THE INFLUENCE OF DYNAMIC VISUALIZATION ON UNDERGRADUATE CALCULUS LEARNING

by

JULIE M. SKINNER SUTTON

Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

August 2015

Acknowledgements

“If I have seen further, it is by standing upon the shoulders of giants” – Sir Isaac Newton.

I want express my deepest gratitude to my advisor, Dr. James Epperson, for his excellent guidance, his caring nature and his encouragement, especially on those days when I was not sure I could continue. I want to thank Dr. Theresa Jorgensen for always having an open door and a patient ear. If it were not for Dr. Minerva Cordero I would never have even thought about attending graduate school; her loving nature and kind words will never be forgotten. To my friend Dr. Katherine Rhoads, your willingness to read my drafts and to help me with expressing my thoughts succinctly helped make me a better written and, as a result, this document a better dissertation.

I want to thank my family, particularly my parents. Without their constant love, support, stern reminders, and childcare I know that completing my dissertation would have been much, much harder. Garrison, thank you for your selflessness to understand those long nights where I came home late and left early the next morning. You are always my biggest cheerleader. I love you more than I can express.

Thank you Jennifer Anderson Rich, Scott Lacy, Cavender Campbell, Rose Walczak, Amanda Sonnenburg Milani, Sarah Langford, and Jason & Staci Hammer for your humor, support, boardgames and calls checking in on progress.

I also want to acknowledge the National Science Foundation for the partial support of the Arlington Undergraduate Research Based Achievement in STEM (AURAS) program (DUE#08056796) which, in turn, supported this research.

July 13, 2015

Abstract

THE INFLUENCE OF DYNAMIC VISUALIZATION ON

UNDERGRADUATE CALCULUS LEARNING

Julie M. Skinner Sutton, PhD

The University of Texas at Arlington, 2015

Supervising Professor: James A. Mendoza Epperson

The urgent need in the United States to produce an additional one million graduates in the areas of science, technology, engineering and mathematics and previous research into why students choose to abandon their quest to earn such degrees motivates exploration of the appropriate use of technology that engages students in the first-year calculus classroom. Technology enhances our capacity to incorporate visualization into the teaching of mathematics, yet research suggests that high school mathematics students who prefer to visualize when solving mathematics tasks are not the “super stars” of their classrooms.

In this qualitative study, nine students, five of whom prefer to visualize when solving mathematics tasks, completed a series of four individual interviews focusing on topics covered in their calculus course. Four of the nine students used dynamic visualization software (DVS) during interview sequences whereas the remaining five students participated in analogous static interviews. Research questions guiding the study design focus on how student interactions with DVS result in differences in student views of derivative as a rate of change at a point, if students exploring mathematical relationships using DVS hold multi-representational views of derivative when compared to their peers who were not offered DVS as a learning tool, and how exploration with DVS influences student understanding of the Extreme Value Theorem and the relationship between continuity and differentiability.

Using grounded theory, results suggest that focusing features must accompany DVS exploration in the undergraduate calculus classroom. In addition, results raise questions about accurate assessment of transfer of knowledge when DVS is presented as a learning tool and not made available during testing. There were several instances where DVS exploration clarified mathematical relationships. Although findings suggest a higher incidence of this for visualizers, DVS exploration also enhanced the reasoning of those students who prefer not to visualize when solving mathematics tasks.

Table of Contents

Acknowledgements iii

Abstract iv

List of Tables xii

Chapter 1 Introduction 1

Chapter 2 Literature Review 4

2.1 Procedural vs. Conceptual Knowledge in Calculus 4

2.2 Cognitive Obstacles to Success in Calculus 8

2.3 Visualization in Calculus Learning 15

2.4 Intervention Programs: The Emerging Scholars Program 18

Chapter 3 Methodology 23

3.1 Student Population 23

3.2 Classification of Students by preference to Visualize 24

3.3 Interview Protocol Creation 25

3.3.1 Static Interview Protocol I 26

3.3.2 DVS Interview Protocol I 27

3.3.3 Static Interview Protocol II 29

3.3.4 DVS Interview Protocol II 31

3.3.5 Static Interview Protocol III 33

3.3.6 DVS Interview Protocol III 35

3.3.7 Exit Interview Protocol 35

3.4 Theoretical Perspective 37

3.4.1 Common Coding Schemes 40

Chapter 4 Results 53

4.1 Student Vignettes 56

4.1.1 Corbin 56

4.1.2 Beverly 62

4.1.3 Amy 67

4.1.4 Evan 71

4.1.5 David 77

4.1.6 Felicity 81

4.1.7 George 87

4.1.8 Harry 93

4.1.9 Ian 97

4.2 Interview Groups 101

4.2.1 DVS Interview Group 101

4.2.2 Static Interview Group 119

4.3 Comparison Between Interview Groups 141

4.4 Comparison Between Visualizers and Non-Visualizers 147

4.5 Student Group Comparisons 150

4.5.1 Beverly and Amy 150

4.5.2 Evan, George and David 153

4.5.3 Harry and Ian 158

Chapter 5 Discussion and Conclusions 161

5.1 Procedural Knowledge vs. Conceptual Knowledge. 161

5.1.1 Focusing Phenomena 162

5.1.2 Unsupported Conceptual Knowledge 165

5.2 An Across-Time view 170

5.3 Concept Image and Concept Definition 171

5.4 Supporting Visualizers 174

5.5 Insufficient Mathematical Vocabulary 179
5.6 Geometric and Algebraic Approaches 181
5.7 The EVT and Differentiability 182
5.8 Study Limitations 183
5.8.1 Study Participants 183
5.8.2 Delay of Interviews 184
5.9 Conclusion 184
5.9.1 Implications 184
5.9.2 Research Questions 186
Appendix A Student Participant Information 190
Appendix B Static Interview Protocol I 192
Appendix C Dynamic Interview Protocol I 196
Appendix D Static Interview Protocol II 199
Appendix E Dynamic Interview Protocol II 202
Appendix F Static Interview Protocol III 205
Appendix G Dynamic Interview Protocol III 208
Appendix H Exit Interview Protocol 211
Appendix I Interview Invitation 213
Appendix J Fall 2013 MATH 1421 Course Syllabus 215
Appendix K Fall 2013 ESP Syllabus 224
Appendix L Baseline Demographic Survey 229
References 234
Biographical Information 240

List of Figures

Figure 3.1 Snapshot of Sketch I from DVS Interview I. 27

Figure 3.2 Snapshot of Sketch 2 from DVS Interview I. 28

Figure 3.3 Snapshot of Sketch 3 from DVS Interview I. 29

Figure 3.4 Task 1 from static Interview II. 30

Figure 3.5 Snapshot of Sketch 3 from DVS Interview II. 32

Figure 3.6 Graph of the piecewise-defined function used for both static and DVS Interview III. 34

Figure 3.7 The graphs presented to student for analysis and comparison in the Exit Interview. 36

Figure 3.8 Amy's drawing illustrating her understanding of why the derivative value does not exist for a point making a sharp corner. 42

Figure 4.1 Flow chart illustrating the comparisons made between interview participants. 53

Figure 4.2 Flow chart illustrating how each type of interview was analyzed and how the different types of interview were compared. 54

Figure 4.3 Evidence of Corbin Visualizing during Interview I. 58

Figure 4.4 Corbin's visualizations as he reasons about the existence of the derivative at a point making a sharp corner. 60

Figure 4.5 Beverly's sketch describing her understanding of derivative. 64

Figure 4.6 Amy's sketch relating known values of the instantaneous rate of change at discrete points to the derivative graph. 69

Figure 4.7 Evan's sketch describing why the derivative at a cusp does not exist. 72

Figure 4.8 Evan's sketch of the "limit of the secant lines." 76

Figure 4.9 David's illustration of a tangent line. 81

Figure 4.10 Felicity's illustration describing her understanding of the relationship between average rate of change and instantaneous rate of change at a point. 84

Figure 4.11 George's sketch accompanying his explanation of the relationship between average rate of change and instantaneous rate of change at a point. 90

Figure 4.12 George's sketch as he reasons about $f(x+h)$. 93

Figure 4.13 Ian's sketch relating secant and tangent lines. 99

Figure 4.14 Screenshot of DVS Interview II. 106

Figure 4.15 Sketch of a discontinuous function, defined on a closed interval, which does not achieve its minimum function value. 113

Figure 4.16 Screenshot of DVS Interview III. 117

Figure 4.17 Corbin's work to find $f'x$. 122

Figure 4.18 Corbin's incorrect sketch of $f(x)$ and $f'(x)$ from Task 2 of static Interview I. 124

Figure 4.19 Evan's work from Interview I. 126

Figure 4.20 George's illustration about the relationship between average rate of change and instantaneous rate of change at a point. 128

Figure 4.21 George's notations from Interview III. 130

Figure 4.22 Evan's work with $f(x) = \sin x$. 132

Figure 4.23 Evan's drawings from Exit Interview task 4. 133

Figure 4.24 Interviewer illustration of a function defined on a closed interval that fails to achieve either its maximum or minimum function values. 135

Figure 4.25 The result of Evan's reasoning about the difference quotient. 137

Figure 4.26 Markings from Evan's interview III. 139

Figure 4.27 Corbin's markings for Interview III. 146

Figure 4.28 Beverly's example of a maximum function when the derivative is undefined. 153

Figure 4.29 David's sketch as he reasons about a function's derivative value at the point he circled. 156

Figure 4.30 George's work as he struggles to decide if an endpoint of an open interval can correspond to a function's maximum value. 157

Figure 4.31 Ian's sketch of a continuous function that does not attain its maximum of minimum function value. 159

Figure 5.1 Screenshot of Sketch 1 from DVS Interview II. 176

Figure 5.2 David's illustration when asked what he recalled in Interview I. 177

List of Tables

Table 3.1 Student participant distribution 25

Table 4.1 Reference for commonly discussed codes 55

Table 4.2 Seven most common codes within the DVS Interviews 102

Table 4.3 List of codes co-occurring with student understanding of instantaneous rate of change in the DVS interview group. 104

Table 4.4 Co-occurring codes in DVS interviews with procedural knowledge. 110

Table 4.5 Co-occurring codes with conceptual knowledge within DVS interviews. 115

Table 4.6 Five most common coding themes present in the static interviews. 120

Table 4.7 Common co-occurring codes with procedural knowledge within the static interviews. 122

Table 4.8 Common co-occurring codes with evidence of visualization for the static interview group.. 129

Table 4.9 Common codes co-occurring with concept image within the static interview group. 136

Table 4.10 Codes most commonly co-occurring with conceptual knowledge within the static interviews. 139

Table 4.11 List of high frequency codes for the DVS and Static interview group comparison 141

Table 4.12 High frequency codes for visualizer vs. non-visualizer comparison. 148

Table 5.1 Co-Occurring codes with student understanding of instantaneous rate of change 162

Table A.1 Student participant information 191

Chapter 1

Introduction

Improving student success in calculus is crucial in light of a predicted shortage of science, technology, engineering and mathematics (STEM) graduates in the United States. Statistics provided in 2012 by The President's Council of Advisors on Science and Technology suggest that fewer than 40% of students who begin their college careers intending to study a STEM field successfully complete such a degree. One reason these students give for changing from a STEM major is that they find their introductory courses to be uninspiring (The President's Council of Advisors on Science and Technology, 2012). Students who struggle to gain a deep understanding of calculus leave the sciences altogether (Carlson, Oehrtman, & Thompson, 2007). I propose that the use of dynamic visualization in the undergraduate classroom may provide insight into student learning while offering an opportunity for students to remain engaged with the topics presented in this vital introductory STEM course.

Research suggests that high school mathematics students who visualize when learning and doing mathematics are not necessarily the star students in their classrooms, but their understanding of the concepts and ideas of mathematics is stronger than their non-visualizing peers (Presmeg, 2006). Presmeg defines visualization as including, "the processes of constructing and transforming both visual and mental imagery and all the inscriptions of a spatial nature that may be implicated when doing mathematics" (Presmeg, 2006, p. 3). I investigate how student interactions with dynamic visualization software (DVS) influence the learning and understanding of calculus for undergraduate STEM majors.

Software that allows students to explore mathematical relationships and constructs in a dynamic environment may give insight into student learning and understanding of calculus, thus providing a pathway to enhance undergraduate calculus through student engagement. DVS facilitates visual investigation of important mathematical relationships and may assist students in exploring topics in a way that promotes conceptual understanding. In this study I investigate how the use of such software influences student views of derivative as a rate of change of one quantity with respect to another and how the experiences with DVS affect student understanding of derivative at a point as well a student's graphical, analytical and conceptual understanding of derivative.

Some of the issues associated with poor student performance in introductory undergraduate calculus include few opportunities for students to develop the necessary conceptual knowledge that ties together the procedures linking topics within the course

(White & Mitchelmore, 1996; Baker, Cooley, & Trigueros, 2003; Gray, Loud & Sokolowski, 2009; Moore & Carlson, 2012; Szydlik, 2000; Hardy, 2009), a failure for students to transfer their knowledge from previous mathematical experiences to newly presented tasks (Lobato, 2008) and the struggle for students to accurately tend to the relationship between covarying quantities (Carlson, Oehrtman, & Thompson, 2007). The compounding of these issues results in students struggling to gain a deep understanding of calculus and, in turn, students abandoning their efforts to earn a STEM degree. There is little research on how DVS may help support first year calculus students as they strive to develop the necessary calculus knowledge required by their course of study.

Though DVS may aid those students who prefer to visualize when working on mathematical tasks, the impact of this software on those who choose not to visualize is unknown. DVS is not meant to replace the need for either proof or rigorous analytical thinking (Cory & Garofalo, 2011). In fact, the introduction of misconceptions when using DVS, either through uncontrollable mental imagery or student mis-understandings should be carefully teased out and addressed to ensure that students do not use this incorrect information as a building bock for further misconceptions. Visualizations, particularly dynamic ones, serve as powerful referents for visualizing students and any places where conceptual connections may be obscured should be discussed with students to avoid future misconceptions. When students focus on items considered peripheral to the concept being explored they should be redirected to focus on the desired patterns or connections (Aspinwall, Shaw, & Presmeg, 1997; Lobato & Burns-Ellis, 2002).

There is no current research linking the use of visual investigations with DVS, whether these investigations are student-led, instructor-led or a used in conjunction with specific tasks, to their impact on student learning in calculus. Through one-on-one interviews with undergraduates taking a first-semester calculus course I aim to determine how student engagement with DVS impacts student learning and understanding of several topics in calculus. Specifically I examine how students working with the software view derivative as rate of change, their multi-representational views of derivative as well as their understanding of the Extreme Value Theorem (EVT) and differentiability. This is then compared to the same views held by students working on similar tasks without the aid of DVS.

Chapter 2

Literature Review

This study investigates the differences in student performance and knowledge when students interact with DVS (such as Geometer's Sketchpad® published by Key Curriculum Press) compared to students who work traditional problems similar to those presented in a college calculus textbook without the opportunity to use DVS. I will refer to these activities presented without the DVS interaction as static tasks. Several factors may influence a student's interaction with DVS including their preference to visualize and participation in an intensive, Treisman-style intervention program. I now turn to the existing body of literature that serves as a framework for the study. I first discuss student struggles in the course including a discussion about procedural and conceptual knowledge as well as how students develop well-connected schema, or ideas, about concepts; I examine the importance of student understanding of function, particularly the need for a student to consider how one quantity may change in response to a change in another. I then highlight some work on visualization in mathematics learning and understanding, including problems that may arise from such activities. Lastly I present a discussion about activities and methods that encourage learning concepts in-depth and how a particular intervention program aims to equip students with the tools necessary to succeed in a calculus course.

2.1 Procedural vs. Conceptual Knowledge in Calculus

Research within the mathematics education community suggests that students do not have a deep, fluent understanding of the concept of function when they complete a first-year calculus course (Carlson, Oehrtman, & Thompson, 2007; Schwarz & Hershkowitz, 1999; Szydlik, 2000; Carlson, Oehrtman, & Engelke, 2010; Tall & Vinner, 1981). The reasons for this lack of understanding vary, but the community generally accepts that this lack of understanding ultimately affects the students' life experiences, as many choose not to continue with a STEM major. Carlson, Oehrtman, & Thompson (2007, p. 150) assert that, "this impoverished understanding of a central concept of secondary and undergraduate mathematics likely results in many students discontinuing their study of mathematics." While the explanations for this lack of deep understanding range from delayed algebraic thinking to an underdeveloped understanding of the underlying meaning and sense of variable expression, many students do not achieve deep conceptual understanding (i.e., students are not able to relate mathematical objects to one another). A student who achieves a deep level of conceptual knowledge should be able to provide examples of concepts. The student should also possess knowledge rich in relationships regarding the understanding of how the conceptual relationships are connected to one another in a fluid fashion (White & Mitchelmore, 1996). This contrasts with students who use overly formal language, almost as if it was memorized, have only a limited understanding of procedures to implement in certain situations or can only express relationships in a broken fashion. These students have only procedural knowledge of a concept (White & Mitchelmore, 1996). A student who solely possesses procedural knowledge typically cannot identify the relationships that the material they are studying contains; they have a limited toolbox containing definitions, theorems, terms, and formulas but do not have the ability to discuss the relationships between these items. For students with only a sense of procedural knowledge, the ability to implement several ideas in order to solve a problem is impossible (Baker, Cooley, & Trigueros, 2003; White & Mitchelmore, 1996; Gray, Loud, & Sokolowski, 2009; Moore & Carlson, 2012; Szydlik, 2000; Hardy 2009). "Procedural knowledge may, or may not, be supported by conceptual knowledge; when unsupported, a student might know the rules of calculus, without knowing the reasons why these rules work" (White & Mitchelmore, 1996, p. 81).

Students who have unsupported procedural knowledge fail to notice the nuances when questions are fashioned differently from what they previously have seen; that is, they have a set of rules which they must follow. Hardy (2009) describes that when students are asked to find limits of rational functions, those with unsupported procedural knowledge immediately begin by factoring both the numerator and denominator, even if this step is unnecessary. If the student possesses conceptual knowledge regarding limits of rational functions, she would understand when to implement procedures based on the attributes of the problem rather than simply to start with the same procedure each time. Students without conceptual knowledge of a particular topic may never feel confident in their ability to solve problems grounded in this topic because the ideas possibly remain unconnected and difficult to understand. When the interconnection of relationships is not a natural process for a student, problem solving becomes a daunting, difficult task.

The emphasis on procedures within a calculus course may be a norm from within the academic institution (Hardy, 2009), from instructors (Szydlik, 2000) or from a student's prior experiences (such as a high school math course) (St. Jarre, 2008). In a survey of beliefs regarding learning calculus, Szydlik (2000), found that a number of mathematics faculty members noted that the best way to learn mathematics was to practice problems from the textbook. This possibly reveals a procedural focus in instruction and assignment, since many calculus textbooks contain routine problems which focus on the procedure taught in the corresponding section of the text. Lithner's (2004) large study of undergraduate calculus textbooks found that the exercises presented to students are procedurally disconnected and generally fail to highlight the intrinsic similarities between problems that students might use to attain conceptual knowledge. Students studying calculus from the textbooks examined in Lithner's study gave no consideration to the mathematical properties of the question at hand, instead they mentally searched for the ‘right’ procedure to implement and follow (2004). Carlson, Oehrtman and Thompson (2007) claim the following.

> The strong emphasis on procedures without accompanying activities to develop deep understanding of the concept has not been effective for building foundational function conceptions – ones that allow for meaningful interpretation and use of function in various representational and novel settings (p. 151)

This focus on procedure may result in a failure for the student to achieve the necessary level of abstraction for transfer to occur. Transfer, in this context, refers to “shared mental representations” (Lobato, Rhodehamel, & Hohensee, 2012, p. 435) as a similarity between tasks wherein a learner recognizes the commonalities and is able to sufficiently proceed with completing the newly presented task using the knowledge from the previous tasks. The consideration of a learner’s prior experiences becomes central, then, to how she approaches novel problems. Lobato, et al.’s actor-oriented perspectives (AOT) refine the view of transfer and allow researchers to consider the “influence of prior experience on learners’ activity in novel situations (2012, p. 437).” This may lead to gaining more understanding of how students develop conceptual knowledge by combining their prior experiences as they forge ahead with newly encountered ones.

To further examine conceptual understanding, in calculus in particular, I refer to the work of Tall & Vinner (1981) and their anatomy of conceptual knowledge into two related terms: concept image and concept definition. Concept image is the total cognitive structure that a student associates with the concept, including any visualizations and associated properties and procedures. The concept definition, by contrast, will refer to the words used by the student to specify this concept. This may be a personal definition held by the student used to explain the concept image (Tall & Vinner, 1981). The repetitive task of working rote problems may lead to yet another issue for young students, which is the opportunity to experience cognitive conflict due to concept images and concept definitions that are not in sync with one another. When the student is faced with a conflict between portions of the concept image a cognitive change must occur. Such conflicts may occur when a student fails to correctly classify something as a function due to the limitations of her concept image (Williams, 1991; Carlson, Oehrtman, & Engelke, 2010). For example, if a student does not consider a constant function to be a function, then when they are asked to find the limit of the function at a given point, a conflict arises. Students whose concept image does not recognize that a constant function satisfies the formal definition of function found this task to be difficult, if not impossible. In several cases, students' ability to determine the limit of a function at a point arose only after they developed a clear and correct concept image of function (Williams, 1991).

2.2 Cognitive Obstacles to Success in Calculus

Using Piaget and Garcia's (1989) work as a basis, Dubinsky laid the groundwork for a theory describing how a learner synthesizes knowledge from existing knowledge. This theory is referred to by the acronym APOS (Action, Process, Object, Schema) Theory and suggests "students build concepts through a standard process of set steps" (Bennett, 2009). Students first have an action view of the concept, followed with a process view, then an object view and finally a schema is formed. A student who holds an action view of function may rely on "plugging in" values or manipulating variables in less than meaningful ways. This compares to a student who views function as the process of relating one value (the input) to another value (the output) (Dubinsky & McDonald, 2002). Once a function is viewed as an object processes can be performed on it, such as taking a derivative or a limit (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & McDonald, 2002; Baker, Cooley, & Trigueros, 2003). Schema, in this context, refers to something that is repeatable and generalizable as an action. It is the relationships and coordination between individual schemes that result in well-formed schema (Baker, Cooley, & Trigueros, 2003). In order to access each scheme, the working memory must call for the information from the long-term memory. Any manipulations or adjustments to the scheme occur in the working-memory (Sweller, Van Merrienboer, & Paas, 1998).

Before a student refines their view of function to encompass the process of relating one value to another, they first hold an action view of function. This implies that a student views a function only in terms of symbolic manipulations and procedural techniques. Dubinsky and McDonald (2002) make the distinction between an action view and a process view of a concept as the individual reflection of a repeated action resulting in an internal mental construction. This student, most likely, would not have a developed conceptual knowledge about functions and would continue to operate within their limited procedural knowledge, probably in a computational fashion (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Alternately, a student who possesses a process view of a function is able to consider a "continuum of input values in the domain of the function resulting in the production of a continuum of output values" (Carlson, Oehrtman, & Engelke, 2010, p. 116). Students who have an action view of function often struggle to apply the necessary algebraic manipulations needed to fully show the relationships between variables (Carlson, Oehrtman, & Engelke, 2010; Gray, Loud, & Sokolowski, 2009). Students are considered to possess algebraic thinking skills if they can exhibit the ability to work with algebraic expressions in a general manner, and if they can interpret the mathematical relationships between these expressions. In contrast, students who simply view variables as objects or labels do not have a developed understanding of the relationships between objects and are considered to possess arithmetic thinking.

It is important that students understand that the procedural manipulations involved in solving a mathematical task have a meaning within the problem context. When posed a word problem, an arithmetic thinker may struggle with inappropriate use of variables as labels (Gray, Loud, & Sokolowski, 2009). Although it is easy for an algebraic thinker to consider the procedure implemented in a manipulation, the underlying variable relationships are what drives the manipulations. In their 2009 study, Gray, et. al. developed a tool that attempted to predict students' success in calculus based upon their development of algebraic reasoning. The results of this study showed more than just a simple correlation, but that for many calculus students the ability to generate an algebraic response to a statement is a struggle. In fact, students who did not have a developed sense of algebraic thinking struggled with covariation of quantities and were likely to struggle in calculus with the concepts of limit and derivative. Interestingly, the completion of a one-semester calculus course may not be enough to sufficiently develop this sense of variables acting as varying quantities (Gray, Loud, & Sokolowski, 2009). The tendency for students was to approach a problem arithmetically and to use questionable methods and notation while doing so. This is evident in calculus classrooms, especially when introducing the idea of an interval of the real number line. Students who struggle with underdeveloped algebraic thinking also struggle with ideas of inequalities, because they tend to regress to thinking in terms of natural numbers or integers (Szydlik, 2000; Gray, Loud, & Sokolowski, 2009). Indeed this can be seen when students make what the mathematical community might consider a 'trivial' mistake, such as incorrectly distributing the expression $-(x + a)$; it is suggested that students who make this error consistently are incorrectly viewing $x + a$ as a single variable (Carlson, Oehrtman, & Thompson, 2007). Helping students develop a stable, rich and deep understanding of the nature of variables and covarying quantities can help students develop a better understanding of the concept of limit of a function, continuity, derivative and other calculus concepts.

Once a student is able to consider a function as a process they may continue to struggle due to a lack of understanding covariation (Carlson, Oehrtman, & Thompson, 2007; Gray, Loud, & Sokolowski, 2009; White & Mitchelmore, 1996; Carlson, Oehrtman, & Engelke, 2010; Baker, Cooley, & Trigueros, 2003). Students gravitate toward linear and quadratic functions because they are more familiar with functions of this type. However, when students are introduced to the concepts of limit of a function, derivative of a function or rate of change, these linear and quadratic functions no longer suffice for what qualifies as a function. In order to fully develop an understanding of how functions operate, a student should develop an ability to consider how the output values of functions vary as the input values change. This ability allows students to consider how functions model dynamic situations and illustrates covariational reasoning as defined by Carlson, Oehrtman and Thompson (2007).

Covariational reasoning, as defined by Carlson, Jacobs, Coe, Larson and Hsu (2002), is “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p. 354), and the authors state that this is a necessary refinement of a process view of function. In order for this sense of reasoning to develop, students must leave the comfort of the integers and begin to operate within the realm of the real number system (Szydlik, 2000). In cases where students do not do this, their modeling results may conflict with the constraints of the situation being modeled (Schwarz & Hershkowitz, 1999). Students who demonstrate an ability to use variables as “varying quantities” earn higher final course grade in calculus that those who are considered to have moderate or basic levels of understanding of variables (Gray, Loud, & Sokolowski, 2009). When given a graph representing velocity versus time and another representing distance versus time the ability to relate the two graphs (and ultimately to understand the depth of information given through them) is a struggle for students, but “developing an understanding of function in such real-world situations that model dynamic change is an important bridge for success in advanced mathematics” (Carlson, Oehrtman, & Thompson, 2007, p. 154). When asked to relate the functions for a square’s area and the same square’s perimeter, a student must first realize the need to compose one function and invert the other. The goal of such a task is to gain insight into the way a student relates two varying quantities. Previous studies suggest that students who view a function as a process that accepts inputs and returns outputs of real numbers possess a well-developed sense of function (Carlson, Oehrtman, & Engelke, 2010). In this way, a student could be limited due not only by how deeply they understand the dynamic view of the covariation of variable, but also by that student’s algebraic maturity.

Carlson, Jacobs, Coe, Larsen and Hsu (2002) developed a covariational reasoning framework aligning student mental actions (MA1-MA5) to behaviors exhibited by the student while solving tasks involving the interpretation and representation of dynamic function situations. Interestingly, their study mentions the need for students to exhibit not only the behaviors associated with the mental action level but also those of lower levels for a clear classification to be evident. This means that a student who is able to verbalize an awareness of amount of change with regard to the output when considering an amount of change to the input (MA3) would also need to verbally exhibit an awareness to the direction of change with regard to the output when considering changes to the input (MA2). This makes sense and serves a guidepost for considering student levels of classification involving a clear hierarchy. When the underlying behaviors are not evident, even after asking probing questions to the student, this is labeled as pseudoanalytical behavior. Carlson, et. al. (2002) also created a corresponding list of five covariational reasoning levels (L1-L5); this is aligned directly with the mental ability structure and students must exhibit a corresponding MA rating for their covariational reasoning level.   When a student exhibits a high MA level (MA5), but is unable to then apply the expected covariational reasoning level of L5 it suggests that the student is relying on memorized rules from calculus and not truly demonstrating an ability to understand covariation of quantities. The framework presented in the study allows for classification of student understanding in relation to student application while solving problems of a dynamic nature.

Students would need to have an object view of function in order to take the derivative of the function (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & McDonald, 2002). This is similar to the way a student might view a function in order to understand what it is means for the function to have a limit at point. Numerous studies focus on how well first-year students understand the concept of limit; and the consensus of these studies suggest that many first-year students possess an underdeveloped concept of limit, at best (Bezuidenhout, 2001; Cetin, 2009; Hardy, 2009; Tall & Vinner, 1981). When given a list of statements regarding the limit concept and asked to categorize them, mark them as true or false and then to give a personal definition of the limit of a function, Williams (1991), found that "students often describe their understandings of limit in terms of two or more" (p. 225) informal models. When the students were presented with conceptual conflicts regarding their view of limit they were often slow to revise their personal views. In fact, regarding methods for finding the limit of a given function, Williams (1991) found that:

> No student was willing to give up the view that plugging in a finite number of values was essentially correct, or that moving along the graph was a good way to view a limit problem... In large part, limits are seen as

> a process performed on a function, an idealized form of evaluating the function at a series of points successively closer to a given value (p. 230)

This procedural view of the limit concept may be related to calculus textbooks containing exercises that emphasize an algebraic approach regarding finding limits. If instructors also emphasize this approach without equal emphasis on other methods then students may develop the idea that algebraic means are the most efficient, or even the only way, to calculate a limit correctly (Bezuidenhout, 2001). Hardy (2009) suggests that when provided with problems regarding the limit of a function at a point, students immediately attempted an algebraic procedure without first checking to see if the procedure was necessary. When given a rational function where the numerator or denominator was an irreducible polynomial, the students did not know how to complete the task. One explanation for the students' struggle might be that they had never seen a task where the algebraic procedure led them astray. In an interview regarding the exercise, one student stated "...They never gave me a problem that wasn't factorable" (Hardy, 2009, p. 355). If a student perceives that a problem is of a certain type, a type that they are 'normally' provided with on an exam or a homework assignment, then they will fail to see that the characteristics of the problem are different than that norm. Without a deep understanding of functions and algebraic knowledge, students are often left unable to complete problems that are not routine in nature. When students have a weak concept definition, but a strong concept image, these evoked conflicts can discourage and intimidate students (Tall & Vinner, 1981). Students may develop a better understanding of the limit concept if they are provided with a wider variety of examples, problems and methods (Cetin, 2009; Williams, 1991; Hardy, 2009).

2.3 Visualization in Calculus Learning

As many students closely link their concept image and concept definitions of certain topics in a visual manner, it is important to understand the role visualization plays in mathematics learning. For example, a student whose concept definition of continuity is "never picking up my pencil" when they sketch the function's graph may struggle when presented with the graph of the continuous function $f(x) = \frac{1}{x}$ (Tall & Vinner, 1981; Williams, 1991; Graham & Ferrini-Mundy, 1994). In this case, the student's personal concept definition, concept image and the formal definition of continuity are not in alignment. Visual imagery is a strong referent that often guides student intuition; so, it is important that we understand the role both visual imagery and visualization play in the learning of calculus.

A visual image is taken to be a mental construct depicting visual or spatial information where as visualization refers to, "the processes of constructing and transforming both visual and mental imagery and all the inscriptions of a spatial nature that may be implicated in doing mathematics" (Presmeg, 2006, p. 3). Using this language, the moniker visualizer refers to a person who prefers to visualize when there is a choice in doing so. The imagery created by the individual refers to any menial construct depicting visual or spatial imagery. Presmeg also asserts that while students who are "super stars" of the classroom are those who prefer not to visualize when soling mathematical tasks, their visualizing peers have a stronger understanding of mathematical concepts and ideas (2006). Presmeg identifies six modalities of imagery, but suggests that auditory, kinesthetic and (particularly) visual imagery are the most common. These notions are the basis of Presmeg's MPI.

It is possible for student to possess strong mental dynamic images that do more to obscure than to explain. For example, a student's graphical understanding could be a powerful referent as she processes the graphical and symbolic understanding of the derivative of a function who's shape is a parabola (Aspinwall, Shaw, & Presmeg, 1997; Aspinwall & Shaw, 2002). In their 2002 article, Aspinwall, and Shaw highlight the struggles of a calculus student who believes the graph of $y = f(x) = x^2$, "… [when it] goes up far enough its slope is undefined" (p. 715). The student used his belief that the function has vertical asymptotes to guide him as he attained incorrect knowledge about the graph of $f'(x)$. Thus, this graphical understanding is a powerful referent for this student and the misconception will obscure the fact that exponential functions are differentiable across the real numbers. This type of imagery is referred to as uncontrollable mental imagery (Aspinwall, Shaw, & Presmeg, 1997). The possibility of a student experiencing uncontrollable mental imagery when working with DVS is, most likely, elevated. It is important that the student's misconception, as it relates to imagery especially, be teased out and discussed in order to combat future experiences.

One way to achieve the latter is through a focusing effect, or "the regular direction of the users' attention toward certain properties or patterns," as they interact with technology (Lobato & Burns-Ellis, 2002, p. 298). When a student begins to focus on an unintended aspect of the visual, the instructor or teacher should gently refocus the student on the desired connection. For example, when presented with a table to data and asked about what relationships she sees, a student may focus on changes seen within one column of the table when the interviewer intended for the student to compare values in the same row. The interviewer should, according to Lobato & Burns-Ellis, bring the student's attention to rows of the table and ask what similarities she sees. It is important to note that identifying what a student will attend to prior to the student's interaction with the visualization is virtually impossible. However, after sufficient student interactions with the DVS the visualizations may be altered to remove or redesign those items there were most often obscuring the conceptual goals of the visualization.

Lobato, et al. (2012) connect the moments of refocusing to enhanced transfer of knowledge. Through the actions of other students, educators or researchers, a learner's focus can lead to them "noticing" the important attributes of a mathematically posed situation or task. These moments of noticing can lend themselves to a student's ability to make generalizations about situations and, as a result, the incorporation of conceptual knowledge into a student's understanding of calculus.

How then, do students make connections between mathematical concepts when visualizing? Eisenberg & Dreyfus (1991) suggest that students must attain an "across-time" understanding of function in order to be successful in calculus. This understanding may be viewed as a refinement of the commonly held "pointwise" understanding of graphical presentations. This failure to transition from a pointwise view of functions to the more dynamically rich across –time view may stem from the mechanical understanding that students are presented with in their secondary mathematics courses. Professional mathematicians may prefer dynamic views of their subject; whereas secondary mathematics teachers prefer more analytic methods. This is true even of teachers who prefer to visualize when working on mathematical tasks (Graham & Ferrini-Mundy, 1994; Presmeg, 2006). The use of dynamic visualization in the teaching of undergraduate calculus may help bridge this gap and foster a sense of visual understanding with regard to the basic underlying notions that serve as the building blocks of calculus. This suggests that we should strive to gain understanding of how students who are exposed to dynamic interactions within calculus understand common calculus topics such as derivative. The dynamic visualizations used in a calculus classroom should facilitate visual processes and clarify the solution pathway for some mathematical problems (Johnson, 2012; Cory & Garofalo, 2011; Schwarz & Hershkowitz, 1999).

2.4 Intervention Programs: The Emerging Scholars Program

The university where this study took place offers an intensive problem-solving workshop designed to support freshmen taking several high-loss classes. The strategies employed during workshop meetings are intended to improve performance in the course. The strategies used during the workshops are modeled after similar programs offered at the University of California, Berkeley and the University of Texas at Austin (Asera, 2001). I will now discuss the existing literature about these intervention programs.

The Emerging Scholars Program (ESP) arises from Uri Treisman's ethno-graphic study of successful versus unsuccessful minority students enrolled in first-year calculus courses at UC Berkeley. He studied twenty mandarin-speaking Chinese American students (the "successful group") and twenty African American students (the "unsuccessful group"). The "successful" minority students formed a peer group with one another and supported each other while studying; in contrast, the majority of "unsuccessful" minority students studied alone or with just one other person. Fullilove and Treisman (1990) assert that:

> The Chinese Americans' study groups facilitated the exchange of information between group members, and this exchange became a critical component of each student's mastery of calculus. Students checked each other's work, pointed out errors in each others' solutions, and freely offered each other any insights they had obtained – as a result of their own efforts or through conversations with TAs or professors – about how to manage difficult problems (pp. 466-467).

As the semester continued the student communication and problem solving methods began to streamline in the Chinese American study groups. The students in these study groups became familiar with one another, and each other's work habits and idiosyncrasies, as the continued with their mathematical studies. As a result of

Treisman's investigation, UC Berkeley began a Professional Development Program (PDP) and the Mathematics Workshop (MWP) in an effort to support calculus students and foster a community of mathematics.

When the MWP started, the overreaching goal was to "create a mathematically rich setting in which students could learn calculus" (Asera, 2001, p. 19) which was achieved by providing calculus students with challenging, well-crafted problems. These problems were like nothing the students had seen previously and required time and deep thinking in order to solve them. The MWP also provided a space for students to collaborate on these problems on their own time and schedule. TA's were available for assistance but an emphasis on peer groups was made clear to students. While challenging problems and peer groups represent the heart of any ESP group, there are non-academic components that support the students, especially minority or first generation students (Fullilove & Treisman, 1990; Asera, 2001). These non-academic components range from opportunities to foster a sense of community, such as picnics, service projects in the surrounding communities, pizza parties, and offering on-campus resources for counseling, to providing an open and welcome atmosphere for all students. Students who chose to participate in a MWP or ESP group find that the worksheets are challenging, but the support groups they join allow them to enjoy greater success than their classmate who does not participate in the programs (Asera, 2001; Fullilove & Treisman, 1990; Moreno & Muller, 1999).

All groups that are modeled on MWP/ESP principles place a strong emphasis on student work groups, which was not, at least at first, a pedagogical instrument in the initial models. What has evolved, through time and student feedback, is a model where student work is public and refined by peers through questioning and conceptual support. Facilitators for MWP/ESP groups are present at each meeting but their role is to question, challenge and in general, prompt students to think deeply about the mathematics they are working on (Asera, 2001). While not every group was homogeneous with regarding to student talent, ability or mathematics exposure, the groups work together to support one another in this way; they see that others make mistakes just as they do. Students feel encouraged by others' successes and strive to contribute to their group's successes as well. Although much data is reported about 'passing rates,' the goal of a MWP/ESP group is not for a student to simply succeed. The goal is for them to understand deeply the material taught in their calculus course and thus be prepared to tackle subsequent courses.

Each campus throughout the United States that creates a model of MWP/ESP adapts it to the specific needs of its student population. At the University of Kentucky (UK), MathExcel: Calculus Among Friends reported in the Fall 2009 semester 81% of the program participants earned an A, B or C in freshman calculus while only 69% of regular students achieved this goal. Through the success that UK has seen they have assisted over 20 other academic institutions to begin similar programs (Tilley, 2011). The University of Texas at Austin (UT) began the initial ESP group in 1988 and, though the program has ceased to exist today, saw that students succeeded by earning an A or B in their freshman calculus courses over 80% of the time. ESP participants at UT were "more than five times as likely to earn either an A or B in calculus than non-ESP students" (Epperson, 1998, p. 270).

Though peer group collaboration, student support and facilitator interactions are important parts of the MWP/ESP model, and the challenging problems posed on the assigned worksheets serve as the meat of the programs.

…the real core was the problem sets which drove the group interaction. One of the greatest challenges that we faced, and still face today was figuring out suitable mathematical tasks for the students that not only would help them to crystallize their emerging understanding of the calculus, but that also would show them the beauty of the subject (Treisman, 1992, p. 388)

The problems presented to the students for solution are often routine problems that have been tweaked, or changed in a way that the solution requires reasoning about and employment of multiple procedures and theorems. In some cases students are presented with counter-examples and asked about the validity regarding a mathematical concepts. Scaffolding is utilized to minimize intimidation on the part of the student, and to model novel problem-solving approaches. This scaffolding technique allows for students to succeed at high-level tasks within their groups while reinforcing concepts from calculus (Henningsen & Stein, 1997). Often students employ methods that they have not yet seen in their course. Through the exchange of ideas within their peer groups, and the development of conceptual knowledge, students discover the root of future concepts on their own, thus encouraging further conceptual knowledge growth (Walter & Barros, 2011). Often questions include a heavy writing component through problems that begin with a prompt such as “in your own words…”, “describe how...”, or “predict what will happen if…”; students are expected to write their thoughts and convictions in terms that could teach someone else. These writing exercises reinforce student concept images and allow for others to discuss and question student thoughts regarding concepts (Porter & Masingila, 2000; Henningsen & Stein, 1997; Pugalee, 2004). Students in non-ESP classrooms receive assignments with routine problems, while ESP students are exposed to a wide variety of examples, ideas and problem-solving techniques along with a peer group community to support them. Through these activities and experiences, ESP students have a higher rate of retention, earn high course grades in first-year calculus and should develop a deeper, more meaningful understanding of calculus than their non-ESP counterparts.

Chapter 3

Methodology

3.1 Student Population

This study took place at a large university (38,000 students) in the Southwestern United States where one-fourth of the student population are graduate students. During the Fall 2013 semester, I selected a section of calculus with 110 students to include in this study. These students attended two eighty-minute lectures per week as well as two fifty-minute recitation sessions. During one of the recitation sessions a graduate teaching assistant (TA) worked requested homework problems and answered questions a short quiz was also administered. In the second session, called “lab,” students worked in small groups on problem solving activities focused on topics from the previous week’s class.

Fifty students enrolled in this section also participated in the Emerging Scholars Program (ESP) and attended weekly, two-hour problem solving workshops. Funding restrictions limit enrollment in ESP to first-time freshman students who are United States citizens or permanent residents majoring in engineering, mathematics, chemistry or physics. Students outside of this group expressing interest in ESP are allowed to participate when space is available. During workshop, students work in small groups of 3-4 students on sets of challenging calculus problems presented on worksheets. A senior TA facilitates the workshop meetings, writes the problems and assigns a subset of the worksheet problems for grading. Assignments collected from the ESP workshops are graded and feedback is given to students as they develop problem-solving skills. While in workshop, the students communicate their views and approaches for solving each problem. When there is a difference of opinion, or approach, the students are expected to discuss and defend their points of view. Eventually a consensus is reached within the group and a full solution is written out. Through this peer-feedback process, students hone their personal concept definitions for the topics presented. ESP students are exposed to approaches for problem solving that may, at first, seem foreign or esoteric. However, as the finer points for each approach are discussed and refined within the group, the students also refine their personal preferences for problem solving. The activities collected, along with any quizzes given, contribute to six percent of the ESP students' final grades in this course. The ESP workshop is partially supported by a NSF Science Talent Expansion Program (STEP) grant (DUE#08056796). The Arlington Undergraduate Research-based Achievement in STEM (AURAS) project aims to increase the number of students at this university successfully earning a degree in engineering, mathematics, chemistry or physics.

3.2 Classification of Students by preference to Visualize

During the first week of classes for the Fall 2013 semester, all students enrolled in the selected section of calculus completed Presmeg's (1985) Mathematical processing Instrument (MPI) during one of the fifty-minute recitation sessions. Scoring of the MPI only provides information about the student's preference to visualize when solving a mathematical task. However, when reviewing the MPI it was revealed that not all students sufficiently completed the instrument and I decided to assign each student an additional score indicating the number of tasks that were solved correctly. I made this decision in an attempt to select the students most likely to successfully complete the calculus course. Thus, students scoring less than an 8 (out of 12) for correctness were not considered for further participation in the study. Presmeg states that the MPI section B (the section administered) was designed for use in a high school classroom, but has been validated with undergraduate students. Upon scoring the MPI for visualization preference I assigned each student a visualization number. Students classified as strong visualizers (V) earned a score greater than 14 (out of 24) on the MPI while those earning less than 9 are classified as strong non-visualizers (NV). Vs exhibit a strong preference to visualize when solving mathematical tasks while NVs exhibit a similarly strong preference not to visualize. Fifteen students representing these two extremes were invited to participate in a series of individual interviews held over the course of the Fall 2103 semester. Table 3.1 lists the distribution of students invited to participate in the interviews as those who accepted the invitation. Once students agreed to participate in interviews they were placed into one of two interview groups: those working with DVS and those working on static-type tasks. Six students participated in each group and students experienced the same interview type for the entire series of interviews. Three ESP students experienced DVS enhanced interviews and one experienced static interviews. Unfortunately, the ESP student from the static interview group did not complete the series of interviews.

Table 3.1 Student participant distribution

	Visualizers (15>, MPI)	Non-Visualizers (<8, MPI)
ESP Students Invited (accepted)	3(2)	4(2)
Non-ESP Students Invited (accepted)	5(5)	3(3)

3.3 Interview Protocol Creation

Prior to administering the MPI, I created two interview protocols, one for each of the two interview groups. The questions and tasks selected were analogous to one another and covered topics from undergraduate calculus. Tasks presented during the static interviews were adapted from traditional calculus textbook exercises. The rationale for creating textbook-like tasks for the static interview groups was to minimize the argument that any gains seen in the DVS interview group, when compared to students not in the study, were due to the additional time spent working on calculus. Students experiencing this interview type were allowed to use a basic scientific calculator like the one allowed for use during course exams, a writing utensil and paper. DVS was neither offered nor allowed for exploration during these interviews. During DVS interviews, questions posed required students to manipulate mathematical objects within the DVS, make conjectures about mathematical relationships, collect and interpret data in a table format and discuss predicted outcomes. I selected three topics for interviews: secant and tangent line relationships, tangent lines as rate of change and the EVT.

3.3.1 Static Interview Protocol I

Static Interview Protocol I (see Appendix B) consisted of tasks that I chose from a typical calculus textbook (Stewart, 1995) and made necessary modifications. For example, the first set of interviews focused on the relationship between secant and tangent lines. I selected a task showing data relating the outside temperature to the time of day and modified the original problem to include a graph of the data. As students worked on the task, they found the average rate of change of temperature between specified times of day and then postulated as to how they might find the instantaneous change in temperature at noon. Then students estimated the instantaneous rate of change using the provided graphical representation of the data. The second task included in this static interview asked students to complete a table of values for the function $f(x) = x^2 - 5x - 6$. The table, shown in Appendix B, included a fixed x_1 value of 1 and students sometimes found the value of x_2 when Δx was given or vice versa. Students also found the average rate of change for each pair of x_1, x_2 of values as well as the difference quotient. The final task for the first static interview asked students to relate the average speed of a particle over a one second interval when given the equation describing the particle's position with respect to time; students also discussed the relationship between the particle's average speed and its instantaneous speed. In the final task students were directly asked about the geometric interpretation of average rate of change and instantaneous rate of change.

