Nilpotent Lie Algebras and Nilmanifolds Constructed from Graphs

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Abstract

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The interaction between graph theory and differential geometry has been studied previously, but S. Dani and M. Mainkar brought a new approach to this study by associating a two-step nilpotent Lie algebra (and thereby a two-step nilmanifold) with a simple graph. We present a new construction that associates a two-step nilpotent Lie algebra to an arbitrary (not necessarily simple) directed edge-labeled graph. We then use properties of a Schreier graph to determine necessary and sufficient conditions for this Lie algebra to extend to a three-step nilpotent Lie algebra.

After considering the curvature of the two-step nilmanifolds associated with the graphs, we show that if we start with pairs of non-isomorphic Schreier graphs coming from Gassmann-Sunada triples, the pair of associated two-step nilpotent Lie algebras are always isometric. In contrast, we use a well-known pair of Schreier graphs to show that the associated three-step nilpotent extensions need not be isometric.

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Chapter 1

Introduction

Research in the areas of both graph theory and differential geometry has been done for some time, and some interaction between these two areas has been studied by R. Brooks [3] and P. Buser [5] with graphs serving as the discrete analogue of manifolds. More specifically, Brooks and Buser showed that T. Sunada's method for constructing isospectral nonisomorphic manifolds, see [23], could also be used to produce isospectral graphs. Also of interest for this paper is the previous study of the geometry of two-step nilmanifolds by P. Eberlein in [9].

In 2004, S.G. Dani and M.G. Mainkar first presented a method for constructing two-step nilpotent Lie algebras from simple graphs [8]. They used the two-step nilpotent construction to find properties of a graph that would result in the constructed manifold admitting Anosov automorphisms. J. Lauret and C. Will used this construction to find examples of nonisometric Einstein solvmanifolds [17], and H. Pouseele and P. Tirao used the construction to consider symplectic nilmanifolds [22]. Mainkar also proved that for simple graphs, the resulting Lie algebras are isomorphic if and only if the graphs are isomorphic [21], and in [20], she extended this construction to k-step nilpotent Lie algebras. Also, V. Grantcharov is currently working on extending the Dani-Mainkar construction on simple graphs to three-step solvable Lie algebras [12].

In the Dani-Mainkar construction, each vertex and each edge of the graph corresponds to a distinct element in the Lie algebra; therefore for large graphs, the corresponding dimension of the Lie algebra is also large. For the higher-step construction, the dimension of the constructed Lie algebras grows more rapidly.

Much of this thesis will focus on Schreier graphs because of their inherent group structure. J.L. Gross proved that every connected regular graph of even degree is a Schreier graph, [13]. Schreier graphs, however, are often non-simple directed graphs, in which case the Dani-Mainkar construction is not defined. We therefore introduce a new method for associating Lie algebras with Schreier graphs, as suggested by C.S. Gordon.

In Chapter 2, we discuss the definitions and notation that will be used in this paper. For more detail on graph theory, see e.g. [6, 10], for the study of Lie algebras, see [15, 16], and for the areas of differential and Riemannian geometry, I will be following the definitions and notation of [18, 19]. In §3.2, we detail the new construction of a two-step nilpotent Lie algebra associated with an arbitrary Schreier graph. We also provide necessary and sufficient conditions on the graph for this construction to extend to a three-step nilpotent Lie algebra in §3.3. In Chapter 4, we look at the geometry of the constructed nilmanifolds. In particular, §4.2 looks at the curvature of the resulting two-step nilmanifolds and in §4.3, we prove that for any pair of Schreier graphs associated to a Gassmann-Sunada triple, the resultant two-step nilpotent Lie algebras are isometric. We then give an example where the pair of three-step nilpotent Lie algebras are non-isometric.

Chapter 2

Preliminary Concepts

In this thesis, all groups are finite and vector spaces are finite dimensional. Also, all graphs have a finite number of vertices and edges.

2.1 Graph Theory

Since we will be investigating the connection between the areas of differential geometry and graph theory, we will first consider the basic definitions and concepts of graph theory that will be used throughout this thesis.

2.1.1 Graphs

Definition 2.1.1.1. A *graph*, $\mathcal{G} = (V, E)$, consists of two sets, V and E, called the vertex set and edge set, respectively, where E is a set of unordered pairs (α, β), where $\alpha, \beta \in V$. We denote these unordered pairs as simply $\alpha\beta$. Visually, we represent each element in V as a point or vertex, and then if $\alpha\beta \in E$, we connect vertex α to vertex β by an edge. If $\alpha\beta$ is an edge, we say that α is *adjacent* to β and write $\alpha \sim \beta$.

If we need to specify which graph we are considering, we will denote the vertex set and edge sets of graph, $\mathcal{G}$, as $V(\mathcal{G})$ and $E(\mathcal{G})$, respectively.

Definition 2.1.1.2. A *directed graph* $\mathcal{G} = (V, E)$ is related to a graph, but the edge set E consists of ordered pairs (α, β) where $\alpha, \beta \in V$. To distinguish from undirected edges, we always write directed edges (also called arcs) as ordered pairs. Graphically, the directed edge (α, β) is represented by an edge with an arrow pointing from vertex α to vertex β.

Note that for an undirected graph, the edge $\alpha\beta$ is the same as the edge $\beta\alpha$. That is not true, however, with directed graphs. Every undirected graph can be converted to a directed graph by exchanging each undirected edge, $\alpha\beta$, for the two directed edges (α, β) and (β, α).

Remark 2.1.1.3. In this thesis, we use the term graph for an undirected graph.

In general, these graphs may have multiple edges between vertices; they are sometimes referred to as multigraphs, and they can also have loops - an edge connecting the vertex to itself, $\alpha\alpha$.

Definition 2.1.1.4. A *simple (directed) graph* is a (directed) graph that has no multiple edges or loops.

Definition 2.1.1.5. An *edge-labeled graph* is a graph where each edge receives a label from some set X. We can then think of each edge is an ordered triple (α, β, x) where $(\alpha, \beta) \in E$ and $x \in X$.

Often, we will consider these edge labels as colors and refer to the graph as a colored graph. Note that this does not mean that the graph has a proper coloring, where no two edges of the same color share a common vertex.

Definition 2.1.1.6. [5] Two undirected graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are *isomorphic* by the map ϕ if there exists a bijection between the vertex sets, $\phi : V(\mathcal{G}_1) \to V(\mathcal{G}_2)$, such that $\alpha\beta \in E(\mathcal{G}_1) \iff \phi(\alpha)\phi(\beta) \in E(\mathcal{G}_2)$. Two directed edge-labeled graphs are *isomorphic* if $(\alpha, \beta) \in E(\mathcal{G}_1) \iff (\phi(\alpha), \phi(\beta)) \in E(\mathcal{G}_2)$ or $(\phi(\beta), \phi(\alpha)) \in E(\mathcal{G}_2)$. If ϕ also preserves the direction and labeling of the edges, then the graphs are *strongly isomorphic*.

Note that relabeling the vertices of a graph produces a strongly isomorphic graph. [10, 8.1.1]

Definition 2.1.1.7. The *degree* of a graph vertex, v, is the number of edges where v is one of the vertices of that edge. A graph is called *k-regular* if every vertex is of degree k.

Definition 2.1.1.8. For an undirected graph, a *walk* of length q from vertex v to vertex w is a sequence of $q+1$ vertices (and therefore q edges) where successive vertices in the sequence are adjacent to each other. If these vertices are all unique, except possibly the first and last vertex, then this is called a *path* of length q, or a q-path. If $v = w$, then this path is called *closed.* For a directed graph, we require the vertices in the walk to follow the directions of the edges. If all of the edges in a path have the same label, we call this a *same-label path.*

2.1.2 Isospectral Graphs

Definition 2.1.2.1. The *adjacency matrix* of a directed graph $\mathcal{G}$ is $A(\mathcal{G})$ where the (i, j)-entry in the matrix is the number of arcs connecting v_i to v_j. For undirected graphs, since the edge $v_i\, v_j$ is the same as the edge $v_j\, v_i$, the adjacency matrix is symmetric because $A_{i,j} = A_{j,i}$. Let $D(\mathcal{G})$ be the diagonal matrix where the (i, i)-entry is the degree of v_i. Then, the *Laplacian matrix* of a directed graph is $\mathcal{L}(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$.

Definition 2.1.2.2. Two graphs are called *(Laplacian) isospectral* if the set of eigenvalues with multiplicities of their adjacency (Laplacian) matrix are equal. This collection of eigenvalues is called the *(Laplace) spectrum* of the graph.

Example 2.1.2.3. The following two figures show examples of pairs of isospectral graphs. The adjacency matrices of the two graphs in Figure 2.1 are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the Laplacian matrices are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively making these graphs adjacency isospectral but not Laplacian isospectral because the adjacency spectrum is $\{-2, 2, 0, 0, 0\}$ for both while the Laplace spectra are $\{5, 1, 1, 1, 0\}$ and $\{4, 2, 2, 0, 0\}$, respectively. The adjacency matrices for the pair of graphs in Figure 2.2 are

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with Laplacian matrices

$$\begin{pmatrix} 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 4 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 4 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 4 & -1 & 0 & 0 & -1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 4 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 3 \end{pmatrix},$$

which means that this set of graphs is both adjacency and Laplacian isospectral.

Figure 2.1: Small isospectral graphs

Claim 2.1.2.4. [10, 13.1.2] In fact, any pair of regular graphs is adjacency isospectral if and only if the graphs are Laplacian isospectral.

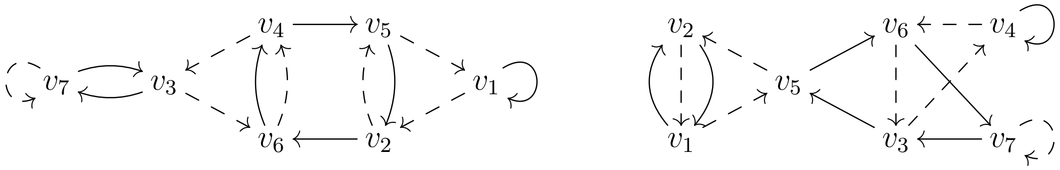


Figure 2.2: Brooks', Buser's isospectral graphs, [3, 5]

Proof. Let a graph be regular of degree k, let λ be an eigenvalue of the adjacency matrix A with eigenvector x, and let I denote the identity matrix.

$$\begin{aligned} \text{We know } Ax = \lambda x \quad &\text{AND} \quad (kI)x = kx \\ \implies (kI)x - Ax \quad &= \quad kx - \lambda x \\ \iff Dx - Ax \quad &= \quad (k-\lambda)x \\ \iff \mathcal{L}x \quad &= \quad (k-\lambda)x \end{aligned}$$

So λ is an eigenvalue of the adjacency matrix if and only if $k - \lambda$ is an eigenvalue of the Laplacian matrix. $\square$

In this paper, we work mostly with regular graphs. We therefore discuss isospectrality in general and only specify the spectrum if necessary.

2.1.3 Schreier Graphs

Most of the graphs examined in this paper are Schreier graphs because they have an inherent group structure.

Definition 2.1.3.1. Let G be a finite group and H a subgroup of G. Let $C := \{z_1, \ldots, z_c, z_1^{-1}, \ldots, z_c^{-1}\}$ be a generating set of G that does not contain the identity and that is closed under inverses, and let $C_{pos} := \{z_1, \ldots, z_c\}$. The *Schreier graph* of G relative to H and C, written $\mathcal{G}(G, H, C)$ or simply $\mathcal{G}$ if understood in context, is a directed edge-labeled graph defined by the following. The vertices of $\mathcal{G}$ consist of the set of right cosets, $V(\mathcal{G}) = \{Hg : \ g \in G\}$. The edges consist of the set of ordered pairs $E(\mathcal{G}) = \{(Hg, Hgz_i^{-1}) : z_i \in C_{pos}\}$, and each edge (Hg, Hgz_i^{-1}) is given the label z_i.

Note that a Schreier graph relative to the identity subgroup is the same as the Cayley graph of the group. Moreover, the Schreier graph of G relative to H is the same is the quotient graph of the Cayley graph of G mod H, i.e. $\mathcal{G}(G, H, C) \cong H\backslash\mathcal{G}(G, \{e\}, C)$ for any generating set C of G.

Example 2.1.3.2. Let $G = S_4$, $H = S_3$, and $C_{pos} = \{(123), (1234)\}$. Then the Schreier graph $\mathcal{G}$ with respect to H and C is given by the following figure where the solid lines correspond to edges formed by the first generator, (123), in C_{pos} and dotted lines to the second generator, (1234).

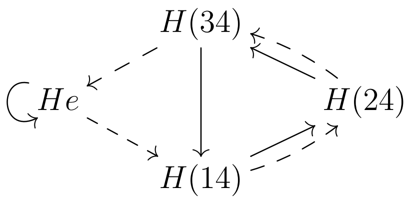


Figure 2.3: Schreier graph of $S_3\backslash S_4$

The following properties of a Schreier graph are important in the proofs of the main theorems in §3.3 and §4.3.

Remark 2.1.3.3. Note that while a Schreier graph is defined for an element of C_{pos} of order 2, the edges associated to those elements will become trivial elements in the Lie algebras we construct in §3.2. Hence, in what follows, we assume that the generating set C does not contain order 2 elements, i.e. $z \neq z^{-1}$ for all $z \in C$.

Remark 2.1.3.4. The structure of a Schreier graph implies that the group G acts on $V(\mathcal{G})$ by right inverse multiplication. To see this, we define $\alpha(z_i) : V(\mathcal{G}) \to V(\mathcal{G})$ for $z_i \in C$ by

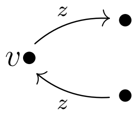
$$\alpha(z_i)(Hg) = Hgz_i^{-1} \text{ for all } z_i \in C.$$

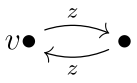
Then α extends from C to G because C generates G.

Remark 2.1.3.5. Because the edges of a Schreier graph are associated with generators of a finite group, each generator produces a union of closed paths that span the vertex set of $\mathcal{G}$, where the length of each closed path is less than or equal to the order of the generating element. When we then take the union over all generators in C_{pos}, we obtain the full Schreier graph.

