Weight Modules of Orthosymplectic Lie Superalgebras

by

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Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

May 2015

In memory of my parents, Travis and Flora Ferguson

ACKNOWLEDGEMENTS

I would like to thank my supervising professor Dr. Dimitar Grantcharov for constantly motivating and encouraging me, and also for his invaluable advice during the course of my doctoral studies. I wish to thank professors Dr. Michaela Vancliff, Dr. Minerva Cordero-Epperson, Dr. David Jorgenson, and Dr. Ruth Gornet for serving on my dissertation committee.

I also want to acknowledge Mrs. Eileen Smith, Mr. Bob Wylie, and Mr. C.C. Turner for being great mathematics teachers during my high school years. Thanks to Dr. Martin Pettet, my undergraduate professor at Texas A&M, who sparked my interest in modern algebra.

I also want to thank my fellow students, including Richard Chandler, Allie Ray, and Dr. Denise Rangel, who were a great support and encouragement to me during this journey.

I would also like to extend my appreciation to the administration of Southwestern Assemblies of God University for providing financial support as well as many prayers and words of encouragement. Finally, thanks to my wife Cynthia and my family for their great patience.

February 6, 2015

ABSTRACT

Weight Modules of Orthosymplectic Lie Superalgebras

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The University of Texas at Arlington, 2015

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A long-standing problem in representation theory is the classification of all simple weight modules of the classical Lie superalgebras. This problem was reduced to the classification of simple bounded highest weight modules. The latter classification has been accomplished for all classical Lie superalgebras except for the orthosymplectic series $\mathfrak{osp}(m|2n)$ where $m = 1, 3, 4, 5, 6$. In this thesis, we complete the classification of the simple bounded highest weight modules of $\mathfrak{osp}(1|2n)$ (i.e., for $m = 1$). The classification is obtained by developing constraints on primitive vectors in tensor products of bounded (Weyl) and finite-dimensional $\mathfrak{osp}(1|2n)$-modules.

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CHAPTER 1

Introduction

1.1 History and Motivation

Lie superalgebras are of great importance in modern physics, in particular, the study of elementary particle theory. Lie superalgebras are a generalization of Lie algebras based on a $\mathbb{Z}_2$-grading. In the 1950s, an important milestone in the development of modern physics occurred when the relationship between representation theory of Lie groups and Lie algebras to elementary particle theory was discovered. In the 1970s, the concept of supersymmetry in particle theory led to specific interest in Lie superalgebras. In 1977, Victor Kac [15] classified all finite-dimensional simple Lie superalgebras over typical fields.

The study of general (not necessarily finite-dimensional) weight representations of Lie algebras emerged in the early 1980s as a part of a fundamental effort in the structure theory of representations of Lie algebras and Lie groups. This effort has been motivated significantly by theoretical physics. Weight representations have been studied in the works of Georgia Benkart, Dan Britten, Suren Fernando, Vyacheslav Futorny, and Frank Lemire over the last 20 years, [2, 8, 9]. A major breakthrough in the representation theory of reductive Lie algebras was made by Olivier Mathieu. In 2000, he classified all irreducible weight representations with finite weight multiplicities [16].

Following Mathieu's classification, it is natural to aim at classifying all simple weight modules with finite weight multiplicities of all classical Lie superalgebras $\mathfrak{g}$. That classification was obtained in [6] for all $\mathfrak{g}$ except for the Lie superalgebra series

$\mathfrak{osp}(m;2n)$, $m = 1,3,4,5,6$; $\mathfrak{psq}(n)$, $D(2,1,\alpha)$. On the other hand, the classification of simple weight modules with finite weight multiplicities was reduced to the classification of the so-called *bounded* highest weight modules [12]. The latter classification was obtained for $\mathfrak{g} = \mathfrak{psq}(n)$ and $\mathfrak{g} = D(2,1,\alpha)$ in [10] and [13], respectively, which leaves the orthosymplectic series as the only remaining classical Lie superalgebras to consider.

In 2006, Dimitar Grantcharov and Vera Serganova [11] discovered that the category of all weight representations of the symplectic Lie algebras $\mathfrak{sp}(2n)$ is wild, i.e., its indecomposable objects cannot be parameterized. On the other hand, some categories of bounded weight representations of $\mathfrak{sp}(2n)$ are tame, i.e., the indecomposable objects can be parameterized. Examples of bounded representations of $\mathfrak{sp}(2n)$ are the so-called Weyl representations, which appear as Laurent polynomial representations. This class arises naturally from the Weyl presentation of $\mathfrak{sp}(2n)$ in terms of differential operators.

In this thesis, we make the first important step toward classifying bounded highest weight $\mathfrak{osp}(m|2n)$-modules. We complete the classification in the case $m = 1$, i.e., for $\mathfrak{osp}(1|2n)$. We employ techniques based on Weyl representations and constraints on primitive vectors within tensor products of Lie superalgebra representations, while exploiting the fact that the symplectic Lie algebra can be considered as the even part of the orthosymplectic Lie superalgebra. More precisely, we consider tensor products of Weyl representations and finite-dimensional representations of $\mathfrak{osp}(1|2n)$. By finding primitive vectors in such a tensor product, we obtain necessary conditions for a simple highest weight module to be bounded. Such a technique was used by Kevin Coulembier [5] for some special finite-dimensional representations of $\mathfrak{osp}(1|2n)$. Our methods can be used to obtain character formulae for bounded highest weight modules. In our recent preprint [7], we find such formulas and also provide an alternative method of classifying the simple bounded highest weight representations of $\mathfrak{osp}(1|2n)$.

The content of the thesis is as follows. In Chapter 2, we record the definition of a Lie algebra and some of the associated basic definitions and results. Chapter 3 introduces the Lie superalgebras and illustrates both the parallelism and distinctions between the structures. In Chapter 4, we provide some background material on the universal enveloping algebra for Lie superalgebras, as it will be needed in many of the later results. In Chapter 5, we introduce the orthosymplectic Lie superalgebra; the remainder of the thesis then focuses on these structures. We develop the concept of a Cartan subalgebra, root, weights, and weights of bounded multiplicities of $\mathfrak{osp}(1|2n)$. In Chapter 6, we exhibit a Weyl representation, and several classification results flow from the associated homomorphism. Finally, in Chapter 7, we present results that classify the bounded weights and simple weight modules of $\mathfrak{osp}(1|2n)$.

1.2 Notation and Conventions

In the following, all vector spaces will be assumed to have ground field $\mathbb{C}$ (the complex numbers). Vector spaces will be denoted by Roman letters (e.g., V, W), and V^* will be used to denote the dual space to V. Gothic characters (e.g., $\mathfrak{g}, \mathfrak{h}$) will be used to represent Lie algebras and Lie superalgebras. Associative algebras and superalgebras will be denoted by Roman letters (e.g, A, B). E_{ij} will be used to denote the $(i, j)th$ elementary matrix. We expect that the reader is familiar with basic notions from linear algebra such as the direct sum of vector spaces, linear transformations, and matrices of linear transformations. Unless otherwise noted, the expression $\pm\alpha\pm\beta$ will be used to represent four possible values, i.e., $\alpha+\beta, \alpha-\beta, -\alpha-\beta, -\alpha+\beta$.

CHAPTER 2

Lie Algebras

We begin with the definition of a Lie algebra and some of the associated concepts. We also give specific examples of Lie algebras that will be useful in the sequel. The goal is to have enough machinery to define modules and weights.

2.1 Definitions

Definition 2.1.1 *[14, 1.1] A vector space* $\mathfrak{g}$ *over* $\mathbb{C}$*, with a binary operation* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, called the* Lie bracket *or* commutator*, is a* Lie algebra *if the following axioms are satisfied:*

(L1) $[ax+by,z] = a[x,z]+b[y,z]$ *and* $[z,ax+by] = a[z,x]+b[z,y]$ *(bilinearity)*

(L2) $[x,y] = -[y,x]$ *(skew symmetry)*

(L3) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$ *(Jacobi identity) for every* $a,b \in \mathbb{C}$ *and for every* $x,y,z \in \mathfrak{g}$

Definition 2.1.2 *[14, 1.1] A vector subspace* $\mathfrak{a}$ *of a Lie algebra* $\mathfrak{g}$ *is called a* Lie subalgebra *if* $\mathfrak{a}$ *is closed under the Lie bracket.*

Definition 2.1.3 *[14, 2.1] A subspace* $\mathfrak{a}$ *of a Lie algebra* $\mathfrak{g}$ *is an* ideal *of* $\mathfrak{g}$ *if for every* $g \in \mathfrak{g}$*, for every* $a \in \mathfrak{a}, [g,a] \in \mathfrak{a}$*. The maximal solvable ideal of* $\mathfrak{g}$ *is known as the* radical of $\mathfrak{g}$ *and is denoted* Rad $\mathfrak{g}$*. If* Rad $\mathfrak{g} = 0$*, then* $\mathfrak{g}$ *is said to be* semisimple *(cf. [14] §3.1).*

Definition 2.1.4 *[14, 2.1] The* center *of a Lie algebra* $\mathfrak{g}$ *is the set of elements that commute with every element in* $\mathfrak{g}$*; that is,*

$$Z(\mathfrak{g}) := \{z \in \mathfrak{g} | [x,z] = 0 \text{ for all } x \in \mathfrak{g}\}.$$

If Rad $\mathfrak{g} = Z(\mathfrak{g})$*, then* $\mathfrak{g}$ *is said to be* reductive.

Definition 2.1.5 *[14, 2.1] A Lie algebra* $\mathfrak{g}$ *is called* simple *if its only ideals are itself and* 0.

Definition 2.1.6 *[14, 2.2] A map* $\phi : \mathfrak{g} \to \mathfrak{a}$ *is called a* Lie algebra homomorphism *if it is a homomorphism of vector spaces and* $\phi([x, y]) = [\phi(x), \phi(y)]$ *for every* $x, y \in \mathfrak{g}$.

2.2 Examples of Lie Algebras

2.2.1 The Lie Algebra $\mathfrak{gl}(V)$

The *general linear Lie algebra* $\mathfrak{gl}(V)$ consists of all endomorphisms of V. If dim $V = n$, then the dimension of $\mathfrak{gl}(V)$ is n^2. By fixing a basis for V, we will show that $\mathfrak{gl}(V)$ is isomorphic to the Lie algebra as $\mathfrak{gl}(n, \mathbb{C})$ (all $n \times n$ matrices with complex entries).We define the Lie bracket as $[x, y] = xy - yx$ for $x, y \in \mathfrak{gl}(V)$, where xy is given by function composition.

Proposition 2.2.7 *[14, 1.1] Let* V *be a vector space over* $\mathbb{C}$ *with dim* $V = n$*, then* $\mathfrak{gl}(V)$ *is isomorphic to* $\mathfrak{gl}(n, \mathbb{C})$.

Proof: The proof is standard but for completeness we outline the important steps. Fix a basis $B{:=}\{v_1, v_2, ..., v_n\}$ of V. Define the linear map $\phi : \mathfrak{gl}(V) \longmapsto \mathfrak{gl}(n, \mathbb{C})$ by: $x \longmapsto [x]_B$, where $[x]_B$ represents the matrix of x with respect to B. Clearly, ϕ is a vector space isomorphism. It remains to show that $\phi([x, y]) = [\phi(x), \phi(y)]$. Considering the left hand side, we have $\phi([x, y]) = \phi(xy - yx) = \phi(xy) - \phi(yx)$ by the linearity of ϕ. On the right hand side, $[\phi(x), \phi(y)] = [x]_B[y]_B - [y]_B[x]_B$ by the definition of ϕ. From the correspondence between matrix multiplication and the composition of endomorphisms, we have $[x]_B[y]_B - [y]_B[x]_B = \phi(xy) - \phi(yx)$. □

2.2.2 The Lie Algebra $\mathfrak{sl}(n)$

Definition 2.2.8 *[14, 1.1]* $\mathfrak{sl}(n) := \{A \in \mathfrak{gl}(n) \mid tr(A) = 0\}$.

The *special linear Lie algebra*, which is denoted by $\mathfrak{sl}(n)$, consisting of all traceless endomorphisms of V(here we rely on the fact that the trace of a matrix is independent of the basis).

Proposition 2.2.9 *$\mathfrak{sl}(n)$ is an ideal of $\mathfrak{gl}(n)$.*

Proof: Let $a \in \mathfrak{sl}(n), g \in \mathfrak{gl}(n)$. Consider $[g, a] = ga - ag$. Since $\mathrm{Tr}(ga) = \mathrm{Tr}(ag)$, it follows that $\mathrm{Tr}([g, a]) = 0$. Thus, $[g, a] \in \mathfrak{sl}(n)$ as required. □

Remark 2.2.10 *It follows that $\mathfrak{gl}(n)$ is not simple but is reductive (cf. [14],§6.4), while $\mathfrak{sl}(n)$ is simple.*

Remark 2.2.11 *Of particular interest in the sequel is $\mathfrak{sl}(2)$ with standard basis e, f, h defined as follows $e = E_{12}, f = E_{21}, h = E_{11} - E_{22}$. Then $[h, e] = 2e, [h, f] = -2f, [e, f] = h$.*

2.2.3 The Lie Algebra $\mathfrak{sp}(2n)$

We define a skew-symmetric bilinear form (i.e., *B(v,w)= −B(w,v))* B on V by the matrix s:

$$s = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

$B(v, w) = v^t sw$.This leads to the definition of $\mathfrak{sp}(2n)$, the *symplectic Lie algebra.*

Definition 2.2.12 *[14, 1.1]* $\mathfrak{sp}(2n)$:=$\{x \in \mathfrak{gl}(2n) \mid sx = -x^t s\}$.

An element of $\mathfrak{sp}(2n)$ in matrix form is $\begin{bmatrix} A & X \\ Y & -A^t \end{bmatrix}$ where $X = X^t$ and $Y = Y^t$. We note that $\dim(\mathfrak{sp}(2n)) = 2n^2 + n$. (cf. [14],§1.2). The following proposition is included as an example of a Lie subalgebra.

Proposition 2.2.13 *$\mathfrak{sp}(2n)$ is a Lie subalgebra of $\mathfrak{gl}(2n)$.*

Proof: Clearly $\mathfrak{sp}(2n)$ is a subspace of $\mathfrak{gl}(2n)$ by dimension. It remains to show that $\mathfrak{sp}(2n)$ is closed under the bracket operation. Let $x, y \in \mathfrak{sp}(2n)$ and let $v, w \in \mathbb{C}^{2n}$.Then

$$\begin{aligned} B([x,y](v), w) &= B((xy - yx)(v), w) \\ &= B((xy)(v), w) - B((yx)(v), w) \\ &= -B(v, (xy)(w)) + B(v, (yx)(w)) \\ &= B(v, (yx - xy)(w)) \\ &= -B(v, [x,y](w)) \end{aligned}$$

as required. □

Definition 2.2.14 *The diagonal matrices (denoted $\mathfrak{d}(n)$), are another example of a subalgebra of $\mathfrak{gl}(n)$. The* diagonal subalgebras *of $\mathfrak{sl}(n)$ and $\mathfrak{sp}(2n)$ are*

$$\mathfrak{h}_{\mathfrak{sl}(n)} := \{A \in \mathfrak{sl}(n) \mid A \in \mathfrak{d}(n)\}$$

and

$$\mathfrak{h}_{\mathfrak{sp}(2n)} := \{A \in \mathfrak{sp}(2n) \mid A \in \mathfrak{d}(2n)\},$$

respectively.

2.3 Representations of Lie Algebras

Definition 2.3.15 *[14, 6.1] Let $\mathfrak{g}$ be a Lie algebra and V a vector space. We say that V is a* $\mathfrak{g}$-module *(or a* representation *of $\mathfrak{g}$) if there is a binary operation $\mathfrak{g} \times V \to V$, $(x, v) \mapsto x \cdot v$ such that the following axioms are satisfied:*

(M1) $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$,

(M2) $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$,

(M3) $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$, *for every $x, y \in \mathfrak{g}$ and $a, b \in \mathbb{C}$ and for every $v, w \in V$.*

Proposition 2.3.16 *[14, 6.1] The vector space V is a* $\mathfrak{g}$-module *if* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, $\rho(g)(v) = g \cdot v$ *is a homomorphism of Lie algebras.*

Proof: The first two axioms follow easily; we verify the third here for completeness. Let $x, y \in \mathfrak{g}$ and $v \in V$, then:

$$\begin{aligned}\rho([x,y](v) &= [\rho(x), \rho(y)](v) \\ &= \rho(x)(\rho(y)(v)) - \rho(y)(\rho(x)(v)) \\ &= x \cdot (y \cdot v) - y \cdot (x \cdot v)\end{aligned}$$

Since $\rho([x, y])(v) = [x, y]v$ we have that V is a $\mathfrak{g}$-module. □

Definition 2.3.17 *[14, 1.3] Let $\mathfrak{g}$ be a Lie algebra. For $x \in \mathfrak{g}$, define the* adjoint map ad $x : \mathfrak{g} \to \mathfrak{g}$ *by* $y \mapsto [x, y], y \in \mathfrak{g}$.

Proposition 2.3.18 *[14, 2.2] The map* ad*:* $\mathfrak{g} \to \text{End}\ \mathfrak{g}$ *given by* $x \mapsto \text{ad}\ x$ *is a representation of* $\mathfrak{g}$ *called the* adjoint representation *of* $\mathfrak{g}$.

Proof: For convenience in this proof we write ad_x for $\text{ad}(x)$. Clearly ad is a linear map so it suffices to show that $ad_{[x,y]} = [ad_x, ad_y]$. Let $x, y, z \in \mathfrak{g}$, then using the definition of ad, the skew-symmetric property, and the Jacobi identity we have:

$$\begin{aligned}[ad_x, ad_y](z) &= ad_x ad_y(z) - ad_y ad_x(z) \\ &= ad_x[y, z] - ad_y[x, z] \\ &= [x, [y, z] - [y, [x, z]] \\ &= [x, [y, z]] + [[x, z], y] \\ &= -[y, [z, x] - [z, [x, y]] + [[x, z], y] \\ &= [[x, y], z] \\ &= ad_{[x,y]}(z).\end{aligned}$$

□

Definition 2.3.19 *A* subrepresentation *(*submodule*) W of V is a vector space which is itself a $\mathfrak{g}$-representation.*

Definition 2.3.20 *Let W be a subrepresentation of V over a Lie algebra $\mathfrak{g}$. Then the quotient vector space V/W is a representation of $\mathfrak{g}$ defined by $x \cdot (v+W) := x \cdot v + W$ and is a* quotient representation.

Definition 2.3.21 *[14, 6.1] A representation is called* simple *(*irreducible*) if its only subrepresentations are itself and 0. A representation M is called* indecomposable *if $M = M_1 \oplus M_2$ implies $M_1 = 0$ or $M_2 = 0$.*

2.4 Cartan Subalgebras

Definition 2.4.22 *[14, 3.2] Define the* derived algebra *of $\mathfrak{g}$, denoted $[\mathfrak{g},\mathfrak{g}]$, as the span of all commutators $[x, y]$. Then the* descending central series *of $\mathfrak{g}$ is a sequence of ideals defined as: $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}]$. $\mathfrak{g}$ is said to be* nilpotent *if $\mathfrak{g}^n = 0$, for some $n \in \mathbb{Z}$.*

Example 2.4.23 *The* Heisenberg Lie algebra *$\mathcal{H}_n$ is the Lie algebra with basis*

$$\{x_1, ..., x_n, y_1, ..., y_n, z\}$$

subject to the relations: $[x_i, y_j] = \delta_{ij} z$; $[x_i, x_j] = [y_i, y_j] = [z, x_i] = [z, y_i] = 0$, for all i, j. Note that the Heisenberg Lie algebras are nilpotent and dim $\mathcal{H}_n$=$2n + 1$.

Example 2.4.24 *The set of strictly upper triangular $n \times n$ matrices, denoted $\mathfrak{n}(n)$ is another example of a nilpotent Lie algebra. When $n = 3$, $\mathfrak{n}(n)$ is isomorphic to the* Heisenberg *Lie algebra $\mathcal{H}_1$. We will later extend this to a Lie superalgebra.*

Definition 2.4.25 *[14, 2.1] The* normalizer *of a subalgebra $\mathfrak{t}$ of a Lie algebra $\mathfrak{g}$ is defined by $N_{\mathfrak{g}}(\mathfrak{t}) := \{x \in \mathfrak{g} | [x, \mathfrak{t}] \subset \mathfrak{t}\}$. If $\mathfrak{t} = N_{\mathfrak{g}}(\mathfrak{t})$ then $\mathfrak{t}$ is said to be* self-normalizing.

With these definitions in hand, we can now define a Cartan subalgebra.