3.3.2 DVS Interview Protocol I

Once I wrote and refined the static tasks relating secant and tangent lines, I created analogous sketches within the DVS (see Appendix C). The three sketches incorporated tabular and graphical components similar to those in the static interview protocol. The first sketch (see Figure 3.1) presented students with the graph of a quadratic function, a fixed point A on the graph, a dynamic point B on the graph, and a dashed line connecting points A and B. Through a series of questions, the students explored the relationships involving the interval length between points A and B and the average rate of change of the function on the interval $[x_A, x_B]$. The DVS used allowed for the placement of buttons to show (or hide) desired information.

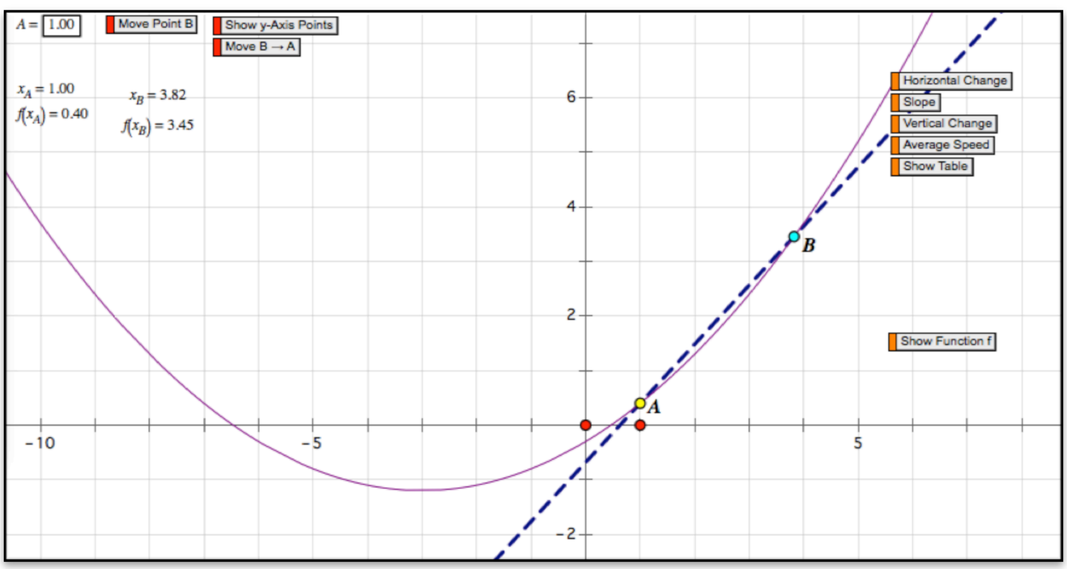


Figure 3.1 Snapshot of Sketch I from DVS Interview I.

Once a student was able to make a connection between the slope of the secant line passing through the point $(x_a, f(x_a))$and $(x_b, f(x_b))$, they would reveal the calculation of this slope. In another step, once the connection between the average rate of change for the function on the interval $[x_A, x_B]$ of the secant line was made the student could reveal the slope calculation labeled as the average rate of change. The students also collected data in a tabular format (e.g. see Figure 3.2) as they manipulated the point B on the sketch. Using this tabular representation the students were asked what they thought was happening as the interval length became smaller and smaller. A button in the sketch animated the points so that the interval length was equal to zero (e.g. see Figure 3.3). When this happened, the slope of the secant line was shown to be "undefined" and the dashed line connecting points A and B disappeared from the screen. This animation allowed students to explore the relationship between the average and instantaneous rates of change. The series of sketches produced for Interview I presented students with three different function graphs, the first a quadratic (see Figure 3.1), the second a fifth degree polynomial (see Figure 3.2) and the final an exponential function graph (see Figure 3.3).

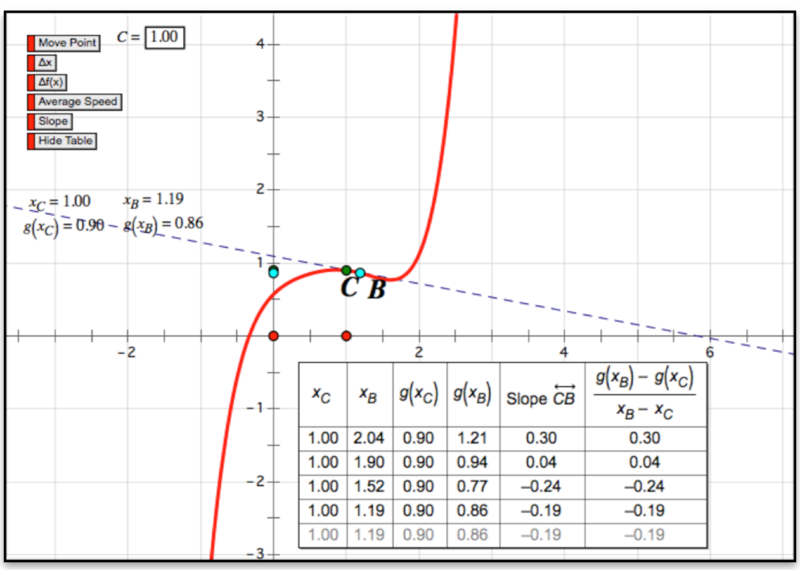


Figure 3.2 Snapshot of Sketch 2 from DVS Interview I.

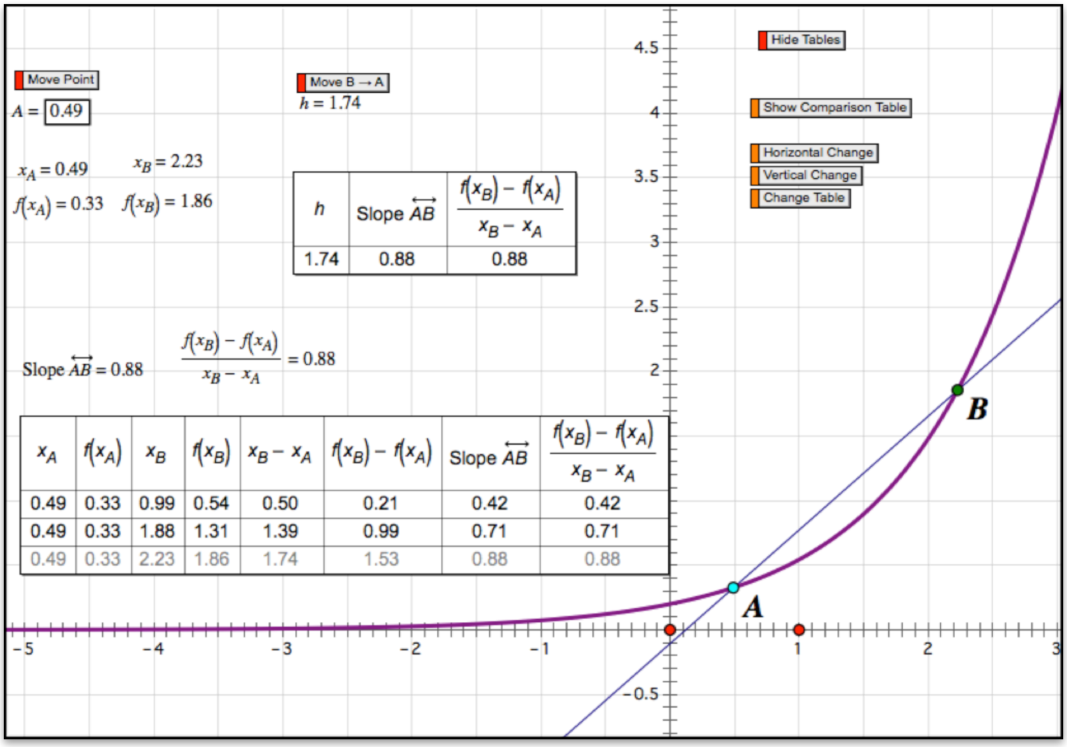


Figure 3.3 Snapshot of Sketch 3 from DVS Interview I.

3.3.3 Static Interview Protocol II

For Interview II, the protocol (see Appendix D) focused on exploring the derivative as a rate of change. The first question for the static group (shown in Figure 3.4) includes the graph of a function and asks students to determine both the function value and derivative value at $x = 1$. The protocol explicitly states that students may use any method they choose for determining the derivative value. In order to explore various representations of derivative held by the students, the questions asked them to express the meaning of the derivative of the function at the given value using three different interpretations of derivative.

In an effort to determine how well the students understood the relationship between the derivative value of a function at a point and the plot of the function's derivative graph, students were asked to name a point that lies on the derivative graph of the function.

Consider the graph below of the function $f(x) = 1.5x^3 - 4.5x^2 - 1.5x - 1.5$.

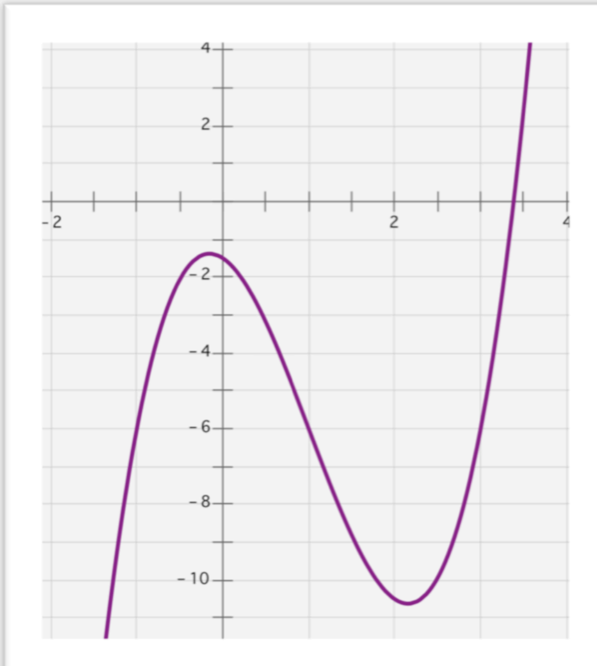


Figure 3.4 Task 1 from static Interview II.

The periodic nature of the sine function may help students make connections regarding the relationship between a function and its derivative. In order to investigate how students use their knowledge of these repeating slopes, the second part of this interview protocol focused the function $f(x) = \sin x$. The question included a graph of the function on the domain $\left[\frac{-3\pi}{2}, \frac{9\pi}{4}\right]$. Initially, students were asked to find $f\left(\frac{\pi}{2}\right)$. This question allowed us to determine if the student used the graph of the function, or if they had enough knowledge about the sine function to know the answer. Next, I asked students to name some points of the graph where the derivative value could be found easily and to explain their choices in terms of the different representations of derivative used in making the decision. Finally, a table was provided for students to complete with $x, f(x)$ and $f'(x)$ values for some predetermined values. These values (with the exception of those in the x column) were either -1, 0 or 1. The students were asked to use the table and the cyclic nature of the function to sketch the derivative graph and to explain what information they used in doing so.

3.3.4 DVS Interview Protocol II

The DVS protocol (see Appendix E) investigating the views that students hold for derivative functions presents many of the same images as the static interview protocol II; in fact, the graphs shown in the static protocol are taken directly from the software used in the DVS interviews. An additional sketch, one continuing the line of questioning from Interview I about exponential functions, was included in the DVS interviews. One main difference in the DVS interviews for this protocol was the instruction for students to form a hypothesis regarding how a point $(x_A, f(x_A))$ lying on the function graph is related to a point $(x_A, f'(x_A))$ on the function's derivative and then validate their hypothesis. For example, a student would have access to the information in Table 3.2 and would be asked to name the point lying on the derivative graph at one of the x values. The student collected the data in the chart as they explored using the DVS. Once the student correctly identifies that the point, for example, $(1.5679, -4.5488)$ is on the derivative graph they are instructed to select a button within the program that validates their statement. If the student was incorrect, a discussion followed to help lead them to the coordinates. Once the point on the derivative graph has been correctly named, the student is asked questions about the derivative graph and then asked to animate the point they manipulated on the original function. At this time, the point on the derivative traces out its path and the student can see the derivative on the screen as the point on the function (and the attached tangent line) moves as well.

Table 3.2 Example of collected data from DVS Interview II.

x	$f(x)$	Instantaneous Rate of Change
.0000	-1.5000	-1.5000
1.5679	-9.2580	-4.5488
1.9986	-10.6578	-1.5126
2.6782	-9.1936	6.6741

In an effort to continue the students' exploration with exponential functions from the first DVS interview, the students were presented with a similar sketch (see Figure 3.5).. The purpose of this sketch is to allow the students to compare the instantaneous rate of change of the function with the function value to see if they recognize that the values are the same. This was followed with a question about the relationship between $f(x) = e^x$ and $f'(x)$. Once students realized that $f'(x) = f(x)$, they were asked if this relationship is unique for the given function or if it occurs for all exponential functions.in hopes that they will recognize that they are the same.

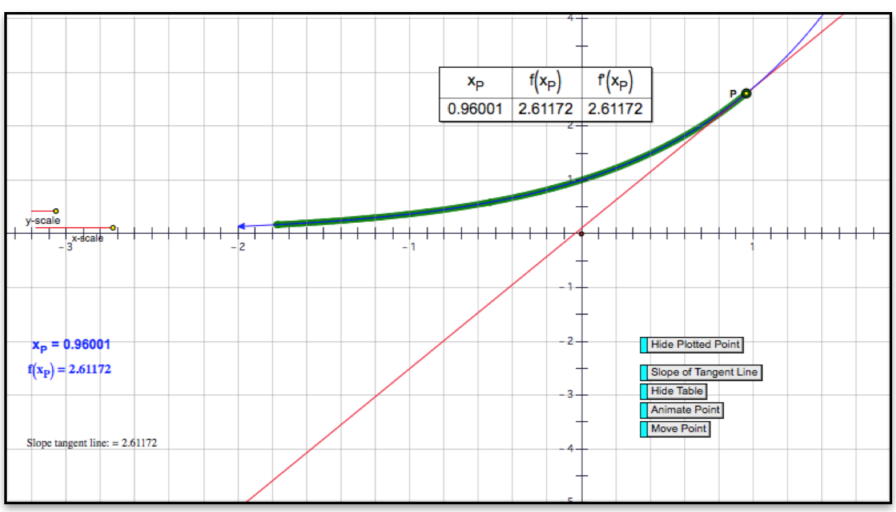
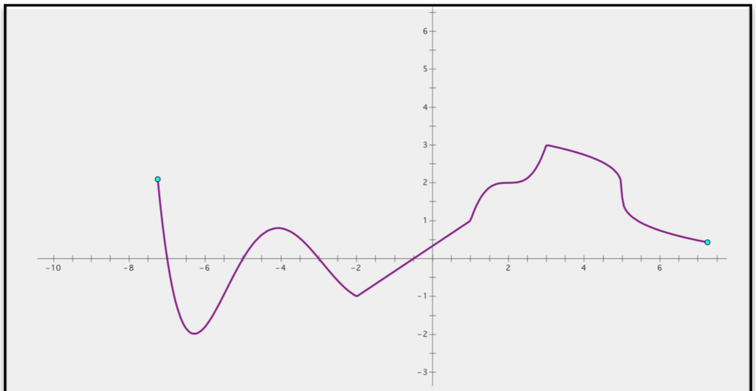


Figure 3.5 Snapshot of Sketch 3 from DVS Interview II.

3.3.5 Static Interview Protocol III

Static Interview III (see Appendix F) focused on student understanding of the EVT. To explore foundational knowledge needed for the EVT, students first worked with examples that included critical points that were not maximum or minimum function values. A piecewise-defined function was developed (see Figure 3.6) in order to investigate student understanding regarding the existence of derivatives at a point where the function graph shows a corner, or where the function appears to be vertical. I included points where the function's maximum or minimum values occurred at points where the function's derivative did not exist in order to see if the students confused the necessary condition of the EVT for function continuity with differentiability. This also allowed me to ask students why the function was not differentiable at such points. Also shown on the graph, at $x = 2$, is an inflection point with $f'(x) = 0$ so the critical value does not yield a maximum or minimum function value.

The students engaged in a series of investigative questions relating to the graph shown in Figure 3.6. In particular, they considered whether the inclusion of the endpoints is important and why that might be the case. They also marked points on the graph where they believed that the derivative value was equal to zero. I included this question for several reasons, first to see if the students indicated that the derivative at a corner was equal to zero and secondly to see if the students only marked points where the function has a local maximum or minimum value. The protocol required that students explain their reasons for choosing each point. In an effort to gain insight into student views of the derivative as a function and how changes to the function relate to changes in the first and second derivative functions, students indicated where, in each closed interval, the instantaneous rate of change was greatest. I followed up this exercise by asking students how these places of greatest instantaneous rate of change appeared on a first derivative graph of the function, and on the graph of the second derivative function. After the exploration regarding instantaneous rates of change, students discussed their knowledge of the EVT and wrote two inequalities guaranteed by the EVT, one for the maximum function value compared to all other function values and an analogous inequality for the minimum function value.



The graph above is of the function

$$f(x) = \begin{cases} \frac{-x^2}{60}(x+7)(x+5)(x+3), & -7.25 \le x \le -2 \\ \frac{1}{3}(2x+1), & -2 < x \le 1 \\ (x-2)^3 + 2, & 1 < x \le 3 \\ -(x-5)^{\frac{1}{3}} + 3 - \sqrt[3]{2}, & 3 < x \le 7.25 \end{cases}$$

Figure 3.6 Graph of the piecewise-defined function used for both static and DVS Interview III.

The last set of questions for this interview protocol investigated student understanding on the relationship between a function's continuity and its differentiability at a point. Students were initially asked if the function was continuous over the interval $[-7.25, 7.25]$ and how they might work to show that their answer was correct. Then the connection of differentiability in relation to continuity was explored. Students were asked why they believe that the function was (or was not) differentiable across its domain and, eventually, to show three places where the function was not differentiable. Since the points selected would be corners or where the tangent line appears vertical, students were asked why the derivative did not exist at these points on the graph classified as corners. At the end of the interview, students were simply asked if continuity implies differentiability.

When students in the DVS interview group responded to a question with "I don't know," they were asked where they might start in order to come to a conclusion. Another line of questioning used was to ask, "where are you confused?" If the student used her hands to indicate that she was visualizing something, she was reminded that she could draw a picture.

3.3.6 DVS Interview Protocol III

As with interview protocol II, the questions included in this protocol (see Appendix G) were identical to those in the static one. The only difference was that students were invited to explore using the software and were, in fact, encouraged to do so when a response of "I'm not sure" was given. At the end of the interview the students were able to click on a button and see the derivative of the function in an effort to validate their ideas of where the function was not differentiable.

3.3.7 Exit Interview Protocol

The Exit Interview protocol (see Appendix H) was the same for both interview groups and no DVS exploration was included in the protocol. As a result, the time limit for the interview was 30 minutes. The purpose of this was to ensure that there was a way to compare the student responses across the two main interview groups. Students were allowed to use a scientific calculator and a pen for this interview. The first series of tasks asked students to list examples, ideas, or concepts that come to mind when they hear the word "derivative." They were given a time limit of one minute to mention as many things as they wished. This was included in the protocol to gain insight into the various ways that students relate derivative to mathematics. If the student did not mention instantaneous rate of change during the first task, they were asked specifically how the instantaneous rate of change at a point and derivative are related.

The second task on the Exit Interview protocol presented students with the graphs of two similar functions (see in Figure 3.7). Students were asked about the long run behavior of the two functions and how they might estimate the instantaneous rate of change for each function at the point $(2,3)$. Then, students were asked to compare the instantaneous rates of change at this point on one graph to the instantaneous rate of change at $(2,3)$ on the other graph. This was included to see how students approached this task: procedurally by sketching a tangent line or by comparing the location of the point on each graph to the local maximum value it is approaching.

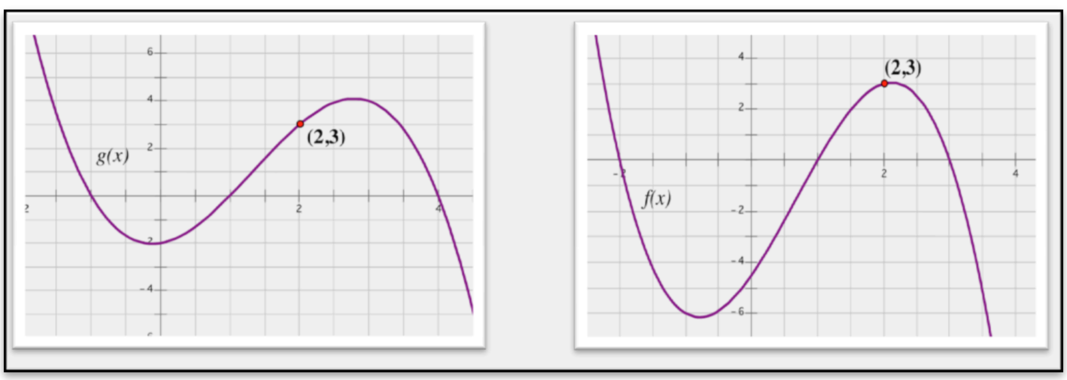


Figure 3.7 The graphs presented to student for analysis and comparison in the Exit Interview.

To see how students understand each necessary condition of the EVT, the third task on the Exit Interview protocol showed three sets of axes, each with an indicated interval for students to illustrate why a function defined on that interval might not achieve both its maximum and minimum values. For example, a student was expected to sketch a function on the interval (a, b) where the function maximum occurred at one of the endpoints and then explain why the function did not achieve this function value on the given interval. On the third set of axes, over the given interval $[a, b]$, students were to draw a function, defined on the given interval, that did not achieve both its maximum and minimum value. The expectation was that students would possibly draw a function with an asymptote or a discontinuity.

The final question on the Exit Interview protocol asked students about how continuity and differentiability are related. Specifically, students were asked if a function is continuous on a closed interval what could they say about the function's differentiability on the interval and how they knew their statement was correct. This segued into a line of questioning about why derivatives at cusp and corner spots on function graphs do not exist.

3.4 Theoretical Perspective

After the semester ended, the interviews were transcribed from the videos taken during each session. The videos were watched in slow motion, all spoken words transcribed and, where applicable, hand gestures or sketches were described. In the cases where a student's words were not clear or when a student's sketch was made with the EchoPen ®, the pencast recording was also used. Each interview was classified, using qualitative analysis software, for interview type and student monikers were assigned to protect anonymity. Any artifacts from the interviews, specifically the interview protocol sheets from each static interview, were electronically scanned. Each interview was tagged for student identification number, interview type and the participant's visualization preference. Once the grades for calculus were made available to researchers they, too were added to the tags. The student's visualization category, gender and ESP participation were also included in each interview tag.

The interview transcripts were coded using grounded theory and open coding. Each interview was reviewed with no pre-determined coding scheme in an effort to let the data speak for itself (Corbin & Strauss, 2007). Corbin and Strauss (2007) define open coding as, "breaking apart data and delineating concepts to stand for blocks of raw data" (p. 195). As possible themes emerged the codes were created and previously reviewed transcripts were re-examined through the lens of the newly added code. Though the first two interview protocols were intimately related, the remaining two varied greatly both from the first two protocols and from each other. Thus, Interview III and Exit Interview were reviewed using very few pre-existing codes related to the other protocols. While analyzing each interview I also created a corresponding memo where I highlighted the major themes present and made note of specific quotations that I could use later in the analysis process. In the memo, the context for many codes was noted as well. For example if a student was asked to "draw a picture," there would not be evidence of visualization, but if the student continued to expand upon the initial sketch throughout the interview, or if they referred back to the sketch, there may be evidence of visualization. It was noted, however, that the interviewer prompted the initial picture.

The code used within a particular episode was not unique; if evidence of multiple codes was present then all codes were used. For example, the episode below is from Interview I with Evan. He is struggling to determine how the difference quotient might be helpful to him as he is asked to find the average rate of change and the difference quotient for different intervals containing $x_1 = 1$ for a given function. "Uh yeah. That's probably why my brain isn't working right now. You could generalize it by removing the x_1 and replacing it with x and that way any value of x would work as long as the function is a -- as long as the uh -- no, no it wouldn't work -- as long as h is not equal to 0." This excerpt was coded as containing both an algebraic approach to rate of change and procedural knowledge. Evan struggles with the algebraic underpinnings of the difference quotient as he works to make sense of a concept for which he, admittedly, has little to no knowledge. Earlier in the interview he states that he, “honestly [has] never heard of it before.” He is also approaching the question of what happens when $\Delta x = 0$ in a procedural manner. Both codes were used for the excerpt.

Once each interview had been reviewed and coded at least once all codes were reviewed and many were refined. For example, the code for “metacognition” was initially used when a student was thinking about her work. When the blocks of data coded as evidence of covariation of quantities were reviewed two distinct themes emerged: “able to covary” and “not able to covary” These codes were added as sub-codes for covariation of quantities and each block was reviewed and the additional code added as necessary. Two codes were created for use solely within the DVS interview transcripts: evidence that DVS obscures and evidence that DVS clarifies.

After the second pass through of coding, the qualitative data analysis software was used to review commonalities between interview codes. This was done using the analysis and coding matrix functions within the software. The user is able to specify which codes should be searched for and in which sources they should search. The result is a matrix indicating the number of instances of each code (rows of the matrix) and the type of source where the instances occurred (columns of the matrix). Further review was possible as the software then linked the entries in the matrix to the instances the spot represented. This allowed researchers to not only review the coding themes but to easily identify those instances that were representative of each group desired.

A randomly chosen list of codes was identified and evidence representative of each code was presented to an outside researcher for validation of the coding. The researchers were found to be in agreement more than 75% of the time. The instances where they were not in agreement were discussed and codes were added or changed as needed to come to an agreement. It should be noted that the disagreement was, in all cases, regarding only one code used in the episode presented. That is, if multiple codes were used within the episode then this corresponded to only one code over which the researchers did not initially agree.

3.4.1 Common Coding Schemes

There are 37 codes used in the analysis of the data, though not all of them arose in all of the interviews. This section will address the coding scheme for each of codes occurring more than 20 times within the interviews. The number of instances for each code as well as common code combinations is presented in chapter 4.

3.4.1.1 Algebraic Approach to Rate of Change

I use the code "algebraic approach to rate of change" to indicate when a student's method for finding rate of change is algebraic in nature. For example, if a student remarks, "to find this I would find the change in temperature from noon to 3 pm and then divide it by 3 hours," I consider the excerpt to contain evidence of an algebraic approach to rate of change. This code may also indicate when a formula is cited such as $\left(\frac{x_2-x_1}{y_2-y_1}\right)$.

3.4.1.2 Geometric Approach to Rate of Change

The code "geometric approach to rate of change" was used when a student mentioned finding the slope of a tangent or secant line, or when they made markings on a graph to indicate they were using such a method. Sometimes a student would work to find a rate of change but would not mention the slope, line or "slope formula" and this code was not used in these cases as the student was not actively intending to approach the task in a geometric manner. However, if a student used the phrase "rise over run," it was assumed that the approach taken was geometric in nature.

3.4.1.3 Concept Definition

Using Tall & Vinner's (1981) definition of concept definition, this code was used to indicate the words a student used to describe a concept, even when the words suggest that a formula is being referenced. For example, when asked what the word derivative brings to mind a student might respond with, "the limit definition, $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$." This would be coded as containing insight into what the student's concept definition is for derivative. However, if the student's words (or, in some cases, hand gestures) suggest imagery is evoked for the student this code was not used.

Concept definition is both a first level and second level code for this project. That is, items containing evidence of a student's concept definition of a concept with its own code were coded twice. For example, when asked what they know about the EVT, a student may respond with, "a function has a maximum and a minimum value." This would be coded as containing evidence of concept definition and, more specifically, evidence of concept definition pertaining to the EVT. The code pertaining to the EVT would be the second-level code and would appear subordinate to the first-level code for EVT. There are several other codes in the project that appear like this and I will use the terms "first-level" and "second-level" to represent these occurrences.

3.4.1.4 Concept Image

Again, using Tall & Vinner's definition of concept image to include the total cognitive structure pertaining to a specific topic, evidence of this code included all instances of concept definition.   The code was used to identify episodes where a student's language, or in some cases sketches, provided insight into the imagery a student associated with a concept. For example, if a student sketched a function graph with a maximum function value occurring at a cusp or a corner this was noted as being part of the student's concept image. For example, during Amy's exit interview she drew a picture to illustrate why the derivative at a corner would not exist as shown in Figure 3.8, below.

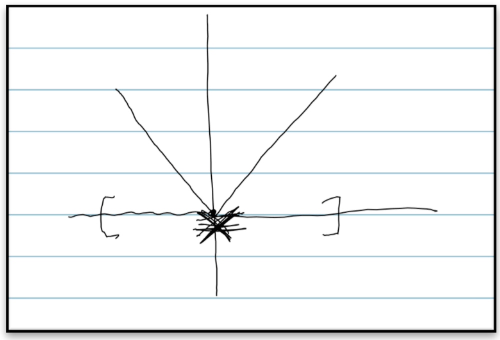


Figure 3.8 Amy's drawing illustrating her understanding of why the derivative value does not exist for a point making a sharp corner.

This is also a first and second-level code. Instances of evidence of concept image pertaining specifically to the EVT were coded under the first-level code EVT.

3.4.1.5 Covariation of Quantities

At times during the interviews students needed to consider how one quantity varies with respect to another quantity. Though evidence occurred within other questioning sequences, the portion of Interview III where a student is asked to consider how the points of greatest instantaneous rate of change of a function will appear on the function's first and second derivative graphs was included to gain information about student's ability to consider covariation. A student who is able to consider such covarying quantities would communicate the relationship between the points on a function graph where the instantaneous rate of change is greatest and the maximum function values for $f'(x)$. Using knowledge about the relationship between points corresponding to maximum function values and how these are related to points on the graph of the function's derivative graph would, in turn, allow a student to make the connection for how maximum function values for $f'(x)$ would correspond to points on the graph of $y = f''(x)$.

Through such a task, evidence of a student's ability to consider covariation of quantities may emerge.

After the first pass of coding was complete, two second-level codes were added beneath this first-level code; they indicate if a student was or was not successful in their endeavor to consider how quantities change to another quantity.

3.4.1.5.1 Evidence a student is able to covary quantities. This second-level code was used to indicate when a student successfully considered how quantities change in relation to one another. For example, this code was used when Beverly, in Interview III, successfully related places where the instantaneous rate of change is greatest to places where the second derivate for the same function would be zero.

3.4.1.5.2 Evidence a student is not able to covary quantities. This code was used to indicate when a student was unable to consider how quantities change with respect to one another. During Ian's interview III, he was asked to write an inequality relating the function's minimum value to all other function values, Ian's response of "everything's positive" [II3p3L2-3] in tandem with his indication of the y −axis above the function minimum was evidence that he was unable to consider how the function values related to one another. This code was used to indicate his struggle.

3.4.1.6 Derivative

To address the research question of how students in the two interview groups view derivative as a rate of change this code was used to indicate evidence of insight into student views of derivative. Answers to the question asked during the Exit Interview regarding "what comes to mind when you hear the word derivative" were coded using this code. After reviewing the places where this code was used, two secondary, and one tertiary code was added to form a more robust picture of how students view derivative.

3.4.1.6.1 *Derivative examples*. This second-level code under derivative was used when students presented examples of derivative. Examples could include a drawing or description, such as “the derivative of a parabola is a line.” This code was also used in the event of a student making a connection to other disciplines, “the derivative of position is velocity.” After reviewing the instances of derivative examples, it was discovered that there was a need for a third-level code to indicate when a student’s example consisted of derivative rules.

3.4.1.6.1.1 Derivative rules. This code was used to indicate that a student’s examples for derivative included statement of the derivative rules, or implementation of such rules. For example, David’s interview included a description that “x^2 is $2x$” by the power rule; other instances included a listing of the rules by name.

3.4.1.6.2 Derivative geometric approach. This code was used to indicate that a student’s view of derivative included a geometric aspect. The student’s use of “slope” or “tangent line” resulted in the use of this code.

3.4.1.7 Derivative with no Mention of Rate of Change

There were several instances of students discussing derivative with no mention of rate of change at a point. For example, David’s exit interview included the statement, “I’m trying to remember how to find the derivative with just the points.” In these cases, the student did not link the derivative to finding the rate of change, or they just mentioned the procedure of finding a derivative when rate of change was not included in the question.

3.4.1.8 Differentiability

Much of Interview III and Exit Interview included questions and discussions regarding the differentiability of a function. This code was used to indicate the presence of evidence of the topic. Even when the interviewer was the one to bring up the topic it was still used to indicate that the concept was being discussed.

3.4.1.9 Evidence of Visualization

The code indicating evidence of visualization was used in several different contexts. If student hand gestures indicated they were visually interacting with either a static sketch or the DVS this code was used. This code was also used if the student used a provided sketch or dynamic situation to answer questions in a way that suggested they were using the sketch in a novel way. For example, when asked to compare the instantaneous rate of change at the point $(2,3)$ on two different (yet similar) graphs (see Figure 3.7), Amy compared the 'placement on the hump' as she made her answer. This interaction with the static graphs was coded as evidence that Amy was visualizing because she was comparing two attributes of the graphs in a manner that was not explicitly requested. This code was also used if a student's description of a situation suggested that they were visualizing; when asked why a particular point was selected to represented a possible maximum function value, Corbin's response of "[*the tangent line*] is just about horizontal" [CIExp2L17] suggests that he is visualizing while working on the task.

It should be noted that there were parameters for using this code. Specifically, there were places where this code would not be used; it was not used when the interviewer requested that a picture be drawn, unless the student continued to interact with the sketch later in the interview. If the student was simply interacting with a provided visual in the requested or expected manner the code was not used. For example, when provided with the graph of a function and asked to estimate the instantaneous rate change a student sketches in a tangent line and begins to find the slope. In this situation, the student interacted with the visual in the expected manner and did not use the visual to further her knowledge about the situation.

3.4.1.9.1 Shape thinking. After reviewing the instances where there is evidence that students were visualizing, it became apparent that there was a type of visualizing occurring that should be delineated from the first-level code. The second-level code for shape thinking is used when a student discusses a function graph or sketch using language to suggest that the shape of graph is being used either to respond to the question asked, to make a judgment about the situation or to gather more information. Note that I use the phrase *shape thinking* in a different sense than Moore & Thompson (2015). It should be noted that the instructor for this class used such language during lecture as he identifies the "hills and valleys" of a function graph as places where the instantaneous rate of change is zero. Several students used this language both for student-generated examples and as support for their responses or answers to a direct question.

3.4.1.10 Evidence that DVS Obscures

After reviewing both the transcripts and video of the interviews it was clear that there were cases where student interaction with the DVS did more to obscure the concept being explored than to clarify.

DVS obscures coded instances when a student focused, either initially or long-term, on the wrong part of the sketch, or when they were unable to make the desire connections due to a distraction within DVS. During Amy's interview II, she was initially unable to connect that the function value and the instantaneous rate of change were the same in the provided table. She was unsure of what she was supposed to focusing on and, instead, focused on the changes within the columns as opposed to across rows. The software only displayed two decimal places on the table and this inexactness lead to confusion for Amy. In this way, the software obscured what Amy should be focusing on. During Beverly's first interview she manipulated two points making a secant line such that the secant line disappeared from view and the table displayed “undefined” for the average rate of change between the two points. Beverly’s assertion that the average rate of change was, “…zero, because there is no change,” is another instance of the software obscuring the concept to be discovered or explored for the student.

3.4.1.11 Evidence that DVS Clarifies

The companion code to evidence that DVS obscures was used to indicate when, with the assistance of the software, a student experienced an “ah ha!” moment, or was able to make a connection that was previously unknown to the student. After exploring more closely with the software, Beverly was able to realize that the secant line mentioned in the above section disappeared because, “you need two points to make an average.” This was coded as an example of the interaction with DVS clarifying a topic for a student.

3.4.1.12 EVT

Since my third research question centers on how students in the two interview groups view the EVT, a code specific to it was used to mark these places. This first-level code served to delineate the second-level codes that described what evidence was present in the interaction. These second-level codes have been previously discussed, in general. The second-level codes beneath EVT are: concept image, concept definition, and examples.

3.4.1.13 Calculus Approach to Rate of Change

At times during the interview students approached, or attempted to approach, rate of change form a calculus perspective. When presented with a function graph, for example, a student response of, “if I had the function definition I would take the derivative,” would be coded as evidence of a calculus approach to rate of change. This code was used even if the student’s approach was neither appropriate nor successful.

3.4.1.14 Conceptual Knowledge

The code conceptual knowledge indicated places where, as White & Mitchelmore (1996) state, a student is able to connect concepts and ideas in a fluid fashion possibly through the use of rich examples. For example, a student's ability to connect the slope of the line tangent to a function at a given point to the function's instantaneous rate of change at that point and further to the function's derivative value at that point suggests that the student possesses conceptual knowledge about instantaneous rate of change at a point.

There were instances where a student's comments suggest the student demonstrates conceptual knowledge about a topic, such as Ian's comparison of instantaneous and average rate of change with regard to varying "window size," but the student was unable to complete the task of calculating said rate of change. These instances were coded as containing evidence of conceptual knowledge.

After analyzing the interview data for the presence of conceptual knowledge the need arose to determine the correctness of the conceptual knowledge present. All instances of conceptual knowledge were reviewed and three second-level codes indicating correct, partially correct and incorrect conceptual knowledge were created. Some episodes containing conceptual knowledge have evidence of a combination of the three second-level codes and these were further analyzed with regard to the transition within the validity of the conceptual knowledge present. For example, a student who states, "the derivative when you reach the function's maximum value has to be zero," but transitions within the episode to consider points corresponding to the function's maximum value but where the derivative does not exist would be described as the student transitioning from partially correct conceptual knowledge to correct conceptual knowledge. There was evidence of this transition from incorrect conceptual knowledge to partially correct conceptual knowledge and from incorrect conceptual knowledge to fully correct conceptual knowledge as well.

3.4.1.15 Procedural Knowledge

The code for procedural knowledge was used in a variety of instances. When a student successfully calculated or found a requested solution in a procedural manner (using a stated formula, etc.) the excerpt was coded as including procedural knowledge. However, there were instances of a student relying on undeveloped procedural knowledge. If a student stated a “fact” with no explanation as to why the “fact” was true the excerpt was coded as an instance of procedural knowledge. For example, if a student were to say, “I just know that the derivative at a cusp does not exist,” the code would be used.

It should be noted that an episode might be coded as containing evidence of either procedural or conceptual knowledge, or both. If a student makes connections between concepts and is able to procedurally compute a solution to a question, or give the procedure to follow in the solution was needed, the excerpt was coded using both conceptual and procedural knowledge. This was done to investigate instances of conceptual knowledge that is unsupported procedurally, that is, when a student appears to make conceptual connections but lacks the procedural knowledge to put this knowledge into action.

Three second-level codes to indicate correct, incorrect and partially correct procedural knowledge, as well as the transitions from partially correct (or incorrect) procedural knowledge to correct procedural knowledge, were created as discussed in section 3.4.1.14.

3.4.1.16 Examples

This first-level code was used to indicate student-generated, unrequested, examples presented during the interviews. If a student was specifically asked for an example this code was not used. It was used, however, if the student returned to a requested example later in the interview to support a comment made or to further a student's knowledge through reasoning.

3.4.1.17 Metacognition

During the first-pass review of the transcripts there were instances noted of students thinking about their work. These instances were coded as evidence of metacognition. Upon further review it was apparent that most of these instances fell into one of two categories: looking back and sense-making. As a result, two second-level codes are included underneath the first-level code metacognition.

3.4.1.17.1 *Looking back at work*. During the static interviews many students would check their work through a variety of methods. Sometimes they simply checked their calculations and sometimes they worked to make sure that their answer was appropriate for the question asked and scenario presented. For example, several students calculated the average rate of change for the temperature problem in Interview I as the average temperature. This resulted in an answer 4 times the expected answer. When students looked to review the answer it was coded as looking back at work.

3.4.1.17.2 *Sense-making*. Several times, mainly in the static interviews, students engaged in sense-making. This means they attempted either to make sense of the situation through reasoning about the situation. For example, Corbin frequently engaged in this type of metacognition. “Um, okay. What properties would the function need to have? It would need to be anything except a horizontal line and since it's a closed interval - - I'm thinking, wait, one second. Okay. It can't be a horizontal line. “He would constantly think about what he said and try to reason about it. In this way, he was engaging in an action described by the code for sense-making.

3.4.1.18 Student Thinking about Rate of Change

At times students would explore their knowledge about rate of change without indicating if they were considering a function's instant or average rate of change. The code to indicate evidence of student thinking about rate of change was used to mark these places within the interviews. I noted that there was a difference in the number or quality of occurrences between the interview groups.

3.4.1.19 Student Understanding of Average Rate of Change

When a student discussed average rate of change, whether it be in a conceptual or procedural manner, or through the use of examples, this code was used. Perhaps a student would remark that, "average rate of change is the secant line," this would be coded as evidence of a geometric approach to rate of change, but specifically it contains student understanding of average rate of change in a geometric manner. The combinations used in tandem with this code, in conjunction with the power of the qualitative analysis software, allowed for these codes to be delineated later in the analysis process and facilitated the emergence of codes from the data in an organic manner.

3.4.1.20 Student Understanding of Instantaneous Rate of Change

Student understanding of instantaneous rate of change codes places in the interviews where students talk about instantaneous rate of change. This code is used in a similar manner to the one described in section 3.4.1.19.

3.4.1.21 Student Understanding of Functions

Though this was not the focus of the study this theme emerged as evidence was found to suggest that how a student understands functions (either a particular function, such a quadratic function, or functions in general) might play a role in student understanding of another concept. When a student presented evidence of their understanding of functions this code was used. For example, Corbin reasoned using the graph of a parabola during Interview I. He attempted to use his knowledge of the symmetry of a parabola about a vertical line to determine the average rate of change of a cubic function as Δx approaches 0. Corbin's incorrect knowledge about functions is what drove his reasoning.

Chapter 4

Results

After completing the transcription process, each interview was analyzed using open coding and grounded theory. In this chapter I report the results of this process. The flow chart shown in Figure 4.1 illustrates the organization of the results. I begin by discussing common themes emerging across the interview series for each student. I then discuss the common themes present among various group of interviews, beginning with the two different types of interviews. I then compare visualizer and non-visualizers. Finally, I compare students who earned the same letter grade in calculus. Within each discussion I also mention the characteristics of the interview participant (such as if they prefer to visualize while solving mathematical tasks)..

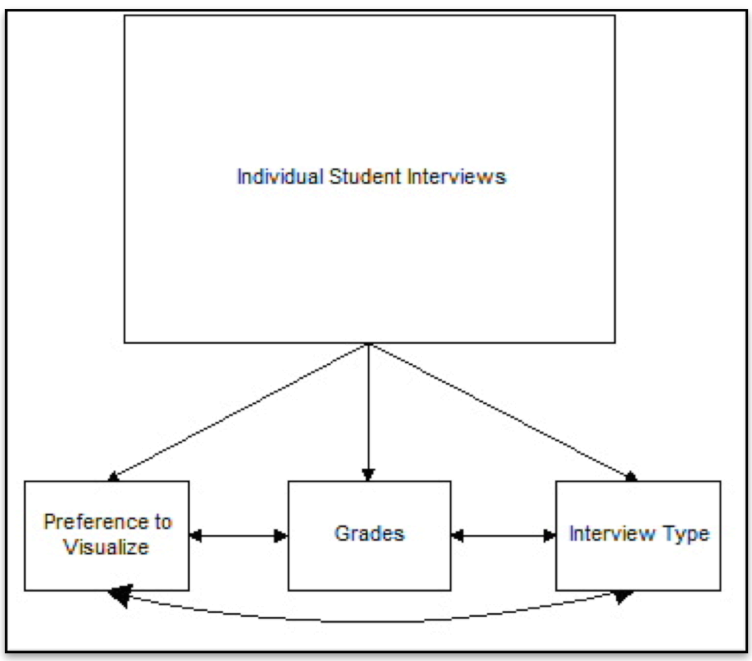


Figure 4.1 Flow chart illustrating the comparisons made between interview participants.

I discuss the most commonly occurring codes present in each type of interview and discuss codes occurring at a high frequency with each other. The flow chart shown in Figure 4.2 illustrates how each type of interview is analyzed. When comparing the two interview types, I discuss the code most commonly occurring in each.

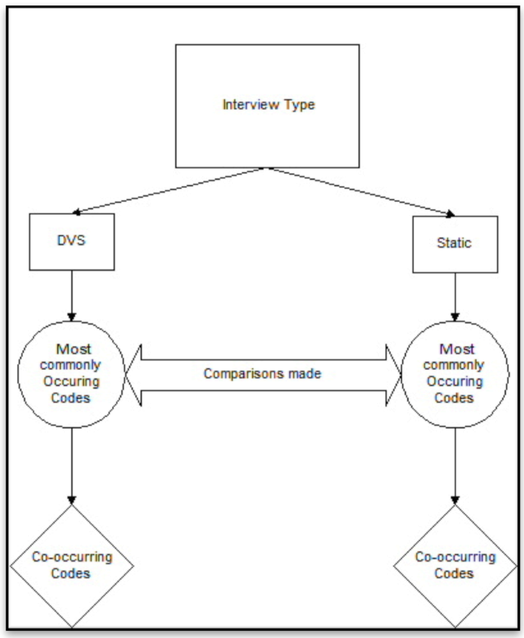


Figure 4.2 Flow chart illustrating how each type of interview was analyzed and how the different types of interview were compared.

The most commonly occurring codes are listed in Table 4.1. This table also gives a short description for each code. It is important to note that in episodes containing for example, procedural knowledge, the knowledge may be correct, partially correct or incorrect. I use Tall & Vinner's (1981) definitions for concept image and concept definition; both terms refer to a student's personal concept image and concept definition for a topic. An episode containing evidence of a student's concept definition may not fully align with the formal concept definition of the topic. A student's concept image and concept definition may be mis-aligned with one another and the formal concept definition.

