

SOME MULTIVARIATE PROCESS CAPABILITY INDICES

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## Abstract

### SOME MULTIVARIATE PROCESS CAPABILITY INDICES

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Process capability indices (PCIs) play an important role in the field of Statistical Process Control. Prior to the last 25 years, PCIs had been formulated to assess the quality of a single product characteristic. Product quality, however, is typically dependent on several related variables. Therefore, there is a great need for multivariate process capability indices (MPCIs). The quality of a product is almost always determined from sample data. Thus, it is imperative that an MPCI has a corresponding confidence interval. In that way, conclusions may be drawn regarding the capability of the process that makes that product. Of the MPCIs proposed over the last 25 years, few have an accompanying confidence interval. Under the assumption of multivariate normality, we propose four new MPCIs, each having a corresponding confidence interval, as well as a decision rule for determining whether a process can be declared capable or not at a given level of significance. Two of the indices are based on principal component analysis (PCA) while the other two rely on non-PCA linear transformations. The indices we propose represent multivariate extensions of the popular univariate index,  $C_p$ . Current indices that extend  $C_p$  to the multivariate domain are referred to as  $MC_p$ s. From this group we select four indices that have accompanying confidence intervals to use in a comparative study with our own. The selected indices have been criticized in the past by

other authors for different reasons. Upon investigation we find that our proposed indices do not suffer from the same limitations as the others and that they may serve as adequate tools for assessing multivariate capability in a manufacturing environment.

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## Chapter 1

### Introduction

A manufacturing process, or *process*, is the making of goods or parts by machinery, typically on a large scale. In a manufacturing setting, capability is the ability of a process to produce output that meets predetermined specifications laid out by customers and engineers. As Kotz and Lovelace (1998) put it “the enemy of perfect output (and high process capability) is variation, which pervades every process and system known to our world.” In other words, “no two things are alike.” The authors further state that “since process variation can never be totally eliminated, the *control* of this variation is the key to product quality.” Manufacturers attempt to control variation in the sense that identifiable causes of variation, once realized, are eliminated and only random, unidentifiable sources of variation are left to contend with. Statistical process control is the science of identifying, eliminating or at least minimizing sources of variation whenever possible. A significant part of statistical process control is the formulation and implementation of process capability indices. A capability index is typically a formula that uses the mean and variance of a particular product characteristic to determine whether the process that makes that product is capable of meeting specifications or not. Often, certain parametric and distributional assumptions are made about the product characteristic’s population. A suitable index will assess whether a given process is truly capable or not. Ideally, it should also be easy to compute and interpret by practitioners who lack a solid statistical background. Indices most commonly used in the field thus far have been primarily based on univariate data, thus they are computed to ascertain the quality of a single product characteristic, such as length, temperature, or porosity, i.e. one measurable attribute. However, determining the quality of a product usually involves several variables. There is a need to ascertain capability by examining multiple characteristics simultaneously. In order to establish procedures that can accurately gauge multivariate capability, several multivariate process capability indices (MPCIs) have been developed that are mostly extensions of univariate counterparts. We will

review some of the MPCIs that have been constructed in the last twenty-five years and present four new MPCIs of our own.

### 1.1 The Simplest Univariate Capability Index, $C_p$

A univariate capability index is typically a ratio that compares the specification range for a particular product characteristic to a measure of spread obtained from the population. The measure of spread is most often an expression involving population standard deviation. When the population standard deviation is unknown, it is usually estimated by the sample standard deviation in the process capability index. Consider a product characteristic with an acceptable upper and lower limit, for instance, the length of a screw. Screws of lengths falling outside the acceptable window are considered “non-conforming”. A ratio such as

$$\frac{\text{size of specification interval (upper limit on screw length – lower limit on screw length)}}{\text{population spread of screw length}}$$

can indicate how many times the specification interval will contain the spread of the distribution of the screw lengths. A ratio value of 1.0 indicates that the spread of the distribution is exactly equal to the length of the specification interval. A value of 1.33 indicates that the specification interval can “hold” or “contain” 1.33 times the distribution spread. Thus, higher ratio values are better. The example above demonstrates the basic idea behind the  $C_p$  index.

After suffering crippling defeat in World War II the Japanese decided that the best way to rebuild their economy and compete in the world market was to produce high quality goods for exportation. They spent considerable time analyzing their manufacturing processes, raising standards and meeting with well-known management consultants like J.M. Juran and E. Deming. The  $C_p$  index is one of five “original capability indices” first developed in Japan during the 1970’s. Juran is credited with bringing

attention to  $C_p$  outside of Japan and in the early 1980's Ford Motor Company was the first to use  $C_p$  and other PCI's in the US.  $C_p$  is calculated as

$$C_p = \frac{USL - LSL}{6\sigma}, \quad (1.1)$$

where USL and LSL are the upper and lower specification limits for a measured characteristic  $X$ , and  $\sigma$  denotes the population standard deviation of  $X$ . It is usually assumed that  $X$  is a normally distributed random variable. An estimator for  $C_p$  is given by

$$\widehat{C}_p = \frac{USL - LSL}{6s}, \quad (1.2)$$

where  $s$  denotes the sample standard deviation of  $X$ . The reason for six in the denominator is that under the assumption of normality, a  $6\sigma$  spread covers 99.73% of the data, leaving only 0.27% in the non-conforming range.

Larger values of  $C_p$  indicate higher capability while smaller values (less than 1.0) indicate lower capability. Typical recommended minimums for  $C_p$  are 1, 1.33 and 1.5. New processes and processes that involve characteristics directly related to human safety such as the manufacture of climbing equipment, or bolts for bridge construction require higher recommended minimum values. New processes directly involving human safety have a recommended minimum value of  $C_p$  equal to 1.67.

The intuition behind the  $C_p$  formula is that it compares *allowable* process spread to *actual* process spread, but it does not bring attention to process center or closeness to target. However, other "original capability indices,"  $C_{pk}$ ,  $C_{pm}$ , and  $C_{pmk}$ , were developed to do just that. We will focus our study on  $MC_p$  (a generic term for any multivariate extension of  $C_p$ ) because it is the most manageable and sufficient index for comparing the strengths and weaknesses of proposed MPCIs. But, if desired, it should be fairly uncomplicated to extend our results for  $MC_p$  to indices like  $MC_{pk}$ ,  $MC_{pm}$  and  $MC_{pmk}$ .

## 1.2 Current Work in MPCIs- Three Main Avenues

Research into multivariate capability indices has followed three main courses of study,

1. constructing a ratio of volumes, i.e. tolerance region volume to process region volume,
2. computing the proportion of non-conforming items, and,
3. employing principal component analysis (PCA), which involves linearly transforming the original process data.

Our main interest is in, but not limited to, indices of the third type. We will propose four new MPCIs. The first two indices involve simple “non-PCA” linear transformations while the last two indices involve PCA.

### 1.3 A Brief Review of PCA

According to Jolliffe (1986), “the central idea of principal component analysis (PCA) is to reduce the dimensionality of a data set which consists of a larger number of interrelated variables, while retaining as much as possible of the variation present in the data set.” Let  $X$  be a  $p \times 1$  random vector with mean  $\mu_{p \times 1}$  and covariance matrix,  $\Sigma_{p \times p}$ . The first step in the PCA procedure is to decompose the covariance matrix,  $\Sigma$ , into its respective eigenvalue/eigenvector pairs. The eigenvalues with their respective eigenvectors are put in descending order, i.e.

$$(e_1, \lambda_1), (e_2, \lambda_2), \dots, (e_p, \lambda_p) \text{ such that } \lambda_1 > \lambda_2 > \dots > \lambda_p.$$

A principal component is simply the product of one of the eigenvectors above with  $X$ . Therefore, it is a linear transformation of the original data. The variance of a principal component is its corresponding eigenvalue. Thus, the “first” principal component will be a linear combination of the first eigenvector with  $X$  ( $Y_1 = e_1'X$ ) and it will have the largest variance. The “second” principal component will be a linear combination of the second

eigenvector with  $X$  ( $Y_2 = e_2'X$ ) and it has the second largest variance, etc. Hence we have  $Y_i = e_i'X, i = 1, 2, \dots, p$ . The  $Y_i$ 's are uncorrelated. When a small set of the largest eigenvalues ( $\lambda_1, \lambda_2, \dots, \lambda_k, k < p$ ) accounts for the majority of the system's variability the remaining eigenvalues ( $\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_{p-1}, \lambda_p$ ) and their corresponding eigenvectors can be disregarded. Transforming the original multivariate data set to a small set of principal components allows us to reduce the dimension of the problem, thus, simplifying future calculations. Besides achieving a lower dimension for our data we also give it a simpler correlation structure, especially when the original data set is normally distributed.

Consider the following-

Suppose  $X$  is a random vector from a  $p$  variate normal distribution with covariance matrix  $\Sigma$ , and, let  $(e_1, \lambda_1), (e_2, \lambda_2), \dots, (e_p, \lambda_p)$  be the corresponding eigenvalue eigenvector pairs. If we let  $Y_i = e_i'X$ , then  $\text{Var}(Y_i) = e_i'\Sigma e_i = \lambda_i e_i'e_i = \lambda_i$ , and  $\text{Cov}(Y_i, Y_j) = e_i'\Sigma e_j = \lambda_j e_i'e_j = 0$ .

(Recall from linear algebra that  $e_i'e_i = 1$  and  $e_i'e_j = 0$  for  $i \neq j$ ). With covariance equal to zero we can say that the principal components are uncorrelated as well as independent since they are normally distributed.

#### 1.4 Extensions of $C_p$ to $M_{C_p}$ with PCA

Current indices based on PCA have been developed by Wang and Chen (1998), Xekalaki and Perakis (2002) and Wang (2005). Wang's index (2005) is primarily used for short run production processes.

##### 1.4.1 Wang and Chen (1998)

Wang and Chen were the first to use PCA to create a new MPCl. They assume that a particular process follows a multivariate normal distribution and apply PCA to the sample covariance matrix (assume  $v$  variables). As a consequence they obtain new variables that are mutually independent and normally distributed. Wang and Chen formulate their index as follows,

$$MC_p = \left( \prod_{i=1}^m C_{p;PC_i} \right)^{1/m}, \quad (1.3)$$

where  $C_{p;PC_i}$  represents the univariate index computed for the  $i$ th principal component and  $m$  represents the number of components selected for analysis from a set of  $v$  principal components. Recall,

$$\widehat{C}_p = \frac{USL - LSL}{6s}.$$

In  $C_{p;PC_i}$   $USL$  and  $LSL$  are replaced by their new values under the transformation of  $PC_i$  and  $s$  is replaced by  $\sqrt{\widehat{\lambda}_i}$ . (We will provide a more detailed discussion of how to calculate  $C_{p;PC_i}$  in a later section). Thus, Wang and Chen's index is the geometric mean of multiple  $C_p$  indices, namely one for each principal component retained in the analysis.

#### 1.4.2 Wang (2005)

Wang (2005) proposes an index similar to that of Wang and Chen but intended for short-run process capability assessment. Short-run process data may not be normal so the formulation of the index is based on Clement's method (1989). Wang's index employs a weighted geometric mean, where the weights are the eigenvalues selected for inclusion in the data analysis. Wang's index is given by

$$MWC_p = \left( \prod_{i=1}^m C_{p;PC_i}^{\lambda_i} \right)^{\frac{1}{\sum_{i=1}^m \lambda_i}}, \quad (1.4)$$

where  $C_{p;PC_i}$  is defined in the same way as it is in Wang and Chen's index except that under Clement's method the denominator in  $C_{p;PC_i}$  is set to 0.9973.

### 1.4.3 Xekalaki and Perakis (2002)

In the same vein as the previous indices, Xekalaki and Perakis transform multivariate data with PCA, however, they combine the univariate indices  $C_{p;PC_i}$  with a weighted *arithmetic* mean. Their index is given by

$$MXC_p = \frac{\sum_{i=1}^m \lambda_i C_{p;PC_i}}{\sum_{i=1}^m \lambda_i}. \quad (1.5)$$

### 1.5 Criticisms of the Three Previous Indices

Shinde and Khadse (2009) claim that Wang and Chen's index transforms the specification region in an inappropriate way which greatly affects the resulting index value. As they explain, principal components may be independent of one another but specification limits transformed with PC's are interrelated. In other words, though the original specification limits may define a hyper-rectangular region, the transformed limits outline a much more complicated specification region (see example below).

Unfortunately, the authors do not present a solution to the transformation of the specification region so that Wang and Chen's index can be improved. Rather, they propose an entirely new index based on empirical methods which becomes quite complicated when more than two variables are involved. Regrettably, if Shinde and Khadse are correct in their criticism of Wang and Chen's index, the indices of Xekalaki and Perakis and Wang suffer the same shortcoming. Consider the following example from Shinde and Khadse (2009) employing process data from Sultan (1986).

Twenty-five sample observations from a bivariate process concerning X1-brinell hardness and X2-tensile strength yield the following sample variance covariance matrix,

$$S = \begin{bmatrix} 337.8 & 85.3308 \\ 85.3308 & 33.6247 \end{bmatrix}.$$

The specification limits for each variable are given by

$$112.7 \leq X_1 \leq 241.3 \text{ and } 32.7 \leq X_2 \leq 73.3.$$



PCA analysis is performed on  $S$  to obtain the following equations,

$$PC_1 = 0.967X_1 + 0.253X_2,$$

and,

$$PC_2 = -0.253X_1 + 0.967X_2.$$

Wang and Chen propose transforming the specification regions as follows,

$$LSL_{PC_i} = \text{evaluate } PC_i \text{ at the lower limits of } X_1 \text{ and } X_2,$$

and,

$$USL_{PC_i} = \text{evaluate } PC_i \text{ at the upper limits of } X_1 \text{ and } X_2.$$

So that

$$LSL_{PC_1} \leq PC_1 \leq USL_{PC_1},$$

and,

$$LSL_{PC_2} \leq PC_2 \leq USL_{PC_2}.$$

Their transformation yields the following rectangular specification region,

$$117.306 \leq PC_1 \leq 251.993,$$

and,

$$3.138 \leq PC_2 \leq 9.899.$$

Xekelaki and Perakis as well as Wang would obtain the exact same region when computing their indices. Shinde and Khadse, however, insist that the correct specification region is given by

$$112.7 \leq 0.967PC_1 - 0.253PC_2 \leq 241.3,$$

and,

$$32.7 \leq 0.253PC_1 + 0.967PC_2 \leq 73.3.$$

The inequalities above form a parallelogram rather than a rectangle. The method in which Shinde and Khadse obtain this region will be explained further in section 2.1.2.

### 1.6. Tano and Vannman's Review of MPCIs

In their paper, Tano and Vannman (2011) recognize the need for multivariate process capability indices as well as the need for construction of confidence intervals and tests that support these indices. Under the assumption of multivariate normality, the authors examine four existing MPCIs that 1) either have corresponding confidence intervals in place already, or 2) have approximate confidence intervals that can be derived. Two of the MPCIs investigated by Tano and Vannman include Wang's index and the index developed by Wang and Chen.

