_j^{n,p,\mathcal{R}}|^2+|\hat{Z}_j^{n,p,\mathcal{R}}|^2)\\ &\le \tfrac{8TK^2}{\gamma(1-\frac{\gamma}{C})}\textstyle\sum_{j=0}^{n-1}e^{-t_j^{n_i}\gamma}E(|\hat{Y}_j^{n_i,p,\mathcal{R}}|^2+|\hat{Z}_j^{n_i,p,\mathcal{R}}|^2)\end{aligned} \tag{4.34}$$

which completes the proof. □

Corollary 1. *Given a finite reflection set $\mathcal{R} \subset \mathbb{Q}$ and $0 < \eta < 1$, there exists $I \in \mathbb{N}$ and a finite constant D such that for all $p \geq 1$ and $i \geq I$*

$$\delta_{n_i} \sum_{j=0}^{n_i-1} E(|Y^{n_i,p,\mathcal{R}}_{t^{n_i}_j} - Y^{n_i,\mathcal{R}}_{t^{n_i}_j}|^2 + |Z^{n_i,p,\mathcal{R}}_{t^{n_i}_j} - Z^{n_i,\mathcal{R}}_{t^{n_i}_j}|^2) \leq D\eta^p. \tag{4.35}$$

Proof. From Lemma 2.3.1 above it follows that for γ as in (2.16)

$$\delta_{n_i} \sum_{j=0}^{n_i-1} E(|Y^{n_i,p,\mathcal{R}}_{t^{n_i}_j} - Y^{n_i,\mathcal{R}}_{t^{n_i}_j}|^2 + |Z^{n_i,p,\mathcal{R}}_{t^{n_i}_j} - Z^{n_i,\mathcal{R}}_{t^{n_i}_j}|^2) \leq \frac{\eta^{p-1}}{(1-\sqrt{\eta})^2} ||(\hat{Y}^{n_i,1,\mathcal{R}}_{t^{n_i}_j}, \hat{Z}^{n_i,1,\mathcal{R}}_{t^{n_i}_j})||^2_\gamma. \tag{4.36}$$

In addition

$$||(\hat{Y}^{n_i,1,\mathcal{R}}_{t^{n_i}_j}, \hat{Z}^{n_i,1,\mathcal{R}}_{t^{n_i}_j})||^2_\gamma \leq M < \infty \tag{4.37}$$

for some i independent constant M, by standard a priori estimates. □

Corollary 2. *Given a finite reflection set $\mathcal{R} \subset \mathbb{Q}$ and $0 < \eta < 1$, there exists $I \in \mathbb{N}$ and a finite constant D such that for all $p \geq 1$ and $i \geq I$*

$$E(\sup_{0 \leq j \leq n_i} |Y^{n_i,p,\mathcal{R}}_{t^{n_i}_j} - Y^{n_i,\mathcal{R}}_{t^{n_i}_j}|^2 + \delta_{n_i} \sum_{j=0}^{n_i-1} |Z^{n_i,p,\mathcal{R}}_{t^{n_i}_j} - Z^{n_i,\mathcal{R}}_{t^{n_i}_j}|^2) \leq D\eta^p. \tag{4.38}$$

Proof. From (4.10)-(2.11) and (2.19) we have

$$\begin{aligned}
\hat{Y}^{n_i,p+1,\mathcal{R}}_j &= (Y^{n_i,p+1,\mathcal{R}}_{t^{n_i}_{j+1}} 1_{\{t^{n_i}_{j+1} \notin \mathcal{R}\}} + [(Y^{n_i,p+1,\mathcal{R}}_{t^{n_i}_{j+1}} \vee \psi^{n_i}_{t^{n_i}_{j+1}}) \wedge \Psi^{n_i}_{t^{n_i}_{j+1}}] 1_{\{t^{n_i}_{j+1} \in \mathcal{R}\}}) \\
&\quad -(Y^{n_i,p,\mathcal{R}}_{t^{n_i}_{j+1}} 1_{\{t^{n_i}_{j+1} \notin \mathcal{R}\}} + [(Y^{n_i,p,\mathcal{R}}_{t^{n_i}_{j+1}} \vee \psi^{n_i}_{t^{n_i}_{j+1}}) \wedge \Psi^{n_i}_{t^{n_i}_{j+1}}] 1_{\{t^{n_i}_{j+1} \in \mathcal{R}\}}) \\
&\quad +(f(t^{n_i}_j, Y^{n_i,p+1,\mathcal{R}}_{t^{n_i}_j}, Z^{n_i,p+1,\mathcal{R}}_{t^{n_i}_j}) - f(t^{n_i}_j, Y^{n_i,p,\mathcal{R}}_{t^{n_i}_j}, Z^{n_i,p,\mathcal{R}}_{t^{n_i}_j}))\delta_{n_i} \\
&\quad -(Z^{n_i,p+1,\mathcal{R}}_{t^{n_i}_j} - Z^{n_i,p,\mathcal{R}}_{t^{n_i}_j}) \epsilon^{n_i}_{j+1} \sqrt{\delta_{n_i}} \\
&= \hat{S}^{n_i,p+1,\mathcal{R}}_{j+1} + \delta_{n_i} \hat{f}^{n_i,p,\mathcal{R}}_j - \hat{Z}^{n_i,p+1,\mathcal{R}}_j \epsilon^{n_i}_{j+1} \sqrt{\delta_{n_i}}.
\end{aligned} \tag{4.39}$$

It follows that

$$|\hat{Y}^{n_i,p+1,\mathcal{R}}_j| = E\left(|\hat{S}^{n_i,p+1,\mathcal{R}}_{j+1} + \delta_{n_i} \hat{f}^{n_i,p,\mathcal{R}}_j| \middle| \mathcal{G}^{n_i}_j\right) \leq E\left(|\hat{Y}^{n_i,p+1,\mathcal{R}}_{j+1}| + \delta_{n_i} |\hat{f}^{n_i,p,\mathcal{R}}_j| \middle| \mathcal{G}^{n_i}_j\right). \tag{4.40}$$

