THE COMPENSATORY BARGAINING SET

OF A COOPERATIVE N-PERSON GAME WITH SIDE PAYMENTS

by

IRINEL DRAGAN

Department of Mathematics
University of Texas at Arlington
Arlington, Texas 76o19

December 1988

Technical Report # 256

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Introduction.

The bargaining sets have been introduced as solution concepts for cooperative n-person games with side payments by R. J. Aumann and M. Maschler (1964). A further study on the relationships between various concepts of solution for such games is due to R. J. Aumann and J. Dreze (1975). The Aumann/Maschler definition of a bargaining set relies upon a stability principle imposed to the payoffs in this set: an admissible payoff belongs to a bargaining set if for every objection against this payoff there is a counter objection. Two modifications of the stability principle have been discussed in earlier papers of the author (Dragan,1985,1987,1988).

The present paper is considering another modification: an objection is valid only if the players who intend to move to new coalitions agree upon a prior commitment, namely that of compensating all partners who join the venture, in case of failure due to a subsequent move. The mathematical description of the model is given in the first section, where the new stability principle and the corresponding compensatory bargaining set M_c are defined. A feasibility theorem for the existence of a flow in a bipartite network associated to a payoff and two partial coalition structures (Th.2.1) is derived in the second section from a similar theorem by D. Gale (1957). The result is used in the third section for proving a combinatorial characterization of the non core payoffs belonging to the compensatory bargaining set (Th.3.5). In the last section, in the set of such payoffs M_c-C(G) the subset of coalitionally rational payoffs has been found, for a 3-person game (Ths. 4.2, 4.6, 4.7, 4.8). This subset is compaired with the Aumann/ Maschler bargaining set M for the same game, in order to illustrate the particularities of the new model by a comparison with a well known one.

1. Bargaining proposals with commitments and the compensatory bargaining set.

Consider $G=(N,v)$ a cooperative n-person game; N is the set of players, $N = \{n\}$ and $v: P(N) \longrightarrow R$ subject to $v(\emptyset) = o$ is the characteristic function. Any partition $\mathcal{S}$ of N is a coalition structure. The set of admissible payoffs for $\mathcal{S}$ is

$$F_{\mathcal{S}} = \{x \mid x \in R^n, \quad x(S) = v(S), \forall S \in \mathcal{S}\} \tag{1.1}$$

where $x(S) = \sum_{i \in S} x_i$ and the set F of admissible payoffs for G is the union of all $F_{\mathcal{S}}$ for all coalition structures $\mathcal{S}$.

Cooperative games with coalition structures have been considered by R. J. Aumann and M. Maschler (1964). The core of the game is defined by means of the excess function $e(x,S) = v(S) - x(S)$, $S \in P(N)$, $x \in F$, as

$$C(G) = \{x \mid x \in F, \quad e(x,S) \leq o, \forall S \subseteq N\}, \tag{1.2}$$

(see Aumann/Dreze,1975,p.224). A new concept of bargaining set for such games will be introduced in this section.

Consider a given coalition structure $\mathcal{S}$ and an admissible payoff $x \in F_{\mathcal{S}}$. If $x \in C(G)$, then x will be assesed as stable, so that in the following we shall assume $x \notin C(G)$, which means that there are coalitions S with a positive excess. A partial partition of N is a set of non empty pairwise disjoint coalitions, that may cover N, or not. Such a partial partition will be called a partial coalition structure.

<u>Definition</u> 1.1: A bargaining proposal of $K \subsetneq N$ against x is a partial coalition structure $\mathcal{T} = (T_1,..,T_q)$ subject to

$$T_h \cap K \neq \emptyset, \ h=1,..,q, \quad T_o \cap K = \emptyset, \quad T_o = N - \bigcup_h T_h , \qquad (1.3)$$

and

$$e(x,T_h) > o, \ h=1,..,q. \qquad (1.4)$$

In words, K is a group of players who could increase their payoffs w.r.t. x by moving to $\mathcal{T}$, if their partners agree.

Definition 1.2: A bargaining distribution for $\mathcal{T}$ initiated by K is any $y \in R^{n-t_o}$, $t_o = |T_o|$, such that

$$y(T_h) = v(T_h), \ h=1,..,q, \quad y_i > x_i, \forall i \in K, \quad y_i \geqslant x_i, \forall i \in N-T_o \quad (1.5)$$

In words, a bargaining distribution for $\mathcal{T}$ is a payoff that could could motivate the players in K to move to $\mathcal{T}$ and the players in $N - T_o - K$ to give their agreement for this move. It is easy to see that $\mathcal{T}$ subject to (1.3) can offer a bargaining distribution if and only if (1.4) holds.

Note that the pair $(y;\mathcal{T})$ is similar to an Aumann/Maschler objection. The difference is that no rationality condition is imposed to x in $F_{\mathcal{S}}$ and K may not be a subset of a block in $\mathcal{S}$. Moreover, $(y;\mathcal{T})$ is not an objection against another group of players, but is an objection against $(x;\mathcal{S})$.

Note also that (1.5) is equivalent to

$$\alpha(T_h) = e(x,T_h), \ h=1,..,q, \quad \alpha_i > o, \ i \in K, \quad \alpha_i \geqslant o, \ i \in N - T_o, \ (1.6)$$

where $\alpha_i = y_i - x_i$, $i \in N - T_o$. In the following, α_i will be called the extra win of player i in $\mathcal{T}$. The set of bargaining distributions for $\mathcal{T}$ initiated by K, defined by (1.5), or (1.6), will be denoted by $B(\mathcal{T},K)$. Bargaining proposals and bargaining distributions have been considered in earlier papers of the author (Dragan,1985,1987,1988) and this terms have been used to avoid any confusion with Aumann/Maschler objections and counter objections.

Now, before going to precise definitions let us describe the specific manner of countering the bargaining proposals introduced in the present paper.