#### 1.6.1 MPC I- Taam et al. (1993)

Assuming multivariate normality Taam et al. constructs a multivariate extension of  $C_p$  that compares the volume of a "modified engineering tolerance region" with the volume of a region containing 99.73% of the process data. The modified tolerance region is simply the largest ellipsoid centered at the target falling within the original rectangular tolerance region. Taam's index is given by

$$MC_p = \frac{\text{Vol(modified tolerance region)}}{(\pi \cdot \chi_{v,0.9973}^2)^{v/2} |\Sigma|^{1/2} \left[ \Gamma\left(\frac{v}{2} + 1\right) \right]^{-1}}, \quad (1.6)$$

where  $v$  is the number of variables. According to Pan and Lee (2010), when the process mean is deviated from the target,  $MC_p$  is multiplied by the correcting factor  $\frac{1}{D}$ , where

$D = (1 + (\mu - T)' \Sigma^{-1} (\mu - T))^{1/2}$ . Santos-Fernandez and Scagliarini (2012) show that the

volume of the modified tolerance region can be calculated as follows,

$$\text{Vol}(\text{modified tolerance region}) = \frac{2\pi^{v/2} \prod_{i=1}^v a_i}{v\Gamma\left(\frac{v}{2}\right)}, \quad (1.7)$$

where  $a_i$  denotes the lengths of the semi-axes for  $i=1, 2, \dots, v$ . Pan and Lee (2010) specify that  $a_i$  is given by

$$a_i = \frac{USL_i - LSL_i}{2} \text{ for } i = 1, 2, \dots, v. \quad (1.8)$$

When there is a random sample of size  $n$  available,  $MC_p$  is estimated by

$$\widehat{MC}_p = \frac{\text{Vol}(\text{modified tolerance region})}{(\pi \cdot \chi_{v,0.9973}^2)^{v/2} |S|^{1/2} \left[\Gamma\left(\frac{v}{2} + 1\right)\right]^{-1}}, \quad (1.9)$$

where  $S$  is the sample covariance matrix.

Pearn et al. (2007) derives a confidence interval for  $MC_p$  that is *quite difficult* to calculate for more than two variables. Hence, Tano and Vannman derive their own approximate confidence interval for  $MC_p$ . The lower confidence bound is given by

$$\widehat{MC}_p \sqrt{\left(1 - \frac{\lambda_{\alpha}\sqrt{2v}}{\sqrt{n}}\right)},$$

where  $\lambda_{\alpha}$  denotes the  $1 - \alpha$  quantile of the standard Normal distribution.

### 1.6.2 MPCII- Pan and Lee (2010)

Pan and Lee's index is actually a revision of Taam et al.'s index given above. The authors claim that the previous index overestimates capability of a process when some or all of the characteristics are not independent. Their suggested index follows,

$$MC_p = \left(\frac{|A^*|}{|\Sigma|}\right)^{1/2}, \quad (1.10)$$

where the elements of  $A^*$  are given by

$$A_{ij}^* = \rho_{ij} \left( \frac{USL_i - LSL_i}{2\sqrt{\chi_{v,0.9973}^2}} \right) \left( \frac{USL_j - LSL_j}{2\sqrt{\chi_{v,0.9973}^2}} \right) \quad (1.11)$$

and  $\rho_{ij}$  is the correlation between the  $i$ th and  $j$ th univariate quality characteristic. The index is estimated by

$$\widehat{MC}_p = \left( \frac{|A^*|}{|S|} \right)^{1/2}. \quad (1.12)$$

Although it is not explicitly mentioned, in their illustrative examples, the authors estimate the elements of  $A^*$  by replacing all values of  $\rho_{ij}$  with  $\widehat{\rho}_{ij}$ .

Pan and Lee derive a lower approximate  $100(1-\alpha)\%$  confidence bound,

$$\widehat{MC}_p \sqrt{w_\alpha},$$

where  $w_\alpha$  is the  $\alpha$  quantile of the distribution  $\frac{\prod_{i=1}^v \chi_{n-i}^2}{(n-1)^v}$ .

### 1.6.3 MPCII III- Wang and Chen (1998)

Tano and Vannman investigate Wang and Chen's index along with its corresponding confidence interval proposed by Wang and Du (2000). The proposed lower approximate  $100(1-\alpha)\%$  confidence bound is

$$\left( \prod_{i=1}^m \widehat{C}_{p,PC1} \sqrt{\frac{\chi_{n-1,\alpha^2}}{n-1}} \right)^{1/m}. \quad (1.13)$$

### 1.6.4 MPCII IV- Wang (2005)

Wang does not present a confidence interval for his index. However, Tano and Vannman (2011) follow the idea of Wang and Du (2000) and present their own approximate lower confidence bound for Wang's index,

$$\left( \prod_{i=1}^m \left( C_{p,PC_1} \sqrt{\frac{\chi_{n-1,\alpha}^2}{n-1}} \right)^{\lambda_i} \right)^{1/\sum \lambda_i} \quad . \quad (1.14)$$

Tano and Vannman's derivation of (1.14) does not use Clement's method. Instead the authors assume that  $C_{p,PC_1}$  is defined exactly as it is in Wang and Chen's index (see section 1.4 (i)).

Tano and Vannman's (2011) conclusions are given as follows:

In examining the two indices based on PCA Tano and Vannman agree with Shinde and Khadse in their criticisms of the method in which the specification region is transformed. The authors concur that a transformed region for bivariate data is in fact a parallelogram rather than a rectangle. They further elucidate the drawbacks of Wang and Chen's and Wang's indices with the following hypothetical example:

Suppose a bivariate process has the following variance covariance matrix,

$$\Sigma = \begin{bmatrix} 0.089 & 0.027 \\ 0.027 & 0.089 \end{bmatrix},$$

and, the specification interval for each variable is [-1,1]. Corresponding eigenvalues and eigenvectors are given by

$$\lambda_1 = 0.1157 \text{ and } e_1' = [0.707 \quad 0.707],$$

and,

$$\lambda_2 = 0.0623 \text{ and } e_2' = [0.707 \quad -0.707].$$

If using both PC's to compute the final index, one would obtain zero for the numerator in  $C_{p,PC_2}$ , leading to an overall value of zero for MPCIs III and IV. Clearly, this is a problem and Xekalaki and Perakis' index will suffer the same consequence.

In comparing MPC I and II Tano and Vannman find that when correlation among variables is low the two indices are nearly equal, however, when correlation among variables increases, Taam et al.'s index may grossly overestimate the capability of a process. Consider Pan and Lee's (2010) findings:

Pan and Lee use the previously mentioned Sultan data to compute their index as well as Taam's and make a comparison. Note that there is high correlation between brinell hardness and tensile strength ( $\rho_{12} = .80$ ).

Table 1-1 Taam's index vs. Pan and Lee's index

Taam's index	Pan and Lee's index
$\overline{MC}_p = 1.88$	$\overline{MC}_p = 1.04$

The estimated conforming rate for Sultan's data is 99.91%. A value close to 1 should indicate that the 99.73% process region fits inside the 99.73% modified tolerance region, however, Taam's index seems to be quite high.

Therefore, Tano and Vannman declare Pan and Lee's index superior to the other three MPCs. However, they do discover one drawback which will be discussed in a future section. We will make use of Tano and Vannman's case study when proposing our own indices.

### 1.7 Additional MPCs

The following indices by Chan, et al. (1991) and Shahriari (1995) lack confidence intervals but have been discussed heavily in the literature. Tano and Vannman's index is more recent and has a corresponding confidence interval, however, it currently only applies to the bivariate case.

#### 1.7.1 Chan, Cheng and Spiring (1991)

The following index developed by Chan, et al. is one of the first attempts at a multivariate version of the "original" univariate index  $C_{pm}$  (assume  $X \sim N_v(\mu, \Sigma)$ ),

$$C_{pm} = \sqrt{\frac{nv}{\sum_{i=1}^n (X_i - T)' \Sigma^{-1} (X_i - T)}} \quad (1.15)$$

where  $n$  represents sample size and  $T$  is the target vector.

### 1.7.2 Shahriari (1995)

Shahriari developed a three dimensional vector to assess multivariate capability.

The first component of Shahriari's capability vector is given by the following index

(assume  $X \sim N_v(\mu, \Sigma)$ ),

$$C_{pM} = \left[ \frac{\prod_{i=1}^v (USL_i - LSL_i)}{\prod_{i=1}^v (UPL_i - LPL_i)} \right]^{1/v} \quad (1.16)$$

$UPL_i$  and  $LPL_i$  are calculated as follows,

$$UPL_i = \mu_i + \sqrt{\frac{\chi_{(v,\alpha)}^2 \det(\Sigma_i^{-1})}{\det(\Sigma^{-1})}} \quad (1.17)$$

$$LPL_i = \mu_i - \sqrt{\frac{\chi_{(v,\alpha)}^2 \det(\Sigma_i^{-1})}{\det(\Sigma^{-1})}}$$

where  $\mu_i$  is the  $i$ th component of the vector  $\mu$  and  $\Sigma_i$  is the matrix obtained by

deleting the  $i$ th row and  $i$ th column of  $\Sigma$ .

### 1.7.3 Tano and Vannman (2013)

Tano and Vannman offer their own multivariate process capability index based on PCA. The authors formulate a lower confidence bound and discuss how to calculate recommended minimum values for their index, however, they restrict their attention to the two dimensional case. Assuming  $X \sim N_v(\mu, \Sigma)$ , Tano and Vannman first standardize their data like so

$$X_{TV} = \frac{X_i - M_i}{d_i}$$

where,

$$M_i = \frac{USL_i + LSL_i}{2} \text{ and } d_i = \frac{USL_i - LSL_i}{2}.$$

Thus, PCA is performed on a new covariance matrix  $\Sigma_{TV} = D\Sigma D$ , where  $D$  is a  $v \times v$  diagonal matrix with  $1/d_i$  on the diagonal. Their index is calculated as follows,

$$C_{p,TV} = \frac{1}{\max|u_{1i}|3\sqrt{\lambda_1}} \text{ for } i = 1, 2, \dots, v, \quad (1.18)$$

where  $\lambda_1$  is the largest eigenvalue and  $\max|u_{1i}|$  denotes the largest component of the first eigenvector in absolute value.



## Chapter 2

### New Proposed Multivariate Process Capability Indices

#### 2.1.1 Proposed MPCIS Using Linear Transformations (Non-PCA)

Besides using principal components there are other ways to linearly transform multivariate data that can lead to adequate process capability indices. For all the indices that follow, we assume that the process data is multivariate normal,  $N_p(\mu, \Sigma)$ . Assume that each component of a random vector  $X$  has a predetermined specification interval and that the target value is the midpoint of that interval, i.e.

$$\begin{aligned}x_{lo_1} &\leq x_1 \leq x_{up_1} \\x_{lo_2} &\leq x_2 \leq x_{up_2} \\&\vdots \\x_{lo_p} &\leq x_p \leq x_{up_p}.\end{aligned}$$

We may choose to write the vectors of upper and lower specification limits as  $X_{up}$  and  $X_{lo}$ , respectively.

##### 2.1.1.1 MC1

MC1 involves multiplying each  $(p \times 1)$  random vector  $X$  by a vector of 1's. The result is the sum of the variables that are contained in each vector.

$$MC1 = \frac{[1 \ 1 \dots 1] \cdot X_{up} - [1 \ 1 \dots 1] \cdot X_{lo}}{6\sqrt{[1 \ 1 \dots 1] \cdot \Sigma \cdot [1 \ 1 \dots 1]'}} \quad (2.1)$$

or, more simply,

$$MC1 = \frac{USL_{MC1} - LSL_{MC1}}{6\sigma_{MC1}}, \quad (2.2)$$

where

$$\sigma_{MC1}^2 = [1 \ 1 \dots \ 1] \cdot \Sigma \cdot [1 \ 1 \dots \ 1]'$$

It is easily observed that the transformed specification region is a simple inequality

$$LSL_{MC1} \leq x_1 + x_2 + \dots + x_p \leq USL_{MC1}.$$

When  $\Sigma$  is unknown we replace it with  $S$  to obtain  $\widehat{MC1}$ . Because  $X$  ( $p \times 1$ ) is distributed multivariate normal we can use the following facts to derive several properties of  $\widehat{MC1}$  as well as its distribution:

$$1) \ [1 \ 1 \dots \ 1] \cdot X = x_{MC1} \sim N(\mu_{MC1}, \sigma_{MC1}^2)$$

$$(\mu_{MC1} = [1 \ 1 \dots \ 1] \cdot \mu)$$

$$2) \ \frac{s_{MC1}^2(n-1)}{\sigma_{MC1}^2} \sim \chi^2_{(n-1)} \text{ when } x \sim N(\mu_{MC1}, \sigma_{MC1}^2).$$

Now, consider the  $r$ th moment of  $\widehat{MC1}$ ,  $E(\widehat{MC1}^r)$ .

$$E(\widehat{MC1}^r) = E\left[\left(\frac{USL_{MC1} - LSL_{MC1}}{6s_{MC1}}\right)^r\right]$$

$$= \left(\frac{USL_{MC1} - LSL_{MC1}}{6}\right)^r E\left(\frac{1}{s_{MC1}^r}\right) \left(\frac{\sigma_{MC1}^r}{\sigma_{MC1}^r}\right)$$

$$= MC1^r \sigma_{MC1}^r E\left(\frac{1}{s_{MC1}^r}\right)$$

If we let  $y = \frac{n-1}{\sigma_{MC1}^2} s_{MC1}^2 \sim \chi_{n-1}^2$  we have

$$\frac{1}{s_{MC1}^r} = \left(\frac{n-1}{\sigma_{MC1}^2}\right)^{r/2} y^{-r/2}.$$

$$E\left(\frac{1}{s_{MC1}^r}\right) = \left(\frac{n-1}{\sigma_{MC1}^2}\right)^{r/2} E(y^{-r/2})$$

Recall,  $y \sim \chi_{n-1}^2$  so that

$$\begin{aligned}
E(y^{-r/2}) &= \int_0^{\infty} y^{-r/2} \frac{y^{\frac{n-1}{2}-1} e^{-y/2}}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} dy \\
&= \frac{2^{\frac{n-r-1}{2}} \Gamma(\frac{n-r-1}{2})}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} \int_0^{\infty} \frac{y^{\frac{n-r-1}{2}-1} e^{-y/2}}{2^{\frac{n-r-1}{2}} \Gamma(\frac{n-r-1}{2})} dy \\
&= \frac{\Gamma(\frac{n-r-1}{2})}{2^{r/2} \Gamma(\frac{n-1}{2})}.
\end{aligned}$$