By iterating this inequality we get

$$|\hat{Y}_j^{n_i,p+1,\mathcal{R}}| \leq E\left(\delta_{n_i} \sum_{j=0}^{n_i} |\hat{f}_j^{n_i,p,\mathcal{R}}| \middle| \mathcal{G}_j^{n_i}\right) \tag{4.41}$$

and from Doob's inequality we get

$$E(\sup_{0\leq j\leq n_i} |\hat{Y}_j^{n_i,p,\mathcal{R}}|^2) \leq 4E(\delta_{n_i} \sum_{j=0}^{n_i} |\hat{f}_j^{n_i,p,\mathcal{R}}|^2) \tag{4.42}$$

and the result follows from Corollary 1. □

By Corollary 2 we have the following

Proposition 4.3.2. *If* $Y_t^{n_i,p,\mathcal{R}}$, $Z_t^{n_i,p,\mathcal{R}}$, $Y_t^{n_i,\mathcal{R}}$, *and* $Z_t^{n_i,\mathcal{R}}$ *are defined respectively by (4.11) and (4.5), then*

$$\sup_{n_i} E(\sup_{0\leq t\leq T} |Y_t^{n_i,p,\mathcal{R}} - Y_t^{n_i,\mathcal{R}}|^2 + \int_0^T |Z_t^{n_i,p,\mathcal{R}} - Z_t^{n_i,\mathcal{R}}|^2 dt) \to 0, \text{ as } p \to \infty. \tag{4.43}$$

4.4 Proof of Discrete Convergence

We proceed by showing convergence to zero of

$$Y_t^{n_i,p,\mathcal{R}} - Y_t^{\infty,p,\mathcal{R}} \text{ and } Z_t^{n_i,p,\mathcal{R}} - Z_t^{\infty,p,\mathcal{R}} \tag{4.44}$$

by employing induction on p.

Proof. By (4.11) it follows that

$$\begin{aligned} Y_t^{n_i,p+1,\mathcal{R}} &= \xi^{n_i} + \textstyle\int_t^T f(s, Y_s^{n_i,p,\mathcal{R}}, Z_s^{n_i,p,\mathcal{R}}) dA_s^{n_i} \\ &\quad - \textstyle\int_t^T (Z_s^{n_i,p+1,\mathcal{R}})^\top dW_s^{n_i} + \sum_{s\in\mathcal{R}^t} \rho^{n_i}(Y_s^{n_i,p,\mathcal{R}}), \end{aligned} \tag{4.45}$$

where $\rho^{n_i}(Y_s^{n_i,p,\mathcal{R}}) = [(\psi_s^{n_i} - Y_s^{n_i,p,\mathcal{R}})^+ - (Y_s^{n_i,p,\mathcal{R}} - \Psi_s^{n_i})^+]$ and $A_s^{n_i} = \lfloor \frac{s}{\delta_{n_i}} \rfloor \delta_{n_i}$. The induction assumption is that $(Y^{n_i,p,\mathcal{R}}, Z^{n_i,p,\mathcal{R}})$ converges to $(Y^{p,\mathcal{R}}, Z^{p,\mathcal{R}})$ in the sense of (4.7) and we will prove that $(Y^{n_i,p+1,\mathcal{R}}, Z^{n_i,p+1,\mathcal{R}})$ converges to $(Y^{p+1,\mathcal{R}}, Z^{p+1,\mathcal{R}})$ in the same sense. We first define the process

$$\begin{aligned} M_t^{n_i} &= Y_t^{n_i,p+1,\mathcal{R}} + \textstyle\int_0^t f(s, Y_s^{n_i,p,\mathcal{R}}, Z_s^{n_i,p,\mathcal{R}}) dA_s^{n_i} + \sum_{s\in\mathcal{R}_t} \rho^{n_i}(Y_s^{n_i,p,\mathcal{R}}) \\ &= E\bigg(\xi^{n_i} + \textstyle\int_0^T f(s, Y_s^{n_i,p,\mathcal{R}}, Z_s^{n_i,p,\mathcal{R}}) dA_s^{n_i} + \sum_{s\in\mathcal{R}_T} \rho^{n_i}(Y_s^{n_i,p,\mathcal{R}}) \bigg| \mathcal{F}_t^{n_i}\bigg) \\ &= E\bigg(M_T^{n_i} \bigg| \mathcal{F}_t^{n_i}\bigg). \end{aligned} \tag{4.46}$$

Furthermore, note that

$$M_t^{n_i} = M_0^{n_i} + \int_0^t (Z_s^{n_i,p+1,\mathcal{R}})^\top dW_s^{n_i}. \tag{4.47}$$

Thus in order to apply Corollary 3.2 from [5], it suffices to prove the L^1 convergence of $M_T^{n_i}$. Since Y^{n_i} and Z^{n_i} are piecewise constant, it follows that

$$\begin{aligned} & |M_T^{n_i} - \xi - \textstyle\int_0^T f(s, Y_s^{p,\mathcal{R}}, Z_s^{p,\mathcal{R}}) ds - \sum_{s\in\mathcal{R}_T} \rho(Y_s^{p,\mathcal{R}})| \\ \leq\ & |\xi^{n_i} - \xi| + \textstyle\int_0^T |f(s, Y_s^{n_i,p,\mathcal{R}}, Z_s^{n_i,p,\mathcal{R}}) - f(s, Y_s^{p,\mathcal{R}}, Z_s^{p,\mathcal{R}})| ds \\ & + \textstyle\sum_{s\in\mathcal{R}_T} |\rho^{n_i}(Y_s^{n_i,p,\mathcal{R}}) - \rho(Y_s^{p,\mathcal{R}})| \\ \leq\ & (1 + |\mathcal{R}| + KT) \sup_{0\leq t\leq T} |Y_t^{n_i,p,\mathcal{R}} - Y_t^{p,\mathcal{R}}| + \textstyle\int_0^T |Z_s^{n_i,p,\mathcal{R}} - Z_s^{p,\mathcal{R}}| ds \\ & + \textstyle\sum_{s\in\mathcal{R}_T} (|\psi_s^{n_i} - \psi_s| + |\Psi_s^{n_i} - \Psi_s|) \end{aligned} \tag{4.48}$$

which tends to zero in probability and thus in L^1 byL^2-boundedness. Then by Corollary 3.2 in [5] M^{n_i} converges to

$$M_t = E\bigg(\xi + \int_0^T f(s, Y_s^{p,\mathcal{R}}, Z_s^{p,\mathcal{R}}) ds \bigg| \mathcal{F}_t\bigg) = Y_t^{p+1,\mathcal{R}} + \int_0^T f(s, Y_s^{p,\mathcal{R}}, Z_s^{p,\mathcal{R}}) ds \tag{4.49}$$

in the sense that

$$\sup_{0\leq t\leq T} |M_t^{n_i} - M_t| + \int_0^T |Z_s^{n_i,p+1,\mathcal{R}} - Z_s^{p+1,\mathcal{R}}| ds \to 0 \text{ in probability.} \tag{4.50}$$