The new assumption of our bargaining model is that before acknowledging their bargaining proposal the players in $N - T_o$ agree upon the following commitment: a group of players $\tilde{K} \subsetneq N - T_o$ will be able to leave $\mathcal{T}$ and join coalitions of another partial coalition structure only if $\tilde{K}$ contains players of all coalitions of $\mathcal{T}$, i.e. $\tilde{K} \cap T_h \neq \emptyset$, h=1,..,q, and for each coalition T_h the players in $\tilde{K} \cap T_h$ will compensate T_h with $e(x,T_h)$ from their extra wins in their new coalitions. In other words, for each T_h the players in $\tilde{K} \cap T_h$ should compensate all members of T_h (including themselves) for losing their extra wins in T_h. If there is a group of players $\tilde{K}$ able to leave under the above stated conditions and move to a new partial coalition structure $\mathcal{U}$, then we shall consider that $\mathcal{U}$ can counter $\mathcal{T}$. This condition can be expressed mathematically as a system of conditions imposed to $\mathcal{U}$, as shown hereafter.

Consider a partial coalition structure $\mathcal{U} = (U_1,..,U_r)$, different of $\mathcal{T}$, such that

$$\tilde{K}_h = (N-U_o) \cap T_h \neq \emptyset, \quad h=1,..,q, \quad U_o = N - \bigcup_j U_j \, , \qquad (1.7)$$

and

$$e(x,U_j) > o, \quad j=1,..,r. \qquad (1.8)$$

Note that " $\mathcal{U}$ different of $\mathcal{T}$ " means that no coalition in $\mathcal{U}$ is a coalition in $\mathcal{T}$. Obviously, $\tilde{K} = \bigcup_h \tilde{K}_h$ will be the group of players who may be willing to move to $\mathcal{U}$.

Consider $z \in R^{n-u_o}$, $u_o = |U_o|$, such that

$$\beta(U_j) = e(x,U_j), \quad j=1,..,r, \quad \beta_i \geq o, \forall i \in N - U_o, \qquad (1.9)$$

where $\beta_i = z_i - x_i$, $i \in N - U_o$.

Clearly, the players in $\tilde{K}$ will be able to leave $\mathcal{T}$ and move to $\mathcal{U}$, if and only if we have

$$\sum_{i \in (N-U_o) \cap T_h} \beta_i \geq e(x,T_h), \quad h=1,..,q. \qquad (1.1o)$$

Now, to make the bargaining model precise, we introduce the following

<u>Definition</u> 1.3: Let $\mathcal{T} = (T_1,..,T_q)$ be a bargaining proposal of some coalition K against $x \in F_{\mathcal{S}}$, $x \notin C(G)$, and consider a partial coalition structure $\mathcal{U} = (U_1,..,U_r)$, different of $\mathcal{T}$, subject to (1.7) and (1.8). Any $z \in R^{n-u_o}$ subject to (1.9) and (1.1o), where $\beta_i = z_i - x_i$ $i \in N - U_o$, is a compensatory bargaining counter distribution for $\mathcal{U}$ against $\mathcal{T}$. If such a counter distribution does exist, then $\mathcal{U}$ is a compensatory bargaining counter proposal aganist $\mathcal{T}$.

Note that $\tilde{K} = \bigcup_h \tilde{K}_h$ may contain K, or not.

A characterization of the compensatory bargaining counter proposals will be given in the third section and examples will be shown in the last section. Now, we introduce the principle of stability.

<u>Definition</u> 1.4: An admissible payoff x for a coalition structure $\mathcal{S}$ is compensatory stable (c-stable), if for every bargaining proposal $\mathcal{T}$ of every coalition $K \subseteq N$, there exists a compensatory bargaining counter proposal $\mathcal{U}$ against $\mathcal{T}$.

Note that the core payoffs are considered to be c-stable, because there is no bargaining proposal of any group of players against such a payoff.

<u>Definition</u> 1.5: The compensatory bargaining set M_c of G is the set of c-stable payoffs.

2. <u>A bipartite network associated with a payoff and a pair of partial coalition structures</u>.

In this section, we consider as given the following elements of the game G: a coalition structure $\mathcal{S}$, a payoff $x \in F_{\mathcal{S}}$, $x \notin C(G)$, a bargaining proposal $\mathcal{T}$ of a coalition K against x and a partial coalition structure $\mathcal{U}$, different of $\mathcal{T}$, subject to (1.7) and (1.8). First, we define a bipartite capacitated network $\mathcal{N}(\mathcal{U},\mathcal{T})$ with demands and supplies associated with x, $\mathcal{T}$ and $\mathcal{U}$. Secondly, we present a feasibility theorem in $\mathcal{N}(\mathcal{U},\mathcal{T})$. In the next section, this theorem will be used to derive a characterization of the compensatory bargaining counter proposals.

A nondirected graph $\Gamma(N) = (N_\Gamma, E_\Gamma)$ can be associated to a game with the set of players N, as follows: for every coalition $S \subseteq N$, a vertex n_S is taken in N_Γ; if $S_1 \cap S_2 \neq \emptyset$, an edge $[n_{S_1}, n_{S_2}]$ is taken in E_Γ. Obviously, any partial coalition structure of the game is represented in $\Gamma(N)$ as an independent vertex set.

For our given x, we can define a weight function on N, namely

$$w(n_S) = e(x,S), \qquad S \subseteq N, \quad S \neq \emptyset . \tag{2.1}$$

Obviously, we have $w(n_S) = o$ for all $S \in \mathcal{S}$. The weight function defines a subgraph $\Gamma^* = (N^*, E^*)$ of $\Gamma(N)$, where $N^* = \{ n_S \mid n_S \in N_\Gamma, w(n_S) > o \}$ and E^* is the set of edges in E_Γ connecting vertices in N^*, i.e. Γ^* is the subgraph of $\Gamma(N)$ generated by the vertices corresponding to coalitions of positive excess.

The coalition K is represented by $n_K \in N_\Gamma$ which could be in N^*, or not. To our given bargaining proposal $\mathcal{T}$ of K corresponds an independent vertex set $N_{\mathcal{T}}$ of Γ^*, consisting of vertices n_{T_h}, $h=1,..,q$, adjacent to n_K. To our given partial coalition structure $\mathcal{U}$, different of $\mathcal{T}$, subject to (1.7) and (1.8), corresponds an independent vertex set $N_{\mathcal{U}}$ of Γ^* consisting of vertices n_{U_j}, $j=1,..,r$, such that every vertex n_{T_h}, $h=1,..,q$, is adjacent to at least one vertex in $N_{\mathcal{U}}$.