Definition 2.4.26 *[14, 15.3] Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ such that $\mathfrak{h}$ is nilpotent and self-normalizing. Then $\mathfrak{h}$ is said to be a* Cartan subalgebra *of $\mathfrak{g}$.*

Remark 2.4.27 *We note that a Cartan subalgebra need not exist and need not be unique (cf. [14],§8.1). The diagonal subalgebras introduced in Definition 2.2.14 are examples of Cartan subalgebras of $\mathfrak{sl}(n)$ and $\mathfrak{sp}(2n)$.*

From now on we fix the Cartan subalgebras of $\mathfrak{g} = \mathfrak{sl}(n)$ to be the diagonal ones:

$$\mathfrak{h}_{\mathfrak{sl}(n)} = \{a_1E_{11} + a_2E_{22} + ... + a_nE_{n,n} | a_1 + a_2 + ... + a_n = 0\}$$

and $\mathfrak{g} = \mathfrak{sp}(2n)$:

$$\mathfrak{h}_{\mathfrak{sp}(2n)} = \{a_1E_{11} + a_2E_{22} + ... + a_nE_{n,n} - a_1E_{n+1,n+1} - ... - a_nE_{2n,2n} | a_i \in \mathbb{C}\}.$$

Definition 2.4.28 *[14, 20.1] Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and M be a $\mathfrak{g}$-module. For $\lambda \in \mathfrak{h}^*$ define $M^\lambda := \{v \in M | hv = \lambda(h)v, \text{ for all } h \in \mathfrak{h}\}$. Then*

(i) We say that λ is a weight *of M if $M^\lambda \neq 0$.*

*(ii) M^λ is the λ–*weight space *of M.*

(iii) M is a weight module *if $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$ and M^λ is finite-dimensional for every weight λ of M.*

2.5 Root Systems of Lie Algebras

Let V be a vector space over $\mathbb{C}$ with a positive definite symmetric bilinear form $\langle , \rangle$ defined on V.

Definition 2.5.29 *[14, 9.2] A subset Φ of V is a* root system *of V if:*

(i) Φ is finite, spans V, and does not contain the zero vector.

(ii) If $\alpha \in \Phi$, the only scalar multiples of $\alpha \in \Phi$ are $\pm\alpha$.

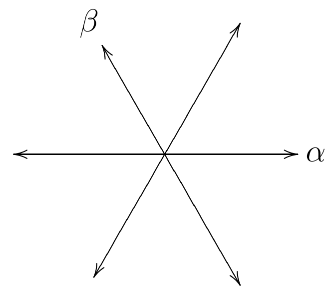

Figure 2.1. A_2.

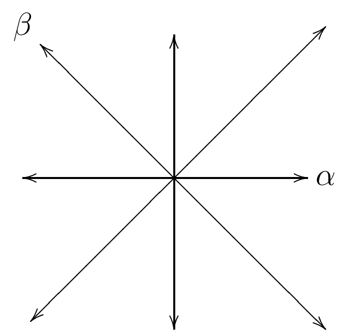

Figure 2.2. C_2.

(iii) If $\alpha \in \Phi$, then the reflection σ_α leaves Φ invariant (σ_α is the reflection in V associated with α).

(iv) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Example 2.5.30 *Let $\Phi = \{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq n\}$ where $\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$ is a basis of $\mathbb{C}^n$. Here $V = \{\sum_{i=1}^n x_i \epsilon_i | \sum_{i=1}^n x_i = 0\}$. We denote Φ by A_n.*

The root system $\Phi = A_2$ is depicted in Figure 2.5.

Example 2.5.31 *Let $\Phi = \{\pm\epsilon_i \pm \epsilon_j, \pm 2\epsilon_i | 1 \leq i < j \leq n\}$, with $V = \mathbb{C}^n$. We denote Φ by C_n.*

The root system $\Phi = C_2$ is depicted in Figure 2.5.

Definition 2.5.32 *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We define the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ of $\mathfrak{g}$ relative to $\mathfrak{h}$ as follows. For $\alpha \in \mathfrak{h}^*$, define $\mathfrak{g}^\alpha := \{x \in \mathfrak{g} | [h, x] = \alpha(h)x$, for all $h \in \mathfrak{h}\}$. Then $\Delta(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}^* | \mathfrak{g}^\alpha \neq 0\}$. The elements of $\Delta(\mathfrak{g}, \mathfrak{h})$ are called the* roots of $\mathfrak{g}$. *A nonzero element $X_\alpha \in \mathfrak{g}^\alpha$ is a* root vector. *We note that $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha)$ (cf. [Hum72], § 8.1). We call this decomposition the* root space decomposition of $\mathfrak{g}$.

Example 2.5.33 *Let $\mathfrak{g} = \mathfrak{sl}(n)$ and $\mathfrak{h} = \mathfrak{d}(n) \cap \mathfrak{sl}(n)$. The root system $\Delta(\mathfrak{g}, \mathfrak{h})$ is $A_n = \{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq n\}$. To relate $\Delta(\mathfrak{g}, \mathfrak{h})$ to A_n, we define $\epsilon_i \in \mathfrak{h}^*$ by: $\epsilon_i(a_1 E_{11} + a_2 E_{22} + ... + a_n E_{nn}) = a_i$ Here, $\sum_{i=1}^n a_i = 0$.*

Example 2.5.34 *Let $\mathfrak{g} = \mathfrak{sp}(2n)$ and $\mathfrak{h} = \mathfrak{d}(2n) \cap \mathfrak{sp}(2n)$. Here the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ is $C_n = \{\pm\epsilon_i \pm \epsilon_j, \pm 2\epsilon_i, i \neq j\}$. Define $\epsilon_i \in \mathfrak{h}^*$ by:*

$$\epsilon_i(a_1 E_{11} + ... + a_n E_{nn} - a_1 E_{n+1,n+1} - ... - a_n E_{2n,2n}) = a_i.$$

Definition 2.5.35 *[2, 1.1] Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with root system Δ. Let M be a weight $\mathfrak{g}$-module.*

(i) M is called bounded *if there is $K > 0$ such that $\dim M^\lambda < K$ for all λ.*

(ii) M is called torsion free *if the action of every root vector $X_\alpha \in \mathfrak{g}^\alpha$ on M; $X_\alpha : M \to M$, is injective.*

(iii) If M is simple, then M is said to be pointed *if M has at least one one-dimensional weight space.*

(iv) M is said to be completely pointed *if all weight spaces of M are one-dimensional.*

CHAPTER 3

Lie Superalgebras

We introduce the definition of a Lie superalgebra and exhibit the general linear Lie superalgebra. We then reprise definitions associated with Lie algebras - some are extended in a natural way through the concept of supersymmetry while others are unchanged.

3.1 Notation

We set $\mathbb{Z}_2$ to be $\mathbb{Z}/2\mathbb{Z}$ and denote the elements of $\mathbb{Z}_2$ by $\overline{0}$ and $\overline{1}$.

3.2 Definitions

Definition 3.2.1 *[17, 1.1] A vector space V is said to be* $\mathbb{Z}_2$-graded *if there exist subspaces $V_{\overline{0}}, V_{\overline{1}}$ such that $V = V_{\overline{0}} \oplus V_{\overline{1}}$. $V_{\overline{0}}$ is the* even *part of V and $V_{\overline{1}}$ is the* odd *part of V*

The vector space $\mathbb{C}^{m|n} = V = V_{\overline{0}} \oplus V_{\overline{1}}$ plays an important role in the sequel. We define it as follows:

Definition 3.2.2 $\mathbb{C}^{m|n}$*:=*$\{(z_1, z_2, ..., z_m; z_{m+1}, ..., z_{m+n}) | z_i \in \mathbb{C}, \textit{for } i = 1, 2, ..., m+n$ *with* $m, n \in \mathbb{Z}_{\geq 0}\}$

We note here that $\mathbb{C}^{m|n}$ is a $\mathbb{Z}_2$-graded vector space by defining:

$$V_{\overline{0}} := \text{span}\{(0, 0, ..., 0; z_{m+1}, ..., z_{m+n}) | z_i \in \mathbb{C}\}$$

and

$$V_{\overline{1}} := \text{span}\{(z_1, z_2, ..., z_m; 0, 0, ..., 0) | z_i \in \mathbb{C}\}.$$

Definition 3.2.3 *[17, 1.1] Let V be a $\mathbb{Z}_2$-graded vector space and $x \in V_{\bar{i}}$. Then $p(x) = \bar{i}$ is the* parity *of x. The element x is* odd *if $p(x)=\bar{1}$ and* even *otherwise. An element that has parity is said to be* homogeneous.

Remark 3.2.4 *Not every element in a $\mathbb{Z}_2$-graded vector space has parity. For example, $(1, 1; 1) \in \mathbb{C}^{2|1}$ is not homogeneous.*

Definition 3.2.5 *Let V, W be $\mathbb{Z}_2$-graded vector spaces. A map $\phi : V \to W$ is* grade-preserving *if for every $x \in V$, $p(x) = p(\phi(x))$ (or, $\phi(V_{\bar{i}}) \subset W_{\bar{i}}$).*

Definition 3.2.6 *V is a $\mathbb{Z}_2$-*graded subspace *of $W = W_{\bar{0}} \oplus W_{\bar{1}}$ if $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and $V_{\bar{i}}$ is a subspace of $W_{\bar{i}}$ for $i \in \mathbb{Z}_2$.*

Definition 3.2.7 *[17, 1.1] A $\mathbb{Z}_2$-graded space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, (called the* Lie superbracket *or* supercommutator*), is a* Lie superalgebra *if the following axioms are satisfied:*

(i) $[\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}] \subset \mathfrak{g}_{\bar{i}+\bar{j}}$, *for every $\bar{i}, \bar{j} \in \mathbb{Z}_2$;*

(ii) $[\ ,\ ]$ *is bilinear;*

(iii) $[x, y] = -(-1)^{p(x)p(y)}[y, x]$; skew-supersymmetry;

(iv) $[x, [y, z]] = [[x, y], z] + (-1)^{p(x)p(y)}[y, [x, z]]$ *for all homogeneous $x, y, z \in \mathfrak{g}$. This is known as the* the super Jacobi identity.

Definition 3.2.8 *A $\mathbb{Z}_2$-graded space $\mathfrak{a}$ of a Lie superalgebra $\mathfrak{g}$ is called a* Lie subsuperalgebra *if $\mathfrak{a}$ is closed under the Lie superbracket.*

Definition 3.2.9 *A map $\phi : \mathfrak{g} \to \mathfrak{a}$ is called a* Lie superalgebra homomorphism *if it is a grade-preserving homomorphism of $\mathbb{Z}_2$-graded spaces and $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$.*

Lemma 3.2.10 *Let $\mathfrak{g}, \mathfrak{a}$ be Lie superalgebras and let $\phi : \mathfrak{g} \to \mathfrak{a}$ be a linear map of $\mathbb{Z}_2$-graded vector spaces. Let $B = \{v_1, v_2, ...v_n\}$ be a set that spans the vector space $\mathfrak{g}$. If $\phi([v_i, v_j]) = [\phi(v_i), \phi(v_j)]$, for all $v_i, v_j \in B$, then ϕ is a Lie superalgebra homomorphism.*

Proof. Let $x, y \in \mathfrak{g}$ with $x := \sum_{i=1}^n \alpha_i v_i$ and $y := \sum_{i=1}^n \beta_i v_i$, where $\alpha_i, \beta_i \in \mathbb{C}$. The result then follows readily from the bilinearity of the Lie superbracket and the linearity of ϕ.

$$\begin{aligned}
\phi([x,y]) &= \phi([\sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \beta_i v_i]) \\
&= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j \phi([v_i, v_j]) \\
&= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j ([\phi(v_i), \phi(v_j)]) \\
&= [\sum_{i=1}^n \alpha_i \phi(v_i), \sum_{i=1}^n \beta_i \phi(v_i)] \\
&= [\phi(x), \phi(y)]
\end{aligned}$$

□

Definition 3.2.11 *Let X be a subset of a Lie algebra (or Lie superalgebra) $\mathfrak{g}$. Then the* Lie subalgebra (subsuperalgebra) of $\mathfrak{g}$ generated by X *is*

$$\langle X \rangle_{\mathfrak{g}} := \left\{ \sum_{i=1}^n \alpha_i y_i |\ y_i = [x_1, [x_2, ...[x_k, x_{k+1}]...]], \textit{for some } x_i \in X, \alpha_i \in \mathbb{C}, n > 0 \right\}.$$

We say that X generates $\mathfrak{g}$ *(X is a* generating set *of $\mathfrak{g}$) if $\langle X \rangle_{\mathfrak{g}} = \mathfrak{g}$.*

Corollary 3.2.12 *Let $\mathfrak{g}, \mathfrak{a}$ be Lie superalgebras and let $\phi : \mathfrak{g} \to \mathfrak{a}$ be a linear map of $\mathbb{Z}_2$-graded vector spaces. Let $X = \{v_1, v_2, ..., v_n\}$ be a generating set of $\mathfrak{g}$. If $\phi([v_i, v_j]) = [\phi(v_i), \phi(v_j)]$, for all $v_i, v_j \in X$, then ϕ is a Lie superalgebra homomorphism.*

Proof. By definition of generating set, the set B consisting of all $[x_1, [x_2, ...[x_k, x_{k+1}]...]]$, $x_i \in X$, spans $\mathfrak{g}$. On the other hand, it is easy to show that $\phi([v, w]) = [\phi(v), \phi(w)]$, for $v, w \in B$. Now we apply Lemma 3.2.10 to complete the proof. □

Definition 3.2.13 *A map $\phi : \mathfrak{g} \to \mathfrak{a}$ is called a* Lie superalgebra isomorphism *if it is a Lie superalgebra homomorphism of vector spaces and ϕ is bijective.*

3.3 The General Linear Lie Superalgebra

The general linear Lie superalgebra $\mathfrak{gl}(V)$ is defined as follows:[17, 2.2]

$$\mathfrak{gl}(V)_{\bar{0}} := \{x : V \longmapsto V \mid x(V_{\bar{i}}) \subset V_{\bar{i}}\}$$

and

$$\mathfrak{gl}(V)_{\bar{1}} := \{x : V \longmapsto V \mid x(V_{\bar{i}}) \subset V_{\overline{i+1}}\}.$$

A particular example of a general linear Lie superalgebra is $\mathfrak{gl}(m|n)$, i.e. the Lie superalgebra consisting of block matrices $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where the size of A is $m \times m$, of B is $m \times n$, of C is $n \times m$, and of D is $n \times n$. The supercommutator is defined as $[f, g] = fg - (-1)^{p(f)p(g)}gf$. We have $\mathfrak{gl}(m|n)_{\bar{0}} \simeq \mathfrak{gl}_m \oplus \mathfrak{gl}_n$ (when $B = C = 0$).

Proposition 3.3.14 *$\mathfrak{gl}(m|n)$ is isomorphic to $\mathfrak{gl}(\mathbb{C}^{m|n})$.*

Proof: The proof is analogous to Proposition 2.2.7 (the Lie algebra case). □

Definition 3.3.15 *Let $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a matrix in $\mathfrak{gl}(m|n)$. The* supertrace *of X is* $\mathrm{str}(X) := \mathrm{tr}(A) - \mathrm{tr}(D)$.

The Lie superalgebra analog of $\mathfrak{sl}(n)$ is the Lie superalgebra $\mathfrak{sl}(m|n)$ consisting of all X in $\mathfrak{gl}(m|n)$ such that $\mathrm{str}\, X = 0$.

Lemma 3.3.16 *The Lie superbracket is defined on basis elements of $\mathfrak{gl}(m|n)$ as follows:*

$$[E_{ij}, E_{kl}] = \delta_{j,k} E_{ij} - (-1)^{p(E_{ij})p(E_{ij})} \delta_{l,i} E_{kj}$$

Proof: Follows by direct computation. □

Definition 3.4.17 *Let $\mathfrak{g}$ be a Lie superalgebra and V a vector space. We say that V is a* $\mathfrak{g}$-module *(or a* representation of $\mathfrak{g}$*) if there is a binary operation $\mathfrak{g} \times V \to V$, $(x, v) \mapsto x \cdot v$ such that the following axioms are satisfied:*

(M1) $\mathfrak{g}_{\bar{i}} \cdot V_{\bar{j}} \subset V_{\bar{i}+\bar{j}}$ *for every* $i, j \in \mathbb{Z}_2$,

(M2) $(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$,

(M3) $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$,

(M4) $[x, y] \cdot v = x \cdot (y \cdot v) - (-1)^{p(x)p(y)} y \cdot (x \cdot v)$, *for all* $x, y \in \mathfrak{g}$ *and* $a, b \in \mathbb{C}$ *and* $v, w \in V$.

Remark 3.4.18 *The first axiom ensures that the action of $\mathfrak{g}$ on V is compatible with the $\mathbb{Z}_2$-gradings of $\mathfrak{g}$ and V.*

Proposition 3.4.19 *The $\mathbb{Z}_2$-graded vector space V is a $\mathfrak{g}$-module if and only if $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, $\rho(g)(v) = g \cdot v$ is a homomorphism of Lie superalgebras.*

Proof: This is a standard statement; we will prove the reverse direction. The first three axioms follow easily. We verify the fourth for completeness. Let $x, y \in \mathfrak{g}$ and $v \in V$, then:

$$\begin{aligned}
\rho([x, y](v) &= [\rho(x), \rho(y)](v) \\
&= \rho(x)(\rho(y)(v)) - (-1)^{p(x)p(y)} \rho(y)(\rho(x)(v)) \\
&= x \cdot (y \cdot v) - (-1)^{p(x)p(y)} y \cdot (x \cdot v)
\end{aligned}$$

Since $\rho([x, y])(v) = [x, y] \cdot v$ we have that V is a $\mathfrak{g}$-module. □

Definition 3.4.20 *Let V be a $\mathbb{Z}_2$-graded vector space and let $\mathfrak{g}$ be a Lie subsuperalgebra of $\mathfrak{gl}(V)$. The inclusion map ι: $\mathfrak{g} \to \mathfrak{gl}(\mathbb{C}^{m|n})$ is the* standard representation *of $\mathfrak{g}$.*

Definition 3.4.21 *Let $\mathfrak{g}$ be a Lie superalgebra. For $x \in \mathfrak{g}$, we define the* adjoint map *as follows:* ad $x : \mathfrak{g} \to \mathfrak{g}$ *by* $y \mapsto [x, y], y \in \mathfrak{g}$.

Proposition 3.4.22 *The map* ad*:* $\mathfrak{g} \to \operatorname{End} \mathfrak{g}$ *given by* $x \mapsto \operatorname{ad} x$ *is a representation of* $\mathfrak{g}$*. It is called the* adjoint representation *of* $\mathfrak{g}$*.*

Proof: The proof is very similar to the one for the Lie algebra case (Proposition 2.3.18). Recall that $\operatorname{ad}_t = \operatorname{ad}(t)$. The result follows readily from the super Jacobi identity.

$$\begin{aligned}[\operatorname{ad}_x, \operatorname{ad}_y](z) &= \operatorname{ad}_x \operatorname{ad}_y(z) - (-1)^{p(x)p(y)} \operatorname{ad}_y \operatorname{ad}_x(z) \\ &= \operatorname{ad}_x[y, z] - (-1)^{p(x)p(y)} \operatorname{ad}_y[x, z] \\ &= [x, [y, z] - (-1)^{p(x)p(y)}[y, [x, z]] \\ &= [[x, y], z] \\ &= ad_{[x,y]}(z)\end{aligned}$$

□

Definition 3.4.23 *A* subrepresentation *(*submodule*)* W *of a representation* V *of* $\mathfrak{g}$ *is a vector space which is itself a* $\mathfrak{g}$*-representation.*

Definition 3.4.24 *Let* W *be a subrepresentation of* V *over a Lie superalgebra* $\mathfrak{g}$*. Then the quotient vector space* V/W *is* $\mathbb{Z}_2$*-graded and is a representation of* $\mathfrak{g}$ *defined by* $x \cdot (v + W) := x \cdot v + W$*. We call* V/W *a* quotient representation*.*

Definition 3.4.25 *A representation of* $\mathfrak{g}$ *is called* simple *(*irreducible*) if its only subrepresentations are itself and* 0*. A representation* M *is called* indecomposable *if* $M = M_1 \oplus M_2$ *implies* $M_1 = 0$ *or* $M_2 = 0$*.*

CHAPTER 4

Universal Enveloping Algebras

We define associative superalgebras and construct the corresponding universal enveloping algebra. The Weyl superalgebra is defined and a key result is provided that will be useful for proving a module is simple.

4.1 Associative Superalgebras

Definition 4.1.1 *An* associative superalgebra over $\mathbb{C}$ *is a* $\mathbb{Z}_2$*-graded vector space* $A = A_{\overline{0}} \oplus A_{\overline{1}}$*, with a binary operation* $(a, b) \longmapsto ab$*, such that:*

(i) $r(a + b) = ra + rb$

(ii) $(r + s)a = ra + sa$

(iii) $r(sa) = rsa$

(iv) $1a = a$

(v) $r(ab) = (ra)b = a(rb)$

(vi) $A_{\overline{i}}A_{\overline{j}} \subset A_{\overline{i}+\overline{j}}$ *(i.e,* $p(ab){=}p(a){+}p(b)$*)*

for all $a, b \in A$ *and for all* $r, s \in \mathbb{C}$*.*

Remark 4.1.2 *Any associative superalgebra* L *over* F *can be considered as a Lie superalgebra over* F *by defining the Lie superbracket as* $[x, y]{=}xy - (-1)^{p(x)p(y)}yx$*.*

Definition 4.1.3 *Let* A *and* B *be associative superalgebras. A map* $\phi : A \to B$ *is a* homomorphism of associative superalgebras *if* ϕ *is a* $\mathbb{Z}_2$*-graded vector space homomorphism and* $\phi(xy) = \phi(x)\phi(y)$*, for all* $x, y \in A$*.*

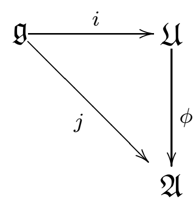

Figure 4.1. Universal Enveloping Algebra diagram.