Thus,

$$E\left(\frac{1}{S_{MC1}^r}\right) = \left(\frac{n-1}{\sigma_{MC1}^2}\right)^{r/2} \frac{\Gamma(\frac{n-r-1}{2})}{2^{r/2} \Gamma(\frac{n-1}{2})}$$

and,

$$E(\widehat{MC1}^r) = MC1^r \frac{\sigma_{MC1}^r}{\sigma_{MC1}^r} \left(\frac{n-1}{2}\right)^{r/2} \frac{\Gamma(\frac{n-r-1}{2})}{\Gamma(\frac{n-1}{2})}.$$

If we let  $r=1$ ,

$$E(\widehat{MC1}) = MC1 \sqrt{\frac{n-1}{2}} \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}.$$

Thus, the bias of  $\widehat{MC1}$  is given by

$$MC1 \left[ \sqrt{\frac{n-1}{2}} \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})} - 1 \right].$$

To get the variance of  $\widehat{MC1}$  we let  $r=2$  in the expression  $E(\widehat{MC1}^r)$ .

$$E(\widehat{MC1}^2) = MC1^2 \left(\frac{n-1}{2}\right) \frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-1}{2})}$$

$$= MC1^2 \left( \frac{n-1}{n-3} \right)$$

Thus,

$$\text{var}(\widehat{MC1}) = MC1^2 \left[ \left( \frac{n-1}{n-3} \right) - \left( \frac{n-1}{2} \right) \left( \frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \right)^2 \right]$$

and,

$$\text{MSE} = \text{var}(\widehat{MC1}) + \left( \text{bias}(\widehat{MC1}) \right)^2.$$

Again, under the assumption that  $X \sim N_p(\mu, \Sigma)$  which implies that

$$[1 \ 1 \dots \ 1] \cdot X = x_{MC1} \sim N(\mu_{MC1}, \sigma_{MC1}^2),$$

we may use the following fact to derive the distribution of  $\widehat{MC1}$ ,

$$y = \frac{(n-1)s_{MC1}^2}{\sigma_{MC1}^2} \sim \chi_{n-1}^2.$$

Start by letting  $w = \widehat{MC1}$ .

$$\begin{aligned} w = \widehat{MC1} &= \frac{USL_{MC1} - LSL_{MC1}}{6s_{MC1}} \\ &= \frac{USL_{MC1} - LSL_{MC1}}{6\sigma_{MC1}} \sqrt{\frac{n-1}{y}} \\ &= MC1 \sqrt{\frac{n-1}{y}} \end{aligned}$$

So that

$$y = \frac{MC1^2(n-1)}{w^2}$$

and,

$$\left| \frac{\partial y}{\partial w} \right| = \frac{2MC1^2(n-1)}{w^3}.$$

Using transformation methods and the fact that  $y \sim \chi_{n-1}^2$  we derive the distribution of  $w$  as follows,

$$\begin{aligned} g_w(w) &= f\left(\frac{MC1^2(n-1)}{w^2}\right) \left| \frac{\partial y}{\partial w} \right| \\ &= \frac{1}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} \left(\frac{MC1^2(n-1)}{w^2}\right)^{\frac{n-1}{2}-1} e^{-\frac{MC1^2(n-1)}{2w^2}} \cdot \frac{2MC1^2(n-1)}{w^3} \text{ for } w > 0. \end{aligned}$$

So that

$$f_{\widehat{MC1}}(c) = \frac{(n-1)^{\frac{n-1}{2}}}{MC1 \Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-3}{2}}} \left(\frac{MC1}{c}\right)^n e^{(-\frac{n-1}{2})\left(\frac{MC1}{c}\right)^2} \text{ for } c > 0.$$

We can derive a confidence interval for  $MC1$  using the following inequality,

$$\chi_{n-1, \alpha/2}^2 < \frac{(n-1)s_{MC1}^2}{\sigma_{MC1}^2} < \chi_{n-1, 1-\alpha/2}^2.$$

Thus,

$$\begin{aligned} \frac{\sqrt{\chi_{n-1, \alpha/2}^2} (USL_{MC1} - LSL_{MC1})}{\sqrt{(n-1)6s_{MC1}}} &< \frac{(USL_{MC1} - LSL_{MC1})}{6\sigma_{MC1}} < \frac{\sqrt{\chi_{n-1, 1-\alpha/2}^2} (USL_{MC1} - LSL_{MC1})}{\sqrt{(n-1)6s_{MC1}}}. \\ \frac{\sqrt{\chi_{n-1, \alpha/2}^2}}{\sqrt{(n-1)}} \widehat{MC1} &< MC1 < \frac{\sqrt{\chi_{n-1, 1-\alpha/2}^2}}{\sqrt{(n-1)}} \widehat{MC1} \end{aligned}$$

Hence a  $100(1-\alpha)\%$  confidence interval for  $MC1$  is given by

$$\left( \frac{\sqrt{\chi_{n-1, \alpha/2}^2}}{\sqrt{(n-1)}} \widehat{MC1}, \frac{\sqrt{\chi_{n-1, 1-\alpha/2}^2}}{\sqrt{(n-1)}} \widehat{MC1} \right).$$

A lower confidence bound for MC1 is given by

$$\frac{\sqrt{\chi_{n-1, \alpha}^2}}{\sqrt{(n-1)}} \widehat{MC1}.$$

### 2.1.1.2 MC2

MC2 computes a weighted arithmetic mean of the variables. Each ( $p \times 1$ ) random vector is multiplied by a vector of weights, the weights being the variances of the corresponding variables.

$$MC2 = \frac{w'X_{up} - w'X_{lo}}{6\sqrt{w'\Sigma w}} \quad (2.3)$$

with

$$w' = \left[ w_1 = \frac{\sigma_1^2}{\sum_{i=1}^3 \sigma_i^2} \quad w_2 = \frac{\sigma_2^2}{\sum_{i=1}^3 \sigma_i^2} \quad \dots \quad w_p = \frac{\sigma_p^2}{\sum_{i=1}^p \sigma_i^2} \right]. \quad (2.4)$$

More simply,

$$MC2 = \frac{USL_{MC2} - LSL_{MC2}}{6\sigma_{MC2}}, \quad (2.5)$$

where

$$\sigma_{MC2}^2 = [w_1 \quad w_2 \quad \dots \quad w_p] \cdot \Sigma \cdot [w_1 \quad w_2 \quad \dots \quad w_p]'$$

When  $\Sigma$  is unknown,  $\Sigma$  and  $\sigma_i^2$  are replaced with  $S$  and  $s_i^2$ , and we obtain  $\widehat{MC2}$ . As with MC1, the transformation of the specification region produces a simple inequality regardless of the number of variables.

$$LSL_{MC2} \leq w_1 x_1 + w_2 x_2 + \dots + w_p x_p \leq USL_{MC2},$$

and when  $\Sigma$  is unknown, we use

$$LSL_{MC2} \leq \widehat{w}_1 x_1 + \widehat{w}_2 x_2 + \cdots + \widehat{w}_p x_p \leq USL_{MC2}.$$

Because  $\widehat{MC2}$  relies so heavily on estimators in the numerator and denominator we cannot easily derive a closed expression for its distribution and properties. Nor can we easily obtain the confidence interval for MC2. We can, however, investigate these characteristics with simulation.

### 2.1.2 Proposed MPCIS Using PCA

For our indices that employ PCA we use the same set of assumptions as before. Namely, we assume  $X \sim N_p(\mu, \Sigma)$  and that we may write the vectors of upper and lower specification limits as  $X_{up}$  and  $X_{lo}$ , respectively.

#### 2.1.2.1 MC3

MC3 is similar to the index developed by Xekelaki and Perakis in that we transform multivariate data with PCA and use a weighted arithmetic mean. However, rather than compute separate values of  $C_{p:PC_i}$  and combine them with a weighted arithmetic mean, we weight the principal components first and then calculate a  $C_p$ -type index with the newly transformed data. Like Xekelaki and Perakis we use the eigenvalues of  $\Sigma$  for our weights.

For a  $(p \times 1)$  random vector  $X$  the weighted transformation by the *first*  $k$  ( $k \leq p$ ) principal components is given by

$$U = |w_1 e_1' X| + |w_2 e_2' X| + \cdots + |w_k e_k' X|, \quad (2.6)$$

with  $w_i = \frac{\lambda_i}{\sum_{i=1}^k \lambda_i}$   $i = 1, 2, \dots, k$  and  $e_i$  denoting the corresponding eigenvectors  $i =$

$1, 2, \dots, k$ . The mean and variance of  $U$  are given by

$$\mu_U = |w_1 e_1' \mu| + |w_2 e_2' \mu| + \cdots + |w_k e_k' \mu|$$

and

$$\sigma_U^2 = \text{var}(U) = (w_1 e_1' + w_2 e_2' + \dots + w_k e_k') \cdot \Sigma \cdot (w_1 e_1' + w_2 e_2' + \dots + w_k e_k')'$$

Under the weighted transformation above,

$$USL_U = |w_1 e_1' X_{up}| + |w_2 e_2' X_{up}| + \dots + |w_k e_k' X_{up}|,$$

and,

$$LSL_U = |w_1 e_1' X_{lo}| + |w_2 e_2' X_{lo}| + \dots + |w_k e_k' X_{lo}|.$$

Thus,

$$MC3 = \frac{USL_U - LSL_U}{6\sigma_U}. \quad (2.7)$$

When  $\Sigma$  is unknown MC3 can be estimated by

$$\widehat{MC3} = \frac{[|\widehat{w}_1 \widehat{e}_1' X_{up}| + |\widehat{w}_2 \widehat{e}_2' X_{up}| + \dots + |\widehat{w}_k \widehat{e}_k' X_{up}|] - [|\widehat{w}_1 \widehat{e}_1' X_{lo}| + |\widehat{w}_2 \widehat{e}_2' X_{lo}| + \dots + |\widehat{w}_k \widehat{e}_k' X_{lo}|]}{6\sqrt{(\widehat{w}_1 \widehat{e}_1' + \widehat{w}_2 \widehat{e}_2' + \dots + \widehat{w}_k \widehat{e}_k') \cdot S \cdot (\widehat{w}_1 \widehat{e}_1' + \widehat{w}_2 \widehat{e}_2' + \dots + \widehat{w}_k \widehat{e}_k')'}}. \quad (2.8)$$

$\widehat{w}_i$  and  $\widehat{e}_i$  represent the sample weights and sample eigenvectors derived from  $S$ , the sample covariance matrix of  $X$ . More simply,

$$\widehat{MC3} = \frac{\widehat{USL}_U - \widehat{LSL}_U}{6s_U}. \quad (2.9)$$

Here, it is pertinent that we demonstrate how the transformation of *our* specification region differs from those of the aforementioned authors who also used PCA. We will use a bivariate example for brevity. Suppose  $E'$  is a matrix of eigenvectors derived from the sample variance covariance matrix  $S$ . Let,  $X$  represent the normally distributed bivariate random vector and,  $w$ , represent the vector of weights.

$$E' = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \quad w' = [w_1 \quad w_2] \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad Y = \begin{bmatrix} PC_1 \\ PC_2 \end{bmatrix}$$



Wang and Chen, Xekelaki and Perakis and Wang transform their specification region as follows,

$$E'X = Y.$$

So that

$$E'X_{lo} = \begin{bmatrix} y_{lo_1} \\ y_{lo_2} \end{bmatrix} \text{ or } \begin{bmatrix} LSL_{PC1} \\ LSL_{PC2} \end{bmatrix}.$$

Similarly,

$$E'X_{up} = \begin{bmatrix} y_{up_1} \\ y_{up_2} \end{bmatrix} \text{ or } \begin{bmatrix} USL_{PC1} \\ USL_{PC2} \end{bmatrix}.$$

They obtain the following rectangular region,

$$LSL_{PC1} \leq PC_1 \leq USL_{PC1},$$

and,

$$LSL_{PC2} \leq PC_2 \leq USL_{PC2}.$$

According to Shinde and Khadse they should use the fact that E and E' are orthonormal matrices and multiply to get

$$EE'X = EY.$$

$$X = EY$$

or, *more explicitly*,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e_{11}PC_1 + e_{21}PC_2 \\ e_{12}PC_1 + e_{22}PC_2 \end{bmatrix}.$$

This yields the non-rectangular region

$$x_{lo_1} \leq e_{11}PC_1 + e_{21}PC_2 \leq x_{up_1},$$

and,

$$x_{lo_2} \leq e_{12}PC_1 + e_{22}PC_2 \leq x_{up_2}.$$

Our transformation, however, does not suffer the same repercussions as those of Wang and Chen and the others. We transform the process data as follows,

$$w'E'X = w'Y.$$

Notice that both sides of the equation are equal to a scalar. Upon substitution of  $X_{I_0}$  for  $X$ , we get  $w_1LSL_{PC1} + w_2LSL_{PC2}$ , which is equal to our  $LSL_U$ . A similar result holds for substitution of  $X_{up}$ . This means that  $LSL_U \leq U \leq USL_U$  (and  $\widehat{LSL}_U \leq \widehat{U} \leq \widehat{USL}_U$ )- a simple inequality. Higher dimensioned data will still produce a simple inequality for our index but an even more complex region for the other authors' indices.

Like  $\widehat{MC2}$ ,  $\widehat{MC3}$  relies heavily on estimation in the numerator and denominator making it difficult to derive certain attributes of interests. We will investigate characteristics of  $\widehat{MC2}$  and  $\widehat{MC3}$  in a later section using simulation and bootstrapping techniques.