Since we want to prove that

$$\sup_{0\leq t\leq T} |Y_t^{n_i,p+1,\mathcal{R}} - Y_t^{p+1,\mathcal{R}}| + \int_0^T |Z_s^{n_i,p+1,\mathcal{R}} - Z_s^{p+1,\mathcal{R}}| ds \to 0 \text{ in probability} \tag{4.51}$$

it remains to show

$$\sup_{0\le t\le T} |\int_0^t f(s, Y_s^{n_i,p,\mathcal{R}}, Z_s^{n_i,p,\mathcal{R}})dA_s^{n_i} - \int_0^t f(s, Y_s^{p,\mathcal{R}}, Z_s^{p,\mathcal{R}})ds| \to 0 \text{ in probability} \tag{4.52}$$

and

$$\sup_{0\le t\le T} |\sum_{s\in\mathcal{R}_t} \rho^{n_i}(Y_s^{n_i,p,\mathcal{R}}) - \sum_{s\in\mathcal{R}_t} \rho(Y_s^{p,\mathcal{R}})| \to 0 \text{ in probability.} \tag{4.53}$$

The convergence (4.52) follows from results in [5], hence for (2.51) we have

$$\begin{aligned} \sup_{0\le t\le T} |\textstyle\sum_{s\in\mathcal{R}_t} \rho^{n_i}(Y_s^{n_i,p,\mathcal{R}}) - \sum_{s\in\mathcal{R}_t} \rho(Y_s^{p,\mathcal{R}})| &\le \textstyle\sum_{s\in\mathcal{R}} |Y_s^{n_i,p,\mathcal{R}} - Y_s^{p,\mathcal{R}}| \\ &\quad + \textstyle\sum_{s\in\mathcal{R}} (|\psi_s^{n_i} - \psi_s| + |\Psi_s^{n_i} - \Psi_s|) \\ &\le |\mathcal{R}| \sup_{0\le t\le T} |Y_s^{n_i,p,\mathcal{R}} - Y_s^{p,\mathcal{R}}| \\ &\quad + \textstyle\sum_{s\in\mathcal{R}} (|\psi_s^{n_i} - \psi_s| + |\Psi_s^{n_i} - \Psi_s|) \end{aligned} \tag{4.54}$$

which converges to zero in probability by the induction assumption on p. This concludes the proof. □

4.5 Proof of Picard Convergence

As a consequence of the above results all that is left to prove is the convergence to zero of $(Y_t^{\infty,p,\mathcal{R}} - Y_t^{\mathcal{R}})$ in the senses that

$$||Y^{\infty,p,\mathcal{R}} - Y^{\mathcal{R}}||_{\mathbb{S}^2}^2 \to 0 \text{ as } p \to \infty. \tag{4.55}$$

Proof. The semimartingale

$$\begin{aligned} \hat{Y}_t^{\infty,p+1,\mathcal{R}} &= (Y_0^{\infty,p+1,\mathcal{R}} - Y_0^{\infty,p,\mathcal{R}}) - \textstyle\int_0^t f(s, Y_s^{\infty,p,\mathcal{R}}, Z_s^{\infty,p,\mathcal{R}})ds \\ &\quad + \textstyle\int_0^t (Z_s^{\infty,p+1,\mathcal{R}} - Z_s^{\infty,p,\mathcal{R}})^\top dW_s - \sum_{s\in\mathcal{R}_t} (\rho(Y_s^{\infty,p,\mathcal{R}}) - \rho(Y_s^{\infty,p-1,\mathcal{R}})) \end{aligned} \tag{4.56}$$

is the sum of an Itô semimartingale

$$\begin{aligned} &Y_0^{\infty,p+1,\mathcal{R}} - \textstyle\int_0^t (f(s, Y_s^{\infty,p,\mathcal{R}}, Z_s^{\infty,p,\mathcal{R}}) - f(s, Y_s^{\infty,p-1,\mathcal{R}}, Z_s^{\infty,p-1,\mathcal{R}}))ds \\ &+ \textstyle\int_0^t (Z_s^{\infty,p+1,\mathcal{R}} - Z_s^{\infty,p,\mathcal{R}})^\top dW_s \end{aligned} \tag{4.57}$$

and the process of finite variation

$$-\sum_{s\in\mathcal{R}_t}(\rho(Y_s^{\infty,p,\mathcal{R}})-\rho(Y_s^{\infty,p-1,\mathcal{R}})). \tag{4.58}$$

We have

$$\begin{aligned}&||-\textstyle\sum_{s\in\mathcal{R}_t}(\rho(Y_s^{\infty,p,\mathcal{R}})-\rho(Y_s^{\infty,p-1,\mathcal{R}}))||_{\mathbb{S}^2}^2\\ &:=E(\sup_{0\le t\le T}|-\textstyle\sum_{s\in\mathcal{R}_t}(\rho(Y_s^{\infty,p,\mathcal{R}})-\rho(Y_s^{\infty,p-1,\mathcal{R}}))|^2)\\ &\le|\mathcal{R}|E(\textstyle\sum_{s\in\mathcal{R}}|Y_s^{\infty,p,\mathcal{R}}-Y_s^{\infty,p-1,\mathcal{R}}|^2)\end{aligned} \tag{4.59}$$

and the maximal inequality of martingales ensures that

$$\begin{aligned}||\textstyle\int_0^{\cdot}(Z_s^{\infty,p+1,\mathcal{R}}-Z_s^{\infty,p,\mathcal{R}})^\top dW_s||_{\mathbb{S}^2}^2 &:= E(\sup_{0\le t\le T}|\textstyle\int_0^t(Z_s^{\infty,p+1,\mathcal{R}}-Z_s^{\infty,p,\mathcal{R}})^\top dW_s|^2)\\ &\le 4E(\textstyle\int_0^T|(Z_s^{\infty,p+1,\mathcal{R}}-Z_s^{\infty,p,\mathcal{R}})^\top|^2dt).\end{aligned} \tag{4.60}$$