Table 4.1 Reference for commonly discussed codes

Code	Characteristics
Concept Definition	Using Tall & Vinner's (1981), this indicates the words that a student uses to define a concept. A personal concept definition often does not align with the formal concept definition.
Concept Image	Using Tall & Vinner's (1981) definition, this indicates the words or images that a student uses to define a concept. A personal concept image may not align with the formal concept definition or with the holder's concept definition.
Conceptual Knowledge	When a student discusses the relationships between concepts and topics presented. The evidence may be correct, partially correct or incorrect. An excerpt where a student communicates incorrect conceptual knowledge but then transitions to partially correct (or fully correct) conceptual knowledge was coded accordingly to capture the transition. There were also transitions from partially correct conceptual knowledge to correct conceptual knowledge.
Evidence of Visualization	Indicates places where a student may be visualizing while solving a mathematical task. This includes sketching an example, using hand gestures or when descriptive words suggesting visualization occurred. For the DVS interviews this code might indicate where the student was interacting with the software in a novel way
Evidence that DVS Clarifies	That a student's interaction with DVS (possibly in conjunction with interviewer questioning) results in a better understanding of the relationship.
Evidence that DVS Obscures	That interaction with DVS may have resulted in the student overlooking the relationship being highlighted. Also indicates when a student's understanding of the relationship is incorrect because of the interaction with DVS.
Geometric Approach to Rate of Change	When the approach used to find, or approximate, a desired rate of change (average over an interval or instantaneous at a point) is geometric in nature. Words used may include slope or rise over run
Procedural Knowledge	When a student mentions, or uses, a procedure to approach a mathematical task. Procedural knowledge may be incorrect. This code is used when a student presents "memorized facts" or says, "I don't know why." An excerpt where a student communicates incorrect procedural knowledge but then transitions to partially correct (or fully correct) procedural knowledge was coded accordingly to capture the transition. There were also transitions from partially correct procedural knowledge to correct procedural knowledge.
Student Understanding of Instantaneous Rate of Change	When a student discusses instantaneous rate of change. This code is not used when a student simply refers to "the derivative at a point" or "the slope of the tangent line" unless the question or student mentions instantaneous rate of change.

I created a referencing code and episode number to use with each interview excerpt discussed. This appears at the end of each quotation in square brackets and consists of a combination of letters and numbers indicating where the material can be found within the interview transcripts. For example, the reference code [E1:CI1p9L5-22] indicates this is episode 1 and that the quotation is from Corbin's interview 1 and can be found on page 9 lines 5 to 22 of the transcript. I chose student monikers that each begin with a different letter, so the C in the reference code only refers to Corbin. Quotations from the exit interviews are coded using IEx. In the event that a quotation used is on two pages of the transcript, the code will show two pages and the corresponding beginning and ending line numbers. For example, the reference code [E10:BI3p6L24-p7L4] indicates that episode 10 begins on page 6, in line 24 of Beverly's interview III and concludes on line 4 of page 7.

4.1 Student Vignettes

I first introduce the interview participants for the study; each student completed a baseline demographic survey (see Appendix L) prior to taking the MPI, thus providing much of the background information provided below. In the event that a student's race or ethnicity that is not specifically mentioned then the student indicated that her race was Caucasian and her ethnicity was non-Hispanic, or that the student did not share this information on the survey.

4.1.1 Corbin

Corbin is an 18 year-old male; he graduated from a small high school, with a student population of 600 students enrolled in grades 9-12. He completed calculus in high school but did not indicate that it was an Advanced Placement (AP) course. His chosen major, as of August 2013, was Civil Engineering. Corbin correctly answered 11/12 items on the MPI and has a visualization score on the MPI of 17/24, indicating that he prefers to visualize when solving mathematical tasks. Corbin earned a grade of F in his first-semester calculus course. He did not participate in the Emerging Scholars Program and was in the static interview group.

Corbin's classification as a visualizer is corroborated with 22 instances during his static interviews where there is evidence of the student visualizing as he worked on a task. In the excerpt below from Interview I, Corbin explains his thinking as he considers filling in the table values for the average rate of change of the function $f(x) = x^2 - 5x - 6$ when $x_1 = x_2 = 1$. Corbin is struggling to reconcile his beliefs that the average rate of change would be zero, because "it doesn't move," his thoughts on the value of $\frac{0}{0}$, as well as what he sees in the pattern of the average rate of change and the difference quotient. He sketches a parabola as a general function (see Figure 4.3) and uses his sketch to reason using the symmetric properties of a parabola. Corbin does not appear to realize that the vertex of graph of f is not at the origin.

> Corbin: I'm just thinking real fast. So if I change from 1 to 0 okay never mind, never mind. I was thinking for some reason - -
>
> Interviewer: What were you thinking?
>
> Corbin: I was thinking, so say we're given a parabola (student draws a parabola) [See figure 4.3]. I was thinking for some reason I was (inaudible) for this point on this side and then corresponding point on this side but we're actually just working with points in the first quadrant, I'm pretty sure. Because the first half - - no we're not. Cuz if x is changing by negative 1 as we're going back this way to, or we're going from further here to here. Okay. One second. I'm getting scatterbrained. Let me come back to this. Let me solve this real fast this way. Uh, so 2 squared minus 5 plus 2 minus 6, this is gonna equal 4 minus 10 minus 6, which would be 6 which would be 0. 4 minus 10, oh oops, 4 minus 10 is gonna be negative 6 minus 6 is gonna be negative 12. Okay. I don't know. I was thinking for some reason we were - - we were using corresponding points along the parabola but we're not. So this was just a random thought that had popped into my head. Okay let's see real fast so negative 10 over 1 minus negative 12 minus 2. Is gonna be negative 1 over 2 which is gonna be negative 2. Okay. Right now. Cuz (inaudible) x_2 okay. All right there has to be something else in here that I'm not seeing. To find an easier way to go. There's gonna be some form of,

um, relation between all of our average rates of change as they go, as they meet at a certain point. [E1:CI1p9L5-22]

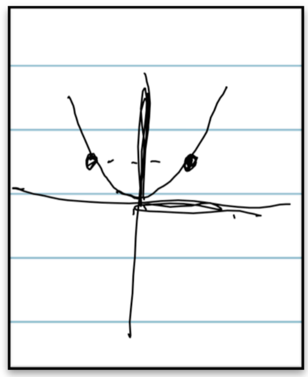


Figure 4.3 Evidence of Corbin Visualizing during Interview I.

In the Exit Interview Corbin was asked to compare the instantaneous rates of change for two functions at the same x value. His description of how he knows that there is a difference also suggests that he is visualizing as he makes the comparisons between the two function graphs. "Um, because on $g(x)$ it's more - - it's gonna be, um, a more slanted, um, tangent line than on $f(x)$. $f(x)$ is almost gonna be a horizontal tangent line." [E2:CIExp2L16-17]

Sometimes Corbin uses the graph of a function with a known feature, like having a sharp corner, as a method of reasoning about another similar situation. In the excerpt below, from the Interview III, he is asked about the differentiability of a function over its domain. The graph of the function given has several points where there is a corner.

Interviewer: Okay. And what does it mean to say that a function is differentiable everywhere on its domain?

Corbin: Um, oh, let me think - - when you - - okay. You can draw a tangent line anywhere - - your derivative is never gonna equal 0. And I'm just - - on a graph you can never have a cusp, a sharp corner, or a hole in the graph. Those are the three things that make it non-differentiable.

Interviewer: Okay. So why did you say that - - where does the 0 - - the derivative equaling 0 come into that?

Corbin: Um at a sharp corner like the absolute value at (0,0). Which - - um, here I'll just. I draw pictures to make it easy (Student draws graph of $y = f(x) = |x|$) [See Figure 4.4.].

Interviewer: That's fine.

Corbin: Like on this one if you take the derivative at this point right here (student indicated the point (0,0) on his picture shown in Figure 4.4) it's going to be 0 so this is non-differentiable at 0.

Interviewer: Because the derivative is 0?

Corbin: Yes. Wait let me think, let me think that through. Yes.

Interviewer: Okay.

Corbin: When the derivative is 0 it's non-differentiable. No it's not. That's just uh - - oh, wait, let me think that through, because on sine and cosine you have all of those and they're all differentiable.

Interviewer: So say that again.

Corbin: On like sine and cosine, um, if you look at the photograph it's all wavy and stuff but it's differentiable at all - - all of its points where it's zero zero It's - - it's if they come to a point at a sharp angle. I'm trying to think what it is - - how to word it. I know what it looks like. (Student draws the graph of $y = \sin(x)$ and marks places where the derivative is zero.) [E3:CI3p1L9-31]

…

Corbin: Oh, let me think it through. It is - - no, it's not different because it's a cusp. Oh. Not exactly sure. [E4:CI3p2L1-2]

Episodes E3 and E4 are examples of Corbin attempting to rely on procedural knowledge that is seemingly neither well supported conceptually nor well understood. In episode E3, Corbin makes the erroneous statement, "when $f'(x) = 0$ the function is non-differentiable". This appears to be an incorrect recall of a possibly memorized 'fact' that Corbin is relying on as he continues to reason about the mathematical concepts and tasks he encounters.   Throughout the interview series, Corbin makes similar statements as though he is trying to pull from procedures that he neither understands nor had fully memorized. He uses visual representations to reason and sometimes realizes that his statements contradict one another. In episodes E3 and E4 above he leaves this conflict unresolved as if he is unsure which example he drew (see Figure 4.2) applies to the question he is considering.

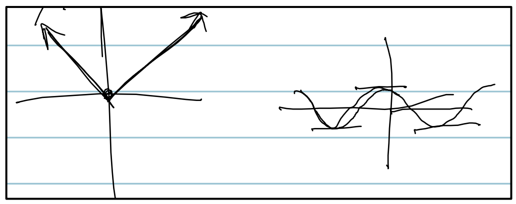


Figure 4.4 Corbin's visualizations as he reasons about the existence of the derivative at a point making a sharp corner.

There are 32 instances of Corbin exhibiting procedural knowledge and only 12 where he shows evidence of conceptual knowledge. Not all of these instances are examples of correct thinking. In many cases Corbin appears to be searching for a procedure to direct his knowledge, and many times this is not a successful endeavor for him. In episode E5 below, from his exit interview, Corbin is asked about the relationship between the instantaneous rate of change of a function at a point and the derivative value of the function at the same point:

> Corbin: Instantaneous rate of change at a point and the derivative? When you set the derivative equal to 0 - - no, one second, let me think about it. Yes. Wait. Instantaneous rate of change. Oh, trying to think of examples so it makes sense to me. Instantaneous rate of change. Okay. The derivative, how are they related? When you set the derivative equal to 0 at a point it's gonna be your x value, no, no, no, that's too simple. [E5:CIExp1L22-26]

In another case, Corbin relies on his procedural knowledge of “rise over run” to guide him as he finds the slope of a tangent line he sketched on to the graph modeling the temperature taken at the top of each hour in Whitefish, Montana (see Appendix B).

Corbin: And, and um, I'm trying to just think how it would - - so for 1 hour - - cuz 1 hour's gonna be the easiest one cuz that's gonna be your change so it'd be cuz your rise over 1 is just gonna be your rise. Okay. 2, okay, so from 12 to 13. Okay. So it's - - oh here I'll bring it back over here.

Interviewer: Okay.

Corbin: Just makes it easier so we're looking for rise over run.

Interviewer: And what is that?

Corbin: That's gonna be the slope for the tangent line. [E6:CI1p5L9-16]

Something to note about Corbin is how he increasingly relied on a set of examples that he generated. He would consider an example function, often sketching the basic graph, and then explore as best he could with his example. In the following episode E7, Corbin is asked about why the instantaneous rate of change cannot be found at a point where the graph has a sharp turn or a cusp. This exchange differs from the other in that Corbin shows evidence of developing conceptual understanding as he thinks, explores and talks about his example, the graph of the function $f(x) = |x|$.

Corbin: Because you cannot solve - - you cannot solve for the instantaneous rate of change at that point - - thinking. Because - - okay. This side - - okay. Um, so I'm trying to think. Okay. So let's use the absolute value and draw the, um, derivative of it. If you draw the derivative of the absolute value on the left side of the y axis it's gonna be -1 and then on the right side of the axis it's gonna be positive 1. [E7:CIExp6L5-9]In general, Corbin's procedural knowledge is not strong enough to yield a correct response to the static questions in the interviews. More interesting is how Corbin uses his limited procedural knowledge to reason about the questions posed. There are eight times in the interviews where Corbin's thoughts are classified as "sense-making." As in the long excerpt above, Corbin attempts to rely on specific memorized facts that are completely disconnected from a procedure or concept, and these serve as the basis for his sense-making. He often utilizes his library of self-generated visual examples as well when he engages in sense-making.

Corbin also struggles with his ability to consider how one quantity changes with respect to (or in response to) and other quantity. In many cases, Corbin struggles to keep the various mathematical ideas (definitions, facts, procedures, etc.) and their relationships clear in his mind. Just prior to episode E8 below from Interview III, Corbin was given five closed intervals and asked to mark the points (or points) in each where he estimates that the instantaneous rate of change for the function is greatest (see Figure 3.6). He struggles to relate how the local maximum values of the first derivative correspond to points on the second derivative function.

> Interviewer: What would those second derivative points look like?
>
> Corbin: They would be - - okay. So that is the greatest. The second derivative - - let me think here for a second, um. Okay, so - - so I'm trying to think. Cuz from there you got here and the derivative of - - the derivative of a parabola is a linear line. Oh. Okay. The derivative value, okay. So those cross - - where it crosses the x axis. The other one - - okay. So the values for 7. What would they be on the second derivative? What would they be for the second derivative?
>
> Interviewer: Yes.
>
> Corbin: Those would be minimums and maximums.
>
> Interviewer: For the second derivative?
>
> Corbin: No. No. No. No. No. [E8:CI3p7L18-28]

Overall, Corbin's reliance on weak procedural knowledge along with his weak ability to understand how related quantities covary results in an inability to reason appropriately about mathematical ideas relating instantaneous rate of change. Corbin also exhibits a very weak foundation of knowledge from which to pull information as he reasons. Many times he states that he does not know common vocabulary words (tangent line, secant line, etc.) and he also lacks a basic understanding of prominent structures presented in calculus, like the definition of the difference quotient.

4.1.2 Beverly

Beverly graduated from a very large urban high school with a student population of 2300 students enrolled in grades 9-12. She is an 18 year-old female of Asian heritage who completed AP Calculus AB as well as three other AP science courses in high school. She correctly answered 10/12 questions on the MPI and has a visualization score of 16/24, indicating that prefers to visualize when solving mathematical tasks. Beverly indicated a major of Chemistry/Biochemistry in August 2013. Beverly participated in the Emerging Scholars Program and was in the DVS interview group; she earned an A in this course.

There are seven instances where Beverly was visualizing (outside of her interactions with DVS) as she worked through the questions in the interviews. During the Exit Interview, where DVS was not offered as a resource, Beverly drew numerous pictures to aid her as she explains how she thinks about mathematics. In excerpt E9 below (from the Exit Interview), she is explaining what first comes to mind when thinking about the word “derivative”.

> Beverly: Okay. So when I think of derivative I think of the, um, slope of the tangent line and just basically, um, a continued slope almost and so, like, let's say we have this graph. (Student draws examples) Um, let's do an easier one, there [See Figure 4.5]. Let's see if that works. But um, okay. Where the derivative, um, that's basically the slope of this line. So at this point the slope is 0 and so at this point it hits 0 and at this point the slope is positive and it's going up, so that's kind of what's going on here. And it kind of goes up higher, so I guess this isn't a perfect curve but like that almost. Um, that's all I can think of for derivative. [E9:BIExp1L4-9]

Beverly is able to communicate the image evoked when she thinks about the word derivative (see Figure 4.5). She is also able to make connections between the derivative of a function, the slope of the tangent line (though she does not precisely call it that) and other procedural feature that seem important to her, such as when the derivative is zero.

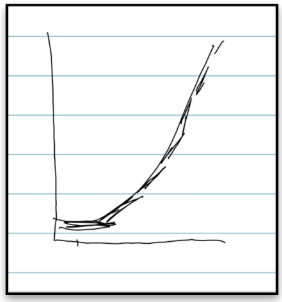


Figure 4.5 Beverly's sketch describing her understanding of derivative.

There are 17 instances where Beverly exhibits conceptual knowledge about the topics being discussed and 13 where procedural knowledge is evident. In Interview III, Beverly explores with the graph of a piecewise-defined function using DVS and investigates the relationships between the slope of a tangent line at a point on the graph of the function and corresponding point on a derivative graph. During this particular exchange (episode E10), Beverly comments about the derivative graph (and, subsequently, the second derivative graph) without looking at the derivative graph. Just prior to this episode, Beverly is given a set of closed intervals and asked to find the place within each interval where the function's instantaneous rate of change is greatest.

> Interviewer: So on the graph$f'(x)$, what would these point that you collected represent? What kind of points would they be?
>
> Beverly: They would be the max and min of $f'(x)$.
>
> Interviewer: Why do you think so?
>
> Beverly: Because those are the greatest um, that's the greatest slope, I mean that's the - - well that's the greatest slope of the original function and those are the greatest points of f'.
>
> Interviewer: Okay. So I'd like you to then tell me what would the value of the second derivative of this function be at each of the points that you've collected?
>
> Beverly: 0 just like how the - - cuz in your first function, when you get the derivative and you find the zeroes of it that your extreme ones which find

the maximum and min with the relative. So it's the same idea or if you have $f'(x)$, and you find the derivative of that and you find the zeroes that's the extrema of $f'(x)$. [E10:BI3p6L24-7L4]

Beverly is able to make a generalization about the relationships between f, f', and f''. Not only that, but she exhibits the ability to keep track of covarying quantities. Beverly is also able to view both the first and second derivatives as functions themselves.

As Beverly interacts with the DVS there are three instances where the software may be obscuring Beverly's investigations; this contrasts with the twenty-two times that the software helped to clarify a relationship for her. During Interview I, Beverly investigates average rate of change. She manipulates a point on the graph and, as she does, she also collects data about the function values using the table features in the DVS. Eventually, the two points are so close together that the line connecting them disappears. In the excerpt from Interview I below, Beverly attempts to explain why she thinks this happened and what the average rate of change might be when $x_a = x_b$.

> Beverly: Okay. Then um it would be 0 because if x_b is the same as x_a that'd be 0. Over $x_b - x_a$ and that'd be 0 too. Which is just the same things or same numbers.
>
> Interviewer: Okay. So you're saying there is no - - is there an average speed there then?
>
> Beverly: Um. There is but it can't be found using the slope.
>
> Interviewer: Why not?
>
> Beverly: Because the slope - - or the slope of the secant line there is no secant line because the secant line needs two points and there's only one point exactly cuz they're on top of each other. [E11:BI1p7L12-20]

Initially the software possibly obscured the relationship between average rate of change, the distance between points A and B, and instantaneous rate of change at a point because Beverly makes a statement that when the secant line disappears from the screen the slope of the secant line, "… would be 0." Beverly believes that the average rate of change, which she previously acknowledged was represented by the slope of the secant line, equals zero when $x_a = x_b$. However, because the secant line disappears from the screen, Beverly was forced to explain why this happened and what its disappearance implies about the existence of both the function's average rate of change and the secant line when $x_a = x_b$. The correct reason that the line disappears from the screen remains elusive for Beverly, "… it would be 0 because if x_b is the same as x_a that would be zero. Over $x_b - x_a$ and that'd be 0, too." Beverly does not notice the flaw in her use of the slope formula and that it is not appropriate because $\frac{0}{0}$ is indeterminate. She continues this logic with her statement that the particle's average speed exists when $x_a = x_b$, but, "…it can't be found using slope." However, once Beverly reviews the data she collects and manipulates point B, she correctly reasons that the secant line does not exist when the points are the same. After some further investigation and questioning from the interviewer, Beverly reasons again about this situation. Episode E12 is from later during Interview I.

> Interviewer: So, you're saying that the slope of the secant line is the average speed?
>
> Beverly: Yes.
>
> Interviewer: And when they're on top of each other you said you can't find?
>
> Beverly: Um, the average. Oh! you find the instantaneous so that means at that instant so technically it's not the average because it's at that instant.
>
> Interviewer: Okay. So do you notice anything as you look at these though did they behave kind of in the same fashion as you saw here? Cuz what did you say as you looked at all of these and you were up here playing with them?
>
> Beverly: Um this - - B was getting closer and so $x_b - x_a$ was getting closer to 0 and then once they were on top of each other there was, uh,

you can not define slope therefore the speed you couldn't define using, um, a secant line where you use just is slope. [E12:BI1p9L12-23]

Although Beverly is initially unsure about the implications of $x_a = x_b$, her initial interactions with DVS do not fully clarify the relationship, once she is able to manipulate the points again and talk through what she sees she makes a conceptual statement relating instantaneous rate of change and the limitations of average rate of change. In this way, DVS exploration is not enough on its own to clarify the relationships, but it undeniably plays a role in Beverly making these connections.

Episodes E11 and E12 above also suggest that Beverly experiences a change in her concept image and concept definition in order to bring the two structures into alignment. She initially believes that the scant line provides information about a function's average rate of change as successive secant lines are approaching coincidence with the tangent line at the point $(x_a, f(x_a))$. After the DVS exploration and questions from the interviewer, Beverly conveys a concept image that includes correspondence to instantaneous rate of change when the interval $[x_a, x_b]$ is sufficiently small.

4.1.3 Amy

Amy graduated in May 2013 from a large private high school. She is an eighteen year-old female who indicated a major of mathematics. Amy completed AP Calculus AB as well as two other AP courses. Amy correctly answered all of the questions on the MPI and has a visualization score of 8/24, that of a non-visualizer. Amy earned an A in first-semester calculus and was a participant in the Emerging Scholars Program. She was in the DVS interview group.

Amy resisted interacting with the DVS and was able to make statements connecting conceptual knowledge, examples and procedural knowledge at a far higher rate than any of the other participants. There were twenty-two instances of conceptual knowledge, nine places where Amy used examples in her reasoning and thirteen instances of procedural knowledge. In the following excerpt, E13 from the Exit Interview, Amy was asked to explain the relationship between the derivative of a function at a point and the instantaneous rate of change at that point. Not only is Amy able to say that they are the same, but she is able to give an example and (later in the Exit interview) connect this to a procedure for finding a derivative using the power rule.

> Amy: They're the same thing, really.
>
> Interviewer: Can you explain that?
>
> Amy: Like to find the derivative you - - when you take the derivative, if you want the instantaneous - - like, say you're given a function -- if you take the derivative and plug in a particular point then you have the instantaneous rate of change at that particular point. [E13:AIExp1L10-14]

Although the MPI places Amy as a non-visualizer, there were ten instances of visualization. Amy appears to be reluctant to sketch or draw her explanations, but uses her hands when speaking and can describe, in vivid detail, how she pictures a certain relationship or scenario. In excerpt E14 from Interview II below, Amy is asked how she might draw a derivative graph if she was given a set of x values and the instantaneous rate of change at those values. Amy begins to talk with her hands, as if she is visualizing. I offer that Amy can draw a picture (see Figure 4.6) if she prefers so that I can gain insight into her thinking. Because this suggestion to sketch a picture was provided to Amy the note of “prompted visualization” is included.

> Amy: Um, I'd graph the - - let's see, cuz - - so you're giving me instantaneous rate of change?
>
> Interviewer: And an x value.
>
> Amy: And an x value. Okay. Well I would just, like - - and draw my axes, and then I'd - -

Interviewer: Can you show me?

Amy: (begins to sketch) [prompted visualization – see Figure 4.6] Oh, yeah, sorry I would like draw my axes. And you said at 0 it was 2?

Interviewer: Let's say at 0 it was 2.

Amy: So I'd, like, put a point there.

Interviewer: At - - what are the coordinates of that point?

Amy: This is, um, (0,2).

Interviewer: Okay. And at 1 it was 4.

Amy: So I would say you have 1 and - - 4.

Interviewer: And at 2 the instantaneous speed was 0.

Amy: Like, that and then I guess I would just draw a - - I guess I could connect the dots, but I'm not really sure if that's actually how it's - - yeah, I think I'd do something like that, I guess.

Interviewer: Okay.

Amy: Just draw a little curve - - a simple sketch. [E14:AI2p2L3-21]

Prior to drawing the above sketch seen in Figure 4.6 Amy describes how she would proceed; she begins with "and draw my axes." Initially, Amy approaches the task in a procedural nature, but the description suggests that she may be visualizing the outcome as she does this.

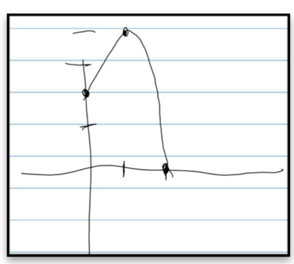


Figure 4.6 Amy's sketch relating known values of the instantaneous rate of change at discrete points to the derivative graph.

Many of the instances where Amy exhibits procedural knowledge are immediately followed by evidence of supporting conceptual knowledge. In episode E15 below, from Interview I, Amy was investigating the effect of manipulating the distance between two points on the graph of a quadratic function. The two points lie on a secant line. Amy states that the slope of this secant line tells us the average rate of change for the function between the two points in question. She has manipulated the points so that the interval is very small.

> Interviewer: Okay. And is there a way that we could estimate our instantaneous speed of this particle at time x_a?
>
> Amy: Time x_a? Well, you would use the definition of a tangent line and or definition of a derivative and, uh, calculate the slope of the tangent line doing the, um, using the h notation doing the, um, $\frac{f(x+h)-f(x)}{h}$. [E15:AI1p5L6-10]

In this excerpt, Amy does not mention that h must be approaching zero; in fact, she does not seem to notice that the procedure she gives could be used to find the slope of a secant line if $h = x_b - x_a$. There is evidence later in the interview that Amy's exploration with the DVS, particularly when she is able to collect values in a table, clarifies Amy's understanding of the relationship between a secant line, average rate of change, tangent line and instantaneous rate of change. The following episode E16, also from Interview II, occurs after the E15. Amy is discussing what she notices about the relationship between average rate of change and the interval $[x_a, x_b]$.

> Amy: Okay. Well the difference between x_b and x_a gets smaller. And the average speed gets smaller.
>
> Interviewer: Okay. Now if we do this did that change things?
>
> Amy: Uh, yeah, the difference between them is now 0 and because you can't have a denominator as 0 it has the slope as undefined.
>
> Interviewer: So what happened to our secant line?
>
> Amy: It's no longer there anymore.

Interviewer: Why do you think that is?

Amy: Well, because the secant line needs to go through two points but when they're right on top of each other it's like there's only one point. [E16:AI1p5L29-6L5].

This episode contains evidence that Amy may have experienced a change in her concept image and definition of secant line and tangent line. It appears that Amy now has some understanding about the relationship between when $x_b - x_a = 0$ and when a secant line does not exist, thus suggesting that she has gained insight into when the instantaneous rate of change can be found. This also suggests that Amy's investigation with DVS was the catalyst for this change. There were six such instances through Amy's interviews where the DVS seems to clarify a mathematical relationship for her.

Amy's interview transcripts contain more instances of evidence that she holds an algebraic view of instantaneous rate of change than any other DVS participant. I found five places suggesting that she may hold a mostly algebraic view of instantaneous rate of change. One such example occurs in excerpt E15 when Amy relates the slope of the tangent line to the difference quotient. During Interview I, when asked to find the average rate of change of a particle whose position graph is shown on the computer screen, Amy gives an answer, and then states that she "just did $\frac{\Delta y}{\Delta x}$ [E17:AI1p4L14]." Available to Amy was the slope of the secant line she was manipulating, but she does not explicitly connect this to dynamically changing slope of the secant line.

4.1.4 Evan

Evan, an eighteen year-old male, graduated from a rural high school with 1000 students in grades 9-12.   AP Calculus AB is the only AP course that Evan completed. He correctly answered 11/12 questions on the MPI and had a visualization score of 6/24, suggesting that his prefers not to visualize when solving mathematical tasks. Evan earned a C in calculus and did not participate in the Emerging Scholars Program. His major, as of August 2013, was Computer Science. Evan participated in the static interview group.

There were nine instances where Evan may have been visualizing. During episode E18 below, from the Exit Interview, Evan attempts to explain why you cannot find the instantaneous rate of change of a function at a cusp for a given graph. He draws the picture seen in Figure 4.7 to illustrate that you can draw infinitely many tangent lines through that point. He states, “There are so many different tangent lines, or you can draw -- tangent lines you can draw on that point that there is no way of determining which one would be the correct one. And you can also draw a - - you - - no, that's not - - no, you can't draw a vertical tangent, that wouldn't work very well.” [E18:EIExp5L29-32]

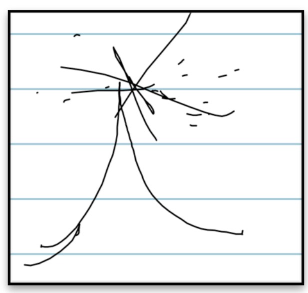


Figure 4.7 Evan's sketch describing why the derivative at a cusp does not exist.

Further evidence of visualization occurred during Interview II. Evan was using a provided graph of a function, as well as the function definition, to find the instantaneous rate of change at specific points. As he completes each part of the task he engages in sense-making as he refers to the graph as he checks the validity of his work.

Interviewer: What were you checking?

Evan: To see how much it looks like it's increasing cuz that just looked like a really weird number to get.

Interviewer: How can you tell how much it's increasing?

> Evan: By how much the -- basically I drew a tangent line in my head where it'd be where the points would be sort of. Like where it'd sort of be cuz I can't get that exact. I can get pretty close it'd be like right there so it's kind of going down, not very much though. I wasn't sure if that was a good number or not, but it looks like it for the y coordinate of that point at least. [E19:EI2p5L14-22]

Though the nature of this static interview was heavily procedural, Evan appears to be checking his work using mental images. Initially, he engages in sense-making because, “that just looked like a really weird number to get,” and he looks back at the provided graph to make comparisons. His statement, “…I drew a tangent line in my head…” further supports this; however, all the work that Evan wrote for this interview was procedural. In fact, there are thirty-three instances coded as procedural knowledge throughout Evan’s interviews and ten instances coded as conceptual knowledge. In excerpt E20 below, from Interview III, he is discussing places on the graph of a continuous piecewise-defined function where he indicated that the instantaneous rate of change would be zero. At $(2, f(2))$ the graph shows an inflection point (see Figure 3.6).

> Evan: The maximum and minimums, but I do not believe the one I marked at 2 would be considered a maximum or minimum.
>
> Interviewer: Why not?
>
> Evan: Because it doesn't really - - because you have a slope that's decreasing and a slope that's increasing approaching the same point. For it to be a maximum or minimum it has to be decreasing or -- you have a slope that’s - - you have - - the graph is increasing at a decreasing rate and decreasing at an increasing rate but both of them are going through the same point and in order to have a maximum and minimum it has to be decreasing to - - like the slope has to be - - the graph has to be decreasing and then start increasing for a minimum, or increase and then start decreasing for a maximum. That one doesn't have either. It's increasing all the way through. It's just a difference of how fast it's increasing. [E20:EI3p5L26-p6L2]

This excerpt suggests that Evan has some conceptual knowledge about when a point on the graph of a function might be a maximum or minimum value. He appears to understand that on the function graph he is analyzing, at the point $(2, f(2))$, the derivative is zero but he incorrectly states that the “rate of change of the rate of change” differs on either side of $x = 2$. Evan is able to further connect how this is not a maximum or minimum function value. He does not, however, mention what he thinks about the function maximum or minimum value occurring at a point where the derivative does not exist. Later in the interview he does state that the function’s maximum value occurs at the point $(3,3)$.

Evan was often able to use his procedural knowledge to determine when his statements or calculations were incorrect. As an example, excerpt E21 below is from Interview I. Evan was asked to find the average rate of change in temperature from noon to 3 p.m. using a provided chart. Initially, Evan uses the wrong procedure and, as a result, he finds the average temperature from noon to 3 p.m. He is able to reason that his answer does not make sense and procedurally make the change necessary to answer the desired question. This episode (E21) also presents evidence that Evan’s conceptual knowledge did not help to correct the procedural error as he was able to ascertain the fact that his initial attempt did not make sense with the data given.

> Evan: Yeah, cuz I actually don't know what I just did now. Okay. So A is 32.5 divided by 3 equals 10.83. A is something like that. Didn't work when I tried to find the fraction earlier.
>
> Interviewer: So what does this represent?
>
> Evan: It is the average rate of change from noon to 3 from -- and my mind just went blank. It's the average rate of change from noon to 3 with the values - - or from noon to 3:00 p.m. with the values given.
>
> Interviewer: Okay.
>
> Evan: So from. No, no, wait, no it's not - - that's the average temperature from then to then. Average rate would be - -
>
> Interviewer: How do you know that's the average temperature? What makes you say that?

Evan: Because that is the temperature at noon. I added the temperature at noon and the temperature at 3. I forgot that you're supposed to subtract the temperature at 3 from the temperature at noon then divide it by however many hours is between the two. 15.3. Yeah, I am not gonna try. 3.9. 3.9 over 3 is the correct - - we have change of - - average change of temperature. That's what I was trying to figure out. [E21:EI1p1L21-p2L3]

In general, Evan was able to provide rules that he followed during his interviews. He would state the power rule, then implement it, when appropriate, or procedurally find the desired answer to a question. His work during the interviews consisted of many calculations.

Evan relied on procedures in an effort to relate average rate of change over an interval and instantaneous rate of change at a point. Also evident in the next excerpt, E22, is that Evan is able to visualize when solving a mathematical task, even when the task does not specifically call for visualization. He uses the illustration shown in Figure 4.8 to explain his “limit of the secant lines” comment. In a previous portion of the interview, he was able to determine that average rate of change and the difference quotient are the same over a given interval, unless "$h = 0$".

Interviewer: To find the average rate of change you would?

Evan: The average rate? Yeah, I would take the 2 points in question and do the slope formula to find the average rate of change like the x -- $\frac{y_2-y_1}{x_2-x_1}$ for the average rate I would, like if the, if this was the function (student begins to sketch a parabola) [see Figure 4.8]. I would take this point and say this point and I would find the slope between those two using the y -- $\frac{y_2-y_1}{x_2-x_1}$ to find the average rate of change between those two points, but to find the instantaneous rate I would do my best to draw the tangent line at whatever point. Whatever the point in question in this case 1 (student indicates one point on the graph of the function that was also on the secant line) and use the same formula for two points on the tangent line to solve for the instantaneous rate of change.

Interviewer: Okay. For the average rate of change you drew a line connecting two points that were on your graph. Do you have a name for that line?

Evan: Oh, that would be - - which line is it? Secant line. Yeah, secant line.

Interviewer: So you could find the average rate of change by finding?

Evan: The limit of the secant lines.

Interviewer: The limit of the secant lines?

Evan: The limit of the slope of the secant lines I guess is the more appropriate phrase. It would be - - you would just take a number as it gets closer to x and find the slope of said line until the change of x is equal to 0 using the difference quotient would be the - - which would be the best bet because then you could plug in the change of x easy - - you could plug in the change of x without having to keep track of all the, three different numbers you only have to keep track of two, or I guess one. [E22:EI1p13L34-p14L20] .

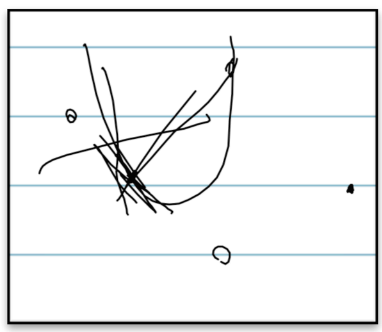


Figure 4.8 Evan's sketch of the "limit of the secant lines."

This episode highlights Evan's reliance on memorized procedural knowledge. His statement that he would, " … do the slope formula to find the average rate of change…" suggests that he has a weak association between the slope of the secant line that passes through the two points in question as he works to find the average rate of change. In fact, Evan suggests that you could find, "the limit of the secant lines," in response to a question concerning average rate of change; however, his description of "change in x getting closer to 0," suggests that he is using the average rate of change to approximate the instantaneous rate of change at a point, though it is unclear if he realizes this. As he began to write out the formula, he stumbles over the variables and mentions "x" and "y". After reviewing the video and transcripts, I do not believe that this has any meaning other than Evan talking to himself as he states the formulas for average rate of change.

4.1.5 David

David is an eighteen year-old, African-American male; he is a 2013 graduate of a large urban high school with a student population of 2100 students enrolled in grades 9-12. The only AP course David completed in high school was Calculus AB. As of August 2013, his indicated major was Mechanical Engineering. David correctly answered 10/12 items on the MPI and has an MPI visualization score of 18/24, classifying him as a visualizer. He participated in the Emerging Scholars Program and earned a course grade of C. David interacted with the DVS during his interviews for this study.

David's interaction with the DVS resulted in one of the four instances of uncontrollable mental imagery in the study. The incident occurred during Interview I; David was about to begin exploring with a sketch centered around the exponential function $f(x) = e^x$ when this episode (E23) occurs:

> David: Oh, what do I know about exponentials?
>
> Interviewer: Yeah.
>
> David: So it's not like it has asymptotes so it's like never touching whatever like what the 4 and 0 (student uses his hands to gesture that the function has a vertical asymptote at $x = 4$ and a horizontal asymptote at $y = 0$).
>
> Interviewer: What do you mean by it never touches 4?
>
> David: It's like it's going up, so there's an asymptote and an asymptote right there. (student again gestures)
>
> Interviewer: Okay. So you're saying there's a horizontal asymptote at?
>
> David: 0.

Interviewer: $y = 0$?

David: And a vertical asymptote at, I'd say 4 cuz it passed through.

Interviewer: So you would say there's a vertical asymptote at $x = 4$?

David: Yes. [E23:DI1p11L21-p12L1]

Prior to this exchange, when working with a different sketch within the DVS, David asked if he could explore a part of the graph of a function that he found interesting. This suggests that he felt comfortable, prior to this episode E23, exploring within the software even when he was not directly directed to do so. However, David did not choose to explore further in this instance. Instead, David appears to believe that what he sees on the computer screen is all there is to the graph. David's assertion that the graph of $f(x) = e^x$ has a vertical asymptote at $x = 4$ is a referent that he continues to hold throughout the interview.

Contained within David's interviews are five instances where the DVS obscured the concept or idea that David was exploring. In excerpt E24 below, also from Interview I, David is attempting to investigate the relationship between average rate of change and instantaneous rate of change for a quadratic function.

David: The change in x is equal. When x - - I mean when $A = B$ the change is 0 cuz … the same thing so … and there's no average speed.

Interviewer: Okay. Do you still think that there's no way to tell the instantaneous speed this way?

David: No.

Interviewer: Okay so no? Expound upon no. No you don't or no you - -

David: Oh, no, well - - I mean - - so you can if they're equal - -

Interviewer: What is the? So if what are equal?

David: The two points.

Interviewer: Okay. So what is the instantaneous speed at A if we can estimate it?

David: It's undefined. [E24:DI1p7L23-p8L1]

The table shown in the DVS listed the slope of the secant line as undefined when the two points, A and B, were sufficiently close together. In fact, the secant line disappeared from the screen when the two points were moved to be in the same place. This caused David to assume that the instantaneous speed of a particle whose position with respect to time was given by the quadratic function graphed was undefined. Later in the same interview, when David is interacting with the graph of the function $f(x) = e^x$, he gains some insight into the relationship between average speed and the instantaneous speed of a particle:

Interviewer: And, what did we say that the secant line could represent since they're really close together.

David: The tangent line, oh the speed, the speed.

Interviewer: Well, wait you said tangent line?

David: Yeah.

Interviewer: What is a tangent line?

David: When it only goes through the - - graph lines.

Interviewer: Okay. So talk to me about that. Why did you say that?

David: Cuz that's what it looks like, a tangent line

Interviewer: And what would the tangent line give you or help you with or why is it important?

David: Uh, find the equation of the tangent line, cuz they ask that in every problem. So it's like, that's the speed of the particle? (Student draws on paper)

Interviewer: What kind of speed?

David: Instantaneous? Cuz it's right at the point, right at - - (student begins to sketch) [see Figure 4.9].

> Interviewer: So you just drew what kind of line?
>
> David: Well, that was supposed to be a tangent line.
>
> Interviewer: Okay. Yeah, that's what I wanted to know. And what about the tangent line would give you this instantaneous speed you mentioned?
>
> David: The slope.
>
> Interviewer: So, looking back at the computer you said this is a pretty good estimate then for what?
>
> David: The speed, or the instantaneous speed?
>
> Interviewer: The instantaneous speed and it's the same, you said it was the same as something.
>
> David: The y value of A.
>
> Interviewer: Okay. Then what is the relationship between the particle's instantaneous speed and its position when it follows an exponential model?
>
> David: They're the same.
>
> Interviewer: So if you were to -- could you generalize this for the rate of change within an exponential model?
>
> David: So, yeah, if you, if you have your y value then it would be equal to the instantaneous speed of the particle. [E25:DI1p13L23-p14L24]

Within episode E25 are several transitions in David's thinking about the relationship between average rate of change and instantaneous rate of change. In the initial excerpt (E24) from Interview I, David believes that the instantaneous rate of change for a function does not exist, or at least that he cannot find it, when the two points forming the secant line coincide. However, upon further exploration he makes this statement, "The tangent line, oh the speed, the speed." This is the first time during his interview that he uses the term "tangent line" and it appears that he has made a connection between the decreasing size of the interval $[x_a, x_b]$ and the slope of the secant line being an appropriate estimate for the instantaneous rate of change at point A. From here David is able to discuss the slope of the tangent line as being the instantaneous rate of change for the function at point A. David uses his drawing, shown in Figure 4.9, to further make this point. David may be developing some previously unseen conceptual knowledge, or the DVS may be evoking this knowledge where as previously it was not. There is little to no evidence of David's procedural knowledge in this episode; he simply discussed these relationships.

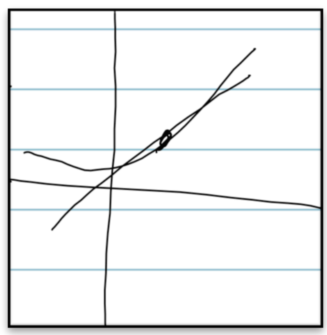


Figure 4.9 David's illustration of a tangent line.

4.1.6 Felicity

Felicity is an eighteen year-old Hispanic female and a 2013 graduate from a very large urban high school with a student population exceeding 3300 students enrolled in grades 9-12. Felicity indicated that she is of mixed racial heritage. Felicity completed AP Calculus AB, as well as two other AP science courses, in high school. As of August 2013, Felicity indicated that she plans to study Industrial Engineering; she was not a participant in the Emerging Scholars Program. She earned a grade of B in her first-semester Calculus I course. Felicity correctly answered 11/12 of the items on the MPI and scored 7/24 on preference to visualize. Felicity is classified as a non-visualizer and she was in the dynamic interview group.

During Felicity's interviews there were five instances where the DVS interactions obscured the connection being highlighted; however, there were twelve instances where the DVS clarified a mathematical object or relationship. During Interview II, Felicity was exploring the relationship between the graph of the function $f(x) = e^x$ and the instantaneous rate of change of this function at a point A. The sketch included the tangent line passing through point A. When asked what she noticed, Felicity initially stated that when the instantaneous rate of change was positive the function was also positive. However, after a slight pause (and moving point A along the function graph) Felicity changes her answer to say that the function is increasing when the instantaneous rate of change is positive. This interaction suggests two things may be at work: Felicity is confusing the generalization that the function is increasing with the specific case that the graph of the function $f(x) = e^x$ is always above the $x-$axis; Felicity, through exploring, was able to make a change in her thinking to include the generalization about increasing functions and positive instantaneous rate of change.

Another instance of evidence that the DVS software clarified a connection for Felicity occurred in Interview I. Episode E26 below occurred during the first two minutes of the interview. Felicity is explaining what she knows about average rate of change, instantaneous rate of change, and the relationship between the two.

> Interviewer: Okay, um. So what would average rate of change mean to you?
>
> Felicity: Average rate of change. The slope of a line, like, um, when I think average rate of change I go back to physics and I think about velocity. So it's more like if you have x andy_b how x and how x, sorry. It's basically how - - I think of it as a slope.
>
> Interviewer: Okay. Of what?
>
> Felicity: Of a line.
>
> Interviewer: Okay. And so then what is instantaneous change?

> Felicity: Instantaneous change, it's like at that certain point what is the change. Um, it's like, think of like acceleration. Um, yeah, I don't know.
>
> Interviewer: Okay, are - - do you think the average rate of change and instantaneous rate of change are related?
>
> Felicity: Yeah. They are cuz the average is for like a line or as the instantaneous for like a point. So the average would be like - - like the whole line I guess. [E26:FI1p1L11-23].

At the end of the interview, Felicity is again asked about her thoughts on the relationship between average rate of change and instantaneous rate of change and she draws the sketch shown in Figure 4.10 as she explains her understanding of this relationship. Felicity indicates that, “average speed, is, you have point A and point B, separately.” She struggles with the appropriate name for a line connecting such points on a graph, vacillating between using secant and tangent line until she decides that “… secant line, yeah, …the slope, like the slope formula. Whereas the instantaneous is just at a certain point.” She relates this to her experience with DVS and moving one point on a graph closer to another, recalling “you have to manipulate the graph in order to get the instantaneous rate of change.” When asked if it is possible that a line might intersect with a function graph just once Felicity replies, “yes, it’s the tangent line.” I did not use the words tangent or secant line in the interview until Felicity mentions them first. The interaction with DVS appears to have helped Felicity make connections with the definitions of these words as well as how they are related to one another. However, when probed about what is important about a tangent line Felicity responds by asking if she can draw a picture (see Figure 4.10). While drawing the graph shown in Figure 4.10, Felicity continues to explain how she understands the relationship between average rate of change and instantaneous rate of change at a point, “the slope of that line (student indicates the tangent line) is the same as the instantaneous rate of change.”

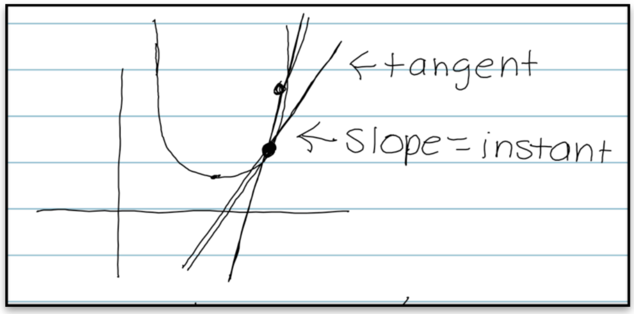


Figure 4.10 Felicity's illustration describing her understanding of the relationship between average rate of change and instantaneous rate of change at a point.

