PROPERTIES OF THE PINCHED TENSOR PRODUCT

by

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Abstract

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For complexes of modules we study a new construction, called the pinched tensor product, which was introduced in [1] by Lars Winther Christensen and David A. Jorgensen to study Tate homology $\widehat{\mathrm{Tor}}$. We explore properties of the pinched tensor product and their comparison to properties of the ordinary tensor product. For example; we show the isomorphisms $\Sigma(C \otimes_R A) \cong (\Sigma C) \otimes_R A \cong C \otimes_R (\Sigma A)$ where A and C are two complexes, no longer holds for the pinched tensor product. Although if we change isomorphism to *quasi-isomorphism* the pinched version holds. Plus if $f : C \to D$ and $g : A \to B$ are morphisms of complexes of R-modules with f homotopic to 0, then $f \otimes g$ homotopic to 0, and this property is not true for the pinched tensor product.

Table Of Contents

List of Illustrations

Chapter 1

Introduction

Tensor products of complexes is a classical construction and has been used in countless applications for many decades. Consequently, their properties are well-known. We refer to them as the ordinary tensor product of complexes. Lars Christensen and David Jorgensen introduced in [1] a variant of the ordinary tensor product of complexes, call the pinched tensor product. They use it to compute Tate homology and to show Tate homology is balanced. Also, they show it yields a complete resolution of the tensor product of two Tate Tor-independent modules. Since it is a brand new construction, very few properties of the pinched tensor product are known. Therefore we investigate the properties of the pinched tensor product and compare them with the analogous properties of the ordinary tensor product.

In Chapter 2 we recall the definition of the ordinary tensor product of modules, complexes and maps, and give proofs of many properties for the ordinary tensor products. We also see that the ordinary tensor product is a functor of complexes, and discuss how it relates to morphisms, shift and homotopy.

In Chapter 3 we give the definition of the pinched tensor product and a precise description of its basic properties, like commutativity and associativity. Also, we compare the basic properties of the pinched tensor product with those of the ordinary tensor product, and conclude which of these properties hold for the pinched tensor product and which do not.

Chapter 4 is focus on the pinched tensor product and shift. We show that the isomorphisms $\Sigma(C \otimes_R A) \cong (\Sigma C) \otimes_R A \cong C \otimes_R (\Sigma A)$ that hold for the ordinary tensor product no longer hold for the pinched tensor product. Although if we change isomorphism to *quasi-isomorphism* the statement for the pinched tensor product holds. In addition, we give some counterexamples for the isomorphisms that no longer hold for the pinched tensor product.

Finally, Chapter 5 examines two other properties: the pinched homotopy and the pinched mapping cone. We see that the pinched tensor product is a functor from the category of complexes, but we cannot extend to it a functor on the homotopy categories. Also, we give some counterexamples for the relevant properties that no longer hold for the pinched tensor product. In addition we show that the isomorphism that implies the mapping cone commutes with tensor product for the ordinary tensor product no longer holds for the pinched tensor product. However, we show there is a morphism.

Chapter 2

Preliminary Concepts

Unless otherwise indicated, we will assume A, B, C and D are complexes of R-modules, R is an associative ring and M, N are R-modules. We assume M and N are either left or right R-modules, depending on the context. For example when discussing $M \otimes_R N$ we are assuming M is a right R-module and N is a left R-module. The same holds for complexes. For example, when discussing $C \otimes_R D$ we are assuming C is a complex of right R-module homomorphisms and D is a complex of left R-module homomorphisms. We will find most of the definitions and results of this chapter in [2] and [4].

2.1 Complexes

Definition 2.1.1. A *chain complex* C of R-modules is a sequence of R-module homomorphisms,

$$C : \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}^C} C_n \xrightarrow{\partial_n^C} C_{n-1} \longrightarrow \cdots$$

such that $\operatorname{Im} \partial_{n+1}^C \subseteq \ker \partial_n^C$ for all n. Equivalently $\partial_n^C \partial_{n+1}^C = 0$, for all n. The maps ∂_n are called the *differentials* of C.

Definition 2.1.2. Given a complex C of R-modules

$$C : \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}^C} C_n \xrightarrow{\partial_n^C} C_{n-1} \longrightarrow \cdots$$

We say that it is exact at C_n if $\operatorname{Im}\partial_{n+1}^C = \ker\partial_n^C$. Moreover, we say C is an *exact sequence* if $\operatorname{Im}\partial_{n+1}^C = \ker\partial_n^C$ for all n.

Definition 2.1.3. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}^C} C_n \xrightarrow{\partial_n^C} C_{n-1} \longrightarrow 0.$$

Definition 2.1.4. Let C be a complex. Then the submodule of n-cycles is $Z_n(C) = \ker\partial_n^C$, and the submodule of n-boundaries is $B_n(C) = \operatorname{Im}\partial_{n+1}^C$.

Remark 2.1.5. The condition that $\operatorname{Im}\partial_{n+1}^C \subseteq \ker\partial_n^C$ yields for all n,

$$B_n(C) \subseteq Z_n(C).$$

Definition 2.1.6. The *nth homology group* of a complex C is

$$H_n(C) = Z_n(C)/B_n(C).$$

Remark 2.1.7. In Definition 2.1.6 we can conclude that $H_n(C) = 0$ if and only if C is exact at C_n.

Remark 2.1.8. It is known that if C and D be complexes. A chain map $f : C \to D$ takes cycles to cycles and boundaries to boundaries.

2.2 Functors

Definition 2.2.1. A *category* $\mathcal{C}$ consists of three ingredients: a class $obj(\mathcal{C})$ of objects, a set of morphisms $\operatorname{Hom}(A, B)$ for every ordered pair (A, B) of objects, and composition $\operatorname{Hom}(A, B) \times \operatorname{Hom}(B, C) \to \operatorname{Hom}(A, C)$ denoted by $(f, g) \to gf$, for every ordered triple A, B, C of objects. These ingredients are subject to the following axioms:

1. The Hom sets are pairwise disjoint: that is , each $f \in \text{Hom}(A, B)$ has a unique domain A and a unique target B;

2. For each object A, there is an identity morphism $1_A \in \text{Hom}(A, A)$ such that $f1_A = f$ and $1_B f = f$ for all $f : A \to B$;

3. Composition is associative: given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then $h(gf) = (hg)f$.

Definition 2.2.2. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a function such that

1. If $A \in obj(\mathcal{C})$ then $F(A) \in obj(\mathcal{D})$.
2. If $f : A \to A'$ in $\mathcal{C}$, then $F(f) : F(A) \to F(A')$ in $\mathcal{D}$.
3. If $A \xrightarrow{f} A' \xrightarrow{g} A''$ in $\mathcal{C}$ then $F(A) \xrightarrow{F(f)} F(A') \xrightarrow{F(g)} F(A'')$ in $\mathcal{D}$ and $F(gf) = F(g)F(f)$.
4. $F(1_A) = 1_{F(A)}$ for every $A \in obj(\mathcal{C})$.

Definition 2.2.3. A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are categories, is a function such that

1. If $C \in obj(\mathcal{C})$ then $F(C) \in obj(\mathcal{D})$.
2. If $f : C \to C'$ in $\mathcal{C}$, then $F(f) : F(C') \to F(C)$ in $\mathcal{D}$ (note the reversal of arrows).
3. If $C \xrightarrow{f} C' \xrightarrow{g} C''$ in $\mathcal{C}$ then $F(C'') \xrightarrow{F(g)} F(C') \xrightarrow{F(f)} F(C)$ in $\mathcal{D}$ and $F(gf) = F(f)F(g)$.
4. $F(1_A) = 1_{F(A)}$ for every $A \in obj(\mathcal{C})$.

2.3 Tensor Products

2.3.1 Tensor Products of Modules

Definition 2.3.1. Let R be a ring, let M be a right R-module, let N be a left R-module, and let G be an additive abelian group. A function $f : M \times N \to G$ is called *R-biadditive* if for all $m, m' \in M, n, n' \in N$, and $r \in R$, we have

$$f(m + m', n) = f(m, n) + f(m', n),$$

$$f(m, n + n') = f(m, n) + f(m, n'),$$

$$f(mr, n) = f(m, rn).$$

Definition 2.3.2. Given a ring R and R–modules M and N, their *tensor product* is an abelian group $M \otimes_R N$ together with an R-biadditive function

$$\tau : M \times N \to M \otimes_R N$$

such that the following universal mapping property holds: for every abelian group G and every R-biadditive function $\varphi : M \times N \to G$, there exists a unique map $\phi : M \otimes_R N \to G$ making the following diagram commute.

$$\begin{array}{ccc} M \times N & \xrightarrow{\tau} & M \otimes_R N \\ {\scriptstyle\varphi}\downarrow & \swarrow{\scriptstyle\phi} & \\ G & & \end{array}$$

That is, $\varphi = \phi\tau$.

Proposition 2.3.3. *If R is a ring, and M is a right R-module and N is a left R-module, then their tensor product exists.*

Proof: let F be the free abelian group with basis $M \times N$: that is, F is free on all ordered pairs (m,n), where $m \in M$ and $n \in N$. Define S to be the subgroup of F generated by all elements of the following three types;

$$(m, n + n') - (m, n) - (m, n');$$

$$(m + m', n) - (m, n) - (m', n);$$

$$(mr, n) - (m, rn).$$

where $m \in M$, , $n \in N$ and $r \in R$. Define $M \otimes_R N = F/S$, denote the coset $(m,n) + S$ by $m \otimes n$, and define $\tau : M \times N \to M \otimes_R N$ by $\tau\big((m,n)\big) = m \otimes n$. It is now obvious that τ is R-biadditive since

$$\begin{aligned} \tau(m + m', n) &= (m + m' \otimes n) \\ &= (m \otimes n) + (m' \otimes n) \\ &= \tau(m,n) + \tau(m',n). \end{aligned}$$

Similarly for the other properties. We now need to verify the universal mapping properties. Let G be an abelian group and $\varphi : M \times N \to G$ be an R-biadditive function. Define $\Psi : F \to G$ by $\Psi\big((m,n)\big) = \varphi\big((m,n)\big)$ and extend by linearity. Since φ is R-biadditive we have $\Psi(S) = 0$, and therefore by the first isomorphism theorem there exists a well-defined homomorphism $\phi : M \otimes_R N \to G$ with $\phi(m \otimes n) = \varphi\big((m,n)\big)$. Also $(\phi\tau)((m,n)) = \phi\big((m \otimes n)\big) = \varphi(m,n)$, therefore $\varphi\tau = f$ exists. Finally, suppose that $\phi' : M \otimes_R N \to G$ satisfies, $\phi'\tau = \varphi$. Then $\phi\Big((m \otimes n\Big) = \phi(\tau(m,n)) = \varphi(m,n) = \phi'\Big(\tau(m,n)\Big) = \phi'\Big((m \otimes n)\Big)$, thus $\phi' = \phi$.

□

Proposition 2.3.4. *The tensor product $M \otimes_R N$ is unique up to isomorphism.*

Proof: Assume that $M \otimes_R N$ and $(M \otimes_R N)'$ are two abelian groups satisfying the definition of the tensor product. Then we have

$$\begin{array}{ccc} M \times N & \xrightarrow{\tau_1} & M \otimes_R N \\ \downarrow{\scriptstyle \tau_2} & \swarrow{\scriptstyle \varphi} & \\ (M \otimes_R N)' & & \end{array}$$

And,

$$\begin{array}{ccc} M \times N & \xrightarrow{\tau_2} & (M \otimes_R N)' \\ \downarrow{\scriptstyle \tau_1} & \swarrow{\scriptstyle \varphi'} & \\ M \otimes_R N & & \end{array}$$

It is clear that $\varphi' \varphi = \mathrm{Id}_{M \otimes_R N}$. Similar work shows $\varphi \varphi' = \mathrm{Id}_{(M \otimes_R N)'}$. Therefore φ' and φ are isomorphisms.

□

Remark 2.3.5. 1. Since $M \otimes_R N$ is generated by the elements of the form $m \otimes n$, every $x \in M \otimes_R N$ has the finite sum form

$$x = \sum_i m_i \otimes n_i.$$

2. The tensor product of two elements of M and N is bilinear by Definition 2.3.2 which means we have $(m + m') \otimes n = m \otimes n + m' \otimes n$, $m \otimes (n + n') = m \otimes n + m \otimes n'$ and $mr \otimes n = m \otimes rn$ for all $m, m' \in M$, $n, n' \in N$ and $r \in R$. When R is commutative, we also have $r(m \otimes n) = rm \otimes n = m \otimes rn$ in $M \otimes N$

3. From 2, the expression for $x \in M \otimes_R N$ is not necessarily unique; for example for all $m, m' \in M$, $n, n' \in N$ and $r \in R$ we have,

$$mr \otimes n - m \otimes rn = 0,$$

$$(m + m') \otimes n - m \otimes n - m' \otimes n = 0,$$

$$m \otimes (n + n') - m \otimes n - m \otimes n' = 0.$$

The following Propositions are basic facts, using the universal mapping property.

Proposition 2.3.6. *Assume R is commutative. For any R-modules M and N there is an isomorphism $M \otimes_R N \cong N \otimes_R M$.*

Proof: The proof is well-known. (Proposition 2.56 in [2].)

□

Proposition 2.3.7. *Let M be right R-module. Then*

$$M \otimes_R R \cong M.$$

Proof: The proof is well-known. (Proposition 2.58 in [2].)

□

2.3.2 Tensor Products of Complexes

Definition 2.3.8. The *tensor product* $C \otimes_R D$ over R of chain complexes C and D is specified by letting

$$(C \otimes_R D)_n = \bigoplus_{i \in \mathbb{Z}} C_i \otimes_R D_{n-i}$$

The differential is defined by

$$\partial_n^{C \otimes D}(c \otimes d) = \partial_i^C(c) \otimes d + (-1)^i c \otimes \partial_{n-i}^D(d)$$

for $c \in C_i$ and $d \in D_{n-i}$. The sign $(-1)^i$ ensures that $\partial_n^{C \otimes D} \partial_{n+1}^{C \otimes D} = 0$ for all n.

Definition 2.3.9. Let C be complex, Then $C_{\geqslant n}$ is defined by

$$(C_{\geqslant n})_i = \begin{cases} C_i & \text{for } i \geqslant n \\ 0 & \text{for } i < n, \end{cases}$$

and

$$\partial_i^{C_{\geqslant n}} = \begin{cases} \partial_i^C & \text{for } i > n \\ 0 & \text{for } i \leqslant n. \end{cases}$$

Similarly, $C_{\leqslant n}$ is defined by

$$(C_{\leqslant n})_i = \begin{cases} C_i & \text{for } i \leqslant n \\ 0 & \text{for } i > n, \end{cases}$$

and

$$\partial_i^{C_{\leqslant n}} = \begin{cases} \partial_i^C & \text{for } i \leqslant n \\ 0 & \text{for } i > n. \end{cases}$$

Remark 2.3.10. From Definition 2.3.8, we can see that

$$(C \otimes_R D)_n = \bigoplus_{i \in \mathbb{Z}} C_i \otimes_R D_{n-i}$$

is not finitely generated even if C_i and D_{n-i} are so for all i (assuming C_i and D_{n-i} are nonzero for all i). Here is a picture illustrating the ordinary tensor product, where each point $(i, n-i)$ in the plane represents the tensor product $C_n \otimes_R D_{n-i}$ of modules and each line $Y = -X + n$ represents $(C \otimes_R D)_n$.

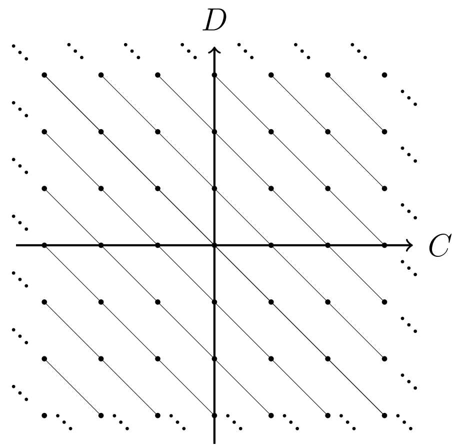


Figure 2.1: Ordinary Tensor Product

And the differentials can be represented by;

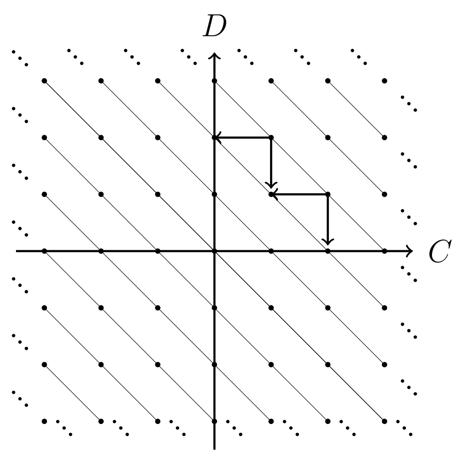


Figure 2.2: Differential on the Ordinary Tensor Product

Theorem 2.3.11. *Let R be a commutative ring and C and D be complexes. Then*

$$C \otimes_R D \cong D \otimes_R C.$$

Proof: The proof is well-known. (Propostion 2.56 in [2].)

□

Theorem 2.3.12. *Let R and S be rings, and A, B and C be complexes and A a right R-module, B a RS-bimodule and C a left S-module. Then*

$$(A \otimes_R B) \otimes_S C \cong A \otimes_R (B \otimes_S C).$$

Proof: The proof is well-known. (Proposition 2.57 in [2].)

□

Proposition 2.3.13. *Let C be a complex. Then*

$$C \otimes_R R \cong C.$$

Proof: The proof follows easily from Proposition 2.3.7.

□

2.3.3 Tensor Products of Maps

Proposition 2.3.14. *Let M and N be R-modules, and Let $f : M \to M'$, $g : N \to N'$ be R-module maps. Then there exists a homomorphism of abelian groups $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ defined by $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$ for all $n \in N, m \in M$.*

Proof: The function $\varphi : M \times N \to M' \otimes_R N'$, given by $(m, n) \to f(m) \otimes g(n)$, is an R-biadditive function. For example,

$$\varphi : (mr, n) \to f(mr) \otimes g(n) = f(m)r \otimes g(n)$$

and

$$\varphi : (m, rn) \to f(m) \otimes g(rn) = f(m) \otimes rg(n);$$

these are equal because of the identity $m'r \otimes n' = m' \otimes rn'$ in $M' \otimes_R N'$. The biadditive function φ yields a unique homomorphism $M \otimes_R N \to M' \otimes_R N'$ taking $m \otimes n \to f(m) \otimes$

$g(n)$.

□

Corollary 2.3.15. *If $f : M \to M'$ and $g : N \to N'$ are, respectively, isomorphisms of right and left R-modules, then $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ is an isomorphism of abelian groups.*

Proof: It is easy to see that $f \otimes 1_N$ is an isomorphism since f is and, similarly, $1_M \otimes g$ is an isomorphism. Then we have $f \otimes g = (f \otimes 1_N)(1_M \otimes g)$. Therefore, $f \otimes g$ is an isomorphism, being the composite of isomorphisms.

□

Theorem 2.3.16. *Let M be a right R-module, and let*

$$N' \xrightarrow{i} N \xrightarrow{p} N'' \longrightarrow 0.$$

be an exact sequence of left R-modules. Then

$$M \otimes_R N' \xrightarrow{1 \otimes i} M \otimes_R N \xrightarrow{1 \otimes p} M \otimes_R N'' \longrightarrow 0$$

is an exact sequence of abelian groups.

Proof: We must check three things

1. $\mathrm{Im}\,(1 \otimes i) \subseteq \ker(1 \otimes p)$.

   It suffices to prove that the composite is 0. We have $(1 \otimes p)(1 \otimes i) = 1 \otimes pi = 1 \otimes 0 = 0$.

2. $\ker(1 \otimes p) \subseteq \mathrm{Im}\,(1 \otimes i)$.

   Let $E = \mathrm{Im}\,(1 \otimes i)$. By part (i), $E \subseteq \ker(1 \otimes p)$, and so $1 \otimes p$ induces a map $p' : (M \otimes N)/E \to M \otimes N''$ with

$$p' : m \otimes n + E \to m \otimes pn,$$

where $m \in M$ and $n \in N$. Now if $\pi : M \otimes N \to (M \otimes N)/E$ is the natural map, then $p'\pi = 1 \otimes p$, for both send $m \otimes n \to m \otimes pn$ where $n \in N$ and $m \in M$.

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\pi} & (M \otimes_R N)/E \\ \downarrow{\scriptstyle 1\otimes P} & \swarrow{\scriptstyle p'} & \\ M \otimes N'' & & \end{array}$$

Suppose we show that p' is an isomorphism. Then

$\ker(1 \otimes p) = \ker(p'\pi) = \ker(\pi) = E = \mathrm{Im}\,(1 \otimes i)$, and we are done. To see that p' is an isomorphism, we construct its inverse $M \otimes N'' \to (M \otimes N)/E$ as follows. If $n'' \in N''$, there is $n \in N$ with $pn = n''$, because p is surjective; let

$$f : (m, n'') \to m \otimes n.$$

Now f is well-defined: if $pn_1 = n''$, then $p(n - n_1) = 0$ and $n - n_1 \in \ker p = \mathrm{Im}\, i$. Thus there is $n' \in N'$ with $in' = n - n_1$, and hence $m \otimes (n - n_1) = a \otimes in' \in \mathrm{Im}\,(1 \otimes i) = E$. Clearly, f is R-biadditive, and so the definition of tensor product gives a homomorphism $f' : M \otimes N'' \to (M \otimes N)/E$ with $f'(m \otimes n'') = m \otimes n + E$, and f' is the inverse of P'.

3. $1 \otimes p$ is surjective.

   If $\sum m_i \otimes n_i'' \in M \otimes N''$, then there exist $n_i \in N$ with $pn_i = n_i''$ for all i, for p is subjective. But $1 \otimes p : \sum m_i \otimes n_i \to \sum m_i \otimes pn_i = \sum m_i \otimes n_i''$.

□

A similar statement holds for the functor $\Box \otimes_R N$: if N is a left R-module and $M' \xrightarrow{l} M \xrightarrow{p} M'' \longrightarrow 0$ is a short exact sequence of right R-modules, then the sequence

$$M' \otimes_R N \xrightarrow{i\otimes 1} M \otimes_R N \xrightarrow{p\otimes 1} M'' \otimes_R N \longrightarrow 0$$

is exact.

2.4 Ordinary Tensor Product and Morphisms

Definition 2.4.1. A *morphism*, or *a degree zero chain map*, $f : C \to D$ between complexes C and D is a family of R-module homomorphisms f_n such that each square in the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial^C_{n+1}} & C_n & \xrightarrow{\partial^C_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & D_{n+1} & \xrightarrow{\partial^D_{n+1}} & D_n & \xrightarrow{\partial^D_n} & D_{n-1} & \longrightarrow & \cdots
\end{array}$$

commutes. In other words, for each n we have $f_{n-1}\partial^C_n = \partial^D_n f_n$.

Also, if $f_n : C_n \to D_n$ is an isomorphism for all n then, C and D are said to be *isomorphic*, denoted by $\cong$.

Example 2.4.2. Suppose M and N are two modules and $f : M \to N$ is a homomorphism. Then the following diagram obviously commutes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \tau} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & N & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

and so is an example of a morphism of complexes.

□

Definition 2.4.3. Let C and D be complexes. A *degree d chain map* $f : C \to D$ is a family of maps $f_n : C_n \to D_{n+d}$ such that $f_{n-1}\partial_n^C = (-1)^d \partial_{n+d}^D f_n$, which is expressed by the diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}^C} & C_n & \xrightarrow{\partial_n^C} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & D_{n+d+1} & \xrightarrow{\partial_{n+d+1}^D} & D_{n+d} & \xrightarrow{\partial_{n+d}^D} & D_{n+d-1} & \longrightarrow & \cdots
\end{array}$$

Proposition 2.4.4. *Let $f : C \to D$ and $g : A \to B$ be morphism of complexes, then there exists a morphism of complexes $f \otimes g : C \otimes_R A \to D \otimes_R B$ defined by $(f \otimes g)_n(c \otimes a) = f_i(c) \otimes g_{n-i}(a)$ for $c \otimes a \in C_i \otimes A_{n-i}$.*

Proof: The goal is to define $(f \otimes g)$ in each degree and show each square commutes

$$\begin{array}{ccc}
(C \otimes_R A)_n & \xrightarrow{\partial_n^{C\otimes A}} & (C \otimes_R A)_{n-1} \\
\downarrow{\scriptstyle (f\otimes g)_n} & & \downarrow{\scriptstyle (f\otimes g)_{n-1}} \\
(D \otimes_R B)_n & \xrightarrow{\partial_n^{D\otimes B}} & (D \otimes_R B)_{n-1}
\end{array}$$

Then for all $n \in \mathbb{Z}$ the diagram will be

$$\begin{array}{ccc}
\bigoplus_{i\in\mathbb{Z}} C_{n-i} \otimes_R A_i & \xrightarrow{\partial_n^{C\otimes A}} & \bigoplus_{i\in\mathbb{Z}} C_{n-1-i} \otimes_R A_i \\
\downarrow{\scriptstyle (f\otimes g)_n} & & \downarrow{\scriptstyle (f\otimes g)_{n-1}} \\
\bigoplus_{i\in\mathbb{Z}} D_{n-i} \otimes_R B_i & \xrightarrow{\partial_n^{D\otimes B}} & \bigoplus_{i\in\mathbb{Z}} D_{n-1-i} \otimes_R B_i
\end{array}$$

Let $c \otimes a \in C_{n-i} \otimes_R A_i$ then we will have,

$$\begin{aligned}
\left(\partial_n^{C\otimes B}\right)(f \otimes g)_n(c \otimes a) &= \left(\partial_n^{C\otimes B}\right)(f_{n-i}(c) \otimes g_i(a)) \\
&= \partial_{n-i}^D(f_{n-i}(c)) \otimes g_i(a) + (-1)^{n-i} f_{n-i}(c) \otimes \partial_i^B(g_i(a)).
\end{aligned}$$

And

$$\begin{aligned}(f\otimes g)_{n-1}\big(\partial_n^{C\otimes A}(c\otimes a)\big) &= (f\otimes g)_{n-1}\big(\partial_{n-i}^{C}(c)\otimes a+(-1)^{n-i}c\otimes\partial_i^A(a)\big)\\ &= f_{n-i-1}\big(\partial_{n-i}^{C}(c)\big)\otimes g_i(a)+(-1)^{n-i}f_{n-i}(c)\otimes g_{i-1}\big(\partial_i^A(a)\big).\end{aligned}$$

Since both f and g are morphisms, we have,

$f_{n-i-1}\big(\partial_{n-i}^{C}(c)\big)=\partial_{n-i}^{D}\big(f_{n-i}(c)\big)$, and $\partial_i^B\big(g_i(a)\big)=g_{i-1}\big(\partial_i^A(a)\big)$. Therefore, $\partial_n^{D\otimes B}(f\otimes g)_n = (f\otimes g)_{n-1}\partial_n^{C\otimes A}$.