Therefore, the same estimate holds for the semimartingale $\hat{Y}^{\infty,p+1,\mathcal{R}}$ since

$$\begin{aligned}||\hat{Y}^{\infty,p+1,\mathcal{R}}||_{\mathbb{S}^2}^2 &\le 4\big(E(|Y_0^{\infty,p+1,\mathcal{R}}-Y_0^{\infty,p,\mathcal{R}}|^2)\\ &\quad+E((\textstyle\int_0^T|f(s,Y_s^{\infty,p,\mathcal{R}},Z_s^{\infty,p,\mathcal{R}})-f(s,Y_s^{\infty,p-1,\mathcal{R}},Z_s^{\infty,p-1,\mathcal{R}})|ds)^2)\\ &\quad+||\textstyle\sum_{s\in\mathcal{R}_t}(\rho(Y_s^{\infty,p,\mathcal{R}})-\rho(Y_s^{\infty,p-1,\mathcal{R}}))||_{\mathbb{S}^2}^2\\ &\quad+||\textstyle\int_0^{\cdot}(Z_s^{\infty,p+1,\mathcal{R}}-Z_s^{\infty,p,\mathcal{R}})^\top dW_s||_{\mathbb{S}^2}^2\big)\\ &\le 4E(|Y_0^{\infty,p+1,\mathcal{R}}-Y_0^{\infty,p,\mathcal{R}}|^2)\\ &\quad+4TE(\textstyle\int_0^T|f(s,Y_s^{\infty,p,\mathcal{R}},Z_s^{\infty,p,\mathcal{R}})-f(s,Y_s^{\infty,p-1,\mathcal{R}},Z_s^{\infty,p-1,\mathcal{R}})|^2dt)\\ &\quad+16E(\textstyle\int_0^T|(Z_s^{\infty,p+1,\mathcal{R}}-Z_s^{\infty,p,\mathcal{R}})^\top|^2dt)\\ &\quad+|\mathcal{R}|E(\textstyle\sum_{s\in\mathcal{R}}|Y_s^{\infty,p,\mathcal{R}}-Y_s^{\infty,p-1,\mathcal{R}}|^2)\end{aligned} \tag{4.61}$$

Using the following representation

$$\begin{aligned}\hat{Y}_t^{\infty,p+1,\mathcal{R}} &= \textstyle\int_t^T f(s,Y_s^{\infty,p,\mathcal{R}},Z_s^{\infty,p,\mathcal{R}})ds+\int_t^T(Z_s^{\infty,p+1,\mathcal{R}}-Z_s^{\infty,p,\mathcal{R}})^\top dW_s\\ &\quad+\textstyle\sum_{s\in\mathcal{R}^t}(\rho(Y_s^{\infty,p,\mathcal{R}})-\rho(Y_s^{\infty,p-1,\mathcal{R}}))\end{aligned} \tag{4.62}$$

the same estimates can be made, since

$$
\begin{aligned}
E((\hat{Y}_t^{\infty,p+1,\mathcal{R}})^2) &= E\Bigg(\Bigg(\big[\int_0^T f(s,Y_s^{\infty,p,\mathcal{R}},Z_s^{\infty,p,\mathcal{R}})ds \\
&\quad + \int_0^T (Z_s^{\infty,p+1,\mathcal{R}} - Z_s^{\infty,p,\mathcal{R}})^\top dW_s \\
&\quad + \textstyle\sum_{s\in\mathcal{R}_T}(\rho(Y_s^{\infty,p,\mathcal{R}}) - \rho(Y_s^{\infty,p-1,\mathcal{R}}))\big] \\
&\quad - \big[\int_0^t f(s,Y_s^{\infty,p,\mathcal{R}},Z_s^{\infty,p,\mathcal{R}})ds + \int_0^t (Z_s^{\infty,p+1,\mathcal{R}} - Z_s^{\infty,p,\mathcal{R}})^\top dW_s \\
&\quad + \textstyle\sum_{s\in\mathcal{R}_t}(\rho(Y_s^{\infty,p,\mathcal{R}}) - \rho(Y_s^{\infty,p-1,\mathcal{R}}))\big]\Bigg)^2\Bigg) \\
&\leq 2E\Bigg(\big[\int_0^T f(s,Y_s^{\infty,p,\mathcal{R}},Z_s^{\infty,p,\mathcal{R}})ds + \int_0^T (Z_s^{\infty,p+1,\mathcal{R}} - Z_s^{\infty,p,\mathcal{R}})^\top dW_s \\
&\quad + \textstyle\sum_{s\in\mathcal{R}_T}(\rho(Y_s^{\infty,p,\mathcal{R}}) - \rho(Y_s^{\infty,p-1,\mathcal{R}}))\big]^2 \\
&\quad + \big[\int_0^t f(s,Y_s^{\infty,p,\mathcal{R}},Z_s^{\infty,p,\mathcal{R}})ds + \int_0^t (Z_s^{\infty,p+1,\mathcal{R}} - Z_s^{\infty,p,\mathcal{R}})^\top dW_s \\
&\quad + \textstyle\sum_{s\in\mathcal{R}_t}(\rho(Y_s^{\infty,p,\mathcal{R}}) - \rho(Y_s^{\infty,p-1,\mathcal{R}}))\big]^2\Bigg).
\end{aligned}
\tag{4.63}
$$

