A METHOD FOR EXACT SOLUTIONS TO INTEGRABLE EVOLUTION EQUATIONS IN 2+1 DIMENSIONS

by

ALICIA MACHUCA

Presented to the Faculty of the Graduate School of

The University of Texas at Arlington in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT ARLINGTON

August 2014

ACKNOWLEDGEMENTS

I would like to thank Professor Tuncay Aktosun for introducing me to the problem and for his help. I would also like to thank Professor Jianzhong Su and Professor Fengxin Chen for their support.

ABSTRACT

A METHOD FOR EXACT SOLUTIONS TO INTEGRABLE EVOLUTION EQUATIONS IN 2+1 DIMENSIONS

Alicia Machuca, Ph.D.

The University of Texas at Arlington, 2014

Supervising Professor: Tuncay Aktosun

A systematic method is developed to obtain solution formulas for certain explicit solutions to integrable nonlinear partial differential equations in two spatial variables and one time variable. The method utilizes an underlying Marchenko integral equation that arises in the corresponding inverse scattering problem. The method is demonstrated for the Kadomtsev-Petviashvili and the Davey-Stewartson equations. A derivation and analysis of the solution formulas to these two nonlinear partial differential equations are given, and an independent verification of the solution formulas is presented. Such solution formulas are expressed in a compact form in terms of matrix exponentials, by using a set of four constant matrices as input. The formulas hold for any sizes of the matrix quadruplets and hence yield a large class of explicit solutions. The method presented is a generalization of a method used to find exact solutions for integrable evolution equations in one spatial variable and one time variable, which uses a constant matrix triplet as input.

TABLE OF CONTENTS

CHAPTER 1

Introduction

The ability to capture the behavior of complex systems is one of the most exciting attributes of nonlinear partial differential equations (PDEs). Unfortunately nonlinear PDEs generally do not yield explicit solutions. We present a technique to obtain an explicit solution formula for certain solutions to a certain class of nonlinear PDEs, known as integrable nonlinear evolutions equations. The study of integrable nonlinear PDEs is interesting to mathematicians, engineers and physicists because they have physically important solutions that can be expressed in terms of elementary functions. Soliton solutions to some integrable nonlinear PDEs can be used to describe the propagation of surface water waves as well as the behavior of quantum particles and their interactions.

In 1967 Gardner, Greene, Kruskal and Miura outlined a method using direct and inverse scattering to solve an initial value problem to the Korteweg-deVries equation $u_t - 6uu_x + u_{xxx} = 0$, where the subscripts signify the respective x and t derivatives. Soliton solutions to the KdV equation are physically important because they can be used to model surface waves in shallow, narrow canals [10]. Since then this method has been shown to be applicable to many other nonlinear PDEs. This method is now referred to as the inverse scattering transform. Those nonlinear PDEs that are solvable by the inverse scattering transform are called integrable. My advisor, Tuncay Aktosun, and his collaborators, in 2006, developed a systematic method to find exact solution formulas to integrable nonlinear PDEs which yields a solution formula that will have as its input a triplet of constant matrices (A, B, C) of sizes $p \times p$, $p \times 1$ and

$1 \times p$, respectively [5], in this case we will say that the quadruplet of matrices has size p. This method has been applied to the Korteweg-deVries equation, sine-Gordon equation, nonlinear Schrödinger equation and the Toda lattice equation

$$u_t - 6uu_x + u_{xxx} = 0, \qquad \text{(Korteweg-deVries eq)}$$

$$iu_t + u_{xxx} + 2|u|^2 u = 0, \qquad \text{(nonlinear Schrödinger eq)}$$

$$u_{xt} = \sin u, \qquad \text{(sine-Gordon eq)}$$

$$u_t(n,t) = e^{u(n-1,t)-u(n,t)} - e^{u(n,t)-u(n-1,t)}, \qquad \text{(Toda lattice eq)}$$

where the subscripts signify the respective x and t derivatives, and is generalizable to other 1+1, one spatial variable and one time variable, integrable nonlinear PDEs [3, 4, 5]. The solution formulas to each of these equations will have as its input a triplet of constant matrices (A, B, C) and be independent of the size of this triplet. By using matrix exponentials as building blocks we are able to construct solutions that are algebraic combinations of elementary functions. The matrices (A, B, C) are able to hold large amounts of information which gives these solution formulas a compact form that can be written and worked with more easily.

The goal of my dissertation is to describe a systematic method to find certain solution formulas that provide a large class of solutions that can be applied to (2+1) dimension, two spatial variables and one time variable, integrable nonlinear PDEs. I have outlined this method in its application to the Kadomtsev-Petvishvili II equation and the generalized Davey-Stewartson II system

$$(u_t - 6uu_x + u_{xxx})_x \pm 3u_{yy} = 0, \qquad \text{(Kadomtsev-Petviashvili II eq)}$$

$$
\begin{cases}
iu_t + u_{xx} - u_{yy} - 2uvu + \phi u = 0, \\
iv_t - v_{xx} + v_{yy} + 2vuv - v\phi = 0, \\
\phi_{xx} + \phi_{yy} - 2(uv + vu)_{yy} = 0,
\end{cases}
\qquad \text{(Davey-Stewartson II system)}
$$

In the case of (2+1) dimension equations, two spatial variables x and y and one time variable t, the solution formulas will have a quadruplet of matrices (A, M, B, C) as input. The solution formula is independent of the size of these matrices and has a compact form, similar to that of the (1+1) dimension case.

The method that I have used is based on the inverse scattering transform and the idea that each integrable nonlinear PDE is associated to a linear differential equation where the solution $u(x,t)$ to the nonlinear PDE appears as a coefficient in the linear differential equation. The initial scattering data $S(\lambda;0,0)$ will be found using this associated linear differential equation and its time evolution would need to be determined. By imposing appropriate restrictions on $u(x,0,0)$ we can have a one-to-one correspondence between $u(x,0,0)$ and $S(\lambda;0,0)$ which also assures a one-to-one correspondence between $u(x,y,t)$ and $S(\lambda;y,t)$. In the classic framework of the inverse scattering transform the time evolution of the scattering data would be used to recover the solution $u(x,y,t)$. The method of the inverse scattering transform is shown in the diagram below.

$$
\begin{array}{ccc}
u(x,0,0) & \xrightarrow{\text{direct scattering}} & S(\lambda;\,0,0) \\
\text{solution to IVP}\ \Big\downarrow & & \Big\downarrow\ \text{time evolution} \\
u(x,y,t) & \xleftarrow[\text{inverse scattering}]{} & S(\lambda;\,t,y)
\end{array}
$$

It is important to note that the evolution of the scattering data will be unique to each nonlinear PDE.

The work presented in this thesis focuses on an associated Marchenko integral equation,

$$K(x,\xi;t,y)+\Omega(x,\xi;t,y)+\int_x^\infty d\eta\,K(x,\eta;t,y)\,\Omega(\eta,\xi;t,y)=0 \quad \text{for } x<\xi.$$

The Marchenko integral equation is a linear integral equation associated to the scattering data through a Fourier transform. To find a certain solution formula we use the relationship between the time evolution of the Marchenko kernel, $\Omega(x,\xi;y,t)$, and $u(x,y,t)$. The relationship between the kernel of the Marchenko equation and inverse scattering transform can be visualized in the following diagram:

$$\begin{array}{ccccc} u(x,0,0) & \xrightarrow{\text{direct scattering}} & S(\lambda;0,0) & \xrightarrow{\text{Fourier transform}} & \Omega(x,\xi;0,0) \\ \downarrow & & \downarrow & & \downarrow \\ u(x,y,t) & \xleftarrow[\text{inverse scattering}]{} & S(\lambda;t,y) & \xleftarrow[\text{inverse Fourier transform}]{} & \Omega(x,\xi;t,y) \end{array}.$$

Using the Marchenko kernel in a separable form, written in terms of a quadruplet of matrices (A,M,B,C), we are able to find a simple relationship between $K(x,x;t,y)$ and $u(x,y,t)$. Through the relationship between $K(x,x:t,y)$ and $u(x,y,t)$ we are ultimately able to obtain a general solution formula for $u(x,y,t)$ in terms of (A,M,B,C). This method can be generalized to all integrable (2+1) nonlinear PDEs with an associated Marchenko integral equation. I have outlined this method in its application to the Kadomtsev-Petviashvili II equation and the generalized Davey-Stewartson II system. The generalized Davey-Stewartson II system is an an interesting example of an integrable nonlinear PDE with a matrix Marchenko integral equation. Furthermore, the solution formulas are independent of the size of the matrices, which allows for operator solutions as well.

The availability of a large class of solutions is important to numerical analyst who may use them as a way to test their numerical methods. With this solution formula the user will be able to choose from a large class of quadruplets of constant matrices (A, M, B, C) to produce an exact solution formula. This resulting exact solution formula can be used to test the accuracy of numerical methods that are intended to solve certain nonlinear partial differential equations. In this paper we will provide the solution formulas for the Kadomtsev-Petviashvili II equation and the generalized Davey-Stewartson II system.

This thesis is organized as follows. In Chapter 2 we describe the application of this method to the Kadomtsev-Petivashvili II equation. A thorough description of the process will be given as well as a study of the properties of the exact solution formulas found. This chapter will conclude with the presentation of an example. In Chapter 3 we describe the application of this method to the generalized Davey-Stewartson II system. This chapter will focus on the differences that occur when the integrable evolution equation has an associated matrix Marchenko integral equation.

CHAPTER 2

An Exact Solution Formula to the Kadomtsev-Petviashvilli Equation

In this chapter we use our unified method to find an explicit solution formula for certain solutions to the Kadomtsev-Petviashvili II (KP II) equation [11]

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \tag{2.1}$$

where the subscripts signify the respective x, y, and t-derivatives. The KP II equation is related to the KP I equation

$$(u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{2.2}$$

through a transformation $y \to iy$. The KP II equation is an integrable nonlinear partial differential equation (PDE) in terms of $u(x, y, t)$ which has two spatial variables x and y, and one time variable t, known as a (2+1) dimension equation. The KP II equation (2.1) is an extension of the well studied Korteweg-deVries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2.3}$$

The KdV equation is an integrable nonlinear PDE in terms of $u(x, t)$ which has one spatial variable x and one time variable t, known as a (1+1) dimension equation. The motivation behind this method comes from the work done by Aktosun and van der Mee in the case of integrable evolution equations in (1+1) dimension [3, 4, 5]. In their work they have developed a systematic method to find exact solutions to certain integrable nonlinear PDEs. This method yields a solution formula to (2.3) that will have as its input a triplet of constant matrices (A, B, C) of sizes $p \times p$, $p \times 1$ and $1 \times p$, respectively, where p is an positive integer.