Definition 4.2.4 *[17, 6.1] Let $\mathfrak{g}$ be a Lie superalgebra. A* universal enveloping algebra *of $\mathfrak{g}$ is a pair $(\mathfrak{U}, i)$ where $\mathfrak{U}$ is an associative superalgebra with identity and i: $\mathfrak{g} \rightarrow \mathfrak{U}$ is a homomorphism of $\mathbb{Z}_2$-graded vector spaces such that:*

$$i([x,y]) = i(x)i(y) - (-1)^{p(x)p(y)}i(y)i(x) \tag{4.2.5}$$

with the following universal property:

For any associative superalgebra $\mathfrak{A}$ with identity and any $\mathbb{Z}_2$-graded vector space homomorphism $j : \mathfrak{g} \rightarrow \mathfrak{A}$ satisfying 4.2.5*, there exists a unique homomorphism of associative superalgebras $\phi : \mathfrak{U} \rightarrow \mathfrak{A}$ such that $\phi(1) = 1$ and $\phi \circ i = j$.*

In other words, there exists unique ϕ such that the diagram in Figure 4.2 commutes (here i is the inclusion map).

One way to explicitly describe the universal enveloping algebra of $\mathfrak{g}$ is using the tensor algebra given as follows.

Definition 4.2.6 $T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} T^k(\mathfrak{g})$*, where $T^k(\mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g} \otimes ... \otimes \mathfrak{g}$, k times, and $T^0(\mathfrak{g}) = \mathbb{C}$. We call $T(\mathfrak{g})$ the* tensor algebra of $\mathfrak{g}$.

If $I(\mathfrak{g}) = \langle x \otimes y - (-1)^{p(x)p(y)} y \otimes x - [x,y] | x, y \in \mathfrak{g}\rangle$, $\pi : T(\mathfrak{g}) \rightarrow T(\mathfrak{g})/I(\mathfrak{g})$ is the natural projection, and $i = \pi|_{\mathfrak{g}}$ and $\mathfrak{U}(\mathfrak{g})$=$T(\mathfrak{g})/I(\mathfrak{g})$, then $(T(\mathfrak{g})/I(\mathfrak{g}), i)$ is a universal enveloping algebra of $\mathfrak{g}$. Hence, for every Lie superalgebra $\mathfrak{g}$ there exists a universal enveloping algebra of $\mathfrak{g}$. On the other hand, any two such algebras are isomorphic.

The proof for the Lie superalgebra case(cf. [17], §6.1) is similar to that of the Lie algebra case (cf. [14], §17.]).

4.3 Weyl Superalgebras

Let $\mathcal{W}_n$ be the subalgebra of End $\mathbb{C}[x_1, x_2, ..., x_n]$ generated by x_i and ∂_j, for $i, j = 1, 2, ..., n$, where $\partial_j := \frac{\partial}{\partial x_j}$. Explicitly,

$$\mathcal{W}_n := \text{span}_{\mathbb{C}}\{1, x_{i_1}...x_{i_k}\partial_{j_1}...\partial_{j_\ell} | i_1, ..., i_k, j_1, ..., j_\ell \leq n \text{ and } k, l \geq 0\}.$$

We call $\mathcal{W}_n$ the *n-th Weyl algebra.* We note that $\mathcal{W}_n$ is an associative algebra with identity. To define parity on $\mathcal{W}_n$, we establish a $\mathbb{Z}$-grading of $\mathcal{W}_n$ as follows. Set deg $x_i = 1$ and deg $\partial_j = -1$ and extend deg on $\mathcal{W}_n$ multiplicatively. Namely,

$$\deg(x_{i1}...x_{ik}\partial_{j_1}...\partial_{j_\ell}) := \sum_{m=1}^{k} i_m - \sum_{t=1}^{\ell} j_t.$$

We then define $(\mathcal{W}_n)_{\overline{0}}$=span$\{D \in \mathcal{W}_n |$deg D is even$\}$ and $(\mathcal{W}_n)_{\overline{1}}$=span$\{D \in \mathcal{W}_n |$deg D is odd$\}$.

Example 4.3.7 $x_3{}^2\partial_1 - x_1\partial_2{}^2 \in (\mathcal{W}_2)_{\overline{1}}$ *because* $\deg(x_3{}^2\partial_1) = 1$ *and* $\deg(x_1\partial_2{}^2) = -1$.

Recall the definition of the Heisenberg Lie algebra, cf. §2.4.23.

Proposition 4.3.8 *The vector space homomorphism* $\phi : \mathcal{H}_n \to \mathcal{W}_n$ *defined by* $\phi(a_i) = x_i, \phi(b_i) = \partial_i, \phi(z) = 1$ *extends to a homomorphism of associative algebras* $\overline{\phi} : \mathfrak{U}(\mathcal{H}_n) \to \mathcal{W}_n$ *(cf. §4.2.4). Furthermore,* $\overline{\phi}$ *is surjective and has a kernel* $\langle z - 1\rangle$*. In particular,* $\mathfrak{U}(\mathcal{H}_n)/\langle z - 1\rangle \simeq \mathcal{W}_n$ *as associative algebras.*

Proof: It is easy to check that ϕ satisfies (4.2.5), that $\overline{\phi}$ is surjective and that $\langle z - 1\rangle \subset \ker\overline{\phi}$. To show that $\ker\overline{\phi} \subset \langle z - 1\rangle$ we take $u \in \ker\overline{\phi}$ with $u = \sum_{I,J,k} z^k a_1^{i_1} a_2^{i_2}...a_n^{i_n} b_1^{j_1} b_2^{j_2}...b_n^{j_n}$ where $I = (i_1, i_2, ..., i_n); J = (j_1, j_2, ..., j_n)$, $k \in \mathbb{Z}$ then $\overline{\phi}(u) = 0$ implies $\sum_{I,J,k} C_{I,J,k} x_1^{i_1} x_2^{i_2}...x_n^{i_n} \partial_1^{j_1}\partial_2^{j_2}...\partial_n^{j_n} = 0$. By comparing coefficients, we have that $\sum_{I,J,k} C_{I,J,k} = 0$, for all I, J. Therefore, $z-1$ is a multiple of u, that is,

$u = (z-1)u_1$ for some $u_1 \in \mathfrak{U}(\mathcal{H}_n)$. Thus, $u \in \langle z-1 \rangle$. It follows that $\langle z-1 \rangle = \ker \overline{\phi}$. $\square$ The following lemma provides a key mechanism for proving that modules are simple and will be used frequently in the sequel.

Lemma 4.3.9 *Let $\mathfrak{g}$ be a Lie superalgebra with universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ and let M be a $\mathfrak{g}$-module. Assume that there is $u \in M$ such that (i) for every $w \in M$ there exists $X \in \mathfrak{U}(\mathfrak{g})$ such that $w = X(u)$; (ii) for every $v \in M$ there exists $Y \in \mathfrak{U}(\mathfrak{g})$ such that $u = Y(v)$. Then M is a simple module.*

Proof. Suppose that K is a non-trivial submodule of M. It then suffices to show that $K = M$. Let $v \in K$, with $v \neq 0$. Then $u = Y(v)$ for some $Y \in \mathfrak{U}(\mathfrak{g})$. Therefore $u \in K$. But then for any $w \in M$ there exists $X \in \mathfrak{U}(\mathfrak{g})$ such that $w = X(u)$. Thus $w \in K$ which implies that $K = M$. $\square$

CHAPTER 5

Orthosymplectic Lie Superalgebras

A class of Lie superalgbras preserving a nondegenerate bilinear form are the *orthosymplectic Lie superalgebras*, which we will denote by $\mathfrak{osp}(m|2n)$. In this chapter we first introduce the necessary general definitions for orthosymplectic Lie superalgebras, and then focus on the case $\mathfrak{osp}(1|2n)$. The latter case is considered in the rest of the thesis. In order to develop certain examples and motivation, in some cases the study will be restricted to $\mathfrak{osp}(1|2)$ or $\mathfrak{osp}(1|4)$.

5.1 Definition of $\mathfrak{osp}(1|2n)$

Definition 5.1.1 *Let* $\mathfrak{g}$ *be a Lie superalgebra. A bilinear form* B *on* $\mathfrak{g}$ *is said to be* supersymmetric *if* $B(x, y) = (-1)^{p(x)p(y)}B(y, x)$ *for all* $x, y \in \mathfrak{g}$.

Definition 5.1.2 *Let* $\mathfrak{g}$ *be a Lie superalgebra. A supersymmetric bilinear form* B *on* $\mathfrak{g}$ *is said to be* nondegenerate *if* $\operatorname{rad} B = 0$ *where*

$$\operatorname{rad} B := \{x \in \mathfrak{g} \mid B(x, y) = 0 \textit{ for all } y \in \mathfrak{g}\}.$$

Definition 5.1.3 *[17, 2.3] The* orthosymplectic Lie superalgebra *is defined as follows:*

$$\mathfrak{osp}(\mathbb{C}^{m|2n}, B) := \{g \in \mathfrak{gl}(\mathbb{C}^{m|2n}) | B(g(x), y) = -(-1)^{p(g)p(x)}B(x, g(y)) \textit{ for all } x, y \in \mathbb{C}^{m|2n}\},$$

where B is a nondegenerate supersymmetric bilinear form. Alternatively, if we use the matrix $s = \begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix}$ *of B relative to the standard basis we will have that* $\mathfrak{osp}(\mathbb{C}^{m|2n}, B)$ *is isomorphic to*

$$\mathfrak{osp}(m|2n) := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{gl}(m|2n) \mid A^tG + GA = B^tG - HC = D^tH + HD = 0 \right\}.$$

We note that for $\mathfrak{g} = \mathfrak{osp}(m|2n)$, we have that $\mathfrak{g}_{\overline{0}} \simeq \mathfrak{o}(m) \bigoplus \mathfrak{sp}(2n)$. In the particular cases examined, the orthogonal part of the sum will be 0, so we will exploit the isomorphism $\mathfrak{osp}(1|2n)_{\overline{0}} \simeq \mathfrak{sp}(2n)$.

In the remainder of the thesis, we will be concerned only with $\mathfrak{osp}(1|2n)$:

$$\mathfrak{osp}(1|2n) := \left\{ \begin{bmatrix} 0 & W \\ U & Y \end{bmatrix} \in \mathfrak{gl}(1|2n) \mid Y \in \mathfrak{sp}(2n), W, U \in \mathbb{C}^{2n} \right\}. \tag{5.1.4}$$

Here W is realized as a row vector formed from $W_1, W_2 \in \mathbb{C}^n$ and U is a column vector formed from W_2^t and $-W_1^t$.

Remark 5.1.5 *From (5.1.4), we can easily compute the dimension of* $\mathfrak{osp}(1|2n)$ *as follows:*

$$\dim(\mathfrak{osp}(1|2n)) = \dim(\mathfrak{sp}(2n)) + 2n = 2n^2 + 3n.$$

In particular, $\dim(\mathfrak{osp}(1|2)) = 5$ *and* $\dim(\mathfrak{osp}(1|4)) = 14$.

One can naturally extend the definitions of nilpotent algebra and normalizer in the superalgebras case. Using these new definitions we can introduce the following.

Definition 5.1.6 *A* Cartan subalgebra $\mathfrak{h}$ *of* $\mathfrak{g} = \mathfrak{osp}(1|2n)$ *is a self-normalizing nilpotent subalgebra of* $\mathfrak{g}$. *In what follows we fix* $\mathfrak{h}$ *to be* the standard Cartan subalgebra *of* $\mathfrak{g}$, *i.e.*

$$\mathfrak{h} = \{a_1E_{11} + a_2E_{22} + ... + a_nE_{n,n} - a_1E_{n+1,n+1} - ... - a_nE_{2n,2n} | a_i \in \mathbb{C}\}.$$

Weight modules of $\mathfrak{osp}(1|2n)$ play a crucial role in this thesis. Although the definition of a weight module is identical to Definition 2.4.28, for the reader's convenience we include it below.

Definition 5.1.7 *[17, 8.2] An $\mathfrak{osp}(1|2n)$-module M is a* weight module *if $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$, and $\dim M^\lambda < \infty$. Here, $M^\lambda = \{v \in M | hv = \lambda(h)v$, for all $h \in \mathfrak{h}\}$ is the λ*–weight space *of M and $\dim M^\lambda$ is the* weight multiplicity *of λ.*

We note here that *for the remainder of the thesis, $\mathfrak{g} = \mathfrak{osp}(1|2n)$ unless otherwise specified.*

Lemma 5.1.8 *Let V be a weight module of $\mathfrak{g}$ and W a submodule of V. Then W is also a weight module.*

Proof. Clearly $\bigoplus_{\lambda \in \mathfrak{h}^*} W^\lambda \subset W$. Let $v \in W$, $v \neq 0$. Since $v \in V$, then $v = \sum_{i=1}^p v_i$ with $v_i \in V^{\lambda_i}$. Assume also that λ_i are pairwise distinct. It remains to show that $v_i \in W$. Choose $h \in \mathfrak{h}$ such that $\lambda_i(h) \neq \lambda_j(h)$ for all i, j. To choose such h, note that for fixed i and j, the set of all $x \in \mathfrak{h}$ such that $(\lambda_i - \lambda_j)(x) = 0$ is a hyperplane in $\mathfrak{h}$. We then construct a linear system by repeated action of h on v as follows:
$h(v) = \sum_{i=1}^p \lambda_i(h) v_i$
$h^2(v) = \sum_{i=1}^p {\lambda_i}^2(h) v_i$
$\vdots$
$h^{p-1}(v) = \sum_{i=1}^p {\lambda_i}^{p-1}(h) v_i$.

We can then write the system as a matrix equation as follows:

$$\begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & \vdots & \vdots \\ {\lambda_1}^{p-1} & \cdots & {\lambda_n}^{p-1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{bmatrix} = \begin{bmatrix} v_1 \\ h^2 v_2 \\ \vdots \\ h^{p-1} v_p \end{bmatrix}$$

Consider the $p \times p$ system as $A\mathbf{v} = \mathbf{w}$. Then using the Vandermonde determinant formula, we have

$$\det A = \prod_{k>\ell}(\lambda_k - \lambda_\ell).$$

Since λ_i are distinct, we have that $\det A \neq 0$. Therefore, A is invertible, hence $\mathbf{v} = A^{-1}\mathbf{w}$, which implies that $v_i \in W$. □

Definition 5.1.9 *Let M be a weight $\mathfrak{g}$-module.*

(i) We say M is a module of bounded multiplicities or bounded $\mathfrak{g}$-module *if there exists $k \in \mathbb{N}$ such that* $\dim M^\lambda \leq k$ *for all weights λ of M. The minimal such k is the* degree of M. *Note that a module of degree 1 is completely pointed. (see Definition* 2.5.35*).*

(ii) If $v \in M^\lambda$ we say that the weight of v is λ and write $\lambda = \mathrm{wt}(v)$.

5.2 Root System of $\mathfrak{osp}(1|2n)$

Recall the definition of a root system of a Lie algebra (Definition 2.5.29). We extend this definition specifically for $\mathfrak{osp}(1|2n)$ as follows. (See [4],§1.2 for details about the more general cases.) We first define $\delta_i \in \mathfrak{h}^*$ by the map

$$\delta_i(a_1, a_2, ..., a_n, -a_1, -a_2, ..., -a_n) = a_i,$$

for $i = 1, 2, ..., n$.

Definition 5.2.10 *[17, 8.1] Let $\alpha \in \mathfrak{h}^*$, with $\alpha \neq 0$, then α is a* root of $\mathfrak{osp}(1|2n)$ *if $\mathfrak{g}^\alpha \neq 0$, where $\mathfrak{g}^\alpha = \{x \in \mathfrak{osp}(1|2n) | [h, x] = \alpha(h)x,$ for all $h \in \mathfrak{h}\}$. The set of all roots of $\mathfrak{osp}(1|2n)$ will be denoted by Δ. The* even roots *are $\Delta_{\bar{0}} = \{\alpha \in \Delta | \mathfrak{g}^\alpha \cap \mathfrak{g}_{\bar{0}}\}$ and* the odd roots *are $\Delta_{\bar{1}} = \{\alpha \in \Delta | \mathfrak{g}^\alpha \cap \mathfrak{g}_{\bar{1}}\}$.*

The roots of $\mathfrak{g}$ are listed in the following proposition.

Proposition 5.2.11 $\Delta_{\bar{0}} = \{\pm\delta_i \pm \delta_j; \pm 2\delta_i\}$ *and* $\Delta_{\bar{1}} = \{\pm\delta_i\}$ *for* $i = 1, 2, 3, ..., n$, $i \neq j$.

Note that, in future, when we identify $\mathfrak{sp}(2n)$ as the even part of $\mathfrak{osp}(1|2n)$, we will use δ_i's instead of ϵ_i's for the roots.

Definition 5.2.12 *[17, 3.4] Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$ and Δ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. A subset Π of Δ is a* base of Δ *if:*

(i) Π is a basis of $\mathfrak{h}^*$,

(ii) Each $\beta \in \Delta$ can be written as $\beta = \Sigma k_\alpha \alpha$, with $\alpha \in \Pi$ such that the coefficients k_α are either all nonnegative or all nonpositive.

If all k_α are nonnegative then β is a positive root. *Otherwise we call β a* negative root. *The sets of positive and negative roots are denoted by Δ^+ and Δ^-, respectively.*

For our convenience, and mostly due to the Weyl homomorphism, (see Proposition 6.1.1), we fix the following bases of the root systems of $\mathfrak{sp}(2n)$ and $\mathfrak{osp}(1|2n)$, respectively:

$$\Pi_{\mathfrak{sp}} = \{-2\epsilon_1, \epsilon_1 - \epsilon_2, ..., \epsilon_{n-1} - \epsilon_n\} \tag{5.2.13}$$

and

$$\Pi_{\mathfrak{osp}} = \{-\delta_1, \delta_1 - \delta_2, ..., \delta_{n-1} - \delta_n\}.$$

We should note that our choice of base of Δ is different from the one in [17]. Once the choice of a base of Δ is fixed, we can identify the positive and negative roots as follows:

$$\Delta^+ = \{-\delta_i, -\delta_i - \delta_j, \delta_i - \delta_j, -2\delta_i \mid 1 \leq i < j \leq n\}$$
$$\Delta^- = -\Delta^+$$

Definition 5.2.14 *[17, 3.2] Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$ and let Δ^+ be the set of positive roots associated with $\Pi_{\mathfrak{osp}}$. Then the subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ with $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ is the* Borel subalgebra of $\mathfrak{g}$ corresponding to Π.

5.3 Root Vectors of $\mathfrak{osp}(1|2n)$

Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$. In this section we fix elements X_α in $\mathfrak{g}^\alpha$ for all $\alpha \in \Delta$.

First, we list the even elements X_α, i.e. those in $\mathfrak{osp}(1|2n)_{\bar{0}} \simeq \mathfrak{sp}(2n)$:

$$X_{\delta_i-\delta_j} = \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{bmatrix}, X_{2\delta_i} = \begin{bmatrix} 0 & E_{ii} \\ 0 & 0 \end{bmatrix}, X_{-2\delta_i} = \begin{bmatrix} 0 & 0 \\ E_{ii} & 0 \end{bmatrix},$$

$$X_{\delta_i+\delta_j} = \begin{bmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{bmatrix}, X_{-\delta_i-\delta_j} = \begin{bmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{bmatrix}$$

The odd elements $X_{\pm\delta_i}$ are of the form $\begin{bmatrix} 0 & W_1 & W_2 \\ W_2^t & 0 & 0 \\ -W_1^t & 0 & 0 \end{bmatrix}$ where

- $W_1 = (0, ..., 1, ..., 0)$ (1 on the ith position) and $W_2 = (0, 0, ..., 0)$ for $X_{-\delta_i}$;
- $W_1 = (0, 0, ..., 0)$ and $W_2 = (0, ..., 1, ..., 0)$ (1 on the ith position) for X_{δ_i}.

Finally we fix the following elements in $\mathfrak{h}$:

$$h_{\delta_i-\delta_j} = \begin{bmatrix} E_{ii}-E_{jj} & 0 \\ 0 & -E_{ii}+E_{jj} \end{bmatrix}, h_{2\delta_i} = \begin{bmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{bmatrix}.$$

Note that $\{h_{2\delta_1}, h_{\delta_1-\delta_2}, ..., h_{\delta_{n-1}-\delta_m}\}$ forms a basis of $\mathfrak{h}$.