This leads to

$$
\begin{aligned}
||\hat{Y}^{\infty,p+1,\mathcal{R}}||_{\mathbb{S}^2}^2 &\leq 4\Bigg(4TE(\int_0^T |f(s,Y_s^{\infty,p,\mathcal{R}},Z_s^{\infty,p,\mathcal{R}}) - f(s,Y_s^{\infty,p-1,\mathcal{R}},Z_s^{\infty,p-1,\mathcal{R}})|^2 dt) \\
&\quad +16E(\int_0^T |(Z_s^{\infty,p+1,\mathcal{R}} - Z_s^{\infty,p,\mathcal{R}})^\top|^2 dt) \\
&\quad +|\mathcal{R}|E(\textstyle\sum_{s\in\mathcal{R}} |Y_s^{\infty,p,\mathcal{R}} - Y_s^{\infty,p-1,\mathcal{R}}|^2)\Bigg)
\end{aligned}
\tag{4.64}
$$

and again since f is Lipschitz with constant K we get

$$
\begin{aligned}
||\hat{Y}^{\infty,p+1,\mathcal{R}}||_{\mathbb{S}^2}^2 &\leq 4\Bigg(4KTE(\int_0^T |Y_s^{\infty,p,\mathcal{R}} - Y_s^{\infty,p-1,\mathcal{R}}|^2 dt) \\
&\quad +KE(\int_0^T |(Z_s^{\infty,p,\mathcal{R}} - Z_s^{\infty,p-1,\mathcal{R}})^\top|^2 dt) \\
&\quad +16E(\int_0^T |(Z_s^{\infty,p+1,\mathcal{R}} - Z_s^{\infty,p,\mathcal{R}})^\top|^2 dt) \\
&\quad +|\mathcal{R}|E(\textstyle\sum_{s\in\mathcal{R}} |Y_s^{\infty,p,\mathcal{R}} - Y_s^{\infty,p-1,\mathcal{R}}|^2)\Bigg).
\end{aligned}
\tag{4.65}
$$

Therefore, for some constant A with the norm $||\cdot||$ defined in (1.62) we have

$$
||\hat{Y}^{\infty,p+1,\mathcal{R}}||_{\mathbb{S}^2}^2 \leq A(||(\hat{Y}^{\infty,p+1,\mathcal{R}},\hat{Z}^{\infty,p+1,\mathcal{R}})|| + ||(\hat{Y}^{\infty,p,\mathcal{R}},\hat{Z}^{\infty,p,\mathcal{R}})||). \tag{4.66}
$$

Moreover, by Proposition 1.4.12 there exists $0 < \eta < 1$ such that

$$||\hat{Y}^{\infty,p+1,\mathcal{R}}||_{\mathbb{S}^2}^2 \quad \leq \quad 2A\eta^{p-1}\ ||(Y^{\infty,1,\mathcal{R}}, Z^{\infty,1,\mathcal{R}})||. \tag{4.67}$$

Furthermore, since

$$Y_t^{\infty,1,\mathcal{R}} = \xi + \int_t^T f(s,0,0)ds - \int_t^T (Z_s^{\infty,1,\mathcal{R}})^\top dW_s + \sum_{s\in\mathcal{R}^t} \rho(0) \tag{4.68}$$

it follows that there exists B such that

$$||(Y^{\infty,1,\mathcal{R}}, Z^{\infty,1,\mathcal{R}})|| \leq B \tag{4.69}$$

which concludes the proof. □

CHAPTER 5

Financial Applications

5.1 Introduction

In this chapter our attention shifts to the very important financial applications of FBSDEs. It is now well know that derivatives play a crucial role in the world's financial markets. Derivatives, financial assets whose values are derived from other financial assets, have long been the focal point of research. Options, also referred to as contingence claims, are a type of derivative. The value of the option is contingent on the future value of some underlying asset; here the option's value is derived from the value of the underlying asset. Furthermore there are two types of options calls and puts. Call options give the owner the right, but not the obligation, to buy a specified amount of the underlying asset at specified times, called the exercise times, for a specified price, called the strike price. Put options give the owner the right, but not the obligation, to sell a specified amount of the underlying asset at specified times, called the exercise times, for a specified price, called the strike price. To exercise a Call option means to buy the specified amount of the underlying asset at the exercise times for the strike price. To exercise a Put option means to sell the specified amount of the underlying asset at the exercise times for the strike price.

For both the Call and Put options one can take either a long or short position. The holder of the long position is the buyer of the option. While the owner of the short position is the seller of the option. Whomever is long the option has the right, but not the obligation, to exercise that option. In turn, if the holder of the long position decides to exercise the option the obligation to fulfill the transaction resides on the owner of the short position. Note that if being long a Call option one decides to exercise the option, the holder of the short position is obligated to sell the specified amount of the underlying asset at the exercise times for the strike price. In a similar fashion whomever is short a Put option is required to buy the specified amount of the underlying asset at at the exercise times for the strike price, if the holder of the long position exercises the option.

We will discuses three styles of options European, American, and Bermudan options. In the most simple of the three, a European option, you are given a strike price usually denoted by K and a future exercise time usually denoted by T. In a European option the exercise time T is one point in time called the time of maturity. In the case of a European call option the payoff to the holder of the long position is

$$max(S_T - K, 0)$$

where S_T represents the price of the underlying asset at maturity. It follows that the payoff to the holder of the short position in a European call option is

$$-max(S_T - K, 0) = min(K - S_T, 0).$$

For put options the payoff to the holder of the long position is

$$max(K - S_T, 0)$$

and the payoff to the holder of the short position is

$$-max(K - S_T, 0) = min(S_T - K, 0).$$

The payoff structures are summarized in Fig. 5.1.