After explaining the relationship between the slope of the secant line and instantaneous rate of change at a point, Felicity adds her thoughts about how this is related to the function’s average rate of change. Her words at the end of the episode E27 suggest that her understanding of the relationship is relies on geometry and is visually dependent on her drawing:

> …it's like we were doing the average rate of change it would be like you have two dots (student draws an additional point on the graph of the function and connects it to the existing point with a line segment) and that's the line and instantaneous rate of change would be that dot and the difference between them would be, but the instantaneous is just for one dot and the average would be for two dots.

Felicity indicates that the slope of the lines she drew in Figure 4.10 represent the average rate of change (“two dots”) and the instantaneous rate of change (“one dot”) [E27:FI1p6L17-p7L13]. Her description and illustration suggest that Felicity has gained some conceptual knowledge about the importance of the interval $[x_a, x_b]$ when relating average rate of change over an interval and instantaneous rate of change at a point. Even though Felicity is considered a non-visualizer, she used a picture to align her thoughts and to aid her in explaining how she relates two mathematical ideas.

Within the interviews there were 16 instances of Felicity exhibiting conceptual knowledge and 14 instances where she did the same with procedural knowledge. In excerpt E28 below, from the Exit Interview, Felicity was asked to explain how derivative at a point and instantaneous rate of change at a point are related. In this exchange, Felicity shows both conceptual as well as procedural knowledge. She also provides an example to strengthen her argument about the relationship.

> Interviewer: Okay. How are instantaneous rate of change at a point and derivative related?
>
> Felicity: Well like I was saying, um, you can find the - - the instantaneous rate of change through the derivative by taking the derivative of a function and plugging in for what like whatever point you want you will find the slope or the instantaneous rate of change.
>
> Interviewer: Okay. When you say you'll find the slope what do you mean?
>
> Felicity: The um, the instantaneous rate of change at a certain point. Like, um, like I was saying, like the example I gave of $x^2 + 1$ for a function and you want to find the slope or the instantaneous rate of change at 1 well you would just derive the function and you plug in 1 and you will have the instantaneous rate of change. [E28:FIExp1L5-19]

Felicity uses some mathematical words incorrectly ("derive"), but she is able to convey her knowledge about this relationship. This contrasts with episode E27 above where she relied on a picture to support her knowledge. Felicity follows a learned procedure that taking the derivative of a function, and "plugging in" the desired x value will result in the numerical value of the function's instantaneous rate of change at that particular x value; however, she does not mention that this is the slope of the line tangent to the function at this particular x value.

At times during the interviews, Felicity's conceptual and procedural knowledge appeared to be weakly linked. In excerpt E29 below, from Interview II, Felicity was using DVS to explore the attributes of a cubic function. Point P on the graph is moveable, and there is a line tangent to the function passing through point P. Felicity is asked how she might graph the derivative function of the function shown using the information on the screen.

> Felicity: So using, like, um - - I know that you can like find the derivative of function by a certain point but I don't know if, like, you mean the whole function.
>
> Interviewer: So let's click on this orange button and look at this table. So I want you to leave your point P where it's at. Can you name a point that is on the derivative - - the derivative graph of this function?
>
> Felicity: The x would be the same; wouldn't it?
>
> Interviewer: So you think the x would be the same?
>
> Felicity: Yeah.
>
> Interviewer: Okay. And then what would your y value be?
>
> Felicity: The function's derivative at that point, so the instantaneous rate of change. [E29:FI2p2L25-p3L1]

In excerpt E29, there is evidence that Felicity has the ability to successfully think about how one quantity changes with respect to another value. She is able to state, with some questioning, that a point lying on the derivative graph will be of the form $\left(x_p, f'(x_p)\right)$. Her ability to equate the derivative value at x_p to the instantaneous rate of change at x_p suggests there is evidence of conceptual knowledge. Felicity does struggle, at least at first, with moving from a point-wise view of function to a continuous view of function for the derivative. This is also evidence of DVS clarifying the topic being discussed as Felicity was able to utilize the traced graph of the derivative dynamically as she moved point P in the next portion of the interview through this Felicity was able to receive immediate validation of her hypothesis.

4.1.7 George

George, an eighteen year-old male, graduated from a large urban high school in 2013; the population of his high school was 1850 students in grades 9-12. George completed AP Calculus BC as well as 2 other AP science courses in high school. In August 2013, George indicated that he planned to study Computer Science. George correctly answered 11/12 items on the MPI and had a visualization score of 8/24. He is classified as a non-visualizer. He did not participate in the Emerging Scholars Program and he earned a final grade of C in this course. George participated in the static interviews for this study.

During the set of interviews, there were 22 instances of evidence of procedural knowledge and 17 of evidence of conceptual knowledge. I also found two instances where George used examples as he discussed a topic or to make a connection between concepts. In episode E30 below, from Interview I, George is asked to estimate the instantaneous rate of change of temperature in Whitefish, MT at noon on a given day. The task provides a table of temperatures and the corresponding times when they were taken (see Appendix B). When the question about estimating the instantaneous rate of change is posed, the students are supplied with a graph of the temperatures and times. George has mentioned needing to find two points in order to complete the task.

> George: Well the two points need to lie on the function so I suppose I could take two points that are really close to 12, find their x and y's, take the average rate of change, and that would be a good approximation. I could do that. I was just trying to draw the line and then try and guess the slope by finding two points that would indicate its slope. Kind of, sort of.
>
> Interviewer: And you're drawing a line you that - - I guess I'm confused about the line. Can you explain the line that you're drawing to me again?
>
> George: The line - - okay, so, normally to find an average rate of change
>
> Interviewer: We're wanting an instantaneous rate of change though.

George: I know, but, I'ma get there, but when you're taking the average rate of change you got two points and then you're just, on a graph anyways, you want like an average rate between those two. (student begins sketching points and lines) [See Figure 4.11]. You just draw the line between those two and then you continue it and that's the average rate of change, but if I want instantaneous and these two dots are really, really close to each other, which makes it like harder to draw a line because you don't have space and so - - and everything's less precise I guess. So that was my goal is to draw a line from these two really close dots. It's not the instantaneous, cuz if it was instantaneous it'd just be one dot.

Interviewer: Can you draw a line that goes through just one point on a graph?

George: Uh, I mean, I could - - I could do that but it wouldn't like have a -- like to find the slope of one point would be - - if they only gave me one point on this graph if they didn't give me anything else, they gave me one point, I wouldn't be able to find the slope of that, because - - it doesn't - - it's not two points. Instantaneous rate is the closest you can get. It almost - - it's like the closest you can get two points together, except even closer than that - - they're all. It's so hard to explain.

Interviewer: So then this - - you don't, you don't like the idea of a line that just goes through one point of a graph?

George: Well, I mean that's what's going to happen. Like when I take the instant rate - - that is the instantaneous rate whatever the point where I have one. Just the point and then it has a slope. A point that's represented by a slope, that's like the instantaneous rate. [E30:GI1p5L19-p6L11]

Episode E30 begins with George explaining the procedure he would follow to find the average rate of change at noon: "I suppose I could take two points that are really close to 12, find their x and y's, take the average rate of change, and that would be a good approximation." He is suggesting that on an small enough interval around noon the average rate of change would be an appropriate estimate for the instantaneous rate of change of the temperature at noon. His sketch, shown in Figure 4.11, lends support to this as he indicates that the interval over which the secant line is drawn is important. He then clarifies that he would need "one dot" to actually find the instantaneous rate of change, but that doing this might be difficult,

> …when you're taking the average rate of change you got two points and then you're just, on a graph anyways, you want like an average rate between those two. You just draw the line between those two and then you continue it and that's the average rate of change, but if I want instantaneous and these two dots are really, really close to each other, which makes it like harder to draw a line because you don't have space and so - - and everything's less precise I guess. So that was my goal is to draw a line from these two really close dots. It's not the instantaneous, cuz if it was instantaneous it'd just be one dot.

This portion of episode E30 is filled with examples of procedural knowledge, as if George wants to follow a set plan to find the solutions requested. While George is able to discern the difference between average rate of change over an interval, and instantaneous rate of change at a point, he appears confused about finding the slope of the line with "one dot", however. At the end of episode E30 is more evidence of George's procedural knowledge, though this time it is more resigned to the idea that he knows something will occur but the path is not certain, "Well, I mean that's what's going to happen. Like when I take the instant rate -- that is the instantaneous rate whatever the point where I have one. Just the point and then it has a slope. A point that's represented by a slope, that's like the instantaneous rate." When questioned about what he means by "a point represented by a slope," George continues to exhibit a reliance on underdeveloped procedural knowledge. Episode E31, below, occurs just after E30. George is asked to elaborate on his statement, "a point that's represented by a slope, that's like the instantaneous rate."

> Um, a point represented by a slope. That doesn't seem to make that much sense, cuz also the point represents x and y. So, um, not that -- more that the slope definitely just has -- or the point definitely just has one slope. That - - it does, it does, but the reason, I don't – cuz you were saying I didn't like the idea, it's that I can't calculate it with just one point. I need two points to calculate it. It makes sense to me that the slope is, like associated with that point but I need to points to calculate it. [E31:GI1p6L14-19]

George's insistence on the need for "two points" suggests that his procedural knowledge limits his ability to complete the task.

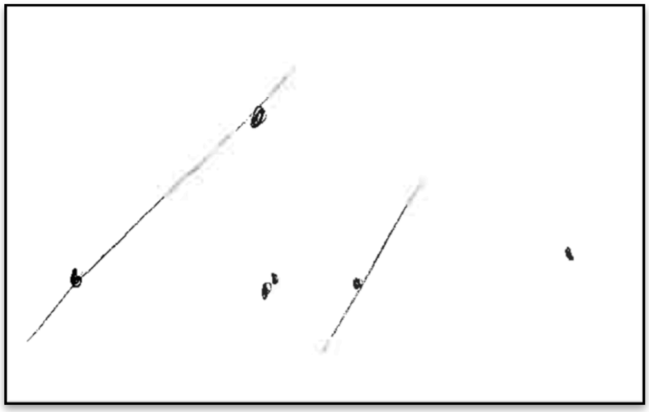


Figure 4.11 George's sketch accompanying his explanation of the relationship between average rate of change and instantaneous rate of change at a point.

During Interview II there is evidence that George is using a visual image, or relying on some type of visual connection, as he works on a task. There are 17 instances in the interviews of George visualizing, even though the MPI visualization score of 8/24 classifies him as one who prefers not to visualize when working on mathematical tasks. In episode E32 below, from Interview II, George is discussing the importance of when $f'(x) = 0$.

George: Well, at, uh, I couldn't find the exact number but a little over 2 so probably like 2.25 maybe, the derivative is 0; and then at negative .25 the derivative's 0.

Interviewer: So what does that mean? How do you know that?

George: Cuz there's a maximum and minimum - - relative maximum and relative minimum and those always have, um, the derivative's always equal to 0 at a max or min

Interviewer: Why does that happen?

> George: Cuz they're, like, the highest point in that location because the derivative changes from increasing to decreasing for maxes or decreasing to increasing for mins. Yep.
>
> Interviewer: But how can you just look at this graph and tell me that at somewhere near $x = -0.25$ the derivative value is 0?
>
> George: Well when you got - - on the graph you got these mountainish shapes, I guess, and at the tips of mountains the derivative is equal to 0 because - - just, I don't know it is. [E:32GI2p2L10-21]

George's use of "mountainish" is evidence of shape thinking in his understanding of the importance of when and why $f'(x) = 0$. It is important to note that the instructor for this course did discuss "hill and valleys" when teaching about minimum and maximum function values. George's thinking about minimum and maximum values is another example of procedural knowledge. He states that, "relative maximum and relative minimum and those always have, um, the derivative's always equal to 0 at a max or min." This suggests that, at least procedurally, he is looking for places that fit a certain description. It also suggests that his concept image for relative minimum and maximum does not include the possibility of $f'(x)$ not existing, but $(x, f(x))$ being a relative maximum or minimum of the function.

During Interview I there is another instance where George uses visualization to reason. This is the only instance in the interviews where he voluntarily draws a picture to help him reason through a task. George is struggling with two ideas. He is having trouble reconciling that on the graph of a function with a fixed point $(x_1, f(x_1))$ another point (x_2, y_2) could be represented as $(x_1 + h, f(x_1 + h))$ for some value h. George struggles to understand why the simplified difference quotient yields a value when $h = 0$, but the average rate of change is undefined for this scenario.

George: …Cuz the change in x in this case was 1 and so – (long pause) I don't know why, I'm having a hard time connecting y_2 and $f(x+h)$. I like, I could swear they're the same.

Interviewer: Why do you think they're the same?

George: Well the formula's really close to each other and when you – (student begins to sketch) [See Figure 4.12] - - so you have y_1, or $f(x)$, I guess, I need to draw this graph a little better, that's not very good, it'd be like up here then. Um, but this is like y_1, or $f(x)$, and then this moves over three spaces or something an it's over here, it's just an example and then this would be y - - well you don't know the y_1, you don't know - - this is $f(x)$ though. Yeah it's the change in x cuz, it's all on a function of some kind, so - - I feel like it's right. There's no way it can't be right, but I'm having a hard time proving that it is right. Like there's no - - like with a lot of other things you can always prove that f - - I'm having a hard time finding like the exact equation where it's like this is equal to this. I feel like this one's more of a - - you're just supposed to look at a graph and you're suppose to determine that they're the same uh, cuz you have - -

Interviewer: So you think they're the same?

George: I do think they're the same.

Interviewer: And they'll always be the same?

George: Well, then yeah if that's - - one's the same then the other has to be the same. Well unless like you do the limit at 0, cuz - - if you do 0, then you got this big problem with dividing by 0 and most of the time with this formula you can somehow cancel out h most of the time if you take the derivative, but this one, since the 0 isn't represented by one variable it's harder to cancel out, I think. [E33:GI1p10L10-31]

George's confusion seems to stem from his need to find a procedure to guide him. He is unable to continue to reason about the difference quotient until he gains some closure on the issue of $y_2 = f(x_1 + h)$. Though his drawing, as seen in Figure 4.12 is messy, and his line of thinking difficult, at times, to follow, he has some basic intuition that the two are the same. This episode is also consistent with the ten instances each of George approaching instantaneous rate of change algebraically and geometrically.

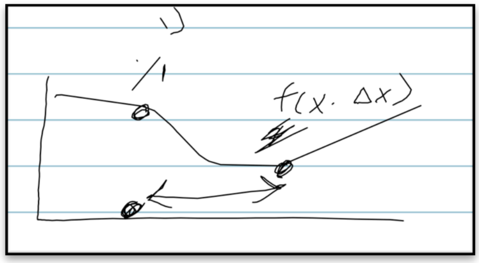


Figure 4.12 George's sketch as he reasons about $f(x + h)$.

4.1.8 Harry

Harry graduated in May 2013 from a very large urban high school with a student population over 3300 students enrolled in grades 9-12. He is an eighteen year-old male majoring in mechanical engineering. Harry completed AP Calculus BC as well as "other AP science courses" while in high school. Harry did not participate in the Emerging Scholars Program, and he earned a D in Calculus I. Harry correctly answered 11/12 tasks on the MPI and has a visualization score of 15/20, classifying him as a visualizer. Harry participated in the static interviews.

During the interviews, there were 18 instances of Harry having procedural knowledge compared to only four instances where there was evidence he possessed conceptual knowledge. Many of the procedural knowledge instances occurred when Harry would say "you plug in" or "I don't know why, but I know." When asked about average rate of change in Interview I, Harry makes this remark, "I mean when it's just between two points. Yeah that's, x, so you just plug each x in and subtract your $f(x)$'s." [E34:HI1p4L23-24] There is evidence that he relies on the procedure of plugging in values. Later, still during Interview I, Harry is completing a table of values finding the average rate of change between two points as well as calculating the difference quotient. It should be noted that Harry stated that he did not know what the term difference quotient meant. When asked about the difference quotient, Harry responds, "I don't know. It’s the same answer [as the average rate of change on the provided table].” The definition of the difference quotient was provided for him in an effort to allow him to continue with the interview.

> Interviewer: Can you see any set of numbers in your table where they might not be the same?
>
> Harry: 1.
>
> Interviewer: Why do you think one is a problem?
>
> Harry: Cuz there is no change.
>
> Interviewer: Because why?
>
> Harry: There's not a change.
>
> Interviewer: Why is that a problem?
>
> Harry: Cuz that is the main factor in your equations.
>
> Interviewer: So what's the average rate of change then when x_1 and x_2 are both 1?
>
> Harry: 0
>
> Interviewer: Your average rate of change is 0?
>
> Harry: Well it's 0 over 0.
>
> Interviewer: So what is 0 over 0?
>
> Harry: Nothing I guess.
>
> Interviewer: It's nothing?
>
> Harry: It's - - there isn't a change so, I guess you could put does not exist. [E35:HI1p8L23-p9L9]

Harry’s reliance on finding the average rate of change with a formula hinders him when he must think about what happens if the two values are equal. His procedural knowledge about $\frac{0}{0}$ does not support his exploration of this situation. When asked about the difference quotient at this point (see excerpt E36, below), Harry is able to use his knowledge about patterns and explore what might happen when $x_1 = x_2$. He does not appear to possess conceptual knowledge relating the difference quotient, when $h = 0$ to the value of the function's instantaneous rate of change at $x = 1$; but, procedurally, he recognizes that the average rate of change does not exist when "the change of x … is 0." He does not mention that, in this case, $h = 0$.

> Harry: Cuz the equation has numbers that it can still use even though the numbers 0.
>
> Interviewer: Why can't you use it in average rate of change?
>
> Harry: Cuz everything depends on the change of x, which is 0.
>
> Interviewer: Okay. So what is the value then of the difference quotient when x_1 and x_2 are equal?
>
> Harry: Negative 3. [E36:HI1p9L14-20]

Immediately after episode E36, Harry is presented with the same function, but it is modeled as representing the position of a particle, with respect to time. When asked for the instantaneous speed of the particle at $t = 2$, Harry states, "the derivative gives me my velocity." He is able to discuss the relationship between the two functions in a conceptual manner, though it unclear whether the relationship was memorized or rote.

During Interview II Harry was presented with the graph of the function $f(x) = \sin x$, and asked to find points on the graph where he knows the value of $f'(x)$. In the next episode E37, there is evidence of procedural knowledge when Harry states that the values he picked are extrema and they have derivative values of zero.

> Interviewer: Okay. Can you name some points on our graph $f(x)$ where it would be relatively easy to determine the derivative value?
>
> Harry: Uh, probably, $\frac{\pi}{2}$, $\frac{3\pi}{2}$ and I'm not for sure about 0, π, 2π.
>
> Interviewer: So why would you say $\frac{\pi}{2}$, and $\frac{3\pi}{2}$?
>
> Harry: Cuz they're extrema.

Interviewer: Okay. So what's the derivative value there?

Harry: Uh, 0. [E37:HI2p5L6-12]

However, when asked why he chose 0, π, and 2π, Harry backtracks and says that he isn't sure. He may be confusing the function values (0) with the derivative values at these values of x. Later in the interview, Harry states that he knows that $f'(x) = \cos x$, again suggesting that he has procedural knowledge about the function $f(x) = \sin x$. He appears to struggle with covarying quantities, which may contribute to his confusion about the derivative values at these points.

Interviewer: Okay. What about - - you said you thought maybe π and 0. What do you think the derivative values are at π and 0?

Harry: I'm not for sure.

Interviewer: Okay. Why did you think that maybe they would be easy to - - ones you might want to calculate?

Harry: Well they're crossing the x axis, so I don't know.

Interviewer: Okay.

Harry: They have value, but they're probably not easy.

Interviewer: Okay. So you say they have value but they might not be easy to calculate.

Harry: Yes. [E38:HI2p6L19-28]

Harry did not make any sketches as he worked on the provided tasks unless the drawings were requested in the instructions, however there are thirteen places where there is evidence that Harry was visualizing. Often these occurred when Harry was describing his thinking. For example, when asked about extrema of a graph during Interview III, Harry makes this statement, "Maxes going up and mins going down." Earlier in the interview, when analyzing a continuous piecewise-defined function, Harry was asked to indicated where on a closed interval the instantaneous rate of change was greatest. His description suggests that he was making visual comparisons (the function definition was given) as he completed the task.

Harry: Where the instantaneous rate of change is the greatest?

Interviewer: Yes, please.

Harry: Possibly it'd be like right here.

Interviewer: So you're saying somewhere near $x = 5$?

Harry: Yeah.

Interviewer: Okay. Why do you think so?

Harry: The graph is the most vertical. [E39:HI3p4L10-16]

Overall, the evidence in the interviews with Harry suggests an overreliance on procedural understanding, though that limits his ability when the procedural

4.1.9 Ian

Ian is an eighteen year-old Hispanic male who graduated in May 2013 from large urban high school with a student population over 2000 students enrolled in grades 9-12. In high school Ian completed AP Calculus AB as well as two other AP science classes. He did not participate in the Emerging Scholars Program and Ian earned a D in Calculus I. Ian indicated that his planned major was Aerospace Engineering. It is important to note that, at times, Ian appears to struggle with language; though it is unconfirmed he may be an English Language Learner even though he graduated from a high school in Texas. Ian correctly answered 8/12 items on the MPI, the lowest of any study participant. However, his visualizer score was 20/24, the highest of any study participant. Ian’s high visualization score is the reason that his low correctness score was overlooked. Ian participated in the DVS interview group.

As we may have expected given Ian's high visualization score, there were twenty-six instances where he was visualizing during the interviews. During Interview II Ian was asked to recap what he remembered from Interview I, namely the connections between secant lines, tangent lines, derivatives, etc. Ian asks if he may draw out his thoughts because "I don't know the definitions, the words."

> Ian: Uh, derivatives are the - - oh, let's go with tangent line it's easier. Tangent lines are - - tangent line, um, instantaneous rate of change is at one point where secant lines are between two points on a graph or function. Uh, derivative is the, um - - um, let's see rate of change of a tangent. Yeah.
>
> Interviewer: Okay.
>
> Ian: Like I said I don't - - I know the math I just don't know the definitions, the words.
>
> Interviewer: Okay. What about a tangent line tells you that rate of change?
>
> Ian: Um - - can I draw that?
>
> Interviewer: Sure.
>
> Ian: (student begins to draw) [See Figure 4.13.] Draw a parabola rather - - tangent lines are when it touches one point it and tells you what the slope is at that point (student sketched in a tangent line and marks the point where it intersects the parabola). Secant lines are between two points (student draws a secant line and marks the two points where it intersects the parabola) which would tell you the average rate between those two derivatives, um, or basically it's just the, uh, equation at that point. [E40:II2p1L5-18]

Ian does not use "average" or "instantaneous" rate of change to delineate between the characteristics of secant or tangent lines. His knowledge is classified as procedural, though the procedure is not formally present in his descriptions. He can state what "it tells you" when talking about a secant or tangent line, though his statements, generally, are not wholly correct. His sketch, shown in Figure 4.13, when considered in tandem with his statements about tangent and secant lines, suggests that his knowledge may be memorized. Later, during the same interview, Ian makes another point that he prefers to visualize, "Um, let's see. I would need to see the work to do it. I'm more of a visual person than speaking with words." [E41:II2p1L32-33] He was asked how he might sketch out the derivative graph of a function if he was given the values of the function's instantaneous rate of change at a set of discrete points.

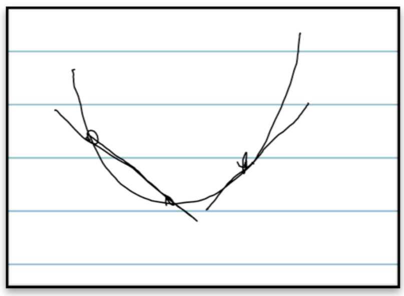


Figure 4.13 Ian's sketch relating secant and tangent lines.

Ian struggles greatly with keeping track of covarying quantities. During Interview III, Ian was asked to write an inequality comparing the minimum value to all other function values for a continuous function defined on a closed interval. Episode E42 (below) highlights his struggle to keep track of how the minimum function value compares to all other function values and suggests that he is unable to consider the relationship between covarying quantities.

> Interviewer: Okay. What then can you tell me about all other function values?
>
> Ian: That there is domain is I guess relative which for those it's relative for those intervals.
>
> Interviewer: Okay. Say that again, I'm sorry.
>
> Ian: Okay. Um - -

Interviewer: I just didn't hear you, I'm sorry.

Ian: I guess that the other function their minimums are each minimum of the, um, each minimum of the, uh, certain of the equation is just a relative minimum.

Interviewer: Okay. But you said you believe the absolute max - - the absolute minimum of this function is -2?

Ian: Yes.

Interviewer: What does that mean for every other function value?

Ian: I don't know.

Interviewer: What does it mean to be an absolute minimum value?

Ian: It means that it's the absolute lowest it can go. It can't go any lower but that means it can go high.

Interviewer: So what does that mean for everything else compared to the absolute minimum of -2?

Ian: It's just positive - - it's higher than the minimum and it's - - compared to the relative minimum, everything's positive.

Interviewer: Everything's positive?

Ian: Compared to the, um, compared to the, um, minimum.

Interviewer: Okay. Okay. So - - go ahead.

Ian: No. No. I'm just saying it's compared to the, um, compared to the minimum everything's positive because it doesn't go below that, it doesn't go below the actual - -

Interviewer: What do you mean by everything's positive?

Ian: Um, I'm thinking as the - - that is meant as the axis and below that would be negative and above would be positive. If you're considering, um, this - - if you talk about the absolute maximum anything below it would be negative because it's nothing can pass that.

Interviewer: Okay.

Ian: When I talk about the positive I'm talking about just the, um, compared to the other. [E42:II3p3L16-p4L18].

In episode E42, Ian continues to make the statement “they are all positive” in reference to function values being compared to the minimum function value, the he means “above.” It is possible that this can be attributed to Ian struggling with the English language. Ian only considers the function values (which are in the positive direction along the y axis) and cannot consider how the corresponding input values affect this. When asked to explain what he means by “all positive,” Ian provides this answer, “Um, I'm thinking as the - - that is meant as the axis and below that would be negative and above would be positive. If you're considering, um, this - - if you talk about the absolute maximum anything below it would be negative because it's nothing can pass that.” It appears that Ian is unable to consider that function across-time and is, instead, viewing it in a point-wise manner.

4.2 Interview Groups

Analysis of the coding for each group type allows for commonalities and themes to emerge. In this section I discuss the various themes present in each group and will conclude with a comparison of the themes present between the two interview groups: static and DVS.

4.2.1 DVS Interview Group

Table 4.2 lists the most common codes found in the interview transcripts of the DVS interview group. The DVS group was comprised of two non-visualizers and three visualizers, there were 69 instances where visualization was evident. The sixty-five places where there is evidence that DVS clarifies a concept, topic or connection contrasts with the twenty-three instances of DVS obscuring. For each of the codes listed in Table 4.2, I discuss the evidence of each as well as those codes commonly occurring simultaneously within the interviews.

Table 4.2 Seven most common codes within the DVS Interviews

	Frequency within DVS Interviews
Student understanding of Instantaneous ROC	83
Procedural Knowledge	73
*Correct Procedural Knowledge	29
*Partially Correct Procedural Knowledge	22
*Incorrect Procedural Knowledge	16
*Partially Correct/Correct Procedural Knowledge	5
*Incorrect/Partially Correct Procedural Knowledge	1
Conceptual Knowledge	71
*Correct Conceptual Knowledge	37
*Partially Correct Conceptual Knowledge	12
*Incorrect Conceptual Knowledge	4
*Partially Correct/Correct Conceptual Knowledge	11
*Incorrect/Partially Correct Conceptual Knowledge	4
*Incorrect/Correct Conceptual Knowledge	3
Evidence of visualization	69
Evidence that DVS clarifies	65
Concept Image	63
Geometric Approach to ROC	62

4.2.1.1 Student Understanding of Instantaneous Rate of Change

The most common theme present within the DVS interview group was “Student Understanding of Instantaneous Rate of Change.” As illustrated in episode E43, below, Beverly is asked to make a prediction about the relationship between a particle’s average rate of change and its instantaneous rate of change when the interval between the particle’s time values is sufficiently small. Beverly had access to a dynamic table when she collected data about the particle’s initial and final positions as well as the rates of change. At the particular point in the interview, the DVS was animating the two points so that the interval length was constant at 0.01.

> Interviewer: And what do you notice about the average speed and the particle's position at that - - at - - where you took the average speed?

Beverly: Um, they're the same. Like –

Interviewer: So what do you think is gonna happen with the instantaneous speed and the position of the particle?

Beverly: They're gonna be even closer because if the average speed is already like close enough like then it's basically the instantaneous speed.

Interviewer: So what do you think that means about functions that follow this model that are exponential in nature?

Beverly: Um, their average rate, like their position can be figured out using the instantaneous speed.[E43:BI1p14L1-11]

In E43, Beverly uses the information she collects in a dynamic table to make a hypothesis about the relationship between the function's average rate of change over an interval of length 0.01 and its instantaneous rate of change at a point. She correctly states that for a particle who's position, s, with respect to time is modeled using $s = f(x) = e^x$ the position and instantaneous speed at a given x-value will be the same. This excerpt is an example of a student discussing instantaneous rate of change.

4.2.1.1.1 *Student understanding of instantaneous rate of change co-occurring codes for the DVS group*. Within the eighty-three instances of students in the DVS interview group experiencing, mentioning or discussing their understanding of instantaneous rate of change, there are several codes occurring with them simultaneously. Table 4.3 below shows the codes that occurred with student understanding of instantaneous rate of change within the DVS interview group.

Table 4.3 List of codes co-occurring with student understanding of instantaneous rate of change in the DVS interview group.

	Student understanding of Instantaneous Rate of Change
Evidence that DVS Clarifies	24
Geometric Approach to Rate of Change	17
Conceptual Knowledge	14
*Correct Conceptual Knowledge	12
*Partially Correct Procedural Knowledge	2
Procedural Knowledge	10
*Correct Procedural Knowledge	4
*Partially Correct Procedural Knowledge	2
*Partially Correct/Correct Procedural Knowledge	4
Evidence that DVS Obscures	6

Within the DVS interviews, 24 (out of 83) instances of student understanding of instantaneous rate of change coincided with the evidence that the software clarified the concept or topic being investigated. This compares with the six instances of the software obscuring a connection to instantaneous rate of change. Episodes E11 and E12 (discussed in Section 4.1.2) highlight such instances.

Seventeen instances of a student using a geometric approach to rate of change occurred simultaneously with student understanding of instantaneous rate of change. An example of this was discussed in episode E15 (from Section 4.1.3).

There are fourteen instances where conceptual knowledge is simultaneously coded with student knowledge of instantaneous rate of change. In the three episodes that follow, E44 – E46, both from Interview II, Ian discusses his thoughts on instantaneous rate of change. Episode E44 occurs prior to Ian's investigation whereas episode E45 happens afterwards. Ian is asked to discuss how instantaneous rate of change at a point is represented in the dynamic sketch in which he is exploring. A screenshot of the sketch can be seen in Figure 4.14 below.

Interviewer: Okay. So if you'll click on the button that says instantaneous rate of change. So geometrically what does this instantaneous rate of change correspond to on our graph?

Ian: On the graph or the tangent line?

Interviewer: Okay. You think - - why do you think it may be the tangent line?

Ian: I don't know. I was asking are you talking about the original graph or the change with the tangent line?

Interviewer: Either. Geometrically what does the instantaneous rate of change correspond to?

Ian: It corresponds to the tangent line and, um, because it's a derivative of the, um, it's the derivative of that, um, of that function at that point.

Interviewer: Okay. Again what about the tangent line though?

Ian: Um, I don't know, I'm sorry. [E44:II2p3L10-22]

In this first excerpt from Interview II, Ian’s knowledge about instantaneous rate of change is discussed, but another code “procedural knowledge’ is present as well. Ian’s words “…it corresponds to the tangent line and, um, because it's a derivative of the, um, it's the derivative of that, um, of that function at that point” suggest possible rote recall of connections between the words “tangent line”, “derivative” and “instantaneous rate of change.” However, in episode E45 below, also from Ian’s interview II, his discussion about instantaneous rate of change is much different than the one from excerpt E44. For reference, we also include a screenshot from the DVS interview II shown in Figure 4.14, below.

Interviewer: Okay. So how might we go from knowing our instantaneous rate of change to being able to plot the derivative?

Ian: We could plot the derivative - - we can plot it by, um, just following the tangent line at certain points.

Interviewer: What do you mean by following the tangent line?

Ian: Um, the tangent line tells us the derivative or instantaneous rate of change at that - - at one point. If you could get multiple points we could plot the, um, the derivative of the graph by itself. [E45:II2p3L23-30]

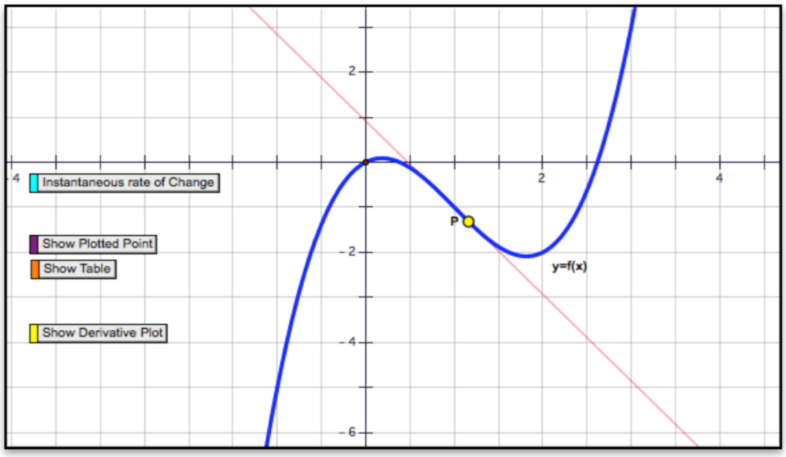


Figure 4.14 Screenshot of DVS Interview II.

In episode E45, Ian explains the relationship between instantaneous rate of change, the tangent line, and the derivative graph. He specifically mentions that the derivative at one point, the instantaneous rate of change at that point and the "tangent line" are the same and that by collecting enough information at "multiple points" one could graph the function's derivative. Ian explored how a point on the derivative graph of a function is related to the slope of the tangent line to the function at the given point and, in turn, the instantaneous rate of change of the function. Though he does not specifically mention that the slope of the tangent line at a point is the same as the derivative value at that point (notice that in episode E44 he was not able to make this statement either) episode (E45) suggests that Ian may be developing some conceptual knowledge connecting these topics. To further make this point, later during the same interview (see episode E46 that follows) Ian is able to make statements regarding the properties of a point that lies on the graph of the derivative function of a given function.

Interviewer: The green shadow is graphing what for the y coordinate?

Ian: The - - the - - y would be graphing the instantaneous rate of change.

Interviewer: Okay. So you said that you think that this green is tracing out the?

Ian: The, um, the derivative of function - - $f'(x)$. [E46:II2p6L26-29]

Looking at the three previous episodes (E44-E46) sequentially, Ian's knowledge about the relationships between the derivative at a point and a function's instantaneous rate of change at that point changes. In E44 he is unable to make any connections between the topics and in E46 he correctly states that the y values of the points on the derivative graph correspond to the function's instantaneous rate of change. Ian's conceptual knowledge is still weak; he is not consistent about noticing that the relationship between the topics occurs at points and he is still unable to fully connect that the slope of the tangent line at a point is the function's instantaneous rate of change at that point. Another example of these co-occurring codes happened during Interview II with Beverly. I asked if it was possible to sketch the graph of the derivative of a function if she were provided with the function's instantaneous rate of change at some discrete points. Beverly's response, "Well if you're given the instantaneous rate of change isn't that the derivative?" suggests that she has some conceptual knowledge about these relationships.

The fourth most common code occurring with student knowledge about instantaneous rate of change is evidence of procedural knowledge. Evidence of this combination was seen in Ian's second excerpt form Interview II in excerpt E45, "…If you could get multiple points we could plot the, um, the derivative of the graph by itself." Further evidence of this combination is from David's exit interview, shown in episode E47, below. David's explanation about the relationship between instantaneous rate of change at a point and the derivative focuses on "… you take the derivative of that function." He clearly has a procedural view of this relationship, though his procedural knowledge appears to be weak. Notice that David does not provide a full procedure to follow. Specifically, he does not state the need to "take the derivative of that function," at the same point where the value of the instantaneous rate of change was requested.

> Interviewer: Okay. So how are instantaneous rate of change at a point and derivative related?
>
> David: Instantaneous rate of change. Aren't they the same? Yes - -
>
> Interviewer: Why do you think so?
>
> David: Because when we have the - - we have like a function and you need to find the instantaneous rate of change at that point you take the derivative of that function. [E47:DIExp1L10-15]

Other examples of student knowledge of instantaneous rate of change occurring simultaneously with procedural knowledge are seen in episode E15 from Amy's first interview (discussed in-depth in Section 4.1.3). Amy is interacting with a sketch that models a particle's position with respect to time and point A is fixed while point B is not. Amy is able to explore with point B, and there is a secant line connecting points A and B visible on the screen. When asked is there could be a way to estimate the instantaneous speed of the particle at time x_A, Amy responds , "…Well, you would use the definition of a tangent line and or definition of a derivative and, uh, calculate the slope of the tangent line doing the, um, using the h notation doing the, um, $\frac{f(x+h)-f(x)}{h}$." Through this statement, Amy is giving the procedure that links these two concepts in her mind and her understanding of instantaneous rate of change.

4.2.1.2 Procedural Knowledge

Within the DVS interviews there were seventy-three instances of students possessing procedural knowledge, or attempting to rely on procedural knowledge that the student did not possess. In some cases "I just know" or a similar statement characterized the latter. In the excerpt [E48] that follows, from David's exit interview, he is relying on underdeveloped procedural knowledge regarding the derivative at a corner on a function graph.

> David: Why? Because it's - - it's a 0 or not 0, undefined so. Never thought about that. Cuz the - - what's it called? The slope at that point is like, no - - I guess it would be 0. And you can't find the derivative of 0 so - -
>
> Interviewer: So the slope at a corner or cusp is 0 and you can't find the derivative of 0.
>
> David: Yeah, yes. Or I just know you can't. You're - - you can't find the derivative at a corner cusp so - - [E48:DIExp6L21-26]

David's final statement of "…Or I just know you can't…" was characteristic of weak procedural knowledge though there is also the presence of weak conceptual knowledge as well.

4.2.1.2.1 *Procedural knowledge co-occurring codes*. Table 4.4 below displays the four most common codes occurring simultaneously with the code procedural knowledge within the DVS interview group. The seven concept definition codes coincide with the concept image codes in this case because while the concept definition refers explicitly the words one associates with a concept, the concept image is the total cognitive structure. Hence, all concept definition codes are also concept image codes as well.

Table 4.4 Co-occurring codes in DVS interviews with procedural knowledge.

	Procedural Knowledge
Concept Image	13
Conceptual Knowledge	11
*Correct Conceptual Knowledge	4
*Partially Correct Conceptual Knowledge	3
*Incorrect Conceptual Knowledge	2
*Partially Correct/Correct Conceptual Knowledge	2
Student understanding of Instantaneous ROC	10
Concept Definition	7

During Interview III, Amy was asked if a function reached its absolute maximum function value over its domain. She was shown a graph (see Figure 3.6) of the piecewise-defined function as well as the function definition. Amy responded that the function did achieve its absolute maximum function value and she was then asked how she knew. “Well looking at the graph it's easy. It'll do all your -- you just find where y is the highest. So here at this point there aren't any y's that are higher than this point right here. Then the same thing with this one, there isn't anything lower [E49:AI3p2L24-26].” Amy is describing the procedure she used to determine the existence of the maximum value but she also describes the images that come to mind when she is doing so. “…there aren’t any y's that are higher than this point right here.” Her concept definition for absolute maximum function value means that no other function values are higher. She also has an analogous concept definition for absolute minimum, “…Then the same thing with this one, there isn't anything lower.” Another example of this occurred during Interview III with Felicity. She was asked what it meant for a function to be continuous on its domain. “That there's no holes, there's no um, there's no holes it's - - it's just continuous everywhere you can go through the line or you wouldn't take off your pencil if you drew a line" [E50:FI3p1L4-6]. Another example of procedural knowledge co-occurring with concept image occurs in Felicity's interview III after episode E50:

> Felicity: When, um, you take that derivative and if its tangent line equals 0 the -- the slope of a tangent line equals 0 and if you take the derivative -- the first derivative test and you look at what the -- when does the, uh -- when does x make the equation of f' negative or positive if it's -- if it changes from negative to positive that's where you can find where the, um, absolute minimum. [E51:FI3p4L24-29]

Felicity's description of the change in sign of f' suggests her concept image includes the visuals of this in relation to the first derivative test as she mentioned.

The eleven instances of conceptual knowledge occurring simultaneously with procedural knowledge occurred primarily in Interview III and the Exit Interview after the students had become comfortable with the software. In episode E52, Interview II, Amy reasons about why the instantaneous rate of change of the function $f(x) = e^x$ for a given value of x is the same as the function value at x.

> Amy: You would take the log, the natural log of, um, well say it was y equals your something to the x, then you would take the log of both sides -- the natural log of both sides and then using your logarithmic rules you can say that the natural log of something to the x is the same thing as taking that x and multiplying and see how -- and so, um, that's how you would take the -- and then you would take the derivative of it when you don't have the exponent anymore it's a lot easier to take the derivative of the natural log of something and multiply that by x cuz it's just product rule and then the derivative of the natural log of something is 1 over that something. Which, the natural log and e are inverses of one another, so they cancel out. [E52:AI2p7L5-12]

Episode E52 shows Amy's reliance on procedures as she lays out each step she would complete to check if an exponential function of the form, $y = b^x$, is its own derivative. However, in the final lines from episode E52, Amy connects the inverse functions $f(x) = e^x$ and $f^{-1}(x) = \ln x$ to make her argument that $f(x) = e^x$ is the only exponential function with this property. Her ability to make these connections across topics is the reason why this was coded as evidence of conceptual knowledge as well a procedural knowledge.

The co-occurrence of student understanding of instantaneous rate of change and procedural knowledge is discussed in section 4.2.1.1.1.

4.2.1.3 Conceptual Knowledge

There are seventy-one instances of evidence of conceptual knowledge within the DVS interviews. This code was used when a student was able to connect topics or mathematical ideas. Sometimes conceptual knowledge indicates when a student connects concepts; even if the connection was weak or incorrect. In episode E48 (discussed in section 4.2.1.2) David attempt's to connect why at a sharp corner on a function graph the derivative does not exist. He is unsuccessful in this endeavor and his reasoning and his statement "you can't find the derivative of 0," indicates that he possesses incorrect conceptual knowledge about this scenario.

In episode E55 from Beverly's interview III is an example of conceptual knowledge. Beverly was discussing (without DVS) the conditions for the EVT and why they are necessary. I drew the function graph shown below in Figure 4.15 for Beverly after her attempts to sketch a function that fails to attain both it's minimum and maximum function values on a closed interval $[a, b]$ were unsuccessful. Beverly is answering the question about why the function must be continuous on the interval in question in order to achieve both its maximum and minimum function values. Beverly connects the concepts of continuity, instantaneous rate of change and extremum values in excerpt E53. Beverly suggests that since there is an "open dot," and that, "this isn't a single point that just approaches to it (student indicates the point on the sketch where there is a discontinuity) [See Figure 4.15.] cuz there's a hole at that point and so that absolute minimum is reaching to that point…" This suggests that Beverly may be considering the limit of the function values at the function approaches the discontinuity from the left and adds further support to the evidence of Beverly's ability to connect the concepts in a meaningful way.

> Interviewer: So what if we had a function on a closed interval from a to a like this (Interviewer draws example)? [See Figure 4.15] So can you tell me anything about what continuity -- what the lack of continuity tells us?
>
> Beverly: Well the lack of continuity -- it's -- you can't find an absolute max or a min that way because, um, like for example for this problem (student points to sketch) [See Figure 4.15], there's no absolute min.
>
> Interviewer: How do you know that?
>
> Beverly: Because this is, um, yeah, this is the largest, I mean, yeah, this is well like the largest negative value and it's, um, an open I guess dot which means that this isn't a point that you -- like this isn't a single point that just approaches to it cuz there's a hole at that point and so the absolute minimum is reaching to that point, but it isn't so there's no definite one. And so that's where you can't find an absolute minimum.
>
> Interviewer: So that's why continuity is needed?
>
> Beverly: Um, yes. Because that's when, um, that's how -- yes, okay. Um, for you to find a maximum and minimum you have to find the extremums … which is when the tangent line is 0 or undefined. But if it's not continuous then there wouldn't be a tangent line and then there wouldn't be any extremums. [E53:BIExp3L16-30]

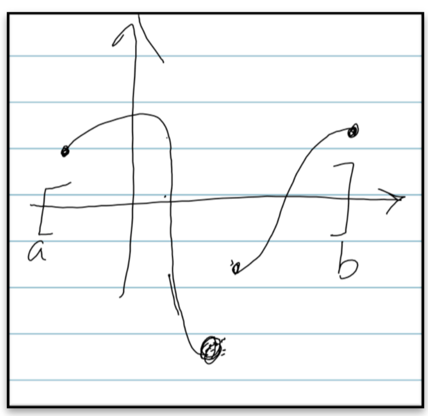


Figure 4.15 Sketch of a discontinuous function, defined on a closed interval, which does not achieve its minimum function value.

Another example of conceptual knowledge present is from Ian's interview I, shown below in episode E54. Ian initially stated that he was not sure how average rate of change and instantaneous rate of change were related. After exploring with some sketches in the DVS, Ian is again asked if he can articulate this relationship.

> Interviewer: …But first I want to ask you, how do you think that average speed is related to instantaneous speed? You said in the beginning you thought they were related, but you weren't sure how. Do you have a better idea of how?
>
> Ian: Um, the right relationship I think would be the difference between times. So it's an average rate would give you a broad answer or an instant rate of change would be a precise, small window, a small difference in time. Where the average is a bigger window. [E54:II1p7L8-14]

Though Ian does not necessarily show the procedural knowledge needed to find either the average rate of change or instantaneous rate of change, his ability to compare the two through a lens of "bigger window" suggests emerging conceptual knowledge about these concepts.

4.2.1.3.1 *Co-occurring codes with conceptual knowledge in the DVS interview group*. Table 4.5 lists the five most common codes occurring simultaneously with conceptual knowledge within the DVS interview group. The code occurring with the highest frequency with conceptual knowledge is student knowledge of instantaneous rate of change. The excerpt immediately above this section from Ian's first interview is an example of this situation. "…So it's an average rate would give you a broad answer or an instant rate of change would be a precise, small window, a small difference in time." Within this quote there is evidence that Ian has some conceptual knowledge about instantaneous rate of change. He is able to discuss instantaneous rate of change both compared to another concept (average rate of change) and in abstract terms of "window size".