#### 2.1.2.2 $C_{pv}$

Authors Kirmani and Polansky (2008) have developed a method for testing multivariate capability which involves assuming (based on prior experience) that a given multivariate process has a particular covariance matrix,  $\Sigma_0$ . For our proposed index,  $C_{pv}$  we will use a similar idea. As noted above, when we calculate  $\widehat{MC3}$  we use sample principal components and sample eigenvalues for the weights. Instead, suppose that from *prior experience* we know that the given multivariate normal process has a particular covariance matrix,  $\Sigma_0$  ( $\Sigma_0$  is  $p \times p$ ). The eigenvalue-eigenvector pairs are

$$(e_{1_0}, \lambda_{1_0}), (e_{2_0}, \lambda_{2_0}), \dots, (e_{p_0}, \lambda_{p_0}) \text{ such that } \lambda_{1_0} > \lambda_{2_0} > \dots > \lambda_{p_0}.$$

So, from this known matrix we compute the principal components and their respective weights (assume  $X \sim N_p(\mu, \Sigma)$ ).

$$Y_{i_0} = e_{i_0}' X, \quad i = 1, 2, \dots, p \quad (2.10)$$

$$w_{i_0} = \frac{\lambda_{i_0}}{\sum_{i=1}^k \lambda_{i_0}} \text{ for } i = 1, 2, \dots, k \text{ } k \leq p$$

For a  $(p \times 1)$  random vector  $X$  the weighted transformation by the *first*  $k$  ( $k \leq p$ ) principal components is given by

$$V = |w_{1_0} e_{1_0}' X| + |w_{2_0} e_{2_0}' X| + \dots + |w_{k_0} e_{k_0}' X|, \quad (2.11)$$

and,

$$C_{pv} = \frac{USL_V - LSL_V}{6\sigma_V}, \quad (2.12)$$

with

$$USL_V = |w_{1_0} e_{1_0}' X_{up}| + |w_{2_0} e_{2_0}' X_{up}| + \dots + |w_{k_0} e_{k_0}' X_{up}|,$$

and,

$$LSL_V = |w_{1_0} e_{1_0}' X_{lo}| + |w_{2_0} e_{2_0}' X_{lo}| + \dots + |w_{k_0} e_{k_0}' X_{lo}|.$$

The mean and variance of  $V$  are given by

$$\mu_V = |w_{1_0} e_{1_0}' \mu| + |w_{2_0} e_{2_0}' \mu| + \dots + |w_{k_0} e_{k_0}' \mu|$$

and,

$$\sigma_V^2 = (w_{1_0} e_{1_0}' + w_{2_0} e_{2_0}' + \dots + w_{k_0} e_{k_0}') \Sigma (w_{1_0} e_{1_0}' + w_{2_0} e_{2_0}' + \dots + w_{k_0} e_{k_0}')$$

In  $\widehat{C}_{pv}$  the only change is that  $\Sigma$  is replaced by  $S$ , the sample covariance matrix, i.e.

$$\widehat{C}_{pv} = \frac{USL_V - LSL_V}{6s_V}. \quad (2.13)$$

$$s_V^2 = (w_{1_0} e_{1_0}' + w_{2_0} e_{2_0}' + \dots + w_{k_0} e_{k_0}') S (w_{1_0} e_{1_0}' + w_{2_0} e_{2_0}' + \dots + w_{k_0} e_{k_0}')'$$

Transformation of the specification region occurs in the same fashion as with MC3.

Holding the eigenvalues and eigenvectors constant makes it possible to derive the distribution and other properties of  $\widehat{C}_{pv}$ . Just like in our derivations for  $\widehat{MC1}$  we make use of similar facts,

$$1) \quad |w_{1_0} e_{1_0}' X| + |w_{2_0} e_{2_0}' X| + \dots + |w_{k_0} e_{k_0}' X| = V \sim N(\mu_V, \sigma_V^2)$$

$$(\text{Assuming } X \sim N_p(\mu, \Sigma) \text{ and } \mu_V = |w_{1_0} e_{1_0}' \mu| + |w_{2_0} e_{2_0}' \mu| + \dots + |w_{k_0} e_{k_0}' \mu|)$$

$$2) \quad \frac{s_V^2(n-1)}{\sigma_V^2} \sim \chi^2_{(n-1)} \text{ when } V \sim N(\mu_V, \sigma_V^2).$$

The steps for each derivation below are complementary to those of  $\widehat{MC1}$  and will not be shown. The distribution of  $\widehat{C}_{pv}$  is given by

$$f_{\widehat{C}_{pv}}(c) = \frac{(n-1)^{(n-1)/2}}{C_{pv} \Gamma\left(\frac{n-1}{2}\right) 2^{(n-3)/2}} \left(\frac{C_{pv}}{c}\right)^n e^{-(n-1)/2 (C_{pv}/c)^2} \text{ for } c > 0.$$

The properties of  $\widehat{C}_{pv}$  follow

$$1) \quad E(\widehat{C}_{pv}) = C_{pv} \sqrt{\frac{n-1}{2} \frac{\Gamma\left(\frac{n-r-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}}$$

$$2) \quad \text{Bias}(C_{pv}) = C_{pv} \left[ \sqrt{\frac{n-1}{2} \frac{\Gamma\left(\frac{n-r-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}} - 1 \right]$$

$$3) \quad \text{var}(\widehat{C}_{pv}) = C_{pv}^2 \left[ \left(\frac{n-1}{n-3}\right) - \left(\frac{n-1}{2}\right) \left(\frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right)^2 \right]$$

$$4) \text{MSE} = \text{var}(\widehat{C}_{pv}) + (\text{bias}(\widehat{C}_{pv}))^2.$$

A 100(1- $\alpha$ ) % confidence interval for  $C_{pv}$  is given by

$$\left( \sqrt{\frac{\chi^2_{n-1, \alpha/2}}{n-1}} \widehat{C}_{pv}, \sqrt{\frac{\chi^2_{n-1, 1-\alpha/2}}{n-1}} \widehat{C}_{pv} \right),$$

and, a lower confidence bound for  $C_{pv}$  is

$$\sqrt{\frac{\chi^2_{n-1, \alpha}}{n-1}} \widehat{C}_{pv}.$$

## 2.2 Designating a Recommended Minimum Value for the New Indices

When  $C_p$  was first introduced outside of Japan in the 1970's the value of  $C_p = 1.0$  was designated the recommended minimum value because it specifically corresponds to a probability of non-conformance  $\{P(\text{NC})\}$  equal to 0.0027 when the process data is normal. Thus, a process with  $C_p > 1$  is considered capable. When a smaller  $P(\text{NC})$  is desired the value of  $C_p$  must increase. For example, to obtain  $P(\text{NC})=0.00006334$ , the  $C_p$  index for a given (normal) process must surpass 1.33. This particular relationship between an index value and the probability of non-conformance leads Tano and Vannman (2013) to point out what they consider to be a flaw in Pan and Lee's (2010) index. Namely this- for Pan and Lee's index,  $MC_p = 1.0$  *does not* guarantee that  $P(\text{NC})=0.00027$ . In fact, Pan and Lee give no indication of how to compute the probability of non-conformance with respect to their index. They simply recommend 1.0 as the desired value of their index because it indicates that "the manufacturing process region falls completely within an engineering tolerance region." In Tano and Vannman (2013) the authors propose their own multivariate capability index,  $C_{p,TV}$  but limit their investigation of recommended minimums to bivariate data. They use several bivariate data sets in which  $MC_p = 1.0$  to calculate probability of non-conformance with respect to

their own index. They show that probability of non-conformance is not constant for Pan and Lee's index even though  $MC_p$  may be fixed. The authors identify several instances where  $P(\text{NC})$  is much larger than 0.00027 although  $MC_p$  equals 1.0. Tano and Vannman also give examples of bivariate data sets where  $MC_p = 1.20$  and  $P(\text{NC})$  becomes as large as 0.03 according to their own index,  $C_{p,TV}$ . The authors insist that a "threshold" value for a particular index should corroborate a specific level of  $P(\text{NC})$ .

Following the recommendation of Tano and Vannman (2011) we will establish the recommended minimums for our proposed indices by confirming that they correspond to a desired level of  $P(\text{NC})$ . We will use  $MC3$  as an illustration. Assume,  $X \sim N_p(\mu, \Sigma)$  and that the components of  $\mu$  are equal to the midpoints of the specification intervals for each quality characteristic. For  $MC3$

$$P(\text{NC}) = 1 - P(\text{LSL}_U < U < \text{USL}_U).$$

Recall,

$$U = |w_1 e_1' X| + |w_2 e_2' X| + \dots + |w_k e_k' X|,$$

so that

$$U \sim N(|w_1 e_1' \mu| + |w_2 e_2' \mu| + \dots + |w_k e_k' \mu|, (w_1 e_1' + w_2 e_2' + \dots + w_k e_k') \Sigma (w_1 e_1' + w_2 e_2' + \dots + w_k e_k)'),$$

or, more simply,

$$U \sim N(\mu_U, \sigma_U^2).$$

Thus,

$$\begin{aligned} P(\text{NC}) &= 1 - P\left(\frac{\text{LSL}_U - \mu_U}{\sigma_U} < \frac{U - \mu_U}{\sigma_U} < \frac{\text{USL}_U - \mu_U}{\sigma_U}\right) \\ &= 1 - \Phi\left(\frac{\text{USL}_U - \mu_U}{\sigma_U}\right) + \Phi\left(\frac{\text{LSL}_U - \mu_U}{\sigma_U}\right). \end{aligned}$$

We make the following substitution

$$\mu_U = \frac{LSL_U + USL_U}{2},$$

so that

$$\begin{aligned} P(\text{NC}) &= 1 - \Phi\left(\frac{USL_U}{\sigma_U} - \frac{LSL_U + USL_U}{2\sigma_U}\right) + \Phi\left(\frac{LSL_U}{\sigma_U} - \frac{LSL_U + USL_U}{2\sigma_U}\right) \\ &= 1 - \Phi\left(\frac{USL_U - LSL_U}{2\sigma_U}\right) + \Phi\left(\frac{LSL_U - USL_U}{2\sigma_U}\right) \\ &= 1 - \Phi\left(\frac{USL_U - LSL_U}{2\sigma_U}\right) + \left[1 - \Phi\left(\frac{USL_U - LSL_U}{2\sigma_U}\right)\right] \\ &= 2 - 2\Phi\left(\frac{USL_U - LSL_U}{2\sigma_U}\right) \\ &= 2[1 - \Phi(3\text{MC3})]. \end{aligned}$$

$$\text{MC3} = \frac{1}{3}\Phi^{-1}\left(1 - \frac{P(\text{NC})}{2}\right)$$

Thus, we can select any value of P(NC) that we desire and obtain a corresponding value for MC3. For example, when P(NC) is equal to 0.0027, MC3 is equal to 1.0. When P(NC) is equal to 0.00006334, MC3 is equal to 1.33. The recommended minimums for our other indices are calculated in a similar fashion. The normality assumption gives us the same values of P(NC) for the other three indices, i.e. for P(NC)=0.0027, MC1, MC2, and, C<sub>pV</sub> are equal to 1.0 and for P(NC)=0.00006334, MC1, MC2, and, C<sub>pV</sub> are equal to 1.33.

## Chapter 3

### Comparative Study

#### 3.1 Hypothesis Tests and Power

To test the performance of our proposed indices we will conduct hypothesis tests and compute power under different scenarios. We will repeat the same calculations for the indices of previously mentioned authors if they have a corresponding lower confidence bound. Thus, we will be comparing our indices with those of Taam, Pan and Lee, Wang and Chen and Wang. Though Wang and Chen and Wang may be transforming the specification limits incorrectly, the list of MPCIs with confidence intervals that we can compare with is brief. By studying their power we may lend further support to Shinde and Khadse's claim. To illustrate this study we will take advantage of a relevant example from industry. Consider the following trivariate normal process data (the data below actually come from a plastics manufacturer in Taiwan and give the specifications for a rectangular-shaped container, see Wang and Chen (1998)),

$$\mu = \begin{bmatrix} 2.16 \\ 304.72 \\ 304.77 \end{bmatrix}, \text{ and } \Sigma = \begin{bmatrix} 0.0021 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}.$$

Assume specifications are the following (units are unknown but assumed to be the same for each variable),

depth,  $D \in (2.1, 2.3)$ ,

length,  $L \in (304.5, 305.1)$ ,

width,  $W \in (304.5, 305.1)$ .

##### 3.1.1 $MC1$ , $MC2$ , $MC3$ and $C_{pv}$

Consider the hypotheses



$H_0$ : MC1 = c (process is not capable) vs.

$H_a$ : MC1 > c (process is capable)

(Similar hypotheses can be constructed for MC2, MC3 and  $C_{pv}$ ). We will use the following decision rules for each index, reject the null hypothesis when the appropriate inequality below holds,

$$\widehat{MC1} > \frac{\sqrt{(n-1)}}{\sqrt{\chi_{n-1,\alpha^2}}} c, \quad \widehat{MC2} > \frac{\sqrt{(n-1)}}{\sqrt{\chi_{n-1,\alpha^2}}} c, \quad \widehat{MC3} > \frac{\sqrt{(n-1)}}{\sqrt{\chi_{n-1,\alpha^2}}} c, \quad \text{or} \quad \widehat{C}_{pv} > \frac{\sqrt{(n-1)}}{\sqrt{\chi_{n-1,\alpha^2}}} c.$$

When we compute the power of the hypothesis test we are computing the probability of rejecting the null hypothesis when a particular value of the index is known to be true. In other words we are computing the probability that we declare a process capable when a particular value of the capability index is true. To say that an index performs well means that its power indicates one of two things:

1. we are very likely to declare a process capable when it is in fact capable (according to that particular index) – or –
2. we are not likely to declare a process capable when in fact it is not capable (according to that particular index).

Because MC1 and  $C_{pv}$  have properties that can be derived analytically (assuming normality of the data) we will discuss them first. MC2 and MC3 will be examined with simulation.

Using the first decision rule above, we may compute the power for any particular value of MC1, like so

$$\begin{aligned} \pi(\text{MC1}) &= P \left\{ \widehat{MC1} > c \frac{\sqrt{n-1}}{\sqrt{\chi_{n-1,\alpha^2}}} \mid \text{MC1} \right\} \quad (3.1) \\ &= P \left\{ \left( \frac{USL_1 - LSL_1}{6s} \right)^2 > c^2 \frac{n-1}{\chi_{n-1,\alpha^2}} \mid \text{MC1} \right\} \end{aligned}$$

$$\begin{aligned}
&= P \left\{ \frac{(n-1)s^2}{\sigma^2} < \left( \frac{USL_1 - LSL_1}{6\sigma} \right)^2 \frac{\chi_{n-1,\alpha}^2}{c^2} \middle| MC1 \right\} \\
&= P \left\{ \chi_{n-1}^2 < MC1^2 \frac{\chi_{n-1,\alpha}^2}{c^2} \right\}.
\end{aligned}$$

For  $C_{pv}$  we use the fourth decision rule above and find that the expression for power is nearly identical to that of MC1.

$$\pi(C_{pv}) = P \left\{ \chi_{n-1}^2 < C_{pv}^2 \frac{\chi_{n-1,\alpha}^2}{c^2} \right\} \quad (3.2)$$

We will obtain the power for MC2 and MC3 using simulation and the second and third decision rules above.

Example 1:

For the first example, consider the original data taken from the plastic's manufacturer in Taiwan

$$\mu = \begin{bmatrix} 2.16 \\ 304.72 \\ 304.77 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 0.0021 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}, \quad (3.3)$$

with specification limits,

$$D \in (2.1, 2.3)$$

$$L \in (304.5, 305.1)$$

$$W \in (304.5, 305.1),$$

i.e.

$$X_{lo} = \begin{bmatrix} 2.1 \\ 304.5 \\ 304.5 \end{bmatrix} \quad \text{and} \quad X_{up} = \begin{bmatrix} 2.3 \\ 305.1 \\ 305.1 \end{bmatrix}.$$

Recall equations 2.1 and 2.11,

$$MC1 = \frac{[1 \ 1 \dots \ 1] \cdot X_{up} - [1 \ 1 \dots \ 1] \cdot X_{lo}}{6\sqrt{[1 \ 1 \dots \ 1] \cdot \Sigma \cdot [1 \ 1 \dots \ 1]'}}$$

and,

$$C_{pv} = \frac{USL_V - LSL_V}{6\sigma_V},$$

i.e.