Remark 2.1.3.6. If $|C_{pos}| = c$, then the Schreier graph $\mathcal{G}$ is $2c$-regular, where each vertex has a directed edge labeled z_i going out of the vertex and one going into the vertex, $i = 1, \ldots, c$. This gives three different possibilities for each vertex v and each generator (and hence each label) z:

1. $\alpha(z)(v) \neq \alpha(z^{-1})(v)$ 2. $\alpha(z)(v) = \alpha(z^{-1})(v) \neq v$, and 3. $\alpha(z)(v) = \alpha(z^{-1})(v) = v$





2.2 Lie Algebras

Another area of research that is used in this thesis is the study of Lie algebras. In this section, we introduce the basic definitions as well some results that are used in later chapters.

2.2.1 Lie Algebras

Definition 2.2.1.1. A *Lie algebra* is a vector space $\mathcal{V}$ over a field $\mathbb{F}$ together with a binary operation $[\cdot,\cdot] : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$, called the *Lie bracket*, that satisfies the following statements $\forall x, y, z \in \mathcal{V}$ and $\forall a, b \in \mathbb{F}$:

1. (a) $[ax + by, z] = a[x, z] + b[y, z]$
   (b) $[x, ay + bz] = a[x, y] + b[y, z]$
2. $[x, x] = 0$
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Note that (1a) and (1b) imply that the bracket operation is bilinear. That along with (2) imply that the bracket is anticommutative, i.e. $[x, y] = -[y, x]$. Condition (3) is known as the *Jacobi identity*.

Example 2.2.1.2. Let $\mathfrak{v} = \text{span}_{\mathbb{R}}\{X, Y, Z\}$, and define the Lie bracket as $[X, Y] = Z$, and all other brackets not defined by linearity or skew-symmetry equal zero. This is referred to as the *Heisenberg Lie algebra*.

Definition 2.2.1.3. Two Lie algebras, $(\mathcal{V}, [\ ,\ ])$ and $(\mathcal{V}', [\ ,\ ]')$, are *isomorphic* if there exists a linear bijection $\phi : \mathcal{V} \to \mathcal{V}'$ such that $\phi([x, y]) = [\phi(x), \phi(y)]'$, $\forall x, y \in \mathcal{V}$.

We will often assign an orthonormal basis to a given Lie algebra in order to obtain a metric $<,>$ on the Lie algebra.

Definition 2.2.1.4. Two metric Lie algebras, $(\mathcal{V}, <,>)$ and $(\mathcal{V}', <,>')$, are *isometric* if they are isomorphic as Lie algebras and if $< x, y >=< \phi(x), \phi(y) >'$, $\forall x, y \in \mathcal{V}$.

2.2.2 Central Series

Definition 2.2.2.1. For a Lie algebra $\mathcal{V}$, we define the *descending central series* recursively as the sequence of ideals $\mathcal{V}^{(0)} = \mathcal{V}$, $\mathcal{V}^{(1)} = [\mathcal{V}, \mathcal{V}]$, and $\mathcal{V}^{(n)} = [\mathcal{V}, \mathcal{V}^{(n-1)}]$. A Lie algebra is called *k-step nilpotent* if $\mathcal{V}^{(k)} = 0$ but $\mathcal{V}^{(k-1)} \neq 0$.

Definition 2.2.2.2. The *center* of a Lie algebra $\mathcal{V}$, denoted $Z(\mathcal{V})$, is defined as $Z(\mathcal{V}) = \{x \in \mathcal{V} : [x, y] = 0, \forall y \in \mathcal{V}\}$. The *ith center* of a Lie algebra is defined as $Z_i(\mathcal{V}) = \{w \in \mathcal{V} : [w, \mathcal{V}] \subseteq Z_{i-1}(\mathcal{V})\}$, where $Z_0 = \{0\}$. The sequence of ith centers is known as the *ascending central series.*

This implies that if a Lie algebra $\mathcal{V}$ is k-step nilpotent then $\mathcal{V}^{(k-1)}$ is a subset of $Z(\mathcal{V})$.

Proposition 2.2.2.3. *A Lie algebra isomorphism preserves the ascending and descending central series.*

Proof. Assume that $\phi : V \to W$ is a Lie algebra isomorphism. We proceed by induction on the sequence of ideals, $V^{(i)}$. Let $v \in V^{(0)} = V$. Then $\phi(v) \in W$ since ϕ is an isomorphism, which implies that $\phi(v) \in W^{(0)} = W$.
Now assume that $\forall v \in V^{(i)},\ \phi(v) \in W^{(i)}$.

$$\begin{aligned} \text{Let } x \in V^{(i+1)} &\implies x \in [V, V^{(i)}] \\ &\implies x = \sum_i [v_i, v_i'] \text{ for some } v_i \in V,\ v_i' \in V^{(i)} \\ &\implies \phi(x) = \sum_i [\phi(v_i), \phi(v_i')] \text{ for some } v_i \in V,\ v_i' \in V^{(i)}. \end{aligned}$$

We know $\forall v_i,\ \phi(v_i) \in W$ since ϕ is an isomorphism, and

$\forall v_i',\ \phi(v_i') \in W^{(i)}$ by induction hypothesis

$$\implies \phi(x) \in [W, W^{(i)}] = W^{(i+1)}.$$ □

The proof for the ascending central series is similar. This Proposition will play a major role in the proof of Theorem 4.3.0.4.

2.3 Differential and Riemannian Geometry

This section will discuss the various definitions and results in the areas of differential and Riemannian geometry used to study the nilmanifolds constructed from graphs in Chapter 4. We will follow the definitions of J.M. Lee in [18] and [19].

2.3.1 Riemannian Manifolds

Definition 2.3.1.1. Given a differentiable manifold M, a *Riemannian metric* on M is a choice of inner product g_p on each tangent space T_pM such that the inner product varies smoothly on M. In this case, (M, g) is called a *Riemannian manifold.*

Example 2.3.1.2. Let $M = \mathbb{R}^n$, and let $X_p = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x^i}$ and $Y_p = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x^i}$ be arbitrary elements of T_pM. Then $g_p(X_p, Y_p) = \sum_{i,j=1}^n \delta_{i,j}\alpha_i\beta_j = \sum_{i=1}^n \alpha_i\beta_i$ defines a Riemannian metric on $\mathbb{R}^n$, called the *Euclidean metric.* This metric is also denoted by $g = \sum_i dx^i \otimes dx^i$.

Example 2.3.1.3. Let $M = S^2 \subseteq \mathbb{R}^3$ with cylindrical coordinates $\varphi : S^2 \to \mathbb{R}^3$ given by $\varphi : (\theta, h) \mapsto (x, y, z) := (\sqrt{1-h^2}\cos\theta, \sqrt{1-h^2}\sin\theta, h)$ with $0 \leq \theta < 2\pi$ and $-1 < h < 1$. Because φ is a submersion, we can give S^2 the *induced metric*, $g = \varphi^*\bar{g}$, where $\bar{g}$ is the Euclidean metric on $\mathbb{R}^3$ and φ^* is the pullback of φ. To see this with respect to coordinates, note that

$$\begin{aligned} dx &= \tfrac{-h}{\sqrt{1-h^2}}\cos\theta dh - \sqrt{1-h^2}\sin\theta d\theta, \quad dy = \tfrac{-h}{\sqrt{1-h^2}}\sin\theta dh + \sqrt{1-h^2}\cos\theta d\theta \\ \text{and } dz &= dh. \\ \text{Then } g &= \varphi^*\bar{g} = \varphi^*(dx \otimes dx + dy \otimes dy + dz \otimes dz) \\ &= \tfrac{h^2}{1-h^2} dh \otimes dh + (1-h^2) d\theta \otimes d\theta + dh \otimes dh \\ &= \tfrac{1}{1-h^2} dh \otimes dh + (1-h^2) d\theta \times d\theta \end{aligned}$$

Definition 2.3.1.4. Two Riemannian manifolds (M, g) and (M', g') are *isometric* if there exists a diffeomorphism $\varphi : M \to M'$ such that $\varphi^*g' = g$.

We will consider the geometry of the various manifolds, including curvature, constructed from graphs in Section 4.2.

Definition 2.3.1.5. Let M be a Riemannian manifold, then the map $R : \mathscr{T}(M) \times \mathscr{T}(M) \times \mathscr{T}(M) \to \mathscr{T}(M)$ defined by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ is called the *curvature endomorphism*, where $\nabla_X Y$ is the *covariant derivative* of Y in the direction of X.

Definition 2.3.1.6. The *Riemann curvature tensor* is defined as the covariant 4-tensor field $Rm = R^\flat$, i.e. $Rm(X,Y,Z,W) =< R(X,Y)Z, W >$.

Because the curvature tensor is quite complicated, we often look at other ideas of curvature that are related to this tensor. Two of these include sectional curvature and Ricci curvature.

Definition 2.3.1.7. Let M be a Riemannian manifold. Given a two-dimensional subspace Π of T_pM, the Gaussian curvature of the surface S_Π at p with the induced metric is called the *sectional curvature* of Π, denoted $K(\Pi)$.

Proposition 2.3.1.8. *[19, Proposition 8.8] If $\{X,Y\}$ is a basis of a two-dimensional subspace Π of T_pM, then the sectional curvature of Π is*

$$K(\Pi) = K(X,Y) = \frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - < X,Y >^2}.$$

Definition 2.3.1.9. *Ricci curvature* or the *Ricci tensor* is the covariant 2-tensor field defined as the trace of the curvature endomorphism on the first and last indices, i.e. for arbitrary X, Y in the Lie algebra $\mathfrak{n}$ and given an orthonormal basis $\{E_i\}$ of $\mathfrak{n}$,

$$Ric(X,Y) = \sum_i Rm(E_i, X, Y, E_i).$$

2.3.2 Lie Groups

Definition 2.3.2.1. A Lie group is a differentiable manifold M that is also an algebraic group where the multiplication and inverse operators are smooth functions on M.

In Section 2.2, we gave the definition for a general Lie algebra, but there is a relationship present between Lie algebras and Lie groups that gives us the ability to study one by looking at the other.

Because we are assuming that each Lie algebra is a finite vector space, we can consider $GL(\mathfrak{v})$ as a subset of $GL(n, \mathbb{F})$ by choosing a basis for $\mathfrak{v}$. In this way, we can define the operator $\exp : GL(n, \mathbb{F}) \to GL(n, \mathbb{F})$. where

$$\exp X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

Remark 2.3.2.2. Given a Lie algebra $\mathfrak{g}$, there exists a unique simply connected Lie group G such that G is generated by elements of the form $\{\exp(tX) : \ X \in \mathfrak{g},\ t \in \mathbb{R}\}$. We therefore call G the Lie group associated with $\mathfrak{g}$, and vice versa.

Remark 2.3.2.3. The Lie algebra $\mathfrak{g}$ associated with a Lie group G is isomorphic to the tangent space at the identity of G, i.e. $\mathfrak{g} \cong T_eG$. Moreover, exp is a local diffeomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of the identity in G. The inverse of this function is denoted by log.

This relationship gives the ability to look at the multiplication and inverse operators on G in terms of elements of its associated Lie algebra $\mathfrak{g}$.

Remark 2.3.2.4 (Baker-Campbell-Hausdorff Formula). Given X, Y in $\mathfrak{g}$, $\exp X, \exp Y \in G$ and $(\exp X)(\exp Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots)$. Also, $(\exp X)^{-1} = \exp(-X)$.

Definition 2.3.2.5. A Lie group is called *k-step nilpotent* if its Lie algebra is k-step nilpotent as given in Definition 2.2.2.1.

Since the Lie algebras discussed here are mostly nilpotent, the right-hand side of the Baker-Campbell-Hausdorff formula will not continue indefinitely and so this formula is a well-defined and smooth operation on G.

Definition 2.3.2.6. A metric on a Lie group G is called *left invariant* if $g_p(X_p, Y_p) = g_{ap}(L_a(X_p), L_a(Y_p)), \forall a \in G, \forall X, Y \in \mathfrak{g}$, where L_a is left multiplication by the group element a.

Given a metric Lie algebra, we can define a left invariant metric on a Lie group by requiring $g_p(X_p, Y_p) =< L_{p^{-1}}(X_p), L_{p^{-1}}(Y_p) >$ where $<,>$ is the inner product on the Lie algebra (T_eG) associated with the Lie group.

2.3.3 Nilmanifolds

Instead of looking at the simply connected Lie group associated with a Lie algebra, we will often want to look at a compact nilmanifold instead, because the geometry of these manifolds is more well understood since they behave similar to a torus.

Definition 2.3.3.1. Given a Lie group G, a subgroup Γ of G is called a *discrete subgroup* if the relative topology of Γ in G is the discrete topology.

For example, $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$.

Definition 2.3.3.2. A subgroup Γ of G is *cocompact* if $\Gamma \backslash G$ is compact.

Definition 2.3.3.3. [7] For a simply connected Lie group G, a subgroup Γ of G is called a *lattice subgroup* if it is a cocompact discrete subgroup of G.

Definition 2.3.3.4. A *compact nilmanifold* is a manifold of the form $\Gamma \backslash G$, where G is a simply connected nilpotent Lie group and Γ is a lattice subgroup of G.

Example 2.3.3.5. Let H denote the Heisenberg Lie group,

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \; x, y, z \in \mathbb{R} \right\},$$

with group operation defined by $(x,y,z)(x',y',z') = (x+x', y+y', z+z'+\frac{1}{2}xy')$, and let

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & \frac{1}{2}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \; x,y,z \in \mathbb{Z} \right\}.$$

Then $\Gamma \backslash H$ is a compact nilmanifold.

2.4 Gassmann-Sunada Triples

Many of the examples of isospectral graphs and manifolds come from constructions based on Gassmann-Sunada triples and the Sunada theorem.

Definition 2.4.0.1. Let G be a finite group, with H_1 and H_2 subgroups of G such that for every $g \in G$,

$$|[g] \cap H_1| = |[g] \cap H_2|,$$

where $[g]$ denotes the conjugacy class of g in G. In this case, H_1 and H_2 are called *almost conjugate* subgroups of G, and (G, H_1, H_2) is called a *Gassmann-Sunada triple*, [5, 23].

Example 2.4.0.2. In [3, 5], the following is shown to be a Gassmann-Sunada triple: let $G = SL(3, \mathbb{F}_2)$, $H_1 = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$, and $H_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}$.

In [2], W. Bosma and B. de Smit found that only 19 Gassmann-Sunada triples existed (up to isomorphism) of index less than or equal to 15, where index is the order of the quotient group, $H_i \backslash G$. Moreover, the smallest index where a nontrivial Gassmann-Sunada triple exists is 7, which is the example given above.