5.4 Root Vector Relations for $\mathfrak{osp}(1|2n)$

It will be convenient to have root vector relations for $[X_\alpha, X_\beta]$ available for later use. They are grouped in the order odd-odd, even-odd, and even-even based on the parity of the root vectors X_α, X_β. Note that if the sum $\alpha+\beta$ of the roots α, β is not a root, then the corresponding Lie superbracket is zero.

The odd-odd relations are symmetric.

$$[X_{\pm\delta_i}, X_{\pm\delta_i}] = \pm 2X_{\pm 2\delta_i}$$

$$[X_{\pm\delta_i}, X_{\pm\delta_{i+1}}] = \pm X_{\pm\delta_i \pm\delta_{i+1}}$$

$$[X_{\delta_i}, X_{-\delta_i}] = h_{2\delta_i}$$

$$[X_{\pm\delta_i}, X_{\mp\delta_{i+1}}] = \pm X_{\pm\delta_i \mp\delta_{i+1}}$$

The even-odd relations.

$$[X_{2\delta_i}, X_{-\delta_i}] = -X_{\delta_i}$$

$$[X_{\delta_i-\delta_{i+1}}, X_{\delta_{i+1}}] = X_{\delta_i}$$

$$[X_{\delta_i-\delta_{i+1}}, X_{-\delta_i}] = -X_{-\delta_{i+1}}$$

$$[X_{-2\delta_i}, X_{\delta_i}] = -X_{-\delta_i}$$

$$[X_{-\delta_i-\delta_{i+1}}, X_{\delta_i}] = -X_{-\delta_{i+1}}$$

$$[X_{-\delta_i-\delta_{i+1}}, X_{\delta_{i+1}}] = -X_{-\delta_i}$$

$$[X_{\delta_i+\delta_{i+1}}, X_{-\delta_i}] = -X_{\delta_{i+1}}$$

$$[X_{\delta_i+\delta_{i+1}}, X_{-\delta_{i+1}}] = -X_{\delta_i}$$

$$[X_{\delta_{i+1}+\delta_i}, X_{\delta_i}] = X_{\delta_{i+1}}$$

$$[X_{\delta_{i+1}-\delta_i}, X_{-\delta_{i+1}}] = -X_{-\delta_i}$$

The even-even relations are antisymmetric.

$$[X_{2\delta_i}, X_{-2\delta_i}] = h_{2\delta_i}$$

$$[X_{2\delta_i}, X_{\delta_{i+1}-\delta_i}] = -X_{\delta_i+\delta_{i+1}}$$

$$[X_{2\delta_{i+1}}, X_{\delta_i-\delta_{i+1}}] = -X_{\delta_i+\delta_{i+1}}$$

$$[X_{-2\delta_{i+1}}, X_{\delta_{i+1}-\delta_i}] = X_{-\delta_i-\delta_{i+1}}$$

$$[X_{\delta_{i+1}-\delta_i}, X_{\delta_i-\delta_{i+1}}] = -h_{\delta_i-\delta_{i+1}}$$

$$[X_{-2\delta_i}, X_{\delta_i-\delta_{i+1}}] = -X_{\delta_i-\delta_{i+1}}$$

The following specific computations will be referred to later in the document:

$$[X_{\delta_1}, X_{-\delta_1}] = h_{2\delta_1} \tag{5.4.15}$$

$$[X_{-\delta_1}, X_{2\delta_1}] = X_{\delta_1} \tag{5.4.16}$$

$$[X_{\delta_k-\delta_{k+1}}, X_{\delta_1}] = 0 \tag{5.4.17}$$

Remark 5.4.18 *We will identify $\mathfrak{h}^*$ with $\mathbb{C}^n$ via the vector space isomorphism $\mathfrak{h}^* \mapsto \mathbb{C}^n$, $\lambda \mapsto (\lambda(h_{2\delta_1}), ..., \lambda(h_{2\delta_n}))$. Note that $\lambda = \sum_{i=1}^n \lambda(h_{2i})\delta_i$.*

5.5 The Space $\mathbb{C}^{1|2n}$ as an $\mathfrak{osp}(1|2n)$-representation

We introduce the standard representation here for use as an example of concepts in the next section. In addition, this representation plays an important role in Chapter 7.

We denote the standard basis of $\mathbb{C}^{1|2n}$ by $\{v_0, v_1, ..., v_n, v_{n+1}, ..., v_{2n}\}$, with v_0 being even and the remaining elements odd. The action of the basis root vectors $X_{-\delta_1}$ and $X_{\delta_i-\delta_{i+1}}$, $i \geq 1$, on $\mathbb{C}^{1|2n}$ is given by matrix multiplication. For completeness, we write down the action of $X_{\pm\delta_i}$, $X_{\delta_i-\delta_{i+1}}$:

$$X_{-\delta_i}(v_j) = \begin{cases} -v_{n+i} \text{ if } j = 0 \\ v_0 \text{ if } j = i \\ 0 \text{ otherwise} \end{cases}$$

$$X_{\delta_i}(v_j) = \begin{cases} -v_i \text{ if } j = 0 \\ v_0 \text{ if } j = n+i \\ 0 \text{ otherwise} \end{cases} \quad X_{\delta_i - \delta_{i+1}}(v_j) = \begin{cases} v_i \text{ if } j = i+1 \\ -v_{n+i+1} \text{ if } j = n+i \\ 0 \text{ otherwise.} \end{cases}$$

5.6 Highest Weight Modules and Primitive Vectors

Definition 5.6.19 *Let M be a $\mathfrak{g}$-module, then $0 \neq v \in M$ is a* primitive vector *if $xv = 0$ for all $x \in \mathfrak{n}^+$. Equivalently, v is a primitive vector if $X_\alpha v = 0$ for $\alpha = -\delta_1, \delta_1 - \delta_2, ..., \delta_{n-1} - \delta_n$.*

Example 5.6.20 *Let $n = 2$ and $M = \mathbb{C}^{1|4}$. In this case, we have*

$$X_{-\delta_1}(v_4) = X_{\delta_2 - \delta_1}(v_4) = 0,$$

thus v_4 is a primitive vector of M of weight $\mathrm{wt}(v_4) = (0, -1)$.

Definition 5.6.21 *[17, 8.2] By $\mathbb{C}_\lambda$ we denote the one-dimensional $\mathfrak{U}(\mathfrak{b})$-module $\mathbb{C}v_\lambda$ with action defined by $hv_\lambda = \lambda(h)v_\lambda$, for $h \in \mathfrak{h}$ and $xv_\lambda = 0$ for $x \in \mathfrak{n}^+$.*

Definition 5.6.22 *[17, 8.2]*

1. *The* Verma module with highest weight $\lambda \in \mathfrak{h}^*$ *is defined by $M(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} \mathbb{C}_\lambda$. Every quotient module M of $M(\lambda)$ is by definition a* highest weight module *of* highest weight λ*. If M is a highest weight module, any $v \in M^\lambda$ is called a* highest weight vector *of M.*
2. *The quotient of $M(\lambda)/M_+$ where M_+ is the maximal proper submodule of $M(\lambda)$ is the* simple highest weight module of weight λ*, and it will be denoted by $L(\lambda)$.*

3. *We say* λ *is a* $\mathfrak{g}$-bounded weight *(or, simply,* bounded weight*) if* $L(\lambda)$ *is a bounded* $\mathfrak{g}$*-module.*

Lemma 5.6.23 *Every highest weight vector of a highest weight module is a primitive vector.*

Proof. Let M be a highest weight module of weight λ. Then $M = M(\lambda)/M'$ for some submodule M' of $M(\lambda)$. Every highest weight vector of M is a multiple of $u = 1 \otimes v_\lambda + M'$. Let $x \in \mathfrak{n}^+$, then we have:

$$\begin{aligned} xu &= x1 \otimes v_\lambda \pm 1 \otimes xv_\lambda + M' \\ &= 1 \otimes xv_\lambda \pm 1 \otimes xv_\lambda + M' \\ &= 0 + M'. \end{aligned}$$

Therefore, u is primitive as required. □

Lemma 5.6.24 *Let* v *be a primitive vector of weight* λ *of a module* V*. Then* $L(\lambda)$ *is isomorphic to a subquotient of* V*.*

Proof. This is a standard fact, but for reader's convenience we outline the proof. We take the submodule M of V generated by v, i.e. $M = \mathfrak{U}(\mathfrak{g}) \cdot v$. This module is a highest weight module. Indeed, the map $M(\lambda) \to M$, $u \otimes v_\lambda \mapsto u \cdot v$, is a surjective module homomorphism. If M' is the kernel of this homomorphism, then $M \simeq M(\lambda)/M'$. On the other hand, $L(\lambda) = M(\lambda)/M_+$, where M_+ is a maximal submodule of $M(\lambda)$. Then $L(\lambda) \simeq (M(\lambda)/M') \,/\, (M_+/M')$. Hence, $L(\lambda)$ is a quotient of the submodule M of V, which completes the proof. □

5.7 Finite-Dimensional Representations

Definition 5.7.25 *A weight* $(\lambda_1, \lambda_2, ..., \lambda_n)$ *is a* dominant integral weight *if* $\lambda_i \in \mathbb{Z}_{\leq 0}$ *and* $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$.

The following result can be found in ([17], §14.1 et seq) and ([4], Lemma 2.8).

Theorem 5.7.26 *A weight module $L(\lambda)$ is finite-dimensional if and only if λ is a dominant integral weight.*

5.8 Tensor Products of Weight Modules

The action of $\mathfrak{g}$ on the tensor product $V \otimes W$ of two $\mathfrak{g}$-modules V, W is given by the formula

$$x(v \otimes w) = xv \otimes w + (-1)^{|x||v|} v \otimes xw \text{ with } x \in \mathfrak{U}(\mathfrak{g}) \text{ and } v \in V, w \in W. \quad (5.8.27)$$

(see, for example, [17], §A.2)

Lemma 5.8.28 *Let $u \in V \otimes W$, with $u = v \otimes w$,* $\mathrm{wt}(v) = \lambda$ *and* $\mathrm{wt}(w) = \mu$. *Then the* $\mathrm{wt}(u) = \lambda + \mu$. *Furthemore, if v and w are highest weight vectors of V and W respectively, then $v \otimes w$ is a primitive vector of $V \otimes W$.*

Proof. Let $h \in \mathfrak{h}$ (the Cartan subalgebra of $\mathfrak{osp}(1|2n)$) and note that $h \in \mathfrak{h}$ is even since $\mathfrak{h} = \mathfrak{h}_0$. Then $h(u) = hv \otimes w + v \otimes hw = (\lambda + \mu)(h)(v \otimes w)$ by definition of the action. Thus $h(u) = (\lambda + \mu)(h)(u)$.

If v, w are highest weight, then both are also primitive (see Lemma 5.6.23). Let $X \in \mathfrak{n}^+$, then $X(v \otimes w) = Xv \otimes w \pm v \otimes Xw = 0$. Therefore, $v \otimes w$ is primitive. $\square$

CHAPTER 6

Weyl Representations of $\mathfrak{osp}(1|2n)$

We provide a homomorphism from $\mathfrak{U}(\mathfrak{osp}(1|2n))$ to the Weyl superalgebra $\mathcal{W}_n$. We use this homomorphism to study $\mathfrak{osp}(1|2n)$-modules corresponding to modules of $\mathcal{W}_n$ of shifted Laurent polynomials.

6.1 Weyl Algebra Homomorphism

Proposition 6.1.1 *The following correspondence defines a homomorphism* $\phi : \mathfrak{U}(\mathfrak{osp}(1|2n)) \to \mathcal{W}_n$ *of associative superalgebras with identity:*

$$\begin{aligned}
X_{\delta_i-\delta_j} &\longmapsto x_i\partial_j; i \neq j;\\
X_{2\delta_i} &\longmapsto \frac{1}{2}x_i^2;\\
X_{-2\delta_i} &\longmapsto \frac{-1}{2}\partial_i^2;\\
X_{\delta_i+\delta_j} &\longmapsto x_ix_j;\\
X_{-\delta_i-\delta_j} &\longmapsto -\partial_i\partial_j;\\
h_{\delta_i-\delta_j} &\longmapsto x_i\partial_i - x_j\partial_j;\\
h_{2\delta_i} &\longmapsto x_i\partial_i + \frac{1}{2};\\
X_{\delta_i} &\longmapsto \frac{1}{\sqrt{2}}x_i;\\
X_{-\delta_i} &\longmapsto \frac{1}{\sqrt{2}}\partial_i.
\end{aligned}$$

Proof. According to Definition 4.2.4, in order to show that ϕ is a homomorphism of associative superalgebras, it is sufficient to show that

$$\phi([x,y]) = \phi(x)\phi(y) - (-1)^{p(x)p(y)}\phi(y)\phi(x)$$

for any basis elements $x, y \in \mathfrak{osp}(1|2n)$ (cf. Lemma 3.2.10). We do this case by case for x, y as follows: *(odd,odd)*, *(odd, even)*, and *(even,even)*.

Case 1: (odd,odd)

Subcase 1: $x = X_{\delta_i}, y = X_{\delta_j}, i = j$.

$$\phi([X_{\delta_i}, X_{\delta_i}]) = \phi(2X_{\delta_{2i}}) = 2(\frac{1}{2}x_i^2) = x_i^2$$

$$[\phi(X_{\delta_i}), \phi(X_{\delta_i})] = [\frac{1}{\sqrt{2}}x_i, \frac{1}{\sqrt{2}}x_i] = x_i^2$$

Subcase 2: $x = X_{\delta_i}, y = X_{\delta_j}, i \neq j$.

$$\phi([X_{\delta_i}, X_{\delta_j}]) = \phi(X_{\delta_i+\delta_j}) = x_i x_j$$

$$[\phi(X_{\delta_i}), \phi(X_{\delta_j})] = [\frac{1}{\sqrt{2}}x_i, \frac{1}{\sqrt{2}}x_j] = x_i x_j$$

Subcase 3: $x = X_{-\delta_i}, y = X_{-\delta_j}, i = j$.

$$\phi([X_{-\delta_i}, X_{-\delta_i}]) = \phi(-2X_{-\delta_{2i}}) = \partial_i^2$$

$$[\phi(X_{-\delta_i}), \phi(X_{-\delta_i})] = [\frac{1}{\sqrt{2}}\partial_i, \frac{1}{\sqrt{2}}\partial_i] = \partial_i^2$$

Subcase 4: $x = X_{-\delta_i}, y = X_{-\delta_j}, i \neq j$.

$$\phi([X_{-\delta_i}, X_{-\delta_j}]) = \phi(-X_{-\delta_i-\delta_j}) = \partial_i\partial_j$$

$$[\phi(X_{-\delta_i}), \phi(X_{-\delta_j})] = [\frac{1}{\sqrt{2}}\partial_i, \frac{1}{\sqrt{2}}\partial_j] = \partial_i\partial_j$$

Subcase 5: $x = X_{\delta_i}, y = X_{\delta_j}, i = j$.

$$\phi([X_{\delta_i}, X_{-\delta_i}]) = \phi(h_{\delta_{2i}}) = x_i\partial_i + \frac{1}{2}$$

$$[\phi(X_{\delta_i}), \phi(X_{-\delta_i})] = [\frac{1}{\sqrt{2}}\partial_i, \frac{1}{\sqrt{2}}x_i] = x_i\partial_i + \frac{1}{2}$$

As an example, we demonstrate the action of $[\frac{1}{\sqrt{2}}\partial_i, \frac{1}{\sqrt{2}}x_i]$ on $f \in \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$.

$$[\frac{1}{\sqrt{2}}\partial_i, \frac{1}{\sqrt{2}}x_i](f) =$$
$$\frac{1}{2}x_i\partial_i(f) + \frac{1}{2}\partial_i x_i(f) =$$
$$\frac{1}{2}x_i\partial_i(f) + \frac{1}{2}(x_i\partial_i(f) + \frac{1}{2}(f)] = (x_i\partial_i + \frac{1}{2})(f)$$

Subcase 6: $x = X_{\delta_i}, y = X_{-\delta_j}, i \neq j$.

$$\phi([X_{\delta_i}, X_{-\delta_j}]) = \phi(X_{\delta_i - \delta_j}) = x_i\partial_j$$
$$[\phi(X_{\delta_i}), \phi(X_{-\delta_j})] = [\frac{1}{\sqrt{2}}x_i, \frac{1}{\sqrt{2}}\partial_j] = x_i\partial_j$$

Case 2: (odd,even)

Subcase 1: $x = X_{2\delta_i}, y = X_{-\delta_j}, i = j$.

$$\phi([X_{2\delta_i}, X_{-\delta_i}]) = \phi(-X_{\delta_i}) = -\frac{1}{\sqrt{2}}x_i$$
$$[\phi(X_{2\delta_i}), \phi(X_{-\delta_i})] = [\frac{1}{2}x_i^2, \frac{1}{\sqrt{2}}\partial_i] = -\frac{1}{\sqrt{2}}x_i$$

Subcase 2: $x = X_{\delta_i - \delta_{i+1}}, y = X_{-\delta_i}$.

$$\phi([X_{\delta_i - \delta_{i+1}}, X_{-\delta_i}]) = \phi(-X_{-\delta_{i+1}}) = \frac{1}{\sqrt{2}}\partial_{i+1}$$
$$[\phi(X_{\delta_i - \delta_{i+1}}), \phi(X_{-\delta_i})] = [x_i\partial_{i+1}, \frac{1}{\sqrt{2}}\partial_i] = \frac{1}{\sqrt{2}}\partial_{i+1}$$

Subcase 3: $x = X_{\delta_i - \delta_{i+1}}, y = X_{\delta_{i+1}}$.

$$\phi([X_{\delta_i - \delta_{i+1}}, X_{\delta_{i+1}}]) = \phi(X_{\delta_i}) = \frac{1}{\sqrt{2}}x_i$$
$$[\phi(X_{\delta_i - \delta_{i+1}}), \phi(X_{\delta_{i+1}})] = [x_i\partial_{i+1}, \frac{1}{\sqrt{2}}x_{i+1}] = \frac{1}{\sqrt{2}}x_i$$

Subcase 4: $x = X_{-2\delta_i}, y = X_{\delta_j}, i = j$.

$$\phi([X_{-2\delta_i}, X_{\delta_i}]) = \phi(-X_{\delta_i}) = -\frac{1}{\sqrt{2}}x_i$$
$$[\phi(X_{-2\delta_i}), \phi(X_{\delta_i})] = [\frac{1}{2}x_i^2, \frac{1}{\sqrt{2}}\partial_i] = -\frac{1}{\sqrt{2}}x_i$$

Subcase 5: $x = X_{-\delta_i-\delta_{i+1}}, y = X_{\delta_i}$.

$$\phi([X_{-\delta_i-\delta_{i+1}}, X_{\delta_i}]) = \phi(-X_{-\delta_{i+1}}) = -\frac{1}{\sqrt{2}}\partial_{i+1}$$
$$[\phi(X_{-\delta_i-\delta_{i+1}}), \phi(X_{\delta_i})] = [-\partial_i\partial_{i+1}, \frac{1}{\sqrt{2}}x_i] = -\frac{1}{\sqrt{2}}\partial_{i+1}$$

Subcase 6: $x = X_{-\delta_i-\delta_{i+1}}, y = X_{\delta_{i+1}}$.

$$\phi([X_{-\delta_i-\delta_{i+1}}, X_{\delta_{i+1}}]) = \phi(-X_{-\delta_i}) = -\frac{1}{\sqrt{2}}\partial_i$$
$$[\phi(X_{-\delta_i-\delta_{i+1}}), \phi(X_{\delta_{i+1}})] = [-\partial_i\partial_{i+1}, \frac{1}{\sqrt{2}}x_{i+1}] = -\frac{1}{\sqrt{2}}\partial_i$$

Subcase 7: $x = X_{\delta_i+\delta_{i+1}}, y = X_{-\delta_i}$.

$$\phi([X_{\delta_i+\delta_{i+1}}, X_{-\delta_i}]) = \phi(-X_{\delta_{i+1}}) = -\frac{1}{\sqrt{2}}x_{i+1}$$
$$[\phi(X_{\delta_i+\delta_{i+1}}), \phi(X_{-\delta_i}] = [x_ix_{i+1}, \frac{1}{\sqrt{2}}\partial_i] = -\frac{1}{\sqrt{2}}x_{i+1}$$

Subcase 8: $x = X_{\delta_i+\delta_{i+1}}, y = X_{-\delta_{i+1}}$.

$$\phi([X_{\delta_i+\delta_{i+1}}, X_{-\delta_{i+1}}]) = \phi(-X_{\delta_i}) = -\frac{1}{\sqrt{2}}x_i$$
$$[\phi(X_{\delta_i+\delta_{i+1}}), \phi(X_{-\delta_{i+1}}] = [x_ix_{i+1}, \frac{1}{\sqrt{2}}\partial_{i+1}] = -\frac{1}{\sqrt{2}}x_i$$

Subcase 9: $x = X_{\delta_{i+1}-\delta_i}, y = X_{\delta_i}$.

$$\phi([X_{\delta_{i+1}-\delta_i}, X_{\delta_i}]) = \phi(X_{\delta_{i+1}}) = \frac{1}{\sqrt{2}}x_{i+1}$$
$$[\phi(X_{\delta_{i+1}-\delta_i}), \phi(X_{\delta_i})] = [x_{i+1}\partial_i, \frac{1}{\sqrt{2}}x_i] = \frac{1}{\sqrt{2}}x_{i+1}$$