□

Corollary 2.4.5. *Let $f:C\to D$ be morphism of complexes, then there exist a morphism of complexes $(f\otimes g):(C\otimes A)\to(D\otimes_R A)$ defined by $(f_i\otimes_R \mathrm{Id})(c\otimes a)=f_i(c)\otimes a$.*

Proof: Let $g=\mathrm{Id}_A$ in the previous Proposition 2.4.4.

□

Theorem 2.4.6. *Let C and D be complexes. Then,*

$$C\otimes_R(\bigoplus_{i\in I}D_i)\cong\bigoplus_{i\in I}(C\otimes_R D_i).$$

Proof: The proof is well-known.(Theorem 2.65 in [2] .)

□

Theorem 2.4.7. *Let C and D be complexes. Then,*

$$(\bigoplus_{i\in I}C_i)\otimes_R D\cong\bigoplus_{i\in I}(C_i\otimes_R D).$$

Proof: The proof is well-known.

□

2.5 Ordinary Tensor Product and Shift

Definition 2.5.1. Let C be a complex of R-modules. Then the *Shift of C*, ΣC is complex of R-modules defined by $(\Sigma C)_n=C_{n-1}$, and $\partial_n^{\Sigma C}=-\sigma_{n-2}\partial_{n-1}^{C}\sigma_{n-1}^{-1}$ for all n. Also, *the canonical map* $\sigma:C\to\Sigma C$ is obtained by shifting degrees of elements, specifically, if $c \in C$. Then $|\sigma(c)| = |c| + 1$.

Remark 2.5.2. Let C and D be complexes, then for the ordinary tensor product we have $\Sigma(C \otimes_R D) \cong (\Sigma C) \otimes_R D \cong C \otimes_R (\Sigma D)$. Since we know by definition
$\Big(\Sigma(C \otimes_R D)\Big)_n = (C \otimes_R D)_{n-1} = \bigoplus_{i \in \mathbb{Z}} C_{n-1-i} \otimes_R D_i$,
$\Big((\Sigma C) \otimes_R D\Big)_n = \bigoplus_{i \in \mathbb{Z}} (\Sigma C)_{n-i} \otimes_R D_i = \bigoplus_{i \in \mathbb{Z}} C_{n-1-i} \otimes_R D_i$, and
$\Big(C \otimes_R (\Sigma D)\Big)_n = \bigoplus_{i \in \mathbb{Z}} C_{n-i} \otimes_R (\Sigma D)_i = \bigoplus_{i \in \mathbb{Z}} C_{n-i} \otimes_R D_{i-1}$. Since these are direct sums over all $i \in Z$, we have $\Sigma(C \otimes_R D) \cong (\Sigma C) \otimes_R D \cong C \otimes_R (\Sigma D)$.

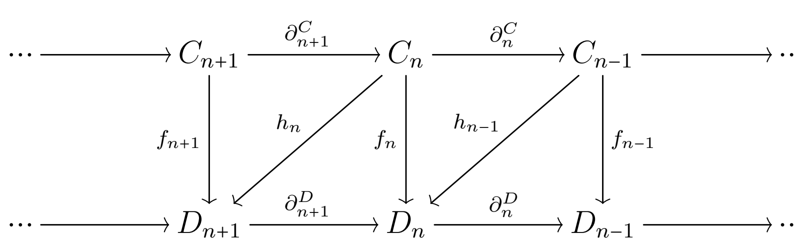
Corollary 2.5.3. *Let $f : C \to \Sigma^{d_1}(D)$ and $g : A \to \Sigma^{d_2}(B)$ be morphisms of complexes. Then there exist a morphism of complexes $(f \otimes g) : (C \otimes_R A) \to \Sigma^{d_1+d_2}(D \otimes_R B)$.*

Proof: By Proposition 2.4.4. We have a morphism $(f \otimes g) : (C \otimes_R A) \to \Sigma^{d_1}(D) \otimes_R \Sigma^{d_2}(B)$. Also, by Remark 2.5.2 there exists an isomorphism $\varphi : \Sigma^{d_1}(D) \otimes_R \Sigma^{d_2}(B) \to \Sigma^{d_1+d_2}(D \otimes_R B)$. Therefore $(f \otimes g) : (C \otimes_R A) \to \Sigma^{d_1+d_2}(D \otimes_R B)$.

□

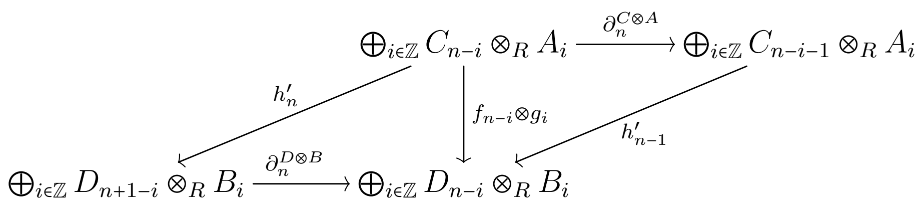
2.6 Ordinary Tensor Product and Homotopy.

Definition 2.6.1. We say that a chain map $f : C \to D$ is *null homotopic* if there are maps $h_n : C_n \to D_{n+1}$ such that $f_n = h_{n-1}\partial_n^C + \partial_{n+1}^D h_n$ for all n. In this case we write $f \sim 0$



Theorem 2.6.2. *Let $f : C \to D$ and $g : A \to B$ be morphisms of complexes of R-modules with $f \sim 0$. Then $f \otimes g \sim 0$.*

Proof: Consider the diagram



Define $h'_n = (h_{n-i} \otimes g_i)_{i\in Z}$. Choose an element $c \otimes a \in C_{n-i} \otimes_R A_i$. Then,

$$\begin{aligned}
&\partial_{n+1}^{D\otimes_R B}(h'_n)(c\otimes a) + (h'_{n-1})\partial_n^{C\otimes_R A}(c\otimes a)\\
&\quad = \partial_{n+1}^{D\otimes_R B}(h'_n)_{i\in\mathbb{Z}}(c\otimes a) + (h'_{n-1})_{i\in\mathbb{Z}}\Big(\partial_{n-i}^C(c)\otimes a + (-1)^{n-i} c\otimes \partial_i^A(a)\Big)\\
&\quad = \partial_{n+1}^{D\otimes_R B}\Big(h_n(c)\otimes g_i(a)\Big) + h_{n-1}(\partial_{n-i}^C(c))\otimes g_i(a) + (-1)^{n-i} h_n(c)\otimes g_{i-1}(\partial_i^A(a))\\
&\quad = \Big(\partial_{n-i+1}^D\Big(h_n(c)\Big)\otimes g_i(a) + (-1)^{n-i+1} h_n(c)\otimes \partial_i^B(g_i(a))\Big)\\
&\qquad\qquad + h_{n-1}(\partial_{n-i}^C(c))\otimes g_i(a) + (-1)^{n-i} h_n(c)\otimes g_{i-1}\Big(\partial_i^A(a)\Big) \qquad (1)\\
&\quad = \partial_{n-i+1}^D\Big(h_n(c)\Big)\otimes g_i(a) + h_{n-1}\Big(\partial_{n-i}^C(c)\Big)\otimes g_i(a)\\
&\quad = \Big(\partial_{n-i+1}^D\big(h_n(c)\big) + h_{n-1}\big(\partial_{n-i}^C(c)\big)\Big)\otimes g_i(a)\\
&\quad = f_{n-i}(c)\otimes g_i(a)\\
&\quad = (f\otimes g)_n(c\otimes a).
\end{aligned}$$

Where in (1) we use the fact that $\partial_i^B g_i = g_{i-1}\partial_i^A$. Therefore $f \otimes g \sim 0$.

□

Corollary 2.6.3. *Let $f : C \to D$ and $\mathrm{Id}_A : A \to A$ with $f \sim 0$, then $f \otimes \mathrm{Id}_A \sim 0$.*

Proof: Repeat the same work as in Theorem 2.6.2: by replacing g with identity map.

□

2.7 Ordinary Tensor Product and Mapping Cones

Definition 2.7.1. Let A, B, C and D be complexes, and let $f : C \to D$ and $g : A \to B$ be chain maps. Then for $c \otimes a \in C_i \otimes A_{n-i}$ we have $(f \otimes g)(c \otimes a) = (-1)^{|c||g|} f_i(c) \otimes g_{n-i}(a)$.

Definition 2.7.2. If $f : C \to D$ is a chain map, then its *mapping cone,* cone(f), is a complex of R-modules whose term of degree n is $\operatorname{cone}(f)_n = (\Sigma C)_n \oplus D_n$ and whose differentials $\partial_n : \operatorname{cone}(f)_n \to \operatorname{cone}(f)_{n-1}$ is given by

$$\partial_n^{\operatorname{cone}(f)} = \begin{bmatrix} \partial_n^{\Sigma C} & 0 \\ f_{n-1}\sigma_{n-1}^{-1} & \partial_n^D \end{bmatrix}.$$

A straightforward computation shows that $\partial_{n-1}^{\operatorname{cone}(f)} \partial_n^{\operatorname{cone}(f)} = 0$:

$$\begin{aligned} \partial_{n-1}^{\operatorname{cone}(f)} \partial_n^{\operatorname{cone}(f)} &= \begin{bmatrix} \partial_{n-1}^{\Sigma C} & 0 \\ f_{n-2}\sigma_{n-2}^{-1} & \partial_{n-1}^D \end{bmatrix} \begin{bmatrix} \partial_n^{\Sigma C} & 0 \\ f_{n-1}\sigma_{n-1}^{-1} & \partial_n^D \end{bmatrix} \\ &= \begin{bmatrix} \partial_{n-1}^{\Sigma C}\partial_n^{\Sigma C} & 0 \\ f_{n-2}\sigma_{n-2}^{-1}\partial_n^{\Sigma C} + \partial_{n-1}^D f_{n-1}\sigma_{n-1}^{-1} & \partial_{n-1}^D \partial_n^D \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

since we know that f is a morphism and $\partial_n^{\Sigma C} = -\sigma_{n-2}\partial_{n-1}^C \sigma_{n-1}^{-1}$.

Theorem 2.7.3. *Let A be an R-complex and $f : C \to D$ be a morphism of complexes of R-modules. Then*

$$A \otimes_R \operatorname{cone}(f) \cong \operatorname{cone}(A \otimes_R f).$$

Proof: The proof is well-known.(Proposition 4.1.12 in [4].)

□

Chapter 3

The Pinched Tensor Product and Morphisms

In this chapter we give the definition of the pinched tensor product and a precise description of its basic properties, like commutativity and associativity. Also, we compare there basic properties of the pinched tensor product with those of the ordinary tensor product, and conclude which of these properties hold for the pinched tensor product and which do not.

3.1 Assumptions and Notation

Definition 3.1.1. (Refrence [1]) Let C and D be complexes. Consider the R-complex $C \otimes_R^{\bowtie} D$ defined by:

$$(C \otimes_R^{\bowtie} D)_n = \begin{cases} (C_{\geqslant 0} \otimes_R D_{\geqslant 0})_n & \text{for } n \geqslant 0 \\ (C_{\leqslant -1} \otimes_R (\Sigma D)_{\leqslant 0})_n & \text{for } n \leqslant -1. \end{cases}$$

and $\partial^{C \otimes_R^{\bowtie} D}$ defined by

$$\partial_n^{C \otimes_R^{\bowtie} D} = \begin{cases} \partial_n^{C_{\geqslant 0} \otimes_R D_{\geqslant 0}} & \text{for } n \geqslant 1 \\ \partial_0^C \otimes_R (\sigma \partial_0^D) & \text{for } n = 0 \\ \partial_n^{C_{\leqslant -1} \otimes_R (\Sigma D)_{\leqslant 0}} & \text{for } n \leqslant -1 \end{cases}$$

where σ denotes the *canonical map* $D \to \Sigma D$. This is a differential on $C \otimes_R^{\bowtie} D$ since

$$(\partial_0^C \otimes_R (\sigma \partial_0^D))\partial_1^{C_{\geqslant 0} \otimes_R D_{\geqslant 0}} = 0 = \partial_{-1}^{C_{\leqslant -1} \otimes_R (\Sigma D)_{\leqslant 0}}(\partial_0^C \otimes_R (\sigma \partial_0^D)).$$

Remark 3.1.2. From Definition 3.1.1, $\partial_n^{C\otimes_R^{\bowtie}D}$ is the differential on $C\otimes_R^{\bowtie} D$. In particular, we show the previous equations. Let $c\otimes d\in C_1\otimes_R D_0$. Then

$$\begin{aligned}
(\partial_0^C\otimes_R(\sigma\partial_0^D))\partial_1^{C_{\geqslant 0}\otimes_R D_{\geqslant 0}}(c\otimes d) &= \Big(\partial_0^C\otimes_R(\sigma\partial_0^D)\Big)\Big(\partial_1^{C_{\geqslant 0}\otimes_R D_{\geqslant 0}}(c\otimes d)\Big)\\
&= \Big(\partial_0^C\otimes_R(\sigma\partial_0^D)\Big)\Big(\partial_1^{C_{\geqslant 0}}(c)\otimes(d)+(-1)^1(c)\otimes\partial_0^{D_{\geqslant 0}}(d)\Big)\\
&= \Big(\partial_0^C\otimes_R(\sigma\partial_0^D)\Big)\Big(\partial_1^{C_{\geqslant 0}}(c)\otimes(d)+0\Big)\\
&= \partial_0^C\partial_1^{C_{\geqslant 0}}(c)\otimes\sigma\partial_0^D(a)\\
&= 0\otimes\sigma\partial_0^D(d)\\
&= 0.
\end{aligned}$$

Now, if $c\otimes d\in C_0\otimes_R D_1$. Then

$$\begin{aligned}
(\partial_0^C\otimes_R(\sigma\partial_0^D))\partial_1^{C_{\geqslant 0}\otimes_R D_{\geqslant 0}}(c\otimes d) &= \Big(\partial_0^C\otimes_R(\sigma\partial_0^D)\Big)\Big(\partial_1^{C_{\geqslant 0}\otimes_R D_{\geqslant 0}}(c\otimes d)\Big)\\
&= \Big(\partial_0^C\otimes_R(\sigma\partial_0^D)\Big)\Big(\partial_0^{C_{\geqslant 0}}(c)\otimes(d)+(-1)^0(c)\otimes\partial_1^{D_{\geqslant 0}}(d)\Big)\\
&= \Big(\partial_0^C\otimes_R(\sigma\partial_0^D)\Big)\Big(0\otimes(d)+(-1)^0(c)\otimes\partial_1^{D_{\geqslant 0}}(d)\Big)\\
&= \Big(\partial_0^C\otimes_R(\sigma\partial_0^D)\Big)\Big((-1)^0(c)\otimes\partial_1^{D_{\geqslant 0}}(d)\Big)\\
&= \Big(\partial_0^C(c)\otimes(\sigma\partial_0^D)\partial_1^{D_{\geqslant 0}}(d)\Big)\\
&= \Big(\partial_0^C(c)\otimes 0\Big)\\
&= 0.
\end{aligned}$$

Similarly, if $c\otimes d\in C_0\otimes_R D_0$. Then

$$\begin{aligned}
\partial_{-1}^{C_{\leqslant -1}\otimes_R(\Sigma D)_{\leqslant 0}}\Big(\partial_0^C\otimes_R(\sigma\partial_0^D)\Big)(c\otimes d) &= \partial_{-1}^{C_{\leqslant -1}\otimes_R(\Sigma D)_{\leqslant 0}}\Big((\partial_0^C\otimes_R(\sigma\partial_0^D))(c\otimes d)\Big)\\
&= \partial_{-1}^{C_{\leqslant -1}\otimes_R(\Sigma D)_{\leqslant 0}}\Big(\partial_0^C(c)\otimes\sigma\partial_0^D(d)\Big)\\
&= \partial_{-1}^{C_{\leqslant -1}}\partial_0^C(c)\otimes\sigma\partial_0^D(d)+(-1)^1\partial_0^C(c)\otimes\partial_0^{(\Sigma D)_{\leqslant -1}}\sigma\partial_0^D(d)\\
&= 0\otimes\sigma\partial_0^D(d)+(-1)^1\partial_0^C(c)\otimes 0\\
&= 0.
\end{aligned}$$

Remark 3.1.3. From Definition 3.1.1 we can see that

$$(C \otimes_R^{\bowtie} D)_n = \begin{cases} \bigoplus_{i=0}^{n} C_i \otimes_R D_{n-i} & \text{for } n \geqslant 0 \\ \\ \bigoplus_{i=-1}^{n} C_i \otimes_R (\Sigma D)_{n-i} & \text{for } n \leqslant -1 \end{cases}$$

is finitely generated for all n (Provided C_i and D_j are finilety generated for all i, j). Here is a picture illustrating the pinched tensor product, where each point $(i, n-i)$ in the plane represents the tensor product $C_i \otimes_R D_{n-i}$ of modules and $C_i \otimes_R (\Sigma D)_{n-i}$ and each line segment $Y = -X + n$ represents $(C \otimes_R^{\bowtie} D)_n$.

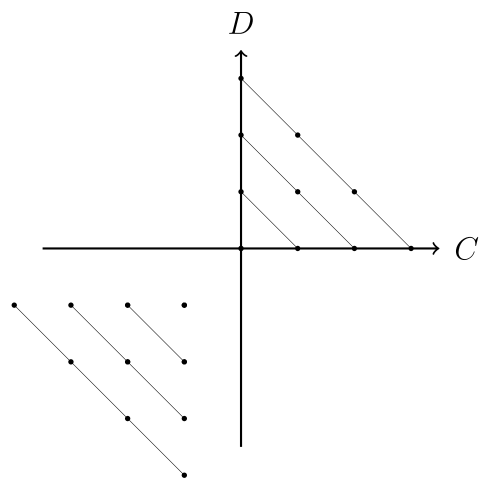


Figure 3.1: Pinched Tensor Product

And the differentials can be represented by;

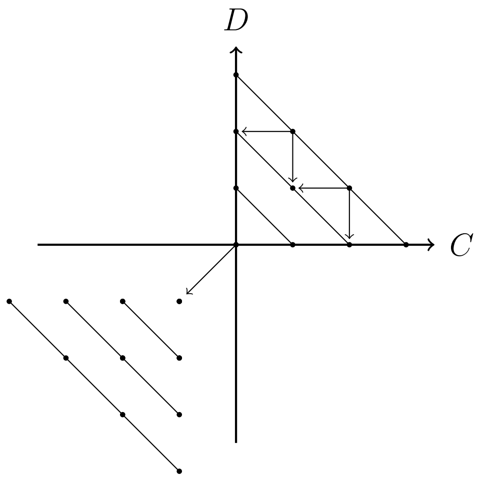


Figure 3.2: Differential on the Pinched Tensor Product

3.2 Properties

Definition 3.2.1. Let $g : A \to B$ be a morphism of complexes between A and B. Then $(\Sigma g)_n = \sigma^B_{n-1} g_{n-1} (\sigma^A{}_{n-1})^{-1}$.

$$\begin{array}{ccc} A_{n-1} & \xrightarrow{\sigma^A_{n-1}} & (\Sigma A)_n \\ \Big\downarrow{\scriptstyle g_{n-1}} & & \Big\downarrow{\scriptstyle (\Sigma g)_n} \\ B_{n-1} & \xrightarrow{\sigma^B_{n-1}} & (\Sigma B)_n \end{array}$$

Theorem 3.2.2. *Let R be a commutative ring and C and D be complexes. Then*

$$C \otimes_R^{\bowtie} D \cong D \otimes_R^{\bowtie} C.$$

Proof: By the definition of the pinched tensor product, Remark 2.3.6 and Remark 2.5.2 we have,

$$\begin{aligned}(C \otimes_R^{\bowtie} D)_{\geq 0} &= C_{\geq 0} \otimes_R D_{\geq 0}\\ &\cong D_{\geq 0} \otimes_R C_{\geq 0}\\ &= (D \otimes_R^{\bowtie} C)_{\geq 0}.\end{aligned}$$

And

$$\begin{aligned}(C \otimes_R^{\bowtie} D)_{\leq -1} &= C_{\leq -1} \otimes_R (\Sigma D)_{\leq 0}\\ &\cong (\Sigma D)_{\leq 0} \otimes_R C_{\leq -1}\\ &= \Sigma(D_{\leq -1}) \otimes_R C_{\leq -1}\\ &\cong D_{\leq -1} \otimes_R \Sigma(C_{\leq -1})\\ &\cong D_{\leq -1} \otimes_R (\Sigma C)_{\leq 0}\\ &\cong (D \otimes_R^{\bowtie} C)_{\leq -1}.\end{aligned}$$

It remains to show in the case when $n = 0$ the diagram

$$\begin{array}{ccc}(C \otimes_R^{\bowtie} D)_0 & \xrightarrow{\partial_0^{C \otimes_R^{\bowtie} D}} & (C \otimes_R^{\bowtie} D)_{-1}\\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_{-1}}\\ (D \otimes_R^{\bowtie} C)_0 & \xrightarrow{\partial_0^{D \otimes_R^{\bowtie} C}} & (D \otimes_R^{\bowtie} C)_{-1}\end{array}$$

commutes. This diagram is equal to:

$$\begin{array}{ccc}C_0 \otimes_R D_0 & \xrightarrow{\partial_0^{C \otimes_R^{\bowtie} D}} & C_{-1} \otimes_R (\Sigma D)_0\\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_{-1}}\\ D_0 \otimes_R C_0 & \xrightarrow{\partial_0^{D \otimes_R^{\bowtie} C}} & D_{-1} \otimes_R (\Sigma C)_0\end{array}$$

Let $c \otimes d \in C_0 \otimes_R D_0$, $\tau_{-1}(c \otimes d) = (\sigma_{-1}^D)^{-1}(d) \otimes \sigma_{-1}^C(c)$ and note that $\partial_0^{C \otimes_R^{\bowtie} D} = \partial_0^C \otimes \sigma_{-1}^D \partial_0^D$.

Then we have,

$$
\begin{aligned}
\tau_{-1}\big(\partial_0^{C\otimes_R^{\bowtie}D}(c\otimes d)\big) &= \tau_{-1}\big(\partial_0^C(c)\otimes\sigma_{-1}^D\partial_0^D(a)\big)\\
&= ((\sigma_{-1}^D)^{-1}\otimes\sigma_{-1}^C)\big(\partial_0^C(c)\otimes\sigma_{-1}^D\partial_0^D(d)\big)\\
&= (\sigma_{-1}^D)^{-1}\sigma_{-1}^D\partial_0^D(d)\otimes\sigma_{-1}^C\partial_0^C(c).\\
&= \partial_0^D(d)\otimes\sigma_{-1}^C\partial_0^C(c).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\big(\partial_0^{D\otimes_R^{\bowtie}C}\big)\big(\tau_0(c\otimes d)\big) &= \big(\partial_0^{D\otimes_R^{\bowtie}C}\big)(d\otimes c)\\
&= \partial_0^D(d)\otimes\sigma_{-1}^C\partial_0^C(c).
\end{aligned}
$$

Therefore, $(C\otimes_R^{\bowtie}D)\cong(D\otimes_R^{\bowtie}C)$.