Gassmann originally studied these pairs to consider whether two algebraic number fields had the same Dedekind Zeta function, Bosma and de Smit were investigating Galois groups of arithmetically equivalent number fields, but the following two theorems show that Gassmann-Sunada triples also produce pairs of isospectral non-strongly isomorphic graphs. In fact, if we take the Gassmann-Sunada triple in Example 2.4.0.2 above and construct the Schreier graphs, $\mathcal{G}(G, H_1, C)$ and $\mathcal{G}(G, H_2, C)$, relative to the generating set $C_{pos} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right\}$, we get the pair of isospectral nonisomorphic Schreier graphs in Figure 2.2.

Theorem 2.4.0.3. *[3, 14, 23] If (G, H_1, H_2) is a Gassmann-Sunada triple, then the Schreier graphs $\mathcal{G}(G, H_1, C)$ and $\mathcal{G}(G, H_2, C)$ will be isospectral graphs for any generating set C of G.*

Theorem 2.4.0.4. *[5, Thm. 11.4.4] Let H_1 and H_2 be almost conjugate subgroups of G and C a generating set of G. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the Schreier graphs of (G, H_1, C) and (G, H_2, C), respectively. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are strongly isomorphic if and only if H_1 and H_2 are conjugate.*

This theorem ensures that the pairs of Schreier graphs associated with a Gassmann-Sunada triple are not strongly isomorphic; however, they still might be isomorphic.

Example 2.4.0.5. Let G be the subgroup of $GL(4, 2)$ generated by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $X = \mathbb{F}_2^4 - \{0\}$ and $Y = X^*$. By [2], (G, X, Y) is shown to be a Gassmann-Sunada triple. Using Magma [1], we found that the set of generators given above resulted in simple Schreier graphs that were isomorphic but not strongly isomorphic. The first generator corresponds to the solid lines on the Schreier graph and the second generator to the dotted lines.

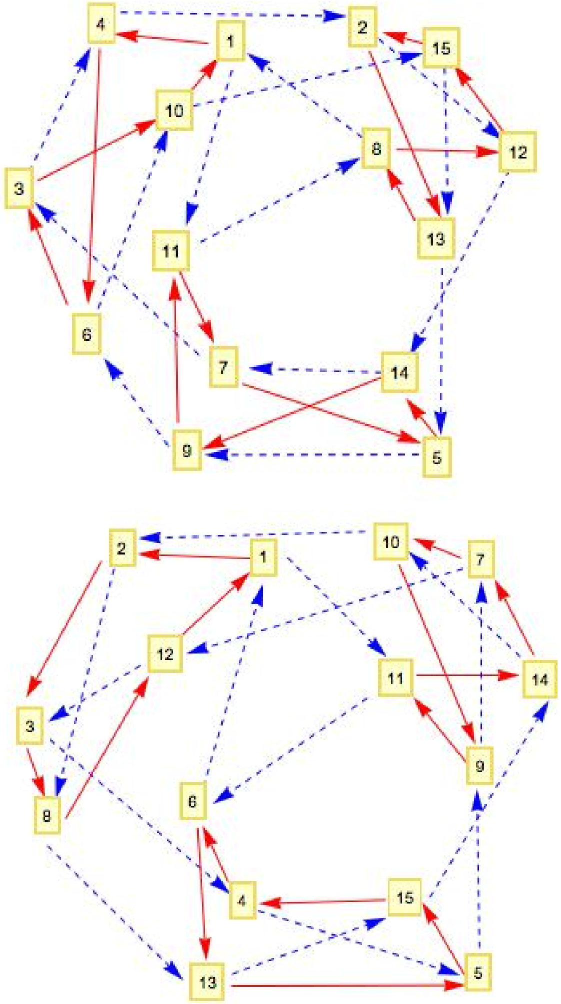


Figure 2.4: Pair of simple Schreier graphs

Chapter 3

Constructions

In this chapter, we will look at the previous constructions from graphs to Lie algebras and Lie groups along with their results. We then consider a new construction.

3.1 Dani-Mainkar Constructions

In [8], S.G. Dani and M.G. Mainkar present the following construction of a two-step nilpotent Lie algebra from a simple graph.

Construction 3.1.0.1 (Dani-Mainkar Two-Step Nilpotent Construction). Let $\mathcal{G}$ be a finite graph without multiple edges. Define $\mathfrak{v}$ to be the space of formal linear combinations over $\mathbb{R}$ of elements in $V(\mathcal{G})$, and let $\mathfrak{z}$ be the subset of $\Lambda^2(\mathfrak{v})$ defined by $\mathfrak{z} = \text{span}_{\mathbb{R}}\{\alpha \wedge \beta : \ \alpha\beta \in E(\mathcal{G})\}$. Then we let $\mathfrak{n}$ be the direct sum of vector spaces, $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, and finally, we define the Lie bracket on a basis of $\mathfrak{n}$ and then extend by linearity by the following: $\forall v_i, v_j \in \mathfrak{v}$ and $\forall z, z' \in \mathfrak{z}$,

$$\begin{aligned}
[(v_i,0),(v_j,0)] &= \begin{cases} (0, v_i \wedge v_j) & \text{if} \quad v_iv_j \in E(\mathcal{G}) \\ (0,0) & \text{if} \quad v_iv_j \notin E(\mathcal{G}) \end{cases} \\
[(0,z),(0,z')] &= (0,0) \\
[(v_i,0),(0,z)] &= (0,0).
\end{aligned}$$

Note that because the wedge product is skew-symmetric, this Lie bracket will also be skew-symmetric. Also because this Lie algebra is two-step nilpotent (because $\mathfrak{z} \subseteq Z(\mathfrak{n})$), the Jacobi identity is trivial, so the above does indeed define a two-step nilpotent Lie algebra.

We use the following graph to contrast the various constructions defined in this chapter.

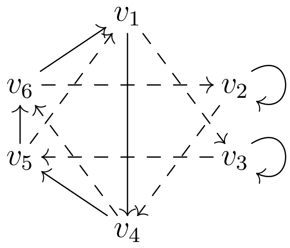


Figure 3.1: Six vertex four regular graph

Example 3.1.0.2. For the graph in Figure 3.1, the Dani-Mainkar construction is valid since the graph does not have multiple edges (the loops are okay). Also, the D-M construction is typically done on undirected graphs, but a direction is arbitrarily chosen in the Lie algebra construction since it is based on the wedge product.

Letting $z_{i,j}$ denote $[v_i, v_j] := v_i \wedge v_j$ in Construction 3.1.0.1, we define

$$\begin{aligned} \mathfrak{v} &= \operatorname{span}_{\mathbb{R}}\{v_1, \ldots, v_6\}, \\ \mathfrak{z} &= \operatorname{span}_{\mathbb{R}}\{z_{1,3}, z_{1,4}, z_{2,4}, z_{3,5}, z_{4,5}, z_{4,6}, z_{5,1}, z_{5,6}, z_{6,1}, z_{6,2}\}, \text{ and} \\ \mathfrak{n} &= \mathfrak{v} \oplus \mathfrak{z}, \end{aligned}$$

and obtain the following Lie brackets:

$$\begin{array}{lllllllllllll} [v_1, v_3] & = & z_{1,3} & [v_1, v_4] & = & z_{1,4} & [v_2, v_4] & = & z_{2,4} & [v_3, v_5] & = & z_{3,5} \\ [v_4, v_5] & = & z_{4,5} & [v_4, v_6] & = & z_{4,6} & [v_5, v_1] & = & z_{5,1} & [v_5, v_6] & = & z_{5,6} \\ [v_6, v_1] & = & z_{6,1} & [v_6, v_2] & = & z_{6,2} & & & & & & \end{array}$$

All other brackets not defined by linearity or skew-symmetry are equal to zero.

Note that $\dim \mathfrak{n} = 16$.

From this construction, Dani and Mainkar found properties on these graphs such that the resulting nilmanifolds would admit Anosov automorphisms, see [8].

Mainkar extended this construction to higher-step nilpotent Lie algebras in [21].

Construction 3.1.0.3 (Mainkar Higher-Step Nilpotent Construction). Given a finite graph without multiple edges, $\mathcal{G}$, we define $\mathfrak{v}$ as in Construction 3.1.0.1. We then take $H_k(\mathfrak{v})$ to be the free k-step nilpotent Lie algebra on $\mathfrak{v}$ and $I_k(\mathfrak{v})$ to be the ideal of $H_k(\mathfrak{v})$ generated by elements that are not in $E(\mathcal{G})$. Then we define the k-step nilpotent Lie algebra, $\widehat{\mathfrak{n}}_k$, by $H_k(\mathfrak{v})/I_k(\mathfrak{v})$.

Example 3.1.0.4. From the graph in Figure 3.1, we obtain the following three-step nilpotent Lie algebra by using Construction 3.1.0.3. We define $\mathfrak{v}$ and $\mathfrak{z}$ as in Construction 3.1.0.1. Letting $\tau_{i,j,k}$ denote $[v_i,[v_j,v_k]]$, we define

$$\begin{aligned}
\mathfrak{t} &= \operatorname{span}_{\mathbb{R}}\{\tau_{1,3,5},\tau_{1,4,5},\tau_{1,4,6},\tau_{1,5,6},\tau_{2,4,6},\tau_{4,5,6},\tau_{3,5,1},\tau_{4,5,1},\tau_{4,6,1},\tau_{5,6,1},\tau_{4,6,2},\tau_{5,6,4}\} \text{ and}\\
\widehat{\mathfrak{n}} &= \mathfrak{v}\oplus\mathfrak{z}\oplus\mathfrak{t}.
\end{aligned}$$

We have all of the brackets listed in Example 3.1.0.2 plus the following nonzero brackets:

$$\begin{aligned}
[v_1,z_{3,5}] &= \tau_{1,3,5} & [v_3,z_{5,1}] &= \tau_{3,5,1} & [v_5,z_{1,3}] &= -\tau_{1,3,5}-\tau_{3,5,1}\\
[v_1,z_{4,5}] &= \tau_{1,4,5} & [v_4,z_{5,1}] &= \tau_{4,5,1} & [v_5,z_{1,4}] &= -\tau_{1,4,5}-\tau_{4,5,1}\\
[v_1,z_{4,6}] &= \tau_{1,4,6} & [v_4,z_{6,1}] &= \tau_{4,6,1} & [v_6,z_{1,4}] &= -\tau_{1,4,6}-\tau_{4,6,1}\\
[v_1,z_{5,6}] &= \tau_{1,5,6} & [v_5,z_{6,1}] &= \tau_{5,6,1} & [v_6,z_{1,5}] &= -\tau_{1,5,6}-\tau_{5,6,1}\\
[v_2,z_{4,6}] &= \tau_{2,4,6} & [v_4,z_{6,2}] &= \tau_{4,6,2} & [v_6,z_{2,4}] &= -\tau_{2,4,6}-\tau_{4,6,2}\\
[v_4,z_{5,6}] &= \tau_{4,5,6} & [v_5,z_{6,4}] &= \tau_{5,6,4} & [v_6,z_{4,5}] &= -\tau_{4,5,6}-\tau_{5,6,4}
\end{aligned}$$

All other brackets not defined by linearity or skew-symmetry are equal to zero.

Note that $\dim\widehat{\mathfrak{n}} = 28$.

Remark 3.1.0.5. The following shows some limitations of this construction and why we decided to use a different construction for our work:

- From the above construction, we find that $\dim\mathfrak{n} = |V(\mathcal{G})| + |E(\mathcal{G})|$ so larger graphs will also produce Lie algebras of much larger dimension.

- The dimension of the higher-step nilpotent Lie algebras constructed in [21] grows even more rapidly.
- This construction is limited to graphs without multiple edges.
- In §4.3, we compare pairs of Schreier graphs of a Gassmann-Sunada triple, which have a given correspondence between elements of the groups. This construction gives no obvious way to relate elements in the resulting Lie algebras.

3.2 Two-Step Nilpotent Construction

The following construction is an adaptation of the Dani-Mainkar construction suggested by C.S. Gordon, and it relieves some of the issues discussed in Remark 3.1.0.5.

Construction 3.2.0.1 (Two-Step Nilpotent Construction). From a Schreier graph $\mathcal{G} = \mathcal{G}(G, H, C)$ given by Definition 2.1.3.1, we let $\mathfrak{v}$ be the space of formal linear combinations over $\mathbb{R}$ of elements in $V(\mathcal{G})$ and $\mathfrak{z}$ be the space of formal linear combinations over $\mathbb{R}$ of elements in C_{pos}. We then define the Lie algebra $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{z}$ as the direct sum of vector spaces; we then require $\mathfrak{z}$ to be contained in the center of $\mathfrak{n}$ and define the Lie bracket by the following: $\forall v_i, v_j \in V(\mathcal{G}) \subseteq \mathfrak{v}$,

$$[v_i, v_j] = \sum_{p=1}^{|C_{pos}|} (\epsilon_p - \epsilon'_p) z_p, \tag{3.2.0.1.1}$$

$$\text{where } \epsilon_p = \begin{cases} 1 & , \quad \text{if } v_j = \alpha(z_p)(v_i) \\ 0 & , \quad \text{otherwise,} \end{cases}$$

$$\text{and } \epsilon'_p = \begin{cases} 1 & , \quad \text{if } v_j = \alpha(z_p^{-1})(v_i) \\ 0 & , \quad \text{otherwise.} \end{cases}$$

All other brackets not defined by linearity or skew-symmetry are set equal to zero.

To see that this does define a Lie algebra, consider the following. First, if $z = z^{-1} \in C_{pos}$, then $[v_i, v_j] = 0 \; \forall v_i, v_j \in \mathfrak{v}$, which is why we exclude such elements from C_{pos} as we mentioned in Remark 2.1.3.3. Also note that $[v_i, v_i] = 0$ because for a fixed label z_p, either v_i has a loop with label z_p in which case $\epsilon_p = \epsilon'_p = 1$, or v_i does not have a loop with label z_p in which case $\epsilon_p = \epsilon'_p = 0$. In either case, $\epsilon_p - \epsilon'_p = 0$ for all p. Furthermore, this bracket will be skew-symmetric because $v_j = \alpha(z_p)(v_i)$ implies $v_i = \alpha(z_p^{-1})(v_j)$. Finally, note that because $\mathfrak{z}$ is contained in the center of $\mathfrak{n}$, the Jacobi identity on the bracket given above is trivial, which makes $(\mathfrak{n}, [\,,\,])$ as defined above a two-step nilpotent Lie algebra.