Table 4.5 Co-occurring codes with conceptual knowledge within DVS interviews.

	Conceptual Knowledge
Student Understanding of Instantaneous Rate of Change	14
Concept Image	12
Procedural Knowledge	11
*Correct Procedural Knowledge	5
*Partially Correct Procedural Knowledge	2
*Incorrect Procedural Knowledge	2
*Partially Correct/Correct Procedural Knowledge	1
*Incorrect/Partially Correct Procedural Knowledge	1
Evidence of Visualization	10
Evidence that DVS Clarifies	10

During Beverly's interview II, in episode E55 (shown below) she was asked to make a prediction about the representation that the path a point with the coordinates $(x_P, f'(x_P))$ would make. Beverly had already indicated that she knew the coordinate of the point $(x_P, f'(x_P))$ prior to this exchange.

> Interviewer: Okay. So I want you to -- this program -- what this computer program is going do is it's actually going draw a path that will follow this point. What do you expect that, that path is going represent?
>
> Beverly: The slope of the -- the tangent -- the instantaneous rate of change and the derivative.
>
> Interviewer: Say that again?
>
> Beverly: The derivative. [E55:BI2p5L4-10]

Here, Beverly's statement suggests that she is able connect the slope of the line tangent to a function at a given point to the function's instantaneous rate of change at that point as well as to the derivative value at said point, thus there is evidence that

Beverly posses conceptual knowledge about instantaneous rate of change. Following this exchange, Beverly was able to verify that her prediction was correct.

There are twelve instances when the codes for conceptual knowledge and concept image coincided in the DVS interviews. The first example of this I will explore comes from Beverly's Interview III and can be read in episode E10 (Section 4.1.2). Beverly was interacting with a sketch of a piecewise-defined, continuous function using a point on the function through which a tangent line was visible. A screen shot of this is shown in Figure 4.16, below. Beverly was asked to find the point within given closed intervals where the derivative value was the greatest. In episode E10, Beverly is able to transition between discussing the derivative value and the slope of the tangent line, the slope of the original function and what the points she was asked to find represent on the derivative graph. Her concept image of f' is related to how the "original function" behaves as she mentions, "well, that's the greatest slope of the original function and those are the greatest points of f'." Not only that, but her statement about "greatest slope" translating to "greatest point" suggest that her concept image for this relationship is robust in nature and she is able to covary between changing quantities. Later in episode E10, Beverly is able to continue expressing these connections between the "original function," f' and f''. She says, "So it's the same idea, or if you have $f'(x)$ and you find the derivative of that and you find the zeroes that's the extrema of $f'(x)$."

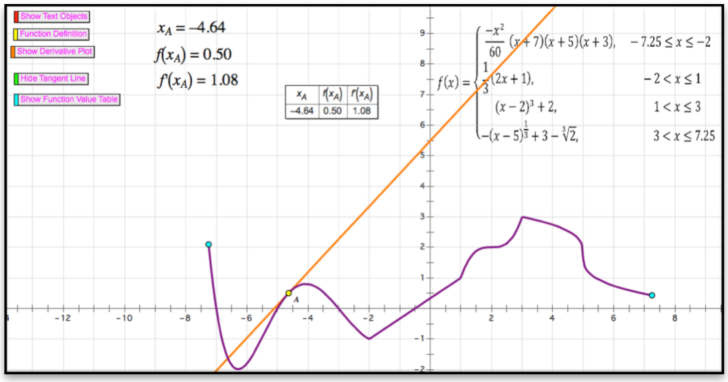


Figure 4.16 Screenshot of DVS Interview III.

Episode E10 contains evidence of conceptual knowledge co-occurring with evidence of Beverly's concept image regarding the second derivative of the function she is working with. It appears that Beverly has made strong connections between how the concept of derivative transfers from the "original function" and can then be applied to taking the derivative of the derivative.

The co-occurrence of student understanding of instantaneous rate of change and procedural knowledge is discussed in section 4.2.1.2.1.

There are ten instances of conceptual knowledge also coded as evidence of visualization. In the excerpt E56, below, from the Exit Interview, Amy was asked why she believes that the instantaneous rate of change at a point making a cusp or corner is undefined. Excerpt E56 is coded as conceptual knowledge because Amy compares this situation to a "smooth curve." She is able to expand upon the differences between the two scenarios and make a comparison of the outcomes. As Amy describes this, she makes a sketch (see Figure 3.8). Her statement, "…from the left it looks like it's negative but coming from the right it looks like it's a positive slope…", suggests she may be visually comparing the function's rates of change as it approaches, from the left and the right, the problem point. Episode E56 concludes with Amy stating, "…I can draw this horizontal or whatever (student draws small line segments indicating the different tangent lines she mentions) [See Figure 3.8], so there's no definite…tangent line." This statement further makes her case that multiple lines tangent to the function at the corner point. This excerpt is also coded as containing evidence of Amy's concept image.

> Amy: Because, let's see -- because I can -- like on a smooth curve I can draw a -- like on these examples you gave me I can draw a line that kind of estimates the instantaneous rate of change but on this one (student being to sketch the function graph of $y = f(x) = |x|$.) [See Figure 3.8.] what it comes to a corner I can draw infinitely many, like I draw coming from the left it looks like it's negative but coming from the right it looks like it's a positive slope or right here I can draw on this horizontal or whatever (student draws small line segments indicating the different tangent lines she mentions), so there's no definite this is what the tangent line would look like. [E56:AIEx4L9-16]

There are ten instances coded *DVS clarifies* that are also coded as conceptual knowledge. This coding occurs when a student, after or while, interacting with DVS was able to make a statement revealing conceptual knowledge when she was not able to do so prior to the interaction with the DVS. For example, in the episode E54 presented in this section (from Ian's second interview) the evidence of conceptual knowledge occurred after Ian explored using DVS. Another example of this is from Felicity's interview III in episode E56 below. As shown in Figure 4.16 above, Felicity was considering the properties of a piecewise defined function graph and she stated that the derivative value at $x = 3$ was zero. After exploring with the DVS sketch, using both a table of values and the point with the attached tangent line, Felicity is able to determine that the instantaneous change as the function approaches $x = 3$ from the right is not the same as the instantaneous rate of change as x approaches 3 from the left. This episode suggests that Felicity now possesses conceptual knowledge about the derivative values of a point at a cusp or corner. She is not only able to say that the derivative does not exist, but she can explain why is does not exist.

Felicity: No, definitely not.

Interviewer: So does the derivative exist there?

Felicity: No.

Interviewer: Why does it not exist?

Felicity: Because in order for it to exist this has to be, like, let's say this was 3 (student positions point A on the let side of $x = 3$) so this has to be 3 and then this has to be 3 (student moves point A to the right side of $x = 3$) and this is not 3. This is -0.21.

Interviewer: So is the derivative at $x = 3$ equal to 0?

Felicity: No, it does not exist. [E57:FI3p7L8-15]

4.2.2 Static Interview Group

Within the static interviews there were 35 different codes and a total of 760 instances of these codes. Table 4.6 lists the five most common codes within this interview group.

Table 4.6 Five most common coding themes present in the static interviews.

	Frequency in the Static Interviews
Procedural Knowledge	105
*Correct Procedural Knowledge	34
*Partially Correct Procedural Knowledge	30
*Incorrect Procedural Knowledge	23
*Partially Correct/Correct Procedural Knowledge	8
*Incorrect/Partially Correct Procedural Knowledge	7
*Incorrect/Correct Procedural Knowledge	3
Evidence of Visualization	61
Concept Image	59
Student Understanding of Instantaneous Rate of Change	46
Conceptual Knowledge	43
*Correct Conceptual Knowledge	13
*Partially Correct Conceptual Knowledge	7
*Incorrect Conceptual Knowledge	1
*Partially Correct/Correct Conceptual Knowledge	10
*Incorrect/Partially Correct Conceptual Knowledge	11
*Incorrect/Correct Conceptual Knowledge	1

4.2.2.1 Procedural Knowledge within the Static Interview group

The most common code in the static interviews is where procedural knowledge was evident. This may be attributed to the procedural nature of textbook-type tasks of which the static interviews were comprised. Though procedural knowledge is present when the code is used the level of procedural knowledge differs. For example, in the temperature task from Interview I, George described the procedure he was using to find the average rate of change of the temperature from noon until 3 p.m. in, excerpt E58 that follows.

> George: Well, I know the slope isn't consistent so I can't just do that, and I guess finding the average rate of change isn't - - from between two times isn't really a derivative, even though I always like taking those. So I'd probably just do the average, or like average, not instantaneous, from noon which would be the 12 to 1 to 15, I guess, so 14.3, or, or wait. Yeah 14.3, I guess. Well it gets larger, so the other way around. 18.2 minus 14.3, divided by the number of hours. 1, 2, 3, 4, 4 hours I guess and then that's 3.9. [E58:GI1p1L12-17]

Within this excerpt George mentions derivative and that the "slope isn't consistent," though his reasoning and intent are not clear. He does not appear to know that the value of the average change in temperature would be same as the value of the slope of the secant line connecting the points $(12, 14.3)$ and $(15, 18.2)$. This episode is coded as containing procedural knowledge, student understanding of average rate of change, student understanding of instantaneous rate of change, and algebraic approach to rate of change. The evidence of student understanding of instantaneous rate of change is coded because George mentions that there is a difference between the two in his statement, "…I'd probably just do the average, or like average, not instantaneous…" though he does not clarify why he makes this distinction.

Another example of procedural knowledge comes from Corbin's interview II. Corbin is given the graph and definition defining expression for a cubic function and told that $x_A = 1$. He is asked to find $f'(x_A)$ and his work for this part of the task is shown in Figure 4.17. Though most of this work is highly procedural in nature. As Corbin writes he states "Take the derivative so it's all power rules", …"then I plug in 1…" There is also evidence of Corbin using a rote calculation approach to rate of change rather than, say a geometric approach. Notice that in Corbin's work he writes that $f'(x_A) = 1 \qquad y = -6$, suggesting that he possibly does not fully understand the relationship between the notation he is using as he works on the task. [E59:CI2p1L15-25]

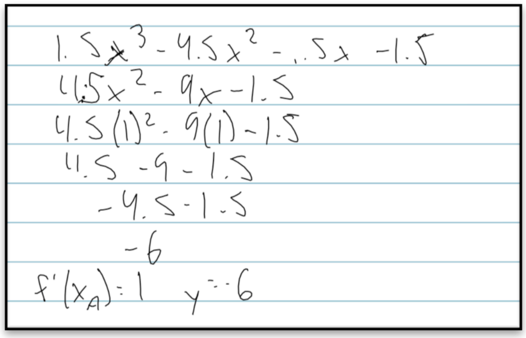


Figure 4.17 Corbin's work to find $f'(x)$.

4.2.2.1.1 *Co-Occurring codes with procedural knowledge within the static interview group*. Table 4.7 lists the four most common codes occurring simultaneously within the static interview group along with procedural knowledge. The most common of these, with twenty-five occurrences, is student understanding of instantaneous rate of change.

Table 4.7 Common co-occurring codes with procedural knowledge within the static interviews.

	Procedural Knowledge
Student Understanding of Instantaneous Rate of Change	25
Algebraic Approach to Rate of Change	20
Concept Image	17
Evidence of Visualization	15

During Evan's exit interview he was asked how instantaneous rate of change and derivative are related. His response was, "They are the same point on the graph because if you were to find the derivative of a graph you plug in the x value for that point, you'll find the instantaneous rate of change at that point on the original graph." [E60:EIExp1L17-19] This episode contains evidence of procedural knowledge as Evan tells the interviewer how to go about finding the desired information, and he is relating two topics: instantaneous rate of change and derivative so it is also coded for student knowledge of instantaneous rate of change.

A different pattern of procedural knowledge was evident at the end of Corbin's interview I. Corbin struggled to complete all the tasks contained in this interview and, at the end, he was still very confused about instantaneous rate of change and average rate of change. Corbin considered a particle whose position $s,$ in feet, is given by the equation $s = f(t) = t^2 - 5t - 6$, where t is measured in seconds. He was then asked to find the average rate of change between $t = 1$ and $t = 2$. Then he is asked if this answer tells you anything about the particle's instantaneous speed at $t = 1$. (See Appendix B) In episode E59, Corbin begins to talk almost to himself in this episode as he tries to reason using his limited knowledge about instantaneous rate of change, "Does it tell you anything about the instantaneous speed – so the speed at one second. No." Figure 4.18, below, shows the sketch of $f(x)$ and $f'(x)$ that Corbin made prior to this exchange when he first encountered the function. The sketch, however, is not correct because the function Corbin was given is quadratic. Corbin has sketched what appears to be a cubic function graph to represent $f(x)$ and a parabola for $f'(x)$; as he makes the sketches he says, "Okay, so it's cubic (sketches basic cubic function), it's derivative is this (sketches parabola)." Corbin's weak procedural knowledge leads him to make the statement, "that's just the nature of the parabola," but it is unclear if he is referencing the function provided or his sketch where he suggests that the derivative function graph is a parabola. In the end, Corbin attempts to relate this to the function's graph shape (a parabola) as he is unsuccessful is answering the question. Corbin's reliance on underdeveloped and, in this case, non-existent, procedural knowledge limits Corbin's reasoning about the situation. [E61:CI1p11L13-26]

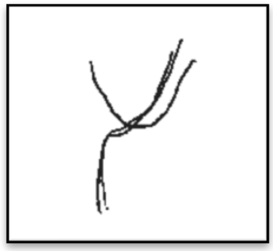


Figure 4.18 Corbin's incorrect sketch of $f(x)$ and $f'(x)$ from Task 2 of static Interview I.

The code indicating when a student's approach to finding a rate of change was algebraic in nature is the second most common co-occurring code with procedural knowledge. This code was used when a student's solution path or reasoning pathway to finding a rate of change included a reliance on algebra or formulas. This code was used regardless of whether instantaneous or average rate of change was being considered. For example, during Interview I Harry was asked how he would find the average change between two points when he was provided with a function definition. "I mean when it's just between two points... (37 second pause) Yeah that's, $f(x)$, so you just plug each x in and subtract your $f(x)$'s." He then writes $f(x_2) - f(x_1)$. Harry's approach does not include finding a slope, or using the slope formula, instead he simply, "… subtract(s) your $f(x)$'s." It does not occur to Harry that he will need to divide by the change in x until he is asked specifically by the interviewer if this is important. When asked, Harry admits that he was simply testing the first values in the table, which are $x_1 = 1$ and $x_2 = 2$. [E62:HI1p4L23-24]

There were seventeen instances of procedural knowledge occurring simultaneously with concept image. Several of Corbin's comments from Interview I, episode E61, illustrate this. "…The speed's gonna be changing at -- differently at different points if we were say from second 2 to second 3 the speeds gonna be -- it's gonna be falling at a -- at a faster speed than negative 2…" Corbin follows this statement with "That's just the nature of the parabola." Though the exact contextual meaning of this last statement is unknown, Corbin's concept image for a parabola includes a relationship like the one he is describing. He reasoned that the graph of the derivative function was a parabola (because the original function was quadratic) and his knowledge about how the average rate of change is changing seems to be related to his understanding of parabolas.

Another example of this combination occurs in Harry's third interview. When asked about how he was able to identify the absolute minimum value of a function he responds with, "Cuz, nothing goes below it [E63:HI3p5L27]." This excerpt is coded as procedural knowledge because Harry is explaining that his method for determining such a value is to use the graph and make a comparison. The concept image code is used because it appears that Harry's concept image of "absolute minimum" includes this picture of a point that "nothing goes below". There is also evidence of conceptual knowledge as Harry is able to discuss characteristics of a function's absolute minimum.

The last code occurring simultaneously with procedural knowledge that I discuss indicates evidence of visualization. Two of the four students participating in the static interviews scored above 15 on the visualization portion of the MPI, suggesting that they prefer to visualize when solving mathematical tasks. Corbin (a visualizer) overwhelmingly exhibited a majority of these instances alone (nine) while Evan (a non-visualizer) had one instance of this coding combination, George (a non-visualizer) had two and Harry (a visualizer) had three.

In Interview I when Corbin describes the "nature of the parabola," (see episode E61 and Figure 4.18) is an example of evidence of visualization on his part. The picture that he drew (see Figure 4.18) seems to guide his thinking on how the average rate of change varies as the interval changes. His final statement, "it's just the nature of the parabola," leads me to believe that he was envisioning this parabola throughout the conversation as his failed attempts to reason about the relationship become less and less mathematically sound.

Evan's single instance of procedural knowledge and evidence of visualization occurs in Interview I. Evan, when presented with a graph modeling temperature change, is asked to estimate the instantaneous rate of change when $t = 12$. Evan eventually sketches a tangent line and finds the slope to estimate this change, but his description of how he was going to do this suggests he was visualizing. However, his language, "…get a visual slope…" and "…a visual line…" [E64:EI1p3L1-11] is odd and he may be recalling bits and pieces of a procedure from class where these words elicit a meaning just for him aside from the normal meaning of visual. Figure 4.19 shows the work that Evan did to find this estimate, including his calculations seen at the bottom of the figure.

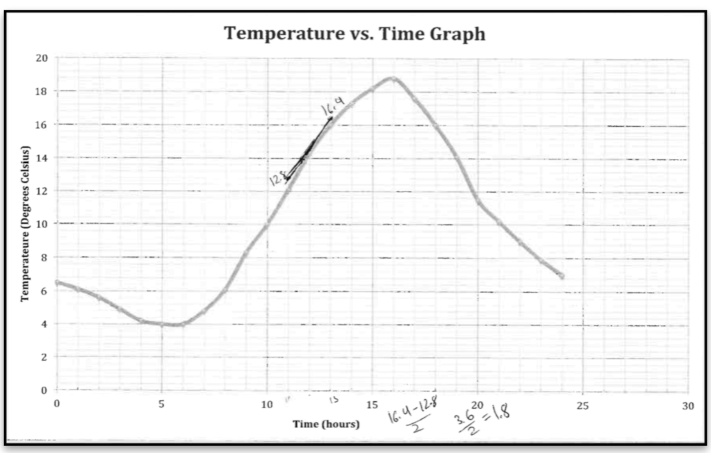


Figure 4.19 Evan's work from Interview I.

4.2.2.2 Evidence of visualization within the Static Interview group

There are 61 instances within the static interview group of evidence of visualization. There is a sub-code used with evidence of visualization for when a student is exhibiting evidence of “shape thinking.” This was used when a student’s words to describe a function (or function graph) suggested that the major shape of the graph was influencing the student. Corbin’s example from episode E61 about the parabola is an example of this code. Another example can be found in episode E18; Evan’s description of why the derivative at a cusp does not exist, along with his drawing (see Figure 4.7), suggest that he was visualizing a situation and this was aiding in his understanding of the concept as he was discussing it.

Episode E65, from George’s exit interview, provides an additional example of evidence of visualization. Initially, George was asked what comes to mind when he hears the word “derivative.” After listing several things, including “the slope at the point,” George is asked to clarify. The result is that George draws an example of what a tangent line is and how it is related to derivative. He then returns to this example, unprompted, to illustrate the relationship between instantaneous rate of change and average rate of change. George uses his illustration (see Figure 4.20) to show that he knows that “…if you were to choose this point right here (student makes point on the first function graph and draws in a tangent line) and make the tangent line that would be there derivative.” Furthermore, there is evidence of George’s conceptual knowledge as he corrects himself by adding, “… that line’s slope, which is whatever the slope formula is (student writes $\frac{y_2-y}{x_2-x}$) … is $f'(x)$ at that x value.” George also states, correctly, that the derivative at that point is the instantaneous rate of change “at that one point.” In order to clarify this connection to the function’s average rate of change over an interval, George continues with his illustration (see Figure 4.20) adding the last three drawings. George’s statements while drawing, “… but if you want to find instantaneous rate of change you get these two points closer and closer together until they are one point,” suggests that

George’s knowledge about this relationship is tied to his visual explanation.

[E65:GIExp1L10-29]

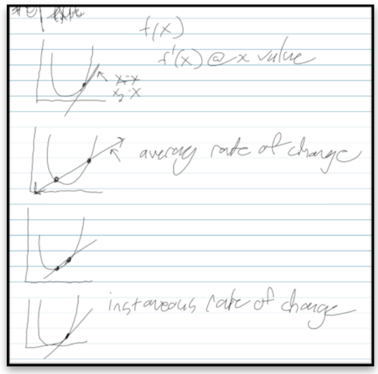


Figure 4.20 George's illustration about the relationship between average rate of change and instantaneous rate of change at a point.

4.2.2.2.1 *Common co-occurring codes with evidence of visualization within the static interview group*. Table 4.8 shows the four most common codes occurring simultaneously with evidence of visualization within the static interviews. The code for student understanding of function indicates episodes within the interviews where a student discusses functions or the presence of how the student understanding of function relationships such as the relationship between a function and its derivative.

Table 4.8 Common co-occurring codes with evidence of visualization for the static interview group.

	Evidence of Visualization
Procedural Knowledge	15
*Correct Procedural Knowledge	4
*Partially Correct Procedural Knowledge	3
*Incorrect Procedural Knowledge	4
*Partially Correct/Correct Procedural Knowledge	2
*Incorrect/Partially Correct Procedural Knowledge	1
*Incorrect/Correct Procedural Knowledge	1
Concept Image	13
Conceptual Knowledge	10
*Correct Conceptual Knowledge	6
*Partially Correct Conceptual Knowledge	3
*Partially Correct/Correct Conceptual Knowledge	1
Student Understanding of Function	10

The co-occurrence of student understanding of evidence of visualization and procedural knowledge is discussed in section 4.2.2.1.1

Episode E65 from George's exit interview contains evidence that his concept image includes the illustrations he made regarding how instantaneous rate of change and average rate of change are related. George's concept image for this relationship includes both the pictures that he drew (see Figure 4.20, above) and the idea of the two points getting "closer and closer together." George also includes an "intermediate picture" where his indicated points appear closer together than in his in Figure 4.18 George writes that the slope of the tangent line is $\frac{y_2-y}{x_2-x}$. He does not, however, indicate the need to use two points on the tangent line to find this slope.

There are ten instances of conceptual knowledge occurring at the same time as evidence of visualization within the static interviews. The next example that I will discuss is also from George, but this occurred during his interview III. George was asked to analyze the graph of a continuous, piecewise-defined function and indicate where the function's instantaneous rate of change would be zero. Figure 4.21, below, shows the graph provided and the marks made by George while working on this task. George says "… it's at the top of the hill…", suggesting that he may be engaging in shape thinking at this point. I should note that the instructor for the section of calculus in which the study participants were enrolled reports that this language was commonly used during lecture when discussing such topics. The evidence of visualization also occurs when George begins talking about the changing of the graph, "… I guess a better explanation was the slope goes from negative and then it -- after that point it goes positive (student marks on graph and uses pen to illustrate the change in slope of the function). So, it must have been zero in between there." This suggests that George has some conceptual knowledge about both derivative functions and continuous functions, in general. [E66:GI3p5L12-20]

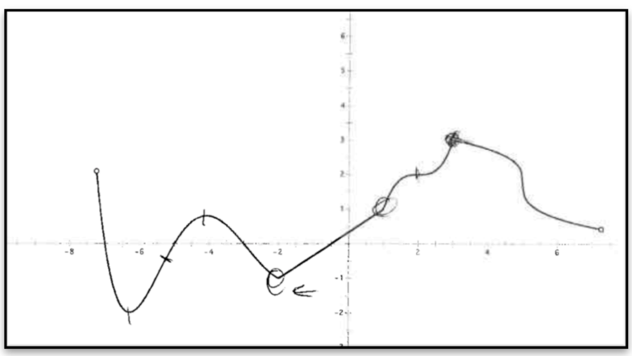


Figure 4.21 George's notations from Interview III.

The final co-occurring code with evidence of visualization that will be discussed is one used when student understanding of functions was present. An example of these co-occurring codes can be seen in the first excerpt E1 from Section 4.1.1. In this excerpt, Corbin is attempting to make sense of the relationship between average rate of change and the difference quotient. When he is asked to explain his thinking, Corbin begins to consider the relationship that the derivative (a quadratic function which Corbin equates to a parabola) function plays. “I was thinking for some reason we were -- we were using corresponding points along the parabola but we're not.” He was attempting to find a pattern in the data; though there is a pattern, Corbin was not able to reason through what it might be. One of his hindrances is that he keeps coming back to the parabola and the symmetry of such a graph. “They're gonna meet at this middle point which is the change of x is 0…” The symmetry of the table and the symmetry of the parabola that Corbin pictures is overwhelming for him. He continues to try and relate these two un-related ideas as he reasons.

Perhaps a more clear example of evidence of visualization and student understanding of function occurring simultaneously occurred in Evan’s second interview. He was provided with the graph of $f(x) = \sin x$ and is asked a variety of questions regarding the relationship between the function and its derivative. In particular, Evan is asked to pick points where he knew that $f'(x) = \pm 1$ and where $f'(x) = 0$. He does this and is asked how he knew which points to pick. Evan’s statement that he used his knowledge that “…the derivative of sine is cosine…” as he identified such points is coded as student knowledge of functions since this knowledge was primarily guiding Evan’s work during this part of the interview. Adding further evidence to this is Evan’s statement that when a function’s derivative graph changes from “increase to decrease, or vice versa” then the derivative graph must, “go through the x-axis, unless it’s undefined and trigonometric equation are defined on all real numbers.” [E67:EI2p6L18-25] Figure 4.22 below shows Evan’s work on this problem.

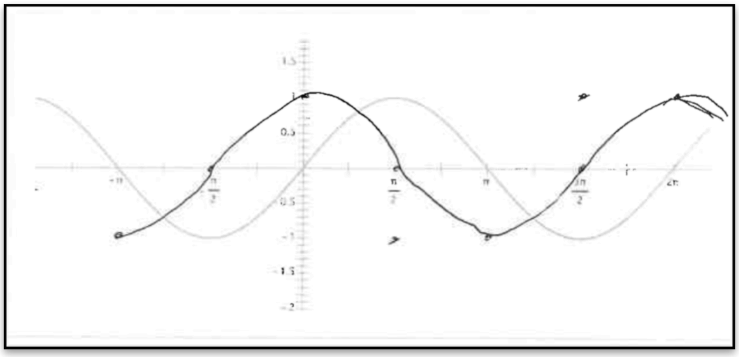


Figure 4.22 Evan's work with $f(x) = \sin x$.

4.2.2.3 Evidence of Concept Image Within the Static Interviews

The 59 instances where there was insight into a student's concept image makes this code the third most commonly occurring one in the static interviews. Concept image was coded when there was evidence of the images or words a student associates with a concept. Consider the excerpt below from Evan's exit interview. Evan was presented with three sets of axes showing intervals, on the x −axis. On the first set of axes, the interval is open (a, b); on the second set it is half-open $(a, b]$, and on the final set it is closed, $[a, b]$ (see Appendix H). Evan is asked to determine on which set of axes any function would attains its maximum and minimum function values. Evan begins to consider each set independently. Evan explains his reasoning for why the open and half-open intervals would fail, "…where $f(a)$ is the greatest point but $f(a)$ is not part of the domain. Therefore it wouldn't reach its maximum." Evan does not appear to notice that he is incorrectly using the word domain in this statement. He continues his statement to include the case seen in the first graph of Figure 4.23, "… if it was the lowest point in the entire interval it would be the same deal or it wouldn't reach it cuz of the – it's not defined at that point." In this statement Evan correctly identifies that the function is not defined at the point $(a, f(a))$.. His concept image (for the EVT) includes an understanding that endpoints not included in the function's domain may cause the EVT to fail. [E68:EIExp3L9-14]

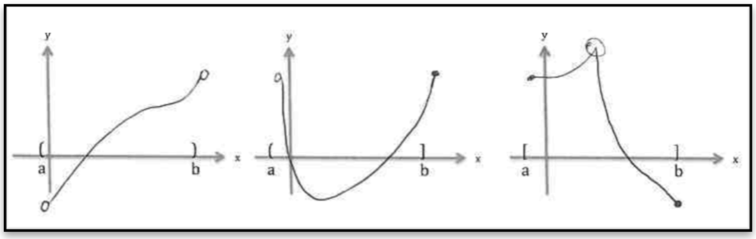


Figure 4.23 Evan's drawings from Exit Interview task 4.

Later in Interview III, Evan discusses his thoughts on a function that may not be differentiable everywhere. There appears to be a conflict between Evan's concept definition and his concept image regarding the need for a function to be continuous in order for the function to achieve both its minimum and maximum function values. Evan reasons about how the derivative graph must look and he is able to determine that a function that is not differentiable at a point could still achieve its extreme function values as he describes in the excerpt below. After this episode E68, Evan draws the third graph shown in Figure 4.23, above.

> Interviewer: Okay. Are there any other properties that are important in order to ensure there is an absolute maximum and absolute minimum?
>
> Evan: The derivative would have to be -- would have to cross the x axis at some point for it to have a maximum or minimum or it has to be undefined at some point. Because of cusps and stuff, even though they don't have a derivative at that point would still be considered a maximum or minimum depending on what side of the point is positive or negative. [E69:EIExp3L28-33]

When asked if there is a similarity between the graphs he drew (see Figure 4.23, above) that Evan has not yet discussed, he suggests that maybe continuity plays a part in this, but he eventually dismisses this idea in episode E70. Evan's reliance on rote procedural knowledge fails to help him reason about the possible importance of continuity in this situation. This further supports that Evan's concept image seems to include the notion but his concept definition does not and he cannot explain why.

> Interviewer: Are there any other properties that all the functions you've driven- -- you have drawn have that we haven't talked about yet?
>
> Evan: They're continuous along their domain.
>
> Interviewer: Is that important?
>
> Evan: (9 second pause) I'm trying to remember which way it is if continuity gives or says it's -- differentiable. Or if it's the other way around, differentiability proves it has continuity. I think that continuity has no say if it's differentiable or not. Which means that it wouldn't matter if it was the -- it wouldn't necessarily mean there's a minimum or maximum if it's continuous or if it's not continuous or -- I know what I'm trying to say it's just not coming out… [E70:EIExp4L6-15]

However, it is not until the interviewer presents Evan with an example that challenges his ideas about continuity that he begins to consider the necessary condition that his function be continuous (see Figure 4.24). Even when presented with an example that should challenge Evan's conflict between his concept image and concept definition, he struggles to reconcile them as his procedural knowledge is not sufficiently supported with conceptual knowledge to bring the conflict to the forefront and to allow for further, deeper, exploration. Evan continues to conflate continuous and differentiable as the interview continues. When asked if the function graph shown in Figure 4.24 Evan is able to say that the function does not achieve its maximum or minim value and that, "… it has a vertical asymptote at that line and it's not continuous along its domain, either." However when I question him about the need for a function to be continuous on a closed domain he agrees but adds, "… you've got to show that it's also differentiable." This directly conflicts with both his words and example from episode E69 and Figure 4.23. Evan's need to rely on a "limit of something" results in his abandoning the question. He says, "… the limit from the left side and the limit from the right side do not equal the same number so it is not differentiable…" When asked to clarify what he is taking the limit of, Evan indicates that he is considering the function values to the left and the right of the point $(c, 0)$ on the graph shown in Figure 4.24. I point out that the third graph he drew (see Figure 4.23) was not differentiable at the point circled, but that the function (according to Evan) achieves its maximum function value. Evan returns to the graph shown in Figure 4.24 and tries to show "….that it's (see Figure 4.24) continuous and that the point (student indicates point c in Figure 4.24)-- at that point it is also -- or at that point it -- that point is exists and is the maximum…"., which is not a true statement. During the Exit Interview, Evan does not come a conclusion about the need for a function to be continuous on a closed interval for it to achieve both its maximum and minimum function values. He remained confused about the implications of differentiability and continuity. [E71:EIExp4L18-p5L16]

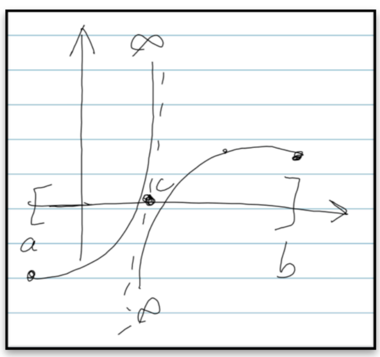


Figure 4.24 Interviewer illustration of a function defined on a closed interval that fails to achieve either its maximum or minimum function values.

4.2.2.3.1 *Co-cccuring codes with concept image*. Table 4.9 lists the code commonly co-occurring with concept image within the static interviews. For a discussion about concept image co-occurring with concept definition, please reference Section 4.2.2.3. Procedural knowledge occurring with concept image was discussed in section 4.2.2.1.1 whereas evidence of both the visualization code and the concept image can be found in section 4.2.2.2.1.

Table 4.9 Common codes co-occurring with concept image within the static interview group.

	Concept Image
Concept Definition	30
Procedural Knowledge	17
*Correct Procedural Knowledge	3
*Partially Correct Procedural Knowledge	5
*Incorrect Procedural Knowledge	4
*Partially Correct/Correct Procedural Knowledge	3
*Incorrect/Partially Correct Procedural Knowledge	2
Evidence of Visualization	13

4.2.2.4 Evidence of Student Understanding of Instantaneous Rate of Change within the Static Interviews

There are forty-six instances of student understanding of instantaneous rate of change within the static interviews. The most common code occurring simultaneously with student understanding of instantaneous rate of change was procedural knowledge, with 24 instances. The next most common co-occurring code (algebraic approach to rate of change) had only 8 occurrences. One example of student understanding of instantaneous rate of change occurring with procedural knowledge occurred during Evan's interview I. When filling out the table given in part of Task #2 for Interview I (see Appendix B), Evan comments that he would find the average rate of change following a procedure of dividing change in $f(x)$ by change in x, but then he mentions the relationship he feels exists between the average rate of change and the difference quotient " … or fill out the rest of the average rate of change and the difference quotient looks like it would be the same."

When Evan was asked specifically about the difference quotient he responded with, "Honestly, I've never heard of it before. As far as I can remember. I feel like I've seen it but I can't think of what it is…" I proceeded to ask Evan some questions about other variables that might represent "change in x," Evan recalls that, "Is the difference quotient the -- is it x_1 or is $\frac{f(x_1+h)-f(x_1)}{h}$? I think that's the difference quotient." His recall is solely procedural, and states that it would, "…work out the be the average rate of change, still." Evan continues to work with the difference quotient, and he comments about "thinking of a letter" that was present in the formula. Eventually, Evan comes to the conclusion that, "you can now have h as being equal to 0 there is no -- there's nothing you couldn't plug in to either value where it wouldn't come up undefined." After making this statement, Evan begins discussing the "immediate change of x," which, in this case, is his way of talking about instantaneous rate of change. Evan continues to reason, using procedural knowledge about the difference quotient, that there is a problem with using $\frac{y_2-y_1}{x_2-x_1}$ when "h was 0." When Evan says "…which is basically this," he draws a box around the formula he used in an earlier part of the interview (see Figure 4.25) for finding the difference quotient of the function $f(x) = x^2 - 5x - 6$. Using procedural knowledge, Evan worked with the difference quotient and come to realize that it differs from the average rate of change of a function over an interval only when $h = 0$. He calls this "the immediate change of x at any given point." At the end of the episode E70, Evan compares the average rate of change, which "would be undefined" if the change in x was 0 to the difference quotient which, would be $2x + 5$." [E72:EI1p5L19-p7L18]

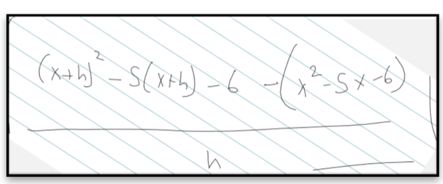


Figure 4.25 The result of Evan's reasoning about the difference quotient.

4.2.2.5 Evidence of Conceptual Knowledge within the Static Interview Group

There are forty-three instances of conceptual knowledge within the static interviews. Episode E40 (see Section 4.1.4) Evan was asked to indicate the points on the graph of a function that corresponded to a zero instantaneous rate of change and the importance of such points. Initially, Evan suggested that these points corresponded to maximum or minimum values for the function, but the made a distinction about the point at $(2, \mathrm{f}(2))$ (See Figure 4.26, below). The excerpt and accompanying picture below are from that interview. Notice that Evan is able to use generic scenarios to describe why a particular point might have an instantaneous rate of change equal to zero but would not be a minimum or maximum function value. He is able to make statement about the function's rate of change as it approaches the point in question, although he does not make distinctions about approaching from the left or right side of the point. Evan does not use the "highest point" (or "lowest point") definition as he discusses what distinguishes a maximum (or minimum) function value from an inflection point. However, Evan does state that he compares, "…a slope that's decreasing and a slope that's increasing approaching the same point." He indicates that for a point on a function graph to be a minimum function value, "… like the slope has to be -- the graph has to be decreasing and the start increasing." He makes an analogous statement for how he might identify a maximum function value. Evan then compares his descriptions of maximum and minimum function values to what he see happening on the graph shown in Figure 4.26. He states that at point $(2, f(2))$, "that one doesn't have either. It's increasing all the way through." He also does not use the words "inflection point" to name point $(2,2)$. This episode, E40, also contains evidence of Evan's conceptual image of maximum, minimum and inflection points, though this code will be discussed in the following section.

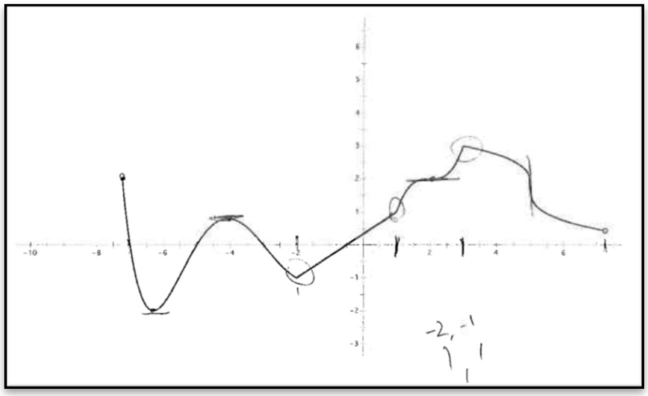


Figure 4.26 Markings from Evan's interview III.

4.2.2.5.1 *Common co-occurring codes with conceptual knowledge within the static interview group*. Table 4.10 below lists the common co-occurring codes with conceptual knowledge within the static interviews. The most common code appearing simultaneously is evidence of visualization; this was discussed in Section 4.2.2.2.1.

Table 4.10 Codes most commonly co-occurring with conceptual knowledge within the static interviews.

	Conceptual Knowledge
Evidence of Visualization	10
Concept Image	9
Student Understanding of Instantaneous Rate of Change	9
Procedural Knowledge	8
*Correct Procedural Knowledge	4
*Partially Correct Procedural Knowledge	3
*Incorrect/Correct Procedural Knowledge	1

Episode E40 (see Section 4.1.4), from Evan's from Interview III contains evidence that his concept image for maximum, minimum and inflection point all include how the rate of change very near the point in question compares to other rate of change. For example, Evan is able to determine that at the point $(2{,}2)$ the shape of the graph does not change. He is comfortable discussing how the graph is "increasing at a decreasing rate" as it approaches $x = 2$ from the left and it is still increasing on the right side of $x = 2$. This suggests that Evan's concept image for these ideas includes more that just "the highest" or "the lowest" points and that he is able to look beyond the basic requirement that $f'(x) = 0$ or is undefined.

The example below of conceptual knowledge evidence co-occurring with student understanding of instantaneous rate of change from a student participating in the static interviews occurred in Corbin's first interview. Corbin was asked if his results about the average rate of change in temperature over varying timeframes were good estimates for the instantaneous rate of change in the temperature at noon. Corbin struggled, at first, with this question, repeating it several times. Finally he stated that, "average is going to be over an extended period of time, instantaneous is going to be at a given point." [E73:CI1p1L34-35] In this statement, which is also coded as evidence of student understanding of instantaneous rate of change, Corbin connects the "period of time" over which the calculation is made as being the difference between the two concepts. Following this statement, Corbin struggles with how small the interval should be and how he would go about shrinking it.

When evidence of both procedural and conceptual knowledge was coded it suggests that the student possesses both types of knowledge regarding the topics being discussed. It does not however, insinuate that the knowledge is equal between the two.

In section 4.1.7 George's excerpt E30 Interview I shows his discussion of how he would find the instantaneous rate of change using a graph. He starts to give the procedure for drawing a secant line between the two points and then finding its slope. However, when George's discussion moves to focus on the instantaneous rate of change at a point he is able to discuss how the two points used in the secant line get closer together and, eventually, meet at one point (see Figure 4.11). This is evidence both of George's procedural knowledge (where he is stating how to find the rate of change) as well as his conceptual knowledge (where he is connecting the two concepts). George was able to show that he could both find the desired answer as well as interpret when an average rate of change might be an acceptable approximation for the instantaneous rate of change.

4.3 Comparison Between Interview Groups

I now discuss the comparisons of codes within the interviews beginning with DVS and static groups. In order to report codes occurring more often in one of the two groups, I chose a threshold of 65%. This means that codes occurring more than 65% in one of the two groups are referred to as "high frequency codes."

Table 4.11, below, lists the six codes I found to be significant for one of the two interview group. Two codes indicating student interactions with DVS (evidence DVS clarifies and Evidence that DVS obscures) are unique to that interview group so they are not listed in table.

Table 4.11 List of high frequency codes for the DVS and Static interview group comparison.

	Total Number of Occurrences	% DVS	% Static
Student Able to Covary	18	0.722	0.278
Student Not Able to Covary	21	0.762	0.238
Concept Definition	26	0.654	0.346
Geometric Approach to Rate of Change	91	0.681	0.319
Algebraic Approach to Rate of Change	56	0.339	0.661
Sense-making	47	0.291	0.702

The first two highest frequency codes both refer to a student's ability to consider the change in one quantity with respect to a change in another. There are 37 instances requiring a student to consider how one quantity changes with respect to another, and two codes to describe the outcome: when the student is able to understand the relationship of the covarying quantities and when the student is not. Both of the codes occurred with highest frequency in the DVS interview group. Beverly's ability to contemplate covariation of quantities occurs seven times during her interviews. David's inability to successfully do so occurs six times. There are eight places within Ian's interviews where he struggles to consider this relationship.

During episode E10 from Beverly's interview III she was asked about the relationship between maximum and minimum values on the graph of a function and the points on the first derivative graph. She was able to state that these places would be zero on the first derivative graph. She was then able to take this a step further when, later in the interview, she was asked about places where the rate of change of the function was the greatest and how this might be shown on the first derivative graph, "…cuz in your first function, when you get the derivative and you find the zeroes of it that your extreme ones which find the maximum and min with the relative. So it's the same idea or if you have $f'(x)$ and you find the derivative of that and you find the zeroes that's the extremes of $f'(x)$." This is discussed in more detail in section 4.2.1.3.1. Ian's struggle to communicate (or to write) an inequality describing the relationship between a function's minimum value and all other values highlighted his struggle to consider the covariation of quantities. This struggle is discussed in section 4.1.9 in episode E42.

From the static interview group, Corbin's interviews contained three instances each of him being able to consider the covariation of quantities and not. During Interview III, Corbin is easily able to make comparisons between the maximum function value of the given function when compared to all other function value, but he does so in a rudimentary manner using "y" in place of function notation. It is unclear if he is even considering that the input values of the function change in order for the resulting function values to change. However, later during the same interview Corbin struggles to track how changes in the function result in changes on the derivative graph but he is able to determine that their changes result in a discontinuity in the derivative. While considering the graph of a piece-wise defined function (see Figure 3.6) Corbin focuses on the point $(-2, -1)$ and says, "From the left it's slowly getting smaller. To -- it's slowly decreasing towards 0 from the left, but on the left it's a very static number." He states that $f'(-2) = 0$ but is struggling that the derivative of the function at $x = -2$ might be undefined. When asked why he thinks the value might not exist he responds, "… because from one side you're approaching 0 from the other side you just end up at 0. You go from, say a slope of .5 to 0 just instantly; it's not a decrease into it. Is it?" [E74:CI3p5L16-26]. Corbin is using the incorrect notion that $f'(-2) = 0$ to reason about what is happening on the graph of $f(x)$.

Evidence of what a student's concept definition occurred 26 times throughout the interviews, but it occurred more in the DVS interviews. There is also a difference in the types of things that appear to be included in the students' concept definitions from the two groups as well. For instance, Evan's concept definition (and image) for a continuous function includes begin able to "draw it without lifting your pencil up from the paper". When Beverly is asked (during Interview III) what it means for a function to be continuous on its domain she responds, "That means, um, for every x, um, okay, for every x there is a y." [E75:BI3p1L3-4] The words that Beverly and Evan use to describe what it means for them to consider a function to be continuous give insight into their concept definition for continuous function. While Evan's is rudimentary and Beverly's more advanced, neither directly accounts for the need to consider the function's domain, though Beverly's is closer.

There are eight instances where I gained insight into Harry's concept definitions and seven for Ian. These two students, one from each interview group, both earned "D" grades in calculus. Even though they were in different interview settings, they both conveyed almost identical concept definitions for a function's absolute minimum corresponding to "lowest point" (Harry) and "lowest it will go" (Ian).

Interestingly, each group had only one significant code with respect to rate of change. For the DVS group, the students were more likely that their counterparts to approach rate of change in a geometric way while the static group was more likely to approach it in an algebraic manner. It is important to note that there are 63% more instances of geometric approach to rate of change (91 instances, total) than algebraic approach to rate of change (56 instances, total). Even though the code indicating a geometric approach to rate of change was significant for the DVS group, there were still George's sketch (See Figure 4.11) and accompanying explanation from excerpts E30 and E31, discussed in section 4.1.7, contain examples of a typical geometric approach to rate of change. A second typical response coded as included instances in which a student referenced finding a "slope" even when they used the "slope formula". In episode E27, Felicity's approach to average rate of change is considered geometric based upon her remark that "…Average speed is, um, just like you have point A you have point B, you have separately. It's the slope between the - - is that what a tangent line is? I think, yeah …" This compares to the highest frequency static interview group code of algebraic approach to rate of change. Felicity was asked about a particle average speed when the function given gave the particle's position with respect to time. "So x, oh - - let me cross that off $x_b - x_a$. $\frac{f(x_b)-f(x_a)}{x_b-x_a}$. And I did that again so now I'll take. I will put each one, just kidding on the 6 and then it's .40 over 2 minus 6 minus. Which I will solve" [E76:FI1p3L20-22]. Though Felicity was finding the slope of the secant line connecting the indicated points, she did not mention slope in her calculations until later. This can be compared to Corbin's description of how he would find the average rate of change between two points, "Rise over run - - well no. Average rate of change. Am I double thinking? The average rate of change is gonna be - - oh, okay it's $x_1, \frac{x_2-x_1}{y_2-y_1}$. Well that's also, I mean, that's rise over run then..." [E77:CI1p7L35-36] Notice that Corbin does not mention the slope of the secant line though he does mention that he needs to find "rise over run" and that he says (and writes, though he later scratches out) the reciprocal of the slope formula.