$C_{pv}$

$$= \frac{(|w_{1_0} e_{1_0}' X_{up}| + |w_{2_0} e_{2_0}' X_{up}| + \dots + |w_{k_0} e_{k_0}' X_{up}|) - (|w_{1_0} e_{1_0}' X_{lo}| + |w_{2_0} e_{2_0}' X_{lo}| + \dots + |w_{k_0} e_{k_0}' X_{lo}|)}{6\sqrt{(w_{1_0} e_{1_0}' + w_{2_0} e_{2_0}' + \dots + w_{k_0} e_{k_0}') \Sigma (w_{1_0} e_{1_0}' + w_{2_0} e_{2_0}' + \dots + w_{k_0} e_{k_0}')'}}$$

where  $w_{i_0}$  and  $e_{i_0}$  are obtained by performing PCA on  $\Sigma_0$  (see(2.10)) and we let  $\Sigma_0 = \Sigma$ .

For the matrix above and all subsequent examples the sum of the first two eigenvalues is close to 90%, thus, k equals 2 for all examples.

To calculate the population index values we plug the example data (including eigenvectors and eigenvalues) into the equations above. For MC1 we obtain 2.20 and for  $C_{pv}$  we obtain 2.46. We use formulas (3.1) and (3.2) above to obtain the power. The table below gives the hypothesized values of c along with the population values and power associated with MC1 and  $C_{pv}$ . Assume a sample size of  $n=50$  and  $\alpha=0.05$ .

Table 3-1 Example 1 MC1 and  $C_{pv}$

H0: process is not capable $\alpha=0.05, n=50$	Population MC1=2.20	Population $C_{pv}=2.46$ $\Sigma_0 = \Sigma$
	Power	Power
c=1.0	100%	100%
c=1.33	100%	100%
c=1.50	99%	99%
c=1.67	84%	99%

For  $C_{pv}$  the power remains high for all values of  $c$ , which is what we would expect since this is a highly capable process according to the population index value of 2.46. The power remains high for MC1 until we get to  $c=1.67$ . The power should perhaps be a bit higher here (over 90%) since the population index value is 2.20. Overall, both indices perform well in that we are highly likely (in all but one case) to declare the process capable when the population index values indicate that it is a highly capable process.

For MC2 and MC3 we will conduct a simulation study of power using the same data, specifications and hypotheses. We obtain population index values by plugging the example data into equations (2.3) and (2.7). Recall,

$$MC2 = \frac{w'X_{up} - w'X_{lo}}{6\sqrt{w'\Sigma w}},$$

with,

$$w' = \left[ w_1 = \frac{\sigma_1^2}{\sum_{i=1}^3 \sigma_i^2} \quad w_2 = \frac{\sigma_2^2}{\sum_{i=1}^3 \sigma_i^2} \quad \dots \quad w_p = \frac{\sigma_p^2}{\sum_{i=1}^p \sigma_i^2} \right],$$

and,

$$MC3 = \frac{USL_U - LSL_U}{6\sigma_U},$$

i.e.

MC3

$$= \frac{(|w_1 e_1' X_{up}| + |w_2 e_2' X_{up}| + \dots + |w_k e_k' X_{up}|) - (|w_1 e_1' X_{lo}| + |w_2 e_2' X_{lo}| + \dots + |w_k e_k' X_{lo}|)}{6\sqrt{(w_1 e_1' + w_2 e_2' + \dots + w_k e_k') \cdot \Sigma \cdot (w_1 e_1' + w_2 e_2' + \dots + w_k e_k')}},$$

where  $w_i$  and  $e_i$  are obtained by performing PCA on  $\Sigma$ , see equations (2.3) and (2.7).

Note, we have already obtained the eigenvectors and eigenvalues corresponding to  $\Sigma$  in our computation of  $C_{pv}$ . As with  $C_{pv}$ ,  $k$  equals 2 for all examples since the first two principal components account for nearly 90% of the total variance in every example.

For MC2 we obtain the population value 2.15 and for MC3 we obtain 2.46. For the simulation study we use SAS to generate 10,000 samples of size  $n=50$  with  $\alpha=0.05$  from a trivariate normal distribution with the parameters from (3.3). The estimates for each index and decision rules are programmed in proc iml. Along with power we compute the variance of the estimates for both indices. Results follow.

Table 3-2 Example 1 MC2 and MC3

H0: process is not capable $\alpha=0.05, n=50$	Population MC2=2.15 $\text{var}(\overline{MC2}) = 0.05$	Population MC3=2.46 $\text{var}(\overline{MC3}) = 0.06$
	Power	Power
$c=1.0$	100%	100%
$c=1.33$	100%	100%
$c=1.50$	94%	100%
$c=1.67$	71%	97%

When computing population index values,  $C_{pv}$  and MC3 will be equal because their expressions are identical and  $\Sigma_0 = \Sigma$ . For this example MC3 performs just as well as  $C_{pv}$ . MC2 also performs well for this example, but comes up a bit low in the case  $c=1.67$ . The power remains relatively high overall, as we would expect with an index value of 2.15.

Example 2:

Now, suppose we adjust the covariance matrix above by doubling the variance of the first quality characteristic, D, keeping  $\mu$  and the specification intervals the same.

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}$$

We calculate the population values of MC1, MC2, MC3 and  $C_{pv}$  in the same manner as in the example above. We derive the power for MC1 and  $C_{pv}$  analytically and use simulation for MC2 and MC3. For the simulation portion we generate 10,000 samples of size  $n=50$

with  $\alpha=0.05$  from a trivariate normal distribution with the parameters from (3.3) but with the adjustments to the covariance matrix given above. (We will follow this same procedure for the remaining examples.) Results are given by

Table 3-3 Example 2 MC1 and  $C_{pv}$

H0: process is not capable $\alpha=0.05, n=50$	Population MC1=2.02  Power	Population $C_{pv}=2.07$ $\Sigma_0 = \Sigma$  Power
c=1.0	100%	100%
c=1.33	100%	100%
c=1.50	89%	93%
c=1.67	55%	65%

Table 3-4 Example 2 MC2 and MC3

H0: process is not capable $\alpha=0.05, n=50$	Population MC2=1.48 $\text{var}(\overline{MC2}) = 0.07$  Power	Population MC3=2.07 $\text{var}(\overline{MC3}) = 0.09$  Power
c=1.0	88%	99%
c=1.33	34%	90%
c=1.50	13%	74%
c=1.67	4%	48%

Here, MC1 and  $C_{pv}$  perform well. MC3's power does not mirror  $C_{pv}$  in this example. As it should, the power for MC3 decreases as the values of  $c$  increase, but it is a bit low for the last two  $c$  values. The power for  $C_{pv}$  is more in line with what we expect for a population value of 2.07.  $C_{pv}$ 's superior performance is likely due to the fact that  $\Sigma_0 = \Sigma$ .  $C_{pv}$ 's performance when  $\Sigma_0 \neq \Sigma$  is something to investigate in the future and may involve developing a test for  $\Sigma$ 's "closeness" to  $\Sigma_0$ . The power for MC2 is not as close to 5% at  $c=1.50$  as it should be. This could be due to the fact that the weights in MC2 are estimated. Hence, the test statistic has a large variance.

Example 3:

Suppose once again that we adjust the original covariance matrix, this time doubling the variance of the first and second quality characteristics, D and L. We recompute the population index values with the new  $\Sigma$ , assuming  $\mu$  and the specification intervals have not changed.

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0034 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}$$

New results follow.

Table 3-5 Example 3 MC1 and  $C_{pv}$

H0: process is not capable $\alpha=0.05, n=50$	Population MC1=1.91  Power	Population $C_{pv}=2.06$ $\Sigma_0 = \Sigma$  Power
c=1.0	100%	100%
c=1.33	97%	100%
c=1.50	74%	93%
c=1.67	34%	63%

Table 3-6 Example 3 MC2 and MC3

H0: process is not capable $\alpha=0.05, n=50$	Population MC2=1.64 $\text{var}(\overline{MC2}) = 0.05$  Power	Population MC3=2.06 $\text{var}(\overline{MC3}) = 0.06$  Power
c=1.0	98%	99%
c=1.33	55%	92%
c=1.50	22%	71%
c=1.67	5%	33%

For this example the power levels are appropriate for MC1 and  $C_{pv}$ . Again, MC3 does not perform as well as  $C_{pv}$ . The power levels are a bit low for the last two values of c

for a process that is highly capable according to  $MC3/C_{pv}$ . MC2 performs well since the power is appropriate for each value of  $c$ .

Example 4:

For the last study we adjust the covariance matrix by tripling the variance of the first quality characteristic and doubling the variance of the third quality characteristic,  $\mu$  and the specifications remain the same.

$$\Sigma = \begin{bmatrix} 0.0063 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0040 \end{bmatrix}$$

Results follow.

Table 3-7 Example 4 MC1 and  $C_{pv}$

H0: process is not capable $\alpha=0.05, n=50$	Population MC1=1.77  Power	Population $C_{pv}=1.65$ $\Sigma_0 = \Sigma$  Power
$c=1.0$	100%	100%
$c=1.33$	87%	65%
$c=1.50$	46%	22%
$c=1.67$	13%	4%

Table 3-8 Example 4 MC2 and MC3

H0: process is not capable $\alpha=0.05, n=50$	Population MC2=1.25 $\text{var}(\overline{MC2}) = 0.05$  Power	Population MC3=1.65 $\text{var}(\overline{MC3}) = 0.09$  Power
$c=1.0$	60%	87%
$c=1.33$	7%	40%
$c=1.50$	1%	13%
$c=1.67$	0.1%	2%



In this last example all three indices, MC1,  $C_{pv}$  and MC2 perform well. The power is in line with what we would expect for the population values of each index. As in previous examples, MC3 does not perform as well as  $C_{pv}$ , the first two powers being lower than what's expected for an index value of 1.65, and at  $c=1.67$  the power should be closer to 5%. This could be attributed to the fact that the weights are only estimates as is the case with MC2. For MC1 and  $C_{pv}$  the index values indicate a capable process and the power steadily decreases to reasonable levels as the value of  $c$  increases. For MC2 the index value indicates a marginally capable process for which the powers are appropriately low and steadily decreasing as  $c$  values increase. We are highly unlikely to declare this process capable except when  $c$  is equal to 1.0.

In comparing the population index values of our indices to one another we see that values for MC1 and  $MC3/C_{pv}$  are similar for each example. MC2 is consistently lower than the other three indices in each example. MC1 and  $MC3/C_{pv}$  steadily decrease for each example as the total variance increases. However, there is an unexpected shift in values for MC2 between examples 2 and 3. The population index value is lower in example 2 than it is in example 3 which does not fall in line with the other three indices. The reason for this shift is the way in which the weighting occurs within the index computation. We examine the calculations of MC2 for examples 2 and 3 below. We only need to look at the numerators since the denominators for both examples are nearly equal.

Example 2-

weight vector=[0.5316 0.2152 0.2532] difference of specification limits

$$\text{vector} = \begin{bmatrix} 0.2 \\ 0.6 \\ 0.6 \end{bmatrix}$$

$$[0.5316 \quad 0.2152 \quad 0.2532] \cdot \begin{bmatrix} 0.2 \\ 0.6 \\ 0.6 \end{bmatrix} = 0.3873$$

Example 3-

weight vector=[0.4375 0.3542 0.2083] difference of specification limits

$$\text{vector} = \begin{bmatrix} 0.2 \\ 0.6 \\ 0.6 \end{bmatrix}$$

$$[0.4375 \quad 0.3542 \quad 0.2083] \cdot \begin{bmatrix} 0.2 \\ 0.6 \\ 0.6 \end{bmatrix} = 0.4250$$

In example 3 the variance of the second variable is double what it is in example 2. This gives the second variable a greater weight in example 3 than what it has in example 2. The greater weight causes the second component of the specification vector to contribute more to the numerator in example 3 than it does in example 2. The result is a larger allowable spread for example 3. With nearly equal denominators, a larger allowable spread means a larger MC2 value.  $\text{var}(\widehat{MC2})$  is consistently lower than  $\text{var}(\widehat{MC3})$  for each example. Both variances are never larger than 0.10.

### 3.1.2 Taam's Index

Tano and Vannman (2011) derive the following lower confidence bound for Taam's index,

$$c < \widehat{MC}_p \sqrt{\left(1 - \frac{\lambda_\alpha \sqrt{2v}}{\sqrt{n}}\right)}.$$

Consider the same examples from above and a similar set of hypotheses,

$H_0$ :  $MC_p = c$  (process is not capable) vs.

$H_a$ :  $MC_p > c$  (process is capable).

We reject  $H_0$  when the following inequality holds,

$$c \sqrt{\frac{\sqrt{n}}{\sqrt{n} - \lambda_\alpha \sqrt{2v}}} < \widehat{MC}_p,$$

where  $\lambda_\alpha$  denotes the  $1 - \alpha$  quantile of the standard Normal distribution.

The following expression gives the power of the test,

$$\pi(\text{MC}_p) = P \left\{ \overline{\text{MC}}_p > c \sqrt{\frac{\sqrt{n}}{\sqrt{n} - \lambda_\alpha \sqrt{2v}}} \text{MC}_p \right\}. \quad (3.4)$$

To study Taam's index we use simulation. For each example, 10,000 samples of size  $n=50$  are generated in SAS from a trivariate normal distribution with the parameters from (3.3). Estimates for each index and power are computed with proc iml.

Example 1:

Consider the original matrix and specification intervals,

$$\mu = \begin{bmatrix} 2.16 \\ 304.72 \\ 304.77 \end{bmatrix} \Sigma = \begin{bmatrix} 0.0021 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}$$

$$D \in (2.1, 2.3)$$

$$L \in (304.5, 305.1)$$

$$W \in (304.5, 305.1).$$

To obtain the population values of Taam's index we apply equation (1.5) to the data above.

$$\text{MC}_p = \frac{\text{Vol}(\text{modified tolerance region})}{(\pi \cdot \chi_{v,0.9973}^2)^{v/2} |\Sigma|^{1/2} \left[ \Gamma\left(\frac{v}{2} + 1\right) \right]^{-1}}$$

where,

$$\text{Vol}(\text{modified tolerance region}) = \frac{2\pi^{v/2} \prod_{i=1}^v a_i}{v\Gamma\left(\frac{v}{2}\right)},$$

and,

$$a_i = \frac{USL_i - LSL_i}{2} \text{ for } i = 1, 2, \dots, v.$$

For the data above we get a population value of  $MC_p$  equal to 2.92. Simulation results are given by

Table 3-9 Example 1 Taam

H0: process is not capable $\alpha=0.05, n=50$	Population $MC_p=2.92$  Power
$c=1.0$	100%
$c=1.33$	99%
$c=1.50$	96%
$c=1.67$	87%

For this example Taam's index performs well except at  $c=1.67$  where the power seems a bit low for a population index value as high as 2.92.