Now, we defined a network $\mathcal{N}(\mathcal{U},\mathcal{T})$ as follows:

The supporting directed graph $\tilde{\Gamma} = (V,A)$ has $V = N_{\mathcal{U}} \cup N_{\mathcal{T}}$ and $A = \{ (n_{U_j}, n_{T_h}) \mid j=1,..,r, \quad h=1,..,q \}$; note that $(n_{U_j}, n_{T_h}) \in A$ whether or not the edge $[n_{U_j}, n_{T_h}]$ belongs to E. The capacity function is

$$c_{jh} = c(n_{U_j}, n_{T_h}) = \begin{cases} \infty & \text{if } [n_{U_j}, n_{T_h}] \in E \\ o & \text{otherwise} \end{cases} \tag{2.2}$$

for $j=1,..,r$ and $h=1,..,q$. We denote

$$\begin{aligned} s(n_{U_j}) &= w(n_{U_j}) = e(x,U_j), \quad j=1,..,r \\ d(n_{T_h}) &= w(n_{T_h}) = e(x,T_h), \quad h=1,..,q \end{aligned} \tag{2.3}$$

a supply and a demand function, respectively. In words, $d(n_{T_h})$, $h=1,..,q$ are the amounts that the groups of players $\widetilde{K}_h$ from all T_h need in order to be able to leave $\mathcal{T}$ and move to $\mathcal{U}$, and $s(n_{U_j})$, $j=1,..,r$ are the amounts that all U_j can offer to attract players in $N - T_o$. Obviously, $\mathcal{N}(\mathcal{U},\mathcal{T})$ is a bipartite capacitated network with supplies and demands (see L. R. Ford and D. R. Fulkerson (1962)).

A flow in $\mathcal{N}(\mathcal{U},\mathcal{T})$ satisfying the demands with the given supplies is a function $f: A \longrightarrow R$ such that

$$\begin{aligned} &\sum_{h=1}^{q} f_{jh} \leq e(x,U_j), \quad j=1,..,r \\ &\sum_{j=1}^{r} f_{jh} \geq e(x,T_h), \quad h=1,..,q \qquad (2.4) \\ &o \leq f_{jh} \leq c_{jh}, \quad j=1,..,r, \quad h=1,..,q \end{aligned}$$

where f_{jh} is the flow on arc (n_{U_j}, n_{T_h}).

In general, it is used to say that a network with demands and supplies is feasible, if there exist a flow satisfying the demands with the given supplies, i.e. the system (2.4) is consistent. Now, we give a theorem similar to a feasibility theorem by D. Gale (1957), (see Ford/Fulkerson,1962,Chapter II,Th.1.1,pp.38-39). This theorem will characterize the networks $\mathcal{N}(\mathcal{U},\mathcal{T})$ which are feasible. We consider only the networks $\mathcal{N}(\mathcal{U},\mathcal{T})$ subject to the assumption

(A) $\quad U_j \cap (N - T_o) \neq \emptyset$, $\quad j=1,..,r.$

In $\mathcal{N}(\mathcal{U},\mathcal{T})$, this assumption means that each vertex in $N_{\mathcal{U}}$ is linked by at least one arc of infinite capacity to some vertex in $N_{\mathcal{T}}$. Note that by (1.7) such arcs are entering every vertex in $N_{\mathcal{T}}$. Of course, there may still exist pairs of vertices n_{U_j}, n_{T_h} corresponding to pairs U_j, T_h of disjoint coalitions.

By extending the network $\mathcal{N}(\mathcal{U},\mathcal{T})$ and using the same procedure as in the proof of Gale's Theorem, we can prove:

<u>Theorem</u> 2.1: There exists a flow satisfying the demands with the

given supplies in a network $\mathcal{N}(\mathcal{U},\mathcal{T})$ subject to assumption (A), if and only if for every subset $X_{\mathcal{U}} \subset N_{\mathcal{U}}$, (including $X_{\mathcal{U}} = \emptyset$), we have

$$\sum_{U_j \notin X_{\mathcal{U}}} e(x,U_j) \geqslant \sum_{T_h \notin \tilde{\Gamma}(X_{\mathcal{U}})} e(x,T_h) \qquad (2.5)$$

where $\tilde{\Gamma}(X_{\mathcal{U}})$ is the set of vertices in $N_{\mathcal{T}}$ adjacent to vertices in $X_{\mathcal{U}}$.

Proof: We should impose the condition that in the extended network the capacity of every cut which contains no arc of infinite capacity is at least equal to the capacity of the cut consisting of all exit arcs. By examining all cuts and eliminating the redundant conditions we end up with (2.5).

3. A combinatorial characterization of the elements of the compensatory bargaining set.

We intend to prove first a characterization of compensatory bargaining counter proposals. Therefore, we consider as given the same elements of the game G as in the previous section, i.e. a coalition structure $\mathcal{S}$, a payoff $x \in F_{\mathcal{S}}$, $x \notin C(G)$, a bargaining proposal $\mathcal{T}$ of a coalition K against x and a partial coalition structure $\mathcal{U}$, different of $\mathcal{T}$, subject to (1.7) and (1.8). By definition 1.3, any $z \in R^{n-u_o}$ subject to

$$\sum_{i \in U_j} \beta_i = e(x,U_j), \quad j=1,..,r, \qquad \beta_i \geqslant o, \quad i \in N - U_o$$
$$\sum_{i \in (N-U_o) \cap T_h} \beta_i \geqslant e(x,T_h), \quad h=1,..,q, \qquad (3.1)$$

where $\beta_i = z_i - x_i$, $i \in N - U_o$, is a compensatory bargaining counter distribution for $\mathcal{U}$ against $\mathcal{T}$. If (3.1) is consistent, then $\mathcal{U}$ is a compensatory bargaining counter proposal against $\mathcal{T}$.

Notice that in (3.1), if $U_j \cap (N-T_o) = \emptyset$ for some j, then the third group of conditions does not contain the unknowns β_i with $i \in U_j$ while the subsystem which contains these unknwons is consistent and contains no other variables. Therefore, in the following it will be enough to consider a partial coalition structure $\mathcal{U}$ subject to assumption (A), as shown by the Lemma 3.3 below.

Lemma 3.1: Let $\mathcal{T}$ be a bargaining proposal of some coalition K against x and consider a partial coalition structure $\mathcal{U}$ subject to (1.7), (1.8) and (A). There exists a compensatory bargaining counter distribution z for $\mathcal{U}$ against $\mathcal{T}$, if and only if there exists a flow satisfying the demands with the given supplies in the network $\mathcal{N}(\mathcal{U},\mathcal{T})$ associated with x, $\mathcal{T}$ and $\mathcal{U}$.