Subcase 10: $x = X_{\delta_{i+1}-\delta_i}, y = X_{-\delta_{i+1}}$.

$$\phi([X_{\delta_{i+1}-\delta_i}, X_{-\delta_{i+1}}]) = \phi(-X_{-\delta_i}) = -\frac{1}{\sqrt{2}}\partial_i$$
$$[\phi(X_{\delta_{i+1}-\delta_i}), \phi(X_{-\delta_{i+1}})] = [x_{i+1}\partial_i, \frac{1}{\sqrt{2}}\partial_{i+1}] = -\frac{1}{\sqrt{2}}\partial_i$$

Case 3: (even,even)

Subcase 1: $[X_{2\delta_i}, X_{\delta_{i+1}-\delta_i}]$.

$$\phi([X_{2\delta_i}, X_{\delta_{i+1}-\delta_i}]) = \phi(-X_{\delta_i+\delta_{i+1}}) = -x_i x_{i+1}$$
$$[\phi(X_{2\delta_i}), \phi(X_{\delta_{i+1}-\delta_i})] = [-\frac{1}{2}x_i^2, x_{i+1}\partial_i] = -x_i x_{i+1}$$

Subcase 2: $[X_{2\delta_{i+1}}, X_{\delta_{i+1}-\delta_i}]$.

$$\phi([X_{2\delta_{i+1}}, X_{\delta_{i+1}-\delta_i}]) = \phi(-X_{\delta_i+\delta_{i+1}}) = -x_i x_{i+1}$$
$$[\phi(X_{2\delta_{i+1}}), \phi(X_{\delta_{i+1}-\delta_i})] = [-\frac{1}{2}x_{i+1}^2, x_i\partial_{i+1}] = -x_i x_{i+1}$$

Subcase 3: $[X_{2\delta_i}, X_{-2\delta_i}]$.

$$\phi([X_{2\delta_i}, X_{-2\delta_i}]) = \phi(h_{2\delta_i}) = x_i\partial_i + \frac{1}{2}$$
$$[\phi(X_{2\delta_i}), \phi(X_{-2\delta_i})] = [\frac{1}{2}x_i^2, -\frac{1}{2}\partial_i^2] = x_i\partial_i + \frac{1}{2}$$

Subcase 4: $[X_{\delta_{i+1}-\delta_i}, X_{-2\delta_{i+1}}]$.

$$\phi([X_{\delta_{i+1}-\delta_i}, X_{-2\delta_i}]) = \phi(-X_{\delta_i-\delta_{i+1}}) = \partial_i\partial_{i+1}$$
$$[\phi(X_{\delta_{i+1}-\delta_i}), \phi(X_{-2\delta_{i+1}})] = [x_{i+1}\partial_i, -\frac{1}{2}\partial_{i+1}^2] = \partial_i\partial_{i+1}$$

Subcase 5: $[X_{\delta_{i+1}-\delta_i}, X_{\delta_i-\delta_{i+1}}]$.

$$\phi([X_{\delta_{i+1}-\delta_i}, X_{\delta_i-\delta_{i+1}}]) = \phi(-h_{\delta_i-\delta_{i+1}}) = -x_i\partial_i + x_{i+1}\partial_{i+1}$$
$$[\phi(X_{\delta_{i+1}-\delta_i}), \phi(X_{\delta_i-\delta_{i+1}})] = [x_{i+1}\partial_i, x_i\partial_{i+1}] = -x_i\partial_i + x_{i+1}\partial_{i+1}$$

Subcase 6: $[X_{-2\delta_i}, X_{\delta_i-\delta_{i+1}}]$.

$$\phi([X_{-2\delta_i}), X_{\delta_i-\delta_{i+1}}]) = \phi(-X_{\delta_i-\delta_{i+1}}) = -x_i\partial_{i+1}$$
$$[\phi(X_{-2\delta_i}), \phi(X_{\delta_i-\delta_{i+1}})] = [-\frac{1}{2}\partial_i^2, x_i\partial_{i+1}] = -x_i\partial_{i+1}$$

□

6.2 The $\mathfrak{osp}(1|2)$-module $M = \mathbb{C}[x^{\pm 1}]$

In this section M=$\mathbb{C}[x^{\pm 1}]$ will be the *ring of Laurent polynomials of one variable* over $\mathbb{C}$. Then $M = M_{\overline{0}} \oplus M_{\overline{1}}$ where $M_{\overline{0}}$ =span$\{x^{2i} \mid i \in \mathbb{Z}\}$ and $M_{\overline{1}}$=span$\{x^{2i+1} \mid i \in \mathbb{Z}\}$. This defines a $\mathbb{Z}_2$-graded vector space structure on M. We define an $\mathfrak{osp}(1|2)$-module structure on M using the homomorphism ϕ defined in Proposition 6.1.1. We set for convenience $x = x_1$ and $\partial = \partial_1$.

Example 6.2.2 $h_{2\delta_1}(x^j) = (x\partial + \frac{1}{2})x^j = (j + \frac{1}{2})x^j$, $X_{2\delta_1}(x^0) = \frac{1}{2}x^2$, $X_{-2\delta_1}(x^2) = (-\frac{1}{2}\partial^2)x^2 = -x^0 = -1$

Remark 6.2.3 *M will be considered as a module of $\mathfrak{osp}(1|2)_{\overline{0}} \cong \mathfrak{sl}(2)$ in a natural way. $M_{\overline{0}}$ is also an $\mathfrak{sl}(2)$-module. Based on Remark 2.2.11 we will identify e with $X_{2\delta_1}$, f with $X_{-2\delta_1}$, and h with $h_{2\delta_1}$.*

Our main theorem in the section is the following.

Theorem 6.2.4 *Let M be defined as above, and define M^+ and M^- as follows: M^+ :=span$_{\mathbb{C}}\{x^i | i \geq 0\}$ and M^-:= M/M^+=span$_{\mathbb{C}}\{x^i + M^+ | i < 0\}$.*

Then we have:

(i) M^+ (and, hence, M^-) is an $\mathfrak{osp}(1|2)$-submodule of M;

(ii) M^+ and M^- are simple $\mathfrak{osp}(1|2)$-modules;

(iii) The sequence $0 \longrightarrow M^+ \longrightarrow M \longrightarrow M^- \longrightarrow 0$ is a non-split exact sequence of $\mathfrak{osp}(1|2)$-modules. In particular, M is indecomposable.

Proof.

(i) Clearly M^+ is a subspace M. It suffices then to show that M^+ is closed under the action of $\mathfrak{osp}(1|2)$. In fact, we need only to verify closure for a generating set of elements of $\mathfrak{osp}(1|2)$ acting on a basis element of M^+, cf. Corollary 3.2.12. We use the basis and representation information from 6.1.1 above and note that X_{δ_1} and $X_{-\delta_1}$ generate the other three basis elements. Let x^i, $i \geq 0$, be a basis element of M^+. Then $X_{\delta_1}(x^i) = (\frac{1}{\sqrt{2}}x)(x^i) = \frac{1}{\sqrt{2}}x^{i+1} \in M^+$. On the other hand, $X_{-\delta_1}(x^i) = (\frac{1}{\sqrt{2}}\partial)(x^i) = \frac{1}{\sqrt{2}}x^{i-1}$. If $i \geq 1$, then $\frac{1}{\sqrt{2}}x^{i-1} \in M^+$. If $i = 0$, then we have $(\frac{1}{\sqrt{2}}\partial)(x^0) = 0 \in M^+$.

(ii) First, we will establish that M^+ is simple. Suppose that K is a non-trivial submodule of M^+. It then suffices to show that $K = M^+$. Let $w \in K$, with $w \neq 0$, then $w = \sum_{i=0}^{n} a_i x^i$ with $a_n \neq 0$. Define $u \in \mathfrak{U}(\mathfrak{osp}(1|2))$ as follows: $u := \frac{(\sqrt{2}X_{-\delta_1})^n}{n!a_n}$. Since $X_{-\delta_1}$ acts as $\frac{1}{\sqrt{2}}\partial$, we have $u(w) = \frac{n!a_n x^0}{n!a_n} = x^0$. Hence $x^0 \in K$. Now let $v \in M^+$ with $v \neq 0$. Let $v = \sum_{i=0}^{m} b_i x^i$ and $b_m \neq 0$. Then $u_1(x^0) = v$ for $u_1 = \sum_{i=0}^{m} b_i(\sqrt{2}X_{\delta_1})^i$. Therefore, $v \in K$, hence, $K = M^+$. Thus, M^+ is simple (by Lemma 4.3.9). Now for the assertion that M^- is simple. Let K be a non-trivial submodule of M^-. Let $w \in K$, $w \notin M^+$, $w := \sum_{i=n}^{0} a_i x^i + M^+$ with $a_n \neq 0$ and $n < 0$. Define $u \in \mathfrak{U}(\mathfrak{osp}(1|2))$ as follows: $u := \frac{(\sqrt{2}X_{\delta_1})^{-n-1}}{a_n}$. Then $u(w) = x^{-1} + M^+$. Hence, $x^{-1} \in K$. Now let $v \in M^-, v \notin M^+$. Let $v = \sum_{i=m}^{-1} b_i x^i + M^+$ and $b_m \neq 0, m < 0$. Then $u_1(x^{-1}) = v + M^+$ for $u_1 = \sum_{i=m}^{-1} b_i \frac{(\sqrt{2}X_{-\delta_1})^{-i-1}}{(-1)^{i+1}(-i-1)!}$. Hence, $v \in K$ and thus $K = M^-$ which implies M^- is simple.

(iii) Since M^+ is a submodule of M, it is a standard fact that the inclusion map ι and canonical projection π make the sequence $0 \longrightarrow M^+ \xrightarrow{\iota} M \xrightarrow{\pi} M/M^+ \longrightarrow 0$ exact. Suppose the sequence does split. By definition, this implies that $M \cong M^+ \bigoplus M/M^+$. This implies that M has a non-trivial submodule N isomorphic to M/M^+. Let $\phi : M/M^+ \to N$ be the corresponding isomorphism and let $m_0 = \phi(x^{-1} + M^+)$. Then $X_{\delta_1}m_0 = X_{\delta_1}\phi(x^{-1} + M^+) = \phi(X_{\delta_1}x^{-1} + M^+) = \phi(x^0 + M^+) = 0$. But X_{δ_1} is injective on M which implies that $m_0 = 0$ contradicting that ϕ is an isomorphism. Hence, the sequence does not split and, in particular, M is indecomposable. □

Proposition 6.2.5 *The even and the odd parts parts $M^+_{\bar{0}}, M^+_{\bar{1}}$ and $M^-_{\bar{0}}, M^-_{\bar{1}}$ of M^+, and M^-, respectively, satisfy the identities: $M^+_{\bar{j}} = \mathrm{span}\{x^{2i+j} | i \geq 0, i \in \mathbb{Z}\}$ and $M^-_{\bar{j}} = M_{\bar{j}}/M^+_{\bar{j}}$. Furthermore:*

(i) $M^+_{\bar{j}}$, and hence, $M^-_{\bar{j}}$, is an $\mathfrak{sl}(2)$-submodule of $M_{\bar{j}}$.

(ii) $M^+_{\bar{j}}$ and $M^-_{\bar{j}}$, $j = 0, 1$, are simple $\mathfrak{sl}(2)$-modules.

(iii) The sequences $0 \longrightarrow M^+_{\bar{j}} \longrightarrow M_{\bar{j}} \longrightarrow M_{\bar{j}}/M^+_{\bar{j}} \longrightarrow 0$, $j = 0, 1$, are non-split exact sequences of $\mathfrak{sl}(2)$-modules. In particular, $M_{\bar{j}}$, $j = 0, 1$, are indecomposable $\mathfrak{sl}(2)$-modules.

Proof.

(i) We prove this item for $j = 0$. The proof in the case $j = 1$ is similar.Clearly $M^+_{\bar{0}}$ is a subspace $M_{\bar{0}}$. It suffices then to show that $M^+_{\bar{0}}$ is closed under the action of $\mathfrak{sl}(2)$. As above, we need only to verify closure for a generating set of $\mathfrak{sl}(2)$ acting on a basis element of $M^+_{\bar{0}}$, cf. Corollary 3.2.12. We use the generating set $\{e, f\}$ of $\mathfrak{sl}(2)$ as described in 2.2.11, cf. Remark 6.2.3. Let x^{2i}, $i \geq 0$, be a basis element of $M^+_{\bar{0}}$. Then $e(x^{2i}) = \frac{1}{2}x^{2i+2} \in M^+_{\bar{0}}$. Similarly, $f(x^{2i}) = -i(2i-1)x^{2i-2} \in M^+_{\bar{0}}$ if $i > 0$. If $i = 0$ then $f(x^0) = 0 \in M^+_{\bar{0}}$.

(ii) We now prove that $M^+_{\bar{0}}$ is simple. Using Lemma 4.3.9, let $u = x^0 \in M^+_{\bar{0}}$. Let $w \in M^+_{\bar{0}}$, with $w \neq 0$, then $w = \sum_{i=0}^n a_{2i}x^{2i}$ with $a_{2n} \neq 0$. Define $X \in \mathfrak{U}(\mathfrak{sl}(2))$ by $X = \sum_{i=0}^m a_{2i}(2X_{2\delta_1})^i$. Then $w = X(u)$. Now let $v \in M^+_{\bar{0}}$, with $v = \sum_{i=0}^m b_{2i}x^{2i}$, $b_{2m} \neq 0$. Define $Y \in \mathfrak{U}(\mathfrak{sl}(2))$ by $Y = \frac{(-2X_{-2\delta_1})^m}{(2m)!b_{2m}}$. Then, $u = Y(v)$. Therefore, $M^+_{\bar{0}}$ is simple.

For $M^+_{\bar{1}}$, let $u = x \in M^+_{\bar{1}}$, $w = \sum_{i=0}^n a_{2i+1}x^{2i+1}$, with $a_{2n+1} \neq 0$ and $v = \sum_{i=0}^m b_{2i+1}x^{2i+1}$, with $b_{2m+1} \neq 0$. Define $X = \sum_{i=0}^m a_{2i+1}(2X_{2\delta_1})^i$ and $Y = \frac{(-2X_{-2\delta_1})^m}{(2m+1)!b_{2m+1}}$ and we have that $w = X(u)$ and $u = Y(v)$ and thus $M^+_{\bar{1}}$ is simple.

For $M^-_{\bar{0}}$, let $u = x^{-2} + M^+_{\bar{0}} \in M^-_{\bar{0}}$. In this case, $w = \sum_{i=-n}^{-1} a_{-2i}x^{2i} + M^+_{\bar{0}}$ and $v = \sum_{i=-m}^{-1} b_{-2i}x^{2i} + M^+_{\bar{0}}$, with $b_{2m} \neq 0$. Let $X = \sum_{i=1}^n a_{2i}\frac{(-2X_{-2\delta_1})^{i-1}}{(2i-1)!}$, then

$w = X(u)$. Let $Y = \frac{(2X_{2\delta_1})^{m-1}}{b_{2m}}$. Then, $u = Y(v)$. Therefore, $M_{\bar{0}}^-$ is simple. For $M_{\bar{1}}^-$, let $u = x^{-1} + M_{\bar{1}}^+ \in M_{\bar{1}}^-$. In this case, $w = \sum_{i=-n}^{-1} a_{-2i-1}x^{2i+1} + M_{\bar{1}}^+$ and $v = \sum_{i=-m}^{-1} b_{-2i-1}x^{2i+1} + M_{\bar{1}}^+$, with $b_{2m-1} \neq 0$. Let $X = \sum_{i=1}^{n} a_{2i-1}\frac{(-2X_{-2\delta_1})^{i-1}}{(2i-2)!}$, then $w = X(u)$. Let $Y = \frac{(2X_{2\delta_1})^{m-1}}{b_{2m-1}}$. Then, $u = Y(v)$. Therefore, $M_{\bar{1}}^-$ is simple.

(iii) Let first $j = 0$. Since $M_{\bar{0}}^+$ is a submodule of $M_{\bar{0}}$, we have that

$$0 \longrightarrow M_{\bar{0}}^+ \longrightarrow M_{\bar{0}} \longrightarrow M_{\bar{0}}/M_{\bar{0}}^+ \longrightarrow 0$$

is an exact sequence. Suppose the sequence does split. By definition, this implies that $M_{\bar{0}} \cong M_{\bar{0}}^+ \bigoplus M_{\bar{0}}/M_{\bar{0}}^+$. This implies that $M_{\bar{0}}$ has a non-trivial submodule N isomorphic to $M_{\bar{0}}/M_{\bar{0}}^+$. Let $\phi : M_{\bar{0}}/M_{\bar{0}}^+ \to N$ be the corresponding isomorphism and let $m_0 = \phi(x^{-2} + M_{\bar{0}}^+)$. Then,

$$\begin{aligned} X_{2\delta_1}(m_0) &= X_{2\delta_1}(\phi(x^{-2} + M_{\bar{0}}^+)) \\ &= \phi(X_{2\delta_1}(x^{-2} + M_{\bar{0}}^+)) \\ &= \phi(0 + M_{\bar{0}}^+) \\ &= 0. \end{aligned}$$

But $X_{2\delta_1}$ is injective on M which implies that $m_0 = 0$ contradicting that ϕ is an isomorphism. Hence, the sequence does not split and, in particular, $M_{\bar{0}}$ is indecomposable.

For $j = 1$ we apply the same reasoning as in the case $j = 0$ to prove that

$$0 \longrightarrow M_{\bar{1}}^+ \longrightarrow M_{\bar{1}} \longrightarrow M_{\bar{1}}/M_{\bar{1}}^+ \longrightarrow 0$$

is a nonsplit exact sequence. □

6.3 Shifted Laurent Polynomials of One Variable

We now turn attention to the ring of shifted Laurent polynomials. For $a \in \mathbb{C}$ we define

$$\mathcal{F}(a) := x^a\mathbb{C}[x^{\pm 1}] = \text{span}\{x^{a+k}|k \in \mathbb{Z}\}.$$

Define $\mathcal{F}(a)_{\bar{0}} := \text{span}\{x^{a+2i} \mid i \in \mathbb{Z}\}$, and $\mathcal{F}(a)_{\bar{1}} := \text{span}\{x^{a+2i+1} \mid i \in \mathbb{Z}\}$. We observe that $\mathcal{F}(a) = \mathcal{F}(a)_{\bar{0}} \oplus \mathcal{F}(a)_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space and that if $a \in \mathbb{Z}$, then $\mathcal{F}(a) = \mathcal{F}(0) = \mathbb{C}[x^{\pm 1}]$.

Lemma 6.3.6 *Let $a, b \in \mathbb{C}$. Then the following hold.*

(i) $\mathcal{F}(a)$ is an $\mathfrak{osp}(1|2)$-module.

(ii) $\mathcal{F}(a) = \mathcal{F}(b)$ if and only if $a - b \in \mathbb{Z}$.

Proof.

(i) Following the approach of Theorem 6.2.4 above we show that $\mathcal{F}(a)$ is closed under the action of $\mathfrak{osp}(1|2)$. We may assume that $a \notin \mathbb{Z}$ since the case of an integer a is considered in Theorem 6.2.4. Consider now a basis vector x^{i+a} in $\mathcal{F}(a)$. Then $X_{\delta_1}(x^{i+a}) = (\frac{1}{\sqrt{2}}x)(x^{a+i+j)} = \frac{1}{\sqrt{2}}x^{a+i+j+1}$ and $X_{-\delta_1}(x^{a+i}) = (\frac{1}{\sqrt{2}}\partial)(x^{a+i}) = \frac{1}{\sqrt{2}}x^{a+i-1}$ are in $\mathcal{F}(a)$. If $i \geq 1$, then $\frac{1}{\sqrt{2}}x^{a+i-1} \in \mathcal{F}(a)$.

(ii) Suppose $\mathcal{F}(a) = \mathcal{F}(b), a, b \in \mathbb{C}$. Let $p \in \mathcal{F}(a)$ then $p = \sum_{i=m}^{n} \alpha_i x^{i+a}$,with $i, m, n \in \mathbb{Z}$. Since $p \in \mathcal{F}(b)$ as well, p can also be written as $p = \sum_{k=r}^{s} \alpha_k x^{k+b}$. Comparing the two expressions of p term-by-term, we have that for every i, there is k such that $x^{a+i} = x^{b+k}$. Hence $a+i-b-k = 0$ and thus $a-b = k-i \in \mathbb{Z}$. Now suppose $a - b \in \mathbb{Z}$. Let $p \in \mathcal{F}(a)$ with $p = \sum_{i=m}^{n} \alpha_i x^{i+a} = \sum_{i=m}^{n} \alpha_i x^{i+b-b+a}$. Since $a - b \in \mathbb{Z}$, we can let $k = i + a - b$ and re-index p as follows: $p = \sum_{k=m}^{n} \alpha_k x^{k+b}$. Thus, we have $p \in \mathcal{F}(b)$ which implies that $\mathcal{F}(a) \subset \mathcal{F}(b)$. An analogous argument shows that $\mathcal{F}(b) \subset \mathcal{F}(a)$ and thus $\mathcal{F}(a) = \mathcal{F}(b)$. □

For $b \in \mathbb{C}$, set $\overline{\mathcal{F}}(b) = \text{span}\{x^{b+2i} \,|\, i \in \mathbb{Z}\}$. Note that $\mathcal{F}(a)_{\bar{0}} = \overline{\mathcal{F}}(a)$ and $\mathcal{F}(a)_{\bar{1}} = \overline{\mathcal{F}}(a+1)$.