Example 2:

With the new matrix (other parameters unchanged)

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix},$$

we find the population value of  $MC_p$  in the same manner as example 1. Specification intervals remain the same. Simulation results and the population value of  $MC_p$  are given by

Table 3-10 Example 2 Taam

H0: process is not capable $\alpha=0.05, n=50$	Population $MC_p=1.96$  Power
c=1.0	96%
c=1.33	55%
c=1.50	30%
c=1.67	13%

For this example the power should be higher for each value of c given a population index value of 1.96. We use the procedure outlined above for the next two examples.

Example 3:

For the following matrix,

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0034 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix},$$

simulation results and the population value of  $MC_p$  appear below.

Table 3-11 Example 3 Taam

H0: process is not capable $\alpha=0.05, n=50$	Population $MC_p=1.17$  Power
c=1.0	13%
c=1.33	0.5%
c=1.50	0.05%
c=1.67	0.01%

For this example the population index value indicates a marginally capable process. The power is appropriately low for all c values. We are unlikely to declare the process capable at any value of c.

Example 4:

For the final matrix

$$\Sigma = \begin{bmatrix} 0.0063 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0040 \end{bmatrix},$$

simulation results and the population value of  $MC_p$  are given by

Table 3-12 Example 4 Taam

H0: process is not capable $\alpha=0.05, n=50$	Population $MC_p=0.95$  Power
c=1.0	1.5%
c=1.33	0.01%
c=1.50	0.01%
c=1.67	0%

In this example the population index value indicates that the process is not capable and the power is appropriately low for all values of  $c$ . We are highly unlikely to deem the process capable at any  $c$  value.

Taam's index for the last example indicates that the process is not capable. All four of our indices, however, indicate that the process is capable.

### 3.1.3 Pan and Lee's Index

Pan and Lee (2010) give the following lower confidence bound for their index  $MC_p$ ,

$$\widehat{MC}_p \sqrt{w_\alpha},$$

where  $\widehat{MC}_p$  is given in (1.11) and  $w_\alpha$  is the  $\alpha$  quantile of the distribution  $\prod_{i=1}^v \chi_{n-i}^2 / (n-1)^v$ .

Tano and Vannman (2013) note that  $w_\alpha$  is the same as  $F_Y^{-1}(\alpha)/(n-1)^v$  where  $Y$  is defined as a product of independent Chi-square random variables with  $n-1, n-2, \dots, n-v$  degrees of freedom, respectively. The quantile  $F_Y^{-1}(\alpha)$  is difficult to calculate for  $v > 2$ , thus, Tano and Vannman show that a good approximation to  $\sqrt{F_Y^{-1}(\alpha)/(n-1)^v}$  is given by

$$\sqrt{(1 - \lambda_\alpha \sqrt{2v/\sqrt{n}})}$$

This is the same approximation that they developed when finding a simpler confidence bound for Taam's index. Again,  $\lambda_\alpha$  denotes the  $(1 - \alpha)$  quantile of the  $N(0,1)$  distribution. In their paper the authors include a brief list of calculations comparing their approximation to the actual quantile (which they find numerically) for  $v=3, \alpha=0.05$  and a handful of sample sizes. The computed values appear in the table below.

Table 3-13 Tano and Vannman's approximation of  $w_\alpha$

n	$\sqrt{F_Y^{-1}(\alpha)/(n-1)^v}$	$\sqrt{(1 - \lambda_\alpha \sqrt{2v/\sqrt{n}})}$
50	0.696	0.656
70	0.745	0.720
100	0.788	0.772
200	0.852	0.845
500	0.907	0.905
1000	0.935	0.934

Since Tano and Vannman (2011) have supplied the actual quantile values for  $v=3$  and  $\alpha=0.05$  for us, we will make use of these in our power study of Pan and Lee's index. Again, consider the same set of hypotheses from the previous studies

$$H_0: MC_p = c \text{ (process is not capable) vs.}$$

$$H_a: MC_p > c \text{ (process is capable).}$$

We reject the null when the following inequality holds,

$$\frac{c}{\sqrt{w_\alpha}} < \widehat{MC}_p.$$

Hence, the expression for power is given by

$$\pi(MC_p) = P\left\{\widehat{MC}_p > \frac{c}{\sqrt{w_\alpha}} \middle| MC_p\right\}. \quad (3.5)$$

For the simulation study of  $MC_p$  we will use the same hypotheses ( $c$  values), mean vector, covariance matrices, and specification intervals from the studies above. For each example we simulate 10,000 samples of size  $n=50$  with  $\alpha=0.05$  from a trivariate normal distribution with the parameters from (3.3) but with the adjustments to the covariance matrix given in each example.

Example 1:

For the following matrix and specification intervals,

$$\Sigma = \begin{bmatrix} 0.0021 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix},$$

$$D \in (2.1, 2.3)$$

$$L \in (304.5, 305.1)$$

$$W \in (304.5, 305.1),$$

we obtain the population value of  $MC_p$  by applying equation (1.9),

$$MC_p = \left( \frac{|A^*|}{|\Sigma|} \right)^{1/2},$$

where the elements of  $A^*$  are given by

$$A_{ij}^* = \rho_{ij} \left( \frac{USL_i - LSL_i}{2\sqrt{\chi_{v,0.9973}^2}} \right) \left( \frac{USL_j - LSL_j}{2\sqrt{\chi_{v,0.9973}^2}} \right) \quad i \text{ and } j = 1, 2, \text{ or } 3,$$



and  $\rho_{ij}$  is computed from  $\Sigma$ . We obtain  $MC_p$  equal to 2.0. Simulation results appear below.

Table 3-14 Example 1 Pan and Lee

H0: process is not capable $\alpha = 0.05, n = 50$	Population $MC_p = 2.0$  Power
c=1.0	96%
c=1.33	64%
c=1.50	41%
c=1.67	24%

For this example the power should be higher for all values of c given a population index value of 2.0.

Example 2:

For the next covariance matrix,

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix},$$

$MC_p$  is calculate as 1.41 (using the same method as in the previous example), and we obtain the following simulation results:

Table 3-15 Example 2 Pan and Lee

H0: process is not capable $\alpha = 0.05, n = 50$	Population $MC_p = 1.41$  Power
c=1.0	52%
c=1.33	10%
c=1.50	3%
c=1.67	1%

In this example the power is in line with what we expect for an index value of 1.41. We repeat the procedure outlined above for the next two examples.

Example 3:

For the following matrix,

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0034 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix},$$

$MC_p$  is calculated as 1.0 and simulation results are the following:

Table 3-16 Example 3 Pan and Lee

H0: process is not capable $\alpha=0.05, n=50$	Population $MC_p=1.0$  Power
$c=1.0$	5%
$c=1.33$	0.15%
$c=1.50$	0.03%
$c=1.67$	0%

For this example the population index value indicates a “barely” capable process. The index performs well since we are unlikely to declare the process capable at any value of  $c$ . Furthermore, the power is 5% at  $c=1.0$ .

Example 4:

For the last variance covariance matrix,

$$\Sigma = \begin{bmatrix} 0.0063 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0040 \end{bmatrix},$$

$MC_p$  is calculated as 0.82 and simulation results are the following:

Table 3-17 Example 4 Pan and Lee

H0: process is not capable $\alpha=0.05, n=50$	Population $MC_p=0.82$  Power
$c=1.0$	0.46%
$c=1.33$	0%
$c=1.50$	0%
$c=1.67$	0%

For this example the population index value indicates an incapable process. The index performs well since we are unlikely to declare the process capable at any value of  $c$ .

Like Taam's index, Pan and Lee's index for example 4 indicates that the process is not capable which contradicts the results for our proposed indices. Pan and Lee's index is lower than Taam's for every example. This supports Pan and Lee's claim that Taam's index may overestimate capability when the variables are not all independent.

### 3.1.4 Wang and Chen's Index

Wang and Du (2000) propose the following lower confidence bound for Wang and Chen's index  $MC_p$ ,

$$\left( \prod_{i=1}^m \widehat{C}_{p,PC_i} \sqrt{\frac{\chi_{n-1, \alpha}^2}{n-1}} \right)^{1/m} .$$

Again, consider the familiar set of hypotheses,

$$H_0: MC_p = c \text{ (process is not capable) vs.}$$

$$H_a: MC_p > c \text{ (process is capable).}$$

We reject the null hypothesis when the following inequality holds,

$$c \sqrt{\frac{n-1}{\chi_{n-1, \alpha^2}}} < \widehat{MC}_p.$$

Hence, the expression for power is given by

$$\pi(MC_p) = P \left\{ \widehat{MC}_p > c \sqrt{\frac{n-1}{\chi_{n-1, \alpha^2}}} MC_p \right\}. \quad (3.6)$$

Again, we use the same hypotheses (c values), covariance matrices,  $\mu$  and specification intervals from the previous studies to do a power study of  $MC_p$ . For each example we simulate 10,000 samples of size  $n=50$  with  $\alpha=0.05$  from a trivariate normal distribution with the parameters from (3.3) but with the adjustments to the covariance matrix given in each example.

Example 1:

Consider the original data,

$$\Sigma = \begin{bmatrix} 0.0021 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix},$$

$$D \in (2.1, 2.3)$$

$$L \in (304.5, 305.1)$$

$$W \in (304.5, 305.1),$$

i.e.

$$X_{lo} = \begin{bmatrix} 2.1 \\ 304.5 \\ 304.5 \end{bmatrix} \text{ and } X_{up} = \begin{bmatrix} 2.3 \\ 305.1 \\ 305.1 \end{bmatrix}.$$

To obtain the population value of  $MC_p$  we perform PCA on the matrix above and apply equation (1.12),

$$MC_p = \left( \prod_{i=1}^m C_{p;PC_i} \right)^{1/m},$$

where

$$C_{p;PC_i} = \frac{USL_{PC_i} - LSL_{PC_i}}{6\sqrt{\lambda_i}} \text{ for } i = 1, 2, \dots, m \text{ (} 1 \leq m \leq v \text{),}$$

with

$$USL_{PC_i} = e_i' X_{up},$$

and,

$$LSL_{PC_i} = e_i' X_{lo}.$$

Recall,  $m$  represents the number of principal components retained for analysis from a matrix of  $v$  dimensions. For this example and subsequent examples  $m$  equals two since the first two principal components account for nearly 90% of the total variance in each matrix.

For the matrix above we obtain  $MC_p$  equal to 1.67. Simulation results are the following:

Table 3-18 Example 1 Wang and Chen

H0: process is not capable $\alpha=0.05, n=50$	Population $MC_p=1.67$  Power
$c=1.0$	87%
$c=1.33$	55%
$c=1.50$	31%
$c=1.67$	11%

For this example the power is reasonable for the first three  $c$  values given an index of 1.67. However, when  $c$  equals 1.67 the probability of declaring the process

capable should perhaps be lower than 5% given that the population value of the index itself is equal to 1.67. This could be attributed to the incorrect transformation of the specification limits. We repeat the procedure described above for examples 2-4.

Example 2:

For the next variance covariance matrix,

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix},$$

$MC_p$  is calculated as 1.75. Simulation results are the following:

Table 3-19 Example 2 Wang and Chen

H0: process is not capable $\alpha=0.05, n=50$	Population $MC_p=1.75$  Power
$c=1.0$	97%
$c=1.33$	74%
$c=1.50$	23%
$c=1.67$	2%

In this example the power is reasonable for the first two values of  $c$  but somewhat low for the last two values of  $c$  given an index value of 1.75.

Example 3:

For the following matrix,

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0034 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix},$$

$MC_p$  is calculated as 1.48 and simulation results are the following:

Table 3-20 Example 3 Wang and Chen

H0: process is not capable $\alpha=0.05, n=50$	Population $MC_p=1.48$  Power
c=1.0	77%
c=1.33	15%
c=1.50	1.4%
c=1.67	0%

In this example the power is in line with what we would expect for all values of  $c$ , except at  $c=1.50$  where the power should be close to 5%.

Example 4:

For the last variance covariance matrix,

$$\Sigma = \begin{bmatrix} 0.0063 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0040 \end{bmatrix},$$

$MC_p$  is calculated as 1.31 and simulation results are the following:

Table 3-21 Example 4 Wang and Chen

H0: process is not capable $\alpha=0.05, n=50$	Population $MC_p=1.31$  Power
c=1.0	58%
c=1.33	0.4%
c=1.50	0%
c=1.67	0%

For this example the power is in line with what we would expect for all values of  $c$ , except at  $c=1.33$  where the power should be close to 5%.

Note the inconsistency in index values between examples 1 and 2. Only the first two eigenvectors are used to compute  $MC_p$  in each example, however, using all three eigenvectors does not remedy this problem. In fact, when using all three eigenvectors the index for example 4 is larger than it is for example 3, and, they are both larger than the index values for examples 1 and 2 (see below). Using two eigenvectors for some examples and three for others is also not a solution. It seems that we cannot attribute the problem to the number of eigenvectors selected for analysis.

Table 3-22 Two vs. Three Eigenvectors

Example	$MC_p$ w/ two eigenvectors	$MC_p$ w/ three eigenvectors
1	1.67	1.11
2	1.75	1.07
3	1.48	1.32
4	1.31	1.39

### 3.1.5 Wang's Index

Tano and Vannman (2011) propose the following lower confidence bound for Wang's index  $MWC_p$ ,

$$\left( \prod_{i=1}^m \left( \widehat{C}_{p,PC_i} \sqrt{\frac{\chi_{n-1,\alpha}^2}{n-1}} \right)^{\lambda_i} \right)^{1/\sum \lambda_i} .$$

Under the following set of hypotheses,

$$H_0: MWC_p = c \text{ (process is not capable) vs.}$$

$$H_a: MWC_p > c \text{ (process is capable),}$$

we reject  $H_0$  when the following inequality holds,



$$c \sqrt{\frac{n-1}{\chi_{n-1, \alpha}^2}} < \widehat{MWC}_p.$$

Thus, the expression for power for any particular value of  $MWC_p$  is given

by the following,

$$\pi(MWC_p) = P \left\{ \widehat{MWC}_p > c \sqrt{\frac{n-1}{\chi_{n-1, \alpha}^2}} MWC_p \right\}. \quad (3.7)$$

We will use the same data from the previous studies to investigate power for  $MWC_p$ . For each example we will simulate 10,000 samples of size  $n=50$  with  $\alpha=0.05$  from a trivariate normal distribution with the parameters from (3.3) but with the adjustments to the covariance matrix given in each example.