Proof: Suppose that there is a compensatory bargaining counter distribution z for $\mathcal{U}$ against $\mathcal{T}$, i.e. (3.1) is consistent. Define

$$f_{jh} = \sum_{i\in U_j\cap T_h} \beta_i \quad , \quad j=1,..,r, \quad h=1,..,q, \tag{3.2}$$

where $\beta_i = z_i - x_i$, $i\in N - U_o$, and $f_{jh} = o$ if $U_j\cap T_h = \emptyset$. From (3.1) and (3.2) we have

$$\begin{aligned} \sum_{h=1}^{q} f_{jh} &= \sum_{i\in U_j\cap(N-T_o)} \beta_i \leq \sum_{i\in U_j} \beta_i = e(x,U_j), \quad j=1,..,r, \\ \sum_{j=1}^{r} f_{jh} &= \sum_{i\in(N-U_o)\cap T_h} \beta_i \geqslant e(x,T_h), \quad h=1,..,q, \\ f_{jh} &\geqslant o, \quad j=1,..,r, \quad h=1,..,q, \end{aligned} \tag{3.3}$$

and if $c_{jh} = o$, i.e. $U_j\cap T_h = \emptyset$, we have $f_{jh} = c_{jh} = o$. Hence, $f: A\rightarrow R$ defined by (3.2) satisfies (2.4), i.e. f is a flow satisfying the demands with the given supplies in $\mathcal{N}(\mathcal{U},\mathcal{T})$.

Conversely, suppose that there is a flow $f: A\rightarrow R$ satisfying the demands with the given supplies in $\mathcal{N}(\mathcal{U},\mathcal{T})$, i.e. (2.4) is consistent. Define β_i, $i\in N - U_o$, as follows

$$\begin{aligned} \beta_i &= o, & &\text{if } i\in N - U_o, \ i\notin N - T_o, \\ \beta_i &= \frac{f_{jh}}{|U_j\cap T_h|} + \frac{d_j}{|U_j\cap(N-T_o)|} & &\text{if } i\in U_j\cap T_h \neq \emptyset \end{aligned} \tag{3.4}$$

where

$$d_j = e(x,U_j) - \sum_{h=1}^{q} f_{jh} \quad , \quad j=1,..,r. \tag{3.5}$$

As all flows and all differences d_j, $j=1,..,r$, are nonnegative by (2.4), we have $\beta_i \geqslant o$, $i \in N - U_o$. For $U_j \cap T_h \neq \emptyset$ we get from (3.4):

$$\sum_{i \in U_j \cap T_h} \beta_i = f_{jh} + \frac{|U_j \cap T_h| \, d_j}{|U_j \cap (N-T_o)|} \quad . \tag{3.6}$$

As $f_{jh} = o$ if $U_j \cap T_h = \emptyset$, for every U_j, $j=1,..,r$, we get from (3.5) and (3.6):

$$\sum_{i \in U_j} \beta_i = \sum_{h/U_j \cap T_h \neq \emptyset} \; \sum_{i \in U_j \cap T_h} \beta_i = \sum_{h=1}^{q} f_{jh} + d_j = e(x,U_j). \tag{3.7}$$

On the other hand, for $U_j \cap T_h \neq \emptyset$ we get from (3.4):

$$\sum_{i \in U_j \cap T_h} \beta_i \geqslant f_{jh} \quad . \tag{3.8}$$

As $\beta_i = o$ if $i \in N - U_o$, $i \notin N - T_o$, for every T_h, $h=1,..,q$, we get from (2.4) and (3.8):

$$\sum_{i \in (N-U_o) \cap T_h} \beta_i \geqslant \sum_{j/U_j \cap T_h \neq \emptyset} f_{jh} = \sum_{j=1}^{r} f_{jh} \geqslant e(x,T_h), \tag{3.9}$$

because $f_{jh} = o$ if $U_j \cap T_h = \emptyset$. It follows that β_i, $i \in N - U_o$, defined by (3.4) satisfy (3.1), hence $z_i = x_i + \beta_i$, $i \in N - U_o$, is a compensatory bargaining counter distribution for $\mathcal{U}$ against $\mathcal{T}$.

From Definition 1.3, Theorem 2.1 and Lemma 3.1, follows

<u>Theorem</u> 3.2: Let $\mathcal{T} = (T_1,..,T_q)$ be a bargaining proposal of some coalition K against $x \in F_{\mathcal{P}}$, $x \notin C(G)$, and consider a partial coalition structure $\mathcal{U} = (U_1,..,U_r)$ subject to

$$\tilde{K}_h = (N-U_o) \cap T_h \neq \emptyset, \quad h=1,..,q, \quad U_o = N - \bigcup_{j=1}^{r} U_j, \tag{3.1o}$$
$$e(x,U_j) > o, \quad j=1,..,r,$$

and

(A) $U_j \cap (N-T_o) \neq \emptyset$, $j=1,..,r$.

Then, $\mathcal{U}$ is a compensatory bargaining counter proposal against $\mathcal{T}$, if and only if for every subset $\mathcal{U}^*$ of blocks of $\mathcal{U}$, (including $\mathcal{U}^* = \emptyset$), we have

(C) $$\sum_{U_j \notin \mathcal{U}^*} e(x,U_j) \geqslant \sum_{T_h \notin \mathcal{T}^*} e(x,T_h),$$

where $\mathcal{T}^*$ is the subset of blocks of $\mathcal{T}$ defined by

$$\text{(D)} \qquad T_h \in \mathcal{T}^* \longleftrightarrow T_h \cap U_j \neq \emptyset \text{ for some } U_j \in \mathcal{U}^* .$$

Note that every $\mathcal{U}$ subject to (3.1o) and (A) should be considered in order to determine whether or not $\mathcal{T}$ can be countered. Note also that one can check whether or not (C) is satisfied by finding a maximum flow in the extended network for $\mathcal{N}(\mathcal{U},\mathcal{T})$; if the maximum flow saturates the exit arcs, then (C) are satisfied and $\mathcal{U}$ is countering $\mathcal{T}$, otherwise a new $\mathcal{U}$, if any, should be considered. Moreover, in the first case the formulas (3.4) define a compensatory bargaining counter distribution for $\mathcal{U}$ against $\mathcal{T}$.