The final highest frequency code for the DVS and static interview group comparison is sense-making. Over 70% of the occurrences of this code were in the static interviews. For example in George's interview I he is asked about average rate of change and he begins to wonder if it matters which calculation he did: $x_2 - x_1$ or $x_1 - x_2$. In fact, he initially stated that the "bigger one" was "normally x_2." George spends over a minute trying to make sense of this notion. He reasons that "the bottom would probably be negative and… it'd cancel out." He then uses the equation $x^2 + x = y$ and the interval $[2,3]$ to test his hypothesis that $\frac{y_2-y_1}{x_2-x_1} = \frac{y_1-y_2}{x_1-x_2}$. In the end, George states, "… so technically it could be $y_1 - y_2$ and $x_1 - x_2$, it's the same thing" [E78:GI1p8L30-p9L20].

George was unsure if the procedure he was using was correct and he chose to "test" his intuition by plugging values into a dummy equation and checking both methods for finding the average rate of change. This was typical of the sense-making instances within the static interview group. George's exploration resulted in confirmation of his intuition and he was able to proceed with the task. That was not always the outcome, however. Prior to E8, from Corbin's interview III, he struggles to mark the points of greatest instantaneous rate of change on the interval [-7.25,-2] for the graph of a piecewise-defined function (see Figure 4.27, below). He first makes small line segments on the graph at local maximum and minimum points within the interval, and then makes a mark at approximately $(-5, f(-5))$and refers to this as the "inflection point between these two (student indicates the points where the drew small line segments).” When asked what he chose the inflection point, Corbin reasons that “… you’re having a change from a -- a -- from going in a greater positive direction to…” Corbin pauses and then compares the changes to the tangent lines just before and after his point marked on the graph is Figure 4.27 “The tangent lines are increasing, increasing, increasing and they start decreasing at the inflection point.” [E79:CI3p6L33-p7L10] After this sense-making, however, Corbin struggled to make connections between the relationships connecting a function, its derivative and second derivative (see episode E8). He did not test his intuition and instead “talked” through his thoughts. Though he did come to a correct conclusion about what points of greatest instantaneous rate of change on the original function would represent, he did not get any validation and continued to question this later in the interview when he was asked a similar question about the second derivative. In this instance Corbin’s limited knowledge yielded an unsatisfying result.

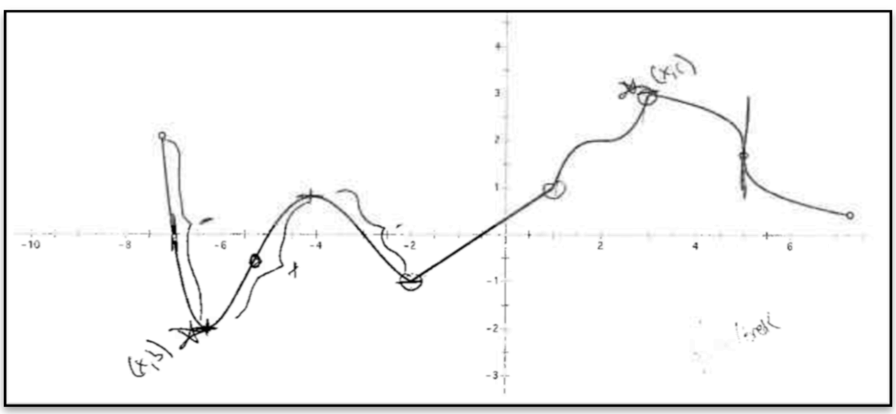


Figure 4.27 Corbin's markings for Interview III.

David, a DVS participant, exhibited sense-making in a different way during Interview II. David was exploring with graph of the function $f(x) = \sin x$ and he was asked about points on the function graph where the instantaneous rate of change was equal to 0, 1 or -1 and the importance of these on the graph of $f'(x)$. David's sense-making occurs after he sees the two graphs on the same set of axes. David's sense-making is that the set of points satisfying $(x, f'(x))$ yields the derivative graph for $f(x)$ and that the "y" value is the slope of the tangent line at the "x" value. He articulates his discovery as making sense of what he sees and he receives instant validation from the DVS about his intuition.

> David: I don't know. I was trying to do it like how I - - it wasn't - - I wasn't picturing it right in my head, no, but I see - - so whatever the x value and then the slope of the - - I guess wherever it is on the $y-$axis so - -
>
> Interviewer: Okay.
>
> David: That makes sense. [E80:DI2p8L6-10]

4.4 Comparison Between Visualizers and Non-visualizers

Since each student in the study was categorized, using the MPI, for their preference to visualize I now compare the two groups. Of the nine participants, five were visualizers (three of these participated in the DVS group) and the remaining four (two of which participated in the DVS group) were non-visualizers. Appendix A also lists this.

Interestingly, when comparing the codes present for both the visualizers and non-visualizers, only three codes met the 65% threshold for being high frequency codes and they only occurred for the visualizer group. These three codes specifically relate to visualization as well as the DVS interactions. Table 4.12 lists these codes as well as the total number of occurrences of each.

Table 4.12 High frequency codes for visualizer vs. non-visualizer comparison.

	Total Number of Occurrences	Visualizer (Occurrence)	Non-Visualizer (Occurrence)
Evidence of Visualization	132	0.652 (86)	0.348 (46)
Evidence DVS Clarifies	66	0.727 (48)	0.273 (18)
Evidence DVS Obscures	23	0.739 (17)	0.261 (6)

An example of a non-visualizer showing evidence of visualization occurred in Felicity's first interview. Felicity was asked about the instantaneous rate of change at a point on a function graph and she stated that the "slope formula doesn't work" to find it. Felicity responds that you'd need to manipulate point B that it is sufficiently close to point A. She is using the visual power of the DVS to test her hypothesis and move the points close together to get an estimate for the instantaneous rate of the change. The way that Felicity describes that interaction suggests that she was using the visualization to make her statements.

> Felicity: Yeah, you could do it, um, you can move it over here and you can, um, get what it would be about just using point B. Not exactly A, but slightly to the right and left. You could also use the other side and you'd get closer and closer until you get to what it should be, but you can't solve it using the slope formula. [E81:FI1p5L5-8]

In episode E9, during her exit interview, Beverly, a visualizer in the DVS group, described the derivative as "… a continued slope…" Figure 4.5 shows the sketch that she drew to clarify and explain her remark. This type of evidence of visualization contrasts sharply with Felicity's in episode E81. Beverly linked the topic to imagery in a concrete manner; that is, the word derivative elicits a picture in Beverly's mind. Felicity uses the power of the visualization to explore but she did not, in this case, have the picture in her head beforehand. Harry, also a visualizer though in the static interview group, explained why the derivative at the same point on two different function graphs (see Figure 3.7) would have different derivative values by describing their “spot” on “the hump.” For Harry this explanation was sufficient explanation for the different rates of change.

The codes for DVS clarifying or obscuring are unique to the DVS interview group. However, the codes were still high frequency codes for the visualization students. For Amy, a non-visualizer, the software clarified how changes in one endpoint of an interval over which average rate of change was being discussed made changes in the overall average rate of change. “Well, because the, like the rate at which the slope is changing is getting smaller and smaller like every time I move it, it just barely changes the slope.” [E82:AI1p9L15-16] She approached most things in a non-visual manner, so her DVS clarifies instances were typically refinements of her existing conceptual knowledge. Beverly (see episode E11], in contrast, approached this with a mix of algebraic and visual approaches. For example, when asked about the average rate of change as the distance between the two points approaches zero. Beverly initially believes that “there’s no change… so it’d just be zero.” However, after exploring with the software she states that, “Because the slope -- or the slope of the secant line there is no secant line because the secant line needs two points and there's only one point exactly cuz they're on top of each other.” Beverly continues to explore with the sketch and make a realization that the change that should be calculated when the two points coincide is the instantaneous rate of change. While Amy’s clarification from working with DVS clarified that on small intervals the average rate of change might serve as an appropriate approximate for the instantaneous rate of change at a point, Beverly realized that when the distance is zero between the two points on a secant line the instantaneous rate of change at that point could be found (though, as she mentions in episode E12, not by using the slope formula).

The final high frequency code for the visualizer group is evidence that DVS obscures. This code was used when there is evidence that the student was focusing on irrelevant, distracting, or unimportant aspects of the sketch. For example, during Interview I while using DVS to explore characteristics of the graph of a quadratic function with a fixed point connected with a secant line to a dynamic point, Ian's description of what he sees focuses on "…more accurate or more average results," and that, "it's getting more bigger between two set points." [E83:II1p2L25-p3L1]

4.5 Student Group Comparisons

After I received student participant grades for calculus I formed three additional comparison groups based on student performance. I will make comparisons between groups delineated by visualizer and non-visualizers or course grade and interview type. The Exit Interview provides a common interview experience for making such comparisons as the format was static and the protocol was the same for all students. Appendix H contains the Exit Interview protocol.

4.5.1 Beverly and Amy

Both Beverly and Amy earned an A in calculus and they both participated in the DVS interview group. However, Amy's score on the MPI categorized her as a non-visualizer. In this section, I compare the interviews for these two students.

During the Exit Interview all students were asked what they think of when they hear the word derivative; they're asked to give as many examples or ideas as they can in one minute. Amy's first response to this question (see episode E84, below) is to state the limit definition of derivative followed by instantaneous rate of change and the slope of the tangent line. She then lists some rules and "shortcuts". This contrasts sharply with Beverly's "continued slope" illustration and dialogue from episode E9, discussed in Section 4.1.2 and seen in Figure 4.5. These two differences highlight the visualizer and non-visualizer styles for each student.

> Amy: Okay. I think of the definition to the $\lim_{h\to 0} f(x-h)$, no , $(x+h) - f(x)$ over h and then I think of instantaneous rate of change and how-- just put simply, the slope of the tangent line at any given point. Um, let's see, and then I just think of all the different rules, the shortcuts for taking the derivative; the power rule, the quotient rule, um, that's it. [E84:AIExp1L4-7]

Even though Amy's preference is to not visualize when working on a mathematical task she often does so and these instances give us insight into her concept images for some topics. For example, at the end of the Exit Interview Amy is asked about the relationship between continuity and differentiability. She states that differentiability implies continuity but the converse does not hold. When probed about why she believes this, Amy offers to present a counterexample. She sketches the graph of the absolute value function (see Figure 3.8) and explains the difficulty in determining which tangent line to use to find a derivative value. Amy connects the example with her knowledge about differentiability and continuity and transfers this to functions whose graphs include sharp corners.

Beverly comes to the same conclusions as Amy, though through a different approach. Whereas Amy's sketch of continuous function that is not differentiable everywhere is used through the lens on "infinitely many tangent lines," Beverly presents an analogous argument using the slopes of the tangent line as the point in question, $(b, f(b))$, is approached from the left and the right (see Figure 4.28). Episode E85 contains evidence that Beverly's concept image of a function's maximum (and minimum) includes those places where the derivative is undefined, not only where $f'(x) = 0$ as she specifically mentions these possibilities. Beverly's language is ambiguous when she says, "… like when you look at limits… this approaches positive $f(b)$ and this is like a negative. Yeah, the slope is negative $f(b)$." Beverly uses her pen to indicate that the portion of the graph to the left of the point $(b, f(b))$ is "positive" and that the portion to the right of the point is "negative." The last comments she makes suggests that she may be considering the limit of the derivative values as they approach $x = b$ from the left and from the right. There is no evidence in the interview of this being included in Amy's concept image, so I cannot know whether it is or not. Neither approach presented by the students is procedural in nature and both present robust examples that connect the topics at hand.

> Beverly: Um, yes. Because that's when, um, that's how -- yes, okay. Um, for you to find a maximum and minimum you have to find the extremums which is when the tangent line is 0 or undefined. But if it's not continuous then there wouldn't be a tangent line and then there wouldn't be any extremums.
>
> Interviewer: Okay. So let's talk a little bit about that. So if you know that a function is continuous on a closed interval, what can you say about that function's differentiability on that interval?
>
> Beverly: Um, just because it's continuous doesn't mean it's differentiable.
>
> Interviewer: Okay.
>
> Beverly: There can be an absolute value and you know that this is the maximum because you know at this point it'd be like, I don't know, b, but that's the maximum, or $f(b)$, that would make more sense and this is b. At $f(b)$ that would be the, um, maximum even though it's not differentiable because it's a sharp point. (Student draws graph, See Figure 4.28)
>
> Interviewer: Okay. Why are sharp points not differentiable?
>
> Beverly: Because, um, as you -- like when you look at limits, uh, yes, when you look at limits, uh, this approaches, I guess, positive whatever b is, $f(b)$ (student uses pen to indicate the portion of her figure to the left of the point $(b, f(b))$. This approaches positive $f(b)$ and this is like a negative (student uses pen to indicate the portion of her figure to the right of the point $(b, f(b))$. Yeah the slope is negative of $f(b)$. [E85:BIExp3L27-p4L8]

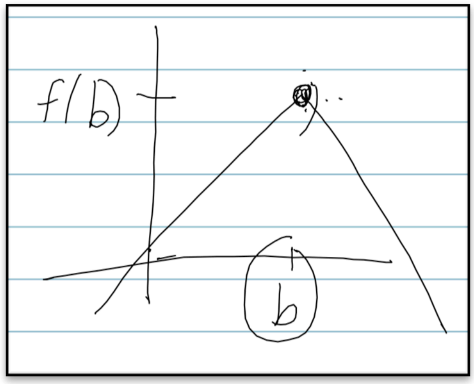


Figure 4.28 Beverly's example of a maximum function when the derivative is undefined.

Amy (a non-visualizer) and Beverly (a visualizer) are able to communicate the connections they possess regarding the calculus topics presented. The only difference between these two students with regard to the coding within their interviews is that DVS clarified concepts in 22 instances for Beverly and only in six instances for Amy.

4.5.2 Evan, George and David

George and Evan are both non-visualizers and both participated in the static interviews. David, a visualizer, experienced the DVS interviews. All three students earned a C in calculus. When asked about what comes to mind when hearing the word “derivative,” both students from the static group respond immediately with instantaneous rate of change. David begins listing examples (“x^2 derivative equals $2x$ … "). Evan and George also mention tangent lines, though neither of them is initially clear about what attribute of the tangent line is the derivative. George immediately offers to draw a picture to clarify his explanation about “slope at a point,” in episode E65 and Figure 4.20. George eventually makes a connection between the slope of the tangent line and the instantaneous rate of change at a point. David never mentions the geometric representation of instantaneous rate of change at a point; in fact, when the derivative at a point is related to instantaneous rate of change at a point (remember, he did not mention this during the first question), David's approach is procedural, "… we have, like, a function and you need to find the instantaneous rate of change at that point you just take the derivative." [E86:DIExp1L14-15] All three students clearly state that the derivative at a point and instantaneous rate of change at a point are the same.

Only George was able to correctly discuss the long run behavior for the two function graphs presented in question three (in the Exit Interviews; see Figure 3.7.); both Evan and David appear to confuse "long run behavior" with "instantaneous rate of change." David's response to a question asking him to compare the long run behavior of the functions provided is, "Um, it's decreasing and increasing. They're both decreasing then increasing." When asked for clarification about this comment, he responds that the "instantaneous rate of change," is what is decreasing then increasing. [E87:DIExp1L18-27] When asked about estimating and comparing the instantaneous rates of change at $(2, f(2))$ and $(2, g(2))$, Evan immediately laments that he was not provided with the function definitions as he, "… would find the derivative and plug in 2 for x." David uses the location of the two points (both function values equal 3) in relation to, "… where it goes from increasing to decreasing, the point where it does that, the point where it switches the 0, the instantaneous rate of change, which is the derivative of the function would be 0" [E88:DIExp2L1-4]. George, on the other hand, chooses a second point for that lies on the graph of $y = g(x)$ and finds the average rate of change across an interval instead of estimating the instantaneous rate of change at the point $\big(2, (g2)\big)$. For the graph of f, however George uses a similar method as David using the function's maximum because, "…it $[(2, f(2))]$ is basically a maximum and any maximum has a slope of 0." [E88:GIExp2L15-16] This last statement suggests that George's concept image for the maximum value of a function does not include a point where the derivative is undefined. However, in his sketches for item four George does place a function maximum at an endpoint included in the interval provided, though it is unclear if George knows that the derivative does not exist at such a point.

When exploring with various intervals in task 4 (see Appendix H) to determine what attributes a function must be guaranteed to attain both its maximum and minimum values, only one of the students, Evan, considered a continuous function with a cusp as the maximum (or minimum) value without prompting from the interviewer. Evan explains why the function he drew is not differentiable at the point he circled, “Since at the point right here it doesn't have a set -- a definite derivative it'd be -- the derivative on the left would be positive and the derivative on the right would be negative making it a maximum.“ [E89:EIExp4L3-5] Evan’s is referring to his illustration in third graph of Figure 4.23. Episode E89 occurs just prior to episode E70 (see Section 4.2.2.3). Evan’s statement in E89 contrasts with his statements that a function must be differentiable as well as continuous over the given closed interval to ensure that the function’s maximum and minimum are achieved (see episode E70). Evan’s overreliance on flawed procedural knowledge results in conflicting statements from him about these concepts.

David also struggles with the idea that a continuous function may not be differentiable over the entire interval. He states that “…it’s continuous and differentiable,” when asked what role continuity plays in knowing about a function’s differentiability. He also struggles with the relationships between the function value at a point and the derivative at a point and, at times, is unsure which he is referencing. After drawing a graph similar to that of the absolute value function, David appears to clear up his confusion and he states that the derivative is undefined a point making a sharp corner. David’s sketch is shown in Figure 4.29. However, David’s understanding of why the derivative does not exist at the point he indicated is limited to an incorrect, rudimentary procedural understanding that, ”… Because it's -- it's a 0 or not 0, undefined so. Never thought about that. Cuz the -- what's it called? The slope at that point is like, no -- I guess it would be 0. And you can't find the derivative of 0 so --." [E90:DIExp6L21-23]

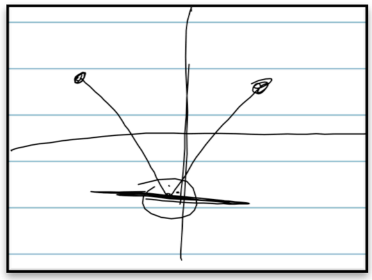


Figure 4.29 David's sketch as he reasons about a function's derivative value at the point he circled.

George struggles with deciding if a function's maximum is attained if the place where it occurs is at an endpoint of an open interval. Figure 4.30 shows his work and sketch as he considers if the function maximum is attained in the function graph he drew on he open interval (a, b). George states that though "if you ask a calculus teacher they'd say no [there is not a maximum function value], but I feel like it's there." This suggests that his concept image for a function's maximum value allows for a non-included endpoint to be an maximum (or minimum) function value and that the function gets "close enough" to attain that value. However, later in the interview when working with the interval $[a, b]$, George states that, "…on the right-side if there's a max or a min you can actually find it." This suggests that he now believes that he cannot determine the maximum function value in Figure 4.30 as previously stated. It is also possible that George's concept definition for the two intervals being discussed (open and closed) are in conflict. Certainly his incorrect concept definition of a function attaining a function value at a point where the function is not defined is in conflict with the formal concept definition.

George: Well if we're only looking inside the interval then it shouldn't matter. I mean, if you don't include the end point I think there might be a problem with actually being able to find the max or min but you can still do it I think.

Interviewer: Can you draw me an example of what you're talking about on the left most set of axes?

George: Left most? Okay. So let's say you had your - - well yeah this will work I guess, I basically messed that up but it's not over yet. So in this case the mins kind of obvious, the mins like right there. But the max it's like right there. But, like if we don't include that end point - - so let's just say the end point's 3. Like right here -- without its maximum, so but we don't have 3, so we want like, it's like all numbers except 3, so we want the number - - like the number that is closest to 3 without being 3 (student draws example) [See Figure 4.30.].

Interviewer: Is there such a number?

George: Uh, no. Um, most of the time you do - - cuz you can always add a 9 to this to get closer and closer but at some point if you have - - if you actually have 3.99 repeating, like a lot of people consider it to be 4 anyways [See Figure 4.30 for work.], so you get the idea that it's like, well I guess it'd be the limit of - - of 3 on the right side. Even though, that's 3.

Interviewer: So does the function that you drew attain its maximum value?

George: Um, it sort of does but at the same time you'll never be able to find it. That's my opinion. I'm pretty sure if you were to ask, like, a calculus teacher they'd say no but I feel like it's there, but it's not possible to get so we say that it - - it can't be found. Like it has no max. [E91:GIExp4L7-27]

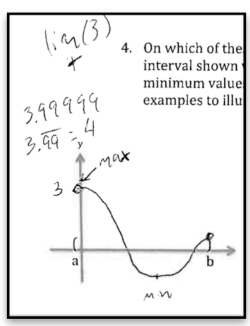


Figure 4.30 George's work as he struggles to decide if an endpoint of an open interval can correspond to a function's maximum value.

When asked about continuity and differentiability, George initially says that a continuous function is also differentiable. However, when probed by the interviewer, George responds with, "… You're right, cusps and corners." He also describes the reason for the discontinuity in the derivative function in a visual manner, "you have two slopes that are, like, colliding, sort of…" Though he later states that the derivative at points making a sharp corner or a cusp is, "infinity." [E92:GIExp5L19-35]

4.5.3 Harry and Ian

Harry and Ian, both visualizers, earned D's in calculus. Ian participated in the DVS interviews while Harry was in the static group. The two students, at different times, expressed varying levels of conceptual and procedural knowledge that would not be present at later points in the Exit Interview. For example, Ian related derivative at a point to change in slope as well as to his knowledge about physics. He makes the connection between the position and velocity vs. time graphs for a particle. He also mentions that the derivative value at a point gives the instantaneous rate of change of the function at that same point. However, later in the Exit Interview, when working with the graph of two similar functions (see Figure 3.7), Ian struggles to communicate why the values $f'(2)$ and $g'(2)$ are different, "They have two different derivatives and are two different graphs entirely. They may have a similar route but, um, the way the equation was ran it has different values too." [E93:IIExp3L7-8] It is as if his insights and evidence of conceptual knowledge are memorized and completely disconnected from any other knowledge he might possess. He makes nonsensical statements throughout the interview and does not have the baseline understanding of how the topics are related to converse about them in a meaningful fashion. For example, Ian is unclear about whether a function's maximum (or minimum) can occur at a cusp or corner. Ian also confuses continuity and differentiability throughout the interview. Ian's concept image does allow for a maximum (or minimum) function value to have an undefined derivative. For example, he states, "Because, um, sometimes, um, well it cannot be derived would be sometimes, um, it could be the, um, could be at a point where it could be the minimum or maximum." [E93:IIExp4L31-33] When referring to Figure 4.31, Ian knows that the function is continuous and that there is a point that is not differentiable, but when asked why this is the case he, "cannot remember."

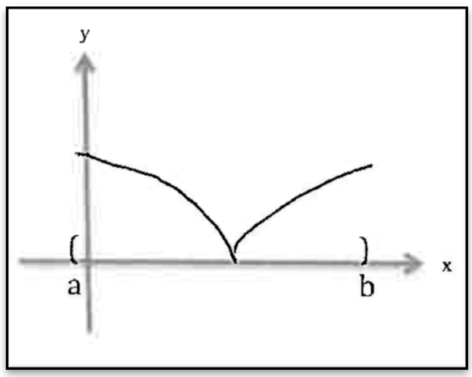


Figure 4.31 Ian's sketch of a continuous function that does not attain its maximum and minimum function value.

Harry's exit interview varies only slightly from Ian's. Harry initially says that his knowledge about derivatives is that, "the derivative of a point is the tangent." He later refers to the power rule as a, "step down." He also discusses the long run behavior of the function graphs shown in Figure 3.7 by first mentioning that, "from the left x goes to infinity and -- so it goes from negative infinity to infinity." He then analyzes the behavior of the function values, say, "…then the y down the left it goes up to infinity and to the right it goes down the negative." [E94:HIExp2L4-6] He may not yet have acquired the necessary across-time view of function to make such comparisons. Harry struggles to estimate the instantaneous rate of change at the indicated points on the graphs shown in Figure 3.7. He is not provided with the "…equation for the line," suggesting that he relies on procedural knowledge that is not fully developed to find such an estimate. Eventually, after I ask some probing questions, Harry says that he could find the slope of a tangent line at the point in question for both functions. Harry's eventual comparison of the instantaneous rates of change for the points $(2, f(x))$ and $(2, g(x))$ are based on, in his words, their "different spots on the hump." This is evidence of Harry's visual engagement with the function graphs.

When asked about the three sets of axes provided in task four of the Exit Interview (see Appendix H), Harry immediately states that the closed interval would guarantee a function's maximum and minimum were attained as, "the endpoints matter." This suggests that he is able to consider that the endpoints should be taken into account when searching for possible minimum and maximum value points but it provides no insight into his knowledge about whether differentiability plays a role in this as well. He is able to compare the different situations that arise when the presented intervals are considered and state that the function must be continuous. When asked about the relationship between continuity and differentiability, Harry quickly sketches an example of a continuous function with a cusp and states that the function is not differentiable at the cusp. He is also comfortable with the cusp being one of the function's extrema. When asked why the graph is not differentiable at the cusp, Harry provides insight on a conceptual nature with his response of, "well, it's drastically changing direction…"

Chapter 5

Discussion and Conclusions

After analyzing common themes emerging from the interview transcripts I use grounded theory (Corbin & Strauss, 2008) to form hypotheses based on these themes and based on the results reported in the previous chapter. Although I had read some of the pertinent literature discussed in Chapter 2 prior to beginning my study, the emerging themes from the data required that I expand the literature review to include additional research to inform my interpretations. In this chapter I explain the relevance of the results reported in Chapter 4, especially those results that may not be consistent with prior research. Although my results are somewhat consistent with previous research, I have extended previous findings as they pertain to undergraduate calculus learners. These new results are compelling for those using, or considering using, DVS in an undergraduate classroom.

5.1 Procedural Knowledge vs. Conceptual Knowledge.

As shown in Table 5.1, the instances of procedural knowledge co-occurring with student understanding of instantaneous rate of change were higher for the static interview group than the DVS interview group. However, the tasks used as the basis for the static interview protocols were crafted from typical exercises presented in classic calculus textbooks and previous research suggests that rote calculus exercises fail to develop conceptual knowledge (Hardy, 2009; Lithner, 2004; Szydlik, 2000). Thus, a prevalence of the procedural knowledge coding for the static group aligns with prior research. However, when comparing the number of instances where conceptual knowledge occurred simultaneously with student understanding of instantaneous rate of change, a majority of the instances (63.6%) are in the DVS interviews. This comparison supports the notion that student interactions with DVS may foster enhanced conceptual understanding for calculus students, whereas students focusing on more typical exercises remain focused on procedures. It is important to note that not all instances of students conceptually discussing instantaneous rate of change involved correct conceptual understanding. When reviewed for correct conceptual knowledge, 12 of the 14 episodes contained correct conceptual knowledge of instantaneous rate of change. Two episodes contained partially correct conceptual knowledge. This compares to six (out of eight) static interview episodes showing correct conceptual knowledge about instantaneous rate of change. Though the number of instances of correct conceptual knowledge about instantaneous rate of change is higher, it is not statistically significant. However, the increased number of instances provides more opportunities for instructors to ask probing questions and address incorrect conceptual knowledge (Aspinwall, Shaw, & Presmeg, 1997).

Table 5.1 Co-occurring codes with student understanding of instantaneous rate of change

	DVS Interview Group	Static Interview Group
Student Understanding of Instantaneous Rate of Change	83	46
Conceptual Knowledge	14(.636)	8 (.364)
*Correct Conceptual Knowledge	12	6
*Partially Correct Conceptual Knowledge	2	2
Examples	4	0
Procedural Knowledge	10 (.294)	24(.706)
*Correct Procedural Knowledge	7	6
*Partially Correct Procedural Knowledge	3	10
*Incorrect Procedural Knowledge	0	8

5.1.1 Focusing Phenomena

It is particularly interesting that many of the moments in which DVS participants discuss instantaneous rate of change conceptually happen to fall in close proximity to instances when exploring with the DVS clarified a relationship between mathematical topics for the student. For example, when Beverly initially investigates the average rate of change between two points on a graph (see episode E11 in Section 4.1.2) she begins to move the points closer and closer together. Once the interval is sufficiently small, the secant line connecting the two points disappears. Beverly surmises that there is no average rate of change in this situation, but her supporting statement for this is that since, "…Then, um, it would be 0 because if x_B is the same as x_A that'd be 0. Over $y_B - y_A$, and that'd be 0 too. Which is just the same things or same numbers." Beverly does not indicate that she understands that $\frac{0}{0}$ is indeterminate. She says that when the distance between points on the function graph is zero, the average rate of change exists but, "it can't be found using the slope." Conceptually, her statement is incorrect as the instantaneous rate of change exists at a point and the idea of calculating the average rate of change does make since as there is no interval over which it could be calculated. Later in the interview, when exploring with the graph of a different function, Beverly makes a statement indicating that her views on the slope of the secant line in episode E11 may be incorrect. While moving dynamic point B, and investigating what happens to the secant line when $f(x_B) = f(x_C)$ but $x_B \neq x_C$, Beverly comments, "that's weird," then manipulates point B so that it coincides with point C on the function graph. When asked what seems amiss, Beverly responds, "I don't know why I was thinking this, but for some reason I thought that if they're at the same points the slope would be zero, but that would make no sense… There is still a slope with the tangent line" [E96:BI1p8L25-33]. After continuing to explore using the software, Beverly changes her mind about the existence of the average rate of change discussed in episodes E11 and E96. Beverly relates the slope of the line to "speed" and then begins to reason about what really happens as the distance between B and C approaches 0, "Oh you find the instantaneous so that means at that instant so technically it's not the average because it's at that instant" (see episode E12 in

Section 4.1.2). Through the investigation Beverly was able to connect the interval around x_B to the type of "speed" she was considering. The connection suggests that Beverly shows a shift to correct conceptual knowledge about the relationship between average rate of change over an interval and instantaneous rate of change at a point contained in the interval (White & Mitchelmore, 1996).

The evidence of Beverly's correct conceptual knowledge occurred after she continued to investigate different relationships using the DVS, but the interviewer also asked a series of probing questions intended to focus Beverly's attention on what was happening within the sketch. That is, the intentional redirection may have precipitated such a conclusion. Another such interaction occurred with Amy during Interview I.

> Amy: Let's see. All right. So, oh, so C and A are moving together and the difference between them is not changing… oh, okay. Um, so this is, this is one. Let's see, I mean they are moving up the graphs. The slope is changing because it's on a different part of the graph. Um, let's see. The y values of C and A are obviously change. Um, I don't know… [E97:AI1p10L24-27]

From this it is evident that Amy is unsure upon which aspects of the sketch she is supposed to focus; as a result, her response to "what do you notice as the two points are animated" is rambling and lacks focus. After being asked to focus on the table of data values Amy comments that, "As the difference in x values change, the difference in y values is the same amount of change in the x values." This indicates that even though Amy's focus was directed to the table, she was making comparisons among the column entries but she missed the nuance that the values for $f(x)$ and $\frac{f(x_C)-f(x_A)}{x_C-x_A}$ were, in many cases, identical. Due to a rounding setting within the DVS, the values were shown with two decimal places and it is possible that the identical values were making it difficult for Amy to notice that they were similar though not exactly the same. However, once Amy's attention was directed toward similarities within the rows of the table, she was able to state, “Oh, so the slope and the y values are the same.” This statement led to Amy’s commenting on the slope of the secant line, when Δx is sufficiently small (in this case $\Delta x = 0.01$), is a good approximation for the function value of the function $f(x) = e^x$.

Without the intentional refocusing of the learners’ attention toward the desired mathematical relationships, neither student may have expressed the conceptual knowledge observed. This supports Lobato & Burns-Ellis (2002) research regarding the necessity of focusing phenomena (when an instructor refocuses a student on the mathematical relationship being highlighted in the investigation) when using technology, in that the focusing provided by the researcher as students used the DVS prompted valid reasoning about the calculus concepts. The need to refocus students’ attention toward highlighted relationships in an effort to guide learning extends to the undergraduate calculus classroom when using (or offering) DVS as a learning tool. Without the interviewer refocusing the students’ attention toward these mathematical relationships, the instances where DVS clarified such relationships would be fewer in number, resulting in less evidence that students’ interactions with DVS can result in the formation of correct conceptual knowledge. This has particular implications when animations are provided without an instructor present (i.e. online homework). An instructor can provide the focusing questions and redirection, but an animation alone would need to have built-in focusing features or voice-overs.

5.1.2 Unsupported Conceptual Knowledge

It is important to discuss those times when a student presented what I call “unsupported conceptual knowledge.” Many research studies report the need for a student’s procedural knowledge to be “conceptually supported” (Baker, Cooley, & Trigueros, 2003; White & Mitchelmore, 1996; Gray, Loud, & Sokolowski, 2009; Moore & Carlson, 2012; Szydlik, 2000; Hardy, 2009). Specifically, White & Mitchelmore (1996) suggest that if a student has memorized the rules of calculus but has no understanding of the "why" behind these rules then the student will have little chance of understanding the nuances of tasks meant to build knowledge and further understanding. There is little research discussing calculus students who are unable to complete a procedural calculus task yet who make conceptual statements suggesting they have the necessary procedural knowledge.

There are several instances within my study where I found evidence that students developed conceptual understanding after working with the DVS but had little procedural knowledge to support completing the task. There is also evidence of students gaining conceptual understanding of mathematical relationships while working with DVS, however in the absence of DVS they are unable to reproduce their conceptual statements, or to show that they have strong procedural knowledge about the topics.

There is evidence in several of the episodes from Ian's interviews that he has conceptual knowledge about topics from calculus. In episodes E44 and E45 (see Section 4.2.1.1) Ian makes the connection between the tangent line and the instantaneous rate of change at a point, though he does not specifically mention the tangent line's slope. In episode E54 (see Section 4.2.1.3), Ian compares a particle's average speed to its instantaneous speed at a given time, "…instantaneous rate of change would be a precise, small window, a small difference in time. Where the average is a bigger window." Ian's ability to make this comparison was profound and the episode was coded as showing that he has achieved a level conceptual understanding. Later in the interview, however, Ian was unable to answer questions of a procedural nature and he often responded to such questions with "I don't know," suggesting that his procedural knowledge was, in many cases unsupported conceptually.

In his exit interview, Ian again makes connections between average rate of change and instantaneous rate of change at a point,

> Derivative would be a change in the slope, so like in physics it'll - - um if you have position graph and you want to find the velocity you would take the derivative. Same thing when you want to find acceleration of the velocity graph you would take the derivative there. It's basically just - - when I think of derivative it's the, um, or different types of derivative instantaneous change of rate would be counted as derivative because it does changes the slope but at a point. [E98:IIExp1L5-9]

After episode E98, Ian indicates that the slope of the line tangent to a function at a point will provide the instantaneous rate of change at that point. However, when asked to find the instantaneous rate of change for $g(x)$ at the point $(2,3)$ (see Figure 3.7) he is unable to successfully do so. Ian's grade of D in the course lends to this argument that, though he did possess some correct conceptual knowledge his weak (and at times incorrect) procedural knowledge hindered his ability to be successful in the course.

Ian's conceptual knowledge was not long-lived; that is, later within the same interview or during subsequent interviews relating to the concept, he was unable to call upon this knowledge. It is possible that he is making conceptual statement in a procedural fashion by mentioning the words that he remembered hearing frequently in class or lab, but his words in episode E54 appear to be a personal description of the relationship. After investigating with DVS during Interview III, Ian determines that, at corners on a function's graph, the slope of the tangent line changes abruptly. "That it does not have an - - it almost immediately changes from - - changes the slope doesn't go directly to 0." He says this as he uses the line tangent to the function (see Figure 4.16) and moves the dynamic point back and forth along an interval surrounding $x = -2$. As Ian does this, the tangent line appears to "jump" and Ian continues to focus on this as he makes the statement above. When asked what this means, Ian indicates the point $(-2, f(-2))$ on the graph and states that, "it isn't differentiable there" [E100:II3p5L28-

p6L3]. However, during the Exit Interview, when DVS was not offered as a mode of exploration, Ian is unable to recall why the derivative does not exist. He simply tries to rely on underdeveloped procedural knowledge and says, “I do not know.” It is possible that Ian may have been able to make a correct statement had he been allowed access to DVS.

Another example of unsupported conceptual knowledge occurred during David’s Interview II. David, after working intensely with the sketches, was able to state that when given a point on the graph of a function, $(a, f(a))$, the corresponding point on the graph of the derivative at $x = a$, would have a “y value” equal to, “whatever the slope is [of the tangent line].” Later in the interview, David was asked to provide the corresponding point on the graph of f' for $x = b$, given a point on the graph of f, $(b, f(b))$, where $f(b)$ is a relative maximum value of the function. He was, again, able to relate $(b, f(b))$ and $(b, f'(b))$, and stated the latter represents a “zero” of the derivative. Throughout the interview, David makes statements relating the topics he is exploring, however, his grade of C in the course suggests that David was weaker at carrying out the procedures asked of him on exams for the course.

During Interview III David struggled to understand why the derivative at a point making a sharp corner on a function graph did not exist. Initially David believes that, “you can’t set the derivative equal to zero,” at such a point. However, after investigating on the graph (see Figure 4.16) near $x = -2$ he realizes, “… so the derivatives from both sides aren’t equal.” David continues to investigate his notion near $x = 1$ and $x = 3$ on the same graph [E99:DI3p11L12-p12L11]. However, in David’s exit interview (see episode E48 in Section 4.2.1.2), he has returned to his incorrect procedural knowledge about such a point and states, again, that “you can’t take the derivative of zero.” This result is important as it supports the research literature that a balance of both procedural and conceptual knowledge may result in student success (Gray, Loud, & Sokolowski, 2009; White & Mitchelmore, 1996; Hardy N. , 2009; Lithner, 2004; Szydlik, 2000). It is important to note that at this particular institution, the calculus exams are highly procedural with few conceptual questions and very little to no visualization.

The instances where a student makes a statement showing evidence of conceptual knowledge that is either not replicable or that does not transfer suggest that the interactions with DVS are possibly not resulting in the creation of connected schema between the concepts; however, this could also be a result of not having the reasoning tools available and may give more insight regarding the need for formative assessment that aligns with learning experiences and tools. It may also be that the isolated conceptual statements, unsupported by a procedural understanding for application, may be statements learned from lectures or labs but forgotten or disregarded due to the lack of connections with which to form schema (Bennett, 2009; Baker, Cooley, & Trigueros, 2003). This situation is, then, the reverse of those reported previously where the focus of procedure-based learning failed to facilitate strong schematic connections (Carlson, Oehrtman, & Thompson, 2007). It may be that DVS enabled the students to communicate their understanding of the conceptual connections presented in the investigations, but in the absence of DVS the students were unable to access these connections. Thus, DVS enables reasoning and conceptual connections for those students who possess weaker procedural skills however in a “regular procedural setting” (i.e. exams or the Exit Interview) when DVS is unavailable these students could not access their understanding.

In these situations there is also evidence that for low performing students, the DVS interactions were not sufficient to result in the transfer of knowledge in a static situation where the DVS tool was not available. Both Ian and David were able to make statements of a conceptual nature in one setting (working with DVS), but when the same topic was presented to them in a different setting (e.g. in the Exit Interview) they were unable to apply their previous knowledge (Lobato, Rhodehamel, & Hohensee, 2012). Although the software provides a tool for exploration when a student is not able to reason about mathematical relationships, student interactions with DVS may not be enough, even with refocusing, for low performers to fully link calculus topics and transfer to situations in which the tools are not available.

5.2 An Across-Time view

It is possible that the work with the DVS allowed students to view each situation in the desired "across-time" frame of mind, whereas the focus for students in the static interviews was a pointwise view of function. That is, the students that interacted with DVS explored how quantities change with regard to each movement whereas the static interview participants did not easily have this capability due to their assigned interview group. Even when Ian was not able to progress on a particular aspect of a task, such as the relationship between the function minimum value and other function values in episode E42 from Interview III (see Section 4.1.9), he was able to refocus with a different line of questioning from the interviewer and compare derivative values across the entire graph. In this way, Ian shifted his focus from working with the task one point at a time and was able to make comparisons across the entire function. This contrasts with students in the static group. In many cases they focused on the "problem spot" and did not consider the bigger picture presented to them. Corbin, as previously examined (see episodes E3, E5, E7, and E8, discussed in Section 4.1.1, episode E61 discussed in 4.2.2.1.1, and episodes E74, and E77, discussed in Section 4.3), was unable to continue working on a task when his procedural knowledge failed to yield the desired result. George struggled in episode (discussed in Section 4.1.7) from Interview I to relate $f(x_2 + h)$ in the difference quotient to the value, y_2, that he preferred to use. After drawing a picture and reasoning, George decides that the two quantities are the same, but his focus on this relationship suggests that his view remains point-wise rather than across time. Thus, the static exercises presented in the interviews may not be sufficient to elevate student thinking to an “across-time” view of function that is necessary to develop a rich understanding of the material (Eisenburg & Dreyfus, 1991). This is related to a student’s ability to consider both a function’s input and output and to understand how changes in one results in changes in the other (Carlson, et. al., 2002).

Two codes occurring with the most frequency for the DVS group indicate when there is evidence of a student considering how one quantity changes with respect to another. One code, “evidence that a student is able to covary,” indicates a student’s successful consideration of changes in quantities and the second code, “evidence that a student is unable to covary,” indicates when a student is unsuccessful tracking changes in quantities. Both codes occurred at a higher frequency for the DVS group (see Table 4.11 in Section 4.3). This may be due to the nature of the DVS interviews requiring that students access more covariational reasoning and thus it may be a natural tool for exposing students’ strengths and weakness in covariation.

5.3 Concept Image and Concept Definition

There were instances within both types of interviews where a student was faced with evidence that their concept image and concept definition were out of alignment. Several of these instances involved a function’s maximum (or minimum) value occurring at a corner. As presented in episodes E86-E71 (see Section 4.2.2.3), Evan experienced such a situation when discussing the EVT; based on the sketches he made (see Figure 4.23), Evan’s concept image included the necessary condition that the function be continuous for the theorem to hold. However, Evan’s concept definition was missing this requirement and, as Evan attempted to reconcile the conflict between his concept image, personal concept definition, and the formal concept definition he conflated continuous and differentiable. Evan showed no pathway to bring his concept image and concept definition into alignment. When the interviewer provided an example where the graph included a vertical asymptote within the closed interval over which the function is defined, Evan made a false argument that the function was continuous. He abandoned his quest with no resolution. His incorrect procedural knowledge and weak conceptual knowledge left him with no pathway in which to continue to reason.

Corbin experienced a similar situation during Interview I when he struggled with how the average rate of change might be found when Δx approaches 0. While working with the table of values (see Appendix B) Corbin reasons that the values for the average rate of change will "meet in the middle" as the change in x approaches 0. Corbin, however, initially wrote that the average rate of change was 0 when $x_1 = x_2$. When this is pointed out, he begins to reason about the value of the average rate of change.

> And that's wrong the average rate of change of this point is actually going to be - - it's gonna be 1. Right? One second cuz that'd be negative 10 over 1 minus negative 10 over 1. So that'd be 0 over 0. Which is undefined. No 000. No. Something, something's different. … Oh, for some reason I'm feeling that this - - this needs to be - - no it doesn't. I was thinking that this wasn't gonna need to be 3, but it's not, or negative 3, but it's not. What - - [E101:CI1p9L25-31]

For Corbin the problem was rooted in incorrect procedural knowledge, specifically his belief that $\frac{0}{0} = 0$. After episode E101 Corbin continues to reason about the pattern he sees in the "progression" of the average rate of change but he does not come to a conclusion. Corbin also abandoned the task and, in frustration, moved to the next question in the interview.

Both Evan and Corbin continued to hold unaligned concept images and definitions, even though both knew that something was off between their personal concept images and concept definitions. Williams (1991) suggests that only through "sufficient" conflict can a cognitive change result in bringing the two into alignment. I argue that, even in cases when a student is aware of the misalignment, they may not possess sufficient conceptual or procedural knowledge to produce a cognitive change bringing their concept image and concept definition into agreement. However, investigation with DVS may provide an avenue for students to make this cognitive change, when necessary.

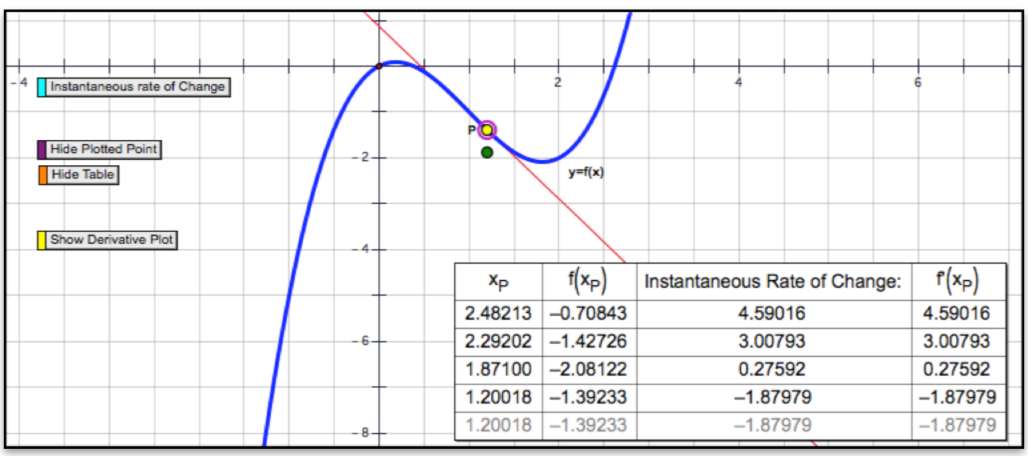
For example, during Felicity's interview III (E50 and 51, discussed in Section 4.2.1.2.1), there is evidence that when given a point on the graph of f, $(a, f(a))$, corresponding to an extreme value of the function, her concept definition only allows for $f'(a) = 0$. In addition, she believes that if $f'(a) = 0$ then $f(a)$ must be an extreme value of f. She mentions that the sign of the derivative value changes on either side of $x = a$. However, she neither mentions the possibility of the derivative being undefined at $x = a$, nor does she consider that there may exist an x-value, say $x = b$, where $f'(b) = 0$, but the function is increasing over an interval containing $x = b$. Felicity does identify a point, $(3, f(3))$, on the graph of the function in Interview III (see Figure 3.6) that corresponds to a corner as possibly yielding the absolute maximum function value; though she struggles, she eventually says that she does believe that $f'(3) = 0$. However, after investigating with the software she states, "…I can't get the tangent line to equal zero" [E102:FI3p3L1-15]. The conflict, for Felicity, appears to reside in her incorrect procedural knowledge; specifically, the belief that $f'(c) = 0$ implies that $f(c)$ is an absolute maximum function value. After investigating the function graph using the software, Felicity comes to the conclusion that the derivative is not zero at x-values where the function graph shows a corner, even if the x-value yields a maximum function value (see episode E57). Felicity amends her concept image and definition to include the possibility that if a maximum (or minimum) function value occurs at $x = c$, then $f'(c)$ may equal 0, or $f'(c)$ may not exist.