Theorem 6.3.7 *If $a \notin \mathbb{Z}$ then the following hold:*

(i) $\mathcal{F}(a)$ is a simple $\mathfrak{osp}(1|2)$-module.

(ii) $\overline{\mathcal{F}}(a)$ and $\overline{\mathcal{F}}(a+1)$ are simple $\mathfrak{sl}(2)$-modules.

(iii) As an $\mathfrak{sl}(2)$-module, $\mathcal{F}(a) \cong \overline{\mathcal{F}}(a) \oplus \overline{\mathcal{F}}(a+1)$.

Proof.

(i) $\mathcal{F}(a)$ is an $\mathfrak{osp}(1|2)$-module as in Lemma 6.3.6. Let $a \in \mathbb{C}$. Suppose that K is a non-trivial submodule of $\mathcal{F}(a)$. It then suffices to show that $K = \mathcal{F}(a)$.

Let $w \in K$, with $w \neq 0$. Want to show that $w \in \mathcal{F}(a)$. Let $w = \sum_{i=1}^{n} a_i x^{t_i+a}$ with $a_i \neq 0$ and the t_i's are distinct integers. Since x^{t_i+a} are in the $(t_i+a+1/2)$-weight space of $\mathcal{F}(a)$ (see Remark 5.4.18), by Lemma 5.1.8, $x^{t_i+a} \in K$. Now using consecutive applications of X_{δ_1} and $X_{-\delta_1}$ on x^{t_i+a} we can obtain $x^{t+a} \in K$ for every integer t. Hence, $K = \mathcal{F}(a)$ by Lemma 4.3.9.

(ii) Let $p \in \overline{\mathcal{F}}(a)$, $p = \sum_{i=1}^{n} \alpha_i x^{a+2t_i}$. Recall $\langle e, f \rangle = \mathfrak{sl}(2)$ and $e = X_{2\delta_1}$, $f = X_{-2\delta_1}$. Then $e(p) = \sum_{i=1}^{n} \frac{\alpha_i}{2} x^{a+2t_i+2} \in \overline{\mathcal{F}}(a)$. Similarly for $f(p)$. Therefore, $\overline{\mathcal{F}}(a)$ is an $\mathfrak{sl}(2)$-module. To show that $\overline{\mathcal{F}}(a)$ is simple, we take a nontrivial $\mathfrak{sl}(2)$-submodule L of $\overline{\mathcal{F}}(a)$ and $\ell \in L$ with $\ell \neq 0$. Then by consecutive applications of $X_{-2\delta_1}$ we conclude that $x^a \in L$. Now using consecutive applications of $X_{2\delta_1}$ on x^a we obtain that $x^{t+a} \in L$ for every $t \in \mathbb{Z}$. Hence, $L = \overline{\mathcal{F}}(a)$ by Lemma 4.3.9. By analogous argument, $\overline{\mathcal{F}}(a+1)$ is also simple.

(iii) Let $p \in \overline{\mathcal{F}}(a)$, $p = \sum_{i=m}^{n} \alpha_i x^{a+2i}, i \in \mathbb{Z}$. We choose $m, n \in 2\mathbb{Z}$ by letting the corresponding $\alpha_m, \alpha_n = 0$ if necessary. Then let $r = m/2$ and $s = n/2$ and rewrite p as follows: $p = \sum_{j=r}^{s} \alpha_{2j} x^{a+2j} + \sum_{j=r}^{r} \alpha_{2j+1} x^{a+2j+1} = p_1 + p_2$ where $p_1 \in$

$\overline{\mathcal{F}}(a)$ and $p_2 \in \overline{\mathcal{F}}(a+1)$. Let $p \in \overline{\mathcal{F}}(a) \cap \overline{\mathcal{F}}(a+1)$. Comparing coefficients term by term, it is clear the p is the zero polynomial. Therefore, $\mathcal{F}(a) \cong \overline{\mathcal{F}}(a) \bigoplus \overline{\mathcal{F}}(a+1)$.

□

6.4 The $\mathfrak{osp}(1|2n)$-module of Shifted Laurent Polynomials of n Variables

We now turn to considering shifted Laurent polynomials as $\mathfrak{osp}(1|2n)$-modules. Since the polynomials will now have multiple variables and exponents, we will extensively use the following multi-index notation: for $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{a} = (a_1, ..., a_n)$ we write $\mathbf{x}^{\mathbf{a}} = x_1^{a_1}...x_n^{a_n}$. Also, we write $\mathbb{C}[\mathbf{x}]$ and $\mathbb{C}[\mathbf{x}^{\pm 1}]$ for $\mathbb{C}[x_1, ..., x_n]$ and $\mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$, respectively.

Definition 6.4.8 *Let $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{C}^n$. We define the following:*

(i) $\mathcal{F}(\mathbf{a}) := \mathbf{x}^{\mathbf{a}}\mathbb{C}[\mathbf{x}^{\pm 1}]$.

(ii) $\mathcal{F}(\mathbf{a})_{\bar{0}} := \text{span}\{x^{\mathbf{a}+\mathbf{i}} \mid i \in \mathbb{Z}^n, i_1 + ... + i_n \textit{ is even}\}$.

(iii) $\mathcal{F}(\mathbf{a})_{\bar{1}} := \text{span}\{x^{\mathbf{a}+\mathbf{i}} \mid i \in \mathbb{Z}^n, i_1 + ... + i_n \textit{ is odd}\}$.

(iv) $\mathcal{F}(\mathbf{0})^+ := \mathbb{C}[x_1, x_2, ..., x_n]$.

We note that the monomials $\mathbf{x}^{\mathbf{i}+\mathbf{a}}$, $\mathbf{i} \in \mathbb{Z}^n$, form a basis of $\mathcal{F}(\mathbf{a})$ and that $\mathcal{F}(\mathbf{a}) = \mathcal{F}(\mathbf{a})_{\bar{0}} \oplus \mathcal{F}(\mathbf{a})_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space.

Lemma 6.4.9 *Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$.*

(i) *$\mathcal{F}(\mathbf{a})$ is an $\mathfrak{osp}(1|2n)$-module.*

(ii) *$\mathcal{F}(\mathbf{a}) = \mathcal{F}(\mathbf{b})$ if and only if $a_i - b_i \in \mathbb{Z}$ for every i (equivalently, $\mathbf{a} - \mathbf{b} \in \mathbb{Z}^n$).*

(iii) *The monomials $\mathbf{x}^{\mathbf{t_i}+\mathbf{a}}$ are weight vectors of $\mathcal{F}(\mathbf{a})$ of weight $(\mathbf{t_i} + \mathbf{a}) + (1/2, ..., 1/2)$.*

(iv) *As an $\mathfrak{sp}(2n)$-module, $\mathcal{F}(\mathbf{a}) = \mathcal{F}(\mathbf{a})_{\bar{0}} \oplus \mathcal{F}(\mathbf{a})_{\bar{1}} = \overline{\mathcal{F}}(\mathbf{a}) \oplus \overline{\mathcal{F}}(\mathbf{a} + \boldsymbol{\delta_1})$.*

Proof.

(i) We already saw that $\mathcal{F}(\mathbf{a})$ is a $\mathbb{Z}_2$-graded vector space. Following the approach of Theorem 6.2.4 above we show that $\mathcal{F}(\mathbf{a})$ is closed under the action of the generators X_{δ_k} and $X_{-\delta_k}$, $1 \le k \le n$, of $\mathfrak{osp}(1|2n)$. Consider now a basis vector $\mathbf{x}^{\mathbf{i}+\mathbf{a}}$ in $\mathcal{F}(\mathbf{a})$. Here $\mathbf{i} = (i_1, ..., i_n) \in \mathbb{Z}^n$.

Then

$$X_{\delta_k}(\mathbf{x}^{\mathbf{i}+\mathbf{a}}) = (\frac{1}{\sqrt{2}}x_k)(\mathbf{x}^{\mathbf{i}+\mathbf{a}}) = \frac{1}{\sqrt{2}}x_1^{a_1}...x_k{}^{a_k+i_k+1}...x_n^{a_n} \in \mathcal{F}(\mathbf{a})$$

and

$$X_{-\delta_k}(\mathbf{x}^{\mathbf{i}+\mathbf{a}}) = (\frac{1}{\sqrt{2}}\partial_k)(\mathbf{x}^{\mathbf{i}+\mathbf{a}}) = \frac{1}{\sqrt{2}}(x_i + a_i)x_1^{a_1}...x_k{}^{a_k+i_k-1}...x_n^{a_n} \in \mathcal{F}(\mathbf{a})$$

are in $\mathcal{F}(\mathbf{a})$. Note that, alternatively, we can show that $\mathcal{F}(\mathbf{a})$ is an $\mathfrak{osp}(1|2n)$-module by proving that it is an $\mathcal{W}_n$-module.

(ii) Suppose $\mathcal{F}(\mathbf{a}) = \mathcal{F}(\mathbf{b})$,with $\mathbf{a}, \mathbf{b} \in \mathbb{C}^{\mathbf{n}}$. Let $p \in \mathcal{F}(\mathbf{a})$ then $p = \sum_{r=0}^{m} \alpha_r \mathbf{x}^{\mathbf{i_r}+\mathbf{a}}$,where $i_r \in \mathbb{Z}^n, r \in \mathbb{Z}$. Since $p \in \mathcal{F}(\mathbf{b})$ as well, p can also be written as $p = \sum_{\ell=0}^{s} \alpha_\ell \mathbf{x}^{\mathbf{k}_\ell+\mathbf{b}}$ where $k_\ell \in \mathbb{Z}^n, \ell \in \mathbb{Z}$. Comparing the two expressions of p term-by-term, we have that for every $\mathbf{i_r}$, there is $\mathbf{k}_\ell$ such that $\mathbf{x}^{\mathbf{a}+\mathbf{i_r}} = \mathbf{x}^{\mathbf{b}+\mathbf{k}_\ell}$. Hence $\mathbf{a} + \mathbf{i_r} - \mathbf{b} - \mathbf{k}_\ell = \mathbf{0}$ and thus $\mathbf{a} - \mathbf{b} = \mathbf{k}_\ell - \mathbf{i_r} \in \mathbb{Z}^{\mathbf{n}}$. The converse follows from an argument analogous to Lemma 6.3.6.

(iii) Note that $h_{2\delta_k}(\mathbf{x}^{\mathbf{i}+\mathbf{a}}) = (\mathbf{x_k}\partial_{\mathbf{k}} + \frac{\mathbf{1}}{\mathbf{2}})(\mathbf{x}^{\mathbf{i}+\mathbf{a}}) = (\mathbf{i_k} + \mathbf{a_k} + \frac{\mathbf{1}}{\mathbf{2}})\mathbf{x}^{\mathbf{i}+\mathbf{a}}$. See Remark 5.4.18.

(iv) Let $p \in \mathcal{F}(\mathbf{a})$, $p = \sum_{i=m}^{n} \alpha_i \mathbf{x}^{\mathbf{a}+\mathbf{2t_i}}$, $i \in \mathbb{Z}$. We choose $m, n \in 2\mathbb{Z}$ by letting the corresponding $\alpha_m, \alpha_n = 0$ if necessary. Then let $r = m/2$ and $s = n/2$ and rewrite p as follows: $p = \sum_{j=r}^{s} \alpha_{2j} x^{a+2j} + \sum_{j=r}^{r} \alpha_{2j+1} x^{a+2j+1} = p_1 + p_2$ where $p_1 \in \mathcal{F}(\mathbf{a})_{\bar{0}}$ and $p_2 \in \mathcal{F}(\mathbf{a})_{\bar{1}}$. Let $p \in \mathcal{F}(\mathbf{a})_{\bar{0}} \cap \mathcal{F}(\mathbf{a})_{\bar{1}}$. Comparing coefficients term by term, it is clear the p is the zero polynomial. Therefore, $\mathcal{F} \cong \mathcal{F}(\mathbf{a})_{\bar{0}} \bigoplus \mathcal{F}(\mathbf{a})_{\bar{1}}$.

□

For $\mathbf{a} \in \mathbb{C}^n$, we set $\overline{\mathcal{F}}(\mathbf{a}) = \text{span}\{\mathbf{x}^{\mathbf{a}+\mathbf{i}} \mid i \in \mathbb{Z}^n, i_1 + ... + i_n \text{ is even}\}$. Note that $\mathcal{F}(\mathbf{a})_{\bar{0}} = \overline{\mathcal{F}}(\mathbf{a})$ and $\mathcal{F}(\mathbf{a})_{\bar{1}} = \overline{\mathcal{F}}(\mathbf{a} + \boldsymbol{\delta}_{\mathbf{1}})$ (recall that $\boldsymbol{\delta}_1 = (1, 0, ..., 0)$).

Lemma 6.4.10 *Let* $\mathbf{a} \in \mathbb{C}^{\mathbf{n}}$ *with* $a_i \notin \mathbb{Z}$*, and let* $\mathbf{i} \in \mathbb{Z}^{\mathbf{n}}$*.*

(i) There exists $X \in \mathfrak{U}(\mathfrak{osp}(1|2n))$ *such that* $\mathbf{x}^{\mathbf{a}+\mathbf{i}} = X(\mathbf{x}^{\mathbf{a}})$ *in* $\mathcal{F}(\mathbf{a})$*.*

(ii) There exists $Y \in \mathfrak{U}(\mathfrak{osp}(1|2n))$ *such that* $\mathbf{x}^{\mathbf{a}} = Y(\mathbf{x}^{\mathbf{a}+\mathbf{i}})$ *in* $\mathcal{F}(\mathbf{a})$*.*

Proof. Recall that the action of $\mathfrak{osp}(1|2n)$ on $\mathcal{F}(\mathbf{a})$ is defined through the correspondences $X_{\delta_j} \longmapsto \frac{1}{\sqrt{2}}x_j$, $X_{-\delta_j} \longmapsto \frac{1}{\sqrt{2}}\partial_j$.

(i) We let $X = X_1X_2...X_n$ where X_j is given as follows, for $j = 1, 2, ..., n$:

Case 1: $i_j > 0$. $X_j = (\sqrt{2}X_{\delta_i})^{i_j}$.

Case 2: $i_j = 0$. $X_j = 1$.

Case 3: $i_j < 0$. $X_j = \frac{(\sqrt{2}X_{-\delta_i})^{-i_j}}{(a_j)(a_j-1)...(a_j+i_j+1)}$.

(ii) We let $Y = Y_1Y_2..Y_n$ where Y_j is given as follows, for $j = 1, 2, ..., n$:

Case 1: $i_j > 0$. $Y_j = \frac{(\sqrt{2}X_{-\delta_i})^{i_j}}{(a_j+i_j)(a_j+i_j-1)...(a_j+1)}$.

Case 2: $i_j = 0$. $Y_j = 1$.

Case 3: $i_j < 0$. $Y_j = (\sqrt{2}X_{\delta_i})^{-i_j}$.

□

We now generalize Theorem 6.3.7 to the $\mathfrak{osp}(1|2n)$ case. Recall $\mathfrak{osp}(1|2n)_{\bar{0}} \simeq \mathfrak{sp}(2n)$. We previously have taken advantage of the fact that $\mathfrak{sp}(2) \cong \mathfrak{sl}(2)$.

Theorem 6.4.11 *If* $\mathbf{a} \in \mathbb{C}^n$ *with* $a_i \notin \mathbb{Z}$ *for all* i*. Then*

(i) $\mathcal{F}(\mathbf{a})$ *is a simple* $\mathfrak{osp}(1|2n)$*-module,*

(ii) $\mathcal{F}(\mathbf{a})_{\bar{0}}$ *and* $\mathcal{F}(\mathbf{a})_{\bar{1}}$ *are simple* $\mathfrak{sp}(2n)$*-modules.*

Proof.

(i) $\mathcal{F}(\mathbf{a})$ is an $\mathfrak{osp}(1|2n)$-module as in Lemma 6.4.9. Let $\mathbf{a} \in \mathbb{C}^n$. Suppose that K is a non-trivial submodule of $\mathcal{F}(\mathbf{a})$. It then suffices to show that $K = \mathcal{F}(\mathbf{a})$. Let $w \in K$, with $w \neq 0$. Want to show that $w \in \mathcal{F}(\mathbf{a})$. Let $w = \sum_{i=1}^{n} a_i \mathbf{x}^{\mathbf{t_i}+\mathbf{a}}$ with $a_i \neq 0$ and the t_i's are distinct multi-indices. Since $\mathbf{x}^{\mathbf{t_i}+\mathbf{a}}$ are weight vectors of $\mathcal{F}(\mathbf{a})$ of weight $(\mathbf{t_i} + \mathbf{a}) + (1/2, ..., 1/2)$ (by Lemma 6.4.9), we have $\mathbf{x}^{\mathbf{t_i}+\mathbf{a}} \in \mathbf{K}$ by Lemma 5.1.8. Now using consecutive applications of X_{δ_i} and $X_{-\delta_i}$ on $\mathbf{x}^{\mathbf{t_i}+\mathbf{a}}$

and that $a_i \notin \mathbb{Z}$, we can obtain $\mathbf{x}^{\mathbf{t}+\mathbf{a}} \in \mathbf{K}$ for every multiindex $\mathbf{t}$ as in Lemma 6.4.10. Since $\mathbf{x}^{\mathbf{t}+\mathbf{a}}$ span $\mathcal{F}(\mathbf{a})$, we have $K = \mathcal{F}(\mathbf{a})$.

(ii) Let $p \in \mathcal{F}(\mathbf{a})_{\bar{0}}$, $p = \sum_{i=1}^{n} \alpha_i \mathbf{x}^{\mathbf{a}+\mathbf{t_i}}$, where $\mathbf{t_i} = (t_{i1}, ..., t_{in})$ is in $\mathbb{Z}^n$ are such that $t_{i1} + ... + t_{in}$ is even. Then $X_{2\delta_i}(p) = \sum_{i=1}^{n} \frac{\alpha_i}{2} \mathbf{x}^{\mathbf{a}+\mathbf{t_i}+2\boldsymbol{\delta}_i} \in \overline{\mathcal{F}}(a)$. Similarly for $X_{-2\delta_i}(p)$. Therefore, $\mathcal{F}(\mathbf{a})_{\bar{0}}$ is an $\mathfrak{sp}(2n)$-module. To show that $\mathcal{F}(\mathbf{a})_{\bar{0}}$ is simple, we take a nontrivial $\mathfrak{sp}(2n)$-submodule L of $\mathcal{F}(\mathbf{a})_{\bar{0}}$ and $\ell \in L$ with $\ell \neq 0$. Then, as in Lemma 6.4.10, applying consecutive applications of $X_{2\delta_i}$ and $X_{2\delta_i}$ to ℓ, we have that $\mathbf{x}^{\mathbf{a}+\mathbf{t}} \in \mathbf{L}$ for all $\mathbf{t}$ such that $t_1 + \cdots t_n$ is even. Hence $L = \mathcal{F}(\mathbf{a})_{\bar{0}}$. By analogous argument, $\mathcal{F}(\mathbf{a})_{\bar{1}}$ is also simple.

□

Definition 6.4.12 *A $\mathfrak{g}$-module M is a* weight module*, respectively* torsion free module *if it is a weight module, respectively torsion free module, as a $\mathfrak{g}_{\bar{0}}$-module.*

Remark 6.4.13 *If M is a torsion free module it is easy to check that X_α acts injectively on M for all roots α (not only for the even ones). Indeed, this follows from the identities $X_{\pm 2\delta_i} = X_{\pm\delta_i}^2$.*

Remark 6.4.14 *For a more general definition of a torsion free module the reader is referred to [6].*

Theorem 6.4.15 *Let $a_i \notin \mathbb{Z}$ for every i. Then $\mathcal{F}(\mathbf{a})$ is a simple torsion free $\mathfrak{osp}(1|2n)$-module. Moreover, $\mathcal{F}(\mathbf{a})$ is a pointed weight module.*

Proof. The nonzero weight spaces of $\mathcal{F}(\mathbf{a})$ are those of weights $\mathbf{t} + \mathbf{a} + (1/2, ..., 1/2)$, $\mathbf{t} \in \mathbb{Z}^n$ (see Lemma 6.4.9). Moreover the weight space of $\mathcal{F}(\mathbf{a})$ of weight $\mathbf{t} + \mathbf{a} + (1/2, ..., 1/2)$ is spanned by $\mathbf{x}^{\mathbf{t}+\mathbf{a}}$. In particular, $\mathcal{F}(\mathbf{a})$ is a weight module and since $\mathbf{x}^{\mathbf{t}+\mathbf{a}}$ is a weight vector, $\mathcal{F}(\mathbf{a})$ is pointed.