The KdV equation is related to the linear ordinary differential equation [10]

$$-\frac{d^2}{dx^2}\psi + u(x,t)\psi = k^2\psi, \tag{2.4}$$

which is known as the Schrödinger equation and $u(x,t)$ is usually called the potential. An analysis of the spectral problem for (2.4) determines the scattering data. The underlying (1+1) Marchenko integral equation associated with the KdV equation is

$$K(x,\xi;t) + \Omega(x,\xi;t) + \int_x^\infty d\eta\, K(x,\eta;t)\,\Omega(\eta,\xi;t) = 0, \quad x < \xi. \tag{2.5}$$

The Marchenko kernel $\Omega(\eta,\xi;t)$ appearing in (2.5) is related to the scattering data for (2.4) through a Fourier transform. Using the scattering data in the form of a rational function of the spectral parameter k, Aktosun and van der Mee were able [5] to show that the Marchenko kernel of the is given by

$$\Omega(\eta,\xi;t) = Ce^{8A^3t - A\eta}e^{-A\xi}B. \tag{2.6}$$

The form of $\Omega(\eta,\xi;t)$ in (2.6) possesses the symmetry $\eta \leftrightarrow \xi$, namely

$$\Omega(\eta,\xi;t) = \Omega(\xi,\eta;t). \tag{2.7}$$

The underlying (2+1) Marchenko integral equation assotiad with the KP II equation is

$$K(x,\xi;t,y) + \Omega(x,\xi;t,y) + \int_x^\infty d\eta\, K(x,\eta;t,y)\Omega(\eta,\xi;t,y) = 0, \quad x < \xi. \tag{2.8}$$

We are able to motivate the form of the (2+1) initial Marchenko kernel by using a form similar to (2.6). In the case of the (2+1) dimension KP II equation we incorporate an additional matrix M to account for the spatial variable y. As in the (1+1) case the Marchenko kernel will be written in terms of matrix exponentials. In the (2+1) case we relax the symmetry $\eta \to \xi$ given in (2.7) and choose the form of the initial Marchenko kernel to be

$$\Omega(\eta,\xi;0,0) = Ce^{-M\eta}e^{-A\xi}B,$$

where the matrices A and M in general need not commute. We expect the evolution of $\Omega(x, \xi; t, y)$ in t and y to occur in the following way:

$$\Omega(\eta, \xi; t, y) = Ce^{f_1(M)t+f_2(M)y}e^{-M\eta}e^{-A\xi}e^{g_1(A)t+g_2(A)y}B, \tag{2.9}$$

where $f_1(M)$, $f_2(M)$, $g_1(A)$, and $g_2(A)$ are functions corresponding to the unique evolution of t and y for the KP II equation. We expect the functions $f_1(M)$, $f_2(M)$, $g_1(A)$, and $g_2(A)$ to be simple functions of A and M, respectively.

A system of linear PDEs is used to determine the functions $f_1(M)$, $f_2(M)$, $g_1(A)$, and $g_2(A)$ that correspond to the KP II equation. With the evolution in t and y of $\Omega(\eta, \xi; t, y)$ given in (2.9) the kernel $\Omega(\eta, \xi; t, y)$ becomes separable in η and ξ. As a result, the corresponding Marchenko equation (2.8) can be solved explicitly by using techniques from linear algebra. The solution to the Marchenko integral equation $K(x, \xi; t, y)$ is related to $u(x, y, t)$, which is the unknown quantity in the KPII equation, in a simple way [2], namely

$$u(x, y, t) = -\frac{d}{dx}K(x, \xi; t, y). \tag{2.10}$$

We begin this chapter with a section that describes a systematic method to construct a solution formula for certain solutions to the KP II equation. One solution formula is given in (2.33) and the equivalent formulas are given in (2.37), (2.40), and (2.42). The input for the solution formula $u(x, y, t)$ will be a constant matrix quadruplet (A, M, B, C) of sizes $p \times p$, $p \times p$, $p \times 1$ and $1 \times p$, respectively, and we say that this quadruplet of matrices has size p. Here p can be chosen as any positive integer. For each choice of matrix quadruplet (A, M, B, C) we obtain a solution to the KP II equation. With certain restrictions on the constant matrix quadruplet (A, M, B, C) we will have solutions that are globally analytic in the xyt-space. For other choices there will be singularities that occur on a surface in the xyt-space. The second section in this chapter will be dedicated to studying the properties of our solution formula. In the third section we verify that our solution formula satisfies the KP II equation by providing a detailed independent verification. In the fourth section we outline how to construct certain solutions, using our solution formula, by starting with the input quadruplet (A, M, B, C). In section 2.5 we present an illustrative example.

2.1 Exact Solution Formula to the KP II Equation

In this section we make use of the temporal and spatial evolution of the kernel $\Omega(\eta, \xi; t, y)$ of the Marchenko integral equation given in (2.8). It is already known in [15] that the Marchenko kernel $\Omega(\eta, \xi; t, y)$ in (2.9) satisfies the system of linear partial differential equations

$$\begin{cases} \Omega_y + \Omega_\eta - \Omega_\xi = 0, \\ \Omega_t + 4\,\Omega_{\eta\eta\eta} + 4\,\Omega_{\xi\xi\xi} = 0. \end{cases} \tag{2.11}$$

The first equation in (2.11) describes the y evolution in terms of η and ξ in $\Omega(\eta, \xi; t, y)$. The second equation in (2.11) describes the t evolution in terms of η and ξ in $\Omega(\eta, \xi; t, y)$. To determine the temporal and spatial evolution, i.e. in t and in y, respectively, of the Marchenko kernel we use the Marchenko kernel found in (2.9). To determine the functions $f_1(M)$, $f_2(M)$, $g_1(A)$, and $g_2(A)$ we substitute (2.9) into (2.11). We begin the process of determining the functions $f_1(M)$, $f_2(M)$, $g_1(A)$, and $g_2(A)$ by defining

$$E_1 := e^{f_1(M)t + f_2(M)y} e^{-M\eta}, \quad E_2 := e^{-A\xi} e^{g_1(A)t + g_2(A)y},$$

so that $\Omega(\eta, \xi; t, y)$ in (2.9) can be written as

$$\Omega(\eta, \xi; t, y) = C E_1 E_2 B.$$

Taking the appropriate derivatives of (2.9), we obtain

$$\Omega_t = C[f_1(M)E_1E_2 + E_1E_2g_1(A)]B, \tag{2.12}$$

$$\Omega_y = C[f_2(M)E_1E_2 + E_1E_2g_2(A)]B. \tag{2.13}$$

Taking the derivatives of $\Omega(\eta, \xi; t, y)$ given in (2.9) with respect to η we obtain

$$\begin{aligned}\Omega_\eta =& C[-ME_1E_2]B, && (2.14)\\ \Omega_{\eta\eta} =& C[M^2E_1E_2]B, \\ \Omega_{\eta\eta\eta} =& C[-M^3E_1E_2]B, && (2.15)\end{aligned}$$

Similarly, the ξ-derivative of $\Omega(x, \xi; t, y)$ in (2.9) gives us

$$\begin{aligned}\Omega_\xi =& C[-E_1E_2A]B, && (2.16)\\ \Omega_{\xi\xi} =& C[E_1E_2A^2]B, \\ \Omega_{\xi\xi\xi} =& C[-E_1E_2A^3]B. && (2.17)\end{aligned}$$

Using (2.12)-(2.17) in (2.11), we obtain

$$\begin{cases} C[f_2(M)E_1E_2 + E_1E_2g_2(A) - ME_1E_2 + E_1E_2A]B = 0, \\ C[f_1(M)E_1E_2 + E_1E_2g_1(A) + 4M^2E_1E_2 + 4E_1E_2A^2]B = 0. \end{cases} \tag{2.18}$$

The equations in (2.18) are used to solve for $f_1(M)$, $f_2(M)$, $g_1(A)$, and $g_2(A)$ in terms of A and M respectively. We find

$$f_1(M) = 4M^3, \quad f_2(M) = -M^2, \quad g_1(A) = 4A^3, \quad g_2(A) = A^2. \tag{2.19}$$

With the help of (2.19) the kernel $\Omega(\eta, \xi; t, y)$ given in (2.9) can be written as

$$\Omega(\eta, \xi; t, y) = Ce^{-M\eta + 4M^3t - M^2y}e^{-A\xi + A^2y + 4A^3t}B. \tag{2.20}$$

It will be helpful to define the evolution in t and y as

$$C(t,y) := Ce^{4M^3t-M^2y}, \quad B(t,y) := e^{4A^3t+A^2y}B. \tag{2.21}$$

The evolution in t and y of $C(t,y)$ and $B(t,y)$ specified in (2.21) is unique to the KP II equation.

Note that the scalar function $\Omega(\eta,\xi;t,y)$ in (2.20) is written in terms of matrix exponentials and as the vector product of the row p-vector $C(t,y)e^{-M\eta}$ and the column p-vector $e^{-A\xi}B(t,y)$. Thanks to the use of matrix exponentials, the form of $\Omega(\eta,\xi;t,y)$ given in (2.20) allows us to take advantage of the separability of $\Omega(\eta,\xi;t,y)$ in η and ξ.

It can be seen from (2.8) that the separability of $\Omega(x,\xi;t,y)$ leads to the separability of $K(x,\xi;t,y)$ in x and ξ. We can show this separability by substituting (2.20) into (2.8). We then obtain

$$K(x,\xi;t,y)+C(t,y)e^{-Mx}e^{-A\xi}B(t,y)+\int_x^\infty d\eta\, K(x,\eta;t,y)C(t,y)e^{-M\eta}e^{-A\xi}B(t,y)=0. \tag{2.22}$$

Isolating $K(x,\xi;t,y)$ on the left side of the equation (2.22), we get

$$\begin{aligned} K(x,\xi;t,t) = &- C(t,y)e^{-Mx}e^{-Ax}B(t,y) \\ &- \left[\int_x^\infty d\eta\, K(x,\eta;t,y)C(t,y)e^{-M\eta}\right] e^{-A\xi}B(t,y). \end{aligned}$$

Equivalently, we can write $K(x,\xi;t,y)$ as

$$K(x,\xi;t,y) = H(x;t,y)e^{-A\xi+4A^3t+A^2y}B, \tag{2.23}$$

where the $1\times p$ matrix valued function $H(x;t,y)$ is defined to be

$$H(x;t,y) := C(t,y)e^{-Mx} + \int_x^\infty d\eta\, K(x,\eta;t,y)C(t,y)e^{-M\eta}.$$

From (2.23) we see that $K(x, \xi; t, y)$ can be written as the product of a $1 \times p$ matrix valued function of x, namely $H(x; t, y)$, and a $p \times 1$ matrix valued function of ξ, namely $e^{-A\xi+4A^3t+A^2y}B$. Notice that the separability of $K(x, \xi; t, y)$ in x and ξ is unaffected by the presence of t and y.