Example 3.2.0.2. Note that by [13], the graph in Figure 3.1 is a Schreier graph because it is a connected four-regular graph; therefore, this two-step nilpotent Lie algebra construction is valid for this graph. We denote the generator that corresponds to the solid line by z_r and the dotted line by z_b. We then define

$$\begin{aligned} \mathfrak{v} &= \operatorname{span}_{\mathbb{R}}\{v_1, \ldots, v_6\}, \\ \mathfrak{z} &= \operatorname{span}_{\mathbb{R}}\{z_r, z_b\}, \text{ and} \\ \mathfrak{n} &= \mathfrak{v} \oplus \mathfrak{z}, \end{aligned}$$

and obtain the following Lie brackets:

$$\begin{array}{llllllllllll} [v_1, v_3] & = & z_b & [v_1, v_4] & = & z_r & [v_2, v_4] & = & z_b & [v_3, v_5] & = & z_b \\ [v_4, v_5] & = & z_r & [v_4, v_6] & = & z_b & [v_5, v_1] & = & z_b & [v_5, v_6] & = & z_r \\ [v_6, v_1] & = & z_r & [v_6, v_2] & = & z_b & & & & & & \end{array}$$

All other brackets not defined by linearity or skew-symmetry are equal to zero.

We see that in $\mathfrak{n}$, $[v_1, v_4] = z_r$ because in the Schreier graph, $\mathcal{G}$, there is an edge labeled z_r connecting v_1 to v_4, while $[v_1, v_6] = -z_r$ because the directed edge connects v_6 to v_1. Loops will bracket to zero because they have a directed edge connected from the vertex to itself. For example, for v_2 in $\mathcal{G}$, $[v_2, v_2] = z_r - z_r = 0$. Continuing in this manner, we obtain all of the above bracket relations of $\mathfrak{n}$.

This constructed Lie algebra differs from the Dani-Mainkar construction not only in that we use directed instead of undirected graphs but also in the dimension of the constructed Lie algebras. The Dani-Mainkar construction states that each edge of the graph corresponds to a unique element in the basis of $\mathfrak{z}$ making $\dim \mathfrak{z}$ the number of edges in $\mathcal{G}$, where in this new construction $\dim \mathfrak{z}$ is the size of the generating set C_{pos}. For the example given, the Dani-Mainkar construction produces a Lie algebra $\mathfrak{n}$ of dimension 16, while the new construction produces one of dimension 8.

Remark 3.2.0.3. In this paper, when needed, we specify an inner product on $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ by requiring $\{V(\mathcal{G}), C_{pos}\}$ to be an orthonormal basis.

Remark 3.2.0.4. The two-step nilpotent Lie algebra defined in Construction 3.2.0.1 does not rely on the fact that the graph was a Schreier graph. A two-step nilpotent Lie algebra can be constructed similarly from any directed, labeled (colored) graph by having a set of graph labels (colors), $C_{pos} = \{z_1, \ldots, z_c\}$, instead of having a set of generators of a group acting on the graph. The Lie bracket on $\mathfrak{n} := \mathfrak{v} \oplus \mathfrak{z}$ is then defined as in Construction 3.2.0.1, except now

$$\epsilon_p = \begin{cases} 1 \quad , & \text{if } (v_i, v_j) \text{ is an edge labeled } z_p \\ 0 \quad , & \text{otherwise,} \end{cases}$$

$$\text{and } \epsilon'_p = \begin{cases} 1 \quad , & \text{if } (v_j, v_i) \text{ is an edge labeled } z_p \\ 0 \quad , & \text{otherwise.} \end{cases}$$

In order to find necessary and sufficient conditions on a Schreier graph for the two-step nilpotent Lie algebra construction to extend to a three-step nilpotent Lie algebra, we must introduce the following definition.

Definition 3.2.0.5. For a Schreier graph $\mathcal{G} = \mathcal{G}(G, H, C)$, a label $z \in C_{pos}$ is called *admissible* if there exists a single closed same-label path of length 3 or 4 with label z, and all other closed same-label paths with label z are of length 1 or 2. Otherwise, z is called *inadmissible*. We denote the set of admissible labels by $\{z_{r_1}, \ldots, z_{r_m}\}$ and the set of inadmissible labels by $\{z_{b_1}, \ldots, z_{b_n}\}$. A path is called *admissible* if it is the single closed same-label path of length 3 or 4 for an admissible label z_r.

Example 3.2.0.6. From Example 3.2.0.2 above, we see that z_r is an admissible label because there is a closed same-label path of length 4, namely $(v_1, v_4, v_5, v_6, v_1)$ and the other closed same-label paths of label z_r are of length 1. On the other hand, z_b is an inadmissible label because there are two closed same-label paths of length 3, (v_1, v_3, v_5, v_1) and (v_2, v_4, v_6, v_2).

3.3 Three-Step Nilpotent Construction

The following is the main theorem of this thesis.

Theorem 3.3.0.1. *Let G be a finite group, H a subgroup of G, C a generating set of G, and $\mathcal{G}$ the Schreier graph of G with respect to H and C as in Definition 2.1.3.1. Let $\mathfrak{n}$ be the two-step nilpotent Lie algebra associated with $\mathcal{G}$ by Construction 3.2.0.1. Then $\mathfrak{n}$ extends to a three-step nilpotent Lie algebra $\widehat{\mathfrak{n}}$ if and only if there exists at least one admissible label in C_{pos}. Moreover, up to the variations allowed in Construction 3.3.0.2 below, this is the only 3-step nilpotent extension of $\mathfrak{n}$.*

Construction 3.3.0.2 (Three-Step Nilpotent Construction). For each admissible label z_{r_k}, we define new elements $\tau_{r_{k,1}}$ and $\tau_{r_{k,2}}$ (at least one $\tau_{r_{k,\ell}} \neq 0$) such that the 3-step nilpotent extension of $\mathfrak{n}$ is $\widehat{\mathfrak{n}} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{t}$, where $\mathfrak{v}$ and $\mathfrak{z}$ are defined as before and $\mathfrak{t} = \operatorname{span}_{\mathbb{R}}\{\tau_{r_{k,1}}, \tau_{r_{k,2}} : z_{r_k} \text{ is admissible}\}$. The Lie bracket is then defined as in Construction 3.2.0.1 with the following additional nonzero brackets, and then extend by linearity and skew-symmetry:

If the admissible path with label z_{r_k} is of length 4 and has successive vertices $(v_1, v_2, v_3, v_4, v_1)$, we set

(3.3.0.2.1)
$$\begin{aligned} [v_1, z_{r_k}] = -[v_3, z_{r_k}] &= \tau_{r_{k,1}}, \text{ and} \\ [v_2, z_{r_k}] = -[v_4, z_{r_k}] &= \tau_{r_{k,2}} \end{aligned}$$

If the admissible path with label z_{r_k} is of length 3 and has successive vertices (v_1, v_2, v_3, v_1), we set

(3.3.0.2.2)
$$\begin{aligned} [v_1, z_{r_k}] &= \tau_{r_{k,1}}, \\ [v_2, z_{r_k}] &= \tau_{r_{k,2}}, \text{ and} \\ [v_3, z_{r_k}] &= -(\tau_{r_{k,1}} + \tau_{r_{k,2}}) \end{aligned}$$

For any other vertex v_i not in the admissible 3- or 4-path, we set

(3.3.0.2.3)
$$[v_i, z_{r_k}] = 0$$

For any edge with inadmissible label z_b, we set

(3.3.0.2.4)
$$[v_j, z_b] = 0 \; \forall v_j \in \mathfrak{v}.$$

Remark 3.3.0.3. In order for $\widehat{\mathfrak{n}}$ to be 3-step nilpotent, we must set at least one $\tau_{r_{k,\ell}} \neq 0$. The 3-step nilpotent extension of $\mathfrak{n}$ is not unique. Distinct Lie algebra extensions can be obtained by defining relations between the various elements $\tau_{r_{k,\ell}}$, namely these elements may be linearly dependent. Because of these variations, we get $1 \leq \dim \mathfrak{t} \leq 2m$, where m is the number of admissible labels.

Remark 3.3.0.4. This paper does not address extensions where $[v_i, v_j] \in \mathfrak{t}$ since these do not seem to intuitively arise from graph properties, nor do they contribute to the extension being 3-step nilpotent.

Example 3.3.0.5. From the graph in Figure 3.1, we obtain the following three-step nilpotent Lie algebra by using Construction 3.3.0.2. We define $\mathfrak{v}$ and $\mathfrak{z}$ as in Construction 3.2.0.1. Then, we define

$$\begin{aligned} \mathfrak{t} &= \operatorname{span}_{\mathbb{R}}\{\tau_{r,1}, \tau_{r,2}\} \text{ and} \\ \widehat{\mathfrak{n}} &= \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{t}. \end{aligned}$$

We have all of the brackets listed in Example 3.2.0.2 plus the following nonzero brackets:

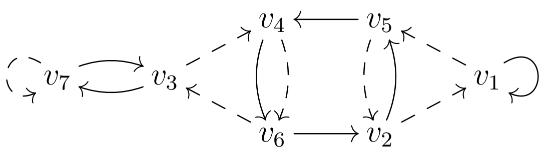
$$\begin{aligned} [v_1, z_r] &= -[v_5, z_r] &= \tau_{r,1} \\ [v_4, z_r] &= -[v_6, z_r] &= \tau_{r,2} \end{aligned}$$

All other brackets not defined by linearity or skew-symmetry are equal to zero.

Note that $\dim \widehat{\mathfrak{n}} = 9$ or 10, depending on if we choose $\tau_{r,1}$ and $\tau_{r,2}$ to be linearly dependent on each other, as per the choices allowed in Construction 3.3.0.2.

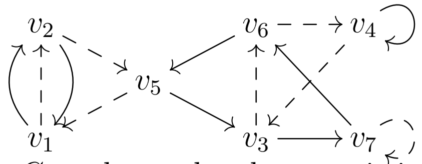
Example 3.3.0.6. The following is a three-step nilpotent extension of the Lie algebras associated with the Schreier graphs in Figure 2.2:

$\mathcal{G}_1$:



and

$\mathcal{G}_2$:



The solid lines correspond to the first generator in C_{pos}, denoted z_r because it is admissible, and the dotted lines correspond to the second generator, denoted z_b because it is inadmissible. If we delete the last column of bracket relations below, we have the two-step nilpotent Lie algebra as defined in Construction 3.2.0.1.

$$\begin{array}{lll|l}
\widehat{\mathfrak{n}_1}: & [v_1, v_2] = -z_b & [v_3, v_4] = z_b & [v_2, z_r] = \tau \\
 & [v_1, v_5] = z_b & [v_3, v_6] = -z_b & [v_4, z_r] = -\tau \\
 & [v_2, v_5] = z_r - z_b & [v_4, v_5] = -z_r & [v_5, z_r] = 0 \\
 & [v_2, v_6] = -z_r & [v_4, v_6] = z_r + z_b & [v_6, z_r] = 0 \\
 & & & \\
\widehat{\mathfrak{n}_2}: & [v_1, v_2] = z_b & [v_3, v_6] = z_b & [v_3, z_r] = \tau \\
 & [v_1, v_5] = -z_b & [v_3, v_7] = z_r & [v_5, z_r] = 0 \\
 & [v_2, v_5] = z_b & [v_4, v_6] = -z_b & [v_6, z_r] = -\tau \\
 & [v_3, v_4] = -z_b & [v_5, v_6] = -z_r & [v_7, z_r] = 0 \\
 & [v_3, v_5] = -z_r & [v_6, v_7] = -z_r &
\end{array}$$

All other brackets not defined by skew-symmetry or linearity are equal to zero.

Note that in $\widehat{\mathfrak{n}_1}$, $[v_4, v_6] = z_r + z_b$ because there are two edges connecting v_4 to v_6, one with label z_r and the other with label z_b. Similarly, $[v_3, v_7] = 0$ because there is a directed edge connecting v_3 to v_7 and one from v_7 to v_3 both with the label z_r. Also, note that in $\widehat{\mathfrak{n}_1}$, we could have defined $[v_5, z_r] = [-v_6, z_r] = \tau$ or set it equal to $\tau_2 \in \mathfrak{t}$ where $\tau_2 \neq 0, \tau$ by the variations allowed in the construction above. These would still produce three-step nilpotent extensions of $\mathfrak{n}_1$.

Proof. Proof of Thm. 3.3.0.1 (sufficiency):

$$\text{Define } \epsilon^{r_k}_{i,j} = \begin{cases} 1 \quad , & \text{if there is a } z_{r_k}\text{-edge connecting } v_i \text{ to } v_j \\ -1 \quad , & \text{if there is a } z_{r_k}\text{-edge connecting } v_j \text{ to } v_i \\ 0 \quad , & \text{otherwise} \end{cases} ,$$

and similarly define $\epsilon^{b_\ell}_{i,j}$. We proceed by induction on the number of admissible labels. Assume that the Schreier graph has only one admissible label z_r, and the inadmissible labels, if any exist, are denoted z_{b_ℓ}, $\ell = 1, \ldots, n$. If we pick any three vertices from the graph, say v_1, v_2, v_3, then the following possibilities occur for the Jacobi identity on those three vertices:

Case 1: There are no edges labeled z_r connecting v_1, v_2, or v_3, in which case the Jacobi identity will be satisfied because

$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]]$

$= [v_1, \epsilon^r_{2,3} z_r] + \sum_{\ell=1}^n [v_1, \epsilon^{b_\ell}_{2,3} z_{b_\ell}] + [v_2, \epsilon^r_{3,1} z_r] + \sum_{\ell=1}^n [v_2, \epsilon^{b_\ell}_{3,1} z_{b_\ell}] + [v_3, \epsilon^r_{1,2} z_r] + \sum_{\ell=1}^n [v_3, \epsilon^{b_\ell}_{1,2} z_{b_\ell}]$

by linearity of the bracket

$= [v_1, 0] + 0 + [v_2, 0] + 0 + [v_3, 0] + 0$ by Equation 3.3.0.2.4 and definition of $\epsilon^r_{i,j}$

$= 0.$

Note that by the linearity of the Lie bracket, we can always take the Jacobi identity and separate the brackets containing z_{b_ℓ} terms, which will equal zero by Equation 3.3.0.2.4, so we only need to consider the Jacobi identity in relation to brackets containing z_{r_k} terms.