Through the exploration with DVS, Felicity was able to gain conceptual knowledge about why derivatives do not exist at x-values where a function graph makes a corner. As a result, Felicity brings her concept image and concept definition about extrema into alignment with both each other and with the formal concept definition for the extrema of a function over a closed interval. Felicity did this not through the use of her procedural knowledge, but through the exploration about what happens at a point $(c, f(c))$ on the graph of f that did not "look like [she] expected." This suggests that the use of DVS may enhance student understanding and allow for realignment of concept image and concept definition even when a student does not have sufficient procedural knowledge to do so in a static setting.

5.4 Supporting Visualizers

The results reported in Section 4.4 (see Table 4.12) regarding the comparison of visualizers and non-visualizers suggest that the use of DVS clarified concepts more for the visualizers than for their non-visualizing peers; though there was evidence for every DVS interview participant that DVS clarified some relationships or topics. It should be noted that at times, Amy was able to make these conceptual connections prior to working with the sketches. Though she did show enhanced knowledge after investigating with DVS, it does not appear that Amy's gains were as great as for students with more limited starting conceptual knowledge. This may explain why students with a limited initial conceptual knowledge experienced a higher rate of DVS obscuring the concept in question as the students were not sure of the relationships being presented and did not know where to focus within the software. When students did not know what to focus on within the interviews they often made general statements (e.g. "It gets bigger") that indicated they were not making the desired connections. This is an important result as it supports the idea that the software should be used to facilitate a solution pathway and not for unsupported exploration (Johnson, 2012; Cory & Garofalo, 2011). That is, students experienced clarification using the software both through the exploration itself and discussing what they saw with the interviewer. Had I not continued to probe and question the students it may be that the instances of DVS clarifying a concept would be less frequent and possibly less profound or deep. The DVS environment enabled this type of questioning and probing whereas the static environment made it more difficult.

This is consistent with scaffolding presented during a problem-solving task and it plays just as vital a role in DVS exploration (Henningsen & Stein, 1997). As previously mentioned, when presented with only a sketch and little guidance students were unsure on which mathematical relationship to focus. By introducing scaffolding in the form of incremental questions, students were able to focus on the desired topic. For example, in Interview II the DVS students investigated the relationship between a function's instantaneous rate of change at a point, derivative value at a point, and the slope of a line tangent to the function at the point. A screenshot of this is shown in Figure 5.1, below. I asked students for the coordinates of a point on the graph of $y = f'(x)$ when $x = x_P$. Ian, David and Felicity struggled with this question. After answering some probing questions about what might be equal to $f'(x_P)$, all three students were able to state that the point $(x, f'(x))$ was on the graph $y = f'(x)$. Once a student hypothesized that this point was on the derivative graph, they were told, "Click on the purple button. What does this point represent?". The students' hypotheses were validated because, on the computer screen, a green point appeared at the point $(x_P, f'(x_P))$ (see Figure 5.1, below). During the static Interview II, I asked students to calculate the function's derivative value at several points and to name at least five points lying on the graph of f' (see Appendix D). Students participating in the DVS interviews were able to receive immediate validation when they checked their work and were then able to continue on with the exploration knowing that they correctly identified the relationship. For some students this took longer and they used the coordinates of the green point (see Figure 5.1) as a way to reason about what they should be focusing on. This was especially true for Ian. This scaffolding not only provided information and focused students, but allowed for students to reason about why the relationship existed. Though this relationship was not with peers, but with the software and the interviewer, students were able further their conceptual understanding via this scaffolding and validation routine (Walter & Barros, 2011). Although Presmeg's (2006) work suggests that visualizers are not the "super star" students in high school mathematics classroom, these results indicate that student investigation with DVS may provide an environment where visualizers can work to clarify conceptual connections in calculus.



x_P	$f(x_P)$	Instantaneous Rate of Change:	$f'(x_P)$
2.48213	−0.70843	4.59016	4.59016
2.29202	−1.42726	3.00793	3.00793
1.87100	−2.08122	0.27592	0.27592
1.20018	−1.39233	−1.87979	−1.87979
1.20018	−1.39233	−1.87979	−1.87979

Figure 5.1 Screenshot of Sketch 1 from DVS Interview II.

The power of visual imagery became evident during the analysis of the interview transcripts. Even for non-visualizers, the experience of working with DVS remained a part of their schema. Before the beginning of Interview II, Amy was asked what she remembered from the previous interview, "… how since the y values and the, um, derivative values are the same, e^x is the derivative of e^x. So, it's like seeing that graphically and seeing how, um, how the graphs were the same. That was really cool" [E103:AI2p1L5-8]. David's response suggests that he, too, was recalling imagery from Interview I. David's hand motions and his sketch (see Figure 5.2) suggest the power of dynamic visuals for David.

> David: Dang it that was two weeks ago. Okay. So tangent lines if you have a function, the tangent line shows the slope of the point on the function (student makes hand gestures).
>
> Interviewer: You can draw a picture, cuz you're making the motions, so you're welcome to draw a picture.
>
> David: So this is the graph (student draws parabola) [See Figure 5.2], the function and the line that would be the slope of that point, like, at the instant right or the instances. [E104:DI2p1L4-9]

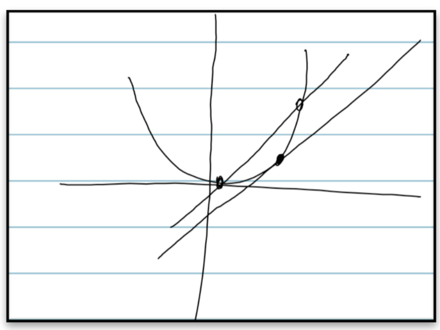


Figure 5.2 David's illustration when asked what he recalled in Interview I.

David, a high visualizer, experienced uncontrollable mental imagery during the DVS interactions (Aspinwall, Shaw, & Presmeg, 1997). In episode E23 from David's interview I (see Section 4.1.5), when presented with the graph of the function $f(x) = e^x$, David states that the graph has a vertical asymptote at $x = 4$. This visual persists with David throughout the interviews. Earlier in the same interview David asked if he could investigate a relationship using the software, so his inexperience with DVS was not hindering his ability to validate (or refute) his intuition. Felicity experienced a similar experience of uncontrollable mental imagery during Interview II while exploring the attributes the graph $y = e^x$. The limited power of DVS made it appear as though the graph of the function stops at $x = -2$ even though the graph shows an arrow at the endpoint. Felicity investigates this spot on the graph by moving a point to the area, yet she still states that the graph is “discontinuous after $x = -2$" [E105:FI2p6L13-27]. It is not until I ask Felicity to move the point to the left and look closely at the graph that she realizes the presence of the arrow and that the graph continues as x approaches negative infinity. Instances such as these should be fully explored by the student and possibly discussed with peers or with their instructor to address possible instances of uncontrollable mental imagery so they do not influence student thinking or introduce further misconceptions (Aspinwall, Shaw, & Presmeg, 1997).

Non-visualizers in the static group also occasionally relied on visual imagery to describe or unpack concepts. For example, during Evan’s interview III, he used the sketch of two functions, one concave up and the other concave down to illustrate his concept image of what it means for a point on the graph of a function f, $(b, f(b))$, to correspond to the absolute maximum value of a function (see Figure 5.3). Initially, Evan uses his hands as he talks, suggesting that he is visualizing. He states, “It’s [a function’s absolute minimum value] the lowest point on the entire function. Like, for example, on the parabola just before it starts - - like it’s going down and going back up.” He also comments that he uses these basic sketches to recall “… which is going the right way…. I always forget which way is going up and down on a parabola unless I have some kind of indication” [E106:EI3p1L10-23]. Through the use of the visualization shown in Figure 5.3, Evan is able to recall how changes in the parabola’s shape (“…going down and back up…”) might indicate that a minimum function value has been achieved. This is evidence of a non-visualizer relying on visualization as he explores the relationship between mathematical concepts. Evan accesses this information in a visual manner even though he is a non-visualizer.

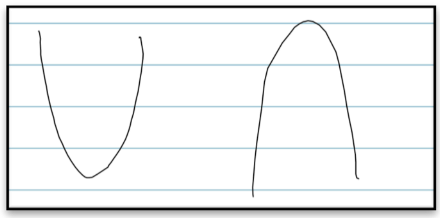


Figure 5.3 Evan's sketch to recall the properties of a function's extreme values.

5.5 Insufficient Mathematical Vocabulary

There were many instances, in both interview groups, where students were unable to recall definitions. When creating each interview protocol, I intentionally used terms that calculus students would be expected to know. However, when a student's concept definition was incorrect or incomplete it was difficult for them to *discover* what a term meant. For instance, a student may discover that a property of a tangent line is that "it only intersects with a graph once." However, since this discovery was made using a limited number of examples the student may struggle when working with the graph of $y = \sin x$ if they notice that the line $y = 1$ intersects the function graph an infinite number of times. The student's incomplete concept image of tangent line is not the same as possessing a rigorous formal concept definition.

Throughout the interviews it was apparent that some students in both groups did not possess the minimum threshold of knowledge to reason in a meaningful way about the topics presented. For example, only one student in the static group was able to use the difference quotient without it being provided and deciphered for him. Without this knowledge, the comparison of a function's average rate of change over an interval and its difference quotient would be meaningless. Students would not be able to explore the case as Δx approaches 0 and how this corresponds to the function's instantaneous rate of change at a given point. Similarly, when a student did not recall the terms tangent line, secant line or critical point during the DVS interviews, the protocol was not designed to allow time for a student to "discover" or learn the basic terms. Once students began exploring, their recall of these concepts or terms surfaced spontaneously. At the beginning of Interview I, David described a tangent line as, "Sine over cosine? I know I'm supposed to know these, but I - - I just know that what sine and cosine look like, so sine over cosine" [E107:DI1p1L12-14]. However, later in the interview (see episode E25 in Section 4.15) David uses tangent line in relation to a particle's, "…Oh, the speed. The speed." He also comments that he should, "find the equation of the tangent line, cuz they ask that in every problem." David is able to further explain his understanding of the importance of the slope of a tangent line by making a drawing (see Figure 4.9) and saying that it gives the speed of the particle "at that point." Tall & Vinner (1981) refer to an evoked concept image as, "the portion of the concept image which is activated at a particular time" (p. 152). David's inability to recall the definition of tangent line in a rote fashion was overcome once he was interacted with DVS, possibly because his explorations with DVS evoked part of his concept image for tangent line which, in turn, allowed him to recall the definition. This finding aligns with Tall & Vinner's (1981) that evoked concept images are closely related to a student's personal concept definitions and extends the results to include how explorations with DVS may have provided a pathway for David to recall what a tangent line is, but had it been a completely unfamiliar term for him the outcome of the investigation would likely have been different.

5.6 Geometric and Algebraic Approaches

Students experiencing the DVS interviews discussed rate of change and derivative at a point in a geometric manner more often than their static interview peers. Perhaps the procedural focus of the static interviews resulted in a more algebraic focus and did not encourage a geometric point of view. The DVS students were rarely asked to calculate average rate of change over an interval or instantaneous rate of change at a point by hand, instead they utilized the power of the software to make comparisons between collected data or by manipulating a dynamic point on a function graph. Again, this focus on procedures in the static group mimics the classic exercises presented in textbooks and may account for the failure of these students to develop a deeper connection between instantaneous rate of change at a point and its geometric representation, the slope of the tangent line at a point (Lithner, 2004; Szydlik, 2000; Hardy N. , 2009).

When asked what hearing the word "derivative" brought to mind, Amy (a non-visualizer) was the only student in the DVS group to mention the limit definition of derivative (see episode E84). Amy also mentions "instantaneous rate of change … the slope of the tangent line at any given point," suggesting that her view of derivative is both algebraic and geometric in nature. The only student who was unable to, when directly asked, connect instantaneous rate of change at a point and the derivative value at a point was Corbin (a visualizer). When given a point $(d, f(d))$ on the graph of f, DVS interactions may lessen the algebraic focus of the relationships between the instantaneous rate of change at $(d, f(d))$ and average rate of change over an interval containing $(d, f(d))$ and connect them to a geometric viewpoint focused on the slope of the tangent line at $(d, f(d))$.

5.7 The EVT and Differentiability

The students in both groups explored the EVT using a graph of the same function. Students in both groups struggled with the idea of a point $(x_A, f(x_A))$ corresponding to an extreme value when $f'(x_A)$ did not exist, though all students eventually stated that it was possible for $f(x_A)$ to be the maximum (or minimum) function value. The struggle, however, continued when students were asked about the existence of $f'(x_A)$ when the point $(x_A, f(x_A))$ appeared on the function graph at a sharp corner. The students in the DVS group explored the instantaneous rate of change at such points and all were able to determine that the derivative did not exist at that point because the slope of the tangent line approaching x_A from the left was not equal to the slope of the tangent line approaching x_A from the right. However, in the Exit Interview both David and Ian were unable to recall this and, while they both stated $f'(x_A)$ would not exist, neither was able to explain why this was so. In fact, Ian stated (during the Exit Interview) that if an extreme value for a continuous function, defined on a closed interval, corresponds to a point making cusp or corner that value is "not attain[ed]."

All four students participating in the static group were able to explain why the derivative at a point occurring on the graph at a corner did not exist, though through different methods. Evan (a non-visualizer) suggested that there are "so many tangent" lines that you could not pick just one (see episode E18 and Figure 4.7) whereas Harry (a visualizer) and George (a non-visualizer) both examined the behavior of the slope of tangent lines approaching the point $(x_A, f(x_A))$ from the left and discussed how they differ from the slopes of tangent lines when approaching $(x_A, f(x_A))$ from the right. Corbin used the absolute value graph as an example and showed that the derivative graph had a discontinuity at $x = 0$ (see Figure 4.4). Although students in each interview group struggled to explain why each necessary condition of the EVT is included in the theorem, the static group appeared to have the necessary knowledge to explain why $f'(x_A)$ does not exist when the point $(x_A, f(x_A))$ appears at a corner on the function graph.

5.8 Study Limitations

Limitations of this study are related to study participant recruitment and the timing of planned interviews in relation to the course lecture schedule.

5.8.1 Study Participants

Initially I extended interview invitations to 15 students with 12 students accepting. Seven of the initial students invited participated in the ESP intervention, but only four of these seven accepted. After Interview I, one of these students failed to return to the interviews. As a result of this, only three of the nine participants who completed the series of interviews also attended the ESP workshops. This imbalance makes it difficult to make comparisons about the impacts DVS may have on students already engaged in deep thinking and problem solving as part of their calculus curriculum such as ESP. This limitation may be corrected though the use of incentives for participants as well as additional explanation to students about the benefit of participating in such a study.

Only one student participating in the interviews dropped the course and this student also failed to complete the interview series. However, Corbin considered dropping the course yet pledged to continue with the interviews after speaking with the course instructor. Though he did not drop calculus, Corbin continued to participate in the study. His reasons for continuing included a hope that the knowledge he gained from the interviews would be beneficial when he took the course a second time. The instructor involvement was instrumental in Corbin's decision.

5.8.2 Delay of Interviews

Due to unavoidable circumstances, initial interviews were delayed by four weeks. The resulting delay meant that the interviews were not held immediately after the concept being discussed was covered in class. For example, for Interview I students were asked to make connections between average rate of change over an interval and instantaneous rate of change at a point. In the writing of the interview protocol the goal was to hold the interview before the class was introduced to the concept of derivative at a point. Since this did not happen, the interviewer instructed students to refrain from taking a derivative when asked about a function's instantaneous rate of change at a point during that interview but to, instead, consider the relationship to of instantaneous rate of change at a point and average rate of change over an interval. This may have been difficult for those students who focused on memorizing the derivative rules and may be a reason that some students failed to make the connections in a meaningful way as desired.

5.9 Conclusion

After analyzing the coding themes and comparing the results among different groups, several hypotheses arose about using DVS in the undergraduate classroom. I now pose these hypotheses and address the research questions that guided the study design.

5.9.1 Implications

Through the analysis of the major themes emerging from my interview data I propose instructors approach the use of DVS in undergraduate classrooms recognizing the amount of scaffolding and focusing that must accompany the use of DVS. In this study, I observed that student interactions with DVS may foster conceptual understanding in the moment; however, I found evidence that these interactions were not sufficient to result in a transfer of knowledge to other situations without access to the DVS. This was especially true for students possessing weak or incorrect conceptual knowledge about the highlighted mathematical relationships present in the DVS investigations. Therefore I hypothesize that students learning with DVS or other technology make conceptual connections that may be lost if these tools are not used in subsequent assessment situations. It can be argued that transfer may have occurred, but only within the static group and that the reason for the failed transfer within the DVS group is related to the lack of DVS availability during the Exit Interview.

I also propose that students possessing conceptual knowledge that is not supported by procedural knowledge have not fully appropriated their knowledge into well-connected schema; though this may be related to DVS not being offered as a tool during the Exit Interview. In my study this occurred when a student was unable to successfully complete procedural tasks but was able to make true conceptual statements about the topics from calculus. Thus, I hypothesize that each DVS investigation must present a balance of opportunities for students to connect their conceptual and procedural knowledge into schema.

Student explorations with DVS, when paired with well-written, scaffolded tasks, may create sufficient conflict between a student's concept image and concept definition to yield a cognitive re-alignment with the two structures. This is important because previous work has reported that exercises in calculus textbooks offer few opportunities for this to happen (Williams, 1991). I suggest that this happens when a student is refocused towards the relationships being highlighted and the instructor carefully considers where problems may occur prior to the implementation of DVS exploration. Thus, DVS use enables enhanced opportunities for conceptual connections that static exercises cannot provide.

My findings regarding the need to refocus student attention toward the mathematical relationships highlighted in a DVS exploration suggest that without the refocusing moments present in the DVS interviews, the outcomes from the DVS group may have been different. Student DVS explorations resulted in more frequent student comments of a conceptual nature. I suspect that without the refocusing moments this evidence of correct or partially correct conceptual knowledge may have gone undetected or may have been less prominent. I propose DVS explorations must be accompanied by refocusing phenomena to achieve the intended goals for the exploration.

In this study, interacting with DVS resulted in more moments of conceptual clarification for visualizers. However, non-visualizers in the DVS group all experienced clarification on at least one mathematical relationship. Although there is a higher incidence for visualizers, DVS exploration also enhanced the reasoning of those students who prefer not to visualizer when solving mathematics tasks.

5.9.2 Research Questions

The first research question I address asks how the use of DVS impacts student understanding of derivative as rate of change at a point. The DVS student interviews contain more episodes of students discussing instantaneous rate of change at a point with evidence of correct conceptual knowledge than do the interviews from the static group. There is evidence of correct, or partially correct, procedural knowledge in the presence of student discussion of instantaneous rate of change at a point in both interview groups. In the static interviews, nearly one-third of the instances of procedural knowledge about instantaneous rate of change at a point were incorrect. The students in the DVS may hold a more conceptual view of derivative as a rate of change at a point, though the evidence of transfer was weak for this group as discussed in Section 5.2. DVS was not offered for investigation during the Exit Interview; this is a possible explanation for the lack of evidence that DVS participants experienced transfer of knowledge and underscores the fact that careful assessment must align to instruction.

Though students categorized as visualizers in the DVS interview group experienced more instances of conceptual clarification (48 occurrences for visualizers and 18 for non-visualizers, see Table 4.12) when working with the software they also experienced more instances of the software obscuring relationships (17 occurrences for visualizers and six for non-visualizers). However, the ratios of instances where DVS obscured a concept to the total number of instances where DVS influenced students are nearly identical for visualizers (17/65) and non-visualizers (6/24). In order to combat DVS obscuring the mathematical relationship highlighted by the investigation, instructors must consider the baseline conceptual and procedural knowledge students must possess to successfully interact with the exploration. The crafting of such visualization tasks must attend to key areas of focus. When students appear to be focusing on a different relationship, whether fictitious or real, the embedded questions within explorations or the instructor, when present, should refocus the student's attention toward specific concept being highlighted. Modification of the DVS exploration may be required if students continue to shift their focus away from the mathematical relationship highlighted in the visualization. This need to refocus undergraduate students should not be overlooked or ignored as students who fail to make the desired connections when working with DVS will continue to hold on to these powerful images and continue to build their knowledge on erroneous images or generalizations. This work suggests that the interactions with DVS are more beneficial for visualizers, though there was no negative impact reported among the non-visualizers.

The second research question I address focuses on students interacting with DVS holding multi-representational views of derivative at a point. The students in the static interview group focused more on algebraic interpretations and procedural relationships rather than geometric and conceptual ones. One possible cause for this is that the static exercises modeled common tasks from calculus textbooks. Students in the DVS group, discuss the geometric interpretation of the derivative more than the static interview students. In the end, all students presented connections between the instantaneous rate of change at a point and the derivative value, though the students in the DVS group preferred to access a geometric viewpoint while those in the static group were focused on the algebraic aspects of the relationship.

There are concerns about students in the DVS group not appropriately transferring discovered conceptual relationships into appropriate schema. Conceptual statements made following refocusing and exploration with DVS were not replicated during the Exit Interview. Students who earned a C or below in the course reverted to their reliance on underdeveloped (and possibly incorrect) procedural knowledge characterized with responses of, "I don't know why, I just know…" One explanation for this is the lack of DVS exploration during the Exit Interview, though one could argue that the procedural components of the DVS interviews play a role as well.

The final research question focuses on how students interacting with DVS understand the EVT as compared to students who do not experience DVS explorations. Both interview groups struggled with the exploration of the EVT. During Interview III, all student participants, when given a point $(c, f(c))$ corresponding to a sharp corner on the graph of f, explained why $f'(c)$ did not exist. However, in the DVS group, only those who earned high grades in the course recalled this knowledge during the Exit Interview. This contrasts with the static group, where all students, even those failing to pass the course, made some conceptual connections regarding why $f'(c)$ does not exist when the point

$(c, f(c))$ on the graph of f corresponds to a corner.. The reasons for this failure to properly transfer knowledge are addressed earlier in this chapter.

Instructors of undergraduate calculus considering the use of animations or DVS, for either individual or whole-class exploration, should ensure that focusing features accompany such investigations. In the case of individual student explorations (i.e. online homework) tasks should be scaffolded in a way to encourage students to focus on the highlighted mathematical relationships. This may be accomplished using voice-over technology within a video or animation. When exploration with DVS is emphasized during lecture or homework, instructors should consider incorporating an aspect of this into assessment situations (i.e. exams or quizzes) to accurately determine student understanding of the topics explored.

In conclusion, student DVS explorations of calculus topics influence their conceptual knowledge about derivative and instantaneous rate of change at point. The students who explored using DVS tended to access a geometric, rather than algebraic, view of derivative more often than the non-DVS group, yet not all of their conceptual knowledge transferred into well-connected schema when the DVS tool was not offered during interviews concerning the EVT, continuity and differentiability. Future research should explore instances of seemingly procedurally unsupported conceptual knowledge on student learning in undergraduate calculus. The connections between students acquiring correct conceptual knowledge about a mathematical relationship and the need to refocus students' attention toward the highlighted mathematical concept being explored is another important implication from this study that warrants further examination.

Appendix A

Student Participant Information

Table A.1 Student participant information

	Major	ESP Classification	MPI Score (Visualizing)	Gender	Interview Type	1426 Grade
Corbin	Civil Engineering	Non ESP	17 (V)	M	Static	F
Beverly	Biochemistry	ESP	16 (V)	F	DVS	A
Amy	Mathematics	ESP	8 (NV)	F	DVS	A
Evan	Computer Science	Non ESP	6 (NV)	M	Static	C
David	Mechanical Engineering	ESP	18 (V)	M	DVS	C
Felicity	Industrial Engineering	Non ESP	7 (NV)	F	DVS	B
George	Computer Science	Non ESP	8 (NV)	M	Static	C
Harry	Engineering (Undeclared)	Non ESP	15 (V)	M	Static	D
Ian	Aerospace Engineering	Non ESP	20 (V)	M	DVS	D

Appendix B

Static Interview Protocol I

Secants to Tangents

(Stewart, 1995, p 106)
Temperature reading T, in degrees Celsius, were recorded every hour starting at midnight on a day in April in Whitefish, MT. The time x is measured in hours with t=0 representing midnight. The data were given in the table below.

x(hr)	0	1	2	3	4	5	6	7	8
T (^oC)	6.5	6.1	5.6	4.9	4.2	4.0	4.0	4.8	6.1

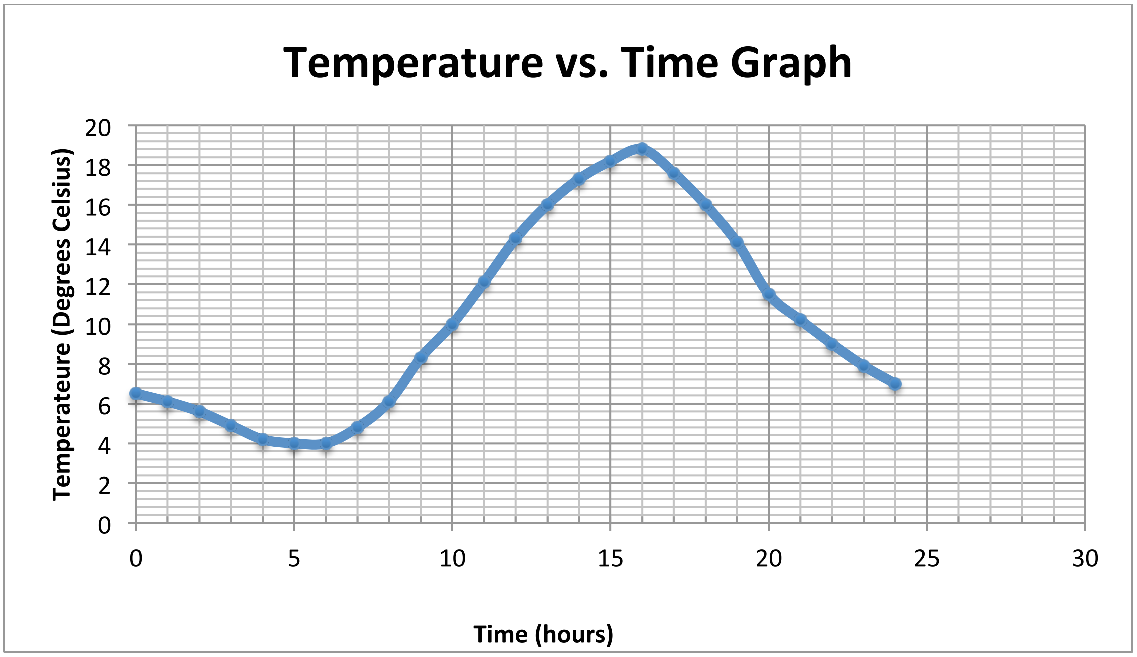
x(hr)	9	10	11	12	13	14	15	16	17
T (^oC)	8.3	10.0	12.1	14.3	16.0	17.3	18.2	18.8	17.6

x(hr)	18	19	20	21	22	23	24		
T (^oC)	16.0	14.1	11.5	10.2	9.0	7.9	7.0		

Find the average change of temperature with respect to time from
a. noon to 3 p.m.
b. noon to 2 p.m.
c. noon to 1p.m.
d. 11 a.m. to noon

How would you find the instantaneous change of temperature at noon?

Use the graph on the next page to approximate the instantaneous change of temperature at noon.


Temperature vs. Time Graph
Temperateure (Degrees Celsius)
0
2
4
6
8
10
12
14
16
18
20
0
5
10
15
20
25
30
Time (hours)

2. Consider the function $f(x) = x^2 - 5x - 6$. Use the table below to find the missing values.

x_1	x_2	Δx	Average rate of Change	Difference Quotient
1	0	-1	-4	-4
1	.25			
1	.50	-0.5		
1	0.75			
1		-0.125		
1		-0.065		
1	0.96875			
1	0.99			
1	0.999			
1	0.9999			
1	1			
1		.00001		
1	1.0001			
1	1.001			
1	1.01			
1		0.03125		
1	1.065			
1	1.125			
1.	1.25			
1		0.5		
1		.75		
1	2			

Consider a particle whose position s, in feet, is given by the function $s = f(t) = t^2 - 5t - 6$, where t is measured in seconds. What is the average speed of this particle from one second to two seconds? Does the latter tell you anything about the instantaneous speed of this particle at 1 second?

What is a geometric interpretation of the average rate of change of this function on the interval $[1,2]$? What is the geometric interpretation of the instantaneous rate of change at $t = 1$?

Appendix C

Dynamic Interview Protocol

1. Describe a secant line.
2. Describe a tangent line.
3. Do you think that they are related in some way? Explain.
4. What does average rate of change mean to you?
    a. How would you find this?
5. What about instantaneous rate of change?
    a. Do you think these rates of change are related in some way? Why?

Beginning with visualization #1 (Secants_to_tangents.gsp)

6. Describe the line passing through points A and B.
7. Point B on this sketch is moveable, use the mouse to move the point along the graph of the function. What do you notice?
8. This graph represents the position of a particle with respect to time. Based on the graph, when $t = 1$, what is the position of the particle?
9. How would we find the particle's average speed with respect to time over the interval $[x_A, x_B]$?
10. Leaving B where it is, please calculate the particle's average speed as it travels from $f(x_A)$to point $f(x_B)$.
11. For this graph, as B gets closer to A what do you notice about the average speed of the particle with respect to time? Why?
12. Is it possible to estimate the instantaneous speed of the particle? Use the graph and the manipulation of point B to illustrate this?
13. What do you notice about the value of the slope of secant line and the value of the average speed for this particle?
14. Now click on the button to show the table, vary B to record the data in the table. What do you notice about the relationships between the values listed?

Continuing with visualization #2 (Secants_to_tangents.gsp)

15. We will continue with this exploration much as we did the last one, how do you think that the average speed is related to the instantaneous speed?
16. Again you can manipulate point B, move it along the path of the graph and notice how the values in the table change. Does this path appear to behave as the previous one did?
17. Now look at the table that also shows the slope of the line BA and the difference quotient. Do you notice anything?
18. When might it be appropriate to estimate instantaneous speed of this particle using its average speed?

Continuing with visualization #3 (Secants_to_tangents.gsp)

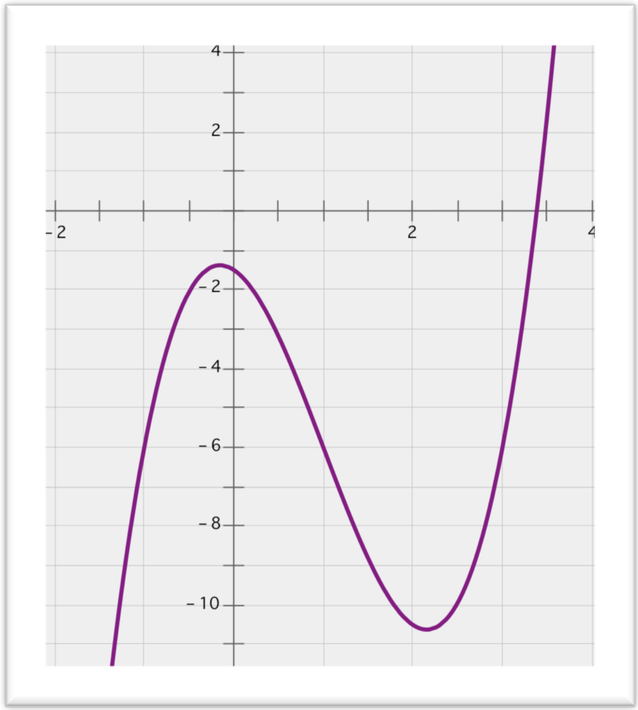
19. This time we will explore a function belonging to the exponential family. What do you know about exponential functions?
20. Again you can manipulate point B, move it along the path of the graph, do you notice any similarities to the previous sketches you have interacted with today? Any differences?

21. As you calculate the average speed of a particle whose position with respect to time follows this model what are you calculating?
22. What do you notice about the function value and the particle's instantaneous speed?
23. Now look at the last sketch; animate the point and look at the table showing the particle's average speed. You can capture the table's content by clicking on it; do this several times to capture some data. What do you notice?
24. What is the relation between the particle's instantaneous speed and it's position when following this model? Can you generalize this for the rate of change within an exponential mode

Appendix D

Static Interview Protocol II

Consider the graph below of the function $f(x) = 1.5x^3 - 4.5x^2 - 1.5x - 1.5$.

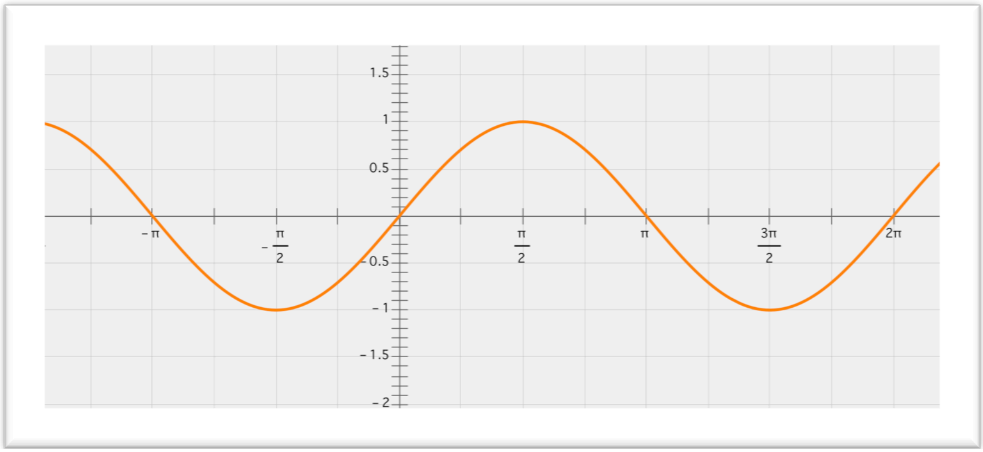


1. Let $x_A = 1$, what is the value of $f(x_A)$? $f'(x_A)$? *Note: you may use any method you choose to determine $f'(x_A)$.
2. What does $f'(x_A)$ mean? Give at least three interpretation of $f'(x_A)$.
3. Based on the work you did for #1, name at least 5 points on the graph of $f'(x)$. For example, as you observed, when $x_A = 1$ you computed $f'(x_A)$. Name the point corresponding to these values that lie on the graph of $f'(x)$.

4. Complete the table below, then plot the graph of $f'(x)$ above.

x	$f(x)$	$f'(x)$
-1		
-0.15		
0		
2		
2.15		
3.38		

Consider the graph below of the function $f(x) = \sin x$.



5. Let $x_B = \frac{\pi}{2}$, what is $f(x_B)$?
6. Name some points on the graph of $f(x)$ where you can easily determine the value of $f'(x)$. Why did you choose these points? What different representations of derivative might you consider when choosing these points?
7. Complete the table of values below and sketch a graph of $f'(x)$ above. How can the cyclic nature of this function aide in plotting the graph of its derivative?

x	$f(x)$	$f'(x)$
$\frac{\pi}{2}$		
$-\frac{\pi}{2}$		
	0	
		0

Appendix E

Dynamic Interview Protocol II

1. Last time we explored the relationships between secant lines, tangent lines and derivatives of functions. What do you remember?
2. How do instantaneous rates of change correspond to the derivative of a function?
3. If you were given the instantaneous rate of change a set of discrete points could you create the graph of the derivative? How might you do that?

Using Visualization #1 (Polynomial)

4. On this graphic you can move point P all along the graph of the function. Feel free to explore this. What do you see? What does the tangent line tell you about the function?
5. Click on the button to show you the instantaneous rate of change. What does this represent, geometrically, on the graph?
6. How might we go from knowing the instantaneous rate of change to being able to plot the derivative? What would be the x value and what would be the y-value?
7. Click on the orange button to show the table; how does the information shown correspond to your answer from the previous question?
8. Now click on the purple button that reads "show plotted point." What does this point represent? This computer program will draw the path that this point follows, what do you expect that will be?
9. Again manipulate point P and watch what happens as you do this. Was this what you expected?
10. Click on the Green Button labels "Hide Objects" what relationship do you notice on the table shown? What does this tell you about the trace of the point?
11. Click on the yellow button labeled "Derivative Plot." Does this confirm your thoughts about constructing a derivative graph?

Using Visualization #2 (Trigonometric)

12. What can you tell me about trigonometric functions?
13. On this graphic you can move point P all along the graph of the function. Feel free to explore this. What do you notice about the instantaneous rate of change at point P along this function? Where is the instantaneous rate of change equal to zero? How can you tell this? What does this tell you about the derivative of f at these points?
14. Click on the pink button labeled "Show instantaneous rate of change and table". Use the table (by clicking on it to capture values) to mark where the instantaneous rate of change is zero and where it is ± 4). Why do you think that we are interested in where the instantaneous rate of change is equal to ± 4?
15. Use the values in your chart (you may need to look at the values in terms of Pi) and make a prediction about the derivative graph for this function.
16. Now click on the dark green button to show point, then on the orange animate button. Were your predictions correct.
17. What is the parent function of the derivative graph?
18. By selecting the aqua "Show Objects" graph you can check your prediction.

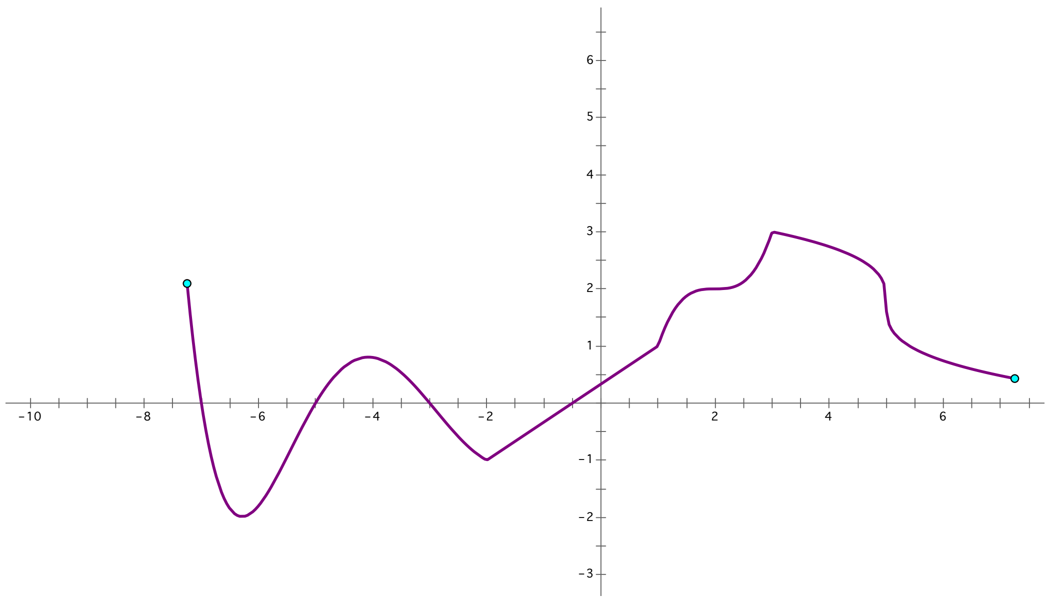
Using Visualization #3 (Exponential)

19. On this graphic you can move point P all along the graph of the function. Feel free to explore this. What do you see?
20. Now, look at the slope of the tangent line and continue to explore moving point P around. What does the slope of the tangent line tell us about the derivative of the function?
21. How might we go from knowing this slope to being able to plot the derivative? What would be the x value and what would be the y-value?
22. Do you notice anything about the slope of the tangent line that look interesting? What does this suggest?
23. Now, use the animate point button and watch what the derivative trace looks like. Are you surprised? What does this tell you about the derivative of the exponential parent function?

Appendix F

Static Interview Protocol III

1. What does it mean to say that a function is continuous on its domain?
2. What does it mean to say that a function is differentiable everywhere on its domain?
3. What is the absolute minimum value of a function? How do you know that it has been achieved? What about the absolute maximum value?
4. What is the EVT?



The graph above is of the function

$$f(x) = \begin{cases} \frac{-x^2}{60}(x+7)(x+5)(x+3), & -7.25 \leq x \leq -2 \\ \frac{1}{3}(2x+1), & -2 < x \leq 1 \\ (x-2)^3 + 2, & 1 < x \leq 3 \\ -(x-5)^{\frac{1}{3}} + 3 - \sqrt[3]{2}, & 3 < x \leq 7.25 \end{cases}$$

5. On what interval is this function defined? Is this important to know, why or why not? What are the function values at the endpoints of the interval?
6. Use the graph above to mark places where the instantaneous rate of change of the function is zero. How did you know where to mark?
7. Now mark where you think the instantaneous rate of change is the greatest, if this value exists, on the following intervals:
   a. $[-7.25, -2]$
   b. $[-2,1]$
   c. $[1,3]$
   d. $[3, 7.25]$

8. Think about the derivative graph of this function; approximately where would the values marked in #6 fall on this graph? What would the second derivative values of these points be? What kind of points are these on the initial graph of $f(x)$?
9. Looking again at the graph of the function above does the function have an absolute minimum value? How do you know? What is it? What about an absolute maximum value?
10. Write an inequality for the maximum value of the function compared to all other function values. Do the same for the minimum function value.
11. How does what you wrote in #9 compare the EVT?
12. Is this function continuous on its domain? How do you know? What points should you look at in-depth to prove continuity? Why did you pick those points? What do you need to show in order to show continuity?
13. Show that this function is, in fact, continuous on it's domain.
14. Do you think that this function is differentiable on it's domain? Why or why not?
15. What points should be checked to show this? Why?
16. Show at least 3 points where $f'(x)$ does not exist.
17. What can you say about continuity ⇒ differentiability?

Appendix G

Dynamic Interview Protocol III

What does it mean to say that a function is continuous on its domain?

1. What does it mean to say that a function is differentiable everywhere on its domain?
2. What is the absolute minimum value of a function? How do you know that it has been achieved? What about the absolute maximum value?
3. What is the EVT?

    a. Sketch Investigation
4. This sketch shows a piecewise function. You can investigate the function values by moving the yellow point A along the function graph.
5. Does this function have an absolute maximum value? What about an absolute minimum value? What information do you need to know in order to determine this? Move point A and estimate the possible values for these. You can double click to record the values in the table.
6. Now click on the yellow button that reads "Function Definition." Can you now say for sure if this function has absolute minimum/maximum values? Why or why not?
7. Over what interval is this function defined? Is it an open interval or a closed interval? How do you know? Does this function achieve its maximum/minimum function values?
8. Write two inequalities that the EVT guarantees for this function, one for the minimum function value compared to all other function values and the other for the maximum function value.
9. What are the $f'(x)$ values for the absolute maximum/minimum values of this function? Are there other places where the value of the derivative is the same as this? Where? How do you know? What do we call these points?
    a. Now click on the green button that reads "Show Tangent Line". Again you can explore with point A. (You may hide the function definition at any time). Mark the points on the following intervals where $f'(x)$ has the greatest value.
    b. $[-7.25, -2]$
    c. $[-2,1]$
    d. $[1,3]$
    e. $[3, 7.25]$
10. What kind of points would these be on the graph of $f'(x)$? Why? What do you think the value of $f''(x)$ would be at each of these points? What kind of points are these?
11. Do you think that $f(x)$ is continuous? Why or why not? Manipulate point A to investigate this; where should you first check? Why? What are you using for comparison?
12. Is $f(x)$ differentiable everywhere? Why do you think this?
13. Use point A to again investigate this. What points will you look at? How will you know if the function is not differentiable at that point? Use point A to show this; what do you see?
14. At how many points is $f(x)$ not differentiable? What are they?

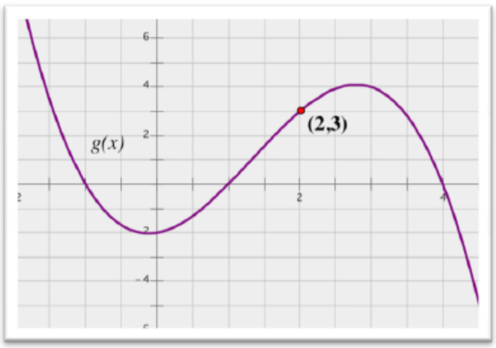
15. Click on the orange button that reads “Show Derivative Plot” to check you response to question 15.
16. What can you say about continuity $\Rightarrow$ differentiability?

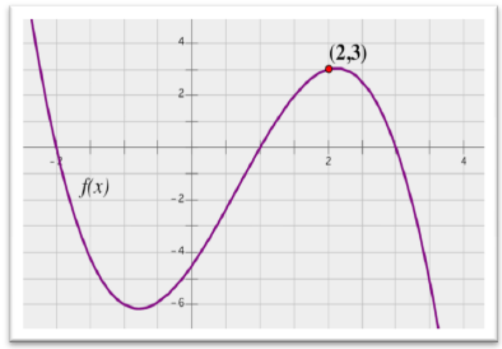
Appendix H

Exit Interview Protocol

Exit Interview Questions

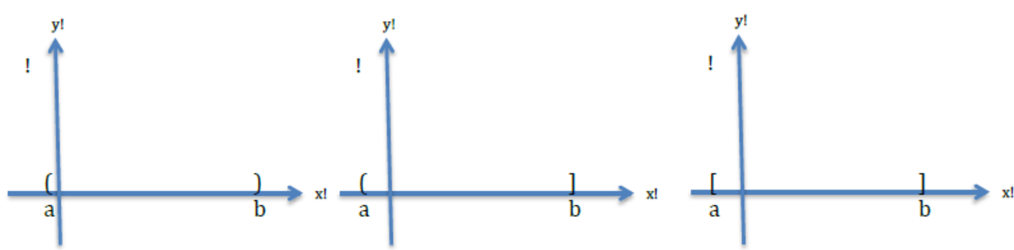
1. When I say the word "derivative" what comes to mind? Give me as many examples or ideas that you can in one-minute or less.
2. How are instantaneous rate of change at a point and derivative related? Explain.
3. Consider the graphs below of two different functions $f(x)$ and $g(x)$.