We look for solutions to the Marchenko integral equation in the form of (2.23) by substituting (2.23) into (2.22). This procedure gives us

$$\begin{aligned} &H(x; t, y)e^{-A\xi}B(t, y) + C(t, y)e^{-Mx}e^{-Ax}B(t, y) \\ &\qquad + \int_x^\infty d\eta\, H(x; t, y)e^{-A\eta}B(t, y)C(t, y)e^{-Mx}e^{-A\eta}B(t, y) = 0. \end{aligned} \tag{2.24}$$

Factoring $e^{-A\xi}B(t, y)$ on the right from (2.24) we obtain

$$\left(H(x; t, y) + C(t, y)e^{-Mx} + \int_x^\infty d\eta H(x; t, y)e^{-A\eta}B(t, y)C(t, y)e^{-M\eta}\right) e^{-A\xi}B(t, y) = 0. \tag{2.25}$$

Note that (2.25) is equivalent to

$$H(x; t, y) + C(t, y)e^{-Mx} + \int_x^\infty d\eta\, H(x; t, y)e^{-A\eta}B(t, y)C(t, y)e^{-M\eta} = 0. \tag{2.26}$$

Factoring $H(x; t, y)$ on the left, from (2.26), we obtain

$$H(x; t, y) \left[I + \int_x^\infty d\eta\, e^{-A\eta}B(t, y)C(t, y)e^{-M\eta}\right] = -C(t, y)e^{-Mx}, \tag{2.27}$$

where I is the $p \times p$ identity matrix. When solving (2.27) for $H(x; t, y)$ it will be helpful to define the $p \times p$ matrix-valued function $\Gamma(x; t, y)$ as

$$\Gamma := I + \int_x^\infty d\eta\, e^{-A\eta}B(t, y)C(t, y)e^{-M\eta}. \tag{2.28}$$

Equivalently, using (2.21) we can write (2.28) as

$$\Gamma = I + \int_x^\infty d\eta\, e^{-A\eta+A^2y+4A^3t}BCe^{-M\eta-M^2y+4M^3t}. \tag{2.29}$$

The form of $H(x;t,y)$ from (2.23) can now be written explicitly in terms of the matrix quadruplet (A, M, B, C) as

$$H(x;t,y) = -Ce^{-Mx-M^2y+4M^3t}\Gamma^{-1}, \tag{2.30}$$

where Γ is the matrix given in (2.29).

Finally, using (2.30) in (2.23), we obtain the solution $K(x,\xi;t,y)$ to (2.22) as

$$K(x,\xi;t,y) = -Ce^{-Mx-M^2y+4M^3t}\Gamma^{-1}e^{-A\xi+4A^3t+A^2y}B, \tag{2.31}$$

where $\Gamma(x;t,y)$ is the matrix given in (2.29).

The relationship between the solution $K(x,\xi;t,y)$ to the Marchenko integral equation (2.8) and the solution $u(x,y,t)$ to the KP II equation (2.1) is already known [15], and it is given by

$$u(x,y,t) = -2\frac{d}{dx}K(x,x;t,y). \tag{2.32}$$

Using (2.31) in (2.32) we obtain $u(x,y,t)$ as

$$u(x,y,t) = 2C\frac{\partial}{\partial x}\left(e^{-Mx-M^2y+4M^3t}\Gamma^{-1}e^{-Ax+A^2y+4A^3t}\right)B, \tag{2.33}$$

where $\Gamma(x;t,y)$ is the matrix appearing in (2.29).

It is possible to express our solution formula given in (2.33) as various equivalent forms. For example we can express it in terms of the matrix Γ appearing in (2.29). The solution $u(x,y,t)$ appearing in (2.33) is a scalar quantity. As a scalar quantity $u(x,y,t)$ is equal to the matrix trace of the quantity on the right hand side of (2.33). Thus,

$$u(x,y,t) = \mathrm{tr}(2C\frac{\partial}{\partial x}\left(e^{-Mx-M^2y+4M^3t}\Gamma^{-1}e^{-Ax+A^2y+4A^3t}\right)B), \tag{2.34}$$

where tr denotes the matrix trace (sum of the diagonal entries). We will next use certain properties of the matrix trace, namely

$$\operatorname{tr}(\frac{\partial \alpha}{\partial x}) = \frac{\partial}{\partial x}\left[\operatorname{tr}(\alpha)\right], \tag{2.35}$$

$$\operatorname{tr}(\alpha\beta) = \operatorname{tr}(\beta\alpha), \tag{2.36}$$

which are valid for any matrices α and β, where the products $\alpha\beta$ and $\beta\alpha$ are meaningful. Using (2.35) and (2.36) in (2.34), we write our solution formula (2.33) as

$$u(x,y,t) = 2\frac{\partial}{\partial x}\operatorname{tr}\left(\Gamma^{-1}e^{Ax+A^2y+4A^3t}BCe^{-Mx-M^2y+4M^3t}\right). \tag{2.37}$$

Notice that from (2.29) we obtain Γ_x, the derivative of Γ, as

$$\Gamma_x = -e^{-Ax+A^2y+4A^3t}BCe^{-Mx-M^2y+4M^3t}.$$

Recall the identity that [7]

$$\operatorname{tr}[\alpha^{-1}\alpha_x] = \frac{(\det\alpha)_x}{\det\alpha}, \tag{2.38}$$

for any square matrix α with entries that are functions of x. This allows us to have an alternate form of our solution (2.37), namely

$$u(x,y,t) = -2\frac{d}{dx}\operatorname{tr}(\Gamma^{-1}\Gamma_x). \tag{2.39}$$

Using (2.38) in (2.39), we obtain

$$u(x,y,t) = -2\frac{d}{dx}\left[\frac{(\det\Gamma)_x}{\det\Gamma}\right]. \tag{2.40}$$

Recall the identity that

$$\frac{f_x}{f} = (\log f)_x \tag{2.41}$$

for any scalar function of x. Using (2.41) in (2.40), we obtain

$$u(x,y,t) = -2\frac{d}{dx}\left[\frac{d}{dx}\log(\det\Gamma)\right]. \tag{2.42}$$

In the context of $K(x,x;t,y)$ and $\Gamma(x;t,y)$ we consider t and y as parameters, which establishes the use of the regular derivative rather than the partial derivative. The form of $u(x,y,t)$ in (2.42) can be compared to the solution of an equation related to the KP II equation in (2.1) found in [14]. The form of $u(x,y,t)$ in (2.42) can also be compared to the solution of the KP II equation in [12] where $\Gamma(x;t,y)$ is related to the so called τ function.

2.2 Properties of the Auxiliary Matrix Γ

In the previous section we have obtained a solution formula for certain solutions to the KP II equation (2.1). Namely, we obtained the formula (2.33) in terms of the matrix quadruplet (A,M,B,C). In this section we analyze our solution formula. For certain choices of (A,M,B,C) we will have solutions that are valid on the entire xyt-space. This solution formula will also provide solutions that are analytic everywhere except on the surface $\det[\Gamma(x;t,y)]=0$. The solution formula given in (2.33) will provide solutions that can be expressed as a Taylor series for all $x,y,,t\in\mathbb{R}$ such that $\det[\Gamma(x;t,y)]\neq 0$. In this section we present a generalized expression for $\Gamma(x;t,y)$ that is no longer dependent on an integral as in (2.28). A thorough verification of the solution formula is given in the following section to show that this solution formula is indeed a solution to the KP II equation.

We begin by considering $\Gamma(x;t,y)$ in (2.28) which can also be written as

$$\Gamma(x;t,y)=I+N(x;t,y), \tag{2.43}$$

where

$$N(x;t,y):=\int_x^\infty d\eta\, e^{-A\eta+A^2y+4A^3t}BCe^{-M\eta-M^2y+4M^3t}. \tag{2.44}$$

The integral defining $N(x;t,y)$ exists for certain choices of the matrix quadruplet (A,M,B,C). It is known that (2.44) exists when A and M are chosen such that the eigenvalues of A and M have positive real parts. The proof of this is similar to the proof given for Proposition 4.1 in [3]. Our goal will then be to write $N(x;t,y)$ in a generalized form that is not dependent on an integral.

Theorem 2.1. *Let A and M be real constant matrices of size $p \times p$. Assume that the eigenvalues of A and M have positive real parts. Then, for every x, y and $t \in \mathbb{R}$ the matrix $N(x;t,y)$ defined in (2.44) satisfies*

$$N(x;t,y) = e^{-Ax-A^2y+4A^3t}Pe^{-Mx-M^2y+4M^3t}, \tag{2.45}$$

with the constant $p \times p$ matrix P defined as

$$P := N(0;0,0) = \int_0^\infty d\eta\, e^{-A\eta}BCe^{-M\eta}. \tag{2.46}$$

Proof. We know that the integral defining $N(x;t,y)$ in (2.44) exists. Our goal is to write $N(x;t,y)$ in a more general form that is not dependent on an integral. Using the definition of $\Omega(\eta,\xi;t,y)$ in (2.21) the integral $N(x;t,y)$ from (2.44) can be written as

$$N(x;t,y) = e^{A^2y+4A^3t}\left(\int_x^\infty d\eta\, e^{-A\eta}BCe^{-M\eta}\right)e^{-M^2y+4M^3t}. \tag{2.47}$$

Using the change of variable $s = \eta - x$ we now have

$$\begin{aligned} N(x;t,y) &= e^{A^2y+4A^3t}\left[\int_0^\infty ds\, e^{-A(s+x)}BCe^{-M(s+x)}\right]e^{-M^2y+4M^3t} \\ &= e^{-Ax+A^2y+4A^3t}\left(\int_0^\infty ds\, e^{-As}BCe^{-Ms}\right)e^{-Mx-M^2y+4M^3t} \\ &= e^{-Ax+A^2y+4A^3t}N(0;0,0)e^{-Mx-M^2y+4M^3t}. \end{aligned}$$

We have shown $N(x;t,y)$ in a more general form in terms of the constant matrix quadruplet (A,M,B,C) and the constant matrix $P = N(0;0,0)$ as in (2.45) and (3.70). □

We now have a formulation for $\Gamma(x;t,y)$ that no longer depends on an integral but rather we can write $\Gamma(x;t,y)$ in terms of the matrix quadruplet (A,M,B,C) and the constant matrix P. Using (2.45) in (2.43) we obtain an alternate form of $\Gamma(x;t,y)$ as

$$\Gamma(x;t,y) = I + e^{-Ax+A^2y+4A^3t} P e^{-Mx-M^2y+4M^3t}. \tag{2.48}$$

We can now also write $K(x,\xi;t,y)$ appearing in (2.31) in terms of the matrix quadruplet (A,M,B,C) and the constant matrix P. Using the matrix identity for the product of two matrices and (2.31) we rewrite $K(x,\xi;t,y)$ as

$$K(x,\xi;t,y) = \left(e^{Ax-A^2y-4A^3t} \Gamma e^{Mx+M^2y-4M^3t}\right)^{-1}. \tag{2.49}$$