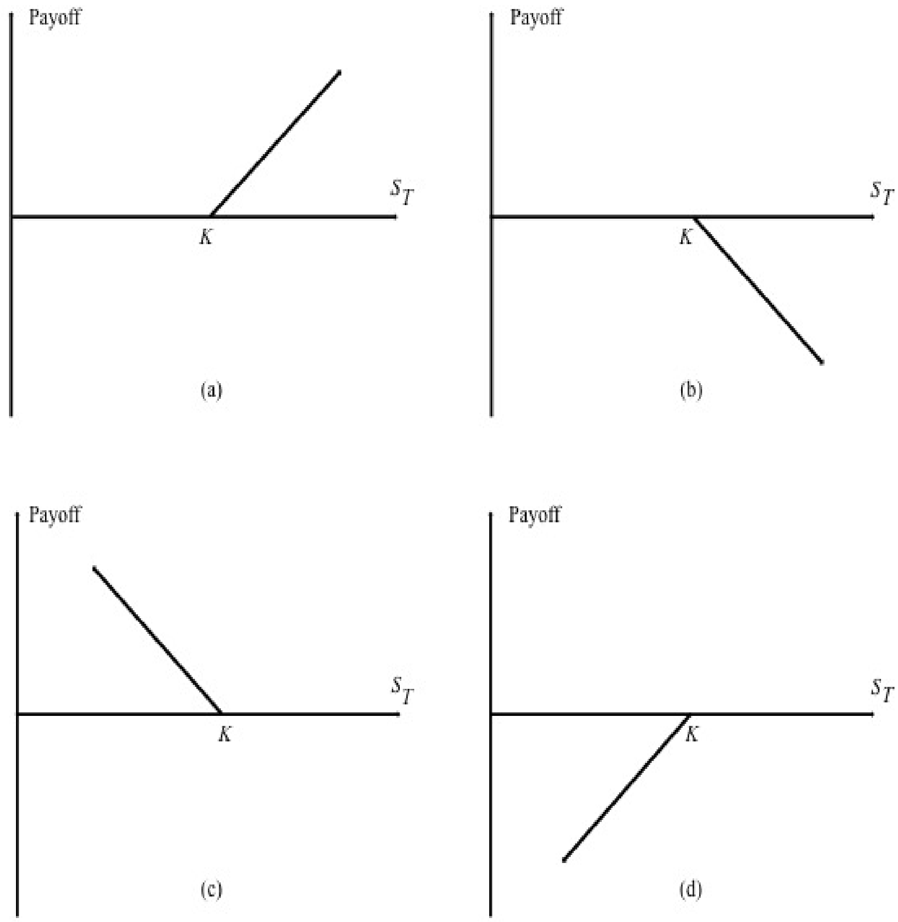

Figure 5.1. Payoffs from positions in European options: (a) long call, (b) short call, (c) long put, (d) short put. Strike price = K; price of asset at maturity = S_T.

We can denote the payoff of a European option as a function of the terminal price of the underlying asset

$$g_T = g(S_T).$$

In the case of a European call option

$$g_T = g(S_T) = max(S_T - K, 0) = (S_T - K)^+.$$

The difference between a European and American option is the exercise times. Unlike in a European option, where the exercise time occurs at the time of maturity T, an American option's exercise times are given by the set $[0, T]$ here 0 represent, as usual, the initial time of the life of the option. That is an American option can be exercised up to the time of expiration T. So in an American option the payoff process is a function of the price process of the underlying asset

$$g_t = g(S_t).$$

In the case of an American call option

$$g_t = g(S_t) = max(S_t - K, 0) = (S_t - K)^+.$$

As mentioned in [27] just as the Bermuda islands are situated between Europe and America, Bermudan options take an intermediate place between American and European options. The exercise times for a Bermudan option are given by a set $\mathcal{R}$ which is a subset of the set $[0, T]$. It follows that the payoff process of a Bermudan option has a value of zero for any $t \notin \mathcal{R}$. Thus the payoff process of a Bermudan option is given by

$$\tilde{g}_t = g_t 1_{t \in \mathcal{R}} = g(S_t) 1_{t \in \mathcal{R}}$$

where g_t, the payoff process, is a function of the price process of the underlying asset. In the case of a Bermudan call option

$$\tilde{g}_t = g(S_t) 1_{t \in \mathcal{R}} = max(S_t - K, 0) 1_{t \in \mathcal{R}} = (S_t - K)^+ 1_{t \in \mathcal{R}}.$$

5.2 Option Pricing

A very important question in financial mathematics is: What is the correct price for an option today? In particular we are concerned with pricing Bermudan options.

To this end we first present an overview of the fundamental components of arbitrage pricing theory. An arbitrage opportunity is a trading possibility which yields riskless profits. If we are to assume the Efficient-market hypothesis, which states that it is impossible to "beat the market" because stock market efficiency causes existing share prices to always incorporate and demonstrate all relevant information, we must suppose that there does not exist any arbitrage opportunity in the stock market.

Using the replicating portfolio approach it follows that in an arbitrage-free market model the correct price for a option is given by the initial price of a portfolio which generates the same payoff. If we further assume the arbitrage-free market model to be complete then all contingent claims can be attained using the replicating portfolio approach. It is worth noting that in this approach there is no mention of the probabilities for upward or downward movements of any assets.

As mentioned in [16], a market model is arbitrage-free if there exists an equivalent martingale measure. Under such a measure, trading in the market's assets is the same as playing a fair game because the discounted price of the assets hold the martingale property. This amounts to assuming a risk-neutral world, that is a world where investors do not require higher returns for higher risk. In general this can not be assumed of financial markets, notwithstanding in pricing options this is irrelevant. In fact the price of options remain the same under the risk-neutral probabilities, $\tilde{P}$, and the actual probabilities, P.