Case 2: Without loss of generality, there is precisely one z_r-edge connecting v_1 to v_2, which implies that v_3 is not contained in the admissible path with label z_r. In this case,

$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]]$

$= [v_1, 0] + [v_2, 0] + [v_3, z_r]$ by Equation 3.3.0.2.4 and definition of $\epsilon^r_{i,j}$

$= 0$ by Equation 3.3.0.2.3.

Case 3a: There are precisely two edges labeled z_r between the vertices v_1, v_2, v_3. The first way that this may occur is with a closed same-label 2-path. Without loss of generality, assume there is a closed same-label 2-path with label z_r between the vertices v_1 and v_2. This means that $[v_1, v_2] = 0$ by the definition of the Lie bracket in Equation 3.2.0.1.1. Therefore, the Jacobi identity is satisfied because

$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]]$

$= [v_1, 0] + [v_2, 0] + [v_3, 0]$ by Equation 3.3.0.2.4 and 3.2.0.1.1

$= 0.$

Case 3b: The second way that two edges labeled z_r may appear is, without loss of generality, one edge connects v_1 to v_2 and the other from v_2 to v_3. Since there is no z_r-edge connecting v_3 to v_1, this implies that the path with labels z_r must be an admissible 4-path so $[v_1, z_r] = -[v_3, z_r]$ by Equation 3.3.0.2.1. Again the Jacobi identity is satisfied because

$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]]$

$= [v_1, z_r] + [v_2, 0] + [v_3, z_r]$ by Equation 3.3.0.2.4 and definition of $\epsilon^r_{i,j}$

$= 0$ by Equation 3.3.0.2.1.

Case 4: There are three edges labeled z_r connecting v_1 to v_2 to v_3 back to v_1. So the path here is an admissible 3-path with label z_r. The Jacobi equation becomes

$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]]$

$= [v_1, z_r] + [v_2, z_r] + [v_3, z_r]$ by Equation 3.3.0.2.4 and definition of $\epsilon^r_{i,j}$

$= 0$ by Equation 3.3.0.2.2.

These four cases cover all possibilities because of the properties of a Schreier graph discussed in Remark 2.1.3.6. Therefore, no matter which three vertices we pick in the graph and by the linearity of the Lie bracket, the Jacobi identity is always satisfied, making $\widehat{\mathfrak{n}}$ a Lie algebra.

Now using induction, assume that we have a Lie algebra associated with a graph with admissible labels, $z_{r_1}, \ldots, z_{r_m}$, and inadmissible labels, $z_{b_1}, \ldots, z_{b_n}$. If we add an additional admissible label $z_{r_{m+1}}$ in C_{pos}, then the Jacobi identity for any three vertices becomes

$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]]$

$= [v_1, \epsilon^{r_{m+1}}_{2,3} z_{r_{m+1}}] + \sum_{k=0}^{m} [v_1, \epsilon^{r_k}_{2,3} z_{r_k}] + [v_2, \epsilon^{r_{m+1}}_{3,1} z_{r_{m+1}}] + \sum_{k=0}^{m} [v_2, \epsilon^{r_k}_{3,1} z_{r_k}] + [v_3, \epsilon^{r_{m+1}}_{1,2} z_{r_{m+1}}] + \sum_{k=0}^{m} [v_3, \epsilon^{r_k}_{1,2} z_{r_k}]$ by Equation 3.3.0.2.4 and linearity of the bracket

$= ([v_1, \epsilon^{r_{m+1}}_{2,3} z_{r_{m+1}}] + [v_2, \epsilon^{r_{m+1}}_{3,1} z_{r_{m+1}}] + [v_3, \epsilon^{r_{m+1}}_{1,2} z_{r_{m+1}}]) + \sum_{k=0}^{m} ([v_1, \epsilon^{r_k}_{2,3} z_{r_k}] + [v_2, \epsilon^{r_k}_{3,1} z_{r_k}] + [v_3, \epsilon^{r_k}_{1,2} z_{r_k}])$

$= [v_1, \epsilon_{2,3}^{r_{m+1}} z_{r_{m+1}}] + [v_2, \epsilon_{3,1}^{r_{m+1}} z_{r_{m+1}}] + [v_3, \epsilon_{1,2}^{r_{m+1}} z_{r_{m+1}}] + 0$ by the induction hypothesis. $= 0$ because the proof of the base case of the induction proof showed that the Jacobi identity is satisfied for any single admissible label. □

Proof. Proof of Thm. 3.3.0.1 (necessity): Assume now that the Schreier graph $\mathcal{G}$ has no admissible labels in C_{pos}. This means that for each label z_{b_ℓ}, at least one of the following occur:

1. Each closed same-label path of label z_{b_ℓ} is of length 1 or 2.
2. There are at least two closed same-label paths of length 3 or 4, with label z_{b_ℓ}.
3. There exists a closed same-label path with label z_{b_ℓ} that is of length q, $q \geq 5$.

We continue by induction on the number of inadmissible labels z_{b_ℓ} in the Schreier graph. Assume that $\mathcal{G}$ only has one inadmissible label z_b.

Case 1: If each closed same-label path in $\mathcal{G}$ with label z_b is of length 1 or 2, then $[v_i, v_j] = 0$ for all $v_i, v_j \in \mathfrak{v}$ by how the Lie bracket is defined in Equation 3.2.0.1.1. Therefore, $\dim \mathfrak{z} = 0 \implies \dim \mathfrak{t} = 0$ so there does not exist a three-step nilpotent extension of $\mathfrak{n}$.

Case 2: Assume that $\mathcal{G}$ has at least two closed paths of length 3 or 4, with edges labeled z_b. Let v_i be a vertex in one of these paths and (v_j, v_k) be an edge in one of the other paths of length 3 or 4. Note that these two paths will not have any vertices in common by Remark 2.1.3.6. Because we are assuming that $\mathfrak{n}$ is a Lie algebra and only considering when there is a 3-step nilpotent extension, we may assume that the Jacobi identity is satisfied for all $v \in \mathfrak{v}$. Therefore,

$$[v_i, [v_j, v_k]] + [v_j, [v_k, v_i]] + [v_k, [v_i, v_j]] = 0$$

$$\implies [v_i, z_b] + [v_j, 0] + [v_k, 0] = 0 \text{ (by Equation 3.2.0.1.1)}$$

$$\implies [v_i, z_b] = 0 \text{ for all } v_i \text{ in the closed path.} \tag{3.3.0.6.1}$$

Since this was for an arbitrary v_i in a path of length 3 or 4, we can conclude that $[v_i, z_b] = 0$ for all v_i in any path of length 3 or 4. Now, let v_i be another vertex on this graph not contained in a closed path of length 3 or 4, and again let (v_j, v_k) be an edge in one of the closed paths of length 3 or 4. Then,

$$[v_i, [v_j, v_k]] + [v_j, [v_k, v_i]] + [v_k, [v_i, v_j]] = 0$$

$$\implies [v_i, z_b] + [v_j, 0] + [v_k, 0] = 0 \text{ (by Equation 3.2.0.1.1)}$$

$$\implies [v_i, z_b] = 0 \ \forall \, v_i \text{ not in the closed path of length 3 or 4.} \tag{3.3.0.6.2}$$

Therefore, $[v_i, z_b] = 0$ for all $v_i \in \mathfrak{v}$ (by Equations 3.3.0.6.1 and 3.3.0.6.2), which implies that $\dim \mathfrak{t} = 0$ so a three-step extension of $\mathfrak{n}$ of the type assumed does not exist.

Case 3: Assume that $\mathcal{G}$ has a closed path of length q, $q \geq 5$, with edges labeled z_b. Let the successive vertices of this closed path be $(v_0, v_1, \ldots, v_{q-1}, v_0)$. Because $\mathfrak{n}$ is a Lie algebra, we assume that the Jacobi identity is satisfied for $v_i, v_{(i+2) \bmod q}$, and $v_{(i+3) \bmod q}$. This implies that $\forall i = 0, \ldots, q-1$,

$$[v_i, [v_{(i+2) \bmod q}, v_{(i+3) \bmod q}]] + [v_{(i+2) \bmod q}, [v_{(i+3) \bmod q}, v_i]] + [v_{(i+3) \bmod q}, [v_i, v_{(i+2) \bmod q}]] = 0$$

$\implies [v_i, z_b] + [v_{(i+2) \bmod q}, 0] + [v_{(i+3) \bmod q}, 0] = 0$ because two nonconsecutive points in a closed path with labels z_b on a Schreier graph cannot have a z_b-edge connecting them.

$$\implies [v_i, z_b] = 0 \ \forall i = 0, \ldots, q-1. \tag{3.3.0.6.3}$$

Now let v_j be a vertex not in this closed path of length q. Then

$$[v_j, [v_0, v_1]] + [v_0, [v_1, v_j]] + [v_1, [v_j, v_0]] = 0$$

$$\implies [v_j, z_b] + [v_0, 0] + [v_1, 0] = 0 \text{ (by Equation 3.2.0.1.1)}$$

$$\implies [v_j, z_b] = 0 \ \forall \, v_j \text{ not in the closed path of length } q. \tag{3.3.0.6.4}$$

Therefore, $[v_j, z_b] = 0$ for all $v_j \in \mathfrak{v}$ (by Equations 3.3.0.6.3 and 3.3.0.6.4), which implies that $\dim \mathfrak{t} = 0$ so a three-step extension of $\mathfrak{n}$ does not exist.

Now, assume that $\mathcal{G}$ has inadmissible labels $z_{b_1}, \ldots, z_{b_n}$, and also assume that a three-step extension of $\mathfrak{n}$ does not exist, i.e. $[v_i, z_{b_\ell}] = 0 \ \forall v_i \in \mathfrak{v}$ and $\forall \ell = 1, \ldots, n$. Now if we add an inadmissible label $z_{b_{n+1}} \in C_{pos}$, we see that for any $v_1, v_2, v_3 \in \mathfrak{v}$,

$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$

$\implies ([v_1, \epsilon_{2,3}^{b_{n+1}} z_{b_{n+1}}] + [v_2, \epsilon_{3,1}^{b_{n+1}} z_{b_{n+1}}] + [v_3, \epsilon_{1,2}^{b_{n+1}} z_{b_{n+1}}]) + \sum_{\ell=0}^{n} ([v_1, \epsilon_{2,3}^{b_\ell} z_{b_\ell}] + [v_2, \epsilon_{3,1}^{b_\ell} z_{b_\ell}] + [v_3, \epsilon_{1,2}^{b_\ell} z_{b_\ell}]) = 0$ by linearity of the bracket

$\implies [v_1, \epsilon_{2,3}^{b_{n+1}} z_{b_{n+1}}] + [v_2, \epsilon_{3,1}^{b_{n+1}} z_{b_{n+1}}] + [v_3, \epsilon_{1,2}^{b_{n+1}} z_{b_{n+1}}] = 0$ by induction hypothesis

$\implies [v_i, z_{b_{n+1}}] = 0 \ \forall v_i \in \mathfrak{v}$ because the proof of the base case of this induction hypothesis showed that this is the result if the Jacobi identity is satisfied for any inadmissible label. $\square$

Proof. Proof of Thm. 3.3.0.1 (nilpotency): $[\widehat{\mathfrak{n}}, \widehat{\mathfrak{n}}] = \mathfrak{z} \oplus \mathfrak{t}$ and $\widehat{\mathfrak{n}}^{(2)} = [\widehat{\mathfrak{n}}, [\widehat{\mathfrak{n}}, \widehat{\mathfrak{n}}]] = \mathfrak{t} \subseteq Z(\widehat{\mathfrak{n}}) \implies \widehat{\mathfrak{n}}^{(3)} = 0$. Therefore, $\widehat{\mathfrak{n}}$ is a 3-step nilpotent Lie algebra. $\square$

Chapter 4

Geometry of Constructed Manifolds

In this chapter, we take $\mathfrak{n}$ (or $\widehat{\mathfrak{n}}$) to be the two-step (respectively three-step) nilpotent metric Lie algebra constructed from a Schreier graph $\mathcal{G}$ by Construction 3.2.0.1 (respectively Construction 3.3.0.2) with the orthonormal basis given by $V(\mathcal{G})\bigcup C_{pos}\bigcup\{\tau_{r_{k,1}},\ \tau_{r_{k,2}}\ :\ \ z_{r_k}$ is admissible and $\tau_{r_{k,i}}\ \neq 0\}$. We then take N and $\widehat{N}$ to be the simply connected Lie group associated with the Lie algebras $\mathfrak{n}$ and $\widehat{\mathfrak{n}}$, respectively with left-invariant metrics induced by the inner product on the metric Lie algebra as discussed in Section 2.3.2.

Definition 4.0.0.1. Let $\Gamma = \mathrm{span}_{\mathbb{Z}}\{v_1, \ldots, v_{|V(\mathcal{G})|}\}\cup\{\frac{1}{2}z_i :\ z_i \in C_{pos}\}$. Then $\exp(\Gamma)$ is a cocompact discrete subgroup of N, and we call call $\exp(\Gamma)\backslash N$ the *two-step nilmanifold associated with the Schreier graph* $\mathcal{G}$, with induced metric from N as discussed in Section 2.3.1.

4.1 The j-Operator

To discuss the geometry of the two-step nilmanifolds constructed from a Schreier graph, we look at the following operator on the Lie algebra.

Definition 4.1.0.1. [9] Given a two-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{v}\oplus\mathfrak{z}$, where $\mathfrak{z} = [\mathfrak{n},\mathfrak{n}]$ and where $\mathfrak{v}$ and $\mathfrak{z}$ are inner product spaces, we can define the *j-operator* by $j : \mathfrak{z} \to so(\mathfrak{v})$ given by $j(z)(v) = (ad\ v)^*z$ where $(ad\ v)(w) = [v, w]$ and * denotes the adjoint operator with respect to the given inner product. In other words $j(z)v$ is the unique element in $\mathfrak{v}$ such that

$$< j(z)v, w >=< z, [v, w] > \text{ for all } w \text{ in } \mathfrak{v}.$$

Because the Lie algebra is isomorphic to the tangent space at the identity of the associated Lie group, knowing the j-operator on the Lie algebra gives us the ability to examine the curvature of the related nilmanifold.