Substituting (2.48) in (2.49) we get

$$K(x,\xi;t,y) = \left(e^{Ax-A^2y-4A^3t}[I + e^{-Ax-A^2y+4A^3t} P e^{-Mx-M^2y+4M^3t}] e^{Mx+M^2y-4M^3t}\right)^{-1}. \tag{2.50}$$

Further simplifications of (2.50) using the product of matrices lead to this final form of $K(x,\xi;t,y)$ as

$$K(x,\xi;t,y) = \left(e^{Ax-A^2y-4A^3t} e^{Mx+M^2y-4M^3t} + P\right)^{-1}. \tag{2.51}$$

Let $\{a_1,\ldots,a_p\}$ and $\{m_1,\ldots,m_p\}$ be the eigenvalues of the $p\times p$ matrices A and M, respectively. By writing Γ as (2.48) we now need only to restrict ourselves to matrices A and M where $a_l + m_j \neq 0$ for all $l = 1,\ldots,p$ and $j = 1,\ldots,p$. We can use equation (2.33) to write a solution formula for the KP II equation that depends on the matrix quadruplet (A,M,B,C) and the constant matrix P as input. The matrix P can be found as the solution to the auxiliary equation

$$AP + PM = BC.$$

We will use the formulations for $\Gamma(x;t,y)$ in (2.29) and (2.48) to show that we also obtain an auxiliary equation that involves P when using the formulation for

$u(x, y, t)$ involving (2.29). This linear equation will allow us to construct P in terms of the matrix quadruplet (A, M, B, C). The linear matrix equation will have the form of $YX - XZ = W$ where X, Y, W and Z are matrices of sizes $p \times q$, $p \times p$, $p \times q$ and $q \times q$, respectively. If $z_1, \ldots, z_p$ and $y_1, \ldots, y_q$ are the eigenvalues of Z and Y, respectively, then it is known that the equation $YX - XZ = W$ has a unique solution X if and only if $y_l - z_j \neq 0$ for any choice of l and j. The reader can reference pp.383-387 in [9] for a proof of this.

Theorem 2.2. *Let A and M be real constant matrices of size $p \times p$. Assume that the eigenvalues of A and M have positive real parts. Let $a_1, \ldots, a_p$ be the eigenvalues of A and $m_1, \ldots, m_p$ be the eigenvalues of M. The matrix $P = N(0; 0, 0)$ from (2.45) will satisfy the linear matrix equation*

$$AP + PM = BC, \tag{2.52}$$

which will be uniquely solvable if $a_l + m_j \neq 0$ for all $l = 1, \ldots, p$ and $j = 1, \ldots, p$.

Proof. Investigating the partial derivative of $\Gamma(x; t, y)$ defined in (2.29) we have

$$\Gamma_x(x; t, y) = -e^{-Ax + A^2 y + 4A^3 t} BC e^{-M\xi - M^2 y + 4M^3 t}. \tag{2.53}$$

Equivalently, we can differentiate (3.102) with respect to x to obtain

$$\Gamma_x(x; t, y) = -Ae^{-Ax + A^2 y + 4A^3 t} P e^{-Mx - M^2 y + 4M^3 t} - e^{-Ax + A^2 y + 4A^3 t} P e^{-Mx - M^2 y + 4M^3 t} M.$$

We can equate the expressions for $\Gamma_x(x; t, y)$ in (2.53) and (2.2) which yields the equation

$$-e^{-Ax + A^2 y + 4A^3 t} BC e^{-M\xi - M^2 y + 4M^3 t} = e^{-Ax + A^2 y + 4A^3 t} (AP - PM) e^{-Mx - M^2 y + 4M^3 t}. \tag{2.54}$$

Multiplying by $e^{Ax - A^2 y - 4A^3 t}$ on the left side and $e^{M\xi + M^2 y - 4M^3 t}$ on the right side of the equation in (2.54) reveals the relation

$$AP + PM = BC. \tag{2.55}$$

This equation will be useful to construct the auxiliary matrix P given the matrix quadruplet (A, M, B, C). Given a constant matrix quadruplet (A, M, B, C), we can write this auxiliary equation as

$$AP - P(-M) = BC. \tag{2.56}$$

which is uniquely solvable if $a_l + m_j \neq 0$ for all $l = 1, \ldots, p$ and $j = 1, \ldots, p$. □

Let $\{a_1, \ldots, a_p\}$ and $\{m_1, \ldots, m_p\}$ be the eigenvalues of the $p \times p$ matrices A and M, respectively. We now need only to restrict ourselves to matrices A and M where $a_l + m_j \neq 0$ for all $l = 1, \ldots, p$ and $j = 1, \ldots, p$. We can use equation (2.33) to write a solution formula for the KP II equation that depends on the matrix quadruplet (A, M, B, C) and the constant matrix P as input.

2.3 Independent Verification of the Solution Formula for the KP II Equation

The method used to find the solution formula in (2.33) is based on the inverse scattering transform. The inverse scattering transform relies on the idea that each integrable nonlinear PDE is associated with a linear differential equation [13] containing a spectral parameter. The linear system associated to the KP II equation is given by

$$\psi_{xx} + \psi_y + (\lambda - u)\psi = 0,$$

where λ is the spectral parameter. An analysis of the associated spectral problem will provide the scattering data, $S(\lambda; 0, 0)$. The inverse scattering transform then relates the time evolution of the scattering data

$$S(\lambda; 0, 0) \mapsto S(\lambda; t, y)$$

to the solution of the nonlinear PDE, $u(x, y, t)$. In our method of obtaining exact solution formulas to integrable evolution equations, we emphasize the time evolution

$$\Omega(\eta, \xi; t, y) \mapsto \Omega(\eta, \xi; t, y)$$

of the Marchenko kernel. The Marchenko kernel $\Omega(x, \xi; t, y)$ is in general related to the scattering data $S(\lambda; t, y)$ via a Fourier transform. The relationship between the inverse scattering transform and the method outlined can be seen below.

$$\begin{array}{ccccc}
u(x,0,0) & \xrightarrow{\text{direct scattering}} & S(\lambda;0,0) & \xrightarrow{\text{Fourier transform}} & \Omega(\eta,\xi;0,0) \\
\downarrow & & \downarrow & & \downarrow \\
u(x,y,t) & \xleftarrow[\text{inverse scattering}]{} & S(\lambda;t,y) & \xleftarrow[\text{inverse Fourier transform}]{} & \Omega(\eta,\xi;t,y)
\end{array}$$

An independent verification is provided to show that the function (2.33) is indeed a solution to the KP II equation (2.1).

Theorem 2.3. *Let A, M, B, and C be real constant matrices of sizes $p \times p$, $p \times p$, $p \times 1$, and $1 \times p$, respectively. Let $\{a_1, \dots, a_n\}$ be the eigenvalues of A and $\{m_1, \dots, m_n\}$ be the eigenvalues of M such that $a_l + m_j \neq 0$ for all $l = 1, \dots, p$ and $j = 1, \dots, p$. The function*

$$u(x, y, t) = 2C(G^{-1})_x B,$$

where G is defined to be

$$G := e^{Ax - A^2y - 4A^3t} e^{Mx + M^2y - 4M^3t} + P, \tag{2.57}$$

and P is the solution the linear matrix equation

$$AP + PM = BC,$$

satisfies (2.1) for those $x, y, t \in \mathbb{R}$ where G^{-1} exists.

Proof. Using (2.33) and (2.51) the solution formula to the KP equation (2.1) can be written as

$$u(x, y, t) = 2C(G^{-1})_x B, \tag{2.58}$$

where G is defined to be

$$G := e^{Ax - A^2 y - 4A^3 t} e^{Mx + M^2 y - 4M^3 t} + P, \tag{2.59}$$

and P is the solution the linear matrix equation

$$AP + PM = BC.$$

Using the function $u(x, y, t)$ from (2.58) in the KP II equation (2.1) we obtain

$$[2C(G^{-1})_{xt}B - 12C[(G^{-1})_x BC(G^{-1})_x B]_x + 2C(G^{-1})_{xxxx}B]_x + 6C(G^{-1})_{xyy}B = 0. \tag{2.60}$$

Factoring $2C$ from the left and B from the right of the equation (2.60) gives

$$2C[(G^{-1})_{xt} - 6[(G^{-1})_x BC(G^{-1})_x]_x + (G^{-1})_{xxxx} + 3(G^{-1})_{yy}]_x B = 0. \tag{2.61}$$

We can write (2.61) as

$$2C\left[Z_x + 3(G^{-1})_{yy}\right]_x B = 0, \tag{2.62}$$

where

$$Z := (G^{-1})_t - 6(G^{-1})_x BC(G^{-1})_x + (G^{-1})_{xxx}. \tag{2.63}$$

We use the identities

$$(G^{-1})_t = -G^{-1}G_t G^{-1}, \quad (G^{-1})_y = -G^{-1}G_y G^{-1}, \quad (G^{-1})_x = -G^{-1}G_x G^{-1}, \tag{2.64}$$

to express the derivatives of G^{-1} in terms of G^{-1} and the appropriate derivatives of G. We find the derivatives of G in (3.102) with respect to x, y, and t to be

$$G_t = -4A^3 G - 4GM^3 + 4(A^3 P + PM^3), \tag{2.65}$$

$$G_x = AG + GM - (AP + PM), \tag{2.66}$$

$$G_y = -A^2 G + GM^2 + (A^2 P - PM^2). \tag{2.67}$$

It will be helpful to find a formulation for $-A^2P + PM^2$ and $A^3P + PM^3$ in terms of the matrix quadruplet (A, M, B, C). The auxiliary equation (2.52) is used to simplify $-A^2P + PM^2$ by first adding and subtracting APM, which yields

$$-A^2P + PM^2 = -A^2P - APM + APM + PM^2. \tag{2.68}$$

We can write (2.68) in the equivalent form

$$-A^2P + PM^2 = -A(AP + PM) + (AP + PM)M. \tag{2.69}$$

Using the auxiliary equation (2.52) in (2.69) leads the helpful equation

$$-A^2P + PM^2 = -ABC + BCM. \tag{2.70}$$

The auxiliary equation (2.52) is also used to simplify $A^3P + PM^3$ by adding and subtracting $A^2PM + APM^2$ which yields

$$A^3P + PM^3 = A^3P + A^2PM - A^2PM + APM^2 - APM^2 + PM^3. \tag{2.71}$$

We can write the equation (2.71) in the equivalent form

$$A^3P + PM^3 = A^2(AP + PM) - A(AP - PM)M + (AP + PM)M^2. \tag{2.72}$$

Using (2.52) in (2.72), we obtain

$$A^3P + PM^3 = A^2BC - ABCM + BCM^2. \tag{2.73}$$

Using (2.65)-(2.67), (2.70), and (2.73) in (2.64), we find the derivatives of G^{-1} to be

$$(G^{-1})_t = 4G^{-1}A^3 + 4M^3G^{-1} - 4G^{-1}[A^2BC - ABCM + BCM^2]G^{-1}, \tag{2.74}$$

$$(G^{-1})_y = G^{-1}A^2 - M^2G^{-1} + G^{-1}(-A^2P + PM^2)G^{-1}, \tag{2.75}$$

$$(G^{-1})_x = -G^{-1}A - MG^{-1} + G^{-1}BCG^{-1}. \tag{2.76}$$

To simplify (2.63) we use (2.76) to write $-6(G_x^{-1}BC(G^{-1})_x$ in terms of A, M, B, C, and G^{-1} as

$$\begin{aligned}-6(G^{-1})_xBC(G^{-1})_x =& -6G^{-1}ABCG^{-1} - 6G^{-1}ABCMG^{-1} + 6G^{-1}ABCG^{-1}BCG^{-1}\\ &- 6MG^{-1}BCG^{-1}A - 6MG^{-1}BCMG^{-1}\\ &+ 6MG^{-1}BCG^{-1}BCG^{-1} + 6G^{-1}BCG^{-1}BCG^{-1}A\\ &+ 6G^{-1}BCG^{-1}BCMG^{-1} - 6G^{-1}BCG^{-1}BCG^{-1}BCG^{-1}.\end{aligned}$$