Example 1:

For the following matrix and specification intervals,

$$\Sigma = \begin{bmatrix} 0.0021 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix},$$

$$D \in (2.1, 2.3)$$

$$L \in (304.5, 305.1)$$

$$W \in (304.5, 305.1),$$

we calculate the population value of  $MWC_p$  by performing PCA on  $\Sigma$  (the PCA calculations have already been made while computing Wang and Chen's index) and applying equation (1.13),

$$MWC_p = \left( \prod_{i=1}^m C_{p:PC_i} \lambda_i \right)^{\frac{1}{\sum_{i=1}^m \lambda_i}}.$$

Because our data is multivariate normal we do not use Clement's method for the denominator of  $C_{p;PC_i}$ . As with Wang and Chen's index,

$$C_{p;PC_i} = \frac{USL_{PC_i} - LSL_{PC_i}}{6\sqrt{\lambda_i}},$$

with

$$USL_{PC_i} = e_i' X_{up},$$

and,

$$LSL_{PC_i} = e_i' X_{lo}.$$

Note, we have already obtained the values of  $C_{p;PC_i}$  by calculating Wang and Chen's index.

For this example we obtain  $MWC_p$  equal to 1.90. Simulation results are the following:

Table 3-23 Example 1 Wang

H0: process is not capable $\alpha=0.05, n=50$	Population $MWC_p=1.90$  Power
c=1.0	97%
c=1.33	85%
c=1.50	61%
c=1.67	21%

For this example the power seems to be in line with what we expect for a population index value of 1.90. We repeat the procedure above for examples 2-4.

Example 2:

For the next covariance matrix,

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix},$$

$MWC_p$  is calculated as 1.65. Simulation results are the following:

Table 3-24 Example 2 Wang

H0: process is not capable $\alpha=0.05, n=50$	Population $MWC_p=1.65$  Power
c=1.0	94%
c=1.33	46%
c=1.50	9%
c=1.67	0.49%

For this example power is a bit small for the second and third value of c considering the population index value. For c=1.67 the power is not as close to 5% as it should be. As in the previous index, this could be attributed to the incorrect transformation of the specification limits or the fact that the weights are estimated.

Example 3:

For the following matrix,

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0034 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix},$$

$MWC_p$  is calculated as 1.52 and simulation results are the following:

Table 3-25 Example 3 Wang

H0: process is not capable $\alpha=0.05, n=50$	Population $MWC_p=1.52$  Power
c=1.0	83%
c=1.33	12%
c=1.50	0.7%
c=1.67	0%

In this example the power is appropriately low for the last value of c but should be larger for the first two values of c. For c=1.50 the power should be closer to 5%.

Example 4:

For the last variance covariance matrix,

$$\Sigma = \begin{bmatrix} 0.0063 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0040 \end{bmatrix},$$

$MWC_p$  is calculated as 1.24 and simulation results are the following:

Table 3-26 Example 4 Wang

H0: process is not capable $\alpha=0.05, n=50$	Population $MWC_p=1.24$  Power
c=1.0	41%
c=1.33	0.10%
c=1.50	0%
c=1.67	0%

For this last example the power is in line with what we expect for an index value of 1.24.

### 3.2 Additional Index Computations

Though not all previously mentioned indices are included in the power study above, we may still calculate their population index values and compare them with the values of our own proposed MPCIs. Using the data from examples 1-4 above, we determine the values for the indices by Xekelaki and Perakis, Chan et al., Shahriari and Tano and Vannman (see equations (1.5) and (1.15)-(1.17)). The population value of these indices, along with all indices previously discussed, appear in the table below.

Table 3-27 Additional Index Computations

Example	Xekelaki and Perakis	Chan, Cheng and Spiring*	Shahriari	Tano and Vannman	MC1	MC2	MC3	$C_{pv}$	Taam	Pan and Lee	Wang and Chen	Wang
1	1.96	1.0447	2.14	0.73	2.20	2.15	2.46	2.46	2.92	2.00	1.67	1.90
2	1.67	1.0436	1.86	0.51	2.02	1.48	2.07	2.07	1.96	1.41	1.75	1.65
3	1.53	1.0441	1.49	0.51	1.91	1.64	2.06	2.06	1.17	1.00	1.48	1.52
4	1.27	1.0447	1.39	0.42	1.77	1.25	1.65	1.65	0.95	0.82	1.31	1.24

\*Chan, Cheng and Spiring's index requires data to compute so we simulate a single data set of size 50 with the given parameters ( $\mu$  and  $\Sigma$ ) for each example.

The values for Chan et al.'s index stand out from the rest for two reasons. For one, they follow no apparent trend while the other indices decrease consistently from one example to another, except in two cases. Second, unlike the other index values, Chan et al.'s index values are equal to one another when rounded to the nearest hundredth. This suggests that Chan et al.'s index is not sensitive to changes in parameter. Shahriari's index values follow a trend similar to that of our own as well as Wang and Chen and Wang. In other words, at each example we would draw similar conclusions from examining our indices, Shahriari's index, Wang and Chen's index or Wang's index. Finally, Tano and Vannman's index values appear to follow a trend, however, the authors have given us no guidelines for interpreting the results.

### 3.3 Bootstrap Confidence Intervals

When the theoretical distribution of a statistic is complicated one may use what are known as “bootstrap techniques” to derive confidence intervals and other properties of interest. The bootstrap method is free from distributional assumptions and will allow us to compute approximate 95% confidence intervals for MC2 and MC3. Bootstrapping methods involve using sample data to approximate populations and were first introduced by Bradley Efron in the 1970’s. Since then many different versions of the bootstrap have been developed. The first step in a bootstrap method is to resample an original sample with replacement many times. For each resample a statistic of interest is computed. For our study the statistics of interest are MC2 and MC3 and we will use the same covariance matrices,  $\mu$  and specification intervals that we did in the previous chapter to obtain 95% bootstrap confidence intervals. Again, we are simulating from a trivariate distribution with the parameters specified in (3.3) but for adjustments to the covariance matrix made in each example.

Formulation of the confidence intervals depends on the particular bootstrap method that we choose. We intend to use the standard bootstrap method and the bias corrected percentile bootstrap (BCPB) method. The standard bootstrap confidence interval is calculated as follows:

1. Collect an original sample and calculate  $\widehat{MC2}$  (or  $\widehat{MC3}$ ).
2. Resample the original sample with replacement many times, computing  $\widehat{MC2}$  (or  $\widehat{MC3}$ ) for each resample, these estimates are denoted  $\widehat{MC2}^*$  (or  $\widehat{MC3}^*$ ).
3. Compute the standard deviation of the  $\widehat{MC2}^*$ s (or  $\widehat{MC3}^*$ s).

A single  $100(1-\alpha)\%$  standard bootstrap confidence interval for MC2 is given by

$$\widehat{MC2} \pm z_{\alpha/2} \cdot (\text{standard deviation of } \widehat{MC2}^* \text{'s}) \quad (3.8)$$

where  $z_{\alpha/2}$  is the  $(1 - \alpha/2)$  percentile of the  $N(0,1)$  distribution.

The formula for the confidence interval for MC3 is similar to equation (3.7). Simply replace  $\widehat{MC2}$  and  $\widehat{MC2}^*$  with  $\widehat{MC3}$  and  $\widehat{MC3}^*$ .

We calculate a BCPB confidence interval as follows:

(note steps 1 and 2 are the same as in the standard bootstrap method)

1. Collect an original sample and calculate  $\widehat{MC2}$  (or  $\widehat{MC3}$ ).
2. Resample the original sample with replacement many times, computing  $\widehat{MC2}$  (or  $\widehat{MC3}$ ) for each resample, these estimates are denoted  $\widehat{MC2}^*$  (or  $\widehat{MC3}^*$ ).
3. Determine what percent ( $p$ ) of estimates  $\widehat{MC2}^*$  (or  $\widehat{MC3}^*$ ) are less than  $\widehat{MC2}$  (or  $\widehat{MC3}$ ), respectively.
4. Compute  $\Phi^{-1}(p)$ , where  $\Phi^{-1}$  is the inverse of the standard normal cumulative distribution function.
5. Calculate  $2\Phi^{-1}(p) \pm z_{\alpha/2}$  and then determine the values of  $\Phi(2\Phi^{-1}(p) \pm z_{\alpha/2})$ ,  
where  $\Phi$  is the standard normal cumulative distribution function. Denote these values  $z_1$  and  $z_2$ .
6. Multiply  $z_1$  and  $z_2$  by the number of estimates of  $\widehat{MC2}^*$  (or  $\widehat{MC3}^*$ ) and then locate these positions in the ordered list (ascending) of values of  $\widehat{MC2}^*$  (or  $\widehat{MC3}^*$ ). The values at each location are denoted  $P_L$  and  $P_U$ , for lower and upper, respectively.

A single  $100(1-\alpha)\%$  BCPB bootstrap confidence interval for MC2 or MC3 is given by

$$(P_L, P_U). \quad (3.9)$$

Under the standard method we will draw 1000 “original” samples of size 50 from each population and resample each “original” sample 2000 times. Under the BCPB method we will draw 1000 samples of size 50 from each population and resample each sample 1000 times. The population index values for each example have already been

computed in chapter 3. We will determine the percent of 95% bootstrap confidence intervals that cover the population value of the index. We will also determine the average width and standard deviation of the widths of the 1000 estimated intervals. The bias, variance and MSE for 1,000,000 bootstrap estimates of both indices are included.

Results follow.

Example 1:

$$\Sigma = \begin{bmatrix} 0.0021 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}$$

Table 3-28 Bootstrap Example 1

Index	Pop value	Percent coverage		Average width of confidence intervals		Standard deviation of the widths		Bias	Var	MSE
		Standard	BCPB	Standard	BCPB	Standard	BCPB			
MC2	2.15	94%	93%	0.93	0.93	0.20	0.20	0.00	0.11	0.11
MC3	2.46	93%	93%	1.00	1.01	0.20	0.19	-0.05	0.12	0.12

$\alpha = 95\%$ ,  $n = 50$ , 1000 bootstrap confidence intervals

Example 2:

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}$$

Table 3-29 Bootstrap Example 2

Index	Pop value	Percent coverage		Average width of confidence intervals		Standard deviation of the widths		Bias	Var	MSE
		Standard	BCPB	Standard	BCPB	Standard	BCPB			
MC2	1.48	93%	93%	0.99	0.96	0.21	0.19	0.04	0.13	0.13
MC3	2.07	93%	91%	1.26	1.17	0.39	0.34	-0.17	0.18	0.21

$\alpha = 95\%$ ,  $n = 50$ , 1000 bootstrap confidence intervals

Example 3:

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0034 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}$$



Table 3-30 Bootstrap Example 3

Index.	Pop value	Percent coverage		Average width of confidence intervals		Standard deviation of the widths		Bias	Var	MSE
		Standard	BCPB	Standard	BCPB	Standard	BCPB			
MC2	1.64	94%	93%	0.87	0.87	0.19	0.18	0.16	0.10	0.13
MC3	2.06	91%	89%	1.06	0.97	0.35	0.29	-0.24	0.12	0.17

$\alpha = 95\%$ ,  $n = 50$ , 1000 bootstrap confidence intervals

Example 4:

$$\Sigma = \begin{bmatrix} 0.0063 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0040 \end{bmatrix}$$

Table 3-31 Bootstrap Example 4

Index	Pop value	Percent coverage		Average width of confidence intervals		Standard deviation of the widths		Bias	Var	MSE
		Standard	BCPB	Standard	BCPB	Standard	BCPB			
MC2	1.25	94%	93%	0.82	0.81	0.17	0.16	0.03	0.09	0.09
MC3	1.65	93%	88%	1.55	1.03	0.36	0.33	-0.23	0.15	0.20

$\alpha = 95\%$ ,  $n = 50$ , 1000 bootstrap confidence intervals

For MC2 the coverage percentage for bootstrap confidence intervals of both types (standard and BCPB) are close to 95% for all examples. For MC3 the coverage percentage for standard bootstrap confidence intervals is close to 95%, 91% coverage being the smallest. The coverage percentage for MC3 under the BCPB method is not as high as it is with the standard method, with 88% being the smallest coverage percent. Overall, for both indices, the coverage percentage is close to 95% using the standard bootstrap confidence interval. The BCPB confidence interval performs almost as well as the standard interval for MC2, but it does not perform as well as the standard interval for MC3. MSE for MC2 is consistently smaller than it is for MC3, however, MSE of both indices is never larger than 0.20.

## Chapter 4

### Extension to $C_{pk}$

The predominant shortcoming of  $C_p$  is that it does not take process mean into consideration. In an effort to overcome this deficiency the capability index  $C_{pk}$  was developed. When used together  $C_p$  and  $C_{pk}$  can give a good indication of process capability with regard to process center and spread. Consider the following expressions,

$$C_{pu} = \frac{USL - \mu}{3\sigma} \text{ and } C_{pl} = \frac{\mu - LSL}{3\sigma}. \quad (4.1)$$

The expressions above represent the ratio of the distance between the mean of the process and the upper (or lower) specification limit to three times the process standard deviation. In simpler terms, the expressions above give the ratio of the “allowable” upper spread to the “actual” upper spread as well as the “allowable” lower spread to the “actual” lower spread.

$C_{pk}$  is defined as the minimum of these two quantities, i.e.

$$C_{pk} = \min\{C_{pu}, C_{pl}\}. \quad (4.2)$$

“From observing the definition of  $C_{pk}$ , it is apparent that  $C_{pk}$  quantifies capability for the worst half of the data,” Kotz and Lovelace (1998). Gunter (1989) describes  $C_{pk}$  as “a way to measure the ratio of the amount of room needed to the amount of room available to produce product within specifications.”

Extending our proposed indices to calculate a multivariate version of  $C_{pk}$  is a fairly simple task. As before assume that all process data follow the multivariate normal distribution, that is,  $X \sim N_p(\mu, \Sigma)$ . The  $C_{pk}$  counterparts of our indices are as follows:

For MC1 we calculate

$$\min \left\{ \frac{USL_{MC1} - \mu_{MC1}}{3\sigma_{MC1}}, \frac{\mu_{MC1} - LSL_{MC1}}{3\sigma_{MC1}} \right\}, \quad (4.3)$$

where

$$\mu_{MC1} = [1 \quad 1 \dots 1] \cdot \mu.$$

For MC2 we calculate

$$\min \left\{ \frac{USL_{MC2} - \mu_{MC2}}{3\sigma_{MC2}}, \frac{\mu_{MC2} - LSL_{MC2}}{3\sigma_{MC2}} \right\}, \quad (4.4)$$

where

$$\mu_{MC2} = w' \cdot \mu.$$

For MC3 we calculate

$$\min \left\{ \frac{USL_U - \mu_U}{3\sigma_U}, \frac{\mu_U - LSL_U}{3\sigma_U} \right\}, \quad (4.5)$$

where

$$\mu_U = |w_1 e'_1 \mu| + |w_2 e'_2 \mu| + \dots + |w_k e'_k \mu|.$$

And, for  $C_{pv}$  we calculate

$$\min \left\{ \frac{USL_V - \mu_V}{3\sigma_V}, \frac{\mu_V - LSL_V}{3\sigma_V} \right\}, \quad (4.6)$$

where

$$\mu_V = |w_{1_0} e_{1_0}' \mu| + |w_{2_0} e_{2_0}' \mu| + \dots + |w_{k_0} e_{k_0}' \mu|.$$

For their estimators we make the necessary substitutions described in chapter 2 and we naturally replace all  $\mu$ 's with  $\bar{X}$ . We can also feasibly derive properties and develop confidence intervals by looking to their univariate counterparts.