Theorem 4.1.0.2. *The j-operator on the 2-step nilpotent metric Lie algebra, $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, associated with a Schreier graph by Construction 3.2.0.1 is given by, $\forall z \in \mathfrak{z}$ and $\forall v \in \mathfrak{v}$,*

$$j(z)v = \alpha(z)(v) - \alpha(z^{-1})(v).$$

Proof. Fix basis elements $v \in \mathfrak{v}$ and $z \in \mathfrak{z}$. Let w be a basis element in $\mathfrak{v}$. Then,

$$< j(z)v, w > =< z, [v, w] >$$

$$= \begin{cases} < z, z >= 1 & , \quad \text{if } w = \alpha(z)(v) \text{ and } \alpha(z)(v) \neq \alpha(z^{-1})(v) \\ < z, -z >= -1 & , \quad \text{if } w = \alpha(z^{-1})(v) \text{ and } \alpha(z)(v) \neq \alpha(z^{-1})(v) \\ < z, z - z >= 0 & , \quad \text{if } w = \alpha(z)(v) = \alpha(z^{-1})(v) \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Recall from Remark 2.1.3.6 that this covers all cases that can occur on a Schreier graph. On the other hand,

$$< \alpha(z)(v) - \alpha(z^{-1})(v), w >=< \alpha(z)(v), w > - < \alpha(z^{-1})(v), w >$$

$$= \begin{cases} 1 - 0 = 1 & , \quad \text{if } w = \alpha(z)(v) \text{ and } \alpha(z)(v) \neq \alpha(z^{-1})(v) \\ 0 - 1 = -1 & , \quad \text{if } w = \alpha(z^{-1})(v) \text{ and } \alpha(z)(v) \neq \alpha(z^{1})(v) \\ < 0, w >= 0 & , \quad \text{if } w = \alpha(z)(v) = \alpha(z^{-1})(v) \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Since this is true for any basis elements w in $\mathfrak{v}$ and by the uniqueness and linearity of the inner product, this implies that $j(z)v = \alpha(z)(v) - \alpha(z^{-1})(v)$. □

4.2 Curvature of Two-Step Nilmanifolds

Given a two-step nilmanifold associated with a Schreier graph, we obtain formulas for covariant derivative, the curvature tensor, sectional curvature, and the Ricci tensor from P. Eberlein, [9]. These formulas are given in terms of the Lie bracket and the j-operator. We consider two of these formulas below, which can be simplified in terms of properties of the Schreier graph including the group action α on the vertices of the Schreier graph, thereby giving us a way to calculate the curvature of the constructed nilmanifold by properties of the Schreier graph.

4.2.1 Sectional Curvature

Proposition 4.2.1.1. *Let* $\Gamma\backslash N$ *be the two-step nilmanifold associated with the Schreier graph* $\mathcal{G}$. *Let* Π *be a 2-dimensional subspace of* T_pN *that is spanned by orthonormal elements* $X_p, Y_p \in \mathfrak{n}$. *Then the sectional curvature* $K(\Pi) = K(X, Y)$ *gives us the following:*

a) If v_i, v_k *are orthonormal basis elements of* $\mathfrak{v}$, *then*

$$K(v_i, v_k) = -\frac{3}{4}(\#\textit{ of edges connecting } v_i \textit{ to } v_k \textit{ or } v_k \textit{ to } v_i),$$

where the edges counted must be in a closed same-label path of length > 2.

b) If $v \in \mathfrak{v}$ *and* $z \in \mathfrak{z}$ *are orthonormal basis elements, then*

$$K(v, z) = \begin{cases} \frac{1}{2} \ , & \textit{if } v \textit{ is a vertex on a closed path with label } z \textit{ of length} > 2 \\ 0 \ , & \textit{otherwise.} \end{cases}$$

c) If z, z' *are orthonormal elements of* $\mathfrak{z}$, *then*

$$K(z, z') = 0.$$

Remark 4.2.1.2. In this proof, when we say an edge *between* v_i and v_k, this will ignore direction of the edge, i.e. the edge could be from will mean an edge from v_i to v_k or an edge from v_k to v_i or both.

Proof. a) From Eberlein [9, Equation 2.4], we get

$$\begin{aligned} K(v_i, v_k) &= -\tfrac{3}{4}|[v_i, v_k]|^2. \\ &= -\tfrac{3}{4}\left|\textstyle\sum_{p=1}^{|C_{pos}|}(\epsilon_p - \epsilon'_p)z_p\right|^2 \text{ (by Eqn. 3.2.0.1.1)} \end{aligned}$$

For each p, $\epsilon_p - \epsilon'_p = 0$ if there does not exist an edge with label z_p between v_i and v_k or if there exists an edge with label z_p between v_i and v_k that is part of a closed path of label z_p of length 1 or 2. Otherwise $\epsilon_p - \epsilon'_p = \pm 1$ because there will exist an edge of label z_p between v_i and v_k that is part of a closed path of label z_p of length greater than 2. In this case, $|(\epsilon_p - \epsilon'_p)z_p|^2 = 1$. When we sum over all $z_p \in C_{pos}$, we get $K(v_i, v_k) = -\frac{3}{4}(\#$ of edges between v_i and v_k that are part of closed same-label paths of length $> 2)$.

b) From Eberlein [9, Equation 2.4], we get

$$\begin{aligned} K(v,z) &= \tfrac{1}{4}|j(z)(v)|^2 \\ &= \tfrac{1}{4}|\alpha(z)(v) - \alpha(z^{-1})(v)|^2 \text{ (by Thm. 4.1.0.2)} \end{aligned}$$

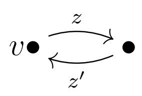
If v is a vertex on a closed path with label z of length > 2, then $\alpha(z)(v) \neq \alpha(z^{-1})(v)$ by Remark 2.1.3.6; and therefore, $\alpha(z)(v)$ and $\alpha(z^{-1})(v)$ will be distinct orthonormal basis elements of $\mathfrak{v} \subseteq \mathfrak{n}$ so $\frac{1}{4}|\alpha(z)(v) - \alpha(z^{-1})(v)|^2 = \frac{1}{4}(2) = \frac{1}{2}$. If v is a vertex on a closed path with label z of length ≤ 2, then $\alpha(z)(v) = \alpha(z^{-1})(v)$ by Remark 2.1.3.6, and $|\alpha(z)(v) - \alpha(z^{-1})(v)|^2 = 0$.

c) This follows directly from [9, Equation 2.4]. □

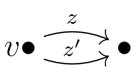
4.2.2 Ricci Curvature Tensor

Definition 4.2.2.1. A pair of edges beginning at vertex v with labels z, z' will be of *Type A* if $\alpha(z)v = \alpha((z')^{-1})v$ and will be of *Type B* if $\alpha(z)v = \alpha(z')(v)$, i.e.

Type A Pair:



Type B Pair:



Proposition 4.2.2.2. *Let* $\Gamma\backslash N$ *be the two-step nilmanifold associated with the Schreier graph* $\mathcal{G}$. *The Ricci tensor gives us the following for all orthonormal basis elements* $v \in \mathfrak{v}, z, z' \in \mathfrak{z}$:

a) *Ric* $(v, z) = 0$

b) Ric $(v, v) = -1 + (\#$ *of closed same-label paths of length 1 or 2 at* $v)$

c) Ric $(z, z) = \frac{1}{2}[|V(\mathcal{G})| - (\#$ *of closed paths of length 1 or 2 of label* $z)]$

d) Ric $(z, z') = -\frac{1}{2}[(\#$ *of type A pairs with labels* $z, z')$

$-(\#$ *of type B pairs with labels* $z, z')]$

Proof. a) This follows directly from [9, Prop. 2.5].

b) Given an orthonormal basis $\{z_1, \ldots, z_m\}$ of $\mathfrak{z}$, [9, Prop. 2.5] states that $T|_{\mathfrak{v}} = \frac{1}{2}\sum_{\ell=1}^{m} j(z_\ell)^2$ where Ric $(v_i, v_k) =< Tv_i, v_k >$, $\forall v_i, v_k \in \mathfrak{v}$. Therefore,

$$\begin{aligned}\text{Ric } (v, v) &= < \tfrac{1}{2}\textstyle\sum_{\ell=1}^{m} j(z_\ell)^2 v, v > \\ &= \tfrac{1}{2}\textstyle\sum_{\ell=1}^{m} < \alpha(z_\ell)^2 v + \alpha(z_\ell^{-1})^2 v - 2v, v > \\ &= \tfrac{1}{2}[-2 + \textstyle\sum_{\ell=1}^{m} < \alpha(z_\ell)^2 v, v > + < \alpha(z_\ell^{-1})^2 v, v >]\end{aligned}$$

Note that

$$\begin{aligned}< \alpha(z_\ell)^2 v, v > &= < \alpha(z_\ell^{-1})^2 v, v > \\ &= \begin{cases} 1 & , \text{ if } \quad \exists 1\text{- or 2-path of label } z_\ell \text{ at } v \\ 0 & , \text{ if } \quad \text{otherwise.} \end{cases}\end{aligned}$$

Summing over all of z_ℓ gives us the number of same-label paths of length 1 or 2. This implies that

$$\begin{aligned}\text{Ric } (v, v) &= \tfrac{1}{2}[-2 + 2\textstyle\sum_{\ell=1}^{m} < \alpha(z_\ell)^2 v, v >] \\ &= -1 + \textstyle\sum_{\ell=1}^{m} < \alpha(z_\ell)^2 v, v > \\ &= -1 + (\#\text{ of closed same-label paths of length 1 or 2 at } v)\end{aligned}$$

c) From [9, Prop. 2.5], we get

$$\begin{aligned}\text{Ric } (z, z) &= -\tfrac{1}{4}\text{trace}(j(z)^2) \\ &= -\tfrac{1}{4}\text{trace}(\alpha(z)^2 + \alpha(z^{-1})^2 - 2Id) \\ &= \tfrac{1}{2}|V(\mathcal{G})| - \tfrac{1}{4}\text{trace}(\alpha(z)^2 + \alpha(z^{-1})^2) \\ &= \tfrac{1}{2}|V(\mathcal{G})| - \tfrac{1}{4}\textstyle\sum_i(\gamma_{i,i} + \gamma'_{i,i})\end{aligned}$$

where $(\gamma_{i,j})$ is the matrix representation of $\alpha(z)^2$ and $(\gamma'_{i,j})$ is the matrix representation of $\alpha(z^{-1})^2$ with respect to the basis $\{v_1, \ldots, v_{|V(\mathcal{G})|}\}$. For each i,

$$\gamma_{i,i} = \gamma'_{i,i} = \begin{cases} 1 & , \quad \text{if } \exists \text{ closed path of label } z \text{ of length 1 or 2 that includes } v_i \\ 0 & , \quad \text{otherwise (by Remark 2.1.3.6).} \end{cases}$$

Summing over all i, we see that

$$\begin{aligned} \textstyle\sum_i (\gamma_{i,i} + \gamma'_{i,i}) &= 2(\#\text{ of closed paths of length 1 or 2 of label } z) \\ \Longrightarrow \text{ Ric } (z,z) &= \tfrac{1}{2}[|V(\mathcal{G})| - (\#\text{ of closed paths of length 1 or 2 of label } z)]. \end{aligned}$$

d) From [9, Prop. 2.5] and Theorem 4.1.0.2 in this thesis, we get

$$\begin{aligned} \text{Ric } (z, z') &= -\tfrac{1}{4}\text{trace}(j(z) \circ j(z')) \\ &= -\tfrac{1}{4}\text{trace}((\alpha(z) - \alpha(z^{-1})) \circ (\alpha(z') - \alpha((z')^{-1}))) \\ &= -\tfrac{1}{4}\text{trace}(\alpha(z)\alpha(z') - \alpha(z)\alpha((z')^{-1}) \\ &\qquad -\alpha(z^{-1})\alpha(z') + \alpha(z^{-1})\alpha((z')^{-1})) \end{aligned}$$

Let $\gamma^1_{i,j}, \ldots, \gamma^4_{i,j}$ be the matrix representations of $\alpha(z)\alpha(z')$, $\alpha(z)\alpha((z')^{-1})$, $\alpha(z^{-1})\alpha(z')$, and $\alpha(z^{-1})\alpha((z')^{-1})$ respectively, all with respect to the basis $\{v_1, \ldots, v_{|V(\mathcal{G})|}\}$. Note that for each i,

$$\gamma^4_{i,i} = \begin{cases} 1 & , \quad \text{if } \exists \text{ Type A pair at } v_i \text{ with labels } z, z' \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Similarly,

$$\gamma^3_{i,i} = \begin{cases} 1 & , \quad \text{if } \exists \text{ Type B pair at } v_i \text{ with labels } z, z' \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Next, note that

$$\begin{aligned} \exists \text{ Type A pair at } v_i \text{ with labels } z, z' &\iff \alpha(z^{-1})\alpha((z')^{-1})v_i = v_i \\ &\iff \alpha((z')^{-1})v_i = \alpha(z)v_i \\ &\iff v_i = \alpha(z')\alpha(z)v_i \end{aligned}$$

$$\begin{aligned} &\iff (\alpha(z)v_i) = \alpha(z)\alpha(z')(\alpha(z)v_i) \\ &\iff v_k = \alpha(z)\alpha(z')\alpha(z)v_k \end{aligned}$$

for some v_k in $V(\mathcal{G})$ since $\alpha(z)$ is a bijection on $V(\mathcal{G})$. This implies that

$$\sum_i \gamma^4_{i,i} = \sum_k \gamma^1_{k,k}.$$

Similarly,

$$\begin{aligned} \exists \text{ Type B pair at } v_i \text{ with labels } z, z' &\iff \alpha(z^{-1})\alpha(z')v_i = v_i \\ &\iff v_k = \alpha(z)\alpha((z')^{-1})\alpha(z)v_k \end{aligned}$$

for some v_k in $V(\mathcal{G})$ since $\alpha(z)$ is a bijection on $V(\mathcal{G})$. This implies that

$$\sum_i \gamma^3_{i,i} = \sum_k \gamma^2_{k,k}.$$

Now we see that

$$\begin{aligned} \text{Ric } (z, z') &= -\tfrac{1}{4}(2\textstyle\sum_i \gamma^4_{i,i} - 2\sum_i \gamma^3_{i,i}) \\ &= -\tfrac{1}{2}[(\#\text{ of type A pairs with labels } z, z') \\ &\qquad -(\#\text{ of type B pairs with labels } z, z')] \end{aligned}$$

□

4.3 Lie Algebras Associated with a Gassmann-Sunada Triple

The Constructions from Chapter 3 do not require us to begin with a Gassmann-Sunada triple, but some interesting results occur when we look at the Lie algebras associated with a pair of Schreier graphs of a Gassmann-Sunada triple. Recall from Remark 3.2.0.3 that in this paper, we will take the union of the set of vertices, the set of labels in C_{pos}, and the set $\{\tau_{r_{k,1}}, \tau_{r_{k,2}} : z_{r_k} \text{ is admissible and } \tau_{r_{k,\ell}} \neq 0\}$ to be an orthonormal basis for $\widehat{\mathfrak{n}} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{t}$.