We continue to simplify the bracketed terms in (2.63). Our goal is to have an expression for $(G^{-1})_{xxx}$ and $3(G^{-1})_{yy}$ in terms of A, M, B, C, P, and G^{-1}. Taking the derivative of (2.76), with respect to x, twice and then using (2.76) to simplify $(G^{-1})_{xxx}$ we obtain

$$\begin{aligned}(G^{-1})_{xxx} =& G^{-1}A^3 - MG^{-1}A^2 + G^{-1}BCG^{-1}A^2 - 3M^2G^{-1}A - M^3G^{-1}\\ &+ M^2G^{-1}BCG^{-1} - 3MG^{-1}A^2 + 5MG^{-1}BCG^{-1}A\\ &+ 2MG^{-1}ABCG^{-1} + 2M^2G^{-1}BCG^{-1} - 6MG^{-1}BCG^{-1}BCG^{-1}\\ &+ 3MG^{-1}BCMG^{-1} + 3G^{-1}ABCG^{-1}A - 4G^{-1}BCG^{-1}BCG^{-1}A\\ &+ 2G^{-1}BCG^{-1}A^2 + 3G^{-1}BCMG^{-1}A + G^{-1}A^2BCG^{-1}\\ &+ MG^{-1}ABCG^{-1} - 3G^{-1}BCG^{-1}ABCG^{-1} + 2G^{-1}ABCMG^{-1}\\ &- 3G^{-1}ABCG^{-1}BCG^{-1} - G^{-1}BCG^{-1}BCMG^{-1} + G^{-1}BCM^2G^{-1}\\ &- 3G^{-1}BCMG^{-1}BCG^{-1} + 4G^{-1}BCG^{-1}BCG^{-1}BCG^{-1}.\end{aligned} \tag{2.77}$$

Taking the derivative of (2.75), with respect to y, and then using (2.75) to simplify $(G^{-1})_{yy}$ we obtain

$$\begin{aligned}3(G^{-1})_{yy} =& 3G^{-1}A^4 - 6M^2G^{-1}A^2 + 3M^4G^{-1} - 6M^2G^{-1}[-A^2P + PM^2]G^{-1} \\ &+ 6G^{-1}[-A^2P + PM^2]G^{-1}A^2 + 3G^{-1}A^2[-A^2P + PM^2]G^{-1} \\ &- 3G^{-1}[-A^2P + PM^2]M^2G^{-1} + 6G^{-1}[-A^2P \\ &+ PM^2]G^{-1}[-A^2P + PM^2]G^{-1}.\end{aligned} \tag{2.78}$$

We will use the identity (2.70) to simplify $3(G^{-1})_{yy}$ from (2.78) as

$$\begin{aligned}3(G^{-1})_{yy} =& 3G^{-1}A^4 - 6M^2G^{-1}A^2 + 3M^4G^{-1} + 6M^2G^{-1}ABCG^{-1} \\ &- 6M^2G^{-1}BCMG^{-1} - 6G^{-1}ABCG^{-1}A^2 + 6G^{-1}BCMG^{-1}A^2 \\ &- 3G^{-1}A^3BCG^{-1} + 3G^{-1}A^2BCMG^{-1} + 3G^{-1}ABCM^2G^{-1} \\ &- 3G^{-1}BCM^3G^{-1} + 6G^{-1}ABCG^{-1}ABCG^{-1} \\ &- 6G^{-1}ABCG^{-1}BCMG^{-1} - 6G^{-1}BCMG^{-1}ABCG^{-1} \\ &+ 6G^{-1}BCMG^{-1}BCMG^{-1}.\end{aligned}$$

The simplified forms of $(G^{-1})_t$, $-6(G^{-1})_xBC(G^{-1})_x$ and $(G^{-1})_{xxx}$ are now substituted into (2.62). Because the matrices A and M are not required to commute the simplification is nontrivial and quite lengthy requiring the use of techniques from linear algebra.

From (2.62) we can see that $C[Z_x + 3(G^{-1})_{yy}]B$ will be independent of x if (2.58) is a solution to (2.1). We investigate the term $C[Z_x + 3(G^{-1})_{yy}]B$ to show that there is no dependence on x. Using (2.74), (2.76), (2.77), and (2.78) we can expand the terms of $C[Z_x + 3(G^{-1})_{yy}]B$ as

$$\begin{aligned}
C[Z_x + 3(G^{-1})_{yy}]B = &-3CM^2G^{-1}BCG^{-1}AB + 6CM^2G^{-1}BCG^{-1}BCG^{-1}B \\
&- 3CM^2G^{-1}BCG^{-1}AB - 6CM^2G^{-1}BCMG^{-1}B \\
&+ 6CMG^{-1}BCG^{-1}ABCG^{-1}B - 6CMG^{-1}BCG^{-1}BCMG^{-1}B \\
&+ 6CMG^{-1}BCM^2G^{-1}B - 6CMG^{-1}BCG^{-1}A^2B \\
&+ 6CG^{-1}BCG^{-1}BCG^{-1}BCMG^{-1}B + 3CG^{-1}BCG^{-1}A^2BCG^{-1}B \\
&- 3CG^{-1}BCG^{-1}A^2BCG^{-1}B - 6CG^{-1}BCMG^{-1}BCG^{-1}BCG^{-1}B \\
&+ 6CG^{-1}ABCG^{-1}ABCG^{-1}B - 6CG^{-1}ABCG^{-1}A^2B \\
&+ 6CG^{-1}A^2BCG^{-1}AB - 3CG^{-1}BCG^{-1}BCM^2G^{-1}B \\
&- 3CG^{-1}BCG^{-1}BCMG^{-1}AB - 12CG^{-1}ABCG^{-1}BCMG^{-1}B \\
&+ 6CG^{-1}ABCG^{-1}BCG^{-1}BCG^{-1}B + 6CG^{-1}BCG^{-1}BCG^{-1}A^2B \\
&- 6CG^{-1}BCG^{-1}BCG^{-1}ABCG^{-1}B - 6CG^{-1}A^2BCG^{-1}BCG^{-1}B \\
&- 6CG^{-1}ABCG^{-1}BCG^{-1}AB + 6CG^{-1}BCMG^{-1}BCMG^{-1}B \\
&+ 6CG^{-1}A^2BCMG^{-1}B + 6CG^{-1}ABCM^2G^{-1}B \\
&- 3CG^{-1}BCG^{-1}BCM^2G^{-1}B + 3CG^{-1}BCG^{-1}BCMG^{-1}AB \\
&+ 6CG^{-1}BCMG^{-1}BCG^{-1}AB.
\end{aligned} \tag{2.79}$$

Because the matrices A and M do not necessarily commute we are unable to simplify this expression any further using algebraic methods. Although our solution $u(x, y, t)$ to (2.1) is a scalar function we have written the solution formula for $u(x, y, t)$ as a product of matrices that do not necessarily commute. The solution formula for $u(x, y, t)$ is written as the product of the $1 \times p$ vector C, the $p \times p$ matrix $(G^{-1})_x$ and the $1 \times p$ vector B. The terms that arise in this verification are algebraic combinations of the matrices found in the matrix quadruplet (A, M, B, C) and G^{-1}. These terms are not easily simplified because these matrices do not necessarily commute. We will now use the property of the trace of matrices in (2.36) applied to (2.79) which yields

$$\begin{aligned}
\mathrm{tr}[C[Z_x{+}3(G^{-1})_{yy}]B] = &-3G^{-1}BCG^{-1}ABCM^2 + 6G^{-1}BCG^{-1}BCG^{-1}BCM^2 \\
&- 3G^{-1}BCG^{-1}ABCM^2 - 6G^{-1}BCMG^{-1}BCM^2 \\
&+ 6G^{-1}BCG^{-1}ABCG^{-1}BCM - 6G^{-1}BCG^{-1}BCMG^{-1}BCM \\
&+ 6G^{-1}BCMG^{-1}BCM^2 - 6G^{-1}BCG^{-1}A^2BCM \\
&+ 3G^{-1}BCG^{-1}BCG^{-1}BCG^{-1}BCM \\
&+ 3G^{-1}BCG^{-1}BCG^{-1}BCG^{-1}BCM \\
&+ 3G^{-1}BCG^{-1}BCG^{-1}A^2BC - 3G^{-1}BCG^{-1}BCG^{-1}A^2BC \\
&- 6G^{-1}BCG^{-1}BCG^{-1}BCG^{-1}BCM + 6G^{-1}BCG^{-1}ABCG^{-1}ABC \\
&- 6G^{-1}ABCG^{-1}A^2BC + 6G^{-1}ABCG^{-1}A^2BC \\
&- 3G^{-1}BCG^{-1}BCG^{-1}BCM^2 - 3G^{-1}BCG^{-1}BCMG^{-1}ABC \\
&- 12G^{-1}BCG^{-1}ABCG^{-1}BCM + 6G^{-1}BCG^{-1}BCG^{-1}BCG^{-1}ABC \\
&+ 6G^{-1}BCG^{-1}BCG^{-1}A^2BC - 6G^{-1}BCG^{-1}BCG^{-1}BCG^{-1}ABC \\
&- 6G^{-1}BCG^{-1}BCG^{-1}A^2BC - 6G^{-1}BCG^{-1}ABCG^{-1}ABC \\
&+ 6G^{-1}BCG^{-1}BCMG^{-1}BCM + 6G^{-1}BCG^{-1}A^2BCM \\
&+ 6G^{-1}BCG^{-1}ABCM^2 - 3G^{-1}BCG^{-1}BCG^{-1}BCM^2 \\
&+ 3G^{-1}BCG^{-1}BCMG^{-1}ABC + 6G^{-1}BCG^{-1}ABCG^{-1}BCM,
\end{aligned} \tag{2.80}$$

We see that all of the terms of (2.80) cancel and

$$\mathrm{tr}\left[2C[Z_x + 3(G^{-1})_{yy}]B\right] = 0. \tag{2.81}$$

We know that (2.62) is a scalar equation and therefore (2.81) implies that

$$2C[Z_x + 3(G^{-1})_{yy}]_x B = 0,$$

and (2.58) is a solution to the KP II equation (2.1). □

2.3.1 Reduction to the (1+1) case

In the absence of y the KP II equation reduces to the Korteweg-deVries (KdV) equation found in [5]. It then follows that any solution $u(x, y, t)$ to the KP II equation is also a solution to the KdV equation if $u_y = 0$ for all x and t in $\mathbb{R}$. We can find a solution formula to the KdV equation by suppressing the y dependence in the solution formula $u(x, y, t)$ for the KP II equation. To accomplish this we only need to make the restriction that $M = A$ on the matrix quadruplet (A, M, B, C).