In [16] it is shown that under the equivalent martingale measure the european option valuation process is

$$Y_t = \tilde{E}(e^{-\int_t^T r(u)du} g_T | \mathcal{F}_t) \tag{5.1}$$

where g_T is the terminal payoff. In [18] we get that under the equivalent martingale measure the american option valuation process is given by the Snell envelope

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathscr{L}_{t,T}} \tilde{E}(e^{-\int_t^\tau r(u)du} g_\tau | \mathcal{F}_t) \qquad a.s. \tag{5.2}$$

where $\mathscr{L}_{t,T}$ denotes the collection of stopping times τ of $(\mathcal{F}_s)_{s\in[0,T]}$ with values in $[t,T]$ and g_τ is the payoff process. From [27] we get that under the equivalent martingale measure the Bermudan option with finite exercise times has a valuation process given by the Snell envelope

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathscr{L}_{t,T}} \tilde{E}(e^{-\int_t^\tau r(u)du} (g_\tau 1_{\tau\in\mathcal{R}}) | \mathcal{F}_t) \qquad a.s. \tag{5.3}$$

where $\mathscr{L}_{t,T}$ denotes the collection of stopping times τ of $(\mathcal{F}_s)_{s\in[0,T]}$ with values in $[t,T]$ and $g_\tau 1_{\tau\in\mathcal{R}}$ is the payoff process. In [14] it is shown that the solution of a Snell envelope corresponds to the solution of an associated RFBSDE. Consequently the solution of an RFBSDE with continuous reflection gives the valuation process of an American option, see [13]. Moreover a FR-FBSDE gives the valuation process of a Bermudan option with finite exercise times. From [18] it follows that the valuation process of a Bermudan option is given by the 1 dimensional Finitely Reflected Forward Backward Stochastic Differential Equation (FR-FBSDE):

$$\begin{aligned} Y_t^{\mathcal{R}} &= g(S_T) + \textstyle\int_t^T (rY_s^{\mathcal{R}} + \sigma\theta Z_s^{\mathcal{R}})ds \\ &\quad - \textstyle\int_t^T \sigma Z_s^{\mathcal{R}} dW_s + \sum_{s\in\mathcal{R}^t}[(g(S_s) - Y_s^{\mathcal{R}})^+] \qquad 0 \le t < T \\ Y_T^{\mathcal{R}} &= g(S_T) \end{aligned} \tag{5.4}$$

where $\mathcal{R}^t = \{s \in \mathcal{R} | s > t\}$ if $0 \le t < T$, S is the price process of the underlying asset which solves the geometric Brownian motion

$$S_t = S_0 + \int_t^T (r + \sigma\theta) S_s ds - \int_t^T \sigma S_s dW_s,$$

r is the risk-free interest rate constant, σ is the volatility constant, and θ is the risk premium constant.

5.3 Implementation

In the above section we discussed the connection of Bermudan option pricing to FR-FBSDEs. It this section our main concern is the numerical implementation of the state-time discretization of the FR-FBSDE (4.4). We first take a Bernoulli sequence $\{\epsilon_j^n\}_{j=1}^n$, $n \in \mathbb{N}$ and the finite terminal time $0 < T \in \mathbb{Q}$. In addition define the 1-dimensional random walk process for a fixed $n \in \mathbb{N}$

$$W_t^n := \sqrt{\delta_n} \sum_{j=1}^{[t/\delta_n]} \epsilon_j^n, \text{ for all } 0 \le t \le T, \delta_n = \frac{T}{n}, \tag{5.5}$$

$\mathcal{G}_j^n := \sigma\{\epsilon_1^n, \ldots, \epsilon_j^n\}$, $t_j^n = j\delta_n$, and $t_0^n = 0$. Using Itô's formula we can solve and then easily discretize S the geometric Brownian motion price process of the underlying asset so that

$$s_j^n = S_0 e^{\sigma W_{t_j^n}^n - \frac{\sigma^2 t_j^n}{2} + (r+\sigma\theta)t_j^n}.$$

Consider the state-time discretization of the backward equation in the FR-FBSDE associated with the Bermudan option valuation processes

$$\begin{aligned} y_j^{n,\mathcal{R}} &= y_{j+1}^{n,\mathcal{R}} 1_{\{t_{j+1}^n \notin \mathcal{R}\}} + [y_{j+1}^{n,\mathcal{R}} \vee g_{t_{j+1}^n}^n] 1_{\{t_{j+1}^n \in \mathcal{R}\}} \\ &\quad + (r y_j^{n,\mathcal{R}} + \sigma\theta z_j^{n,\mathcal{R}})\delta_n - \sigma z_j^{n,\mathcal{R}} \epsilon_{j+1}^n \sqrt{\delta_n} \\ &\quad j = n-1, ..., 0, \\ y_n^{n,\mathcal{R}} &= g_T^n. \end{aligned} \tag{5.6}$$

We start at time T, where $y_n^{n,\mathcal{R}} = g_T^n$, and solve for the process by going backward in time. At each time step for the given $y_{j+1}^{n,\mathcal{R}}$, $g_{t_{j+1}^n}^n$, and $\mathcal{R}$ we want to find G_j^n-measurable $(y_j^{n,\mathcal{R}}, z_j^{n,\mathcal{R}})$. Following [26] we set

$$\begin{aligned} Y_+ &= \left(y_{j+1}^{n,\mathcal{R}} 1_{\{t_{j+1}^n \notin \mathcal{R}\}} + [y_{j+1}^{n,\mathcal{R}} \vee g_{t_{j+1}^n}^n] 1_{\{t_{j+1}^n \in \mathcal{R}\}}\right)\big|_{\epsilon_{j+1}^n = 1} \\ Y_- &= \left(y_{j+1}^{n,\mathcal{R}} 1_{\{t_{j+1}^n \notin \mathcal{R}\}} + [y_{j+1}^{n,\mathcal{R}} \vee g_{t_{j+1}^n}^n] 1_{\{t_{j+1}^n \in \mathcal{R}\}}\right)\big|_{\epsilon_{j+1}^n = -1} \end{aligned}$$

note that both Y_+ and Y_- are G_j^n-measurable. So for $j = n-1, ..., 0$ (5.6) is equivalent to the following algebraic equation

$$\begin{aligned} y_j^{n,\mathcal{R}} &= Y_+ + (ry_j^{n,\mathcal{R}} + \sigma\theta z_j^{n,\mathcal{R}})\delta_n - \sigma z_j^{n,\mathcal{R}}\sqrt{\delta_n} \\ y_j^{n,\mathcal{R}} &= Y_- + (ry_j^{n,\mathcal{R}} + \sigma\theta z_j^{n,\mathcal{R}})\delta_n + \sigma z_j^{n,\mathcal{R}}\sqrt{\delta_n}. \end{aligned} \tag{5.7}$$

This is equivalent to

$$z_j^{n,\mathcal{R}} = \tfrac{1}{2\sigma\sqrt{\delta_n}}(Y_+ - Y_-) \tag{5.8}$$

and

$$y_j^{n,\mathcal{R}} - (ry_j^{n,\mathcal{R}} + \sigma\theta z_j^{n,\mathcal{R}})\delta_n = \tfrac{1}{2}(Y_+ + Y_-). \tag{5.9}$$

Solving (5.8) and (5.9) by starting at time T and going backward in time we get an approximation of the valuation process for the Bermudan Option. Furthermore we obtain an approximation of the correct arbitrage-free price of Bermudan Option which is given by $y_0^{n,\mathcal{R}}$.