□

Theorem 3.2.3. *Let R and S be rings, and A, B and C be complexes and A a right R-module, B a left R-right S-bimodule and C a left S-module. Then*

$$(A\otimes_R^{\bowtie}B)\otimes_S^{\bowtie}C\cong A\otimes_R^{\bowtie}(B\otimes_S^{\bowtie}C).$$

Proof: By the definition of the pinched tensor product and Remark 2.5.2 we have,

$$
\begin{aligned}
\Big((A\otimes_R^{\bowtie}B)\otimes_S^{\bowtie}C\Big)_{\geq 0} &= (A\otimes_R^{\bowtie}B)_{\geq 0}\otimes_S C_{\geq 0}\\
&= (A_{\geq 0}\otimes_R B_{\geq 0})\otimes_S C_{\geq 0}\\
&\cong A_{\geq 0}\otimes_R(B_{\geq 0}\otimes_S C_{\geq 0})\\
&= A_{\geq 0}\otimes_R(B\otimes_S^{\bowtie}C)_{\geq 0}\\
&= \Big(A\otimes_R^{\bowtie}(B\otimes_S^{\bowtie}C)\Big)_{\geq 0}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\Big((A\otimes^{\bowtie}_R B)\otimes^{\bowtie}_S C\Big)_{\le -1} &= (A\otimes^{\bowtie}_R B)_{\le -1}\otimes_S(\Sigma C)_{\le 0}\\
&= \Big(A_{\le -1}\otimes_R(\Sigma B)_{\le 0}\Big)\otimes_S(\Sigma C)_{\le 0}\\
&\cong A_{\le -1}\otimes_R\Big((\Sigma B)_{\le 0}\otimes_S(\Sigma C)_{\le 0}\Big)\\
&= A_{\le -1}\otimes_R\Big(\Sigma(B_{\le -1})\otimes_S(\Sigma C)_{\le 0}\Big)\\
&\cong A_{\le -1}\otimes_R\Sigma\Big(B_{\le -1}\otimes_S(\Sigma C)_{\le 0}\Big)\\
&= A_{\le -1}\otimes_R\Sigma\Big((B\otimes^{\bowtie}_S C)_{\le -1}\Big)\\
&= A_{\le -1}\otimes_R\Big(\Sigma(B\otimes^{\bowtie}_S C)\Big)_{\le 0}\\
&\cong \Big(A\otimes^{\bowtie}_R(B\otimes^{\bowtie}_S C)\Big)_{\le -1}.
\end{aligned}
$$

Now we need to show in the case when $n = 0$ the diagram commutes,

$$
\begin{array}{ccc}
\Big((A\otimes^{\bowtie}_R B)\otimes^{\bowtie}_S C\Big)_0 & \xrightarrow{\partial_0^{(A\otimes^{\bowtie}B)\otimes^{\bowtie}C}} & \Big((A\otimes^{\bowtie}_R B)\otimes^{\bowtie}_S C\Big)_{-1}\\
\downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_{-1}}\\
\Big(A\otimes^{\bowtie}_R(B\otimes^{\bowtie}_S C)\Big)_0 & \xrightarrow{\partial_0^{A\otimes^{\bowtie}(B\otimes^{\bowtie}C)}} & \Big(A\otimes^{\bowtie}_R(B\otimes^{\bowtie}_S C)\Big)_{-1}
\end{array}
$$

which is equivalent to

$$
\begin{array}{ccc}
(A\otimes^{\bowtie}_R B)_0\otimes_S C_0 & \xrightarrow{\partial_0^{(A\otimes^{\bowtie}B)\otimes^{\bowtie}C}} & (A\otimes^{\bowtie}_R B)_{-1}\otimes_S(\Sigma C)_0\\
\downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_{-1}}\\
A_0\otimes_R(B\otimes^{\bowtie}_S C)_0 & \xrightarrow{\partial_0^{A\otimes^{\bowtie}(B\otimes^{\bowtie}C)}} & A_{-1}\otimes_R\Big(\Sigma(B\otimes^{\bowtie}_S C)\Big)_0
\end{array}
$$

which is also equivalent to

$$
\begin{array}{ccc}
(A_0\otimes_R B_0)\otimes_S C_0 & \xrightarrow{\partial_0^{(A\otimes^{\bowtie}B)\otimes^{\bowtie}C}} & \Big((A_{-1}\otimes_R(\Sigma B)_0\Big)\otimes_S(\Sigma C)_0\\
\downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_{-1}}\\
A_0\otimes_R(B_0\otimes_S C_0) & \xrightarrow{\partial_0^{A\otimes^{\bowtie}(B\otimes^{\bowtie}C)}} & A_{-1}\otimes_R\Big(B_{-1}\otimes_S(\Sigma C)_0\Big)
\end{array}
$$

Let $(a \otimes b) \otimes c \in (A_0 \otimes_R B_0) \otimes_S C_0$ and $\tau_{-1}\Big((a \otimes b) \otimes c\Big) = a \otimes \Big((\sigma_{-1}^{B})^{-1}(b) \otimes (\Sigma C)_0(c)\Big)$. Then we have,

$$\begin{aligned}
\tau_{-1}\Big(\partial_0^{(A \otimes^{\bowtie} B) \otimes^{\bowtie} C}((a \otimes b) \otimes c)\Big) &= \tau_{-1}\Big(\Big(\partial_0^{A}(a) \otimes (\sigma_{-1}^{B}\partial_0^{B})(b)\Big) \otimes (\sigma_{-1}^{C}\partial_0^{C})(c)\Big) \\
&= (A_{-1} \otimes (\sigma_{-1}^{B})^{-1} \otimes (\Sigma C)_0)\Big(\Big(\partial_0^{A}(a) \otimes (\sigma_{-1}^{B}\partial_0^{B})(b)\Big) \otimes (\sigma_{-1}^{C}\partial_0^{C})(c)\Big) \\
&= \partial_0^{A}(a) \otimes \Big(((\sigma_{-1}^{B})^{-1}\sigma_{-1}^{B}\partial_0^{B})(b) \otimes (\sigma_{-1}^{C}\partial_0^{C})(c)\Big) \\
&= \partial_0^{A}(a) \otimes \Big((\partial_0^{B})(b) \otimes (\sigma_{-1}^{C}\partial_0^{C})(c)\Big).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_0^{A \otimes^{\bowtie} (B \otimes^{\bowtie} C)}\Big(\tau_{-1}((a \otimes b) \otimes c)\Big) &= \partial_0^{A \otimes^{\bowtie} (B \otimes^{\bowtie} C)}\Big(a \otimes (b \otimes c)\Big) \\
&= \Big(\partial_0^{A} \otimes (\partial_0^{B} \otimes (\sigma_{-1}^{C}\partial_0^{C})\Big)\Big(a \otimes (b \otimes c)\Big) \\
&= \partial_0^{A}(a) \otimes \Big((\partial_0^{B})(b) \otimes (\sigma_{-1}^{C}\partial_0^{C})(c)\Big).
\end{aligned}$$

Therefore, $(A \otimes_R^{\bowtie} B) \otimes_S^{\bowtie} C \cong A \otimes_R^{\bowtie} (B \otimes_S^{\bowtie} C)$.

□

The following Theorem is the pinched analog of Proposition 2.4.4.

Theorem 3.2.4. *Let $f : C \to D$ and $g : A \to B$ be morphisms of complexes. Then there exists a morphism of complexes $f \otimes_R^{\bowtie} g : C \otimes_R^{\bowtie} A \to D \otimes_R^{\bowtie} B$ defined by $f_{\geq 0} \otimes g_{\geq 0} : (C \otimes_R^{\bowtie} A)_{\geq 0} \to (D \otimes_R^{\bowtie} B)_{\geq 0}$ and $f_{\leq -1} \otimes (\Sigma g)_{\leq 0} : (C \otimes_R^{\bowtie} A)_{\leq -1} \to (D \otimes_R^{\bowtie} B)_{\leq -1}$.*

Proof: We only need to show in the case when $n = 0$ that the following diagram commutes.

$$\begin{array}{ccc}
(C \otimes_R^{\bowtie} A)_0 & \xrightarrow{\partial_0^{C \otimes_R^{\bowtie} A}} & (C \otimes_R^{\bowtie} A)_{-1} \\
\Big\downarrow {\scriptstyle f_0 \otimes g_0} & & \Big\downarrow {\scriptstyle f_{-1} \otimes (\Sigma g)_0} \\
(D \otimes_R^{\bowtie} B)_0 & \xrightarrow{\partial_0^{D \otimes_R^{\bowtie} B}} & (D \otimes_R^{\bowtie} B)_{-1}
\end{array}$$

which is equal to,

$$\begin{array}{ccc} C_0 \otimes_R A_0 & \xrightarrow{\partial_0^{C\otimes_R^{\bowtie} A}} & C_{-1} \otimes_R (\Sigma A)_0 \\ \Big\downarrow {\scriptstyle f_0\otimes g_0} & & \Big\downarrow {\scriptstyle f_{-1}\otimes(\Sigma g)_0} \\ D_0 \otimes_R B_0 & \xrightarrow{\partial_0^{D\otimes_R^{\bowtie} B}} & D_{-1} \otimes_R (\Sigma B)_0 \end{array}$$

Let $c \otimes a \in C_0 \otimes_R A_0$ then we will have,

$$\begin{aligned} (f_{-1} \otimes (\Sigma g)_0)\big(\partial_0^{C\otimes_R^{\bowtie} A}(c \otimes a)\big) &= (f_{-1} \otimes (\Sigma g)_0)\big(\partial_0^C(c) \otimes \sigma_{-1}^A \partial_0^A(a)\big) \\ &= \big(f_{-1}(\partial_0^C(c)\big) \otimes (\Sigma g)_0\Big(\sigma_{-1}^A \partial_0^A(a)\Big). \end{aligned}$$

Moreover,,

$$\begin{aligned} \big(\partial_0^{D\otimes_R^{\bowtie} B}\big)(f_0 \otimes g_0)(c \otimes a) &= \big(\partial_0^{D\otimes_R^{\bowtie} B}\big)(f_0(c) \otimes g_0(a)) \\ &= \partial_0^D(f_0(c)) \otimes \sigma_{-1}^B \partial_0^B\Big(g_0(a)\Big). \end{aligned}$$

We note that $\big(f_{-1}(\partial_0^C(c)\big) = \partial_0^D(f_0(c))$ since f is a chain map, and $(\Sigma g)_0\Big(\sigma_{-1}^A \partial_0^A(a)\Big) = \sigma_{-1}^B \partial_0^B\Big(g_0(a)\Big)$ because the following diagram commutes.

$$\begin{array}{ccccc} A_0 & \xrightarrow{\partial_0^A} & A_{-1} & \xrightarrow{\sigma_{-1}^A} & (\Sigma A)_0 \\ \Big\downarrow {\scriptstyle g_0} & & \Big\downarrow {\scriptstyle g_{-1}} & & \Big\downarrow {\scriptstyle (\Sigma g)_0} \\ B_0 & \xrightarrow{\partial_0^B} & B_{-1} & \xrightarrow{\sigma_{-1}^B} & (\Sigma B)_0 \end{array}$$

Therefore, $\big(\partial_0^{D\otimes_R^{\bowtie} B}\big)(f_0 \otimes g_0) = \Big(f_{-1} \otimes (\Sigma g)_0\Big)\big(\partial_0^{C\otimes_R^{\bowtie} A}\big)$.

□

Corollary 3.2.5. *Let $f : C \to D$ be a morphism of complexes. Then there exist a morphism of complexes $(f \otimes_R^{\bowtie} \mathrm{Id}_A) : (C \otimes_R^{\bowtie} A) \to (D \otimes_R^{\bowtie} A)$.*

Proof: Use Proposition 3.2.4 by replacing g with the identity map.

□

Definition 3.2.6. Recall that *a stalk complex* C is one where $C_j = 0$ for $i \neq j$, for some $i \in Z$, and $C_i \neq 0$.

Proposition 3.2.7. *Let C be a complex and M an R-module, considered as a stalk complex concentrated in degree i. Then*

$$C \otimes_R^{\bowtie} M = \begin{cases} C_{\geqslant 0} \otimes_R M & \textit{for } i \geqslant 0 \\ \\ C_{\leqslant -1} \otimes_R (\Sigma M) & \textit{for } i \leqslant -1. \end{cases}$$

Proof: Regarded as a stalk complex, M has the form $\cdots \to 0 \to 0 \to M \to 0 \to \cdots$. Consider cases $i \geqslant 0$ and $i \leqslant -1$.

$i \geqslant 0$:

In this case we have.

$(C \otimes_R^{\bowtie} M)_{\geqslant 0} = C_{\geqslant 0} \otimes_R M_{\geqslant 0} = C_{\geqslant 0} \otimes_R M.$

$i \leqslant -1$:

In this case we have.

$(C \otimes_R^{\bowtie} M)_{\leqslant -1} = C_{\leqslant -1} \otimes_R (\Sigma M)_{\leqslant 0} = C_{\leqslant -1} \otimes_R (\Sigma M).$

Therefore,

$$C \otimes_R^{\bowtie} M = \begin{cases} C_{\geqslant 0} \otimes_R M & \text{for } i \geqslant 0 \\ \\ C_{\leqslant -1} \otimes_R (\Sigma M) & \text{for } i \leqslant -1. \end{cases}$$

□

Corollary 3.2.8. *Let C be a complex and M an R-module, considered as a stalk complex concentrated in degree zero. Then*

$$C \otimes_R^{\bowtie} M = C_{\geqslant 0} \otimes_R M.$$

Proof: The proof follows easily from Proposition 3.2.7.

□

Theorem 3.2.9. *Let C be a complex and $A = 0 \to R \to R \to 0$ where R sits in degrees 0 and -1 . Then*

$$C \otimes_R^{\bowtie} A \cong C.$$

Proof: We want to show $(C \otimes_R^{\bowtie} A)_n \cong C$. Consider three cases: $n \geqslant 0$, $n \leqslant -1$ and $n = 0$.

$n \geqslant 0:$

$(C \otimes_R^{\bowtie} A)_{\geqslant 0} = C_{\geqslant 0} \otimes_R A_{\geqslant 0} = C_{\geqslant 0} \otimes_R R \cong C$, by Proposition 2.3.13.

$n \leqslant -1:$

$(C \otimes_R^{\bowtie} A)_{\leqslant -1} = C_{\leqslant -1} \otimes_R (\Sigma A)_{\leqslant 0} = C_{\leqslant -1} \otimes_R A_{\leqslant -1} = C_{\leqslant -1} \otimes_R R \cong C$, by Proposition 2.3.13.

$n = 0:$

In this case the diagram is

$$\begin{array}{ccc} C_0 \otimes_R A_0 & \longrightarrow & C_{-1} \otimes_R (\Sigma A)_0 \\ \downarrow \cong & & \downarrow \cong \\ C_0 & \longrightarrow & C_{-1} \end{array}$$

Which is equal to

$$
\begin{array}{ccc}
C_0 \otimes_R R & \longrightarrow & C_{-1} \otimes_R R \\
\downarrow \cong & & \downarrow \cong \\
C_0 & \longrightarrow & C_{-1}
\end{array}
$$

Therefore, $C \otimes_R^{\bowtie} A \cong C$.

$\square$

We also have an analogous statement to Theorem 2.4.6.

Theorem 3.2.10. *Let C and D_i, $i \in I$ be complexes. Then,*

$$C \otimes_R^{\bowtie} (\bigoplus_{i \in I} D_i) \cong \bigoplus_{i \in I} (C \otimes_R^{\bowtie} D_i).$$

Proof: We only need to show the case when $n = 0$ the following diagram commutes.

$$
\begin{array}{ccc}
C_0 \otimes_R (\bigoplus_{i \in I} D_i)_0 & \xrightarrow{\partial_0^{C \otimes_R^{\bowtie} \bigoplus_{i \in I} D_i}} & C_{-1} \otimes_R \Sigma (\bigoplus_{i \in I} D_i)_0 \\
\downarrow \tau_0 & & \downarrow \tau_{-1} \\
\bigoplus_{i \in I} (C_0 \otimes_R (D_i)_0) & \xrightarrow{\partial_0^{\bigoplus_{i \in I} (C \otimes_R^{\bowtie} D_i)}} & \bigoplus_{i \in I} (C_{-1} \otimes_R (\Sigma D_i)_0)
\end{array}
$$

Let $c \otimes (d_i) \in C_0 \otimes_R (\bigoplus_{i \in I} D)_0$ with $\tau : c \otimes (d_i) \to (c \otimes d_i)$ and $\partial^{\bigoplus_{i \in I} D_i} = \Big(\partial^{D_i}(d_i) \Big)$.

Then we will have,

$$
\begin{aligned}
\tau_{-1}\Big(\partial_0^{C \otimes_R^{\bowtie} \bigoplus_{i \in I} D_i} (c \otimes (d_i)) \Big) &= \tau_{-1}\Big(\partial_0^C (c) \otimes \sigma_{-1}^{\bigoplus D_i} \partial_0^{\bigoplus D_i} (d_i) \Big) \\
&= \tau_{-1}\Big(\partial_0^C (c) \otimes \big(\sigma_{-1}^{\bigoplus D_i} \partial_0^{\bigoplus D_i} (d_i) \big) \Big) \\
&= \Big(\partial_0^C (c) \otimes \sigma_{-1}^{D_i} \partial_0^{D_i} (d_i) \Big).
\end{aligned}
$$

Moreover,,

$$
\begin{aligned}
\Big(\partial_0^{C \otimes_R^{\bowtie} D} \Big) \tau_0 \Big(c \otimes (d_i) \Big) &= \Big(\partial_0^{\bigoplus (C \otimes_R^{\bowtie} D_i)} \Big) (c \otimes d_i) \\
&= \Big(\partial_0^C (c) \otimes \sigma_{-1}^{D_i} \partial_0^{D_i} (d_i) \Big).
\end{aligned}
$$

Therefore, $C \otimes_R^{\bowtie} (\bigoplus_{i\in I} D_i) \cong \bigoplus_{i\in I}(C \otimes_R^{\bowtie} D_i)$.

□

Theorem 3.2.11. *Let C_i, $i \in I$, and D be complexes. Then,*

$$(\bigoplus_{i\in I} C_i) \otimes_R^{\bowtie} D \cong \bigoplus_{i\in I}(C_i \otimes_R^{\bowtie} D).$$

Proof: We do similar work as in the proof of Theorem 3.2.10.

□

The following proposition was originally stated as Proposition 3.2 in [1], without proof we prove it here.

Proposition 3.2.12. *The pinched tensor product defined in Definition 3.1.1 yields a functor*

$$- \otimes_R^{\bowtie} - : \mathsf{C}(R'\text{–}R) \times \mathsf{C}(R\text{–}S) \longrightarrow \mathsf{C}(R'\text{–}S).$$

Where $\mathsf{C}(R'\text{–}R)$ denotes the category of complex of $R'R$-bimodules , $\mathsf{C}(R\text{–}S)$ denotes the category of complex of RS-bimodules and $\mathsf{C}(R'\text{–}S)$ denotes the category of complex of $R'S$-bimodules.

Proof: Let $C \in \mathsf{C}(R'\text{–}R)$ and $D \in \mathsf{C}(R\text{–}S)$ be two complexes, and define $F(C, D) = C \otimes_R^{\bowtie} D$. We want to show F is a functor.

1. It is clear that $F(C) = C \otimes_R^{\bowtie} D \in \mathsf{C}(R'\text{–}S)$.
2. Let $f : C \to C'$ and $g : D \to D'$ be morphisms. Then $F(f, g) = f \otimes_R^{\bowtie} g$ is a morphism in $\mathsf{C}(R'\text{–}S)$.(Theorem 3.2.4).
3. Let $f : C \to C'$, $f' : C' \to C''$, $g : D \to D'$ and $g' : D' \to D''$ be a morphisms. Then,

   $F\Big((f', g')(f, g)\Big) = F(f'f, g'g) = f'f \otimes_R^{\bowtie} g'g = (f' \otimes_R^{\bowtie} g')(f \otimes_R^{\bowtie} g) = F(f', g')F(f, g).$
   Therefore, $F\Big((f', g')(f, g)\Big) = F(f', g')F(f, g)$.

4. $F(1_C, 1_D) = (1_C \otimes_R^{\bowtie} 1_D) = 1_{F(C,D)}$.

$\square$

Chapter 4

Pinched Tensor Products and Shift

We focus on the pinched tensor product and shift. We show that the isomorphisms $\Sigma(C \otimes_R A) \cong (\Sigma C) \otimes_R A \cong C \otimes_R (\Sigma A)$ that hold for the ordinary tensor product no longer hold for the pinched tensor product. Although if we change isomorphism to *quasi-isomorphism*, the statement for the pinched tensor product holds. In addition, we give some counterexamples for the isomorphisms that no longer hold for the pinched tensor product.

4.1 Shift and Morphisms

Remark 4.1.1. For the pinched tensor product we have in general

$$(\Sigma C \otimes_R^{\bowtie} D) \ncong \Sigma(C \otimes_R^{\bowtie} D) \ncong (C \otimes_R^{\bowtie} \Sigma D),$$

because when $n = 0$
$(\Sigma C \otimes_R^{\bowtie} D)_0 = (\Sigma C)_0 \otimes_R D_0 = C_{-1} \otimes_R D_0$,
$\Sigma(C \otimes_R^{\bowtie} D)_0 = (C \otimes_R^{\bowtie} D)_{-1} = C_{-1} \otimes_R (\Sigma D)_0 = C_{-1} \otimes_R D_{-1}$ and
$(C \otimes_R^{\bowtie} \Sigma D)_0 = C_0 \otimes_R (\Sigma D)_0 = C_0 \otimes_R D_{-1}$.
Now if C and D are complexes of free modules, write $C_{-1} = R^n$, $C_0 = R^{n'}$, $D_{-1} = R^m$ and $D_0 = R^{m'}$, where $m \neq m' = n \neq n'$. Then we get $C_{-1} \otimes_R D_0 \cong R^n \otimes R^{m'} \cong R^{nm\prime}$, $C_{-1} \otimes_R D_{-1} \cong R^n \otimes R^m \cong R^{nm}$ and $C_0 \otimes_R D_{-1} \cong R^{n'} \otimes R^m \cong R^{n'm}$. Then it is clear that $C_{-1} \otimes_R D_{-1} \ncong C_{-1} \otimes_R D_0 \ncong C_0 \otimes_R D_{-1}$ since we assume $m \neq m' \neq n \neq n'$ since the $\operatorname{rank}(C_{-1} \otimes_R D_0) \neq \operatorname{rank}(C_{-1} \otimes_R D_{-1}) \neq \operatorname{rank}(C_0 \otimes_R D_{-1})$. Therefore, $(\Sigma C \otimes_R^{\bowtie} D) \ncong \Sigma(C \otimes_R^{\bowtie} D) \ncong (C \otimes_R^{\bowtie} \Sigma D)$.

The following lemma is a simple fact, which we will use in the proof of the main theorem of this section.

Lemma 4.1.2. *Let C and D be two complexes. If there exists a degree -1 anti-commutative chain map f from C to D, then there exists a morphism g from C to ΣD.*

Proof: It suffices to show that the composition

$$C \xrightarrow{f} D \xrightarrow{\sigma} \Sigma D$$

where $|f| = -1$ and $|\sigma| = 1$ is a morphism. We have that

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n^C} & C_{n-1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ D_{n-1} & \xrightarrow{\partial_{n-1}^D} & D_{n-2} \end{array}$$

anti-commutes for all n. That means $f_{n-1}\partial_n^C = -\partial_{n-1}^D f_n$ for all n. We also have that

$$\begin{array}{ccc} D_n & \xrightarrow{\partial_n^D} & D_{n-1} \\ \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma_{n-1}} \\ (\Sigma D)_{n+1} & \xrightarrow{\partial_{n+1}^{\Sigma D}} & (\Sigma D)_n \end{array}$$

anti-commutes for all n. That means $\sigma_{n-1}\partial_n^D = -\partial_{n+1}^{\Sigma D}\sigma_n$ for all n. Then define $g_n = \sigma_{n-1} f_n$ for all n

$$
\begin{array}{ccc}
C_n & \longrightarrow & C_{n-1} \\
\Big\downarrow{\scriptstyle f_n} & & \Big\downarrow{\scriptstyle f_{n-1}} \\
D_{n-1} & \longrightarrow & D_{n-2} \\
\Big\downarrow{\scriptstyle \sigma_{n-1}} & & \Big\downarrow{\scriptstyle \sigma_{n-2}} \\
(\Sigma D)_n & \longrightarrow & (\Sigma D)_{n-1}
\end{array}
$$

Then

$$
\begin{aligned}
g_{n-1}\partial_n^C &= (\sigma_{n-2} f_{n-1})\partial_n^C \\
&= \sigma_{n-2}(f_{n-1}\partial_n^C) \\
&= \sigma_{n-2}(-\partial_{n-1}^D f_n) \\
&= (-\sigma_{n-2}\partial_{n-1}^D) f_n \\
&= (\partial_n^{\Sigma D}\sigma_{n-1}) f_n \\
&= \partial_n^{\Sigma D}(\sigma_{n-1} f_n) \\
&= \partial_n^{\Sigma D} g_n.
\end{aligned}
$$

Therefore, $g_{n-1}\partial_n^C = \partial_n^{\Sigma D} g_n$.

□

Theorem 4.1.3. *Let A, B, C and D be complexes, and $f : C \to D$ and $g : A \to B$ morphisms. Then there exists a morphism q from $(\Sigma C \otimes_R^{\bowtie} A)$ to $\Sigma(D \otimes_R^{\bowtie} B)$.*

Proof: By Lemma 4.1.2 it suffices to define a degree -1 anti-commutative chain map τ from $\Sigma C \otimes_R^{\bowtie} A$ to $D \otimes_R^{\bowtie} B$, so that the diagram

$$\begin{array}{ccc}
(\Sigma C \otimes_R^{\bowtie} A)_n & \xrightarrow{\partial_n^{\Sigma C \otimes_R^{\bowtie} A}} & (\Sigma C \otimes_R^{\bowtie} A)_{n-1} \\
\Big\downarrow {\scriptstyle \tau_n} & & \Big\downarrow {\scriptstyle \tau_{n-1}} \\
(D \otimes_R^{\bowtie} B)_{n-1} & \xrightarrow{\partial_{n-1}^{D \otimes_R^{\bowtie} B}} & (D \otimes_R^{\bowtie} B)_{n-2}
\end{array}$$

is anti-commutative for all $n \in \mathbb{Z}$. We consider four cases: $n = 1$, $n \geqslant 2$, $n = 0$ and $n \leqslant -1$.