The solution $u(x, y, t)$, from (2.33), to the KP II equation can be written in the following way:

$$\begin{aligned} u(x, y, t) = 2C\frac{d}{dx}[&e^{-Mx-M^2y+4M^3t}(I + e^{-Ax+A^2y+4A^3t}Pe^{-Mx-M^2y+4M^3t})^{-1} \\ &\times e^{-Ax+A^2y+4A^3t}]B. \end{aligned} \tag{2.82}$$

Factoring $e^{A^2y-4A^3t}$ from the left and e^{-M^2y} from the right we can write (2.82) as

$$u(x, y, t) = 2C\frac{d}{dx}[e^{-Mx+4M^3t}(e^{-A^2y-4A^3t}e^{M^2y} + e^{-Ax}Pe^{-Mx+4M^3t})^{-1}e^{-Ax}]B. \tag{2.83}$$

Introducing $I = e^{4M^3t-4M^3t}$ into equation (2.83)

$$\begin{aligned} u(x, y, t) = 2C\frac{d}{dx}[&e^{-Mx+4M^3t}e^{4M^3t-4M^3t} \\ &\times (e^{-4A^3t}e^{-A^2y}e^{M^2y} + e^{-Ax}Pe^{-Mx+4M^3t})^{-1}e^{-Ax}]B, \end{aligned} \tag{2.84}$$

we are able to write (2.84) as

$$u(x, y, t) = 2C\frac{d}{dx}[e^{-Mx+8M^3t}(e^{-4A^3t}e^{-A^2y}e^{M^2y}e^{4M^3t} + e^{-Ax}Pe^{-Mx+8M^3t})^{-1}e^{-Ax}]B. \tag{2.85}$$

If we consider the case $M = A$, we have the following expression for $u(x, y, t)$ in (2.85):

$$u(x, y, t) = 2C\frac{d}{dx}[e^{-Ax+8A^3t}(I + e^{-Ax}Pe^{-Ax+8A^3t})^{-1}e^{-Ax}]B, \qquad (2.86)$$

which is the solution formula for the Korteweg-de Vries equation in (2.3) that was found by Aktosun and van der Mee in [5] using the method for integrable evolution equations in (1+1) dimension. In this case the generalization of this method in the 2+1 case shows itself to be a clear extension of the one used in [5], which depends on the triplet of matrices (A, B, C). The solution formula in (2.86) is known [6] to produce globally analytic solutions for certain restrictions on A, B, and C. When A is chosen to be a $p \times p$ diagonal matrix with real distinct positive entries, B is chosen to be a $p \times 1$ column vector whose entries are all equal to one and C is chosen to be a $1 \times p$ row vector with positive real entries.

We can find globally analytic solutions to the KP II equation by imposing the same conditions on the quadruplet of matrices (A, M, B, C) along with the condition $M = A$. This will produce globally analytic solution to the KP II equation in the form of line solitons.

2.4 Constructing the solution formula $u(x, y, t)$ from (A, M, B, C)

An advantage of the method yielding solution formula (2.33) is that the solution formula found can be used without any knowledge of the method used to derive this solution formula. In this way, a large class of solutions are available to a wide audience, who need only to be familiar with linear algebra and calculus. Below is the summary of our systematic method to construct a solution $u(x, y, t)$ to (2.1) with input of constant matrix quadruplet (A, M, B, C).

(i) Begin by choosing a real valued constant quadruplet of matrices (A, M, B, C) of sizes $p \times p$, $p \times p$, $p \times 1$, and $1 \times p$, respectively, as our input. Let $a_1, \ldots, a_p$ be the eigenvalues of A and $m_1, \ldots, m_p$ be the eigenvalues of M. The only condition on the matrix quadruplet (A, M, B, C) will be that $a_l + m_j \neq 0$ for all $l = 1, \ldots, p$ and $j = 1, \ldots, p$. We require the condition $a_l + m_j \neq 0$ for all $l = 1, \ldots, p$ and $j = 1, \ldots, p$ to ensure the existence of the auxiliary matrix P. Choosing A as a diagonal matrix with positive distinct entries, the entries of B all equal to one and the entries of C as positive we have sufficient conditions for a globally analytic solution.

(ii) Construct the constant auxiliary matrix P by solving the linear matrix equation

$$AP + PM = BC.$$

(iii) Using the quadruplet of matrices (A, M, B, C) and the auxiliary matrix P construct the function $\Gamma(x, y, t)$ in the following way

$$\Gamma(x, y, t) = I + e^{-Ax + A^2 y + 4A^3 t} P e^{-Mx - M^2 y + 4M^3 t}.$$

(iv) We can then formulate the solution $u(x, y, t)$ to the nonlinear PDE (2.1) as

$$u(x, y, t) = 2C \frac{d}{dx} \left(e^{-Mx - M^2 y + 4M^3 t} \Gamma^{-1} e^{-Ax + A^2 y + 4A^3 t} \right) B.$$

2.5 Example

Example 2.4. A one solution solution formula can be found by choosing the matrix quadruplet (A, M, B, C) to be scalar values. Using the matrices A, M, B, and C as input in the auxiliary equation (2.52)

$$AP + PM = BC, \tag{2.87}$$

we solve (2.87) for P and obtain

$$P = \frac{BC}{A + M}. \tag{2.88}$$

Using A, M, B, C, and P from (2.88) in $\Gamma(x, y, t)$ *given by (2.48) we get*

$$\Gamma = I + Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t}. \tag{2.89}$$

We use the solution formula for $u(x, y, t)$ *given in (2.40), namely*

$$u(x, y, t) = -2\frac{d}{dx}\left[\frac{(\det\Gamma)_x}{\det\Gamma}\right],$$

to construct a solution to the KP II equation. In the scalar case, $\det(\Gamma(x; t, y))$ *and* $\Gamma(x; t, y)$ *are equivalent. Taking the derivative of* $\Gamma(x; t, y)$ *in (2.89) we have*

$$\begin{aligned}(\det\Gamma)_x &= -A(\Gamma - I) - (\Gamma - I)M \\ &= A + M - (A + M)\Gamma \\ &= (A + M)(I - \Gamma) \\ &= -(A + M)Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t}.\end{aligned}$$

A one soliton solution formula (2.40) can now be written as

$$u(x, y, t) = 2\frac{d}{dx}\left[\frac{(A + M)Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t}}{I + Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t}}\right]. \tag{2.90}$$

The solution formula in (2.90) has the equivalent forms

$$u(x,y,t) = 2(A+M)\frac{d}{dx}\left[\frac{Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t}}{I+Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t}}\right]$$

$$= 2(A+M)\frac{d}{dx}\left[\frac{1+Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t}-1}{I+Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t}}\right]$$

$$= 2(A+M)\frac{d}{dx}\left(1-\frac{1}{I+Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t}}\right)$$

$$= 2(A+M)\frac{\frac{d}{dx}(I+Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t})}{(I+Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t})^2}$$

$$= 2(A+M)\frac{-(A+M)Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t}}{(I+Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t})^2}.$$

The solution formula $u(x,y,t)$ can now be written as

$$u(x,y,t) = \frac{-2(A+M)^2Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t}}{(I+Pe^{-Ax+A^2y+4A^3t}e^{-Mx-M^2y+4M^3t})^2}. \tag{2.91}$$

Does the solution formula $u(x,y,t)$ contain any singularities? Since A, M, B, and C are all real values $u(x,y,t)$ never becomes singular in the xyt-space when P is positive. From

$$P = \frac{BC}{A+M},$$

we see that the choice

$$\frac{BC}{A+M} > 0,$$

yields globally analytic solutions. Note that if $B=0$ or $C=0$ we get $P=0$, which yields the trivial solution $u(x,y,t)\equiv 0$. It is also important to note that we must choose the input matrix quadruplet (A,M,B,C) such that $A+M\neq 0$ to ensure the existence of P.

CHAPTER 3

An Exact Solution Formula to the Generalized Davey-Stewartson II System

In this chapter we will use the unified method, involving the underlying Marchenko integral equation from the inverse scattering problem, to find an explicit solution formula for certain solutions to the generalized Davey-Stewartson II (DS II) system [1]

$$\begin{cases} iu_t + u_{xx} - u_{yy} - 2uvu + \phi u = 0, \\ iv_t - v_{xx} + v_{yy} + 2vuv - v\phi = 0, \\ \phi_{xx} + \phi_{yy} - 2(uv + vu)_{yy} = 0, \end{cases} \tag{3.1}$$

where the subscripts signify the respective x, y, and t derivatives. The generalized DS II system is an integrable nonlinear partial differential equation (PDE) in terms of $u(x, y, t)$, $v(x, y, t)$, and $\phi(x, y, t)$, which have two spatial variables x and y and one time variable t, known as a 2+1 dimension equation. This equation is related to the well studied nonlinear Schrödinger (NLS) equation and Davey-Stewartson II equation. The generalized DS II system (3.1) is related to the (1+1) dimension focusing nonlinear Schroödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{3.2}$$

by restricting $v(x, y, t)$ and $\phi(x, y, t)$ in the following way:

$$\phi \equiv 0,$$

$$v = -u^*,$$

where the asterisk denotes complex conjugation. The NLS equation is an integrable nonlinear PDE in terms of $u(x, t)$ which has one spatial variable x and one time variable t, known as a (1+1) dimension equation. The motivation behind this method comes from the work done by Aktosun and van der Mee in the case of integrable evolution equations in (1+1) dimension [3, 4, 5]. This method yields a solution formula to (3.2) that will have as its input a triplet of complex constant matrices (A, B, C) of sizes $p \times p$, $p \times 1$, and $1 \times p$, respectively. The NLS equation is related to the linear ordinary differential equation [16]

$$\begin{cases} \xi' = -i\lambda\xi + u(x,t)\eta, \\ \eta' = -u^*(x,t) + i\lambda\eta. \end{cases} \tag{3.3}$$