As far as recommended minimum values of  $C_{pk}$  are concerned the same benchmarks for  $C_p$  apply. That is, 1.33 is still the standard minimal index value and critical processes typically require a value of 1.67 or higher. It is likely that these same values would apply to the multivariate extension of  $C_{pk}$ .

We will demonstrate an extension to  $C_{pk}$  using MC1. As stated above, the extension of MC1 to  $C_{pk}$  (we may call it  $MC1_k$ ) will look like the following,

$$MC1_k = \min \left\{ \frac{USL_{MC1} - \mu_{MC1}}{3\sigma_{MC1}}, \frac{\mu_{MC1} - LSL_{MC1}}{3\sigma_{MC1}} \right\}, \quad (4.7)$$

where

$$\mu_{MC1} = [1 \quad 1 \dots 1] \cdot \mu.$$

The estimator will be

$$\widehat{MC1}_k = \min \left\{ \frac{USL_{MC1} - \bar{X}_{MC1}}{3S_{MC1}}, \frac{\bar{X}_{MC1} - LSL_{MC1}}{3S_{MC1}} \right\}, \quad (4.8)$$

where

$$\bar{X}_{MC1} = [1 \quad 1 \dots 1] \cdot \bar{X}.$$

Another way to write  $\widehat{MC1}_k$  is the following,

$$\widehat{MC1}_k = \frac{d_{MC1} - |\bar{X}_{MC1} - M_{MC1}|}{3S_{MC1}}, \quad (4.9)$$

where

$$d_{MC1} = \frac{USL_{MC1} - LSL_{MC1}}{2} \text{ and } M_{MC1} = \frac{USL_{MC1} + LSL_{MC1}}{2}.$$

We set  $|\bar{X}_{MC1} - M_{MC1}| = \frac{\sigma_{MC1}}{\sqrt{n}} |Z + \delta_{MC1}|$  with  $Z \sim N(0,1)$  and  $\delta_{MC1} = \frac{\mu_{MC1} - M_{MC1}}{\sigma_{MC1}/\sqrt{n}}$ , (Kotz and Johnson (1993)). Assuming multivariate normal data we can derive the moments of

$\widehat{MC1}_k$  using the fact that  $\frac{(n-1)s_{MC1}^2}{\sigma_{MC1}^2} \sim \chi_{n-1}^2$  and that  $|Z + \delta_{MC1}|$  follows the folded normal distribution (see Kotz and Johnson (1993)). The expected value of  $\widehat{MC1}_k$  is given by

$$E(\widehat{MC1}_k) = \frac{1}{3} \sqrt{\frac{n-1}{2\sigma_{MC1}^2}} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \left[ d_{MC1} - \frac{\sigma_{MC1}}{\sqrt{n}} (2\Phi(-\delta_{MC1}) + \delta_{MC1} - 2\delta_{MC1}\Phi(-\delta_{MC1})) \right].$$

$$\text{bias}(MC1_k) = E(\widehat{MC1}_k) - MC1_k \quad (4.10)$$

The variance of  $\widehat{MC1}_k$  is given by

$$\text{var}(\widehat{MC1}_k) = E(\widehat{MC1}_k^2) - [E(\widehat{MC1}_k)]^2, \quad (4.11)$$

with

$$E(\widehat{MC1}_k^2) = \frac{n-1}{9(n-3)\sigma_{MC1}^2} \times \left\{ d_{MC1}^2 - 2d_{MC1} \frac{\sigma_{MC1}}{\sqrt{n}} [2\Phi(-\delta_{MC1}) + \delta_{MC1} - 2\delta_{MC1}\Phi(-\delta_{MC1})] + \frac{\sigma_{MC1}^2}{n} (1 + \delta_{MC1}^2) \right\}.$$

$$\text{MSE}(\widehat{MC1}_k) = \text{var}(\widehat{MC1}_k) + (\text{bias}(MC1_k))^2 \quad (4.12)$$

Using the normal approximation, Bissell (1990) derives a confidence interval for  $C_{pk}$

which we may extend to  $MC1_k$ . It is given by the following

$$\widehat{MC1}_k \pm z_{\alpha/2} \sqrt{\frac{1}{9n} + \frac{1}{2(n-1)} \widehat{MC1}_k^2}. \quad (4.13)$$

We can use the confidence interval above to form a decision rule and tests hypotheses for  $MC1_k$ . For instance, consider the hypotheses

$$H_0: MC1_k = c \text{ (process is not capable) vs.}$$

$H_a: MC1_k > c$  (process is capable).

We reject the null when the following inequality holds

$$c + z_\alpha \sqrt{\frac{1}{9n} + \frac{1}{2(n-1)} \widehat{MC1}_k^2} < \widehat{MC1}_k. \quad (4.14)$$

Using the decision rule and hypotheses above we can conduct another simulation study of power. We will use the same examples from chapter 3 and conduct 10,000 runs for each value of  $c$  ( $n=50$ ). Bias, variance and MSE can be calculated using equations (4.11), (4.10) and (4.12), respectively. Recall, the specification intervals and  $\mu$  remain the same for each example and are given by

$$\mu = \begin{bmatrix} 2.16 \\ 304.72 \\ 304.77 \end{bmatrix},$$

and,

$$D \in (2.1, 2.3),$$

$$L \in (304.5, 305.1),$$

$$W \in (304.5, 305.1),$$

with centers

$$D: 2.20,$$

$$L: 304.80,$$

$$W: 308.80.$$

The distances of the mean from the centers are given by

$$D: 0.04,$$

$$L: 0.08,$$

W: 0.03.

Note that for all examples the process mean will be 0.08 away from the actual center of each specification interval at most.

Example 1:

We have

$$\Sigma = \begin{bmatrix} 0.0021 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}.$$

Table 4-1 Example 1  $C_{pk}$

H0: process is not capable $\alpha=0.05, n=50$	Population $MC1_k=1.73$  Power
c=1.0	100%
c=1.33	79%
c=1.50	37%
c=1.67	10%

Example 2:

We have

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}.$$

Table 4-2 Example 2  $C_{pk}$

H0: process is not capable $\alpha=0.05, n=50$	Population $MC1_k=1.61$  Power
c=1.0	100%
c=1.33	49%
c=1.50	13%
c=1.67	2%

Example 3:

We have

$$\Sigma = \begin{bmatrix} 0.0042 & 0.0008 & 0.0007 \\ 0.0008 & 0.0034 & 0.0012 \\ 0.0007 & 0.0012 & 0.0020 \end{bmatrix}.$$

Table 4-3 Example 3  $C_{pk}$

H0: process is not capable $\alpha=0.05, n=50$	Population $MC1_k=1.50$  Power
c=1.0	99%
c=1.33	28%
c=1.50	5%
c=1.67	1%

Example 4:

We have

$$\Sigma = \begin{bmatrix} 0.0063 & 0.0008 & 0.0007 \\ 0.0008 & 0.0017 & 0.0012 \\ 0.0007 & 0.0012 & 0.0040 \end{bmatrix}.$$

Table 4-4 Example 4  $C_{pk}$

H0: process is not capable $\alpha=0.05, n=50$	Population $MC1_k=1.48$  Power
c=1.0	92%
c=1.33	11%
c=1.50	1%
c=1.67	0%

For examples 1-4 the power is reasonably in line with what we expect for each population index value. The index tends to perform best at c equals 1.0 and 1.67. In example 4, for c=1.50 the power should be closer to 5%. We note that Bissell's



confidence interval for  $C_{pk}$  is an approximation (construction of an exact confidence interval for  $C_{pk}$  is very complicated so most confidence intervals for  $C_{pk}$  are approximate). In each example the population index value indicates a capable process (beyond marginally capable) and at each case of  $c=1.0$  we are highly likely to declare the process capable. In the case of  $c=1.67$  the power is appropriately low. For each example the process is capable but not "highly capable" and we are unlikely to reject the null hypothesis, which is the correct decision.

## Chapter 5

### Conclusion

When a product possesses multiple quality characteristics that are not independent multivariate techniques are required. The need for indices that assess multivariate capability is obvious. Many multivariate process capability indices have been proposed over the last 25 years but few have accompanying confidence intervals. As Tano and Vannman (2011) assert, "In practice, a random sample is used to estimate the process capability index. Hence, conclusions about process capability must be based on a confidence interval or a test for the MPCl." In their paper, Tano and Vannman (2011) review four MPCIs that have corresponding confidence intervals. They find that the index formulated by Pan and Lee is superior to the others, but that it has limitations. In a later paper Tano and Vannman (2013) propose their own index but restrict its application to a bivariate setting.

We propose four new indices that can translate the multivariate situation into the routine univariate capability index. Under the assumption of multivariate normality this simplifies our derivation of properties and confidence intervals, as well as hypothesis tests. Furthermore, we have formulated our indices in such a way that a specific index value can be associated with a particular probability of non-conformance, an attribute which Pan and Lee's index lacks.

We investigate the feasibility of our indices by conducting hypothesis tests and computing power. We include in this study indices from other authors which have accompanying confidence intervals so that we may compare with our own. Results are summarized in the table below.

Table 5-1 Summary of Results

Example 1								
H0: process is not capable $\alpha=0.05, n=50$	Population MC1=2.20	Population MC2=2.15	Population MC3=2.46	Population $C_{pv}=2.46$	Taam Population $MC_p=2.92$	Pan and Lee Population $MC_p=2.0$	Wang and Chen Population $MC_p=1.67$	Wang Population $MWC_p=1.90$
	Power	Power	Power	Power	Power	Power	Power	Power
c=1.0	100%	100%	100%	100%	100%	96%	87%	97%
c=1.33	100%	100%	99%	97%	99%	64%	55%	85%
c=1.50	99%	94%	90%	72%	96%	41%	31%	61%
c=1.67	84%	71%	64%	32%	87%	24%	11%	21%
Example 2								
H0: process is not capable $\alpha=0.05, n=50$	Population MC1=2.02	Population MC2=1.48	Population MC3=2.07	Population $C_{pv}=2.07$	Taam Population $MC_p=1.96$	Pan and Lee Population $MC_p=1.41$	Wang and Chen Population $MC_p=1.75$	Wang Population $MWC_p=1.65$
	Power	Power	Power	Power	Power	Power	Power	Power
c=1.0	100%	88%	100%	100%	96%	52%	97%	94%
c=1.33	100%	34%	89%	100%	55%	10%	74%	46%
c=1.50	89%	13%	72%	94%	30%	3%	23%	9%
c=1.67	55%	4%	46%	66%	13%	1%	2%	0.49%
Example 3								
H0: process is not capable $\alpha=0.05, n=50$	Population MC1=1.91	Population MC2=1.64	Population MC3=2.06	Population $C_{pv}=2.06$	Taam Population $MC_p=1.17$	Pan and Lee Population $MC_p=1.0$	Wang and Chen Population $MC_p=1.48$	Wang Population $MWC_p=1.52$
	Power	Power	Power	Power	Power	Power	Power	Power
c=1.0	100%	98%	97%	100%	13%	5%	77%	83%
c=1.33	97%	55%	75%	100%	0.5%	0.15%	15%	12%
c=1.50	74%	22%	55%	93%	0.05%	0.03%	1.4%	0.7%
c=1.67	34%	5%	27%	63%	0.01%	0%	0%	0%
Example 4								
H0: process is not capable $\alpha=0.05, n=50$	Population MC1=1.77	Population MC2=1.25	Population MC3=1.65	Population $C_{pv}=1.65$	Taam Population $MC_p=0.92$	Pan and Lee Population $MC_p=0.82$	Wang and Chen Population $MC_p=1.31$	Wang Population $MWC_p=1.24$
	Power	Power	Power	Power	Power	Power	Power	Power
c=1.0	100%	60%	85%	100%	1.5%	0.46%	58%	41%
c=1.33	87%	7%	37%	63%	0.01%	0%	0.4%	0.10%
c=1.50	46%	1%	12%	20%	0.01%	0%	0%	0%
c=1.67	13%	0.1%	2%	4%	0%	0%	0%	0%

In the power study of our indices, MC1 and  $C_{pv}$  outperform MC2 and MC3. Recall, performance is based on how likely we are to declare a process capable when the population index takes a particular value. For a large index value we wish for this power to be high and for a small index value we wish for it to be low. Besides outperforming

MC3 in power,  $C_{pv}$  is also simpler to calculate than MC3. In order to use  $C_{pv}$  we must utilize a covariance matrix that we have obtained from prior experience, however, in the absence of a prior covariance matrix, MC3 should make a satisfactory alternative. In our study of  $C_{pv}$  we have assumed that  $\Sigma = \Sigma_0$ , but the case of  $\Sigma \neq \Sigma_0$  may be something to investigate in the future. All of our indices are conservative at  $c=1.67$ , meaning that the probability of declaring a process capable is low, even though the population index value is high. This is not necessarily a drawback since the value 1.67 is typically reserved for new processes involving human safety, i.e. processes in which it is better to err on the side of caution. MC2 performs reasonably well in the power study, although the population index value for example 2 is smaller than it is for example 3. This runs somewhat contrary to what one might expect since the overall variance is higher in example 3 than it is in example 2. However, we are able to explain how this occurs in calculation. A similar situation arises between examples 3 and 4 with Wang and Chen's index, but the reason is not known. The places where MC2 and MC3 come up short in power may be attributed to the fact that the weights are estimated for both of these indices.

With our proposed MPCIs we have overcome some of the disadvantages of the other authors' indices. Namely, the following:

1. the incorrect transformation of specification limits found in Wang and Chen's and Wang's indices,
2. the absence of a threshold index value that pertains to a particular probability of non-conformance (Pan and Lee's index), and,
3. the lack of an accompanying confidence interval/bound ( indices by Chan et al., Shahriari and Tano and Vannman).

Our indices can be interpreted in the same fashion as the familiar univariate indices,  $C_p$ ,  $C_{pk}$ ,  $C_{pm}$ , etc. We have also maintained the same intuition behind our indices that is found in the index  $C_p$ . Each one is basically a ratio of "allowable" process spread

to “actual” process spread. Hence, our indices can be easily understood by those who lack a strong statistical background. Furthermore, we have demonstrated that all of our indices can be extended to a multivariate counterpart of  $C_{pk}$  with little to no difficulty if the need arises.

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## Biographical Information

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