$n = 1$:

In this case the diagram is

$$\begin{array}{ccc}
(\Sigma C)_1 \otimes_R A_0 \oplus (\Sigma C)_0 \otimes_R A_1 & \xrightarrow{\partial_1^{\Sigma C \otimes_R^{\bowtie} A}} & (\Sigma C)_0 \otimes_R A_0 \\
{\scriptstyle \tau_1} \Big\downarrow & & \Big\downarrow {\scriptstyle \tau_0} \\
D_0 \otimes_R B_0 & \xrightarrow{\partial_0^{D \otimes_R^{\bowtie} B}} & D_{-1} \otimes_R (\Sigma B)_0
\end{array}$$

Define τ_1 to be the matrix $[f_0(\sigma_0^C)^{-1} \otimes g_0 \quad 0]$ and $\tau_0 = f_{-1}(\sigma_{-1}^C)^{-1} \otimes (\Sigma g)_0 \sigma_{-1}^A \partial_0^A$. Choose an element $c \otimes a \in (\Sigma C)_1 \otimes_R A_0$. Then,

$$\begin{aligned}
\tau_0\Big(\partial_1^{\Sigma C \otimes_R^{\bowtie} A}(c \otimes a)\Big) &= \tau_0\Big(\partial_1^{\Sigma C}(c) \otimes a\Big) \\
&= f_{-1}(\sigma_{-1}^C)^{-1}\Big(\partial_1^{\Sigma C}(c)\Big) \otimes (\Sigma g)_0 \sigma_{-1}^A \partial_0^A(a) \\
&= f_{-1}(\sigma_{-1}^C)^{-1}\Big(-\sigma_{-1}^C \partial_0^C (\sigma_0^C)^{-1}(c)\Big) \otimes (\Sigma g)_0 \sigma_{-1}^A \partial_0^A(a) \\
&= -f_{-1} \partial_0^C (\sigma_0^C)^{-1}(c) \otimes (\Sigma g)_0 \sigma_{-1}^A \partial_0^A(a).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_0^{D \otimes_R^{\bowtie} B}\Big(\tau_1(c \otimes a)\Big) &= \partial_0^{D \otimes_R^{\bowtie} B}\Big(f_0(\sigma_0^C)^{-1}(c) \otimes g(a)\Big) \\
&= (\partial_0^D \otimes \sigma_{-1}^B \partial_0^B)\Big(f_0(\sigma_0^C)^{-1}(c) \otimes d(a)\Big) \\
&= \partial_0^D f_0(\sigma_0^C)^{-1}(c) \otimes \sigma_{-1}^B \partial_0^B g_0(a).
\end{aligned}$$

We note that $f_{-1}\partial_0^C(\sigma_0^C)^{-1}(c) = \partial_0^D f_0(\sigma_0^C)^{-1}(c)$ since f is a chain map, and $(\Sigma g)_0\sigma_{-1}^A\partial_0^A(a) = \sigma_{-1}^B\partial_0^B g_0(a)$ because the following diagram commutes.

$$\begin{array}{ccccc}
A_0 & \xrightarrow{\partial_0^A} & A_{-1} & \xrightarrow{\sigma_{-1}^A} & (\Sigma A)_0 \\
\downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_{-1}} & & \downarrow{\scriptstyle (\Sigma g)_0} \\
B_0 & \xrightarrow{\partial_0^B} & B_{-1} & \xrightarrow{\sigma_{-1}^B} & (\Sigma B)_0
\end{array}$$

Now choose an element $c \otimes a \in (\Sigma C)_0 \otimes_R A_1$. Then,

$$\begin{aligned}
\tau_0\Big(\partial_1^{\Sigma C \otimes_R^{\bowtie} A}(c \otimes a)\Big) &= \tau_0\Big(c \otimes \partial_1^A(a)\Big) \\
&= f_{-1}(\sigma_{-1}^C)^{-1}(c) \otimes (\Sigma g)_0\sigma_{-1}^A\partial_0^A\partial_1^A(a) \\
&= f_{-1}(\sigma_{-1}^C)^{-1}(c) \otimes 0 \\
&= 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_0^{D \otimes_R^{\bowtie} B}\Big(\tau_1(c \otimes a)\Big) &= \partial_0^{D \otimes_R^{\bowtie} B}(0) \\
&= 0.
\end{aligned}$$

Therefore, $\tau_0\partial_1^{\Sigma C \otimes_R^{\bowtie} A} = -\partial_0^{D \otimes_R^{\bowtie} B}\tau_1$, which is what we wanted to show.

$n \geqslant 2$:

In this case the diagram is

$$\begin{array}{ccc}
\bigoplus_{i=0}^{n}(\Sigma C)_{n-i} \otimes_R A_i & \xrightarrow{\partial_n^{\Sigma C \otimes_R^{\bowtie} A}} & \bigoplus_{i=0}^{n-1}(\Sigma C)_{n-i} \otimes_R A_i \\
\downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \tau_{n-1}} \\
\bigoplus_{i=0}^{n-1} D_{n-1-i} \otimes_R B_i & \xrightarrow{\partial_{n-1}^{D \otimes_R^{\bowtie} B}} & \bigoplus_{i=0}^{n-2} D_{n-1-i} \otimes_R B_i
\end{array}$$

Define

$$\tau_n = \begin{bmatrix} f_{n-1}(\sigma^C_{n-1})^{-1} \otimes g_0 & & & 0 \\ & \ddots & & \vdots \\ & & f_0(\sigma^C_0)^{-1} \otimes g_{n-1} & 0 \end{bmatrix}$$

for $n \geqslant 2$ any element $c \otimes a \in (\Sigma C)_{n-i} \otimes_R A_i$ for $i \geqslant 1$. Then,

$$\begin{aligned}
\tau_{n-1}\Big(\partial_n^{\Sigma C \otimes_R^{\bowtie} A}(c \otimes a)\Big) &= \tau_{n-1}\Big(\partial_{n-i}^{\Sigma C}(c) \otimes a + (-1)^{n-i}(c) \otimes \partial_i^A(a)\Big) \\
&= f_{n-i-2}(\sigma^C_{n-i-2})^{-1}\big(\partial_{n-i}^{\Sigma C}(c)\big) \otimes g_i(a) \\
&\qquad + (-1)^{n-i} f_{n-i-1}(\sigma^C_{n-i-1})^{-1}(c) \otimes g_{i-1}\partial_i^A(a) \\
&= f_{n-i-2}(\sigma^C_{n-i-2})^{-1}\big(-\sigma^C_{n-i-2}\partial^C_{n-i-1}{\sigma^C_{n-i-1}}^{-1}(c)\big) \otimes g_i(a) \\
&\qquad + (-1)^{n-i} f_{n-i-1}(\sigma^C_{n-i-1})^{-1}(c) \otimes g_{i-1}\partial_i^A(a) \\
&= -f_{n-i-2}\partial^C_{n-i-1}(\sigma^C_{n-i-1})^{-1}(c) \otimes g_i(a) \\
&\qquad + (-1)^{n-i} f_{n-1-i}(\sigma^C_{n-i-1})^{-1}(c) \otimes g_{i-1}\partial_i^A(a).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_{n-1}^{D \otimes_R^{\bowtie} B}(\tau_n(c \otimes a)) &= \partial_{n-1}^{D \otimes_R^{\bowtie} B}\Big(f_{n-i-1}(\sigma^C_{n-i-1})^{-1}(c) \otimes g_i(a)\Big) \\
&= \partial^D_{n-i-1} f_{n-i-1}(\sigma^C_{n-i-1})^{-1}(c) \otimes g_i(a) \\
&\qquad + (-1)^{n-i-1} f_{n-i-1}(\sigma^C_{n-i-1})^{-1}(c) \otimes \partial_i^A g_i(a)
\end{aligned}$$

We note that $f_{n-i-2}\partial^C_{n-i-1}(\sigma^C_{n-i-1})^{-1}(c) = \partial^D_{n-i-1} f_{n-i-1}(\sigma^C_{n-i-1})^{-1}(c)$ since f is a chain map. Therefore, $\tau_{n-1}\partial_n^{\Sigma C \otimes_R^{\bowtie} A} = -\partial_{n-1}^{D \otimes_R^{\bowtie} B}\tau_n$, for $c \otimes a \in (\Sigma C)_{n-i} \otimes_R A_i$ for $i \geqslant 1$.

Now choose an element $c \otimes a \in (\Sigma C)_0 \otimes_R A_n$. Then,

$$\begin{aligned}
\tau_{n-1}\Big(\partial_n^{\Sigma C \otimes_R^{\bowtie} A}(c \otimes a)\Big) &= \tau_{n-1}(0) \\
&= 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_{n-1}^{D \otimes_R^{\bowtie} B}\Big(\tau_n(c \otimes a)\Big) &= \partial_n^{D \otimes_R^{\bowtie} B}(0) \\
&= 0.
\end{aligned}$$

Therefore, $\tau_{n-1}\partial_n^{\Sigma C \otimes_R^{\bowtie} A} = -\partial_{n-1}^{D \otimes_R^{\bowtie} B}\tau_n$, which is what we wanted to show.

$n = 0$:

In this case the digram is

$$\begin{array}{ccc}
(\Sigma C)_0 \otimes_R A_0 & \xrightarrow{\partial_0^{\Sigma C \otimes_R^{\bowtie} A}} & (\Sigma C)_{-1} \otimes_R (\Sigma A)_0 \\
\Big\downarrow \tau_0 & & \Big\downarrow \tau_{-1} \\
D_{-1} \otimes_R (\Sigma B)_0 & \xrightarrow{\partial_{-1}^{D \otimes_R^{\bowtie} B}} & D_{-2} \otimes_R (\Sigma B)_0 \otimes D_{-1} \otimes_R (\Sigma B)_{-1}
\end{array}$$

Define $\tau_{-1} = f_{-2}(\sigma_{-2}^C)^{-1} \otimes (\Sigma g)_0$. Choose an element $c \otimes a \in (\Sigma C)_0 \otimes_R A_0$. Then,

$$\begin{aligned}
\tau_{-1}\Big(\partial_0^{\Sigma C \otimes_R^{\bowtie} A}(c \otimes a)\Big) &= \tau_{-1}\Big(\partial_0^{\Sigma C}(c) \otimes \sigma_{-1}^A \partial_0^A(a)\Big) \\
&= f_{-2}(\sigma_{-2}^C)^{-1}\partial_0^{\Sigma C}(c) \otimes (\Sigma g)_0 \sigma_{-1}^A \partial_0^A(a) \\
&= f_{-2}(\sigma_{-2}^C)^{-1}(-\sigma_{-2}^C \partial_{-1}^C (\sigma_{-1}^C)^{-1}(c)) \otimes (\Sigma g)_0 \sigma_{-1}^A \partial_0^A(a) \\
&= -f_{-2}\partial_{-1}^C(\sigma_{-1}^C)^{-1}(c) \otimes (\Sigma g)_0 \sigma_{-1}^A \partial_0^A(a).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_{-1}^{D \otimes_R^{\bowtie} B}\Big(\tau_0(c \otimes a)\Big) &= \partial_{-1}^{D \otimes_R^{\bowtie} B}\Big(f_{-1}(\sigma_{-1}^C)^{-1}(c) \otimes (\Sigma g)_0 \sigma_{-1}^A \partial_0^A(a)\Big) \\
&= \partial_{-1}^D f_{-1}(\sigma_{-1}^C)^{-1}(c) \otimes (\Sigma g)_0 \sigma_{-1}^A \partial_0^A(a).
\end{aligned}$$

We note that $f_{-2}\partial_{-1}^C(\sigma_{-1}^C)^{-1}(c) = \partial_{-1}^D f_{-1}(\sigma_{-1}^C)^{-1}(c)$ since f is a chain map. Thus, $\tau_{-1}\partial_0^{\Sigma C \otimes_R^{\bowtie} A} = -\partial_{-1}^{B \otimes_R^{\bowtie} D}\tau_0$, which it is anti-commutative.

$n \leqslant -1$:

In this case the diagram is

$$\begin{array}{ccc}
\bigoplus_{i=n}^{-1}(\Sigma C)_i \otimes_R (\Sigma A)_{n-i} & \xrightarrow{\partial_n^{\Sigma C \otimes_R^{\bowtie} A}} & \bigoplus_{i=n-1}^{-1}(\Sigma C)_i \otimes_R (\Sigma A)_{n-1-i} \\
\Big\downarrow \tau_n & & \Big\downarrow \tau_{n-1} \\
\bigoplus_{i=n-1}^{-1} D_i \otimes_R (\Sigma B)_{n-1-i} & \xrightarrow{\partial_{n-1}^{D \otimes_R^{\bowtie} B}} & \bigoplus_{i=n-2}^{-1} D_i \otimes_R (\Sigma B)_{n-2-i}
\end{array}$$

Define

$$\tau_n = \begin{bmatrix} f_{n-1}(\sigma^C_{n-1})^{-1} \otimes (\Sigma g)_0 & & \\ & \ddots & \\ & & f_{-2}(\sigma^C_{-2})^{-1} \otimes (\Sigma g)_{n+1} \\ 0 & \cdots & 0 \end{bmatrix}$$

for $n \leqslant -2$ and choose an element $c \otimes a \in (\Sigma C)_i \otimes_R (\Sigma A)_{n-i}$ for $i \geqslant n+1$. Then,

$$\begin{aligned} \tau_{n-1}\Big(\partial_n^{\Sigma C \otimes_R^{\bowtie} A}(c \otimes a)\Big) &= \tau_{n-1}\Big(\partial_i^{\Sigma C}(c) \otimes a + (-1)^i c \otimes \partial^A_{n-i}(a)\Big) \\ &= f_{i-2}(\sigma^C_{i-2})^{-1}\partial_i^{\Sigma C}(c) \otimes (\Sigma g)_{n-i}(a) \\ &\qquad + (-1)^i f_{i-1}(\sigma^C_{i-1})^{-1}(c) \otimes (\Sigma g)_{n-i-1}\partial^A_{n-i}(a) \\ &= f_{i-2}(\sigma^C_{i-2})^{-1}\Big(-\sigma^C_{i-2}\partial^C_{i-1}(\sigma^C_{i-1})^{-1}\Big)(c) \otimes (\Sigma g)_{n-i}(a) \\ &\qquad + (-1)^i f_{i-1}(\sigma^C_{i-1})^{-1}(c) \otimes (\Sigma g)_{n-i-1}\partial^A_{n-i}(a) \\ &= -f_{i-2}\partial^C_{i-1}(\sigma^C_{i-1})^{-1}(c) \otimes (\Sigma g)_{n-i}(a) \\ &\qquad + (-1)^i f_{i-1}(\sigma^C_{i-1})^{-1}(c) \otimes (\Sigma g)_{n-i-1}\partial^A_{n-i}(a) \\ &= -\Big(f_{i-2}\partial^C_{i-1}(\sigma^C_{i-1})^{-1}(c) \otimes (\Sigma g)_{n-i}(a) \\ &\qquad + (-1)^{i-1} f_{i-1}(\sigma^C_{i-1})^{-1}(c) \otimes (\Sigma g)_{n-i-1}\partial^A_{n-i}(a)\Big). \end{aligned}$$

Moreover,

$$\begin{aligned} \partial_{n-1}^{D \otimes_R^{\bowtie} B}\Big(\tau_n(c \otimes a)\Big) &= \partial_{n-1}^{D \otimes_R^{\bowtie} B}\Big(f_{i-1}(\sigma^C_{i-1})^{-1}(c) \otimes (\Sigma g)_{n-i}(a)\Big) \\ &= \partial^D_{i-1} f_{i-1}(\sigma^C_{i-1})^{-1}(c) \otimes (\Sigma g)_{n-i}(a) \\ &\qquad + (-1)^{i-1} f_{i-1}(\sigma^C_{i-1})^{-1}(c) \otimes \partial^D_{n-i}(\Sigma g)_{n-i}(a). \end{aligned}$$

We note that $f_{i-2}\partial^C_{i-1}(\sigma^C_{i-1})^{-1}(c) = \partial^D_{i-1} f_{i-1}(\sigma^C_{i-1})^{-1}(c)$ since f is a chain map. Therefore, $\tau_{n-1}\partial_n^{\Sigma C \otimes_R^{\bowtie} A} = -\partial_{n-1}^{D \otimes_R^{\bowtie} B}\tau_n$, which is anti-commutative. □

Corollary 4.1.4. *Let C and D be complexes. Then there exists a morphism q from $(\Sigma C \otimes_R^{\bowtie} D)$ to $\Sigma(C \otimes_R^{\bowtie} D)$.*

Proof: Let $f = \mathrm{Id}_C$ and $g = \mathrm{Id}_D$ in the previous Theorem 4.1.3.

□

Remark 4.1.5. In Corollary 4.1.4, as far as we know there is no morphism in the opposite direction, from $(C \otimes_R^{\bowtie} D)$ to $(\Sigma C \otimes_R^{\bowtie} D)$. The problem is that there does not necessarily exist a map $C_{-1} \otimes_R (\Sigma D)_0 \to (\Sigma C)_0 \otimes_R D_0$.

Theorem 4.1.6. *Let A, B, C and D be complexes, and $f : C \to D$ and $g : A \to B$ morphisms. Then there exists a morphism q' from $(C \otimes_R^{\bowtie} \Sigma A)$ to $\Sigma(D \otimes_R^{\bowtie} B)$.*

Proof: By Lemma 4.1.2 it suffices to define a degree −1 anti-commutative chain map τ from $C \otimes_R^{\bowtie} \Sigma A$ to $D \otimes_R^{\bowtie} B$, that is, the diagram

$$\begin{array}{ccc} (C \otimes_R^{\bowtie} \Sigma A)_n & \xrightarrow{\partial_n^{C\otimes_R^{\bowtie}\Sigma A}} & (C \otimes_R^{\bowtie} \Sigma A)_{n-1} \\ \downarrow{\scriptstyle \tau'_n} & & \downarrow{\scriptstyle \tau'_{n-1}} \\ (D \otimes_R^{\bowtie} B)_{n-1} & \xrightarrow{\partial_{n-1}^{D\otimes_R^{\bowtie}B}} & (D \otimes_R^{\bowtie} B)_{n-2} \end{array}$$

is anti-commutative for all $n \in \mathbb{Z}$. We consider four cases: $n = 1$, $n \geqslant 2$, $n = 0$ and $n \leqslant -1$.

$n = 1$:

In this case the diagram is

$$\begin{array}{ccc} C_1 \otimes_R (\Sigma A)_0 \oplus C_0 \otimes_R (\Sigma A)_1 & \xrightarrow{\partial_1^{C\otimes_R^{\bowtie}\Sigma A}} & C_0 \otimes_R (\Sigma A)_0 \\ \downarrow{\scriptstyle \tau'_1} & & \downarrow{\scriptstyle \tau'_0} \\ D_0 \otimes_R B_0 & \xrightarrow{\partial_0^{D\otimes_R^{\bowtie}B}} & D_{-1} \otimes_R (\Sigma B)_0 \end{array}$$

Define $\tau'_1 = [0 \quad f_0 \otimes g_0(\sigma_0^A)^{-1}]$ and $\tau'_0 = f_{-1}\partial_0^C \otimes (\Sigma g)_0$. Choose an element $c \otimes a \in C_1 \otimes (\Sigma A)_0$. Then,

$$
\begin{aligned}
\tau_0'\Big(\partial_1^{C\otimes^{\bowtie}_R \Sigma A}(c\otimes a)\Big) &= \tau_0'\Big(\partial_1^C(c)\otimes a\Big)\\
&= f_{-1}\partial_0^C\Big(\partial_1^C(c)\Big)\otimes(\Sigma g)_0(a)\\
&= 0.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\partial_0^{D\otimes^{\bowtie}_R B}\Big(\tau_1'(c\otimes a)\Big) &= \partial_0^{D\otimes^{\bowtie}_R B}\Big(0\Big)\\
&= 0.
\end{aligned}
$$

Now choose an element $c\otimes a\in C_0\otimes_R(\Sigma A)_1$. Then,

$$
\begin{aligned}
\tau_0'\Big(\partial_1^{C\otimes^{\bowtie}_R \Sigma A}(c\otimes a)\Big) &= \tau_0'\Big(c\otimes\partial_1^{\Sigma A}(a)\Big)\\
&= \tau_0'\Big(c\otimes(-\sigma_{-1}^A\partial_0^A(\sigma_0^A)^{-1})(a)\Big)\\
&= f_{-1}\partial_0^C(c)\otimes(\Sigma g)_0(-\sigma_{-1}^A\partial_0^A(\sigma_0^A)^{-1})(a)\\
&= -f_{-1}\partial_0^C(c)\otimes(\Sigma g)_0(\sigma_{-1}^A\partial_0^A(\sigma_0^A)^{-1})(a).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\partial_0^{D\otimes^{\bowtie}_R B}\Big(\tau_1'(c\otimes a)\Big) &= \partial_0^{D\otimes^{\bowtie}_R B}\Big(f_0(c)\otimes g_0(\sigma_0^A)^{-1}(a)\Big)\\
&= \partial_0^D f_0(c)\otimes\sigma_{-1}^B\partial_0^B g_0(\sigma_0^A)^{-1}(a).
\end{aligned}
$$

We note that $f_{-1}\partial_0^C(c)=\partial_0^D f_0(c)$ since f is a chain map, and

$(\Sigma g)_0(\sigma_{-1}^A\partial_0^A(\sigma_0^A)^{-1})(a)=\sigma_{-1}^B\partial_0^B g_0(\sigma_0^A)^{-1}(a)$ because the following diagram commutes.

$$
\begin{array}{ccccc}
A_0 & \xrightarrow{\partial_0^A} & A_{-1} & \xrightarrow{\sigma_{-1}^A} & (\Sigma A)_0\\
\Big\downarrow{\scriptstyle g_0} & & \Big\downarrow{\scriptstyle g_{-1}} & & \Big\downarrow{\scriptstyle (\Sigma g)_0}\\
B_0 & \xrightarrow{\partial_0^B} & B_{-1} & \xrightarrow{\sigma_{-1}^B} & (\Sigma B)_0
\end{array}
$$

Therefore, $f_0\partial_1^{C\otimes^{\bowtie}_R \Sigma A}=-\partial_0^{D\otimes^{\bowtie}_R B}f_1$, which is anti-commutative.

$n \geqslant 2$:

In this case the diagram is

$$\begin{array}{ccc} \bigoplus_{i=0}^{n} C_{n-i} \otimes_R (\Sigma A)_i & \xrightarrow{\partial_n^{C \otimes_R^{\bowtie} \Sigma A}} & \bigoplus_{i=0}^{n-1} C_{n-1-i} \otimes_R (\Sigma A)_i \\ \downarrow \tau'_n & & \downarrow \tau'_{n-1} \\ \bigoplus_{i=0}^{n-1} D_{n-1-i} \otimes_R B_i & \xrightarrow{\partial_{n-1}^{D \otimes_R^{\bowtie} B}} & \bigoplus_{i=0}^{n-2} D_{n-2-i} \otimes_R B_i \end{array}$$

Define

$$\tau'_n = \begin{bmatrix} 0 & (-1)^n f_n \otimes g_{-1}(\sigma_{-1}^A)^{-1} & & \\ \vdots & & \ddots & \\ 0 & & & (-1)^0 f_0 \otimes g_{n-1}(\sigma_{n-1}^A)^{-1} \end{bmatrix}$$

for $n \geqslant 2$ and choose any element $c \otimes a \in C_{n-i} \otimes_R (\Sigma A)_i$ for $i \geqslant 1$. Then,

$$\begin{aligned} \tau'_{n-1}\Big((\partial_n^{C \otimes_R^{\bowtie} \Sigma A}(c \otimes a)\Big) &= \tau'_{n-1}\Big(\partial_{n-i}^C(c) \otimes a + (-1)^{n-i} c \otimes \partial_i^{(\Sigma A)}(a)\Big) \\ &= \tau'_{n-1}\Big(\partial_{n-i}^C(c) \otimes a + (-1)^{n-i} c \otimes (-\sigma_{i-2}^A \partial_{i-1}^A (\sigma_{i-1}^A)^{-1})(a)\Big) \\ &= (-1)^{n-i-1} f_{n-i-1} \partial_{n-i}^C(c) \otimes g_{i-1}(\sigma_{i-1}^A)^{-1}(a) + \\ &\quad (-1)^{n-i}(-1)^{n-i} f_{n-i}(c) \otimes g_{i-2}(\sigma_{i-2}^A)^{-1}(-\sigma_{i-2}^A \partial_{i-1}^A (\sigma_{i-1}^A)^{-1})(a) \\ &= (-1)^{n-i-1} f_{n-i-1} \partial_{n-i}^C(c) \otimes g_{i-1}(\sigma_{i-1}^A)^{-1}(a) + \\ &\qquad - f_{n-i}(c) \otimes g_{i-2} \partial_{i-1}^A (\sigma_{i-1}^A)^{-1}(a) \\ &= -\Big((-1)^{n-i} f_{n-i-1} \partial_{n-i}^C(c) \otimes g_{i-1}(\sigma_{i-1}^A)^{-1}(a) + \\ &\qquad f_{n-i}(c) \otimes g_{i-2} \partial_{i-1}^A (\sigma_{i-1}^A)^{-1}(a)\Big). \end{aligned}$$

Moreover,

$$\begin{aligned} \partial_{n-1}^{D \otimes_R^{\bowtie} B}\Big(\tau'_n(c \otimes a)\Big) &= \partial_{n-1}^{D \otimes_R^{\bowtie} B}\Big((-1)^{n-i} f_{n-i}(c) \otimes g_{i-1}(\sigma_{i-1}^A)^{-1}(a)\Big) \\ &= (-1)^{n-i} \partial_{n-i}^D f_{n-i}(c) \otimes g_{i-1}(\sigma_{i-1}^A)^{-1}(a) \\ &\qquad + (-1)^{n-i}(-1)^{n-i} f_{n-i}(c) \otimes \partial_i^B\Big(g_{i-1}(\sigma_{i-1}^A)^{-1}(a)\Big) \\ &= (-1)^{n-i} \partial_{n-i}^D f_{n-i}(c) \otimes g_{i-1}(\sigma_{i-1}^A)^{-1}(a) \\ &\qquad + f_{n-i}(c) \otimes \partial_{i-1}^B\Big(g_{i-1}(\sigma_{i-1}^A)^{-1}(a)\Big). \end{aligned}$$

We note that $f_{n-i-1}\partial_{n-i}^C(c) = \partial_{n-i}^D f_{n-i}(c)$ since f is a chain map, and

$g_{i-2}\partial^A_{i-1}(\sigma^A_{i-1})^{-1}(a) = \partial^B_{i-1}g_{i-1}(\sigma^A_{i-1})^{-1}(a)$ because the following diagram commutes.

$$\begin{array}{ccccc}
(\Sigma A)_i & \xrightarrow{(\sigma^A_{i-1})^{-1}} & A_{i-1} & \xrightarrow{\partial^A_{i-1}} & A_{i-2} \\
\downarrow{\scriptstyle (\Sigma g)_i} & & \downarrow{\scriptstyle g_{i-1}} & & \downarrow{\scriptstyle g_{i-2}} \\
(\Sigma B)_i & \xrightarrow{(\sigma^B_{i-1})^{-1}} & B_{i-1} & \xrightarrow{\partial^B_{i-1}} & B_{i-2}
\end{array}$$

Now choose an element $c \otimes a \in C_n \otimes_R (\Sigma A)_0$. Then,

$$\begin{aligned}
\tau_{n-1}\Big(\partial_n^{C\otimes_R^{\bowtie}\Sigma A}(c\otimes a)\Big) &= \tau_{n-1}\Big(\partial_n^C(c)\otimes a\Big) \\
&= 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_{n-1}^{D\otimes_R^{\bowtie}B}\Big(\tau_n(c\otimes a)\Big) &= \partial_{n-1}^{D\otimes_R^{\bowtie}B}\Big(0\Big) \\
&= 0.
\end{aligned}$$

Therefore, $\tau_{n-1}\partial_n^{C\otimes_R^{\bowtie}\Sigma A} = -\partial_{n-1}^{D\otimes_R^{\bowtie}B} f_n$, which it is anti-commutative.