Now, we intend to give a combinatorial characterization of the elements of the compensatory bargaining set. Two previous lemmas are needed.

<u>Lemma</u> 3.3: For a bargaining proposal $\mathcal{T}$ of some coalition K against $x \in F_{\mathcal{G}}$, $x \notin C(G)$, there is a compensatory bargaining counter proposal, if and only if there is a compensatory bargaining counter proposal $\mathcal{U}$ against $\mathcal{T}$ satifying condition (A).

<u>Proof</u>: If $\tilde{\mathcal{U}}$ is a compensatory bargaining counter proposal against $\mathcal{T}$ which does not satisfy conditions of type (A), then according to the remark given before Lemma 3.1, the partial coalition structure $\mathcal{U}$ consisting of those blocks of $\tilde{\mathcal{U}}$ that satisfy (A) will also be a compensatory bargaining counter proposal against $\mathcal{T}$.

By Lemma 3.3, in the search for compensatory bargaining counter proposals able to counter various bargaining proposals, we can confine ourselves to consider only those partial coalition structures $\mathcal{U}$ subject to (3.1o) and (A) for every baragining proposal $\mathcal{T}$.

<u>Lemma</u> 3.4: If $\mathcal{T}$ is a bargaining proposal of some coalition K against $x \in F_{\mathcal{G}}$, $x \notin C(G)$, then $\mathcal{T}$ is also a bargaining proposal of every coalition $\mathcal{K}'$ which has members in all blocks of $\mathcal{T}$.

<u>Proof</u>: Follows from Definition 1.1, where (1.4) do not depend

on K.

By Definition 1.3 and Lemma 3.4, if $\mathcal{U}$ is able to counter a bargaining proposal $\mathcal{T}$, then it does not matter what coalition K subject to (1.3) has initiated $\mathcal{T}$. This happens because $\mathcal{U}$ is countering at once all particular distributions for $\mathcal{T}$ (which depend on K). Therefore, in our search for bargaining proposals against x, which may be countered or not, we can confine ourselves to consider all partial coalition structures $\mathcal{T}$ subject to (1.4). A useless waste of computational effort would be to consider for every coalition K all possible bargaining proposals initiated by K.

From Theorem 3.2, Lemmas 3.3 and 3.4 follows

<u>Theorem</u> 3.5: An admissible payoff vector $x \in F_{\mathcal{G}}$, $x \notin C(G)$, is an element of the compensatory bargaining set if and only if, for every partial coalition structure $\mathcal{T} = (T_1,..,T_q)$ subject to

$$e(x,T_h) > o, \quad h=1,..,q, \tag{3.11}$$

there is a partial coalition structure $\mathcal{U} = (U_1,..,U_r)$, different of $\mathcal{T}$, subject to

$$\begin{aligned} &(N-U_o) \cap T_h \neq \emptyset, \quad h=1,..,q, \quad U_o = N - \bigcup_{j=1}^{r} U_j, \\ &U_j \cap (N-T_o) \neq \emptyset. \quad j=1,..,r, \quad T_o = N - \bigcup_{h=1}^{q} T_h, \end{aligned} \tag{3.12}$$

and

$$e(x,U_j) > o, \quad j=1,..,r, \tag{3.13}$$

such that for every subset $\mathcal{U}^*$ of blocks of $\mathcal{U}$ (including $\mathcal{U}^*=\emptyset$) we have

$$\text{(C)} \qquad \sum_{U \notin \mathcal{U}^*} e(x,U) \geq \sum_{T \notin \mathcal{T}^*} e(x,T),$$

where

$$\text{(D)} \qquad \mathcal{T}^* = \{T \mid T \in \mathcal{T}, \; T \cap U \neq \emptyset \text{ for some } U \in \mathcal{U}^*\}.$$

This combinatorial characterization of the elements of the compensatory bargaining set could be used to derive an algorithm for testing whether an admissible payoff which is not in the core belongs or not to the compensatory bargaining set. The algorithm should include as subroutines algorithms for listing the independent vertex sets in a graph, for solving covering set problems and for finding a feasible flow in a bipartite network. Anyway, the problem in the general case is NP hard.

In the next section, Theorem 3.5 will be used for finding admissible payoffs in $M_c - C(G)$ for a 3-person game. In this way, on the one hand, we shall have the opportunity of compairing the compensatory bargaining set with the Aumann/Maschler bargaining set M for the same game and on the other hand, we shall see that the compensatory commitments imposed in our model could be fulfilled for some non core admissible payoffs at least in some games. Of course, the general existence problem is still open.

4. The 3-person games.

Consider the 3-person game

$$(G) \quad v(1)=v(2)=v(3)=o,\ v(12)=a,\ v(23)=b,\ v(13)=c,\ v(123)=d,$$

where a, b, c, d, are nonnegative numbers. This game has been chosen because the bargaining set M of Aumann/Maschler is known (1964). We intend to determine those non core elements of the compensatory bargaining set which are coalitionally rational, because the coalitional rationality has been a requirement in the definition of M. Let us recall that for a coalition structure $\mathcal{S} = (S_1,..,S_p)$, an admissible payoff vector x, i.e. a vector $x \in R^n$ subject to

$$\sum_{i \in S_k} x_i = v(S_k), \quad k=1,..,p, \tag{4.1}$$

is coalitionally rational, if for every $S \subset S_k$, $k=1,..,p$, we have

$$\sum_{i \in S} x_i \geqslant v(S). \tag{4.2}$$

Three remarks about our game (G) will be useful in the following.