Recall from Remark 2.1.3.4, that there exists a group action of G on the vertices of the Schreier graph $V(\mathcal{G})$, and we denoted this action by α. Because $\mathfrak{v}$ is the vector space with orthonormal basis $V(\mathcal{G})$, we can define a group representation of G on $\mathfrak{v}$ by extending by linearity. We will also denote this group representation by α.

Proposition 4.3.0.1. *[11, Lecture 4] Let (G, H_1, H_2) be a Gassmann-Sunada triple and $\mathcal{G}_1$ and $\mathcal{G}_2$ the pair of Schreier graphs associated with this triple. For $i = 1, 2$, let α_i be the group representation of G on $\mathfrak{v}_i$ as in Remark 2.1.3.4, which will be unitary under the assumed metric given in Remark 3.2.0.3. Because H_1 and H_2 are almost conjugate subgroups of G, the representations α_1 and α_2 are unitarily equivalent, i.e. there exists a unitary operator $T : \mathfrak{v}_1 \to \mathfrak{v}_2$ such that $T(\alpha_1(x)(H_1g)) = \alpha_2(x)(T(H_1g))$ for all $x \in G$ and for all $H_1g \in \mathfrak{v}_1$. This operator T is referred to as the transplantation or intertwining operator. For more information, see [11].*

Theorem 4.3.0.2. *Starting with a pair of Schreier graphs coming from a Gassmann-Sunada triple, let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be the associated pair of two-step nilpotent metric Lie algebras determined by Construction 3.2.0.1 with j-operators j_1 and j_2, respectively (denoted as the pair $(\mathfrak{n}_i, j_i)$ from now on). Let T be the unitary transplantation operator guaranteed by the Gassmann-Sunada condition. Then,*

$$T(j_1(z)v) = j_2(z)(Tv)\ \forall z \in \mathfrak{z}_1 \text{ and } \forall v \in \mathfrak{v}_1.$$

Proof.

$$\begin{aligned} T(j_1(z)v) &= T(\alpha_1(z)(v) - \alpha_1(z^{-1})(v)) \\ &= T(\alpha_1(z)(v)) - T(\alpha_1(z^{-1})(v)) \\ &= \alpha_2(z)(Tv) - \alpha_2(z^{-1})(Tv) \\ &= j_2(z)(Tv) \end{aligned}$$

□

Corollary 4.3.0.3. *Starting with a pair of Schreier graphs coming from a Gassmann-Sunada triple, let $(\mathfrak{n}_1, j_1)$ and $(\mathfrak{n}_2, j_2)$ be the associated pair of two-step nilpotent metric Lie algebras determined by Construction 3.2.0.1 with the metric defined in Remark 3.2.0.3. Then, $(\mathfrak{n}_1, j_1)$ is isometric to $(\mathfrak{n}_2, j_2)$.*

Proof. Using [11, Lect. 8, Prop. 4.6], we get $(\mathfrak{n}_1, j_1)$ is isomorphic to $(\mathfrak{n}_2, j_2)$ by $\widetilde{T} := T \oplus Id$. Also $\forall v, v' \in \mathfrak{v}_1$ and $\forall z, z' \in \mathfrak{z}_1$,
$< (v,z),(v',z') >_1 = < v, v' >_1 + < z, z' >_1$
$= < T(v), T(v') >_2 + < Id(z), Id(z') >_2 = < \widetilde{T}(v,z), \widetilde{T}(v',z') >_2.$ □

While the pair of two-step nilpotent Lie algebras associated with a Gassmann-Sunada triple are always isometric, the three-step nilpotent Lie algebra extensions determined by Construction 3.3.0.2 need not be.

Theorem 4.3.0.4. *The pair of three-step nilpotent Lie algebras given in Example 3.3.0.6 from §3.3 are non-isometric.*

Proof. For the full proof, see Section 4.4 below. The idea of the proof is that we assume that there exists ϕ that is an isometry between $\widehat{\mathfrak{n}_1}$ and $\widehat{\mathfrak{n}_2}$. Then using the properties of Lie algebra isometries listed below, we obtain a contradiction. Therefore, the two Lie algebras are non-isometric.

1. $\phi : \mathfrak{v} \to \mathfrak{v}, \mathfrak{z} \to \mathfrak{z},$ and $\mathfrak{t} \to \mathfrak{t}$.
2. ϕ has to preserve the ascending central series.
3. The columns (and rows) of the matrix ϕ must be orthonormal to each other.
4. $\phi([x,y]_1) = [\phi(x), \phi(y)]_2$ for all $x, y \in \widehat{\mathfrak{n}_1}$. □

Note: Because there is a choice in constructing the 3-step nilpotent Lie algebra, a similar argument shows that the following variations on $\widehat{\mathfrak{n}_2}$ are also non-isometric to $\widehat{\mathfrak{n}_1}$:

1. Interchanging τ and $-\tau$, i.e. $[v_3, z_r] = -\tau$ and $[v_6, z_r] = \tau$.
2. Switching the τ and 0 components, i.e. $[v_3, z_r] = 0$, $[v_5, z_r] = \tau$, $[v_6, z_r] = 0$, and $[v_7, z_r] = -\tau$.
3. Switching the τ and 0 components and then interchanging τ and $-\tau$.

4.4 Proof of Theorem 4.3.0.4:

Let $\widehat{\mathfrak{n}_1}$ and $\widehat{\mathfrak{n}_2}$ be the three step nilpotent Lie algebras given in Example 3.3.0.6. Assume that $\phi : \widehat{\mathfrak{n}_1} \to \widehat{\mathfrak{n}_2}$ is an isometry, where the entries of the matrix ϕ with respect to the orthonormal basis $\{v_1, \ldots, v_7, z_r, z_b, t\}$ for both $\widehat{\mathfrak{n}_1}$ and $\widehat{\mathfrak{n}_2}$ are $(\phi_{i,j})_{i,j=1}^{10}$. We begin by computing the ascending central series of the two Lie algebras, obtaining the following:

(4.4.0.0.1)

$$Z(\widehat{\mathfrak{n}_1}) := \{w \in \widehat{\mathfrak{n}_1} : [w, \widehat{\mathfrak{n}_1}] = 0\} = \mathrm{span}_{\mathbb{R}}\{v_1 + v_2 + \cdots + v_6, v_7, z_b, t\}$$

(4.4.0.0.2)

$$Z(\widehat{\mathfrak{n}_2}) := \{w \in \widehat{\mathfrak{n}_2} : [w, \widehat{\mathfrak{n}_2}] = 0\} = \mathrm{span}_{\mathbb{R}}\{v_1 + v_2 + v_5 + v_7, v_3 + v_4 + v_6, z_b, t\}$$

(4.4.0.0.3)

$$Z_2(\widehat{\mathfrak{n}_1}) := \{w \in \widehat{\mathfrak{n}_1} : [w, \widehat{\mathfrak{n}_1}] \subseteq Z(\widehat{\mathfrak{n}_1})\} = \mathrm{span}_{\mathbb{R}}\{v_1, v_2 + v_4, v_3, v_5 + v_6, v_7, z_r, z_b, t\}$$

(4.4.0.0.4)

$$Z_2(\widehat{\mathfrak{n}_2}) := \{w \in \widehat{\mathfrak{n}_2} : [w, \widehat{\mathfrak{n}_2}] \subseteq Z(\widehat{\mathfrak{n}_2})\} = \mathrm{span}_{\mathbb{R}}\{v_1, v_2, v_3 + v_6, v_4, v_5 + v_7, z_r, z_b, t\}$$

Next, we use the assumption that ϕ is an isometry to obtain the following properties about the matrix ϕ:

(4.4.0.0.5) ϕ an isometry $\implies$ the matrix ϕ is orthonormal

(4.4.0.0.6) $\phi : \mathfrak{t} \to \mathfrak{t} \implies \phi_{10,j} = \phi_{j,10} = 0$ for $j = 1, \ldots, 9$

(4.4.0.0.7) $\phi : \mathfrak{z} \to \mathfrak{z} \implies \phi_{8,j} = \phi_{j,8} = \phi_{9,j} = \phi_{j,9} = 0$ for $j = 1, \ldots, 7$

so that ϕ now is of the form

$$\left(\begin{array}{c|c|c} \text{A} & 0 & 0 \\ \hline 0 & \text{B} & 0 \\ \hline 0 & 0 & \text{C} \end{array}\right)$$

where A is of size 7×7, B is 2×2, and C is 1×1.

Finally, we use a combination of the above results along with the property that $\phi([x, y]_1) = [\phi(x), \phi(y)]_2$ for all $x, y \in \widehat{\mathfrak{n}_1}$ to obtain relations about the various entries in ϕ:

(4.4.0.0.8) $\phi_{10,10} = \pm 1$ (by 4.4.0.0.5 and 4.4.0.0.6)

(4.4.0.0.9) $\phi(z_b) \in Z(\widehat{\mathfrak{n}_2}) \implies \phi_{8,9} = 0$ (by 4.4.0.0.1 and 4.4.0.0.2)

(4.4.0.0.10) $\implies \phi_{9,9}, \phi_{8,8} \in \{\pm 1\}$ (by 4.4.0.0.5)

(4.4.0.0.11) and $\phi_{9,8} = 0$ (by 4.4.0.0.5)

(4.4.0.0.12) $\phi(v_7) \in Z(\widehat{\mathfrak{n}_2}) \implies \phi_{1,7} = \phi_{2,7} = \phi_{5,7} = \phi_{7,7}$

(4.4.0.0.13) and $\phi_{3,7} = \phi_{4,7} = \phi_{6,7}$ (by 4.4.0.0.1 and 4.4.0.0.2)

(4.4.0.0.14) $\phi(v_1) \in Z_2(\widehat{\mathfrak{n}_2}) \implies \phi_{3,1} = \phi_{6,1}$ and $\phi_{5,1} = \phi_{7,1}$ (by 4.4.0.0.3 and 4.4.0.0.4)

(4.4.0.0.15) $\phi(v_3) \in Z_2(\widehat{\mathfrak{n}_2}) \implies \phi_{3,3} = \phi_{6,3}$ and $\phi_{5,3} = \phi_{7,3}$ (by 4.4.0.0.3 and 4.4.0.0.4)

$$\phi(v_2 + v_4) \in Z_2(\widehat{\mathfrak{n}_2}) \implies \phi_{7,4} = \phi_{5,2} + \phi_{5,4} - \phi_{7,2} \tag{4.4.0.0.16}$$

(by 4.4.0.0.3 and 4.4.0.0.4)

$$\phi(v_5 + v_6) \in Z_2(\widehat{\mathfrak{n}_2}) \implies \phi_{7,6} = \phi_{5,5} + \phi_{5,6} - \phi_{7,5} \tag{4.4.0.0.17}$$

(by 4.4.0.0.3 and 4.4.0.0.4)

$$[\phi(v_2), \phi(z_r)] = \phi(t) = \phi_{10,10}t \implies \phi_{6,2} = \phi_{3,2} - \phi_{8,8}\phi_{10,10} \tag{4.4.0.0.18}$$

$$[\phi(v_4), \phi(z_r)] = \phi(-t) = -\phi_{10,10}t \implies \phi_{6,4} = \phi_{3,4} + \phi_{8,8}\phi_{10,10} \tag{4.4.0.0.19}$$

$$[\phi(v_5), \phi(z_r)] = \phi(0) = 0 \implies \phi_{3,5} = \phi_{6,5} \tag{4.4.0.0.20}$$

$$[\phi(v_6), \phi(z_r)] = \phi(0) = 0 \implies \phi_{3,6} = \phi_{6,6} \tag{4.4.0.0.21}$$

(row 3) $\cdot$ (row 6) $= 0$ (by 4.4.0.0.5)

$$\implies \phi_{3,4} = \phi_{3,2} - \phi_{8,8}\phi_{10,10} \tag{4.4.0.0.22}$$

$$\implies \phi_{6,4} = \phi_{3,2} \text{ (by 4.4.0.0.13, 4.4.0.0.14, 4.4.0.0.15, 4.4.0.0.18, } 4.4.0.0.19, 4.4.0.0.20, 4.4.0.0.21) \tag{4.4.0.0.23}$$

(row k) $\cdot$ (row 3) $=$ (row k) $\cdot$ (row 6) for $k = 1, 2, 4, 5, 7$ (by 4.4.0.0.5)

$$\implies \phi_{k,4} = \phi_{k,2} \text{ for } k = 1, 2, 4, 5, 7 \text{ (by 4.4.0.0.13, 4.4.0.0.14, } 4.4.0.0.15, 4.4.0.0.18, 4.4.0.0.19, 4.4.0.0.20, 4.4.0.0.21) \tag{4.4.0.0.24}$$

$$\implies \phi_{7,4} = \phi_{7,2} = 2\phi_{5,2} - \phi_{7,2} \text{ (by 4.4.0.0.16)}$$

$$\implies \phi_{5,2} = \phi_{5,4} = \phi_{7,2} = \phi_{7,4} \tag{4.4.0.0.25}$$

$[\phi(v_2), \phi(v_6)] = \phi(-z_r) = -\phi_{8,8}z_r,$ just looking at the z_r-coefficient

$$\implies \sum_{i<j} (\phi_{i,2}\phi_{j,6} - \phi_{j,2}\phi_{i,6})[v_i, v_j] = -\phi_{8,8}z_r$$

$$\begin{aligned} &\implies -(\phi_{3,2}\phi_{5,6} - \phi_{5,2}\phi_{3,6}) + (\phi_{3,2}\phi_{7,6} - \phi_{7,2}\phi_{3,6}) \\ &\quad - (\phi_{5,2}\phi_{6,6} - \phi_{6,2}\phi_{5,6}) - (\phi_{6,2}\phi_{7,6} - \phi_{7,2}\phi_{6,6}) = -\phi_{8,8} \end{aligned}$$

(4.4.0.0.26) $\implies \phi_{7,6} = \phi_{5,6} - \phi_{10,10}$ (by 4.4.0.0.18, 4.4.0.0.21, 4.4.0.0.25)