$n = 0$:

In this case the digram is

$$\begin{array}{ccc}
C_0 \otimes_R (\Sigma A)_0 & \xrightarrow{\partial_0^{C\otimes_R^{\bowtie}\Sigma A}} & C_{-1}\otimes_R (\Sigma(\Sigma A))_0 \\
{\scriptstyle \tau'_0}\downarrow & & \downarrow{\scriptstyle \tau'_{-1}} \\
D_{-1}\otimes_R (\Sigma B)_0 & \xrightarrow{\partial_{-1}^{D\otimes_R^{\bowtie}B}} & D_{-2}\otimes_R (\Sigma B)_0 \oplus D_{-1}\otimes_R (\Sigma B)_{-1}
\end{array}$$

Define $\tau'_{-1} = f_{-1}\otimes(\Sigma g)_{-1}(\sigma^{\Sigma A}_{-1})^{-1}$and choose an element $c\otimes a \in C_0\otimes_R (\Sigma A)_0$. Then,

$$\begin{aligned}
\tau'_{-1}\Big(\partial_0^{C\otimes_R^{\bowtie}\Sigma A}(c\otimes a)\Big) &= \tau'_{-1}\Big(\partial_0^C(c)\otimes \sigma^{\Sigma A}_{-1}\partial_0^{\Sigma A}(a)\Big) \\
&= f_{-1}\partial_0^C(c)\otimes(\Sigma g)_{-1}(\sigma^{\Sigma A}_{-1})^{-1}\sigma^{\Sigma A}_{-1}\partial_0^{\Sigma A}(a) \\
&= f_{-1}\partial_0^C(c)\otimes(\Sigma g)_{-1}\partial_0^{\Sigma A}(a).
\end{aligned}$$

Moreover,

$$
\begin{aligned}
\partial_{-1}^{D\otimes_R^{\bowtie}B}\Big(\tau_0'(c\otimes a)\Big) &= \partial_{-1}^{D\otimes_R^{\bowtie}B}\Big(f_{-1}\partial_0^C(c)\otimes(\Sigma g)_0(a)\Big)\\
&= \partial_{-1}^{D}f_{-1}\partial_0^C(c)\otimes(\Sigma g)_0(a)+(-1)^{-1}f_{-1}\partial_0^C(c)\otimes\partial_0^{\Sigma B}(\Sigma g)_0(a)\\
&= \partial_{-1}^{C}\partial_0^{D}f_0(c)\otimes(\Sigma g)_0(a)+(-1)^{-1}f_{-1}\partial_0^C(c)\otimes\partial_0^{\Sigma B}(\Sigma g)_0(a)\\
&= 0\otimes(\Sigma g)_0(a)+(-1)^{-1}f_{-1}\partial_0^C(c)\otimes\partial_0^{\Sigma B}(\Sigma g)_0(a)\\
&= (-1)^{-1}f_{-1}\partial_0^C(c)\otimes\partial_0^{\Sigma B}(\Sigma g)_0(a).
\end{aligned}
$$

We note that $(\Sigma g)_{-1}\partial_0^{\Sigma A}(a) = \partial_0^{\Sigma B}(\Sigma g)_0(a)$ since g is a chain map. Therefore, $f_{-1}\partial_0^{C\otimes_R^{\bowtie}\Sigma D} = -\partial_{-1}^{C\otimes_R^{\bowtie}D}f_0$, which it is anti-commutative.

$n \leqslant -1$:

In this case the diagram is

$$
\begin{array}{ccc}
\bigoplus_{i=n}^{-1} C_i\otimes_R(\Sigma(\Sigma A))_{n-i} & \xrightarrow{\partial_n^{C\otimes_R^{\bowtie}\Sigma A}} & \bigoplus_{i=n-1}^{-1} C_i\otimes_R(\Sigma(\Sigma A))_{n-1-i}\\
\Big\downarrow{\scriptstyle \tau_n'} & & \Big\downarrow{\scriptstyle \tau_{n-1}'}\\
\bigoplus_{i=n-1}^{-1} D_i\otimes_R(\Sigma B)_{n-1-i} & \xrightarrow{\partial_{n-1}^{D\otimes_R^{\bowtie}B}} & \bigoplus_{i=n-2}^{-1} D_i\otimes_R(\Sigma B)_{n-2-i}
\end{array}
$$

Define

$$
\tau_n' = \begin{bmatrix}
(-1)^n f_n\otimes(\Sigma g)_{-1}(\sigma_{-1}^{\Sigma D})^{-1} & & & \\
 & \ddots & & \\
 & & (-1)^{-1}f_{-1}\otimes(\Sigma g)_n(\sigma_n^{\Sigma D})^{-1} & \\
0 & \cdots & & 0
\end{bmatrix}
$$

for $n \leqslant -2$ and choose any element $c \otimes d \in C_i \otimes_R \Big(\Sigma(\Sigma A)\Big)_{n-i}$ for $i \geqslant n+1$. Then,

$$\begin{aligned}
\tau'_{n-1}\Big(\partial_n^{C\otimes_R^{\bowtie}\Sigma A}(c\otimes a)\Big) &= \tau'_{n-1}\Big(\partial_i^C(c)\otimes a + (-1)^i(c)\otimes \partial_{n-i}^{\Sigma(\Sigma A)}(a)\Big)\\
&= \tau'_{n-1}\Big(\partial_i^C(c)\otimes a + (-1)^i(c)\otimes\big(-\sigma_{n-i-2}^{\Sigma A}\partial_{n-i-1}^{\Sigma A}(\sigma_{n-i-1}^{\Sigma A})^{-1})(a)\Big)\\
&= \tau'_{n-1}\Big(\partial_i^C(c)\otimes a - (-1)^i(c)\otimes\big(\sigma_{n-i-2}^{\Sigma A}\partial_{n-i-1}^{\Sigma A}(\sigma_{n-i-1}^{\Sigma A})^{-1})(a)\Big)\\
&= (-1)^{i-1}f_{i-1}\partial_i^C(c)\otimes(\Sigma g)_{n-i-1}(\sigma_{n-i-1}^{\Sigma A})^{-1}(a)\\
&\quad -((-1)^i)^2 f_i(c)\otimes(\Sigma g)_{n-i-2}(\sigma_{n-i-2}^{\Sigma A})^{-1}(\sigma_{n-i-2}^{\Sigma A}\partial_{n-i-1}^{\Sigma A}(\sigma_{n-i-1}^{\Sigma A})^{-1})(a)\\
&= (-1)^{i-1}f_{i-1}\partial_i^C(c)\otimes(\Sigma g)_{n-i-1}(\sigma_{n-i-1}^{\Sigma A})^{-1}(a)\\
&\qquad\qquad -f_i(c)\otimes(\Sigma g)_{n-i-2}\partial_{n-i-1}^{\Sigma A}(\sigma_{n-i-1}^{\Sigma A})^{-1}(a)\\
&= -\Big((-1)^i f_{i-1}\partial_i^C(c)\otimes(\Sigma g)_{n-i-1}(\sigma_{n-i-1}^{\Sigma A})^{-1}(a)\\
&\qquad\qquad +f_i(c)\otimes(\Sigma g)_{n-i-2}\partial_{n-i-1}^{\Sigma A}(\sigma_{n-i-1}^{\Sigma A})^{-1}(a)\Big).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_{n-1}^{D\otimes_R^{\bowtie}B}\Big(\tau'_n(c\otimes a)\Big) &= \partial_{n-1}^{D\otimes_R^{\bowtie}B}\Big((-1)^i f_i(c)\otimes(\Sigma g)_{n-i-1}(\sigma_{n-i-1}^{\Sigma A})^{-1}(a)\Big)\\
&= (-1)^i\partial_i^C f_i(c)\otimes(\Sigma g)_{n-i-1}(\sigma_{n-i-1-1}^{\Sigma A})^{-1}(a)+\\
&\qquad (-1)^i(-1)^i f_i(c)\otimes\partial_{n-i-1}^{\Sigma B}(\Sigma g)_{n-i-1}(\sigma_{n-i-1}^{\Sigma A})^{-1}(a)\\
&= (-1)^i\partial_i^C f_i(c)\otimes(\Sigma g)_{n-i-1}(\sigma_{n-i-1-1}^{\Sigma A})^{-1}(a)+\\
&\qquad f_i(c)\otimes\partial_{n-i-1}^{\Sigma B}(\Sigma g)_{n-i-1}(\sigma_{n-i-1}^{\Sigma A})^{-1}(a).
\end{aligned}$$

We note that $(\Sigma g)_{n-i-2}\partial_{n-i-1}^{\Sigma A}(\sigma_{n-i-1}^{\Sigma A})^{-1}(a) = \partial_{n-i-1}^{\Sigma B}(\Sigma g)_{n-i-1}(\sigma_{n-i-1}^{\Sigma A})^{-1}(a)$ since g is a chain map. Therefore, $f_{n-1}\partial_n^{C\otimes_R^{\bowtie}\Sigma A} = -\partial_{n-1}^{D\otimes_R^{\bowtie}B} f_n$, which is anti-commutative.

□

Corollary 4.1.7. *Let C and D be complexes. Then there exists a morphism q' from $(C\otimes_R^{\bowtie}\Sigma D)$ to $\Sigma(C\otimes_R^{\bowtie} D)$.*

Proof: Let $f = \mathrm{Id}_C$ and $g = \mathrm{Id}_D$ in the previous Proposition 4.1.6.

□

Remark 4.1.8. In Corollary 4.1.7 as far as we know there is no morphism in the opposite direction, from $\Sigma(C \otimes_R^{\bowtie} D)$ to $(C \otimes_R^{\bowtie} \Sigma D)$. The problem is that there does not necessarily exist a map $C_{-1} \otimes_R (\Sigma D)_0 \to C_0 \otimes_R (\Sigma D)_0$.

Remark 4.1.9. Also, as far as we know, there is no morphism going from $(\Sigma C \otimes_R^{\bowtie} D)$ to $(C \otimes_R^{\bowtie} \Sigma D)$. The problem is that there does not necessarily exist a map $(\Sigma C)_0 \otimes_R D_0 \to C_0 \otimes_R (\Sigma D)_0$. Similarly, as far as we know, there is no morphism going from $(C \otimes_R^{\bowtie} \Sigma D)$ to $(\Sigma C \otimes_R^{\bowtie} D)$. The problem is that there does not necessarily exist a map $C_0 \otimes_R (\Sigma D)_0 \to (\Sigma C)_0 \otimes_R D_0$.

Theorem 4.1.10. *Let $d \geq 0$. If $f : C \to \Sigma^d(D)$ is a morphism and A is a complex. Then there exists a morphism from $(C \otimes_R^{\bowtie} A) \to \Sigma^d(D \otimes_R^{\bowtie} A)$ which is commutative when d is even and anti-commutative when d is odd .*

Proof: We want to use Lemma 4.1.2 and Corollary 2.5.3 to define a morphism $(C \otimes_R^{\bowtie} A) \to \Sigma^d(D \otimes_R^{\bowtie} A)$ such that the following diagram commutes

$$\begin{array}{ccc}
(C \otimes_R^{\bowtie} A)_n & \xrightarrow{\partial_n^{C \otimes_R^{\bowtie} A}} & (C \otimes_R^{\bowtie} A)_{n-1} \\
\downarrow \tau_n & & \downarrow \tau_{n-1} \\
(\Sigma^d(D \otimes_R^{\bowtie} A))_n & \xrightarrow{\partial_n^{\Sigma^d(D \otimes_R^{\bowtie} A)}} & (\Sigma^d(D \otimes_R^{\bowtie} A))_{n-1}
\end{array}$$

when d is even and anti-commutes when d is odd for all $n \in \mathbb{Z}$. Note that $\left(\Sigma^d(D \otimes_R^{\bowtie} A)\right)_n = (D \otimes_R^{\bowtie} A)_{n-d}$ and we consider three cases: $n \geq d, 0 < n < d$ and $n \leq 0$.

$n \geq d:$

Suppose $n > d$. In this case the diagram will be

$$\begin{array}{ccc} (C \otimes_R^{\bowtie} A)_n & \xrightarrow{\partial_n^{C \otimes_R^{\bowtie} A}} & (C \otimes_R^{\bowtie} A)_{n-1} \\ \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \tau_{n-1}} \\ (D \otimes_R^{\bowtie} A)_{n-d} & \xrightarrow{(-1)^d \partial_{n-d}^{D \otimes_R^{\bowtie} A}} & (D \otimes_R^{\bowtie} A)_{n-1-d} \end{array}$$

which is equivalent to

$$\begin{array}{ccc} \bigoplus_{i=0}^{n} C_{n-i} \otimes_R A_i & \xrightarrow{\partial_n^{C \otimes_R^{\bowtie} A}} & \bigoplus_{i=0}^{n-1} C_{n-1-i} \otimes_R A_i \\ \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \tau_{n-1}} \\ \bigoplus_{i=0}^{n-d} D_{n-d-i} \otimes_R A_i & \xrightarrow{(-1)^d \partial_{n-d}^{D \otimes_R^{\bowtie} A}} & \bigoplus_{i=0}^{n-d-1} D_{n-d-1-i} \otimes_R A_i \end{array}$$

Define

$$\tau_n = \begin{bmatrix} f_n \otimes_R A_0 & & & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & f_d \otimes_R A_{n-d} & 0 & \dots & 0 \end{bmatrix}.$$

Choose $c \otimes a \in C_{n-i} \otimes_R A_i$. Then, when $n - i \geq d$ we have

$$\begin{aligned} \tau_{n-1}\big(\partial_n^{C \otimes_R^{\bowtie} A}(c \otimes a)\big) &= \tau_{n-1}\Big(\partial_{n-i}^{C}(c) \otimes a + (-1)^{n-i} c \otimes \partial_i^A(a)\Big) \\ &= f_{n-i-1}\Big(\partial_{n-i}^{C}(c)\Big) \otimes a + (-1)^{n-i} f_{n-i}(c) \otimes \partial_i^A(a). \end{aligned}$$

Moreover,

$$\begin{aligned} \big((-1)^d \partial_{n-d}^{D \otimes_R^{\bowtie} A}\big)\tau_n(c \otimes a) &= \big((-1)^d \partial_{n-d}^{D \otimes_R^{\bowtie} A}\big)\big(f_{n-i}(c) \otimes a\big) \\ &= (-1)^d \partial_{n-d-i}^{D \otimes_R^{\bowtie} A}(f_{n-i}(c)) \otimes a \\ &= (-1)^d\Big(\partial_{n-d-i}^{D}(f_{n-i}(c)) \otimes a + +(-1)^{n-i} f_{n-i}(c) \otimes \partial_i^A(a). \end{aligned}$$

It is clear that $f_{n-i-1}\partial^C_{n-i}(c) = \partial^D_{n-d-i} f_{n-i}(c)$ since the following diagram commutes.

$$\begin{array}{ccc} C_{n-i} & \xrightarrow{\partial^C_{n-i}} & C_{n-i-1} \\ \Big\downarrow{\scriptstyle f_{n-i}} & & \Big\downarrow{\scriptstyle f_{n-i-1}} \\ D_{n-d-i} & \xrightarrow{\partial^D_{n-d-i}} & D_{n-d-i-1} \end{array}$$

Therefore, $\tau_{n-1}\partial_n^{C\otimes^{\bowtie}_R A} = (-1)^d \partial_{n-d}^{D\otimes^{\bowtie}_R A}\tau_n$.

When $n - i < d$ we have

$$\begin{aligned} \tau_{n-1}\big(\partial_n^{C\otimes^{\bowtie}_R A}(c \otimes a)\big) &= \tau_{n-1}\Big(\partial^C_{n-i}(c) \otimes a + (-1)^{n-i} c \otimes \partial^A_i(a)\Big) \\ &= 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \big((-1)^d \partial_{n-d}^{D\otimes^{\bowtie}_R A}\big)\tau_n(c \otimes a) &= \big((-1)^d \partial_{n-d}^{D\otimes^{\bowtie}_R A}\big)\big(0\big) \\ &= 0. \end{aligned}$$

It is clear that $f_{n-i-1}\partial^C_{n-i}(c) = \partial^D_{n-d-i} f_{n-i}(c)$ since the following diagram commutes.

$$\begin{array}{ccc} C_{n-i} & \xrightarrow{\partial^C_{n-i}} & C_{n-i-1} \\ \Big\downarrow{\scriptstyle f_{n-i}} & & \Big\downarrow{\scriptstyle f_{n-i-1}} \\ D_{n-d-i} & \xrightarrow{\partial^D_{n-d-i}} & D_{n-d-i-1} \end{array}$$

Therefore, $\tau_{n-1}\partial_n^{C\otimes^{\bowtie}_R A} = (-1)^d \partial_{n-d}^{D\otimes^{\bowtie}_R A}\tau_n$.

Now suppose $n = d$. Then

$$\begin{array}{ccc} (C \otimes^{\bowtie}_R A)_n & \xrightarrow{\partial_n^{C\otimes^{\bowtie}_R A}} & (C \otimes^{\bowtie}_R A)_{n-1} \\ {\scriptstyle \tau_n}\Big\downarrow & & \Big\downarrow{\scriptstyle \tau_{n-1}} \\ (D \otimes^{\bowtie}_R A)_0 & \xrightarrow{(-1)^d \partial_0^{D\otimes^{\bowtie}_R A}} & (D \otimes^{\bowtie}_R A)_{-1} \end{array}$$

which is equivalent to

$$\begin{array}{ccc}
\bigoplus_{i=0}^{n} C_{n-i} \otimes_R A_i & \xrightarrow{\partial_n^{C\otimes_R^{\bowtie} A}} & \bigoplus_{i=0}^{n} C_{n-1-i} \otimes_R A_i \\
\downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \tau_{n-1}} \\
D_0 \otimes_R A_0 & \xrightarrow{(-1)^d \partial_0^{D\otimes_R^{\bowtie} A}} & D_{-1} \otimes_R (\Sigma A)_0
\end{array}$$

Where

$$\tau_n = \begin{bmatrix} f_n \otimes_R A_0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

and

$$\tau_{n-1} = \begin{bmatrix} f_{n-1} \otimes_R \sigma_{-1}^A \partial_0^A & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}.$$

Choose $c \otimes a \in C_d \otimes_R A_0$. Then,

$$\begin{aligned}
\tau_{n-1}\big(\partial_n^{C\otimes_R^{\bowtie} A}(c \otimes a)\big) &= \tau_{n-1}\Big(\partial_d^C(c) \otimes a\Big) \\
&= f_{d-1}(\partial_d^C(c)) \otimes \sigma_{-1}^A \partial_0^A(a).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\big((-1)^d \partial_0^{D\otimes_R^{\bowtie} A}\big)\tau_d(c \otimes a) &= \big((-1)^d \partial_0^{D\otimes_R^{\bowtie} A}\big)\Big(f_d(c) \otimes a\Big) \\
&= (-1)^d \partial_0^D(f_d(c)) \otimes \sigma_{-1}^A \partial_0^A(a).
\end{aligned}$$

It is clear that $f_{d-1}\partial_d^C(c) = \partial_0^D f_d(c)$ since the following diagram commutes.

$$\begin{array}{ccc}
C_d & \xrightarrow{\partial_d^C} & C_{d-1} \\
\downarrow{\scriptstyle f_d} & & \downarrow{\scriptstyle f_{d-1}} \\
D_0 & \xrightarrow{\partial_0^D} & D_{-1}
\end{array}$$

Therefore, $\tau_{n-1}\big(\partial_n^{C\otimes^{\bowtie}_R A}\big) = \big((-1)^d \partial_{n-d}^{D\otimes^{\bowtie}_R A}\big)\tau_n$.

$0 < n < d$:

In this case the diagram

$$\begin{array}{ccc}
\bigoplus_{i=0}^{n} C_{n-i}\otimes_R A_i & \xrightarrow{\partial_n^{C\otimes^{\bowtie}_R A}} & \bigoplus_{i=0}^{n} C_{n-1-i}\otimes_R A_i \\
\downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \tau_{n-1}} \\
\bigoplus_{i=n-d}^{-1} D_i\otimes_R (\Sigma A)_{n-d-i} & \xrightarrow{(-1)^d\partial_{n-d}^{D\otimes^{\bowtie}_R A}} & \bigoplus_{i=n-d-1}^{-1} D_i\otimes_R (\Sigma A)_{n-d-1-i}
\end{array}$$

Define $\tau_n = f_{n-i}\otimes(\sigma_i^A)^{-1}\partial_i^A$. Choose $c\otimes a \in C_{n-i}\otimes_R A_i$. Then,

$$\begin{aligned}
\tau_{n-1}\big(\partial_n^{C\otimes^{\bowtie}_R A}(c\otimes a)\big) &= \tau_{n-1}\left(\partial_{n-i}^C(c)\otimes a\right)\\
&= f_{n-i-1}\left(\partial_{n-i}^C(c)\right)\otimes \sigma_{i-1}^A\partial_i^A(a).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\big((-1)^d\partial_{n-d}^{D\otimes^{\bowtie}_R A}\big)\tau_n(c\otimes a) &= \big((-1)^d\partial_{n-d}^{D\otimes^{\bowtie}_R A}\big)\big(f_{n-i}(c)\otimes\sigma_{i-1}^A\partial_i^A(a)\big)\\
&= (-1)^d(\partial_{n-d}^{(D\otimes^{\bowtie}_R A)}(f_{n-i}(c))\otimes\sigma_{i-1}^A\circ\partial_i^A(a).
\end{aligned}$$

It is clear that $f_{n-i-1}\partial_{n-i}^C(c) = \partial_{n-d}^D f_{n-i}(c)$ since the following diagram commutes

$$\begin{array}{ccc}
C_{n-i} & \xrightarrow{\partial_{n-i}^C} & C_{n-i-1}\\
\downarrow{\scriptstyle f_{n-i}} & & \downarrow{\scriptstyle f_{n-i-1}}\\
D_{n-d} & \xrightarrow{\partial_{n-d}^D} & D_{n-d-1}
\end{array}$$

Therefore, $\tau_{n-1}\big(\partial_n^{C\otimes^{\bowtie}_R A}\big) = \big((-1)^d \partial_{n-d}^{D\otimes^{\bowtie}_R A}\big)\tau_n$.

$n \le 0$:

Suppose $n = 0$.

In this case the diagram will be

$$\begin{array}{ccc} C_0 \otimes_R A_0 & \xrightarrow{\partial_0^{C \otimes_R^{\bowtie} A}} & C_{-1} \otimes_R (\Sigma A)_0 \\ \downarrow \tau_0 & & \downarrow \tau_{-1} \\ \bigoplus_{i=-d}^{-1} D_i \otimes_R (\Sigma A)_{-d-i} & \xrightarrow{(-1)^d \partial_{-d}^{D \otimes_R^{\bowtie} A}} & \bigoplus_{i=n-d-1}^{-1} D_i \otimes_R (\Sigma A)_{-d-1-i} \end{array}$$

Define

$$\tau_0 = \begin{bmatrix} f_0 \otimes_R \sigma_{-1}^A \partial_0^A \\ \vdots \\ 0 \end{bmatrix}$$

and,

$$\tau_{-1} = \begin{bmatrix} f_{-1} \otimes_R A \\ \vdots \\ 0 \end{bmatrix}.$$

Choose $c \otimes a \in C_0 \otimes_R A_0$. Then,

$$\begin{aligned} \tau_{-1}\big(\partial_0^{C \otimes_R^{\bowtie} A}(c \otimes a)\big) &= \tau_{-1}\Bigg(\partial_0^C(c) \otimes \sigma_{-1}^A \partial_0^A(a)\Bigg) \\ &= \Bigg(f_{-1}(\partial_0^C(c)) \otimes \sigma_{-1}^A \partial_0^A(a)\Bigg). \end{aligned}$$

Moreover,

$$\begin{aligned} \big((-1)^d \partial_{-d}^{D \otimes_R^{\bowtie} A}\big)\tau_0(c \otimes a) &= \big((-1)^d \partial_{-d}^{D \otimes_R^{\bowtie} A}\big)\Big(f_0(c) \otimes \sigma_{-1}^A \partial_0^A(a)\Big) \\ &= (-1)^d (\partial_i^{D \otimes_R^{\bowtie} A}(f_0(c)) \otimes \sigma_{-1}^A \partial_0^A(a) \\ &= (-1)^d \partial_0^D f_0(c) \otimes \sigma_{-1}^A \partial_0^A(a). \end{aligned}$$

It is clear that $f_{-1}\partial_0^C(c) = \partial_0^D f_0(c)$ since the following diagram commutes

$$\begin{array}{ccc} C_0 & \xrightarrow{\partial_0^C} & C_{-1} \\ \Big\downarrow{\scriptstyle f_0} & & \Big\downarrow{\scriptstyle f_{-1}} \\ D_0 & \xrightarrow{\partial_0^D} & D_{-1} \end{array}$$

Therefore, $\tau_{-1}\big(\partial_0^{C\otimes_R^{\bowtie}A}\big) = \big((-1)^d\partial_{-d}^{D\otimes_R^{\bowtie}A}\big)\tau_0$.