First, note that for every coalitionally rational admissible payoff x, from (4.2) we should have $x \geqslant o$, so that all excesses $e(x,i) = -x_i$, $i=1,2,3$, are nonpositive. Therefore, a singleton can not be a coalition in a bargaining proposal or in a compensatory bargaining counter proposal. Secondly, every pair of coalitions of cardinality at least two consists of coalitions with at least one common element. Therefore, if one such coalition is a bargaining proposal against $x \in F$, i.e. its excess is positive, then conditions (3.12) of Theorem 3.5 hold for each of the other three coalitions of cardinality at least two, so that to determine whether x is c-stable, we should check only conditions (3.13) and (C). In this way, for $x \neq o$ we have at most three possible bargaining proposals, because one of the excesses is zero, and for each bargaining proposal against x we have at most two possible compensatory bargaining counter proposals. Third, for every pair of coaltions with positive excesses our condition (C) of Theorem 3.5 is reduced to a very simple one; namely, if $\mathcal{T} = (T)$ with $|T| > 1$ is a bargaining proposal against x, i.e. $e(x,T) > o$, then by Theorem 3.5, $\mathcal{U} = (U)$ with $U \neq T$, $|U| > 1$, $e(x,U) > o$, is a compensatory bargaining counter proposal against $\mathcal{T}$ if and only if we have

$$e(x,U) \geqslant e(x,T). \qquad (4.3)$$

These remarks prove

Theorem 4.1: For the 3-person game (G), an admissible payoff vector x, $x \notin C(G)$, which is coalitionally rational, belongs to the compensatory bargaining set if and only if $H = \max_S e(x,S) > o$ is reached for at least two coalitions.

Theorem 4.1 will be used below, so that it is convenient to call any coalition of highest excess a principal coalition.

Now, we intend to determine the coalitionally rational admissible payoffs belonging to M_c-C(G). Three cases should be considered:

(A) $\mathcal{S} = (1,2,3)$; (B) $\mathcal{S}$ consists of one singleton and one coalition of cardinality two; (C) $\mathcal{S} = (123)$.

Case (A): $\mathcal{S} = (1,2,3)$, $x = (o,o,o)$.

Theorem 4.2: In game (G), $x = (o,o,o)$ belongs to M_c-C(G) if and only if $H = \max(a,b,c,d)$ is positive and at least two of the numbers a,b,c,d, are equal to H.

Proof: Follows from Theorem 4.1 and $e(x,12)=a$, $e(x,23)=b$, $e(x,13)=c$ and $e(x,123)=d$.

Case (B): Three coalition structures should be considered: (B_1) $\mathcal{S} = (23,1)$; (B_2) $\mathcal{S} = (13,2)$; (B_3) $\mathcal{S} = (12,3)$. The first subcase will be discussed in details, for the other two the similar results will be given.

Consider (B_1): $\mathcal{S} = (23,1)$, $x = (o,x_2,x_3)$, where

$$x_2 + x_3 = b, \quad x_2 \geq o, \quad x_3 \geq o. \tag{4.4}$$

If $x \in M_c$-C(G), then besides (4.4) x should satisfy one of the following group of conditions:

(B_1') $H = e(x,13) = e(x,12) > o$ and $e(x,123) \leq H$;

(B_1'') $H = e(x,13) = e(x,123) > o$ and $e(x,12) \leq H$;

(B_1''') $H = e(x,12) = e(x,123) > o$ and $e(x,13) \leq H$.

Note that these groups of conditions may coincide only if $e(x,13) = e(x,12) = e(x,123)$. Note also that each group of conditions could be satisfied only if the game itself satisfies some conditions to be derived hereafter.

(B_1') From $e(x,13) = e(x,12)$ and (4.4) we get

$$x_1 = o, \quad x_2 = 1/2(a+b-c), \quad x_3 = 1/2(-a+b+c), \tag{4.5}$$

and we have $H = 1/2(a-b+c)$, $e(x,123) = d-b$; hence, from $e(x,123) \leq H$, $H > o$ and (4.4), we get

$$a+b+c \geq 2d, \quad b < a+c, \quad c \leq a+b, \quad a \leq b+c. \tag{4.6}$$

Conversely, (4.5) and (4.6) are sufficient for $x = (o,x_2,x_3)$ to belong to M_c-C(G) and to be coalitionally rational.

(B_1'') From $e(x,13) = e(x,123)$ and (4.4) we get

$$x_1 = o, \quad x_2 = d-c, \quad x_3 = b+c-d, \tag{4.7}$$

and we have $H = d-b$, $e(x,12) = a+c-d$; hence, from $e(x,12) \lesssim H$, $H > o$ and (4.4), we get

$$a+b+c \lesssim 2d, \quad b < d, \quad c \lesssim d, \quad d \lesssim b+c. \tag{4.8}$$

Conversely, (4.7) and (4.8) are sufficient for $x = (o,x_2,x_3)$ to belong to M_c-C(G) and to be coalitionally rational.

(B_1''') From $e(x,12) = e(x,123)$ and (4.4) we get

$$x_1 = o, \quad x_2 = a+b-d, \quad x_3 = d-a, \tag{4.9}$$

and we have $H = d-b$, $e(x,13) = a+c-d$; hence, from $e(x,13) \lesssim H$, $H > o$ and (4.4), we get

$$a+b+c \lesssim 2d, \quad b < d, \quad a \lesssim d, \quad d \lesssim b+a. \tag{4.1o}$$

Conversely, (4.9) and (4.1o) are sufficient for $x = (o,x_2,x_3)$ to belong to M_c-C(G) and to be coalitionally rational.

The above considerations prove

<u>Lemma</u> 4.3: For $\mathcal{S} = (23,1)$, there is a coalitionally rational $x \in M_c$-C(G) if and only if (G) satisfies the conditions of one of the columns

$a+b+c \gtrsim 2d$	$a+b+c \lesssim 2d$	$a+b+c \lesssim 2d$
$a \lesssim b+c$	$b < d$	$a \lesssim d$
$b < a+c$	$c \lesssim d$	$b < d$
$c \lesssim a+b$	$d \lesssim b+c$	$d \lesssim a+b$

and x is given, respectively, by

$(o,1/2(a+b-c),1/2(-a+b+c))$, $(o,d-c,b+c-d)$, $(o,a+b-d,d-a)$.

Obviously, (G) may satisfy the conditions of all three columns only if $a+b+c = 2d$; it is easy to see that all three columns coincide in this case. If $a+b+c > 2d$, then (G) may satisfy only the conditions of the first column; note that $b < a+c$ could not be replaced by $b \lesssim a+c$, because in this case for $b = a+c$ we have $x = (o,a,c)$ which is in C(G).