Recall that the asterisk denotes complex conjugation. The linear ordinary differential equation in (3.3) is used to determine the scattering data associated to the NLS equation. The underlying Marchenko matrix integral equation associated with the NLS equation found in [3] is

$$K(x,\xi;t) + F(x,\xi;t) + \int_x^\infty d\eta\, K(x,\eta;t)\, F(\eta,\xi;t) = 0, \tag{3.4}$$

where

$$K(x,\xi;t) = \begin{bmatrix} \bar{K}_1(x,\xi;t) & K_1(x,\xi;t) \\ \bar{K}_2(x,\xi;t) & K_2(x,\xi;t) \end{bmatrix}, \tag{3.5}$$

(3.6)

$$F(\eta,\xi;t) = \begin{bmatrix} 0 & \bar{\Omega}(\eta,\xi;t) \\ \Omega(\eta,\xi;t) & 0 \end{bmatrix}, \tag{3.7}$$

and

$$\bar{\Omega}(\eta,\xi;t) = -\Omega(\eta,\xi;t)^\dagger. \tag{3.8}$$

The bar notation does not denote complex conjugation. In the case of the NLS equation the effect of the bar over $\Omega(\eta,\xi;t)$ becomes the negative complex conjugate of $\Omega(\eta, \xi; t)$. Note that the dagger in (3.8) denotes the complex conjugate of the matrix transpose. We use the dagger in (3.8) which is equivalent to complex conjugation in the scalar case. The use of the dagger in the scalar case will be convenient to take advantage of the matrix properties of our solution formula that is written as the product of matrices.

The Marchenko kernel $F(\eta, \xi; t)$ appearing in (3.4) is related to the scattering data associated with (3.3) through a Fourier transform. Using the scattering data in the form of a rational function of the spectral parameter λ, Aktosun, Demontis, and van der Mee [3] were able to show that the Marchenko kernel appearing in (3.4) is expressed in terms of the matrix triplet (A, B, C) as

$$F(\eta, \xi; t) = \begin{bmatrix} 0 & -B^\dagger e^{-A^\dagger \eta} e^{-A^\dagger \xi + 4i(A^\dagger)^2 t} C^\dagger \\ C e^{-A\eta} e^{-A\xi - 4iA^2 t} B & 0 \end{bmatrix}. \tag{3.9}$$

Comparing (3.9) with (3.7) we get the form of $\Omega(\eta, \xi; t)$ and $\bar{\Omega}(\eta, \xi; t)$ as

$$\Omega(\eta, \xi; t) = C e^{-A\eta} e^{-A\xi - 4iA^2 t} B,$$

$$\bar{\Omega}(\eta, \xi; t) = -B^\dagger e^{-A^\dagger \eta} e^{-A^\dagger \xi + 4i(A^\dagger)^2 t} C^\dagger.$$

From (3.9) we see that the Marchenko kernel in (3.7) possesses symmetry in η and ξ, namely

$$F(\eta, \xi; t) = F(\xi, \eta; t). \tag{3.10}$$

We are able to motivate the form of the (2+1) Marchenko kernel by using a form similar to (3.9). In the case of the (2+1) dimension generalized DS II system we incorporate an additional matrix M to account for the spatial variable y. As in the (1+1) case the Marchenko kernel will be written in terms of matrix exponentials. In the (2+1) case we choose form of the initial Marchenko kernel to be

$$F(\eta, \xi; t, y) = \begin{bmatrix} 0 & \bar{\Omega}(\eta, \xi; t, y) \\ \Omega(\eta, \xi; t, y) & 0 \end{bmatrix}. \tag{3.11}$$

The (2+1) matrix Marchenko integral equation is given by

$$F(x,\xi;t,y)+K(x,\xi;t,y)+\int_x^\infty d\eta\, K(x,\eta;t,y)F(\eta,\xi;t,y)=0, \quad x<\xi, \tag{3.12}$$

with

$$K(x,\xi;t,y)=\begin{bmatrix}\bar{K}_1(x,\xi;t,y) & K_1(x,\xi;t,y)\\ \bar{K}_2(x,\xi;t,y) & K_2(x,\xi;t,y)\end{bmatrix}, \tag{3.13}$$

$$F(\eta,\xi;t,y)=\begin{bmatrix}0 & \bar{\Omega}(\eta,\xi;t,y)\\ \Omega(\eta,\xi;t,y) & 0\end{bmatrix}. \tag{3.14}$$

For a Marchenko integral equation in (2+1) dimension we relax the symmetry given in (3.10) and choose the form of the initial Marchenko kernel to be

$$F(\eta,\xi;0,0)=\begin{bmatrix}0 & \bar{C}e^{-\bar{M}\eta}e^{-\bar{A}\xi}\bar{B}\\ Ce^{-M\eta}e^{-A\xi}B & 0\end{bmatrix}. \tag{3.15}$$

It is important to note that the matrices A and M in general need not commute.

We expect the evolution of t and y in $\Omega(\eta,\xi;t,y)$ to correspond to the matrices A and M as

$$\Omega(\eta,\xi;t,y)=Ce^{f_1(M)t+f_2(M)y}e^{-M\eta}e^{-A\xi}e^{g_1(A)t+g_2(A)y}B, \tag{3.16}$$

where $f_1(M)$, $f_2(M)$, $g_1(A)$, and $g_2(A)$ are simple functions corresponding to the unique evolution of $\Omega(\eta,\xi;t,y)$ in t and y. In the analysis of (1+1) integrable nonlinear PDEs we have seen the evolution of $\Omega(\eta+\xi;t)$ in t to be monomials for the Korteweg-deVries equation and the NLS equation.

A system of linear PDEs is used to determine the functions $f_1(M)$, $f_2(M)$, $g_1(A)$, and $g_2(A)$ appearing in (3.16). With the evolution of $\Omega(\eta,\xi;t,y)$ in t and y known, we use the separability of $\Omega(\eta,\xi;t,y)$ in η and ξ and the underlying matrix

Marchenko integral equation (3.12) to solve for $K(x,\xi;t,y)$. For the generalized DS II system, the solution to the matrix Marchenko integral equation (3.12) is related to the potentials $u(x,y,t)$, $v(x,y,t)$, and $\phi(x,y,t)$ in a simple way [1], namely

$$u(x,y,t) = 2K_1(x,x;t,y), \tag{3.17}$$

$$v(x,y,t) = 2\bar{K}_2(x,x;t,y), \tag{3.18}$$

$$\phi(x,y,t) = 4i[K_2(x,x;t,y) - \bar{K}_1(x,x;t,y)]_y. \tag{3.19}$$

In this way we are able to recover the potentials $u(x,y,t)$, $v(x,y,t)$, and $\phi(x,y,t)$ from the solution of the matrix Marchenko integral equation.

The generalized DS II system in (3.1) can be reduced to the Davey-Stewartson II equation [8]

$$\begin{cases} iu_t + u_{xx} - u_{yy} + 2|u|^2 + \phi u = 0, \\ \phi_{xx} + \phi_{yy} + 4(|u|^2)_{xx} = 0. \end{cases} \tag{3.20}$$

To reduce the generalized DS II system (3.1) to the Davey-Stewartson II equation we require that the solution to (3.1) be restricted by

$$v(x,y,t) = -u^*(x,y,t). \tag{3.21}$$

Recall that an asterisk is used to denote complex conjugation.

With the choice $v(x,y,t) = -u^*(x,y,t)$ we see that the first two equations in (3.1) become

$$iu_t + u_{xx} - u_{yy} + 2uu^*u + \phi u = 0, \tag{3.22}$$

$$-iu_t^* + u_{xx}^* - u_{yy}^* + 2u^*uu^* + u^*\phi = 0. \tag{3.23}$$

It is clear that (3.23) is equal to the conjugate of (3.22). Therefore, (3.22) and (3.23) are equivalent. When $v(x,y,t) = -u^*(x,y,t)$ is used in the third equation of (3.1) we obtain

$$\phi_{xx} + \phi_{yy} - 4(uu^*)_{yy} = 0. \tag{3.24}$$

With the restriction $v(x,y,t) = -u^*(x,y,t)$, we can write (3.22) and (3.24) as the system of equations

$$\begin{cases} iu_t + u_{xx} - u_{yy} + 2|u|^2 + \phi u = 0, \\ \phi_{xx} + \phi_{yy} + 4(|u|^2)_{xx} = 0, \end{cases} \tag{3.25}$$

which is the Davey-Stewartson II equation.

We begin this chapter with a section that describes a systematic method to construct a solution formula for certain solutions to the generalized DS II system. This solution formula has three parts, namely $u(x,y,t)$, $v(x,y,t)$, and $\phi(x,y,t)$. The input for the solution formula will be the constant matrix quadruplets (A,M,B,C) and $(\bar{A},\bar{M},\bar{B},\bar{C})$ of sizes $p\times p$, $p\times p$, $p\times 1$, and $1\times p$, respectively, and we say that these quadruplets of matrices have size p. For each choice of matrix quadruplets (A,M,B,C) and $(\bar{A},\bar{M},\bar{B},\bar{C})$ we obtain the solutions $u(x,y,t)$, $v(x,y,t)$, and $\phi(x,y,t)$ to the generalized DS II system. With certain restrictions on the constant matrix quadruplet (A,M,B,C) and $(\bar{A},\bar{M},\bar{B},\bar{C})$ we will have solutions that are globally analytic on the xyt-space. For other choices there will be singularities that occur on a surface in the xyt-space.

The second section in this chapter will be dedicated to studying the properties of our solution formula. We are able to provide some sufficiency conditions on the matrix quadruplets (A,M,B,C) and $(\bar{A},\bar{M},\bar{B},\bar{C})$ that will provide a globally analytic solution formula in the third section. An independent verification of the solution formula will be given in detail in the fourth section.