Suppose $n < 0$. In this case the diagram will be

$$\begin{array}{ccc} \bigoplus_{i=n}^{-1} C_i \otimes_R (\Sigma A)_{n-i} & \xrightarrow{\partial_n^{C\otimes_R^{\bowtie}\Sigma A}} & C_{-1} \otimes_R (\Sigma A)_0 \\ \Big\downarrow{\scriptstyle \tau_n} & & \Big\downarrow{\scriptstyle \tau_{n-1}} \\ \bigoplus_{i=n-d}^{-1} D_i \otimes_R (\Sigma A)_{-d-i} & \xrightarrow{(-1)^d\partial_{n-d}^{D\otimes_R^{\bowtie}A}} & \bigoplus_{i=n-d-1}^{-1} D_i \otimes_R (\Sigma A)_{n-d-1-i} \end{array}$$

Define

$$\tau_n = \begin{bmatrix} f_n \otimes_R A_0 & & & \\ & \ddots & & \\ & & f_{-1} \otimes_R A_{n+1} & \\ 0 & \cdots & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}.$$

Then,

$$\begin{aligned} \tau_{n-1}\big(\partial_n^{C\otimes_R^{\bowtie}A}(c\otimes a)\big) &= \tau_{n-1}\Big(\partial_i^C(c)\otimes a + (-1)^{|i|}c\otimes\partial_{n-i}^{\Sigma A}(a)\Big) \\ &= f_{n-i}(\partial_i^C(c))\otimes a + (-1)^i f_i(c)\otimes\partial_{n-i}^{\Sigma A}(a). \end{aligned}$$

Moreover,

$$\begin{aligned} \big((-1)^d\partial_{n-d}^{D\otimes_R^{\bowtie}A}\big)\tau_n(c\otimes a) &= \big((-1)^d\partial_{n-d}^{D\otimes_R^{\bowtie}A}\big)\Big(f_i(c)\otimes a\Big) \\ &= \partial_i^D(f_i(c))\otimes a + (-1)^i f_i(c)\otimes\partial_{n-i}^{\Sigma A}(a). \end{aligned}$$

It is clear that $f_{n-1}\partial_i^C(c) = \partial_i^D f_i(c)$ since the following diagram commutes.

$$\begin{array}{ccc} C_i & \xrightarrow{\partial_i^C} & C_{i-1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} \\ D_i & \xrightarrow{\partial_i^D} & D_{i-1} \end{array}$$

Therefore, $\tau_{n-1}\partial_n^{C\otimes_R^{\bowtie}A} = (-1)^d \partial_{n-d}^{D\otimes_R^{\bowtie}A}\tau_n$.

□

Remark 4.1.11. In Theorem 4.1.10 as far as we know there is no map $\Sigma^d(D\otimes_R^{\bowtie}A) \to (C\otimes_R^{\bowtie}A)$.

Remark 4.1.12. In Theorem 4.1.10, as far as we know, if $d < 0$, then there is no map $(C\otimes_R^{\bowtie}A) \to \Sigma^d(D\otimes_R^{\bowtie}A)$.

4.2 A Quasi-isomorphism

Definition 4.2.1. A chain map $f : C \to D$ is a *quasi-isomorphism* if all of its induced maps $H_n(f) : H_n(C) \to H_n(D)$ are isomorphisms. In this case we write $C \simeq D$.

Theorem 4.2.2. *Let C and D be a complexes of free modules. The map on homology induced by q from Corollary 4.1.4 is an isomorphism. In other words, q from Corollary 4.1.4 is a quasiisomorphism.*

Proof: Recall that $q = \sigma^{(C\otimes_R^{\bowtie}D)}\tau$. Since $\sigma : C \to \Sigma C$ induces an isomorphism in homology for any complex C, it suffices to show that the degree -1 chain map $\tau : \Sigma C\otimes_R^{\bowtie}D \to C\otimes_R^{\bowtie}D$ induces an isomorphism in homology. We consider five cases:

$n = 0$, $n = 1$, $n \geqslant 2$, $n = -1$, and $n \leqslant -2$. It suffices to show that $H_n(\tau)$ is one-to-one and onto for all $n \in \mathbb{Z}$.

$n = 0$:

In this case the diagram is

$$\begin{array}{ccccc} (\Sigma C)_1 \otimes_R D_0 \oplus (\Sigma C)_0 \otimes_R D_1 & \xrightarrow{\partial_1^{\Sigma C \otimes_R^{\bowtie} D}} & (\Sigma C)_0 \otimes_R D_0 & \xrightarrow{\partial_0^{\Sigma C \otimes_R^{\bowtie} D}} & (\Sigma C)_{-1} \otimes_R (\Sigma D)_0 \\ \downarrow \tau_1 & & \downarrow \tau_0 & & \downarrow \tau_{-1} \\ C_0 \otimes_R D_0 & \xrightarrow{\partial_0^{C \otimes_R^{\bowtie} D}} & C_{-1} \otimes_R (\Sigma D)_0 & \xrightarrow{\partial_{-1}^{C \otimes_R^{\bowtie} D}} & C_{-2} \otimes_R (\Sigma D)_0 \oplus C_{-1} \otimes_R (\Sigma D)_{-1} \end{array}$$

We first show that $H_0(\tau)$ is one-to-one. Assume $a \in \ker \partial_0^{\Sigma C \otimes_R^{\bowtie} D}$ such that $\tau_0(a) \in \operatorname{Im} \partial_0^{C \otimes_R^{\bowtie} D}$. Choose $z \in C_0 \otimes_R D_0$ such that $\partial_0^{C \otimes_R^{\bowtie} D}(z) = \tau_0(a)$.
Then $\tau_1\left(\left(\sigma_0^C \otimes D_0\right)(z), 0\right) = z$. Consider $\partial_1^{\Sigma C \otimes_R^{\bowtie} D}\left(\left(\sigma_0^C \otimes D_0\right)(z), 0\right) + a$.

$$\begin{aligned} \tau_0\left(\partial_1^{\Sigma C \otimes_R^{\bowtie} D}\Big((\sigma_0^C \otimes D_0)(z), 0\Big) + a\right) &= (\tau_0 \partial_1^{\Sigma C \otimes_R^{\bowtie} D})\left(\left(\sigma_0^C \otimes D_0\right)(z), 0\right) + \tau_0(a) \\ &= -(\partial_0^{C \otimes_R^{\bowtie} D} \tau_1)\left(\left(\sigma_0^C \otimes D_0\right)(z), 0\right) + \tau_0(a) \\ &= -\partial_0^{C \otimes_R^{\bowtie} D}(z) + \tau_0(a) \\ &= 0. \end{aligned}$$

Therefore,
$\partial_1^{\Sigma C \otimes_R^{\bowtie} D}\left(\left(\sigma_0^C \otimes D_0\right)(z), 0\right) + a \in \ker(\tau_0)$. Since

$$(\Sigma C)_0 \otimes_R D_1 \xrightarrow{(\Sigma C)_0 \otimes \partial_1^D} (\Sigma C)_0 \otimes_R D_0 \xrightarrow{\tau_0} C_{-1} \otimes_R (\Sigma D)_0$$

is exact, we have $\partial_1^{\Sigma C \otimes_R^{\bowtie} D}\Big(((\sigma_0^C) \otimes D_0)(z), 0)\Big) + a \in \operatorname{Im}\Big((\Sigma C)_0 \otimes \partial_1^D\Big)$.
Write $\partial_1^{\Sigma C \otimes_R^{\bowtie} D}\Big((\sigma_0^C \otimes D_0)(z), 0)\Big) + a = \Big((\Sigma C)_0 \otimes \partial_1^D\Big)(b)$ for some $b \in (\Sigma C)_0 \otimes_R D_1$.

However,

$$\begin{aligned}\partial_1^{\Sigma C\otimes_R^{\bowtie} D}\left(\left(\left(\sigma_0^C \otimes D_0\right)(-z), b\right)\right) &= -\partial_1^{\Sigma C\otimes_R^{\bowtie} D}\left(\left(\left(\sigma_0^C \otimes D_0\right)(z), 0\right)\right) + \partial_1^{\Sigma C\otimes_R^{\bowtie} D}\left((0,b)\right)\\ &= -\partial_1^{\Sigma C\otimes_R^{\bowtie} D}\left(\left(\left(\sigma_0^C \otimes D_0\right)(z), 0\right)\right)\\ &\qquad + \partial_1^{\Sigma C\otimes_R^{\bowtie} D}\left(\left(\left(\sigma_0^C \otimes D_0\right)(z), 0\right)\right) + a\\ &= a.\end{aligned}$$

Thus a is a boundary, therefore $H_0(\tau)$ is one-to-one.

Now we want want to show $H_0(\tau)$ is onto. Consider $a \in \ker \partial_{-1}^{C\otimes_R^{\bowtie} D}$. Write $\partial_{-1}^{C\otimes_R^{\bowtie} D}(a) = (b,z)$ where $b \in C_{-2}\otimes_R(\Sigma D)_0$ and $z \in C_{-1}\otimes_R(\Sigma D)_{-1}$. Therefore, $\partial_{-1}^{C\otimes_R^{\bowtie} D}(a) = (b,z) = 0$, which implies $z = 0$. Then, $z = (C_{-1}\otimes \partial_0^{\Sigma D})(a) = 0$. Since

$$C_{-1}\otimes_R(\Sigma D)_1 \xrightarrow{C_{-1}\otimes\partial_1^{\Sigma D}} C_{-1}\otimes_R(\Sigma D)_0 \xrightarrow{C_{-1}\otimes\partial_0^{\Sigma D}} C_{-1}\otimes_R(\Sigma D)_{-1}$$

is exact, we have $a \in \ker(C_{-1}\otimes\partial_0^{\Sigma D}) = \operatorname{Im}(C_{-1}\otimes\partial_1^{\Sigma D})$. Therefore there exists $k \in C_{-1}\otimes(\Sigma D)_1$ such that $(C_{-1}\otimes\partial_1^{\Sigma D})(k) = a$. Then $\tau_0\left(\left(\sigma_{-1}^C \otimes (\sigma_0^D)^{-1}\right)(k)\right) = a$. Finally we note that
$\tau_{-1}\partial_0^{\Sigma C\otimes_R^{\bowtie} D}\left(\left(\sigma_{-1}^C\otimes(\sigma_0^D)^{-1}\right)(k)\right) = -\partial_{-1}^{C\otimes_R^{\bowtie} D}\tau_0\left(\sigma_{-1}^C\otimes(\sigma_0^D)^{-1}(k)\right) = -\partial_{-1}^{C\otimes_R^{\bowtie} D}(a) = 0$. Since τ_{-1} is one-to-one we have $\partial_0^{\Sigma C\otimes_R^{\bowtie} D}\left(\left(\sigma_{-1}^C\otimes(\sigma_0^D)^{-1}\right)(k)\right) = 0$, therefore $\left((\sigma_{-1}^C)\otimes(\sigma_0^D)^{-1}\right)(k)$ is a cycle and therefore $H_0(\tau)$ is onto.

$n = 1$:

In this case the diagram is

$$\begin{array}{ccccc}
\bigoplus_{i=0}^{2}(\Sigma C)_{2-i} \otimes_R D_i & \xrightarrow{\partial_2^{\Sigma C \otimes_R^{\bowtie} D}} & \bigoplus_{i=0}^{1}(\Sigma C)_{1-i} \otimes_R D_i & \xrightarrow{\partial_1^{\Sigma C \otimes_R^{\bowtie} D}} & (\Sigma C)_0 \otimes_R D_0 \\
\downarrow \tau_2 & & \downarrow \tau_1 & & \downarrow \tau_0 \\
C_1 \otimes_R D_0 \oplus C_0 \otimes_R D_1 & \xrightarrow{\partial_1^{C \otimes_R^{\bowtie} D}} & C_0 \otimes_R D_0 & \xrightarrow{\partial_0^{C \otimes_R^{\bowtie} D}} & C_{-1} \otimes_R (\Sigma D)_0
\end{array}$$

which is anti-commutative by Theorem 4.1.3. We first want to show $H_1(\tau)$ is one-to-one. Assume $(a,b) \in \ker \partial_1^{\Sigma C \otimes_R^{\bowtie} D}$ such that $\tau_1\Big((a,b)\Big) \in \operatorname{Im} \partial_1^{C \otimes_R^{\bowtie} D}$. Choose $(c,d) \in C_1 \otimes_R D_0 \oplus C_0 \otimes_R D_1$ such that $\partial_1^{C \otimes_R^{\bowtie} D}\Big((c,d)\Big) = \tau_1\Big((a,b)\Big)$.

Then $\tau_2\Bigg(\Big(\big(\sigma_1^C \otimes D_0\big)(c), \big(\sigma_2^C \otimes D_1\big)(d), 0\Big)\Bigg) = (c,d)$.

Consider $\partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Bigg(\Big((\sigma_1^C \otimes D_0)(c), (\sigma_2^C \otimes D_1)(d), 0\Big)\Bigg) + (a,b)$. Then,

$$\begin{aligned}
\tau_1\Bigg(\partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Big(\Big((\sigma_1^C \otimes D_0)(c), (\sigma_2^C \otimes D_1)(d), 0\Big)\Big) + (a,b)\Bigg) \\
&= (\tau_1 \partial_2^{\Sigma C \otimes_R^{\bowtie} D})\Big(\Big((\sigma_1^C \otimes D_0)(c), (\sigma_2^C \otimes D_1)(d), 0\Big)\Big) + \tau_1\Big((a,b)\Big) \\
&= -(\partial_1^{C \otimes_R^{\bowtie} D} \tau_2)\Big(\Big((\sigma_1^C \otimes D_0)(c), (\sigma_2^C \otimes D_1)(d), 0\Big)\Big) + \tau_1\Big((a,b)\Big) \\
&= -\partial_1^{C \otimes_R^{\bowtie} D}\Big((c,d)\Big) + \tau_1\Big((a,b)\Big) \\
&= 0.
\end{aligned}$$

Therefore $\Bigg(\partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Big(\Big((\sigma_1^C \otimes D_0)(c), (\sigma_2^C \otimes D_1)(d), 0\Big)\Big) + (a,b)\Bigg) \in \ker(\tau_1) = (\Sigma C)_0 \otimes_R D_1$.

Then we have

$\partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Big(\Big((\sigma_1^C \otimes D_0)(c), (\sigma_2^C \otimes D_1)(d), 0\Big)\Big) + (a,b) \in \ker\Big((\Sigma C)_0 \otimes_R D_1 \xrightarrow{(\Sigma C)_0 \otimes \partial_1^D} (\Sigma C)_0 \otimes_R D_0\Big)$

Thus gives,

$\partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Big((\sigma_1^C \otimes D_0)(c), (\sigma_2^C \otimes D_1)(d), 0\Big) + (a,b) \in \operatorname{Im}\big((\Sigma C)_0 \otimes_R D_2 \xrightarrow{(\Sigma C)_0 \otimes_R \partial_2^D} (\Sigma C)_0 \otimes$

D_1).

Since

$$(\Sigma C)_0 \otimes_R D_2 \xrightarrow{(\Sigma C)_0 \otimes \partial_2^D} (\Sigma C)_0 \otimes_R D_1 \xrightarrow{(\Sigma C)_0 \otimes \partial_1^D} (\Sigma C)_0 \otimes_R D_0$$

is exact. Write $\partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Big(\big((\sigma_1^C \otimes D_0)(c), (\sigma_2^C \otimes D_1)(d), 0\big)\Big) + (a,b) = \Big((\Sigma C)_0 \otimes \partial_2^D\Big)(k)$ for some $k \in (\Sigma C)_0 \otimes_R D_2$. However,

$\partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Big(\big((\sigma_1^C \otimes D_0)(-c), (\sigma_2^C \otimes D_1)(-d), k\big)\Big)$
$= \partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Big((\sigma_1^C \otimes D_0)(-c), (-d), 0)\Big) + \partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Big((0,0,k)\Big)$
$= -\partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Big((\sigma_1^C \otimes D_0)(c), (d), 0)\Big) + \Bigg(\partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Big((\sigma_1^C \otimes D_0)(c), (\sigma_2^C \otimes D_1)(d), 0)\big) + (a,b)\Bigg)$
$= (a,b)$.

Thus $\partial_2^{\Sigma C \otimes_R^{\bowtie} D}\Big((\sigma_1^C \otimes D_0)(c), (\sigma_2^C \otimes D_1)(d), k\Big) = (a,b)$. Therefore, (a,b) is a boundary. Thus $H_1(\tau)$ is one-to-one.

Now we want want to show $H_1(\tau)$ is onto. Consider $a \in \ker \partial_0^{C \otimes_R^{\bowtie} D}$. Then, $\partial_0^{C \otimes_R^{\bowtie} D}(a) = 0$ and $\tau_1\Big((\sigma_0^C \otimes D_0)(a), 0\Big) = a$. Finally note that

$$\begin{aligned}
\tau_0 \partial_1^{\Sigma C \otimes_R^{\bowtie} D}\Big(\big(\sigma_0^C \otimes D_0\big)(a), 0\Big) &= \tau_0\Big((\partial_{-1}^{\Sigma C} \otimes D_0)\Big((\sigma_0^C \otimes D_0)(a)\Big)\Big) \\
&= \Big((\sigma_{-1}^C)^{-1} \otimes \sigma_{-1}^D \partial_0^D\Big)(\partial_{-1}^{\Sigma C} \otimes D_0)\Big((\sigma_0^C \otimes D_0)(a)\Big) \\
&= 0.
\end{aligned}$$

Since

$$(\Sigma C)_0 \otimes_R D_1 \xrightarrow{(\Sigma C)_0 \otimes \partial_1^D} (\Sigma C)_0 \otimes_R D_0 \xrightarrow{\tau_0} C_{-1} \otimes_R (\Sigma D)_0$$

is exact. There exists $k \in (\Sigma C)_0 \otimes_R D_1$ such that $\Big((\Sigma C)_0 \otimes \partial_1^D\Big)(k) = (\partial_{-1}^{\Sigma C} \otimes D_0)((\sigma_0^{\Sigma C} \otimes D_0)(a))$. Consider now $((\sigma_0^{\Sigma C} \otimes D_0)(a), k) \in \bigoplus_{i=0}^1 (\Sigma C)_{1-i} \otimes_R D_i$. We have $\tau_1((\sigma_0^{\Sigma C} \otimes D_0)(a), k) = a$. Finally we note that

$$\begin{aligned}
\tau_0 \partial_1^{\Sigma C \otimes_R^{\bowtie} D}\Big(\big(\sigma_0^C \otimes (\sigma_1^D)^{-1}\big)(a), k\Big) &= -\partial_0^{C \otimes_R^{\bowtie} D} \tau_1\Big(\big(\sigma_0^C \otimes (\sigma_1^D)^{-1}\big)(a), k\Big) \\
&= -\partial_0^{C \otimes_R^{\bowtie} D}(a) \\
&= 0.
\end{aligned}$$

Since τ_0 is one-to-one we have $\partial_1^{\Sigma C\otimes_R^{\bowtie} D}\left(\left(\sigma_0^C \otimes (\sigma_1^D)^{-1}\right)(a), k\right) = 0$, so $\left(\left(\sigma_0^C \otimes (\sigma_1^D)^{-1}\right)(a), k\right)$ is a cycle. Therefore $H_1(\tau)$ is onto.

$n \geqslant 2$:

We have the diagram, which is anti commutative by Theorem 4.1.3:

$$\begin{array}{ccccc}
\bigoplus_{i=0}^{n+1}(\Sigma C)_{n+1-i} \otimes_R D_i & \xrightarrow{\partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}} & \bigoplus_{i=0}^{n}(\Sigma C)_{n-i} \otimes_R D_i & \xrightarrow{\partial_{n}^{\Sigma C\otimes_R^{\bowtie} D}} & \bigoplus_{i=0}^{n-1}(\Sigma C)_{n-1-i} \otimes_R D_i \\
\downarrow \tau_{n+1} & & \downarrow \tau_n & & \downarrow \tau_{n-1} \\
\bigoplus_{i=0}^{n} C_{n-i} \otimes_R D_i & \xrightarrow{\partial_{n}^{C\otimes_R^{\bowtie} D}} & \bigoplus_{i=0}^{n-1} C_{n-1-i} \otimes_R D_i & \xrightarrow{\partial_{n-1}^{C\otimes_R^{\bowtie} D}} & \bigoplus_{i=0}^{n-2} C_{n-2-i} \otimes_R D_i
\end{array}$$

First we want to show $H_n(\tau)$ is one-to-one. Assume $K = (k_1, k_2, \ldots, k_{n+1}) \in \ker \partial_n^{\Sigma C\otimes_R^{\bowtie} D}$ such that $\tau_n(K) \in \operatorname{Im} \partial_n^{C\otimes_R^{\bowtie} D}$. Choose $Z = (z_0, z_1, \ldots, z_n) \in \bigoplus_{i=0}^{n} C_{n-i} \otimes_R D_i$ such that $\partial_n^{C\otimes_R^{\bowtie} D}(Z) = \tau_n(K)$. Let $Z' = (((\sigma_n^C) \otimes D_0)(z_0), ((\sigma_{n-1}^C) \otimes D_1)(z_1), \ldots, ((\sigma_0^C) \otimes D_n)(z_n), 0)$. Then $\tau_{n+1}(Z') = Z$. Consider $\partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}(Z') + K \in \bigoplus_{i=0}^{n}(\Sigma C)_{n-i} \otimes_R D_i$

Then

$$\begin{aligned}
\tau_n(\partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}(Z') + K) &= (\tau_n \partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D})(Z') + \tau_n(K) \\
&= -(\partial_n^{C\otimes_R^{\bowtie} D} \tau_{n+1})(Z') + \tau_n(K) \\
&= -\partial_n^{C\otimes_R^{\bowtie} D}(Z) + \tau_n(K) \\
&= 0
\end{aligned}$$

Therefore $\partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}(Z') + K \in \ker(\tau_n) = (\Sigma C)_0 \otimes_R D_n$. Since

$$(\Sigma C)_0 \otimes_R D_{n+1} \xrightarrow{(\Sigma C)_0 \otimes \partial_{n+1}^D} (\Sigma C)_0 \otimes_R D_n \xrightarrow{(\Sigma C)_0 \otimes \partial_n^D} (\Sigma C)_0 \otimes_R D_{n-1}$$

is exact, we have $\partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}(Z') + K \in \operatorname{Im}((\Sigma C)_0 \otimes \partial_{n+1}^D)$. Write $\partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}(Z') + K = \left((\Sigma C)_0 \otimes \partial_{n+1}^D\right)(b)$ for some $b \in (\Sigma C)_0 \otimes_R D_{n+1}$ and write $Z'' = \left(-(\sigma_n^C \otimes D_0)(z_0), -(\sigma_{n-1}^C \otimes D_1)(z_1), \ldots, -(\sigma_0^C \otimes D_n)(z_n), b\right) = (-z_0', -z_1', \ldots, -z_n', b)$ where $z_i' = \sigma_{n-i}^C \otimes D_{i-1}$. However,

$$
\begin{aligned}
\partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}(Z'') &= \partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}((-z_0', -z_1', \dots, -z_n', 0)) + \partial_n^{\Sigma C\otimes_R^{\bowtie} D}((0, \cdots, b)) \\
&= -\partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}((z_0', z_1', \dots, z_n', 0)) + \partial_n^{\Sigma C\otimes_R^{\bowtie} D}((0, \cdots, b)) \\
&= -\partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}((z_0', z_1', \dots, z_n', 0)) + \Big(\partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}(Z') + K\Big) \\
&= K.
\end{aligned}
$$

Therefore $\partial_{n+1}^{\Sigma C\otimes_R^{\bowtie} D}(Z'') = K$. Thus K is a boundary. Therefore, $H_n(\tau)$ is one-to-one.