If $a+b+c < 2d$, then (G) could satisfy one of the last two columns or both. For example, if $a=1$, $b=1$, $c=3$, $d=3$, (G) satisfies the second column but not the third and if $a=3$, $b=1$, $c=1$, $d=3$, (G) satisfies the third column but not the second. If $a=1$, $b=2$, $c=1$, $d=3$, (G) satisfies both and gives two different payoffs (o,2,o) and (o,o,2), respectively. Note that $b < d$ could not be replaced by $b \leq d$, because in this case for $b=d$ we have (o,d-c,c) and/or (o,a,d-a) which are in C(G).

Similar results and remarks could be given for (B_2) and (B_3).

<u>Lemma</u> 4.4: For $\mathcal{S} = (13,2)$, there is a coalitionally rational $x \in M_c - C(G)$ if and only if (G) satisfies the conditions of one of the columns

$a+b+c \geq 2d$	$a+b+c \leq 2d$	$a+b+c \leq 2d$
$a \leq b+c$	$a \leq d$	$b \leq d$
$b \leq a+c$	$c < d$	$c < d$
$c < a+b$	$d \leq a+c$	$d \leq b+c$

and x is given, respectively, by

(1/2(a-b+c),o,1/2(-a+b+c)) (a+c-d,o,d-a) (d-b,o,b+c-d).

<u>Lemma</u> 4.5: For $\mathcal{S} = (12,3)$, there is a coalitionally rational $x \in M_c - C(G)$ if and only if (G) satisfies the conditions of one of the columns

$a+b+c \geq 2d$	$a+b+c \leq 2d$	$a+b+c \leq 2d$
$a < b+c$	$a < d$	$a < d$
$b \leq a+c$	$b \leq d$	$c \leq d$
$c \leq a+b$	$d \leq a+b$	$d \leq a+c$

and x is given, respectively, by

(1/2(a-b+c),1/2(a+b-c),o) (d-b,a+b-d,o) (a+c-d,d-c,o).

The results obtained in case (B) can be put together in two theorems

<u>Theorem</u> 4.6: If a nontrivial game (G) satisfies $a+b+c \geq 2d$ and the triangle inequalities $a \leq b+c$, $b \leq a+c$, $c \leq a+b$, then the coalitionally rational payoffs in M_c-C(G) for coalition structures consisting of one singleton and one coalition of cardinality two are

$(o, 1/2(a+b-c), 1/2(-a+b+c))$ for $\mathcal{S} = (23,1)$, if $b < a+c$;

$(1/2(a-b+c), o, 1/2(-a+b+c))$ for $\mathcal{S} = (13,2)$, if $c < a+b$;

$(1/2(a-b+c), 1/2(a+b-c), o)$ for $\mathcal{S} = (12,3)$, if $a < b+c$.

<u>Proof</u>: Follows from Lemmas 4.3, 4.4, 4.5, in which the results given in the first columns have been used, taking into account that all three columns coincide when $a+b+c = 2d$. Moreover, all triangle inequalities can be satisfied with equality signes only for $a=b=c=o$; therefore, except for the trivial game $a=b=c=d=o$, there is always at least one strict inequality, hence the payoffs in M_c-C(G) shown in Theorem 4.6 are exactly those given by the Lemmas.

<u>Theorem</u> 4.7: If (G) satisfies $a+b+c < 2d$ and the inequalities $a \leq d$, $b \leq d$, $c \leq d$, then the coalitionally rational payoffs in M_c-C(G) for coalition structures consisting of one singleton and one coalition of cardinality two are

$(o, d-c, b+c-d)$, $(o, a+b-d, d-a)$ for $\mathcal{S} = (23,1)$, if $b < d$ and $d \leq b+c$, $d \leq a+b$;

$(a+c-d, o, d-a)$, $(d-b, o, b+c-d)$ for $\mathcal{S} = (13,2)$, if $c < d$ and $d \leq a+c$, $d \leq b+c$;

$(d-b, a+b-d, o)$, $(a+c-d, d-c, o)$ for $\mathcal{S} = (12,3)$, if $a < d$ and $d \leq a+b$, $d \leq a+c$.

<u>Proof</u>: Follows from Lemmas 4.3, 4.4, 4.5, in which the results given in the last two columns have been used. In each of the six situations the inequality added is redundant; for example, from $a+b+c < 2d$ and $d \leq b+c$ follows $a < d$. Moreover, among the inequalities $a \leq d$, $b \leq d$, $c \leq d$, at least two should be strict, because $a+b+c < 2d$, hence the payoffs in M_c-C(G) given by Theorem 4.7 are exactly those given

by the Lemmas.

Note that if $a+b+c \geqslant 2d$ we may have at most 3 payoffs in M_c-C(G); for example, when a=3, b=4, c=5, d=2, we get the payoffs (o,1,3), (2,o,3), (2,1,o). If $a+b+c < 2d$ we may have at most 6 payoffs in M_c-C(G); for example, when a=2, b=2, c=3, d=4, we get the payoffs (o,1,1), (o,o,2), (1,o,2), (2,o,1), (2,o,o), (1,1,o). However, it is also possible that no coalitionally rational payoff in M_c-C(G) does exist for case (B). If $a+b+c \geqslant 2d$, this situation occurs when at least one triangle inequality does not hold; for example, when a=4, b=1, c=1, d=2, this happens even though $a+b+c \geqslant 2d$ holds, because we have $a > b+c$. If $a+b+c < 2d$, this situation occurs when $d > a+b$, $d > b+c$, $d > a+c$; for example, this is the case when a=b=c=1, d=3.

Case (C): $\mathcal{S} = (123)$, $x = (x_1, x_2, x_3)$, where

$x_1+x_2+x_3=d$, $x_1 \geqslant o$, $x_2 \geqslant o$, $x_3 \geqslant o$, $x_1+x_2 \geqslant a$, $x_2+x_3 \geqslant b$, $x_1+x_3 \geqslant c$.

As all excesses are nonpositive, we have

Theorem 4.8: In game (G), there is no coalitionally rational payoff in M_c-C(G) for $\mathcal{S} = (123)$.

Theorems 4.2, 4.6, 4.7, 4.8, describe completely the set of coalitionally rational payoffs in M_c-C(G). In Aumann/Maschler paper (1964) is given the bargaining set M of (G), which obviously contains also the core. However, it will be easy to separate the non core elements of M and to compair them with those in M_c-C(G). For the sake of completeness, we give the Aumann/Maschler results hereafter.