Discuss the long run behavior of the two functions. How would you estimate the instantaneous rate of change for each function at the point (2,3)? What would this estimate tell you geometrically? Do you expect the values to be close in value? Why or why not? Please estimate the instantaneous rate of change for each function at the point (2,3)? Is your result as you expected? What does this mean?

4. On which of these would we know that any function graphed over the interval shown would definitely attain its absolute maximum and absolute minimum values? What properties would the function need to have? Draw examples to illustrate why these properties are important.



5. If you know that a function is continuous on a closed interval what can you say about the function's differentiability on this interval? How do you know this? Why do you think that the instantaneous rate of change for a function at a point that makes a corner or cusp is undefined?

Appendix I

Interview Invitation

Dear XXXXXXX,

You have been selected to participate in a series of follow up interviews regarding concepts presented in your MATH 1426 course. You were identified based upon your responses on the Mathematical Processing Instrument completed in your MATH 1426 course on August 22, 2013. This request is for your participation in a series of five (5) interviews, each lasting no more than 60 minutes. These interviews will be held over the course of the semester and you will be asked to schedule a time to meet individually for each interview. All interviews will be video recorded for review. These interviews will cover a variety of topics discussed in Calculus I and you will be asked to explain your thought process for solving calculus problems.

Since Calculus is a gateway course to many science, engineering and technology majors, there is national interest in understanding the factors that lead to successfully completing calculus. Your participation in this study will help us come closer to understanding these factors and possibly help us design courses that would ensure that more students successfully complete Calculus at The University of Texas at Arlington.

Your participation in this study is entirely voluntary and a decision to participate or not participate neither contributes nor detracts from your current standing at The University of Texas at Arlington. Every effort will be made to protect your anonymity throughout the interview process; your responses may be used in an academic publication, presentation or other resource, but you will never be identified by name or student ID number. Interviews will be video recorded for review. With your consent small portions of video may be used when presenting research findings at professional conferences or at other academic institutions. In these cases you will be referred to by a pseudonym (such as Jane or Jack) that is unrelated to your given name or your common nickname.

Because the topics for the interviews are aligned to your course material, it is important that I receive your response regarding participation by Tuesday, September 10. It is expected that you will find the interviews helpful in understanding important concepts in calculus. I would be delighted to hear from you by email at julie.sutton@mavs.uta.edu, or in person (PKH 415) to schedule your interview time or to answer any questions that you may have. If you have any questions about the interviews you may contact Dr. Epperson at Epperson@uta.edu.

Regards,

Julie Sutton
Graduate Teaching Assistant
Department of Mathematics
The University of Texas at Arlington

Office: PKH 415

CC: Dr. James A. Mendoza Epperson, Research Advisor

Appendix J

Fall 2013 MATH 1421 Course Syllabus

MATHEMATICS 1426, CALCULUS I

Instructor:	***Dr. J. Epperson***		
Office:	PKH 423	**e-mail:**	epperson@uta.edu
Phones:	817-272-5047 (office); 817-272-3261 (Mathematics Department)	**Office Hours:**	M 2-3; W 11-12; or by appointment
Website:	http://wweb.uta.edu/faculty/epperson		
Graduate Teaching Assistant:	***Mr. W. Ventura***		
Office:	PKH 416 (Mr. Ventura)	**e-mail:**	wilber.ventura@mavs.uta.edu
Phones:	817-272-5256 (Mr. Ventura's office) 817-272-3261 (Mathematics Department)	**Office Hours:**	Mr. Ventura: Mondays 2:30-4; Wednesdays 5:30-7 or by appt.
Class Meetings:	*Lecture* (1426-700, 1426-710, & 1426-720): Tuesdays & Thursdays 12:30-1:50 in PKH 110 *Labs:* (1426-701) Tuesdays and Thursdays 2:00-2:50 PM in PKH 309 (1426-711) Tuesdays and Thursdays 11:00-11:50 AM in PKH 309 & Wednesdays 1:00-2:50 PM in PKH 305 (1426-721) Tuesdays and Thursdays 11:00-11:50 AM in PKH 309 & Wednesdays 3:00-4:50 PM in PKH 305		
Textbook:	*CALCULUS, EARLY TRANSCENDENTALS, CUSTOM EDITION FOR UT-ARLINGTON, BY SOO T. TAN* ***OR*** *CALCULUS, EARLY TRANSCENDENTALS VOLUME ONE, CUSTOM EDITION FOR UT-ARLINGTON, BY SOO T. TAN** Register** for WebAssign at: http://webassign.net/ NOTE that the Class Key depends upon the lecture section for which you are registered: Class Key for 1426-700: **uta 3224 3129** Class Key for 1426-710: **uta 8805 0591** Class Key for 1426-720: **uta 9491 6086** *The "Volume One" textbook is a cheaper option for those who only take one semester of Calculus. ** If you purchased your book new, you receive an access code for WebAssign. Otherwise, you will need to purchase this. There is a 14-day trial period before action is needed regarding purchasing access.		
Course Prerequisite:	A grade of C or above in Math 1323 (Precalculus II) or a sufficient score on the Math Aptitude Test or sufficient SAT/ACT math scores.		
Course Goals:	**The aim of this course is to develop a conceptually sound understanding of limits, rate, and accumulation.**		

Overview:	The course focuses upon the study of functions, graphs, limits, continuity, and differential and integral calculus. Roughly, we will study Chapters 1 through 4 in your textbook.
Class Format:	The instructor and the GTA will incorporate cooperative learning activities in lecture and lab sections as well as other active learning strategies during the semester. *You are expected to participate fully in these activities*. You will need to have 8-10 hours available weekly to study outside of class in order to succeed in this course.
UT-Arlington Department of Mathematics **Learning Outcomes for M1426**	Upon completion of Math 1426, the students will be able to perform various tasks including (but not limited to) those outlined below with algebraic, trigonometric and transcendental functions. 1. Students will be able to compute the limit of various functions without the aid of a calculator. 2. Students will be able to compute the derivatives and differentials of various functions without the aid of a calculator, and interpret certain limits as derivatives. In particular, they will be able to compute derivatives and differentials using differentiation techniques such as chain rule, implicit differentiation and logarithmic differentiation. 3. Students will be able to find the equation of the tangent line to the graph of a function at a point by using the derivative of the function. They will be able to estimate the value of a function at a point using a tangent line near that point. 4. Students will be able to sketch the graphs of functions by finding and using first-order and second-order critical points, extrema, and inflection points. 5. Students will be able to solve word problems involving the rate of change of a quantity or of related quantities. Students will be able to solve optimization problems in the context of real-life situations by using differentiation and critical points of functions. The problem topics include (but are not limited to) population dynamics, finance, physics, biology, chemistry and sociology. 6. Students will compute the area below the graph of a function by using a limit of a Riemann sum and/or by using a definite integral. 7. Students will be able to compute certain antiderivatives using various antidifferentiation techniques such as integration by substitution. They will be able to apply the Fundamental Theorems of Calculus to compute derivatives, antiderivatives, definite integrals and area. 8. Students will be able to justify and explain their steps in problem solving. In particular, students will be able to construct correct and detailed mathematical arguments to justify their claimed solutions to problems. 9.
Electronic Communication:	UT Arlington has adopted MavMail as <u>its official means to communicate with students</u> about important deadlines and events, as well as to transact university-related business regarding financial aid, tuition, grades, graduation, etc. <u>All students are assigned a MavMail account and are responsible for checking the inbox regularly.</u> There is no additional charge to students for using this account, which remains active even after graduation. Information about activating and using MavMail is available at http://www.uta.edu/oit/cs/email/mavmail.php.

Details About the Course

Grades:

Midterm Exam 1	**Friday, September 20, 2013** **6:00-8:00 PM**	**20%**
Midterm Exam 2	Friday, October 25, 2013 6:00-8:00 PM	25%
Lab grade	Weekly quizzes* Homework* Lab worksheets	5% 5% 10%
Final examination	Saturday, December 7, 2013 12:30-3:00 PM Comprehensive coverage	35%

*Those enrolled in 1426-710/1426-711 and 1426-720/1426-721 your requirements in A-ESP will account for 50% of your average on weekly quizzes and homework, respectively.

Grades will be assigned according to the following scheme (approximately):

90–100	**A**
80– 89	**B**
70– 79	**C**
60– 69	**D**
59 or below	**F**

Midterms and Finals:
These exams are departmental. This means that all sections of Math 1426 take the same midterm and final exams and that the grades on these exams have the same weight in each of the sections of calculus regardless of instructor. All of these exams are comprehensive. The format of each exam will be a mix of multiple-choice problems and free-response problems.

The final exam has a grade weight of 35%; however, **any student who scores below 50 on the final exam cannot receive a grade higher than a D in the course.**

Make-up Policy: If you have a conflict with either midterm or final, you must contact your instructor no later than Census Date (September 9), by using a form provided to you at your request by your instructor & submitting it together with necessary documentation as indicated on the form. If a conflict arises after September 9, contact your instructor immediately. **Delays in submitting a make-up request may mean that your request cannot be approved by the course coordinator.**

All previous midterm exams and some previous final exams can be accessed online at
https://mavspace.uta.edu/xythoswfs/webview/_xy-697804_1.
The solutions to the multiple choice questions are available at
https://mavspace.uta.edu/xythoswfs/webui/_xy-1083634_1-t_jbpAg0IM

Drop Policy: Students may drop or swap (adding and dropping a class concurrently) classes through self-service in MyMav from the beginning of the registration period through the late registration period. After the late registration period, **students must see their academic advisor to drop a class or withdraw**. Undeclared students must see an advisor in the University Advising Center. Drops can continue through a point two-thirds of the way through the term or session. It is the student's responsibility to officially withdraw if they do not plan to attend after registering. **Students will not be automatically dropped for non-attendance**. Repayment of certain types of financial aid administered through the University may be required as the result of dropping classes or withdrawing. For more information, contact the Office of Financial Aid and Scholarships (http://wweb.uta.edu/aao/fao/). Any student who drops this course on or before Wednesday, October 30 at 5 PM will receive a W.

Weekly Quizzes:
Suggested homework will be assigned each day. Online homework assignments have already been made and are already available on WebAssign. Your homework grade* will be based upon your online homework average. You will be given in-class (during lab meetings) and online (via WebAssign) quizzes which assume your having completed and mastered the suggested homework. You are allowed to use your own original handwritten notes (no copies or printouts from the internet) on the in-class quizzes. Your 10 best quiz grades will be used to calculate your quiz average. Although attendance is required, on the occasion that you miss a class please see Dr. Epperson's website http://wweb.uta.edu/faculty/epperson/courses.html for assignments.
*Again, note that online homework will account for only 50% of the homework average for A-ESP (M1426-711 & 721) students.

Attendance:
Attendance for this course and its associated labs is required. Excellent attendance records as well as positive group evaluations will help your grade in that borderline course-grade decisions will be influenced by these records. Arrive on time to class (quizzes take place during the first 10 minutes of class and lab homework is due at the beginning of class).

Lab Information:
Again, *attendance is required*. If you are absent from lab on a problem solving activity day, you will not be part of a lab group for that week and you will be required to submit the missed lab work individually with a 20% reduction of your grade for the missed lab.

In the lab, you will:
- ❑ have the opportunity to ask for guidance on homework questions;
- ❑ take weekly quizzes (except for weeks in which a midterm is scheduled) based upon mastery of the suggested homework assignments; and
- ❑ participate in problem-solving activities from Lab Worksheets (on Thursdays) and submit (on the following Tuesday) group or individual solutions to selected problem-solving activities from the Lab Worksheets—this is 50% of your lab grade (10% of your total course grade).

Instructions for solutions submitted:
- ❑ Work should be done in pencil and erasures should be clean and complete.
- ❑ Problems should be written in order and include the page number and the problem number, i.e. p26 # 5, if appropriate.
- ❑ Write on one side of the paper only.
- ❑ If you tear the page from a spiral notebook, trim the curly edges.
- ❑ Papers <u>must</u> be stapled together (upper left hand corner) and folded in half lengthwise.
- ❑ On the outside write your name, date and assigned problems.
- ❑ If these guidelines are not followed, your paper will not be graded and you will receive 0 points on that work.

Calculators: The only calculators allowed for the midterms and final are TI-30XA and TI-30XIIS.

Help Outside of Class Time:
My office hours are given above. These are times when I will be available in my office to discuss the material/homework/tests. No appointment is necessary for those times. If, however, those times are inconvenient for you, then make an appointment with me for another time (e.g., e-mail me stating the times you prefer). *Please use the subject heading **"Math 1426 Student Question"** when sending Dr. Epperson e-mail and identify yourself (full name) in the communication.*

Student Support Services: UT Arlington provides a variety of resources and programs designed to help students develop academic skills, deal with personal situations, and better understand concepts and information related to their courses. Resources include tutoring, major-based learning centers, developmental education, advising and mentoring, personal counseling, and federally funded programs. For individualized referrals, students may visit the reception desk at University College (Ransom Hall), call the Maverick Resource Hotline at 817-272-6107, send a message to resources@uta.edu, or view the information at www.uta.edu/resources.

START STRONG Freshman Tutoring Program
University Tutorial and Supplemental Instruction (UTSI)/University College
All first time freshmen can receive six FREE hours of tutoring for this course and other selected subjects for this semester. **Students must sign up and complete their first hour of tutoring by September 20th.** To sign up, visit UTSI in 205 Ransom Hall/University College. Upon completion of your first tutoring appointment, you will receive five hours of additional free tutoring. Flexible tutoring hours are available from 7:00am – 9:00pm, seven days a week at secure locations on campus. All tutors receive extensive training. Find out more at www.uta.edu/Startstrong

The Math Department operates the **Math Clinic**, a tutoring service staffed by upper level undergraduate students. The Math Clinic is on the 3rd floor of Pickard Hall; the phone number is 817-272-5674; and the hours of operation for fall and spring are

Monday – Thursday	8am to 9pm
Friday	8am to 1pm
Saturday	1pm to 6pm
Sunday	1pm to 9pm

Go to the Math Clinic webpage http://www.uta.edu/math/clinic/ to get more information or to access assignment sheets for the courses for which tutoring is offered.

All previous midterm exams and some previous final exams are available to students in the **Science Education and Career Center (SECC)**, 106 Life Science Building. The fall and spring hours of operation are

Monday-Thursday	8am - 8pm
Friday	8am - 5pm
Saturday	12pm - 5pm
Sunday	Closed

You need a Mav ID Card to check out these exams. A copy machine is available for you to make copies. There are also video tapes of lectures on calculus topics that can be viewed in the SECC. For more information, go to https://www.uta.edu/cos/SECC/login.php.

The Math Department maintains a list of people who have expressed an interest in tutoring. These persons are not necessarily recommended by the Math Department and they set their own fees. You may obtain a copy of the tutor list in the Math Office, 478 PKH.

My web page will list the homework as the term progresses as well as other miscellaneous information pertinent to this course. My web-page address is above.

Cell Phone, Laptops, Beeper, & Chiming Watch Etiquette:

- Cellular phones should be either switched off or set to "silent" mode during all classes. Cellular-phone use will not be permitted in class. If you must take an important call, please leave the classroom.
- Cellular phones are prohibited during exams.
- Beepers should be either switched off or set to "silent" mode during all classes and during tests.
- You should assure that watches with alarms and chirps will not sound during class.
- <u>Since lecture and lab focus on interpersonal communication, students must request permission to use a laptop during class or lab time.</u>

Final Review Week: A period of five class days prior to the first day of final examinations in the long sessions shall be designated as Final Review Week. The purpose of this week is to allow students sufficient time to prepare for final examinations. During this week, there shall be no scheduled activities such as required field trips or performances; and no instructor shall assign any themes, research problems or exercises of similar scope that have a completion date during or following this week *unless specified in the class syllabus*. During Final Review Week, an instructor shall not give any examinations constituting 10% or more of the final grade, except makeup tests and laboratory examinations. In addition, no instructor shall give any portion of the final examination during Final Review Week. During this week, classes are held as scheduled. In addition, instructors are not required to limit content to topics that have been previously covered; they may introduce new concepts as appropriate.

Emergency Exit Procedures: Should we experience an emergency event that requires us to vacate the building, students should exit the room and move toward the nearest exit. When exiting the building during an emergency, one should never take an elevator but should use the stairwells. Faculty members and instructional staff will assist students in selecting the safest route for evacuation and will make arrangements to assist handicapped individuals.

Americans with Disabilities Act: The University of Texas at Arlington is on record as being committed to both the spirit and letter of all federal equal opportunity legislation, including the *Americans with Disabilities Act (ADA)*. All instructors at UT Arlington are required by law to provide "reasonable accommodations" to students with disabilities, so as not to discriminate on the basis of that disability. Any student requiring an accommodation for this course must provide the instructor with official documentation in the form of a letter certified by the staff in the Office for Students with Disabilities, University Hall 102. Only those students who have officially documented a need for an accommodation will have their request honored. Information regarding diagnostic criteria and policies for obtaining disability-based academic accommodations can be found at www.uta.edu/disability or by calling the Office for Students with Disabilities at (817) 272-3364.

Student responsibility primarily rests with informing faculty **<u>at the beginning of the semester</u> and in providing authorized documentation through designated administrative channels.**

If you require an accommodation based on disability, I would like to meet with you in the privacy of my office, during the first week of the semester, to make sure you are appropriately accommodated.

Academic Integrity: Students enrolled in this course are expected to adhere to the UT Arlington Honor Code:

> *I pledge, on my honor, to uphold UT Arlington's tradition of academic integrity, a tradition that values hard work and honest effort in the pursuit of academic excellence.*
>
> *I promise that I will submit only work that I personally create or contribute to group collaborations, and I will appropriately reference any work from other sources. I will follow the highest standards of integrity and uphold the spirit of the Honor Code.*

UT Arlington faculty members may employ the Honor Code as they see fit in their courses, including (but not limited to) having students acknowledge the honor code as part of an examination or requiring students to incorporate the honor code into any work submitted. Per UT System *Regents' Rule* 50101, §2.2, suspected violations of university's standards for academic integrity (including the Honor Code) will be referred to the Office of Student Conduct. Violators will be disciplined in accordance with University policy, which may result in the student's suspension or expulsion from the University.

Grade Replacement and Grade Exclusion Policies: These policies are described in detail in the University catalog and can also be founded online at
http://wweb.uta.edu/catalog/content/general/academic_regulations.aspx#10
(scroll about half way down the page).

Student Disruption: The University reserves the right to impose disciplinary action for an infraction of University policies. For example, engagement in conduct, alone or with others, intended to obstruct, disrupt, or interfere with, or which in fact obstructs, disrupts, or interferes with, any function or activity sponsored, authorized by or participated in by the University.

Drop for Non-Payment of Tuition: If you are dropped from this class for non-payment of tuition, you may secure an Enrollment Loan through the Bursar's Office.

Important Dates:

September 2	Labor Day
September 9	Census Date, **Deadline for makeup requests for <u>all</u> exams**
Friday, September 20	Midterm 1, 6 – 8 pm
Friday, October 25	Midterm 2, 6 - 8 pm
Wednesday, October 30	Last day to drop a class
November 28-29	Thanksgiving Holidays
Wednesday, December 4	Last day of classes
Saturday, December 7	Final Exam, 12:30 - 3 pm

Student Feedback Survey: At the end of each term, students enrolled in classes categorized as "lecture," "seminar," or "laboratory" shall be directed to complete an online Student Feedback Survey (SFS). Instructions on how to access the SFS for this course will be sent directly to each student through MavMail approximately 10 days before the end of the term. Each student's feedback enters the SFS database anonymously and is aggregated with that of other students enrolled in the course. UT Arlington's effort to solicit, gather, tabulate, and publish student feedback is required by state law; students are strongly urged to participate. For more information, visit http://www.uta.edu/sfs.

Instructor: Dr. J. Epperson
GTA: Mr. Ventura
A-ESP Instructor: Mr. Chandler

Course Schedule[1]

Date	Topic or Activity
22-Aug	1st Day Handouts; Overview of Course; Precalculus Review
27-Aug	An Intuitive Introduction to Limits
29-Aug	Techniques for Finding Limits
3-Sep	Continuous Functions
5-Sep	Continuous Functions, Tangent Lines and Rates of Change
9-Sep	**CENSUS DATE (Deadline for makeup requests for all exams)**
10-Sep	The Derivative
12-Sep	Rules of Differentiation
17-Sep	The Role of the Derivative in the Real World
19-Sep	Review (Sections 2.1-2.3)
20-Sep	**Midterm 1, 6-8 p.m.**
24-Sep	Derivatives of Trigonometric Functions
26-Sep	The Chain Rule
1-Oct	Implicit Differentiation, Derivatives of Logarithmic Functions
3-Oct	Related Rates
8-Oct	Differentials and Linear Approximations
10-Oct	Extreme of Functions
15-Oct	The Mean Value Theorem
17-Oct	Increasing and Decreasing Functions and the First Derivative Test
22-Oct	Limits Involving Infinity
24-Oct	Review (Sections 2.4-3.3)
25-Oct	**Midterm 2, 6-8 p.m.**
29-Oct	Concavity and Inflection Points, Curve Sketching
30-Oct	**Last day to Drop the course**
31-Oct	Concavity and Inflection Points, Curve Sketching
5-Nov	Optimization Problems
7-Nov	Indeterminate Forms and L'Hôpital's rule, Indefinite Integrals
12-Nov	Indefinite Integrals, Integration by Substitution
14-Nov	Area
19-Nov	The Definite Integral
21-Nov	The Fundamental Theorem of Calculus
26-Nov	Numerical Integration, Area Between Curves
3-Dec	Area Between Curves, Review for Final Exam
7-Dec	**FINAL EXAM 12:30-3:00 p.m.**

[1] *As the instructor for this course, I reserve the right to adjust this schedule in any way that serves the educational needs of the students enrolled in this course. –Dr. James A. M. Epperson*

Appendix K

Fall 2013 ESP Syllabus

MATH 1426-711/721: Calculus I A-Emerging Scholars Program

INSTRUCTOR:	Richard Chandler	Richard.Chandler@mavs.uta.edu
PALS:	Lynda Dijeh	Lynda.Dijeh@mavs.uta.edu
	Zachry Engel	Zachry.Engel@mavs.uta.edu
	Colbey Hair	Colbey.Hair@mavs.uta.edu

OFFICE:	Pickard Hall (PKH) 463	**CLASS PLACE/TIME:**	PKH 305
OFFICE HOURS:	Tues/Thurs 2:00pm – 3:00pm		(711) Wed 1:00pm – 2:50pm
	Wed 11:00am – 12:00pm		(721) Wed 3:00pm – 4:50pm
PHONE:	817-272-0008 (Office)	**MATH CLINIC:**	PKH 325
	817-272-3261 (Math Office)	**STUDY ROOM:**	PKH 114

WORKSHOP DESCRIPTION

The Calculus I Workshop for the Arlington Emerging Scholars Program (A-ESP), MATH 1426-711/721, is an extension of Dr. Epperson's regular Calculus I (MATH 1426-710/720) lecture/lab course. The main goal of this workshop is to enrich your calculus experience in a way that is not possible with the format and expectations of the traditional lab sections. To this end, we will be asking much more of you as an emerging scholar.

WORKSHOP OBJECTIVE

In addition to aiding in your course objectives outlined in your MATH 1426-720 syllabus, upon completion of MATH 1426-721,

1. Students will be able to identify concepts which are critical to their understanding;
2. Students will become more confident with problem solving techniques and the application of their knowledge to challenging problem sets;
3. Students will become more confident with mathematical language and key counter-examples;
4. Students will be able to interact effectively in a collaborative learning environment;

Students will be connected with a community of other students whom share similar academic interests.

EXPECTATIONS

In order to make the workshop as effective as possible for you, you will need to put in a lot of time outside of class. Calculus I is not difficult, but the MATH 1426 course is fast paced and covers material that will be brand new to most of you. In addition, as always in math courses, the concepts are cumulative in nature; so, if you fall behind, you damage your chances to fully understand the current material. You should strive to study at least 10 hours a week outside of class (lecture, lab and workshop): this means working on calculus after your homework is done!

Group work is encouraged, but an essential part of learning mathematics is the time you spend alone just thinking about the material. You will find the worksheets very difficult if you don't study the current material before coming to workshop. You should also read the section in the book **before** it is lectured on.

Satisfactory performance in this workshop is required to continue in the ESP program [See **GRADING**]. Examples of **non-satisfactory performance** include: obsessive absences, failure to actively participate or a workshop grade below 70%.

WORKSHOP ENVIRONMENT

In keeping with the goal of providing an academically challenging and collaborative atmosphere in these workshops, the following guidelines will be implemented during workshop sessions:

1. No laptops/tablets are to be out or open.
2. No texting or phone calls will be permitted.
3. Only **TI-30XA** or **TI-30IIS** calculators will be permitted
4. You should be taking rigorous notes during lecture. You may use these notes and any other hand-written materials you have created. **Your textbook should not be used unless given explicit permission**. I will tell you in advance if you need to bring your book to workshop.

These restrictions serve two purposes: to emphasize the necessity for you to read and study the text before the workshop begins as well as the importance in relying on one another to work through any gaps in knowledge that may be present.

GRADING

Your A-ESP grade will count for 25% of the lab component which is equivalent to 5% of your overall course grade. The breakdown is as follows:

A-ESP Workshop Grade	5%
Quizzes & Homework	5%
Course Projects	10%
Midterm 1	20%
Midterm 2	25%
Final Exam	35%

Your A-ESP workshop grade will be calculated as follows:

Attendance & Participation	50%
Homework & Worksheets	30%
Quizzes	20%

Attendance & Participation

Absences are detrimental to one's performance in a given course. You are expected to attend this workshop regularly in order to increase your chances of success in both this workshop and your MATH-1426 course. Tardiness is strongly discouraged as it is disruptive to the workshop atmosphere. If for some reason you cannot make it to the workshop, I would like to know **in advance**, whether it by email, phone call or in person.

Everyone should strive to miss **no more than one** workshop meeting during the semester. Each absence will negatively affect your attendance/participation grade. Each tardy (defined as being more than 10 mins late) will count as one-third of an absence and will thus also affect your attendance/participation grade. In addition, your active participation while in the workshop is expected; attending workshop but not engaging in the day's activities may result in a reduction in your participation grade for the day.

It is also important that you attend both the TR 12:30 lecture with Dr. Epperson and the TR 11:00 lab; this workshop is not a replacement for either of these components and you will find it difficult to be successful in workshop if you are not attending both lecture and lab regularly. To encourage your attendance in these components, I will monitor the attendance records kept by Dr. Epperson and your GTA and will consider this when determining your attendance/participation grade for the workshop. If you have excellent attendance in these components, I will replace your lowest attendance/participation grade with a 100.

Homework & Remediation

Homework will be assigned every week during workshop. I will select a subset of the homework assigned in lecture for that week to be collected at the beginning of the next workshop. Each student is responsible for neatly writing up solutions to the problems and turning them in **at the beginning** of the workshop. **No late work will be accepted** so if you know that will be tardy of absent, it is your responsibility to get the work to me **prior** to the time it is due; in this situation, work may be given to me during office hours, slipped under my door, left with the Math Department front desk (PKH 487) or sent to workshop with a classmate. Each problem will be graded out of 5 points.

Students submitting a problem, but whose work would earn a grade of less than a 4, will need to do remediation on that problem. These corrections are to be turned in at the beginning of class the following week. The maximum grade that can be obtained on a correction is a 4. Failure to complete a remediation will result in a score of 1 for that problem.

The total points available in each week's homework is 11, 5 for each problem and 1 for the online response discussed below. The average out of 100 will establish your base homework grade for ESP

Quizzes

Quizzes will be announced the week prior to being given (I do not give pop-quizzes). The quizzes will last for 10 minutes at the beginning of class and will be over definitions and theorems that you learned the previous week in lecture (or possibly from the previous quiz). That is, you will not work any problems on the quizzes; you will be stating definitions and concepts! The quizzes will be graded out of 10 points and your total score at the end of the semester will be the percentage taken out of the total possible points.

Online Resources

In addition to written homework, there will be a number of online resources for you to view throughout the year. These will include extra examples, animations, interactive websites, etc; the links will be posted on Blackboard (elearn.uta.edu).

Some of the content will require you to make a free account on www.educreations.com and enroll in my class using the Class Code: 8D9429. You may also want to download the free software, Geogebra, from www.geogebra.org. Some of the content will be interactive and this software will allow you to view it.

After viewing the online material each week, you should write a short response to be turned in with your homework that answers the following questions:

What was the main idea of the material?

Did this material help you answer any questions or fix any misconceptions? What were they?

Did this cause any new questions? If so, what are they?

Worksheets

The majority of the time in workshop will be spent in groups working on worksheets comprised of challenging problems related to the material you are learning in lecture. As a group, you will submit 1-2 problems from each worksheet; the number will vary each week and which problems are assigned will vary as well. Problems are due at the beginning of the workshop the week after the worksheet is given. Each worksheet will be graded as follows:

Excellent	Correct Answer, Few to no mistakes, Clear steps	1 point
Good	Some (minor) mistakes, Few missing steps	.5 points
Acceptable	An effort is made but the solution is incorrect or incomplete	0 points
Unacceptable	Work is either not turned in or no effort is put into the solution	-1 points

Should you be absent from a workshop, it is your responsibility to obtain a copy of the worksheet, complete the required problem(s) and turn it in at the beginning of the following workshop. Your work will be graded individually with the same scale as above.

At the end of the semester, I will sum all the points you have earned and add it to you Homework Grade **AFTER AVERAGING**. This means, that should you turn in excellent work each week you can raise the Homework portion of your A-ESP grade by a full letter! This also means that if you are struggling with a problem, your grade will not be negatively affected as long make your best effort.

Exams

In the week prior to each exam, we will hold a review outside of workshop for all A-ESP students (these reviews are closed to students from other sections). These reviews are optional, but we will generally not have time during workshop to review so this will be your time to have questions answered and work on old exams and homework. Attendance is highly encouraged.

In addition to the reviews, on the Friday of the midterm and the Friday before the final, we will have an Extended Study Group. We will typically rent out a classroom and we use it as a place for everyone to come and study together.

Emergency Exit Plans

In the event of an evacuation of the building, we will exit the building using the stair located down the hall to the right of the classroom. The stairs lead directly outside so proceed all the way to the first floor and out.

In the event of a tornado and it becomes necessary to take shelter. Proceed down the same stairs above to the 2nd Floor and then use the interior stairs located down the hall to the right to reach the 1st Floor. Tornado shelters are in PKH 104 and PKH 110.

The A-Emerging Scholars Program is partially funded by a grant (DUE #0856796) from the National Science Foundation for increasing the talent pool in Science, Technology, Engineering and Mathematics.

Appendix L

Baseline Demographic Survey

Name: ____ UTA ID: 1000_______________

I. BACKGROUND AND DEMOGRAPHICS

1. As of today, are you a: (**CIRCLE ONE RESPONSE ONLY**)

1	Freshman, first semester	3	Sophomore, first semester
2	Freshman, second semester	4	Sophomore, second semester
5	Other (**SPECIFY**): ________________________________		

2. What is your age _________ and birthdate MM/DD/YY___________________________

3. What is your gender? (**CIRCLE ONE RESPONSE ONLY**)

1 Male 2 Female

4a. What is your ethnicity? (**CIRCLE ONE RESPONSE ONLY**):

1 Hispanic, Latino
2 Non-Hispanic
3 Do not wish to report ethnicity

4b. What is your racial heritage? (**CIRCLE ALL THAT APPLY**)

1	Black or African-American	4	Native American or Alaska Native
2	White	5	Native Hawaiian or Other Pacific Islander
3	Asian	6	Do not wish to report racial heritage

4c. Did you apply for financial aid with the FAFSA? Circle: Yes / No / Do not wish to report

4d. Your citizenship is:

1	US citizen	3	International citizen with valid US visa
2	US permanent resident	4	Other / Do not wish to report

4e. High school name, city, state: ____

5. In what month and year did you graduate from high school? _____ (month)/____ (year)

5a. In what month and year did you enroll at UTA? _____ (month)/____ (year)

5b. What did you do after you graduated from high school and before you started UTA? (**CIRCLE ALL THAT APPLY**)

1 Worked full-time or part-time
2 Served in the military
3 Attended community college
4 Attended another 4-year college/university
5 Cared for a family member(s)
6 Did volunteer service in the community
7 Traveled
8 Other (**SPECIFY**): ____
9 Nothing

6. What are the highest education levels that your parents attained? (**CHECK THE APPROPRIATE ROW FOR YOUR FATHER AND YOUR MOTHER**):

Education Level	Mother	Father
1. Less than high school		
2. Some high school		
3. High school graduate		
4. Community college or technical/vocational school		
5. Some college		
6. College graduate		
7. Graduate school – MA/MS		
8. Graduate school – Ph.D., MD, Law degree		
9. Other (**SPECIFY**):		

7. Do you have any siblings who are in college or old enough to have gone to college (i.e. 18 or older)?

1 Yes
2 No (**SKIP TO Q.8**)

7a. Is this sibling/(Are these siblings) in college or graduated from college? (**CIRCLE ALL THAT APPLY**)

1 Yes, in college
2 Yes, college graduate
3 None of my siblings currently attends or went to college
4 No, attended college but did not graduate

II. HIGH SCHOOL EXPERIENCES

8. What is the most advanced math class that you took in high school? (**CIRCLE ONE RESPONSE ONLY**)

1 Calculus
2 Pre-Calculus
3 Algebra 2
4 Algebra 1
5 Trigonometry
6 Geometry
7 Other (**SPECIFY**) ______________________________

9. How many classes did you take in high school in: (**RECORD THE APPROPRIATE NUMBER OF CLASSES FOR EACH SUBJECT AREA**)

Subject Areas	Number of Classes
Mathematics	
Physics	
Chemistry	
Computer Science	
Statistics	
Engineering	
Other math, science or technology classes	

10. Did you take any Advanced Placement (AP), International Baccalaureate (IB) or dual credit classes in high school: (**CIRCLE "Y" OR "N" FOR EACH AREA**)

AP Classes Taken	**Yes**	**No**	**IB Classes Taken**	**Yes**	**No**
AP Biology	Y	N	IB Biology	Y	N
AP Chemistry	Y	N	IB Chemistry	Y	N
AP Physics B	Y	N	IB Physics	Y	N
AP Physics C	Y	N	IB Math SL II	Y	N
AP Calculus AB	Y	N	IB Math HL I	Y	N
AP Calculus BC	Y	N	IB Math HL II	Y	N
AP Computer Science A	Y	N	IB Computer Science	Y	N
Other AP Science Courses	Y	N	Other IB Sciences	Y	N
Dual credit math, science or engineering college classes: (**LIST**):					

11. How well did your high school prepare you for college in the following areas? (**CIRCLE ONE NUMBER FOR EACH ROW**)

Areas	**Very Well**	**Well**	**Somewhat**	**Poorly**	**Not At All**
Study skills	1	2	3	4	5
Writing skills	1	2	3	4	5
Oral presentation skills	1	2	3	4	5
Interpersonal communications	1	2	3	4	5
Laboratory skills	1	2	3	4	5
Computer literacy (MSWord, Excel)	1	2	3	4	5
Computer programming, advanced	1	2	3	4	5
Mathematics	1	2	3	4	5
Sciences	1	2	3	4	5
Engineering	1	2	3	4	5

III. COLLEGE EXPECTATIONS AND PLANS

12. Think back to high school; which one of the following statements best describes your high school experience? (**CIRCLE ONE RESPONSE ONLY**)

1 It was very easy for me to get the grades I wanted in all my classes
2 With a few exceptions, it was easy for me to get the grades I wanted in my classes
3 I had to work some, but not at all hard to get the grades I wanted in my classes
4 I had to work hard to get the grades I wanted in my classes

12a. What grade point average (GPA) did you want to get in high school? (**CIRCLE ONE LETTER ONLY**)

A B C D

13. As a first year college student, how hard do you expect to work in college to get the grades you want? Do you expect to: (**CIRCLE ONE RESPONSE ONLY**)

1 Work less than you did in high school to get the grades you want
2 Work the same as you did in high school to get the grades you want
3 Work harder than you did in high school to get the grades you want

13a. What grade point average (GPA) do you strive to get in college? (**CIRCLE ONE LETTER ONLY**)

A B C D

14. What is your intended major? (**CIRCLE ONE RESPONSE ONLY**)

1 Aerospace Engineering
2 Bioengineering
3 Biological Chemistry
4 Chemistry/Biochemistry
5 Civil Engineering
6 Computer Engineering
7 Computer Science
8 Electrical engineering
9 Industrial Engineering
10 Mathematics
11 Mechanical Engineering
12 Physics
13 Software Engineering

15. How confident are you that you will keep this major through college? (**CIRCLE ONE RESPONSE ONLY**)

1	2	3	4	5
Very Confident	Confident	50% Confident	Not Confident	Not at all confident

16. What sources of information did you use to decide what major to pursue in college? (**CIRCLE ALL THAT APPLY**)

1 University advisors
2 University classes
3 University "open house" or campus visit days
4 Other university activities
5 National ranking data on the college or department
6 High school teacher
7 High school counselor
8 Suggestion(s) from peers
9 Parents' advice
10 Suggestion(s) from sibling, family member or family friend
11 Employer
12 Future employment prospects
13 Other (**SPECIFY**) ____

17. How supportive are your parents/guardians of your decision to study the major you specified above? (**CIRCLE ONE RESPONSE ONLY**)

1 Very supportive
2 Supportive
3 Somewhat supportive
4 Neutral
5 Not supportive
6 Against my choice of major
7 Did not discuss choice with them

References

Asera, R. (2001). *Calculus and Community: A History of the Emerging scholars Program.* The College Board, National Task Force on Minority High Achievement. New York: The College Board.

Aspinwall, L. & Shaw, K. (2002). When Visualization is a Barrier to Mathematical Understanding. *Mathematics Teacher*, 5(9), 714-717.

Aspinwall, L., Shaw, K., & Presmeg, N. (1997). Uncontrollable Mental Imagery: Graphical Connections between a Function and its Derivative. *Educational Studies in Mathematics, 33*, 301-317.

Baker, B., Cooley, L., & Trigueros, M. (2003). Thematization of the Calculus Graphing Schema. *International Group for the Psychology of Mathematics Education*, 57-64.

Bennett, A. (2009 September). From http://web.phys.ksu.edu/current/seminar/f09/09-14-09-bennett.pdf

Bezuidenhout, J. (2001). Limits and Continuity: Some concpetions of first-year students. *International Journal of Mathematical Education in Science and Technology, 32*(4), 487-500.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics,* 23(3), 247-285.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002,). Applying Covariational Reasoning while Modeling Dynamic Events: A Framework and a Study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The Precalculus Concept Assessment: A Tool for Assessing Students' Reasoning Abilities and Understanding. *Cognition and Instruction, 28*(2), 113-145.

Carlson, M., Oehrtman, M., & Thompson, P. W. (2007). Foundational Reasoning Abilities that Promote Coherence in Students' Understanding of Function. In M. P. Carlson, & C. Rasmussen, *Making the Connection: Research and teaching in undergraduate mathematics* (pp. 150-171). Washington, DC: Mathematical Association of America.

Cetin, N. (2009, April 15). The Performance of Undergraduate Students in the Limit Concept. *International Journal of Mathematical Education in Science and Technology, 40*(3), 323-330.

Corbin, J., & Strauss, A. (2007). Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory (3rd ed.). Thousand Oaks, CA: SAGE Publications.

Cory, B., & Garofalo, J. (2011). Using Dynamic Sketches to Enhance Preservice Secondary Mathematics Teachers' Understanding of Limits of Sequences. *Journal for Research in Mathematics Education, 42*(1), 65-97.

Dubinsky, E., & McDonald, M. A. (2002). The Teaching and Learning of Mathematics at University Level: An ICMI Study. In D. Holton, D. Holton, M. Artigue, U. Kirchgrader, J. Hillel, M. Niss, & A. Schoenfeld, *APOS: A Constructivist Theory of Learning in Undergraduate Mathematics Education Research* (pp. 275-282). Heidelberg: Springer Netherlands.

Eisenburg, T., & Dreyfus, T. (1991). On the Reluctance to Visualize in Mathematics. In *Visualization in Teaching and Learning in Mathematics - MAA Notes Number 19* (pp. 25-37). Washington, DC: Mathematical Association of America.

Epperson, J. A. (1998). Supporting Students as Emerging Scholars. *Developing mathematically Promising Students*, 269-272.

Fullilove, R. E., & Treisman, P. U. (1990). Mathematics Achievement among African American Undergraduates at the University of California, Berkeley: An Evaluation of the Mathematics Workshop Program. *The Journal of Negro Education, 59*(3), 463-478.

Graham, K., & Ferrini-Mundy, J. (1994). Research in Calculus Learning: Understanding Limits, Derivative & Integrals. In E. D. J. Kaput (Ed.), *Research Issues in Undergraduate Mathematics Learning: MAA Notes Number 33* (pp. 31-45). Washington, DC: Mathematical Association of America.

Gray, S. S., Loud, B. J., & Sokolowski, C. P. (2009). Calculus Students' Use and Interpretation of Variables: Algebraic vs. Arithmetic Thinking. *Canadian Journal of Science, Mathematics and Technology Education, 9*(2), 59-72.

Hardy, N. (2009). Students' perceptions of institutional practices: the case of limits of functions in college level Calculus Courses. *Educational Studies in Mathematics, 72*, 341-358.

Henningsen, M., & Stein, M. K. (1997). Mathematical Tasks and Student Cognition: Classroom-Based Factors that Support and Inhibit High-Level Mathematical Thinking and Reasoning. *Journal for Research in Mathematics Education, 28*(5), 524-549.

Johnson, H. (2012). Reasoning about Variation in the Intensity of Change in Covarying Quantities Involved in Rate of Change. *Journal of Mathematical Behavior, 31*, 313-330.

Lithner, J. (2004). Mathematical Reasoning in Calculus Textbook Exercises. *Journal of Mathematical Behavior, 23*, 405-427.

Lobato, J., & Burns-Ellis, A. (2002). Focusing Effects of Technology: Implications for Teacher Education. J*ournal of Technology and Teacher Education*, 10(2), 297-314.

Lobato, J., Rhodehamel, B., & Hohensee, C. (2012). “Noticing” as an Alternative Transfer of Learning Process. *The Journal of the Learning Sciences*, 21, 433-482.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *Journal of Mathematical Behavior, 31*, 48-59.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students’ graphing activity. In T. Fukawa-Connelly (Ed.), Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education. Pittsburgh, PA: RUME.

Moreno, S. E., & Muller, C. (1999, November). Success and Diversity: The Transition through the First-Year Calculus in the University. *American Journal of Education, 108*, 30-56.

Piaget, J., & Garcia, R. (1989). *Psychogenesis and the History of Science*. Columbia University Press. New York.

Porter, M. K., & Masingila, J. O. (2000). Examining the Effects of Writing on Conceptual and Procedural Knowledge in Calculus. *Educational Studies in Mathematics, 42*, 165-177.

The President’s Council of Advisors on Science and Technology. (2012). Engage to Excel: Producing One Million Additional College Graduates with Degrees in Science, Technology, Engineering and Mathematics. Washington, DC: United States.

Presmeg, N. C. (1985). The Role of Visually Mediated Processes in High School Mathematics: A Classroom Investigation. UK.

Presmeg, N. C. (2006). Research on Visualization in Learning and Teaching in Mathematics: Emergence from Psychology. In A. Gutierrez, & P. Boero, *Handbook of Tesearch on the Psychology of Mathematics Education: Past, Present and Future* (pp. 205-235). Rotterdam, Netherlands: Sense Publishers.

Pugalee, D. K. (2004). A Comparison of Verbal and Written Descriptions of Students' Problem Solving Processes. *Educational Studies in Mathematics, 55*, 27-47.

Schwarz, B. B., & Hershkowitz, R. (1999). Prototypes: Brakes or Levers in Learning the Function Concept? The Role of Computer Tools. *Journal for Research in Mathematics Education, 30*(4), 362-389.

St. Jarre, K. (2008, October). They Knew Calculus When they Left: The thinking Disconnect between High School and University. *The Phi Delta Kappan*, 123-126.

Sweller, J., Van Merrienboer, J., & Paas, F. (1998). Cognitive architecture and instructional design. *Educational Psychology Review, 10*(3), 251-296.

Szydlik, J. E. (2000). Mathematical Beliefs and Conceptual Understanding of the Limit of a Function. *Journal for Research in Mathematics Education, 31*(3), 258-276.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Contnuity. *Educational Studies in Mathematics, 12*, 151-169.

Tilley, R. (2011, July 21). *The University of Kentucky*. Retrieved June 16, 2012, from The University of Kentucky College of Arts & Sciences: http://math.as.uky.edu/calculated-excellence

Treisman, U. (1992 йил November). Studying Students Studying calculus: A Look at the Lives of Minority Mathematics Students in College. (M. A. America, Ed.) *The College Mathematics Journal, 23*(5), 362-372.

Walter, J. G., & Barros, T. (2011). Students build mathematical theory: semantic warrants in argumentation. *Educational Studies in Mathematics*(78), 323-342.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education, 27*(1), 79-95.

Williams, S. R. (1991). Models of Limit Held by College Calculus Students. *Journal for Research on Mathematics Education, 22*(3), 219-236.

Biographical Information

Julie Marie Skinner Sutton was born on August 5, 1979 in San Antonio, Texas, and graduated from Hallsville High School in Hallsville, Texas in 1997. She earned a Bachelor of Science in Mathematics in 2009 and a Masters of Science in Mathematics in 2012, both from The University of Texas at Arlington. Julie served as a Graduate Research Assistant for the Arlington Undergraduate Research based Achievement in STEM program while completing her PhD. She received the Outstanding Graduate Student Award in 2013 from the Mathematics Department at the University of Texas at Arlington and the Outstanding Graduate Teaching Award in 2015.