Now we want want to show $H_n(\tau)$ is onto. Let $Z = (z_0, z_1, \dots, z_{n-1}) \in \ker \partial_{n-1}^{C\otimes_R^{\bowtie} D}$. Consider $Y = (z_0, z_1, \dots, z_{n-1}, 0) \in \bigoplus_{i=0}^{n+1} (\Sigma C)_{n-i} \otimes_R D_i$. we have $\tau_n(Y) = Z$. Therefore $\tau_{n-1}\partial_n^{\Sigma C\otimes_R^{\bowtie} D}(Y) = 0$. Since $\ker \tau_{n-1} = (\Sigma C)_0 \otimes_R D_{n-1}$, we have $\partial_n^{\Sigma C\otimes_R^{\bowtie} D}(Y) \in (\Sigma C)_0 \otimes_R D_{n-1}$. Now $0 = \partial_{n-1}^{\Sigma C\otimes_R^{\bowtie} D}\partial_n^{\Sigma C\otimes_R^{\bowtie} D}(Y) = \Big((\Sigma C)_0 \otimes \partial_{n-1}^D\Big)(\partial_n^{\Sigma C\otimes_R^{\bowtie} D}(Y))$. Since

$$(\Sigma C)_0 \otimes_R D_n \xrightarrow{(\Sigma C)_0\otimes\partial_n^D} (\Sigma C)_0 \otimes_R D_{n-1} \xrightarrow{(\Sigma C)_0\otimes\partial_{n-1}^D} (\Sigma C)_0 \otimes_R D_{n-2}$$

is exact and $\partial_n^{\Sigma C\otimes_R^{\bowtie} D}(Y) \in \ker\Big((\Sigma C)_0 \otimes \partial_{n-1}^D\Big) = \operatorname{Im}\Big((\Sigma C)_0 \otimes \partial_n^D\Big)$. Write $\partial_n^{\Sigma C\otimes_R^{\bowtie} D}(Y) = \Big((\Sigma C)_0\otimes\partial_n^D\Big)(k)$ for some k $\in (\Sigma C)_0\otimes_R D_n$. Then $Y-(0, 0, \dots, k) = (z_0, z_1, \dots, z_{n-1}, -k)$ maps to 0 under $\partial_n^{\Sigma C\otimes_R^{\bowtie} D}$, and therefore is a cycle, and $\tau_n\Big((z_1, z_2, \cdots, -k)\Big) = Z$. Thus $H_n(\tau)$ is onto.

$n = -1$:

We have the diagram,

$$
\begin{array}{ccccc}
(\Sigma C)_0 \otimes_R D_0 & \xrightarrow{\partial_0^{\Sigma C\otimes_R^{\bowtie} D}} & (\Sigma C)_{-1} \otimes_R (\Sigma D)_0 & \xrightarrow{\partial_{-1}^{\Sigma C\otimes_R^{\bowtie} D}} & \bigoplus_{i=-2}^{-1} (\Sigma C)_i \otimes_R (\Sigma D)_{-2-i} \\
\downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_{-1}} & & \downarrow{\scriptstyle \tau_{-2}} \\
C_{-1} \otimes_R (\Sigma D)_0 & \xrightarrow{\partial_{-1}^{C\otimes_R^{\bowtie} D}} & \bigoplus_{i=-2}^{-1} C_i \otimes_R (\Sigma D)_{-2-i} & \xrightarrow{\partial_{-2}^{C\otimes_R^{\bowtie} D}} & \bigoplus_{i=-3}^{-1} C_i \otimes_R (\Sigma D)_{-3-i}
\end{array}
$$

which is anti-commutative by Theorem 4.1.3. We first want to show that $H_{-1}(\tau)$ is one-to-one. Assume $a \in \ker \partial_{-1}^{\Sigma C\otimes_R^{\bowtie} D}$ such that $\tau_{-1}(a) \in \operatorname{Im} \partial_{-1}^{C\otimes_R^{\bowtie} D}$. Choose $c \in C_{-1} \otimes_R (\Sigma D)_0$ such that $\partial_{-1}^{C\otimes_R^{\bowtie} D}(c) = \tau_{-1}(a)$. It follows that $(C_{-1} \otimes_R \partial_0^{\Sigma D})(c) = 0$ since

$$C_{-1} \otimes_R (\Sigma D)_1 \xrightarrow{C_{-1}\otimes_R\partial_1^{\Sigma D}} C_{-1} \otimes_R (\Sigma D)_0 \xrightarrow{C_{-1}\otimes\partial_0^{\Sigma D}} C_{-1} \otimes (\Sigma D)_{-1}$$

is exact. Then there exists $c \in \ker C_{-1} \otimes_R \partial_1^{\Sigma D} = \mathrm{Im}\,(C_{-1} \otimes_R \sigma_{-1}^{D} \partial_0^{\Sigma D}) = \mathrm{Im}\,(\tau_0)$. Therefore there exists $c' \in (\Sigma C)_0 \otimes_R D_0$ such that $\tau_0(c') = c$. Consider $\partial_0^{\Sigma C \otimes_R^{\bowtie} D}(c') + a$, then we have,

$$\begin{aligned} \tau_{-1}\Big(\partial_0^{\Sigma C\otimes_R^{\bowtie} D}(c') + a\Big) &= (\tau_{-1}\partial_0^{\Sigma C\otimes_R^{\bowtie} D})(c') + \tau_{-1}(a) \\ &= -\partial_{-1}^{C\otimes_R^{\bowtie} D}\Big(\tau_0(c')\Big) + \tau_{-1}(a) \\ &= -\partial_{-1}^{C\otimes_R^{\bowtie} D}(c) + \tau_{-1}(a) \\ &= 0. \end{aligned}$$

Thus, $\partial_0^{\Sigma C\otimes_R^{\bowtie} D}(c') + a \in \ker(\tau_{-1}) = 0$.

Therefore, $a = \partial_0^{\Sigma C\otimes_R^{\bowtie} D}(-c')$. Thus $a \in \mathrm{Im}\,\partial_0^{\Sigma C\otimes_R^{\bowtie} D}$. Thus D is a boundary, and therefore $H_{-1}(\tau)$ is one-to-one.

Now we want to show that $H_{-1}(\tau)$ is onto. let $(a, b) \in \ker \partial_{-2}^{C\otimes_R^{\bowtie} D}$. Then, $(C_{-1} \otimes \partial_{-1}^{\Sigma D})(b) = 0$. Since

$$C_{-1} \otimes_R (\Sigma D)_0 \overset{C_{-1}\otimes_R \partial_0^{\Sigma D}}{\longrightarrow} C_{-1} \otimes_R (\Sigma D)_{-1} \overset{C_{-1}\otimes \partial_{-1}^{\Sigma D}}{\longrightarrow} C_{-1} \otimes (\Sigma D)_{-2}$$

is exact, we have $\ker(C_{-1} \otimes \partial_{-1}^{\Sigma D}) = \mathrm{Im}\,(C_{-1} \otimes \partial_0^{\Sigma D})$. Thus there exists $c \in C_{-1} \otimes_R (\Sigma D)_0$ such that $\Big(C_{-1} \otimes \partial_0^{\Sigma D}\Big)(c) = b$. However,

$$\begin{aligned} \partial_{-1}^{C\otimes_R^{\bowtie} D}(c) &= \Big((\partial_{-1}^{C} \otimes (\Sigma D)_0)(c), (C_{-1} \otimes \partial_0^{\Sigma D})(c)\Big) \\ &= \Big((\partial_{-1}^{C} \otimes (\Sigma D)_0)(c), b\Big). \end{aligned}$$

Therefore,

$[(a, b)] = [(a, b) - \Big(\partial_{-1}^{C} \otimes \Sigma D)(c), b\Big)] = [(a - (\partial_{-1}^{C} \otimes \Sigma D)(c), 0]$ as cosets modulo the $\mathrm{Im}\,(C_{-1} \otimes \partial_0^{\Sigma D})$. Consider $Z = \Big((\sigma^{C}{}_{-2})^{-1} \otimes_R (\Sigma D)\Big)(a - (\partial_{-1}^{C} \otimes_R \Sigma D)(c) \in (\Sigma C)_{-1} \otimes_R (\Sigma D)_0$. We have $[\tau_{-1}(z)] = [(a, b)]$, and $\partial_{-1}^{\Sigma C\otimes_R^{\bowtie} D}(z) = 0$ since $\tau_{-2}\partial_{-1}^{\Sigma C\otimes_R^{\bowtie} D}(z) = 0$ and τ_{-2} is one-to-one. Thus z is a cycle and so $H_{-1}(\tau)$ is onto.

$n \leqslant -2$:

We have the diagram,

$$\begin{array}{ccccc}
\bigoplus_{i=n+1}^{-1}(\Sigma C)_i \otimes_R (\Sigma D)_{n+1-i} & \xrightarrow{\partial_{n+1}^{\Sigma C \otimes_R^{\bowtie} D}} & \bigoplus_{i=n}^{-1}(\Sigma C)_i \otimes_R (\Sigma D)_{n-i} & \xrightarrow{\partial_{n}^{\Sigma C \otimes_R^{\bowtie} D}} & \bigoplus_{i=n-1}^{-1}(\Sigma C)_i \otimes_R (\Sigma D)_{n-1-i} \\
\downarrow \tau_{n+1} & & \downarrow \tau_n & & \downarrow \tau_{n-1} \\
\bigoplus_{i=n}^{-1} C_i \otimes_R (\Sigma D)_{n-i} & \xrightarrow{\partial_{n}^{C \otimes_R^{\bowtie} D}} & \bigoplus_{i=n-1}^{-1} C_i \otimes_R (\Sigma D)_{n-1-i} & \xrightarrow{\partial_{n-1}^{C \otimes_R^{\bowtie} D}} & \bigoplus_{i=n-2}^{-1} C_i \otimes_R (\Sigma D)_{n-2-i}
\end{array}$$

which is anticommutative by Theorem 4.1.3. First we want to show $H_n(\tau)$ is one-to-one. Assume $K = (k_n, k_{n+1}, \ldots, k_{-1}) \in \ker \partial_n^{\Sigma C \otimes_R^{\bowtie} D}$ such that $\tau_n(K) \in \operatorname{Im} \partial_n^{C \otimes_R^{\bowtie} D}$. Choose $Z = (c_n, c_{n+1}, \ldots, c_{-1}) \in \bigoplus_{i=n}^{-1} C_i \otimes_R (\Sigma D)_{n-i}$ such that $\partial_n^{C \otimes_R^{\bowtie} D}(Z) = \tau_n(K)$. Then, $(C_{-1} \otimes \partial_{n-1}^{\Sigma D})(c_{-1}) = 0$. Therefore, $c_{-1} \in \ker(C_{-1} \otimes \partial_{n-1}^{\Sigma D}) = \operatorname{Im}(C_{-1} \otimes \partial_n^{\Sigma D})$ since

$$C_{-1} \otimes_R (\Sigma D)_n \overset{C_{-1} \otimes \partial_n^{\Sigma D}}{\longrightarrow} C_{-1} \otimes_R (\Sigma D)_{n-1} \overset{C_{-1} \otimes \partial_{n-1}^{\Sigma D}}{\longrightarrow} C_{-1} \otimes_R (\Sigma D)_{n-2}$$

is exact. Choose c' such that $(C_{-1} \otimes \partial_n^{\Sigma D})(c') = c_{-1}$. Then $\partial_n^{C \otimes_R^{\bowtie} D}\Big(Z - \partial_{n+1}^{\Sigma C \otimes_R^{\bowtie} D}(c')\Big) = \partial_n^{C \otimes_R^{\bowtie} D}(Z) = \tau_n(K)$. Therefore we can assume $Z = (c_n, c_{n+1}, \ldots, c_{-2}, 0)$ and choose $Z' = \Big(\big((\sigma_n^C)^{-1} \otimes D_0\big)(c_0), \big((\sigma_{n-1}^C)^{-1} \otimes D_1\big)(c_1), \ldots, \big((\sigma_0^C)^{-1} \otimes D_{n-1}\big)(c_{n-1})\Big) \in \bigoplus_{i=n+1}^{-1}(\Sigma C)_i \otimes_R (\Sigma D)_{n+1-i}$. Then $\tau_{n+1}(Z') = Z$. Now consider $\partial_{n+1}^{\Sigma C \otimes_R^{\bowtie} D}(C') + K \in \bigoplus_{i=n}^{-1}(\Sigma C)_i \otimes_R (\Sigma D)_{n-i}$. Then

$$\begin{aligned}
\tau_n(\partial_{n+1}^{\Sigma C \otimes_R^{\bowtie} D}(Z') + K) &= (\tau_n \partial_{n+1}^{\Sigma C \otimes_R^{\bowtie} D})(Z') + \tau_n(K) \\
&= -(\partial_n^{C \otimes_R^{\bowtie} D} \tau_{n+1})(Z') + \tau_n(K) \\
&= -\partial_n^{C \otimes_R^{\bowtie} D}(C) + \tau_n(K) \\
&= 0.
\end{aligned}$$

Therefore, $\partial_{n+1}^{\Sigma C \otimes_R^{\bowtie} D}(Z') + K \in \ker(\tau_n) = 0$, and so $K = \partial_{n+1}^{\Sigma C \otimes_R^{\bowtie} D}(-Z')$, which implies that $K \in \operatorname{Im} \partial_{n+1}^{(\Sigma C \otimes_R^{\bowtie} D)}$. Thus K is a boundary, and so $H_n(\tau)$ is one-to-one.

Now want to show $H_n(\tau)$ is onto. Let $K = (k_{n-1}, k_n, \ldots, k_{-1}) \in \ker \partial_{n-1}^{C \otimes_R^{\bowtie} D}$. Since

$$C_{-1} \otimes_R (\Sigma D)_n \overset{C_{-1} \otimes \partial_n^{\Sigma D}}{\longrightarrow} C_{-1} \otimes_R (\Sigma D)_{n-1} \overset{C_{-1} \otimes \partial_{n-1}^{\Sigma D}}{\longrightarrow} C_{-1} \otimes_R (\Sigma D)_{n-2}$$

is exact, we have $\ker(C_{-1} \otimes \partial_{n-1}^{\Sigma D}) = \operatorname{Im}(C_{-1} \otimes \partial_n^{\Sigma D})$. Thus there exists $c \in C_{-1} \otimes_R (\Sigma D)_n$ such that $\left(C_{-1} \otimes \partial_n^{\Sigma D}\right)(c) = k_{-1}$. Define $Z = (0, \ldots, 0, c) \in \bigoplus_{i=n}^{-1} C_i \otimes_R (\Sigma D)_{n-i}$. Then, $K - \partial_n^{C \otimes_R^{\bowtie} D}(Z) = \left(k_{n-1}, \ldots, k_{-2} - \left(\partial_{-1}^C \otimes (\Sigma D)_{n+1}\right)(c), 0\right)$. Therefore, $[K] = [\left(k_{n-1}, \ldots, k_{-2} - \left(\partial_{-1}^C \otimes (\Sigma D)_{-1}\right)(c), 0\right)]$. Therefore, $[K] = [\tau_n\left(k_{n-1}, \ldots, k_{-2} - \left(\partial_{n+1}^C \otimes (\Sigma D)_{-1}\right)(c)\right)]$ and $\left(k_{n-1}, \ldots, k_{-2} - \left(\partial_{n+1}^C \otimes (\Sigma D)_{-1}\right)(c)\right) \in \ker(\partial_n^{\Sigma C \otimes_R^{\bowtie} D})$. Therefore, $H_n(\tau)$ is onto.

□

Theorem 4.2.3. *Let C and D be a complexes of free modules. The map on homology induced by q' from Corollary 4.1.7 is an isomorphism. In other words, q' from Corollary 4.1.7 is a quasiisomorphism.*

Proof: The proof is similar to that of Theorem 4.2.2.

□

Chapter 5

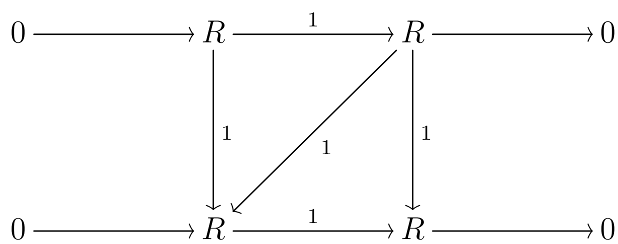
Pinched Homotopy and Cones

In this chapter we examine two other properties: the pinched homotopy and the pinched mapping cone. We see that the pinched tensor product is a functor from the category of complexes, but we cannot extend to it a functor on the homotopy categories. Also, we give some counterexamples for the relevant properties that no longer hold for the pinched tensor product. In addition we show that the isomorphism that implies the mapping cone commutes with tensor product for the ordinary tensor product no longer holds for the pinched tensor product. However, we show there is a morphism.

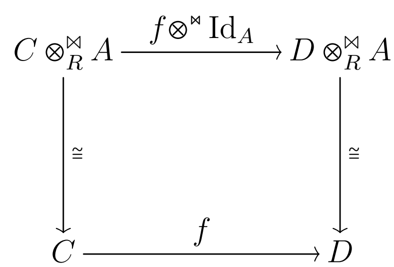
5.1 Homotopy

Remark 5.1.1. In Theorem 2.6.2 if we replace the ordinary tensor product with the pinched tensor product, then there is no map going from $(C \otimes_R^{\bowtie} A)_{-1} = C_{-1} \otimes_R (\Sigma A)_0 \to (D \otimes_R^{\bowtie} B)_0 = D_0 \otimes_R B_0$. That means the pinched tensor product is a functor from the category of complexes to complexes, but we cannot extend to it a functor on the homotopy categories, as we can see in the next example.

Example 5.1.2. Let C and D be complexes and $f : C \to D$ a morphisim such that $f \nsim 0$. Let $A = 0 \to R \to R \to 0$ where the degrees of the $R's$ are 0 and -1, respectively and $g : A \to A$ the identity map. Then clearly we see that $g \sim 0$ from the diagram



Also, by Proposition 3.2.9 we have $C \otimes_R^{\bowtie} A \cong C$ and $D \otimes_R^{\bowtie} A \cong D$. Then $f \otimes^{\bowtie} g = f \otimes^{\bowtie} \mathrm{Id}_A : C \otimes_R^{\bowtie} A \to D \otimes_R^{\bowtie} A$. However, we know



is commutative. Therefore $f \otimes^{\bowtie} g \nsim 0$.

□

However, we do have a positive result involving homotopy.

Theorem 5.1.3. *Let $f : C \to D$ with $f \sim 0$ and $\sigma\partial^A : A_n \to (\Sigma A)_n$ be morphisms of complexes of R-modules . Then $f \otimes^{\bowtie} \sigma\partial^A : C \otimes_R^{\bowtie} A \to D \otimes_R^{\bowtie} \Sigma A$ satisfies $f \otimes \sigma\partial_n^A \sim 0$.*

Proof: We consider three cases: $n = 0, n \geqslant 1$ and $n \leqslant -1$.

$n = 0:$

In this case the diagram is

$$\begin{array}{ccccc} & & C_0 \otimes_R A_0 & \xrightarrow{\partial_0^{C \otimes_R^{\bowtie} A}} & C_{-1} \otimes_R (\Sigma A)_0 \\ & \swarrow^{h'_0} & \downarrow{\scriptstyle f_0 \otimes \sigma^A_{-1}\partial^A_0} & \swarrow_{h'_{-1}} & \\ \bigoplus_{i=0}^{1} D_{1-i} \otimes_R (\Sigma A)_i & \xrightarrow{\partial_1^{D \otimes_R^{\bowtie} \Sigma A}} & D_0 \otimes_R (\Sigma A)_0 & & \end{array}$$

Define $h'_0 = h_0 \otimes \sigma^A_{-1}\partial^A_0$ and $h'_{-1} = h_{-1} \otimes (\Sigma A)_0$. Choose an element $c \otimes a \in C_0 \otimes_R A_0$.

Then,

$$\begin{aligned} & \partial_1^{D \otimes_R^{\bowtie} \Sigma A} h'_0(c \otimes a) + h'_{-1}\partial_0^{C \otimes_R^{\bowtie} A}(c \otimes a) \\ &= \partial_1^{D \otimes_R^{\bowtie} \Sigma A}(h_0 \otimes \sigma^A_{-1}\partial^A_0)(c \otimes a) + h'_{-1}(\partial^C_0 \otimes \sigma^A_{-1}\partial^A_0)(c \otimes a) \\ &= \partial_1^{D \otimes_R^{\bowtie} \Sigma A}(h_0(c) \otimes \sigma^A_{-1}\partial^A_0(a)) + h'_{-1}(\partial^C_0(c) \otimes \sigma^A_{-1}\partial^A_0(a)) \\ &= (\partial^D_1 \otimes (\Sigma A)_0)(h_0(c) \otimes \sigma^A_{-1}\partial^A_0(a)) + (h_{-1} \otimes (\Sigma A)_0)(\partial^C_0(c) \otimes \sigma^A_{-1}\partial^A_0(a)) \\ &= \partial^D_1 h_0(c) \otimes \sigma^A_{-1}\partial^A_0(a) + h_{-1}\partial^C_0(c) \otimes \sigma^A_{-1}\partial^A_0(a) \\ &= [\partial^D_1 h_0(c) \otimes + h_{-1}\partial^C_0(c)] \otimes \sigma^A_{-1}\partial^A_0(a) \\ &= f_0(c) \otimes \sigma^A_{-1}\partial^A_0(a). \end{aligned}$$

$n \geqslant 1:$

In this case the diagram is

$$\begin{array}{ccccc} & & \bigoplus_{i=0}^{n} C_{n-i} \otimes_R A_i & \xrightarrow{\partial_n^{C \otimes_R^{\bowtie} A}} & \bigoplus_{i=0}^{n-1} C_{n-i-1} \otimes_R A_i \\ & \swarrow^{h'_n} & \downarrow{\scriptstyle f_n \otimes \sigma^A_{i-1}\partial^A_i} & \swarrow_{h'_{n-1}} & \\ \bigoplus_{i=0}^{n+1} D_{n+1-i} \otimes_R (\Sigma A)_i & \xrightarrow{\partial_{n+1}^{D \otimes_R^{\bowtie} \Sigma A}} & \bigoplus_{i=0}^{n} D_{n-i} \otimes_R (\Sigma A)_i & & \end{array}$$

Define

$$h'_n = \begin{bmatrix} 0 & & \cdots & 0 \\ & h_n \otimes \sigma^A_{-1}\partial^A_0 & & \\ & & \ddots & \\ & & & h_0 \otimes \sigma^A_{n-1}\partial^A_n \end{bmatrix},$$

and

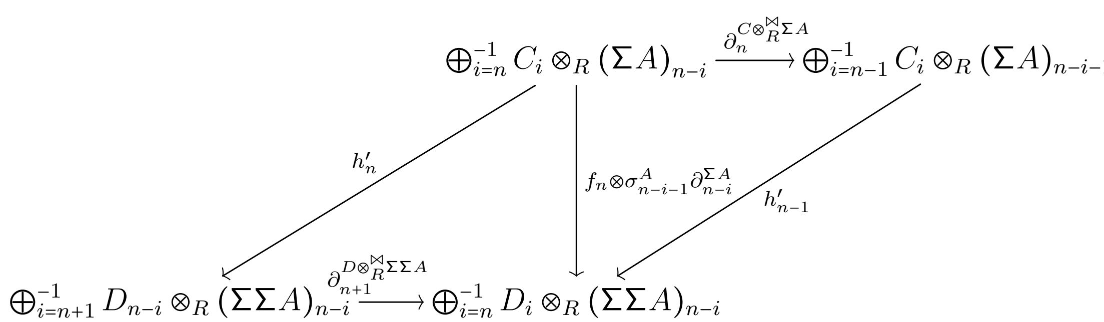
$$f_n \otimes \sigma^A_{i-1}\partial^A_i = \begin{bmatrix} f_n \otimes \sigma^A_{-1}\partial^A_0 & & \\ & \ddots & \\ & & f_0 \otimes \sigma^A_{n-1}\partial^A_n \end{bmatrix}.$$

Choose an element $c \otimes a \in C_{n-i} \otimes_R A_i$, Then,

$$\begin{aligned}
&\partial^{D\otimes^{\bowtie}_R \Sigma A}_{n+1} h'_n(c\otimes a) + h'_{n-1}\partial^{C\otimes^{\bowtie}_R A}_n (c\otimes a)\\
&= \partial^{D\otimes^{\bowtie}_R \Sigma A}_{n+1}(h_n \otimes \sigma^A_{i-1}\partial^A_i)(c\otimes a) + h'_{n-1}(\partial^C_n \otimes A_i)(c\otimes a)\\
&= \partial^{D\otimes^{\bowtie}_R \Sigma A}_{n+1}(h_n(c) \otimes \sigma^A_{i-1}\partial^A_i(a)) + h'_{n-1}(\partial^C_n(c)\otimes a)\\
&= (\partial^D_{n+1} \otimes (\Sigma A)_i)(h_n(c)\otimes \sigma^A_{i-1}\partial^A_i(a)) + (h_{n-1}\otimes \sigma^A_{i-1}\partial^A_i)(\partial^C_n(c)\otimes a)\\
&= \partial^D_{n+1} h_n(c) \otimes \sigma^A_{i-1}\partial^A_i(a) + h_{n-1}\partial^C_n(c)\otimes \sigma^A_{i-1}\partial^A_i(a)\\
&= [\partial^D_{n+1}h_n(c) \otimes + h_{n-1}\partial^C_n(c)] \otimes \sigma^A_{i-1}\partial^A_i(a)\\
&= f_n(c) \otimes \sigma^A_{i-1}\partial^A_i(a).
\end{aligned}$$

$n \leqslant -1$:

In this case the diagram is



Define $h'_n = h_n \otimes \sigma^{\Sigma A}_{n-i-1}\partial^{\Sigma A}_{n-i}$

$$h'_n = \begin{bmatrix} 0 & & & \\ & h_n \otimes \sigma^{\Sigma A}_{-1}\partial^{\Sigma A}_0 & & \\ \vdots & & \ddots & \\ 0 & & & h_0 \otimes \sigma^{\Sigma A}_{n-1}\partial^{\Sigma A}_n \end{bmatrix},$$

and

$$f_n \otimes \sigma^{A}_{n-i-1}\partial^{\Sigma A}_{n-i} = \begin{bmatrix} f_n \otimes \sigma^{A}_{-1}\partial^{\Sigma A}_0 & & \\ & \ddots & \\ & & f_{-1} \otimes \sigma^{A}_{n}\partial^{\Sigma A}_{n+1} \end{bmatrix}.$$

Choose an element $c \otimes a \in C_i \otimes_R (\Sigma A)_{n-i}$, Then,

$$\begin{aligned}
&\partial^{D\otimes^{\bowtie}_R \Sigma\Sigma A}_{n+1} h'_n(c \otimes a) + h'_{n-1}\partial^{C\otimes^{\bowtie}_R \Sigma A}_n(c\otimes a) \\
&\quad = \partial^{D\otimes^{\bowtie}_R \Sigma\Sigma A}_{n+1}(h_n \otimes \sigma^{\Sigma A}_{n-i-1}\partial^{\Sigma A}_{n-i})(c\otimes a) + h'_{n-1}(\partial^C_n \otimes \Sigma A_i)(c \otimes a) \\
&\quad = \partial^{D\otimes^{\bowtie}_R \Sigma\Sigma A}_{n+1}(h_n(c) \otimes \sigma^{\Sigma A}_{n-i-1}\partial^{\Sigma A}_{n-i}(a)) + h'_{n-1}(\partial^C_n(c) \otimes a) \\
&\quad = (\partial^D_{n+1} \otimes (\Sigma\Sigma A)_i)(h_n(c) \otimes \sigma^{\Sigma A}_{n-i-1}\partial^{\Sigma A}_{n-i}(a)) + (h_{n-1} \otimes \sigma^{\Sigma A}_{n-i-1}\partial^{\Sigma A}_{n-i}(a))(\partial^C_n(c) \otimes a) \\
&\quad = \partial^D_{n+1} h_n(c) \otimes \sigma^{\Sigma A}_{n-i-1}\partial^{\Sigma A}_{n-i}(a) + h_{n-1}\partial^C_n(c) \otimes \sigma^{\Sigma A}_{n-i-1}\partial^{\Sigma A}_{n-i}(a) \\
&\quad = [\partial^D_{n+1} h_n(c) \otimes + h_{n-1}\partial^C_n(c)] \otimes \sigma^{\Sigma A}_{n-i-1}\partial^{\Sigma A}_{n-i}(a) \\
&\quad = f_n(c) \otimes \sigma^{\Sigma A}_{n-i-1}\partial^{\Sigma A}_{n-i}(a).
\end{aligned}$$

□

5.2 Mapping Cones

Theorem 5.2.1. *Let A be an R-complex and $f : C \to D$ be a morphism of complexes of R-modules . Then there exists a morphism from $A \otimes_R^{\bowtie} \operatorname{cone}(f)$ to $\operatorname{cone}(A \otimes_R^{\bowtie} g)$, where*

$$g_n = \begin{cases} f_n & \text{for } n \geqslant -1 \\ \\ -f_n & \text{for } n < -1. \end{cases}$$

Proof: We consider three cases: $n = 0, n \geqslant 1$ and $n \leqslant -1$.