5.4 Example

In this section we will demonstrate some results for a very simple example of a Bermudan Put Option. Take $T = 2$(time is in unite of years), exercise times to be quarterly, the risk-free interest rate $r = .1$, the volatility $\sigma = .6$, the risk-premium $\theta = 0$, the initial price of the underlying asset $S_0 = \$50$, and a constant strike price $K = \$52$. Recall that the time discretization must include all reflection points therefore we choose $n = 400$.

First we solve the price process and obtain a surface which is generated by all possible random paths of W^n. In Fig. 5.4 the green surface represents the price process of the underlying asset and the gray surface represents the strike price process in logarithmic scale. The blue line is simply one realization of the random walk and the corresponding realization of the price process.

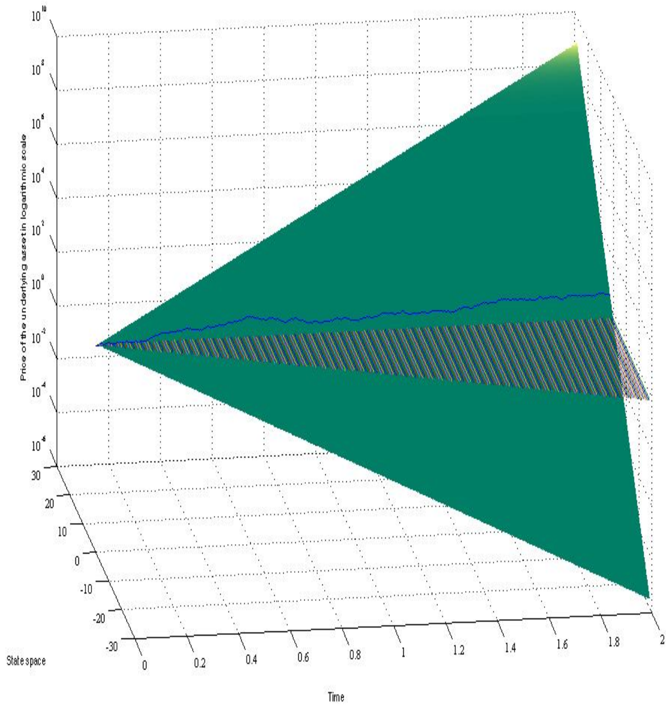

Figure 5.2. Price Process and Strike price Process in logarithmic scale .

Having the price process we can solve for $(y^{n,\mathcal{R}}, z^{n,\mathcal{R}})$. In Fig. 5.4 the surface represents the valuation process of the Bermudan Put option. The blue line is simply one realization of the the random walk and the corresponding valuation process realization.

Since our example is a Bermudan Put Option the valuation process is zero for all points where the price process of the underlying asset is above the strike price process. Furthermore note that reflections occur in an upward manner this is intuitive because the more exercise times and option has the more valuable it is. Finally we get that the price of this Bermudan Put Option is \$13.2012 since $y_0^{n,\mathcal{R}} = 13.2012$, which is consistent with other pricing models for this Bermudan Put Option.

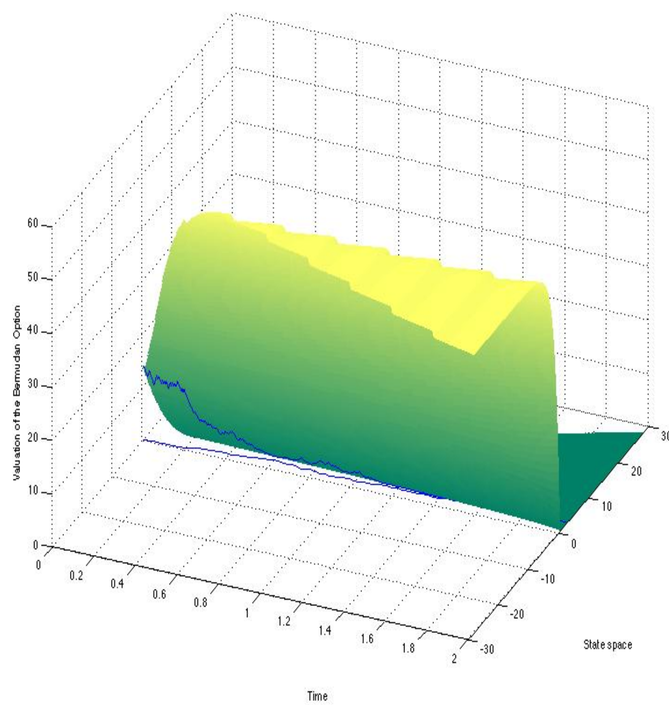

Figure 5.3. Bermudan Put Option Surface.

CHAPTER 6

Future Work

In what follows we describe possible future directions for research concerning topics of FR-BSDEs.

As mentioned above RBSDEs with continuous obstacles can yield the value of an American options. In [28] the authors propose a random walk type numerical scheme for RBSDEs with constant continuous obstacles. We believe that our results could lead a random walk type numerical scheme for RBSDEs with non-continuous random obstacles.

In the general theory of BSDEs much work has been done considering non-Lipschitz generating functions. In applications this is an important development. Therefore a possible future direction for research is the development of numerical techniques which approximate FR-BSDEs with non-Lipschitz generating functions.

Finally we consider that an important next step for research would be to develop computational schemes that could establish L^p convergence to the exact solution with some p-depended tractable rates.

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BIOGRAPHICAL STATEMENT

Wilber Ventura was born in Dallas, Texas, in 1989. He received his B.S. and Ph.D. degrees from The University of Texas at Arlington all in Mathematics.