Theorem 4.9, (Aumann/Maschler, Th.4.1): In (G), if the grand coalition can not be formed, essentially two cases arise:

Case (a): If a, b, c, satisfy the triangle inequalities $a \leqslant b+c$, $b \leqslant a+c$, $c \leqslant a+b$, then the bargaining set M is:

(o,o,o;1,2,3), (1/2(a-b+c),1/2(a+b-c),o;12,3),

(1/2(a-b+c),o,1/2(-a+b+c);13,2), (o,1/2(a+b-c),1/2(-a+b+c);23,1).

Case (b): If, e.g. $a > b+c$, then the bargaining set M is:

(o,o,o;1,2,3), $(c \leq x_1 \leq a-b, a-x_1, o; 12,3)$

(c,o,o;13,2), (o,b,o;23,1).

<u>Theorem</u> 4.1o, (Aumann/Maschler,Th.5.1): In (G), the bargaining set M consists of the admissible payoff configurations given by Theorem 4.9 and also the admissible payoff configurations $(x_1,x_2,x_3;123)$ which satisfy $x_1+x_2+x_3=d$, $x_1 \geq o$, $x_2 \geq o$, $x_3 \geq o$, $x_1+x_2 \geq a$, $x_2+x_3 \geq b$, $x_1+x_3 \geq c$. The latter payoff configurations exist if and only if

$$a \leq d, \quad b \leq d, \quad c \leq d, \quad a+b+c \leq 2d. \qquad (4.11)$$

Now, we intend to compair our results (Ths. 4.2, 4.6, 4.7, 4.8) with Aumann/Maschler results (Ths. 4.9, 4.1o), taking into account that the latter contain also core payoffs. We consider separately each coalition structure.

For $\mathcal{S}$ = (1,2,3), x = (o,o,o) is always in M, while this payoff belongs to M_c-C(G) only in those cases explained in Theorem 4.2. The reason is that Aumann/Maschler definition of M does not allow any objection against (o,o,o;1,2,3).

Consider the coalition structures consisting of one singleton and one coalition of cardinality two. Let us discuss only the coalition structure $\mathcal{S}$ = (12,3), because the payoff configurations in M are completely shown in Aumann/Maschler proof of Theorem 4.9, (1964,p.456); in fact, they can be obtained from Theorem 4.9, (including the circular permutation of results given in case (b)). These payoff configurations are

$(1/2(a-b+c), 1/2(a+b-c), o; 12,3)$ if $a \leq b+c$, $b \leq a+c$, $c \leq a+b$ (4.12)

$(c \leq x_1 \leq a-b, a-x_1, o; 12,3)$ if $a > b+c$ (4.13)

(o,a,o;12,3) if $b > a+c$ (4.14)

(a,o,o;12,3) if $c > a+b$ (4.15)

As shown by Theorem 4.6, (4.12) belongs also to M_c-C(G), however, $a+b+c \geq 2d$ and $a < b+c$ should also be satisfied. The reason is that Aumann/Maschler definition of M does not allow an objection based upon the grand coalition and this fact eliminates our condition

$a+b+c \geqslant 2d$. The second condition, $a < b+c$ is eliminating the payoffs belonging to the core which are included in Aumann/Maschler results.

It is easy to see that the payoffs (4.13) give $e(x,23)=x_1-(a-b) \leqslant o$, $e(x,13)=c-x_1 \leqslant o$. As the grand coalition can not be used for objection and all excesses are nonpositive, the payoffs (4.13) are not threatened by any Aumann/Maschler objection. Our approach is either including them in the core, if $e(x,123)=d-a \leqslant o$, or excluding them from $M_c-C(G)$ because (123) is a bargaining proposal which can not be countered, if $e(x,123)=d-a > o$.

For (4.14), i.e. $\mathcal{S}=(12,3)$, $x_1=o$, $x_2=a$, $x_3=o$, any $(\varepsilon,o,c-\varepsilon;13,2)$ with $o < \varepsilon \leqslant c$ is an objection of 1 against 2; if $b > a+c$, any $(o,a+\eta,b-a-\eta;23,1)$ with $o \leqslant \eta \leqslant b-a-c+\varepsilon$ is a counter objection of 2 against 1. Further, any $(o,a+\varepsilon,b-a-\varepsilon;23,1)$ with $o < \varepsilon \leqslant b-a$ is an objection of 2 against 1, which can be countered by 1 playing alone. As no objection by means of the grand coalition is allowed, $(o,a,o;12,3)$ is in M. In our approach, we have $e(x,13)=c$, $e(x,23)=b-a$, $e(x,123)=d-a$; if $b > a+c$, (13) is a bargaining proposal and (23) is a compensatory bargaining counter proposal because $e(x,23) > e(x,13)$. However, (23) is also a bargaining proposal which can not be compensated by (13) or a singleton. If $b=d$, then $(o,a,o) \in M_c-C(G)$ and $(o,a,o) \notin C(G)$, because $e(x,23)=e(x,123)$, otherwise either (23) or (123) can not be countered. Similarly, for (4.15) if $c > a+b$ then $(a,o,o) \in M$, but $(a,o,o) \notin M_c-C(G)$ and $(a,o,o) \notin C(G)$, except if $c=d$. Clearly, the compensatory commitments are eliminating (4.14) and (4.15) from the compensatory bargaining set, together with the possibility of using the grand coalition as a bargaining proposal.

Note that Theorem 4.7 is giving other payoffs which belong to $M_c-C(G)$ and do not belong to M because a counter objection can not use the grand coalition, so that some objections can not be countered.

For $\mathcal{S}$ = (123), if (G) satisfies (4.11) the payoffs are belonging to the core when they are in M, so that they can not be in M_c-C(G).

The above discussion shows some of the differences between the c-stable payoffs and the payoffs included in the Aumann/Maschler stable payoff configurations. Some c-stable payoffs can not be included in stable payoff configurations because the grand coalition can be used for compensatory countering, but can not be used for counter objections. Some objections can be countered because no compensation is needed in M, while the corresponding bargaining proposals are valid because no compensation is possible. Other relevant differences have not been illustrated because (G) is a 3-person game and/or has a particular form, so that those situations can not occur.

Note that we have exhibited a game for which there are c-stable admissible payoffs outside the core at least for some coalition structures when the parameters of the game satisfy some conditions, hence the concept of c-stability is meaningful. Of course, the existence problem in general games is interesting.

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