It remains to show that $\mathcal{F}(\mathbf{a})$ is torsion free. It is enough to check that all root elements X_α are injective on the weight spaces of $\mathcal{F}(\mathbf{a})$. However , every weight space is one dimensional and spanned by an element $\mathbf{x}^{\mathbf{i}+\mathbf{a}}$ for some $\mathbf{i} \in \mathbb{Z}^n$. We will denote by $\boldsymbol{\delta}_k$ the kth standard basis vector of $\mathbb{R}^n$. Then we have the following:

$$\begin{aligned}
X_{\delta_k-\delta_j}(\mathbf{x}^{\mathbf{i}+\mathbf{a}}) &= (i_j + a_j)\mathbf{x}^{\mathbf{i}+\mathbf{a}+\boldsymbol{\delta}_\mathbf{k}-\boldsymbol{\delta}_\mathbf{j}} \\
X_{2\delta_k}(\mathbf{x}^{\mathbf{i}+\mathbf{a}}) &= \frac{1}{2}\mathbf{x}^{\mathbf{i}+\mathbf{a}+2\boldsymbol{\delta}_k} \\
X_{-2\delta_k}(\mathbf{x}^{\mathbf{i}+\mathbf{a}}) &= \frac{-1}{2}(i_k + a_k)(i_k + a_k - 1)\mathbf{x}^{\mathbf{i}+\mathbf{a}-2\boldsymbol{\delta}_k} \\
X_{\delta_k+\delta_j}(\mathbf{x}^{\mathbf{i}+\mathbf{a}}) &= \mathbf{x}^{\mathbf{i}+\mathbf{a}+\boldsymbol{\delta}_k+\boldsymbol{\delta}_j} \\
X_{-\delta_k-\delta_j}(\mathbf{x}^{\mathbf{i}+\mathbf{a}}) &= -(i_j + a_j)(i_k + a_k)\mathbf{x}^{\mathbf{i}+\mathbf{a}-\boldsymbol{\delta}_k-\boldsymbol{\delta}_j}.
\end{aligned}$$

Since none of the $\mathbf{i} + \mathbf{a}$ are integers, it follows that X_α is injective for each α. Therefore, $\mathcal{F}(\mathbf{a})$ is torsion free. □

CHAPTER 7

Simple Weight $\mathfrak{osp}(1|2n)$-modules

By identifying constraints on primitive vectors in $\mathfrak{osp}(1|2n)$-modules, we are able to establish necessary and sufficient conditions for $\mathfrak{osp}(1|2n)$-weights to be bounded. Using these conditions we complete the classification of simple weight $\mathfrak{osp}(1|2n)$-modules.

7.1 Primitive Vectors of Tensor Products of $\mathfrak{osp}(1|2n)$-modules

Recall the definition of $M = \mathcal{F}(0)$ and $M^+ = \mathcal{F}(0)^+$ (Definition 6.4.8). We begin by considering the tensor products of M^+ with $\mathbb{C}^{1|2n}$. Recall that the action of the $\mathfrak{osp}(1|2n)$ on $M^+ \otimes V$ is given by $X_\alpha(f \otimes v) = X_\alpha(f) \otimes v + (-1)^{|X_\alpha||f|} f \otimes X_\alpha(v)$ with $f \in M^+$ and $v \in \mathbb{C}^{1|2n}$, see (5.8.27).

Theorem 7.1.1 *Let $W = M^+ \otimes V$ with $V = \mathbb{C}^{1|2n}$. Then any primitive vector in W is a linear combination of w_1 and w_2 defined as follows:*

$$w_1 := 1 \otimes v_{2n},\ w_2 := 1 \otimes v_0 + \sqrt{2} \sum_{i=1}^{n} x_i \otimes v_{n+i}.$$

Proof. Recall that M^+ is a highest weight module with highest weight $\lambda = (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$. The module V has $2n+1$ weights: $\mathrm{wt}\,(V) = \{\delta_1, \delta_2, ..., \delta_n, 0, -\delta_1, -\delta_2, ..., -\delta_n\}$ as follows:

$$\mathrm{wt}(v_i) = \begin{cases} 0 \text{ if } i = 0 \\ \delta_i \text{ for } i = 1 \text{ to } n \\ -\delta_i \text{ for } i = n+1 \text{ to } 2n. \end{cases}$$

To prove vector w in W is primitive, it will suffice to show that $X_{\delta_i - \delta_{i+1}}(w) = X_{-\delta_i}(w) = 0$ for $i = 1, 2, ..., n$ because of the specific Borel subalgebra chosen. First, we show that $1 \otimes v_j$ is primitive if and only if $j = 2n$. $X_{-\delta_i}(1 \otimes v_j) = 0$ unless $j = 0$ or $j = i$, that is, $X_{-\delta_i}(1 \otimes v_j) = 0$ for $j = n + 1$ to $2n$. $X_{\delta_i - \delta_{i+1}}(1 \otimes v_j) = 0$ unless $j = 0$ or $j = n + i$. It follows that $X_{-\delta_i}(1 \otimes v_j) = X_{\delta_i - \delta_{i+1}}(1 \otimes v_j) = 0$ for $i = 1$ to n if and only if $j = 2n$.

Let $w_2 = 1 \otimes v_0 + \sqrt{2} \sum_{i=1}^{n} x_i \otimes v_{n+i}$. We consider the action of $X_{-\delta_j}$ on w_2 for $j = 1, 2, ..., n$:

$$\begin{aligned} X_{-\delta_j}(w_2) &= 1 \otimes (-v_{n+j}) - 2 \sum_{i=1}^{n} (X_{-\delta_j}(\frac{1}{\sqrt{2}} x_i) \otimes (-v_{n+i}) + \frac{1}{\sqrt{2}} x_i \otimes X_{-\delta_j}(-v_{n+j})) \\ &= 1 \otimes (-v_{n+j}) - 2(\sum_{i=1}^{n} \frac{1}{2} \otimes (-v_{n+j}) + 0) \\ &= 1 \otimes (-v_{n+j}) + 1 \otimes (v_{n+j}) \\ &= 0 \end{aligned}$$

Similarly, $X_{\delta_j - \delta_{j+1}}(w_2) = 0$, for $j = 1, 2, ..., n - 1$. Therefore, w_2 is primitive.

Assume now that

$$w = \sum_{i=1}^{n} a_i x_1^{\mu_1} ... x_i^{\mu_i - 1} .. x_n^{\mu_n} \otimes v_i + \sum_{i=1}^{n} b_i x_1^{\mu_1} ... x_i^{\mu_i + 1} .. x_n^{\mu_n} \otimes v_{n+i} + a_0 x_1^{\mu_1} ... x_i^{\mu_i} .. x_n^{\mu_n} \otimes v_0$$

is primitive. We then derive the following sets of equations. First, from the constraint that $X_{-\delta_i}(w) = 0$ for $i = 1, 2, ..., n$:

$$a_0 \frac{1}{\sqrt{2}} \mu_i - a_i = 0 \text{ for } i = 1, 2, ..., n \tag{7.1.2}$$

$$b_i \frac{1}{\sqrt{2}} \mu_i - a_0 = 0 \text{ for } i = 1, 2, ..., n \tag{7.1.3}$$

$$a_k \mu_i = 0 \text{ for } k = 1, 2, ..., n \tag{7.1.4}$$

$$b_k \mu_i = 0 \text{ for } k = 1, 2, ..., n, k \neq i. \tag{7.1.5}$$

Then, from the constraint that $X_{\delta_i-\delta_{i+1}}(w)=0$ for $i=1,2,...,n-1$:

$$a_i\mu_{i+1}+a_{i+1}=0 \text{ for } i=1,2,...,n \tag{7.1.6}$$

$$b_{i+1}\mu_i-b_i=0 \text{ for } i=1,2,...,n$$

$$a_k\mu_i=0 \text{ for } k=1,2,...,n, k\neq i$$

$$b_k\mu_{i+1}=0 \text{ for } k=1,2,...,n, k\neq i+1.$$

These equations yield three cases for μ as follows.

Case 1: $\mu=\delta_i$. In this case, 7.1.4 implies $a_1=a_2=...=a_n=0$ except for $i=n$ and 7.1.6 implies $a_i+a_{i+1}=0$; therefore, $a_i=0$. This fact and 7.1.2 yield $a_0=0$ and then 7.1.3 gives $b_i=0$ for all i. We thus have no primitive vectors in this case.

Case 2: $\mu=0$. In this case, 7.1.2 yields $a_1=a_2=...=a_n=0$ and 7.1.3 gives $b_i=\sqrt{2}a_0$. Choose $a_0=1$ and then $b_i=\sqrt{2}$; this is vector w_2 above, which we have already shown to be primitive.

Case 3: $\mu=-\delta_i$. The only vectors in W with weight $\lambda-\delta_i$ are scalar multiples of $1\otimes v_i$, see Lemma 5.8.28. We have already shown that the only vector of this form that is primitive is $1\otimes v_{2n}$. □

Our next goal is to find explicitly a primitive vector of the $\mathfrak{osp}(1|2n)$-module

$$L(\frac{1}{2},...,\frac{1}{2})\otimes L(-\beta,...,-\beta),$$

with $\beta\in\mathbb{Z}_{>0}$.

Lemma 7.1.7 *Let v be a highest weight vector of the $\mathfrak{osp}(1|2n)$-representation:*

$$L(-\beta,-\beta,...,-\beta),\beta,j\in\mathbb{Z}_{>0}.$$

Then the following hold.

(i) $[h_{2\delta_1},X^j_{\delta_1}]=jX^j_{\delta_1}$ *for* $j>0$.

(ii) $X_{-\delta_1}X^j_{\delta_1}(v)=\frac{j}{2}X^{j-1}_{\delta_1}(v)$ *if j is even.*

$X_{-\delta_1}X^j_{\delta_1}(v)=(\frac{j-1}{2}-\beta)X^{j-1}_{\delta_1}(v)$ *if j is odd.*

(iii) $X_{\delta_k-\delta_{k+1}} X_{\delta_1}^j(v) = 0$ *for* $k = 1, 2, ..., n-1$.

Proof.

(i) For $j = 1$, $[h_{2\delta_1}, X_{\delta_1}] = X_{\delta_1}$ by direct computation. If $j > 1$, then we use the super Jacobi identity as follows:

$$[h_{2\delta_1}, [X_{\delta_1}^{j-1}, X_{\delta_1}]] = [h_{2\delta_1}, X_{\delta_1}^{j-1}], X_{\delta_1}] + [X_{\delta_1}^{j-1}, [h_{2\delta_1}, X_{\delta_1}]]$$

The left hand side is $2[h_{2\delta_1}, X_{\delta_1}^j]$. On the right hand side, the induction hypothesis gives us: $(j-1)X_{\delta_1}^{j-1} + 2X_{\delta_1}^j$. Divide both sides by 2 and we have:

$$[h_{2\delta_1}, X_{\delta_1}^j] = jX_{\delta_1}^j$$

as required.

(ii) *Case 1:* $j = 1$. We have $[X_{-\delta_1}, X_{\delta_1}](v) = X_{-\delta_1}X_{\delta_1}(v) + X_{\delta_1}X_{-\delta_1}(v)$. The left-hand side is $h_{2\delta_1}(v)$ see (5.4.15). From the fact that v is highest weight, the second term on the right hand side is 0; therefore we have:

$$X_{-\delta_1}X_{\delta_1}(v) = -\beta(v).$$

Case 2: $j = 2$. We have $[X_{-\delta_1}, X_{\delta_1}^2](v) = X_{-\delta_1}X_{\delta_1}^2(v) - X_{\delta_1}^2 X_{-\delta_1}(v)$. From (5.4.16), the left-hand side is $[X_{-\delta_1}, X_{2\delta_1}](v) = X_{\delta_1}(v)$. From the fact that v is highest weight, the second term on the right hand side is 0; therefore we have:

$$X_{-\delta_1}X_{\delta_1}^2(v) = X_{\delta_1}(v)$$

Case 3: $j > 2$. Finally,

$$\begin{aligned} X_{-\delta_1}X_{\delta_1}^j &= (-X_{\delta_1}X_{-\delta_1} + h_{2\delta_1})X_{\delta_1}^{j-1} \\ &= -X_{\delta_1}X_{-\delta_1}X_{\delta_1}^{j-1} + X_{\delta_1}^{j+1}h_{2\delta_1} + [h_{2\delta_1}, X_{\delta_1}^{j-1}] \end{aligned}$$

If j is even, then $j-1$ is odd, we use the induction hypothesis and apply to v:

$$\begin{aligned} X_{-\delta_1} X_{\delta_1}^j(v) &= -X_{\delta_1}\left(\frac{j-2}{2}-\beta\right) X_{\delta_1}^{j-2}(v) + X_{\delta_1}^{j-1} h_{2\delta_1}(v) + (j-1) X_{\delta_1}^{j-1}(v) \\ &= \left(\beta - \frac{j-2}{2}\right) X_{\delta_1}^{j-1}(v) + X_{\delta_1}^{j-1} h_{2\delta_1}(v) + (j-1) X_{\delta_1}^{j-1}(v) \\ &= \left(\beta - \frac{j-2}{2}\right) X_{\delta_1}^{j-1}(v) - \beta X_{\delta_1}^{j-1}(v) + (j-1) X_{\delta_1}^{j-1}(v) \end{aligned}$$

Thus: $X_{-\delta_1}^j X_{\delta_1}(v) = \frac{j}{2} X_{\delta_1}^{j-1}(v)$. If j is odd, $j-1$ is even, the induction hypothesis yields and apply to v:

$$\begin{aligned} X_{-\delta_1} X_{\delta_1}^j(v) &= -X_{\delta_1}\left(\frac{j-2}{2}\right) X_{\delta_1}^{j-1}(v) + X_{\delta_1}^{j-1} h_{2\delta_1}(v) + (j-1) X_{\delta_1}^{j-1}(v) \\ &= -\left(\frac{j-2}{2}\right) X_{\delta_1}^{j-1}(v) + X_{\delta_1}^{j-1} h_{2\delta_1}(v) + (j-1) X_{\delta_1}^{j-1}(v) \\ &= -\left(\frac{j-2}{2}\right) X_{\delta_1}^{j-1}(v) - \beta X_{\delta_1}^{j-1}(v) + (j-1) X_{\delta_1}^{j-1}(v). \end{aligned}$$

Thus:

$$X_{-\delta_1}^j X_{\delta_1}(v) = \left(\frac{j-1}{2} - \beta\right) X_{\delta_1}^{j-1}(v).$$

(iii) Consider

$$[X_{\delta_k-\delta_{k+1}}, X_{\delta_1}](v) = X_{\delta_k-\delta_{k+1}} X_{\delta_1}(v) + X_{\delta_1} X_{\delta_k-\delta_{k+1}}(v).$$

The final term is 0, and also by (5.4.17) the Lie superbracket is 0, thus we have $X_{\delta_k-\delta_{k+1}} X_{\delta_1}(v) = 0$. Now for $j > 1$,

$$\begin{aligned} X_{\delta_k-\delta_{k+1}} X_{\delta_1}^j &= (X_{\delta_1} X_{\delta_k-\delta_{k+1}} + [X_{\delta_k-\delta_{k+1}}, X_{\delta_1}]) X_{\delta_1}^{j-1} \\ &= X_{\delta_k-\delta_{k+1}} X_{\delta_1}^j(v) \\ &= X_{\delta_1} X_{\delta_k-\delta_{k+1}} X_{\delta_1}^{j-1}(v) + [X_{\delta_k-\delta_{k+1}}, X_{\delta_1}] X_{\delta_1}^{j-1}(v). \end{aligned}$$

By the induction hypothesis, $X_{\delta_1}X_{\delta_k-\delta_{k+1}}X_{\delta_1}^{j-1}(v) = 0$ and $[X_{\delta_k-\delta_{k+1}}, X_{\delta_1}] = 0$. Therefore

$$X_{\delta_k-\delta_{k+1}}X_{\delta_1}^{j}(v) = 0.$$

□

The next theorem plays an important role in this thesis, as it will lead to the classification of the $\mathfrak{osp}(1|2n)$-bounded weights.

Theorem 7.1.8 *Let $\beta \in \mathbb{Z}_{>0}$ and v be a highest weight vector of $L(-\beta, -\beta, ..., -\beta)$. Then the vector*

$$u = x_1^{2\beta} \otimes v + \sum_{k=1}^{2\beta} c_{2\beta-k}x_1^{2\beta-k} \otimes X_{\delta_1}^{k}(v)$$

is a primitive vector of $U = L(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) \otimes L(-\beta, -\beta, ..., -\beta)$, where the scalars c_i are defined as follows:

$$c_{2\beta} = 1$$
$$c_{2\beta-2j} = \frac{(2\beta-1)(2\beta-3)...(2\beta-(2j-1))}{j!}, j > 0$$
$$c_{2\beta-(2j+1)} = -\sqrt{2}c_{2\beta-2j}, j \geq 0$$

Proof. First let $\beta = 1$. Then

$$u = {x_1}^2 \otimes v + c_1x_1 \otimes X_{\delta_1}(v) + c_0 1 \otimes X_{\delta_1}^2(v).$$

It suffices to show that $X_{-\delta_1}(u) = X_{\delta_1-\delta_2}(u) = 0$.

$$X_{-\delta_1}(u) = \sqrt{2}x_1 \otimes v + \frac{c_1}{\sqrt{2}} \otimes X_{\delta_1}(v) - c_1x_1 \otimes X_{-\delta_1}X_{\delta_1}(v) + c_0 1 \otimes X_{-\delta_1}X_{\delta_1}^2(v).$$

Applying Lemma 7.1.7 we have:

$$X_{-\delta_1}(u) = \sqrt{2}x_1 \otimes v + c_1x_1 \otimes v + c_1\sqrt{2} \otimes X_{\delta_1}(v) + c_0 1 \otimes X_{\delta_1}(v).$$

Since $c_2 = 1, c_1 = -\sqrt{2}, c_0 = 1$, we have $X_{-\delta_1}(u) = 0$.

Since $X_{\delta_1-\delta_2}$ acts as $x_1\partial_2$, each power of x_1 is annihilated $X_{\delta_1-\delta_2}$. Hence:

$$X_{\delta_1-\delta_2}(u) = x_1^2 \otimes X_{\delta_1-\delta_2}(v) + c_1 x_1 \otimes X_{\delta_1-\delta_2} X_{\delta_1}(u) + c_0 1 \otimes X_{\delta_1-\delta_2} X_{\delta_1}^2(u).$$

By Lemma 7.1.7, $X_{\delta_1-\delta_2}X_{\delta_1}^j(v) = 0$, for all $j \in \mathbb{Z}_{\geq 0}$. Therefore $X_{\delta_1-\delta_2}(u) = 0$.

Assume now that $\beta > 1$. We must show that $X_{\delta_k-\delta_{k+1}}(u) = 0$ for $k = 1, ..., n-1$ and $X_{-\delta_1}(u) = 0$. Because $x_k\partial_{k+1}(x_1^j) = 0$, we have:

$$X_{\delta_k-\delta_{k+1}}(u) = \sum_{k=1}^{2\beta-1} c_{2\beta-k} x_1^{2\beta-k} \otimes X_{\delta_k-\delta_{k+1}} X^k{}_{\delta_1}(v).$$

Applying Lemma 7.1.7 once again, we find $X_{\delta_k-\delta_{k+1}}(u) = 0$.

Now, we show $X_{-\delta_1}(u) = 0$. We note that $X_{-\delta_1}(u)$ contains even number of terms, two of which are always zero. Namely, $x_1^{2\beta} \otimes X_{-\delta_1}(v) = 0$ since v is a highest weight vector, and $X_{-\delta_1} 1 \otimes X_{-\delta_1}^{2\beta}(v) = 0$ because the action of $X_{-\delta_1}$ on 1 is zero. This leaves a sequence with an even number of nonzero terms with coefficients of the form: $c_{2\beta}, c_{2\beta-1}, -c_{2\beta-1}, ..., -c_1, c_1, c_0$ with $c_{2\beta} = 1$. The claim is that this sequence telescopes to zero since each pair of consecutive terms cancel. We have to consider two cases; one for the case of even-odd subscripts and then odd-even subscripts.

Case 1: Even-odd subscript pair $(2\beta - 2j, 2\beta - (2j + 1))$.