(4.4.0.0.27) $\implies \phi_{7,5} = \phi_{5,5} + \phi_{10,10}$ (by 4.4.0.0.17)

$[\phi(v_2), \phi(v_4)] = \phi(0) = 0,$ just looking at the z_b-coefficient

$$\begin{aligned} &\implies \sum_{i<j}(\phi_{i,2}\phi_{j,4} - \phi_{j,2}\phi_{i,4})[v_i, v_j] = 0 \\ &\implies (\phi_{1,2}\phi_{2,4} - \phi_{2,2}\phi_{1,4}) - (\phi_{1,2}\phi_{5,4} - \phi_{5,2}\phi_{1,4}) \\ &\quad + (\phi_{2,2}\phi_{5,4} - \phi_{5,2}\phi_{2,4}) - (\phi_{3,2}\phi_{4,4} - \phi_{4,2}\phi_{3,4}) \\ &\quad + (\phi_{3,2}\phi_{6,4} - \phi_{6,2}\phi_{3,4}) - (\phi_{4,2}\phi_{6,4} - \phi_{6,2}\phi_{4,4}) = 0 \end{aligned}$$

(4.4.0.0.28) $\implies \phi_{4,2} = \phi_{3,2} - 1/2\phi_{8,8}\phi_{10,10}$ (by 4.4.0.0.18, 4.4.0.0.22, 4.4.0.0.23, 4.4.0.0.24)

(row k) $\cdot$ (row 5) = (row k) $\cdot$ (row 7) for $k = 1, 2, 3, 4, 6$ (by 4.4.0.0.5)

(4.4.0.0.29) $\implies \phi_{k,5} = \phi_{k,6}$ for $k = 1, 2, 3, 4, 6$ (by 4.4.0.0.13, 4.4.0.0.14, $4.4.0.0.15, 4.4.0.0.25, 4.4.0.0.26, 4.4.0.0.27$)

$||$row 5$|| - 1 =$ (row 5) $\cdot$ (row 7) (by 4.4.0.0.5)

(4.4.0.0.30) $\implies \phi_{5,6} = \phi_{5,5} + \phi_{10,10}$ (by 4.4.0.0.13, 4.4.0.0.14, 4.4.0.0.15, $4.4.0.0.25, 4.4.0.0.26, 4.4.0.0.27$)

(4.4.0.0.31) $\implies \phi_{7,6} = \phi_{5,5}$ (by 4.4.0.0.26)

$$
\begin{aligned}
[\phi(v_5), \phi(v_6)] = \phi(0) = 0,&\ \text{just looking at the } z_b\text{-coefficient}\\
\implies &\sum_{i<j}(\phi_{i,5}\phi_{j,6} - \phi_{j,5}\phi_{i,6})[v_i, v_j] = 0\\
\implies &(\phi_{1,5}\phi_{2,6} - \phi_{2,5}\phi_{1,6}) - (\phi_{1,5}\phi_{5,6} - \phi_{5,5}\phi_{1,6})\\
&+ (\phi_{2,5}\phi_{5,6} - \phi_{5,5}\phi_{2,6}) - (\phi_{3,5}\phi_{4,6} - \phi_{4,5}\phi_{3,6})\\
&+ (\phi_{3,5}\phi_{6,6} - \phi_{6,5}\phi_{3,6}) - (\phi_{4,5}\phi_{6,6} - \phi_{6,5}\phi_{4,6}) = 0\\
\implies &\phi_{1,5} = \phi_{2,5} \text{ (by 4.4.0.0.20, 4.4.0.0.21, 4.4.0.0.29, 4.4.0.0.30)} \qquad (4.4.0.0.32)
\end{aligned}
$$

$$
\begin{aligned}
[\phi(v_4), \phi(v_5)] = \phi(-z_r) = -\phi_{8,8}z_r,&\ \text{just looking at the } z_b\text{-coefficient}\\
\implies &\sum_{i<j}(\phi_{i,4}\phi_{j,5} - \phi_{j,4}\phi_{i,5})[v_i, v_j] = -\phi_{8,8}z_r\\
\implies &(\phi_{1,4}\phi_{2,5} - \phi_{2,4}\phi_{1,5}) - (\phi_{1,4}\phi_{5,5} - \phi_{5,4}\phi_{1,5})\\
&+ (\phi_{2,4}\phi_{5,5} - \phi_{5,4}\phi_{2,5}) - (\phi_{3,4}\phi_{4,5} - \phi_{4,4}\phi_{3,5})\\
&+ (\phi_{3,4}\phi_{6,5} - \phi_{6,4}\phi_{3,5}) - (\phi_{4,4}\phi_{6,5} - \phi_{6,4}\phi_{4,5}) = 0\\
\implies &\phi_{1,2}\phi_{1,5} - \phi_{2,2}\phi_{1,5} - \phi_{1,2}\phi_{5,5} + \phi_{2,2}\phi_{5,5}\\
&= -\phi_{4,5}\phi_{8,8}\phi_{10,10} + \phi_{3,5}\phi_{8,8}\phi_{10,10} \text{ (by 4.4.0.0.19, 4.4.0.0.20,} \qquad (4.4.0.0.33)\\
&4.4.0.0.24, 4.4.0.0.32)
\end{aligned}
$$

$$
\begin{aligned}
[\phi(v_4), \phi(v_6)] = \phi(z_r + z_b) = \phi_{8,8}z_r + \phi_{9,9}z_b,&\ \text{just looking at the } z_b\text{-coefficient}\\
\implies &\sum_{i<j}(\phi_{i,4}\phi_{j,6} - \phi_{j,4}\phi_{i,6})[v_i, v_j] = -\phi_{8,8}z_r + \phi_{9,9}z_b\\
\implies &(\phi_{1,4}\phi_{2,6} - \phi_{2,4}\phi_{1,6}) - (\phi_{1,4}\phi_{5,6} - \phi_{5,4}\phi_{1,6})\\
&+ (\phi_{2,4}\phi_{5,6} - \phi_{5,4}\phi_{2,6}) - (\phi_{3,4}\phi_{4,6} - \phi_{4,4}\phi_{3,6})\\
&+ (\phi_{3,4}\phi_{6,6} - \phi_{6,4}\phi_{3,6}) - (\phi_{4,4}\phi_{6,6} - \phi_{6,4}\phi_{4,6}) = \phi_{9,9}
\end{aligned}
$$

$$\Longrightarrow \phi_{1,2}\phi_{1,5} - \phi_{2,2}\phi_{1,5} - \phi_{1,2}\phi_{5,5} + \phi_{2,2}\phi_{5,5}$$

(4.4.0.0.34) $= \phi_{1,2}\phi_{10,10} - \phi_{2,2}\phi_{10,10} - \phi_{4,5}\phi_{8,8}\phi_{10,10} + \phi_{3,5}\phi_{8,8}\phi_{10,10} + \phi_{9,9}$

(by 4.4.0.0.18, 4.4.0.0.20, 4.4.0.0.22, 4.4.0.0.24, 4.4.0.0.29, 4.4.0.0.30)

$[\phi(v_2), \phi(v_5)] = \phi(z_r - z_b) = \phi_{8,8}z_r - \phi_{9,9}z_b,$ just looking at the z_b-coefficient

$$\Longrightarrow \sum_{i<j}(\phi_{i,2}\phi_{j,5} - \phi_{j,2}\phi_{i,5})[v_i, v_j] = -\phi_{8,8}z_r - \phi_{9,9}z_b$$

$$\Longrightarrow (\phi_{1,2}\phi_{2,5} - \phi_{2,2}\phi_{1,5}) - (\phi_{1,2}\phi_{5,5} - \phi_{5,2}\phi_{1,5})$$

$$+ (\phi_{2,2}\phi_{5,5} - \phi_{5,2}\phi_{2,5}) - (\phi_{3,2}\phi_{4,5} - \phi_{4,2}\phi_{3,5})$$

$$+ (\phi_{3,2}\phi_{6,5} - \phi_{6,2}\phi_{3,5}) - (\phi_{4,2}\phi_{6,5} - \phi_{6,2}\phi_{4,5}) = -\phi_{9,9}$$

$$\Longrightarrow \phi_{1,2}\phi_{1,5} - \phi_{2,2}\phi_{1,5} - \phi_{1,2}\phi_{5,5} + \phi_{2,2}\phi_{5,5}$$

(4.4.0.0.35) $= -\phi_{3,5}\phi_{8,8}\phi_{10,10} + \phi_{4,5}\phi_{8,8}\phi_{10,10} - \phi_{9,9}$ (by 4.4.0.0.18, 4.4.0.0.20, 4.4.0.0.32)

(4.4.0.0.36) So, $\phi_{2,2} = \phi_{1,2} + \phi_{9,9}\phi_{10,10}$ (by 4.4.0.0.33, 4.4.0.0.34)

(4.4.0.0.37) and $\phi_{4,5} = \phi_{3,5} + 1/2\phi_{8,8}\phi_{9,9}\phi_{10,10}$ (by 4.4.0.0.33, 4.4.0.0.35)

$[\phi(v_1), \phi(v_5)] = \phi(z_b) = \phi_{9,9}z_b,$ just looking at the z_b-coefficient

$$\Longrightarrow \sum_{i<j}(\phi_{i,1}\phi_{j,5} - \phi_{j,1}\phi_{i,5})[v_i, v_j] = \phi_{9,9}z_b$$

$$\Longrightarrow (\phi_{1,1}\phi_{2,5} - \phi_{2,1}\phi_{1,5}) - (\phi_{1,1}\phi_{5,5} - \phi_{5,1}\phi_{1,5})$$

$$+ (\phi_{2,1}\phi_{5,5} - \phi_{5,1}\phi_{2,5}) - (\phi_{3,1}\phi_{4,5} - \phi_{4,1}\phi_{3,5})$$

$$+ (\phi_{3,1}\phi_{6,5} - \phi_{6,1}\phi_{3,5}) - (\phi_{4,1}\phi_{6,5} - \phi_{6,1}\phi_{4,5}) = \phi_{9,9}$$

$$\Longrightarrow \phi_{1,1}\phi_{1,5} - \phi_{2,1}\phi_{1,5} - \phi_{1,1}\phi_{5,5} + \phi_{2,1}\phi_{5,5}$$

(4.4.0.0.38) $= \phi_{9,9}$ (by 4.4.0.0.14, 4.4.0.0.20, 4.4.0.0.32)

$[\phi(v_1), \phi(v_6)] = \phi(0) = 0,$ just looking at the z_b-coefficient

$$\implies \sum_{i<j} (\phi_{i,1}\phi_{j,6} - \phi_{j,1}\phi_{i,6})[v_i, v_j] = 0$$

$$\implies (\phi_{1,1}\phi_{2,6} - \phi_{2,1}\phi_{1,6}) - (\phi_{1,1}\phi_{5,6} - \phi_{5,1}\phi_{1,6})$$

$$+ (\phi_{2,1}\phi_{5,6} - \phi_{5,1}\phi_{2,6}) - (\phi_{3,1}\phi_{4,6} - \phi_{4,1}\phi_{3,6})$$

$$+ (\phi_{3,1}\phi_{6,6} - \phi_{6,1}\phi_{3,6}) - (\phi_{4,1}\phi_{6,6} - \phi_{6,1}\phi_{4,6}) = 0$$

$$\implies \phi_{1,1}\phi_{1,5} - \phi_{2,1}\phi_{1,5} - \phi_{1,1}\phi_{5,5} + \phi_{2,1}\phi_{5,5}$$

(4.4.0.0.39) $= \phi_{1,1}\phi_{10,10} - \phi_{2,1}\phi_{10,10}$ (by 4.4.0.0.14, 4.4.0.0.21, 4.4.0.0.29, 4.4.0.0.30, 4.4.0.0.32)

(4.4.0.0.40) So, $\phi_{2,1} = \phi_{1,1} - \phi_{9,9}\phi_{10,10}$ (by 4.4.0.0.38, 4.4.0.0.39)

(4.4.0.0.41) and $\phi_{5,5} = \phi_{1,5} - \phi_{10,10}$ (by 4.4.0.0.38, 4.4.0.0.39, 4.4.0.0.40)

$[\phi(v_2), \phi(v_5)] = \phi(z_r - z_b) = \phi_{8,8}z_r - \phi_{9,9}z_b,$ just looking at the z_b-coefficient

$$\implies \sum_{i<j} (\phi_{i,2}\phi_{j,5} - \phi_{j,2}\phi_{i,5})[v_i, v_j] = -\phi_{8,8}z_r - \phi_{9,9}z_b$$

$$\implies (\phi_{1,2}\phi_{2,5} - \phi_{2,2}\phi_{1,5}) - (\phi_{1,2}\phi_{5,5} - \phi_{5,2}\phi_{1,5})$$

$$+ (\phi_{2,2}\phi_{5,5} - \phi_{5,2}\phi_{2,5}) - (\phi_{3,2}\phi_{4,5} - \phi_{4,2}\phi_{3,5})$$

$$+ (\phi_{3,2}\phi_{6,5} - \phi_{6,2}\phi_{3,5}) - (\phi_{4,2}\phi_{6,5} - \phi_{6,2}\phi_{4,5}) = -\phi_{9,9}$$

$\implies -3/2\phi_{9,9} = -\phi_{9,9}$ (by 4.4.0.0.18, 4.4.0.0.20, 4.4.0.0.32, 4.4.0.0.36, 4.4.0.0.37, 4.4.0.0.41)

$\implies \phi_{9,9} = 0$ which contradicts equation 4.4.0.0.10

Therefore, $\widehat{\mathfrak{n}_1}$ is not isometric to $\widehat{\mathfrak{n}_2}$. □

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Biographical Statement

Allie Denise Ray was born in Oklahoma City, OK in 1985 to Darrell and Cecilia Ray. She has one older brother, Adam. Her early years were spent living in northern Texas, but then she moved back to Oklahoma where she received her high school diploma from Eufaula High School in 2003.

Allie then attended Oklahoma Baptist University in Shawnee and received her Bachelor of Science degree in Mathematics Education in 2007. After graduation, she taught high school mathematics for three years in the Oklahoma City area, also taking a couple of graduate-level mathematics courses at the University of Oklahoma in 2009. After this, she moved back to Texas and began doctoral studies at the University of Texas at Arlington in 2010. In May 2015, she earned a Ph.D. in Mathematics under the direction of Ruth Gornet.

Allie's research interests lie in the areas of differential geometry and graph theory, with particular interest in isospectrality issues in both areas and constructions from graphs to Lie groups.