$n = 0:$

In this case the diagram is

$$\begin{array}{ccc} (A \otimes_R^{\bowtie} \operatorname{cone}(f))_0 & \xrightarrow{\partial_0^{A \otimes_R^{\bowtie} \operatorname{cone}(f)}} & (A \otimes_R^{\bowtie} \operatorname{cone}(f))_{-1} \\ \downarrow \mu_0 & & \downarrow \mu_{-1} \\ (\operatorname{cone}(A \otimes_R^{\bowtie} f))_0 & \xrightarrow{\partial_0^{\operatorname{cone}(A \otimes_R^{\bowtie} f)}} & (\operatorname{cone}(A \otimes_R^{\bowtie} f))_{-1} \end{array}$$

Define $\mu_0\Big(a\otimes(c,d)\Big) = \Big(\sigma_{-1}^{A\otimes_R^{\bowtie}C}(\partial_0^A(a)\otimes c), a\otimes d\Big)$ and $\mu_{-1}\Big(a\otimes\sigma_{-1}^{\text{cone}(f)}(c,d)\Big) = \Big(\sigma_{-2}^{A\otimes_R^{\bowtie}C}(a\otimes b), a\otimes\sigma_{-1}^D(d)\Big)$. Choose an element $(a\otimes(c,d))\in A_0\otimes_R((\Sigma C)_0\oplus D_0)$. Then,

$$
\begin{aligned}
&\partial_0^{cone(A\otimes_R^{\bowtie}f)}\mu_0\Big(a\otimes(c,d)\Big)\\
&=\partial_0^{cone(A\otimes_R^{\bowtie}f)}\Big(\Big(\sigma_{-1}^{A\otimes_R^{\bowtie}C}(\partial_0^A(a)\otimes c), a\otimes d\Big)\Big)\\
&=\Big[\partial_0^{\Sigma(A\otimes^{\ltimes}C)}\Big(\sigma_{-1}^{A\otimes_R^{\bowtie}C}\Big(\partial_0^A(a)\otimes c\Big)\Big), \Big(A_{-1}\otimes_R(\Sigma f)_0\Big)(\sigma_{-1}^{A\otimes_R^{\bowtie}C})^{-1}\sigma_{-1}^{A\otimes_R^{\bowtie}C}\Big(\partial_0^A(a)\otimes(c)\Big)\\
&\qquad\qquad+\partial_0^{A\otimes^{\ltimes}D}(a\otimes d)\Big]\\
&=\Big[-\sigma_{-2}^{A\otimes_R^{\bowtie}C}\partial_{-1}^{A\otimes^{\ltimes}C}(\sigma_{-1}^{A\otimes_R^{\bowtie}C})^{-1}\Big(\sigma_{-1}^{A\otimes_R^{\bowtie}C}\Big(\partial_0^A(a)\otimes c\Big)\Big), \Big(A_{-1}\otimes_R(\Sigma f)_0\Big)\Big(\partial_0^A(a)\otimes(c)\Big)\\
&\qquad\qquad+\partial_0^A(a)\otimes\sigma_{-1}^D\partial_0^D(d)\Big]\\
&=\Big[-\sigma_{-2}^{A\otimes_R^{\bowtie}C}\partial_{-1}^{A\otimes^{\ltimes}C}\Big(\partial_0^A(a)\otimes c\Big), \Big(\partial_0^A(a)\otimes_R(\Sigma f)_0(c)\Big)\\
&\qquad\qquad+\partial_0^A(a)\otimes\sigma_{-1}^D\partial_0^D(d)\Big]\\
&=\Big[-\sigma_{-2}^{A\otimes_R^{\bowtie}C}\Big(\partial_{-1}^A\partial_0^A(a)\otimes c+(-1)\partial_0^A(a)\otimes\partial_{-1}^{\Sigma C}(c)\Big), \Big(\partial_0^A(a)\otimes_R(\Sigma f)_0(c)\Big)\\
&\qquad\qquad+\partial_0^A(a)\otimes\sigma_{-1}^D\partial_0^D(d)\Big]\\
&=\Big[\sigma_{-2}^{A\otimes_R^{\bowtie}C}\Big(\partial_0^A(a)\otimes\partial_{-1}^{\Sigma C}(c)\Big), \partial_0^A(a)\otimes_R(\Sigma f)_0(c)+\partial_0^A(a)\otimes\sigma_{-1}^D\partial_0^D(d)\Big].
\end{aligned}
$$

However,

$$
\begin{aligned}
&\mu_{-1}\partial_0^{A\otimes_R^{\bowtie}cone(f)}\Big(a\otimes(c,d)\Big)\\
&=\mu_{-1}\Big[\partial_0^A(a)\otimes\partial_0^{Cone(f)}(c,d)\Big]\\
&=\mu_{-1}\Big[\partial_0^A(a)\otimes\Big(\partial_{-1}^{\Sigma C}(c), f_{-1}(\sigma_{-1}^C)^{-1}(c)+\partial_0^D(d)\Big)\Big]\\
&=\Big[\sigma_{-2}^{A\otimes_R^{\bowtie}C}\Big(\partial_0^A(a)\otimes\partial_{-1}^{\Sigma C}(c)\Big), \partial_0^A(a)\otimes_R\sigma_{-1}^D\Big(f_{-1}(\sigma_{-1}^C)^{-1}(c)+\partial_0^D(d)\Big)\Big].
\end{aligned}
$$

It is clear that $(\Sigma f)_0(c) = \sigma^D_{-1} f_{-1} (\sigma^C_{-1})^{-1}(c)$, since the following diagram commutes

$$\begin{array}{ccc} C_{-1} & \xrightarrow{\ f_{-1}\ } & D_{-1} \\ \Big\downarrow{\scriptstyle \sigma^C_{-1}} & & \Big\downarrow{\scriptstyle \sigma^D_{-1}} \\ (\Sigma C)_0 & \xrightarrow{\ (\Sigma f)_0\ } & (\Sigma D)_{-1} \end{array}$$

Therefore, $\partial_0^{cone(A \otimes^{\bowtie}_R f)} \mu_0 = \mu_{-1} \partial_0^{A \otimes^{\bowtie}_R cone(f)}$, which is what we wanted to show.

$n \geqslant 1$:

In this case the diagram is

$$\begin{array}{ccc} (A \otimes^{\bowtie}_R \text{cone}\,(f))_n & \xrightarrow{\ \partial_n^{A \otimes^{\bowtie}_R \text{cone}\,(f)}\ } & (A \otimes^{\bowtie}_R \text{cone}\,(f))_{n-1} \\ \Big\downarrow{\scriptstyle \mu_n} & & \Big\downarrow{\scriptstyle \mu_{n-1}} \\ (\text{cone}\,(A \otimes^{\bowtie}_R f))_n & \xrightarrow{\ \partial_n^{\text{cone}\,(A \otimes^{\bowtie}_R f)}\ } & (\text{cone}\,(A \otimes^{\bowtie}_R f))_{n-1} \end{array}$$

Define

$$\mu_n\Big(a \otimes (c, d)\Big) = \begin{cases} \Big(\sigma_{n-1}^{A \otimes^{\bowtie}_R C}(a \otimes \sigma_i^{-1}(c)), a \otimes d\Big) & \text{for } i \geqslant 0 \\ \Big(0, a \otimes d\Big) & \text{for } i = 0, \end{cases}$$

$\mu_n\Big(a \otimes (c, d)\Big) = \Big(\sigma_{n-1}^{A \otimes^{\bowtie}_R C}(a \otimes \sigma_i^{-1}(c)), a \otimes d\Big)$. We have two subcases $i = 0$ and $i \geqslant 1$.

$i = 0$:

Choose an element $(a \otimes (c, d)) \in A_n \otimes_R ((\Sigma C)_0 \oplus D_0)$. Then,

$$\begin{aligned} &\partial_n^{\text{cone}\,(A \otimes^{\bowtie}_R f)} \mu_n\Big(a \otimes (c, d)\Big) \\ &= \partial_n^{\text{cone}\,(A \otimes^{\bowtie}_R f)}\Big((a \otimes d)\Big) \\ &= \Big[\partial_n^{A \otimes^{\bowtie} D}(a \otimes d)\Big] \\ &= \Big[\partial_n^{A}(a) \otimes d\Big]. \end{aligned}$$

However,

$$\begin{aligned}
&\mu_{n-1}\partial_n^{A\otimes_R^{\bowtie}\mathrm{cone}\,(f)}\Big(a\otimes(c,d)\Big)\\
&\quad=\mu_{n-1}\Big[\partial_n^A(a)\otimes(c,d)+(-1)^n a\otimes\partial_0^{\mathrm{cone}\,(f)}(c,d)\Big]\\
&\quad=\mu_{n-1}\Big[\partial_n^A(a)\otimes(c,d)+(-1)^n a\otimes 0\Big]\\
&\quad=\Big[\partial_n^A(a)\otimes d\Big].
\end{aligned}$$

Therefore, $\partial_n^{cone(A\otimes_R^{\bowtie}f)}\mu_n=\mu_{n-1}\partial_n^{A\otimes_R^{\bowtie}\mathrm{cone}\,(f)}$, which is what we wanted to show.

$i\geqslant 1$:

Choose an element $(a\otimes(c,d))\in A_{n-i}\otimes_R((\Sigma C)_i\oplus D_i)$. Then,

$$\begin{aligned}
&\partial_n^{\mathrm{cone}\,(A\otimes_R^{\bowtie}f)}\mu_n\Big(a\otimes(c,d)\Big)\\
&=\partial_n^{\mathrm{cone}\,(A\otimes_R^{\bowtie}f)}\Big(\sigma_{n-1}^{A\otimes_R^{\bowtie}C}(a\otimes(\sigma_i^C)^{-1}(c),a\otimes d\Big)\\
&=\Big[\partial_n^{\Sigma(A\otimes^{\ltimes}C)}\Big(\sigma_{n-1}^{A\otimes_R^{\bowtie}C}\Big(a\otimes(\sigma_i^C)^{-1}(c)\Big)\Big),\\
&\qquad\Big(A_{n-i}\otimes_R f_{i-1}\Big)(\sigma_{n-1}^{A\otimes_R^{\bowtie}C})^{-1}\sigma_{n-1}^{A\otimes_R^{\bowtie}C}\Big(a\otimes(\sigma_{i-1}^C)^{-1}(c)\Big)+\partial_n^{A\otimes^{\ltimes}D}(a\otimes d)\Big]\\
&=\Big[-\sigma_{n-2}^{A\otimes_R^{\bowtie}C}\partial_{n-1}^{A\otimes^{\ltimes}C}(\sigma_{n-1}^{A\otimes_R^{\bowtie}C})^{-1}\Big(\sigma_{n-1}^{A\otimes_R^{\bowtie}C}\Big(a\otimes(\sigma_i^C)^{-1}(c)\Big)\Big)\\
&\qquad,\Big(A_{n-i}\otimes_R f_{i-1}\Big)\Big(a\otimes(\sigma_{i-1}^C)^{-1}(c)\Big)+\partial_{n-i}^A(a)\otimes d+(-1)^{n-i}a\otimes\partial_i^D(d)\Big]\\
&=\Big[-\sigma_{n-2}^{A\otimes_R^{\bowtie}C}\partial_{n-1}^{A\otimes^{\ltimes}C}\Big(a\otimes(\sigma_i^C)^{-1}(c)\Big),\Big(a\otimes_R f_{i-1}(\sigma_{i-1}^C)^{-1}(c)\Big)\\
&\qquad\qquad+\partial_{n-i}^A(a)\otimes d+(-1)^{n-i}a\otimes\partial_i^D(d)\Big]\\
&=\Big[-\sigma_{n-2}^{A\otimes_R^{\bowtie}C}\Big(\partial_{n-i}^A(a)\otimes(\sigma_i^C)^{-1}(c)+(-1)^{n-i}a\otimes\partial_{i-1}^C(\sigma_{i-1}^C)^{-1}(c)\Big),\\
&\qquad\Big(a\otimes_R f_{i-1}(\sigma_{i-1}^C)^{-1}(c)\Big)+\partial_{n-i}^A(a)\otimes d+(-1)^{n-i}a\otimes\partial_i^D(d)\Big].
\end{aligned}$$

However,

$$
\begin{aligned}
&\mu_{n-1}\partial_n^{A\otimes_R^{\bowtie}\mathrm{cone}\,(f)}\big(a\otimes(c,d)\big)\\
&\qquad = \mu_{n-1}\Big[\partial_{n-i}^{A}(a)\otimes(c,d)+(-1)^{n-i}a\otimes\partial_i^{\mathrm{cone}\,(f)}(c,d)\Big]\\
&\qquad = \mu_{n-1}\Big[\partial_{n-i}^{A}(a)\otimes(c,d)+(-1)^{n-i}a\otimes\Big(\partial_i^{\Sigma C}(c),f_{i-1}(\sigma_{i-1}^{C})^{-1}(c)+\partial_i^{D}(d)\Big)\Big]\\
&\qquad = \Big[-\sigma_{n-2}^{A\otimes_R^{\bowtie}C}\Big(\partial_{n-i}^{A}(a)\otimes(\sigma_i^{C})^{-1}(c)+(-1)^{n-i}a\otimes(\sigma_{i-1}^{C})^{-1}\partial_i^{\Sigma C}(c)\Big),a\otimes f_{i-1}(\sigma_{i-1}^{C})^{-1}(c)\\
&\qquad\qquad\qquad +\partial_{n-i}^{A}(a)\otimes d+(-1)^{n-i}a\otimes\partial_i^{D}(d)\Big].
\end{aligned}
$$

It is clear that $\partial_{i-1}^{C}(\sigma_{i-1}^{C})^{-1}(c)=(\sigma_{i-2}^{C})^{-1}\partial_i^{\Sigma C}(c)$, since the following diagram commutes

$$
\begin{CD}
C_{i-1} @>{\partial_{i-1}^{C}}>> C_{i-2}\\
@V{\sigma_{i-1}^{C}}VV @VV{\sigma_{i-2}^{C}}V\\
(\Sigma C)_i @>{\partial_i^{\Sigma C}}>> (\Sigma C)_{i-1}
\end{CD}
$$

Therefore, $\partial_n^{cone(A\otimes_R^{\bowtie}f)}\mu_n=\mu_{n-1}\partial_n^{A\otimes_R^{\bowtie}\mathrm{cone}\,(f)}$, which is what we wanted to show.

$n\leqslant -1$:

In this case the diagram is

$$
\begin{CD}
(A\otimes_R^{\bowtie}\mathrm{cone}\,(f))_n @>{\partial_n^{A\otimes_R^{\bowtie}\mathrm{cone}\,(f)}}>> (A\otimes_R^{\bowtie}\mathrm{cone}\,(f))_{n-1}\\
@V{\mu_n}VV @VV{\mu_{n-1}}V\\
(\mathrm{cone}\,(A\otimes_R^{\bowtie}f))_n @>{\partial_n^{\mathrm{cone}\,(A\otimes_R^{\bowtie}f)}}>> (\mathrm{cone}\,(A\otimes_R^{\bowtie}f))_{n-1}
\end{CD}
$$

Define

$\mu_n\Big(a \otimes \sigma^{\text{cone}(f)}_{n-i-1}(c,d)\Big) = \Big((-1)^{|a|}\sigma^{A\otimes^{\bowtie}_R C}_{n-1}(a\otimes c), a\otimes \sigma^{D}_{n-i-1}(d)\Big)$ where $a \otimes \sigma^{\text{cone}(f)}_{n-i-1}(c,d) \in A_i \otimes_R (\Sigma\text{cone}(f))_{n-i}$. with $a \in A_i$, $(c,d) \in (\Sigma C)_{n-i-1} \otimes_R D_{n-i-1}$.

$$\begin{aligned}
&\partial_n^{\text{cone}(A\otimes^{\bowtie}_R f)}\mu_n\Big(a \otimes \sigma^{\text{cone}(f)}_{n-i-1}(c,d)\Big)\\
&\qquad = \partial_n^{\text{cone}(A\otimes^{\bowtie}_R f)}\Big((-1)^{|a|}\sigma^{A\otimes^{\bowtie}_R C}_{n-1}(a\otimes c), a\otimes \sigma^{D}_{n-i-1}(d)\Big)\\
&\qquad = \Big[\partial_n^{\Sigma(A\otimes^{\ltimes} C)}\Big((-1)^{|a|}\sigma^{A\otimes^{\bowtie}_R C}_{n-1}\Big(a\otimes c\Big)\Big),\\
&\qquad\quad \Big(A_i \otimes_R -(\Sigma f)_{n-i-1}\Big)(\sigma^{A\otimes^{\bowtie}_R C}_{n-1})^{-1}(-1)^{|a|}\sigma^{A\otimes^{\bowtie}_R C}_{n-1}\Big(a\otimes c\Big) + \partial_n^{A\otimes^{\ltimes} D}(a\otimes \sigma^{D}_{n-i-1}(d))\Big]\\
&\qquad = \Big[-\sigma^{A\otimes^{\bowtie}_R C}_{n-2}\partial^{A\otimes^{\ltimes} C}_{n-1}(\sigma^{A\otimes^{\bowtie}_R C}_{n-1})^{-1}\Big((-1)^{|a|}\sigma^{A\otimes^{\bowtie}_R C}_{n-1}(a\otimes c)\Big),\\
&\quad \Big(A_i \otimes_R -(\Sigma f)_{n-i-1}\Big)(-1)^{|a|}(a\otimes c) + \partial_i^A(a)\otimes \sigma^D_{n-i-1}(d) + (-1)^i a\otimes \partial^{\Sigma D}_{n-i}\sigma^D_{n-i-1}(d)\Big]\\
&\qquad = \Big[-\sigma^{A\otimes^{\bowtie}_R C}_{n-2}\partial^{A\otimes^{\ltimes} C}_{n-1}(-1)^{|a|}(a\otimes c), (-1)^{|a|}a\otimes_R -(\Sigma f)_{n-i-1}(c)\\
&\qquad\qquad + \partial_i^A(a)\otimes \sigma^D_{n-i-1}(d) + (-1)^i a\otimes \partial^{\Sigma D}_{n-i}\sigma^D_{n-i-1}(d)\Big]\\
&\qquad = \Big[(-1)^{|a|+1}\sigma^{A\otimes^{\bowtie}_R C}_{n-2}\Big(\partial_i^A(a)\otimes c + (-1)^i a\otimes \partial^{\Sigma C}_{n-i-1}(c)\Big), (-1)^{|a|}a\otimes_R -(\Sigma f)_{n-i-1}(c)\\
&\qquad\qquad + \partial_i^A(a)\otimes \sigma^D_{n-i-1}(d) + (-1)^i a\otimes \partial^{\Sigma D}_{n-i}\sigma^D_{n-i-1}(d)\Big].
\end{aligned}$$

However,

$$\begin{aligned}
&\mu_{n-1}\partial_n^{A\otimes^{\bowtie}_R\text{cone}(f)}\Big(a\otimes \sigma^{\text{cone}(f)}_{n-i-1}(c,d)\Big)\\
&= \mu_{n-1}\Big[\partial_i^A(a)\otimes \sigma^{\text{cone}(f)}_{n-i-1}(c,d) + (-1)^i a\otimes \partial^{\Sigma\text{cone}(f)}_{n-i}\sigma^{\text{cone}(f)}_{n-i-1}(c,d)\Big]\\
&= \mu_{n-1}\Big[\partial_i^A(a)\otimes \sigma^{\text{cone}(f)}_{n-i-1}(c,d) + (-1)^i a\otimes -\sigma^{\text{cone}(f)}_{n-i-2}\partial^{\text{cone}(f)}_{n-i-1}(\sigma^{\text{cone}(f)}_{n-i-1})^{-1}\sigma^{\text{cone}(f)}_{n-i-1}(c,d)\Big]\\
&= \mu_{n-1}\Big[\partial_i^A(a)\otimes \sigma^{\text{cone}(f)}_{n-i-1}(c,d) + (-1)^i a\otimes -\sigma^{\text{cone}(f)}_{n-i-2}\partial^{\text{cone}(f)}_{n-i-1}(c,d)\Big]\\
&= \mu_{n-1}\Big[\partial_i^A(a)\otimes \sigma^{\text{cone}(f)}_{n-i-1}(c,d)\\
&\qquad\qquad + (-1)^i a\otimes -\sigma^{\text{cone}(f)}_{n-i-2}\Big(\partial^{\Sigma C}_{n-i-1}(c), f_{n-i-2}\sigma^{-1}_{n-i-2}(c) + \partial^D_{n-i-1}(d)\Big)\Big]\\
&= \Big((-1)^{|a|-1}\sigma^{A\otimes^{\bowtie}_R C}_{n-2}(\partial_i^A(a)\otimes c), \partial_i^A(a)\otimes \sigma^D_{n-i-1}(d)\Big) +\\
&\quad \Big((-1)^{|a|}\sigma^{A\otimes^{\bowtie}_R C}_{n-2}((-1)^{i+1}a\otimes \partial^{\Sigma C}_{n-i-1}(c)\Big), (-1)^i a\otimes -\sigma^D_{n-i-2}\Big(f_{n-i-2}\sigma^{-1}_{n-i-2}(c) + \partial^D_{n-i-1}(d)\Big)\Big).
\end{aligned}$$

Therefore, $\partial_n^{cone(A\otimes_R^{\bowtie} f)} \mu_n = \mu_{n-1} \partial_n^{A\otimes_R^{\bowtie}\text{cone}(f)}$, which is what we wanted to show.

□

Remark 5.2.2. In Theorem 5.2.1 we note that we will not have an isomorphism between $A \otimes_R^{\bowtie} cone(f)$ and $cone(A \otimes_R^{\bowtie} f)$ because $\left(C \otimes_R^{\bowtie} \text{cone}\,(f)\right)_n$ has one more term than $\left(\text{cone}\,(A \otimes_R^{\bowtie} f)\right)_n$ for $n > 0$ and vice versa for $n < 0$.

Theorem 5.2.3. *Let A be an R-complex and $f : C \to D$ be a morphism of complexes of R-modules . Then there exists a morphism from* $\text{cone}(A \otimes_R^{\bowtie} g)$ *to* $A \otimes_R^{\bowtie} \text{cone}(f)$, *where*

$$g_n = \begin{cases} f_n & \text{for } n \geqslant -1 \\ \\ -f_n & \text{for } n < -1. \end{cases}$$

Proof: The proof is similar to that of Theorem 5.2.1.

□

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Yousuf enrolled in Kuwait University in September 1999 and received a Bachelor degree in Mathematics in May 2004 and a Master degree from Bowling Green State University in December 2010. He began the Ph.D. program at the University of Texas at Arlington in August of 2011. In December 2014, he was awarded a Ph.D. in Mathematics under the direction of David Jorgensen.

Yousuf's research interests lie in Algebra (Pure Mathematics), specifically in commutative algebra, with focus on properties of the pinched tensor product.