Note that the sign between the two terms is negative due to parity considerations(the second term came from the action of $X_{-\delta_1}$ on $x_1^{2\beta-(2j+1)}$). We apply Lemma 7.1.7 on the right-hand side of the tensor product (using the odd case):

$$\begin{aligned} & c_{2\beta-2j} X_{-\delta_1}(x_1^{2\beta-2j}) \otimes X_{\delta_1}{}^{2j}(v) - c_{2\beta-(2j+1)} x_1^{2\beta-(2j+1)} \otimes X_{-\delta_1} X_{\delta_1}^{2j+1}(v) \\ &= c_{2\beta-2j} \frac{2\beta-2j}{\sqrt{2}} (x_1^{2\beta-(2j+1)}) \otimes X_{\delta_1}^{2j}(v) - c_{2\beta-(2j+1)} x_1^{2\beta-(2j+1)} \otimes (j-\beta) X_{\delta_1}^{2j}(v). \end{aligned}$$

By hypothesis, $c_{2\beta-(2j+1)} = -\sqrt{2}c_{2\beta-2j}$, thus:

$$\begin{aligned}
&c_{2\beta-2j}\frac{2\beta-2j}{\sqrt{2}}(x_1^{2\beta-(2j+1)}) \otimes X_{\delta_1}^{2j}(v) + \sqrt{2}c_{2\beta-(2j)}x_1^{2\beta-(2j+1)} \otimes (j-\beta)X_{\delta_1}^{2j}(v) \\
&= c_{2\beta-2j}\frac{2\beta-2j}{\sqrt{2}}(x_1^{2\beta-(2j+1)}) \otimes X_{\delta_1}^{2j}(v) + \sqrt{2}\frac{(2j-2\beta)}{2}c_{2\beta-(2j)}x_1^{2\beta-(2j+1)} \otimes X_{\delta_1}^{2j}(v) \\
&= 0
\end{aligned}$$

as required.

Case 2: Odd-even subscript pair $(2\beta-(2j+1), 2\beta-(2j+2))$.

$$\begin{aligned}
&c_{2\beta-(2j+1)}X_{-\delta_1}(x_1^{2\beta-(2j+1)}) \otimes X_{\delta_1}^{2j+1}(v) + c_{2\beta-(2j+2)}x_1^{2\beta-(2j+2)} \otimes X_{-\delta_1}X_{\delta_1}^{2j+2}(v) \\
&= -\frac{(2\beta-1)(2\beta-3)...(2\beta-(2j-1)(2\beta-(2j+1))}{j!}(x_1^{2\beta-(2j+2)}) \otimes X_{\delta_1}^{2j+1}(v) + \\
&\frac{(2\beta-1)(2\beta-3)...(2\beta-(2j+1))}{j!}x_1^{2\beta-(2j+2)} \otimes X_{\delta_1}^{2j+1}(v) \\
&= 0
\end{aligned}$$

as required. Therefore, u is a primitive vector. □

Lemma 7.1.9 *Let V be a finite-dimensional $\mathfrak{g}$-module and W be a bounded $\mathfrak{g}$-module, then $V \otimes W$ is bounded.*

Proof. Let λ be a weight of $V \otimes W$. The vectors in $(V \otimes W)^\lambda$ are of the form $v = \sum_i v_i \otimes w_i$, where $v_i \in V^{\mu_i}$, $w_i \in W^{\nu_i}$, where $\mu_i + \nu_i = \lambda$. Therefore,

$$(V \otimes W)^\lambda \subset \sum_{\mu \in \mathrm{wt}(V)} \left(V^\mu \otimes W^{\lambda-\mu}\right).$$

The sum above is over all weights μ of V and, since $\mathrm{wt}(V)$ is finite, the sum is finite. Also, we have that

$$V = \bigoplus_{\mu \in \mathrm{wt}(V)} V^\mu$$

and therefore $\dim V = \sum_{\mu \in \mathrm{wt}(V)} \dim V^\mu$. We thus have

$$\dim (V \otimes W)^{\lambda} \leq \dim \left(\sum_{\mu \in \mathrm{wt}(V)} \left(V^{\mu} \otimes W^{\lambda-\mu}\right) \right) \leq \sum_{\mu \in \mathrm{wt}(V)} \dim \left(V^{\mu} \otimes W^{\lambda-\mu}\right).$$

Since $\dim \left(V^{\mu} \otimes W^{\lambda-\mu}\right) = \dim V^{\mu} \dim W^{\lambda-\mu}$ and $\dim W^{\nu_i} \leq C$ for some C, we obtain

$$\dim (V \otimes W)^{\lambda} \leq \sum_{\mu \in \mathrm{wt}(V)} \dim V^{\mu} \dim W^{\lambda-\mu} \leq C \sum_{\mu \in \mathrm{wt}(V)} \dim V^{\mu} = C \dim V.$$

Therefore the weight multiplicities of $V \otimes W$ are bounded by $C \dim V$. □

Corollary 7.1.10 *If $\beta \in \mathbb{Z}_{>0}$, then the weight $(\beta, -\beta, ..., -\beta) + (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$ is $\mathfrak{osp}(1|2n)$-bounded.*

Proof. Let v be a highest weight vector of $L(-\beta, -\beta, ..., -\beta), \beta \in \mathbb{Z}_{>0}$ and consider the action of $\mathfrak{osp}(1|2n)$ on the tensor product $M = L(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) \otimes L(-\beta, -\beta, ..., -\beta)$. Let u be as specified in Theorem 7.1.8. Using Lemma 5.8.28, we have:

$$\begin{aligned}
\mathrm{wt}(u) &= \mathrm{wt}(x_1^{2\beta}) + \mathrm{wt}(v) \\
&= \left(2\beta + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right) + (-\beta, -\beta, ..., -\beta) \\
&= \left(\beta + \frac{1}{2}, -\beta + \frac{1}{2}, ..., -\beta + \frac{1}{2}\right)
\end{aligned}$$

Since u is primitive, $L(\beta + \frac{1}{2}, -\beta + \frac{1}{2}, ..., -\beta + \frac{1}{2})$ is a subquotient of M by Lemma 5.6.24. By Lemma 7.1.9, we have that $(\beta + \frac{1}{2}, -\beta + \frac{1}{2}, ..., -\beta + \frac{1}{2})$ is $\mathfrak{osp}(1|2n)$-bounded as required. □

7.2 Classification of Bounded Weights of $\mathfrak{osp}(1|2n)$

In this section we will classify all $\mathfrak{osp}(1|2n)$-bounded weights. We first recall the classification of the $\mathfrak{sp}(2n)$-bounded weights obtained by Britten and Lemire in [3], Theorem 2.15 and Lemma 2.1.

Definition 7.2.11 *Let λ be a weight. By $\dot{L}(\lambda)$ we denote the simple highest weight $\mathfrak{sp}(2n)$-module of highest weight λ relative to the base $\Pi_{\mathfrak{sp}}$ of $\mathfrak{sp}(2n)$ (see (5.2.13)).*

We note that the choice of simple root system in [3] is different from the one in the present thesis. We will write $\dot{L}_{BL}(\lambda)$ for the simple highest module of $\mathfrak{sp}(2n)$ of weight λ relative to the choice of simple roots in [3]. The "change of base formula" is

$$\dot{L}_{BL}(-\lambda_n, -\lambda_{n-1}, ..., -\lambda_1)^{\theta} = \dot{L}(\lambda_1, \lambda_2, ..., \lambda_n).$$

where θ is an automorphism of $\mathfrak{sp}(2n)$ defined on the generators by $X_{\pm\delta_i} \mapsto X_{\mp\delta_{n+1-i}}$, and M^{θ} stands for the $\mathfrak{sp}(2n)$-module obtained from M after twisting by θ.

Definition 7.2.12 *Let λ be a weight. We say that λ is $\mathfrak{sp}(2n)$*-bounded *if the $\mathfrak{sp}(2n)$-module $\dot{L}(\lambda)$ is bounded.*

Theorem 7.2.13 (Theorem 2.15, [3]) *If the simple highest weight module $\dot{L}_{BL}(\nu_1 + \cdots \nu_n, \nu_2 + \cdots \nu_n, ..., \nu_n)$ of $\mathfrak{sp}(2n)$ is infinite-dimensional, then it is $\mathfrak{sp}(2n)$-bounded if and only if*

(A1) $\nu_n = \frac{2k+1}{2}$ with $k \in \mathbb{Z}$,

(A2) $\nu_{n-1} + 2\nu_n + 3 \in \mathbb{Z}_{>0}$.

(A3) $\nu_i \in \mathbb{Z}_{\geq 0}$ for every $i, 1 \leq i \leq (n-1)$,

Proposition 7.2.14 *If λ is an $\mathfrak{osp}(1|2n)$-bounded weight and μ is a dominant integral weight, then $\lambda + \mu$ is an $\mathfrak{osp}(1|2n)$-bounded weight.*

Proof. Since λ is bounded, $L(\lambda)$ is also bounded. Also, $L(\mu)$ is finite-dimensional by Theorem 5.7.26 since μ is dominant integral weight. Thus, $L(\lambda) \otimes L(\mu)$ is bounded by Lemma 7.1.9. On the other hand, $L(\lambda+\mu)$ is a subquotient of $L(\lambda) \otimes L(\mu)$. Indeed this follows from the fact that $v \otimes w$ is a highest weight vector of $L(\lambda) \otimes L(\mu)$, whenever v and w are highest weight vectors of $L(\lambda)$ and $L(\mu)$, respectively (see Lemma 5.8.28). Hence, $L(\lambda + \mu)$ is bounded. □

Lemma 7.2.15 *Let $k_i \in \mathbb{Z}_{>0}$ for $i = 1, 2, ..., n-1$. Then the weight $\lambda = (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) + \beta(1, -1, ..., -1) + k_1(-1, -1, ..., -1) + k_2(0, -1, ..., -1, -1) + ... + k_n(0, 0, ..., 0, -1)$ is $\mathfrak{osp}(1|2n)$-bounded.*

Proof. By Corollary 7.1.10, we have that $(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) + \beta(1, -1, ..., -1)$ is a bounded weight. Define

$$(\gamma_1, \gamma_2, ..., \gamma_n) = k_1(-1, -1, ..., -1) + k_2(0, -1, ..., -1, -1) + ... + k_n(0, 0, ..., 0, -1).$$

By construction, $\gamma_i = -\sum_{j=1}^{i} k_j$, hence, $\gamma_i \leq 0$. Also, $\gamma_i - \gamma_{i+1} = k_{i+1} \geq 0$, so we have $\gamma_i \geq \gamma_{i+1}$. Thus, $\gamma = (\gamma_1, ..., \gamma_n)$ is a dominant integral weight and $L(\gamma)$ is finite-dimensional. Thus, $L((\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) + \beta(1, -1, ..., -1)) \otimes L(-\gamma)$ is a bounded module by Lemma 7.1.9. Now applying Proposition 7.2.14 we complete the proof. □

Theorem 7.2.16 *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ be such that*

(B1) $\lambda_i - \frac{1}{2} \in \mathbb{Z}$, *for* $i = 1, 2, ..., n$

(B2) $\lambda_1 + \lambda_2 \leq 1$

(B3) $\lambda_i - \lambda_{i+1} \geq 0$ *for* $i = 1, 2, ..., n-1$

Then λ is bounded.

Proof. For convenience, we set $\widetilde{\lambda_i} := \lambda_i - \frac{1}{2}$ and $\alpha_k = \sum_{j=k}^{n} \delta_j$. Using Lemma 7.2.15, it suffices to show that there exist $k_i, \beta \in \mathbb{Z}_{\geq 0}$ for $i = 1, 2, ..., n$ such that:

$$(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) + (\widetilde{\lambda_1}, \widetilde{\lambda_2}, ..., \widetilde{\lambda_n}) = (\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) + \beta(1, -1, ..., -1) - \sum_{i=1}^{n} k_i \alpha_k.$$

We treat the above identity as a set of n equations in the $n+1$ variables $\beta, k_1, ..., k_n$. We proceed with a case-by-case consideration.

Case 1: $\widetilde{\lambda_1} + \widetilde{\lambda_2}$ is even. We need to solve the system:

$$\beta - k_n = \widetilde{\lambda}_1$$

$$-\beta - k_n - k_{n-1} = \widetilde{\lambda}_2$$

$$-\beta - k_n - k_{n-1} - ... - k_1 = \widetilde{\lambda}_n$$

We will find $\beta, k_1, k_2, ..., k_n$ with $k_{n-1} = 0$. We first easily find $k_n = \frac{-\widetilde{\lambda}_1 - \widetilde{\lambda}_2}{2}$. Since $\widetilde{\lambda}_1 + \widetilde{\lambda}_2$ is even and $\lambda_1 + \lambda_2 \leq 1$, $k_n \in \mathbb{Z}_{\geq 0}$ as required. Furthermore, we obtain:

$$k_{n-i} = \widetilde{\lambda}_i - \widetilde{\lambda}_{i+1}.$$

By hypothesis, $k_{n-i} \in \mathbb{Z}_{\geq 0}$, for all i. Also, $\beta = \frac{\widetilde{\lambda}_1 - \widetilde{\lambda}_2}{2}$, and since, $\widetilde{\lambda}_1 + \widetilde{\lambda}_2$ is even, we have $\beta \in \mathbb{Z}_{\geq 0}$.

Case 2: $\widetilde{\lambda}_1 + \widetilde{\lambda}_2$ is odd. Contrary to Case 1, we will find $\beta, k_1, k_2, ..., k_n$ but with $k_{n-1} = 1$. Again, we obtain $k_n = \frac{-\widetilde{\lambda}_1 - \widetilde{\lambda}_2 + 1}{2}$. Since $\widetilde{\lambda}_1 + \widetilde{\lambda}_2$ is odd and $\lambda_1 + \lambda_2 \leq 1$, $k_n \in \mathbb{Z}_{\geq 0}$ as required. As above, we obtain $k_{n-i} = \widetilde{\lambda}_i - \widetilde{\lambda}_{i+1}$; therefore by condition ii., $k_{n-i} \in \mathbb{Z}_{\geq 0}$. Finally, $\beta = \frac{\widetilde{\lambda}_1 - \widetilde{\lambda}_2 + 1}{2}$. $\widetilde{\lambda}_1 + \widetilde{\lambda}_2$ is odd implies $\widetilde{\lambda}_1 - \widetilde{\lambda}_2 + 1$ even. $\square$

Theorem 7.2.17 *If $L(\lambda)$ is infinite-dimensional bounded $\mathfrak{osp}(1|2n)$-module, then λ and $\lambda + \delta_1$ are $\mathfrak{sp}(2n)$-bounded.*

Proof. Since $\dot{L}(\lambda)$ is an $\mathfrak{sp}(2n)$-subquotient of $L(\lambda)$, $\dot{L}(\lambda)$ is $\mathfrak{sp}(2n)$-bounded. It remains to prove that $\dot{L}(\lambda + \delta_1)$ is $\mathfrak{sp}(2n)$-bounded. Since $\dot{L}(\lambda)$ is an infinite-dimensional bounded $\mathfrak{sp}(2n)$-module, by Theorem 7.2.13, we have that $\lambda_1 \neq 0$. But then one easily checks that $X_{\delta_1} v$ is a nonzero $\mathfrak{sp}(2n)$-primitive vector in $L(\lambda)$, where v is a highest weight vector of $L(\lambda)$ and X_{δ_1} is in $\mathfrak{g}^{\delta_1}$. Hence, by Lemma 5.6.24, $L(\lambda)$ has an $\mathfrak{sp}(2n)$-subquotient isomorphic to $\dot{L}(\lambda + \delta_1)$. $\square$

Theorem 7.2.18 *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$. Then the following are equivalent.*

(i) λ satisfies conditions (B1), (B2), and (B3), in Theorem 7.2.16.

(ii) $L(\lambda)$ is infinite-dimensional bounded $\mathfrak{osp}(1|2n)$-module.

(iii) $\dot{L}(\lambda)$ and $\dot{L}(\lambda + \delta_1)$ are infinite-dimensional bounded $\mathfrak{sp}(2n)$-modules.

Proof. We have that (i) implies (ii) by Theorem 7.2.16 and that (ii) implies (iii) by Theorem 7.2.17.

It remains to show that (iii) implies (i). Let $\dot{L}(\lambda)$ and $\dot{L}(\lambda + \delta_1)$ be infinite-dimensional $\mathfrak{sp}(2n)$-bounded modules. By [3] Theorem 2.15, we have conditions (A1), (A2), and (A3) for ν_i, where $\lambda_i = -\sum_{k=n-i+1}^{n} \nu_k$. Since ν_n a half-integer by (A2), we easily obtain (B1). For (B2), we use crucially the fact that $\lambda + \delta_1$ is $\mathfrak{sp}(2n)$-bounded. We apply (A3) for $\lambda + \delta_1$ and obtain $\nu_n + (\nu_{n-1} + \nu_n) + 3 > 0$. From here we have $\lambda_1 + \lambda_2 \leq 1$, as needed in (B2). For (B3), we have

$$\lambda_i - \lambda_{i+1} = -\sum_{k=n-i+1}^{n} \nu_k + \sum_{k=n-i+2}^{n} \nu_k = \nu_{n-i}$$

which is in $\mathbb{Z}_{\geq 0}$ (by A1) as required. □

Example 7.2.19 *The set of $\mathfrak{osp}(1|4)$-bounded weights and $\mathfrak{sp}(4)$-bounded weights corresponding to infinite-dimensional highest weight modules are pictured on Figure 7.1. The grey part corresponds to the $\mathfrak{osp}(1|4)$-bounded weights.*

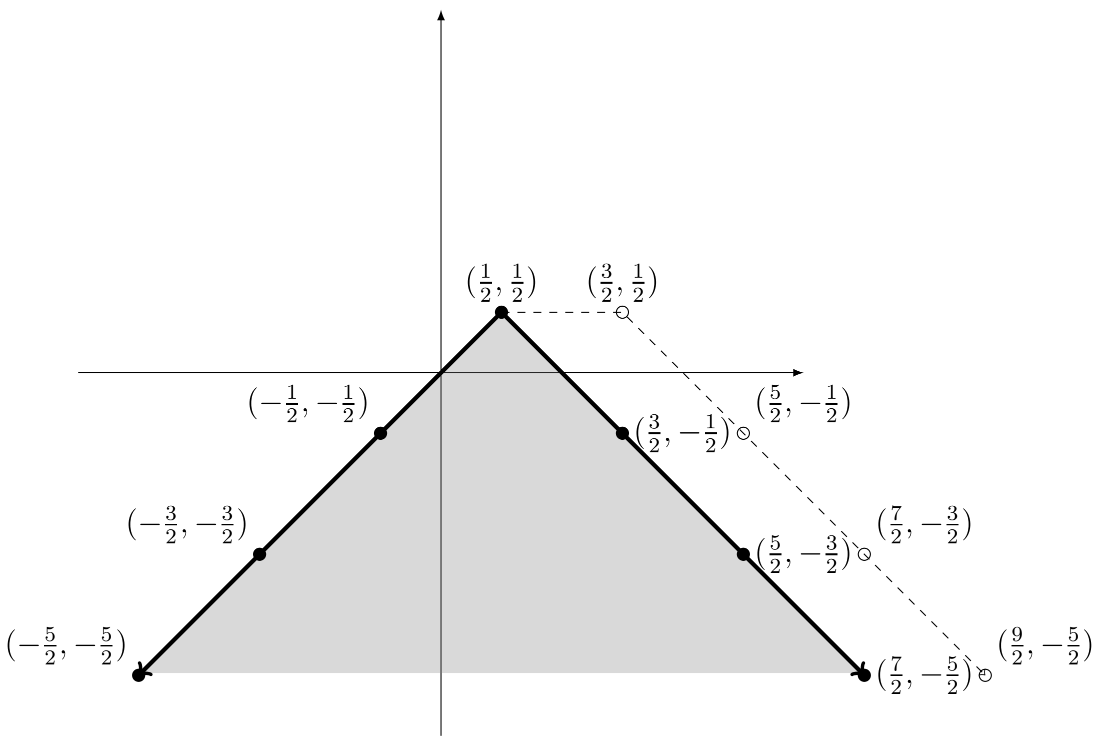

Figure 7.1. Bounded Weights for $\mathfrak{osp}(1|4)$.

7.3 Classification of Simple Weight Modules of $\mathfrak{osp}(1|2n)$

The classification of the infinite-dimensional bounded simple $\mathfrak{osp}(1|2n)$-modules obtained in the previous section leads to another important classification. Namely we obtained a classification of all simple weight modules M of $\mathfrak{osp}(1|2n)$. This follows from the theorem that every such M is isomorphic to a twisted localization of a bounded infinite-dimensional module $L(\lambda)$, see Theorem 5.10 in [12].

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BIOGRAPHICAL STATEMENT

Thomas Lynn Ferguson was born in Longview, Texas, in 1958 to Travis and Flora Ferguson. He is the middle son among a family of three sons and three daughters. He attended Judson Junior High and graduated from Longview High School in 1976. He received his B.S. degree in mathematics from Texas A&M University in 1979. Pursuing further graduate work, he received his M.A. degree in mathematics from the University of Wisconsin in 1981 while also earning teacher certification. He met his wife Cynthia; the couple married in 1982 and have four children and one grandchild. Thomas taught public school for two years and then spent the next twenty- four years in the business world. He was a software test engineer at General Dynamics in Fort Worth from 1986 to 1989. He and his family then moved to Carrollton, Texas. Thomas was employed for nineteen years by Alcatel SA (formerly DSC Communications) first as a software engineer and later as an engineering manager and product manager. In 2008, he left the business world to take a position as assistant professor of mathematics at Southwestern Assemblies of God University in Waxahachie, Texas. The next fall, he began the Ph.D. mathematics program at the University of Texas in Arlington. In May 2015, he earned his Ph.D. in mathematics under the direction of Dr. Dimitar Grantcharov. Thomas' research interest are in algebra with particular interest in the orthosymplectic Lie superalgebras.