3.1 Exact Solution Formula to the Generalized DS II System

In this section we make use of the temporal and spatial evolution of the kernel $F(\eta,\xi;t,y)$ of the matrix Marchenko integral equation given in (3.12). By using

(3.13) in (3.12) one can show that the matrix Marchenko equation given in (3.12) is equivalent to the four scalar equations given by

$$K_1(x,\xi;t,y) + \bar{\Omega}(x,\xi;t,y) + \int_x^\infty d\eta\, \bar{K}_1(x,\eta;t,y)\,\Omega(\eta,\xi;t,y) = 0, \tag{3.26}$$

$$\bar{K}_1(x,\xi;t,y) + \int_x^\infty d\eta\, K_1(x,\eta;t,y)\,\Omega(\eta,\xi;t,y) = 0, \tag{3.27}$$

$$\bar{K}_2(x,\xi;t,y) + \Omega(x,\xi;t,y) + \int_x^\infty d\eta\, K_2(x,\eta;t,y)\,\bar{\Omega}(\eta,\xi;t,y) = 0, \tag{3.28}$$

$$K_2(x,\xi;t,y) + \int_x^\infty d\eta\, \bar{K}_2(x,\eta;t,y)\,\bar{\Omega}(\eta,\xi;t,y) = 0, \tag{3.29}$$

for $x < \xi$. The scalar equations in (3.26) and (3.27) are coupled and similarly the scalar equations given in (3.28) and (3.29) are coupled. It will be helpful to consider $\bar{K}_1(x,\xi;t,y)$ as

$$\bar{K}_1(x,\xi;t,y) = -\int_x^\infty d\eta\, K_1(x,\eta;t,y)\,\Omega(\eta,\xi;t,y), \tag{3.30}$$

and $K_2(x,\xi;t,y)$ as

$$K_2(x,\xi;t,y) = -\int_x^\infty d\eta\, \bar{K}_2(x,\eta;t,y)\,\bar{\Omega}(\eta,\xi;t,y). \tag{3.31}$$

We use (3.30) and (3.31) to uncouple the equations in (3.26) and (3.28), respectively. By uncoupling those equations, we obtain

$$K_1(x,\xi;t,y) + \bar{\Omega}(x,\xi;t,y) - \int_x^\infty ds \int_x^\infty d\eta\, K_1(x,s;t,y)\,\Omega(s,\eta;t,y)\,\bar{\Omega}(\eta,\xi;t,y) = 0, \tag{3.32}$$

$$\bar{K}_2(x,\xi;t,y) + \Omega(x,\xi;t,y) - \int_x^\infty ds \int_x^\infty d\eta\, \bar{K}_2(x,s;t,y)\bar{\Omega}(s,\eta;t,y)\Omega(\eta,\xi;t,y) = 0. \tag{3.33}$$

It is known in [1] that the evolution of the Marchenko kernel $F(\eta,\xi;t,y)$ given in (3.13) is determined by the linear PDEs

$$\begin{cases} iF_y + JF_\eta + F_\xi J = 0, \\ iF_t + 2JF_{\eta\eta} - 2F_{\xi\xi}J = 0, \end{cases} \tag{3.34}$$

where

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.35}$$

Using (3.15) in (3.34), we obtain the evolution of $\Omega(\eta, \xi; t, y)$ and $\bar{\Omega}(\eta, \xi; t, y)$ in t and y as

$$\begin{cases} i\Omega_y - \Omega_\eta + \Omega_\xi = 0, \\ i\Omega_t - 2\Omega_{\eta\eta} - 2\Omega_{\xi\xi} = 0, \end{cases} \tag{3.36}$$

$$\begin{cases} i\bar{\Omega}_y + \bar{\Omega}_\eta - \bar{\Omega}_\xi = 0, \\ i\bar{\Omega}_t + 2\bar{\Omega}_{\eta\eta} + 2\bar{\Omega}_{\xi\xi} = 0. \end{cases} \tag{3.37}$$

The system of linear PDEs in (3.36) describes the t and y evolution of $\Omega(\eta, \xi; t, y)$ with respect to η and ξ. To determine the temporal and spatial evolution, i.e. in t and y, we use the general form of the Marchenko kernel found in (3.16). Substituting the general form of the Marchenko kernel (3.16) into the linear PDEs (3.36), we determine the functions $f_1(M)$, $f_2(M)$, $g_1(A)$, and $g_2(A)$. We begin by defining

$$E_1 := e^{f_1(M)t + f_2(M)y} e^{-M\eta}, \qquad E_2 := e^{-A\xi} e^{g_1(A)t + g_2(A)y},$$

so that $\Omega(\eta, \xi; t, y)$ in (3.16) can be written as

$$\Omega(\eta, \xi; t, y) = C E_1 E_2 B.$$

Taking the appropriate derivatives in (3.16) with respect to t and y, we have

$$\Omega_t = C[f_1(M)E_1E_2 - E_1E_2 g_1(A)]B, \tag{3.38}$$

$$\Omega_y = C[f_2(M)E_1E_2 - iE_1E_2 g_2(A)]B. \tag{3.39}$$

Taking the derivatives of $\Omega(\eta, \xi; t, y)$ given in (3.16) with respect to η and ξ, we obtain

$$\Omega_\eta = C[-ME_1E_2]B, \qquad \Omega_\xi = C[-E_1E_2A]B, \tag{3.40}$$

$$\Omega_{\eta\eta} = C[M^2E_1E_2]B, \qquad \Omega_{\xi\xi} = C[E_1E_2A^2]B. \tag{3.41}$$

Using (3.39) and (3.40) in the first line of (3.36) we have

$$C[if_2(M)E_1E_2 - iE_1E_2g_2(A) + ME_1E_2 - E_1E_2A]B = 0. \tag{3.42}$$

Equivalently,

$$C[(if_2(M) + M)\,E_1E_2 - E_1E_2\,(-ig_2(A) + A)]B = 0. \tag{3.43}$$

Similarly, by using (3.38) and (3.41) in the second line of (3.36) we obtain

$$C[if_1(M)E_1E_2 - iE_1E_2g_1(A) - 2M^2E_1E_2 - 2E_1E_2A^2]B = 0. \tag{3.44}$$

Equivalently,

$$C[(if_1(M) - 2M^2)E_1E_2 - E_1E_2(-ig_1(A) + 2A^2))]B = 0. \tag{3.45}$$

The equations in (3.43) and (3.45) are used to solve for $f_1(M)$, $f_2(M)$, $g_1(A)$, and $g_2(A)$ in terms of M and A respectively, and we find that

$$f_1(M) = -2iM^2, \quad f_2(M) = iM, \quad g_1(A) = -2iA^2, \quad g_2(A) = -iA. \tag{3.46}$$

By using (3.46) in (3.16), we obtain the kernel $\Omega(\eta, \xi; t, y)$ of the Marchenko integral equation as

$$\Omega(\eta, \xi; t, y) = Ce^{-M\eta + iMy - 2iM^2t}e^{-A\xi - iAy - 2iA^2t}B. \tag{3.47}$$

The evolution of $\Omega(\eta, \xi; t, y)$ in t and y shown in (3.47) is unique to the generalized DS II system.

In a similar way, we are able to find $\bar{\Omega}(\eta, \xi; t, y)$ in the separable form

$$\bar{\Omega}(\eta, \xi; t, y) = \bar{C}e^{-\bar{M}\eta - i\bar{M}y + 2i\bar{M}^2t}e^{-\bar{A}\xi + i\bar{A}y + 2i\bar{A}^2t}\bar{B}. \tag{3.48}$$

The matrix quadruplet $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$, of size p, may be chosen independently of the matrices A, M, B, and C. However, by choosing $\bar{A}$, $\bar{M}$, $\bar{B}$, and $\bar{C}$ in a specific manner, so that they are dependent on A, M, B, and C in a simple way, we have sufficient conditions on the input matrices so that we obtain globally analytic solutions. Later in this chapter we discuss a specific dependence of the matrix quadruplet $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$ on A, M, B, and C which provides sufficient but not necessary conditions for globally analytic solutions.

The scalar function $\Omega(\eta, \xi; t, y)$ given in (3.47) is written in terms of matrix exponentials as the vector product of the row p-vector $C(t,y)e^{-M\eta}$ and the column p-vector $e^{-A\xi}B(t,y)$. Writing $\Omega(\eta, \xi; t, y)$ in the form (3.47) allows us to take advantage of the compact form of matrix exponentials and the separability of $\Omega(\eta, \xi; t, y)$ in η and ξ. From the separability of $\Omega(\eta, \xi; t, y)$ and $\bar{\Omega}(\eta, \xi; t, y)$ we can see that the solution $K_1(x, \xi; t, y)$ to the Marchenko integral equation in (3.32) will be separable in terms of x and ξ. We solve the Marchenko integral equation in (3.12) by substituting (3.47) into (3.32) to obtain

$$\begin{aligned} K_1(x,\xi;t,y) &+ \bar{C}(t,y)e^{-\bar{M}x}e^{-\bar{A}\xi}\bar{B}(t,y) \\ &- \left(\int_x^\infty d\eta \int_x^\infty ds\, K_1(x,s;t,y)\Omega(s,\eta;t,y)\bar{C}(t,y)e^{-\bar{M}\eta}\right) e^{-\bar{A}\xi}\bar{B}(t,y) = 0. \end{aligned}$$

Isolating $K_1(x, \xi; t, y)$ on the left side of the equation allows us to find a general form for $K_1(x, \xi; t, y)$, namely

$$\begin{aligned} K_1(x,\xi;t,y) = &- \bar{C}(t,y)e^{-\bar{M}x}e^{-\bar{A}\xi}\bar{B}(t,y) \\ &+ \left(\int_x^\infty d\eta \int_x^\infty ds\, K_1(x,s;t,y)\Omega(s,\eta;t,y)\bar{C}(t,y)e^{-\bar{M}\eta}\right) e^{-\bar{A}\xi}\bar{B}(t,y). \end{aligned}$$

Equivalently we can write $K_1(x, \xi; t, y)$ as

$$K_1(x,\xi;t,y) = H(x;t,y)e^{-\bar{A}\xi + i\bar{A}y + 2i\bar{A}^2 t}\bar{B}, \tag{3.49}$$

where the $1 \times p$ matrix valued function $H(x; t, y)$ is defined as

$$H(x;t,y) := -\bar{C}(t,y)e^{-\bar{M}x} + \int_x^\infty d\eta \int_x^\infty ds\, K_1(x,s;t,y)\Omega(s,\eta;t,y)\bar{C}(t,y)e^{-\bar{M}\eta}. \tag{3.50}$$

Using the definition of $H(x;t,y)$ in (3.50) we see that $K_1(x,\xi;t,y)$ from (3.49) can be written as the matrix product of $H(x;t,y)$, the $1\times p$ matrix valued function of x, and $e^{-\bar{A}\xi+i\bar{A}y+2i\bar{A}^2t}\bar{B}$, a $p\times 1$ matrix valued function of ξ. Notice that the separability of $K_1(x,\xi;t,y)$ in x and ξ is unaffected by the presence of t and y.

We look for solutions to the Marchenko integral equation given in (3.26) in the form of (3.49) by substituting (3.49) into (3.32) and use the equation

$$\begin{aligned}&H(x;t,y)e^{-\bar{A}\xi}\bar{B}(t,y)+\bar{C}(t,y)e^{-\bar{M}x}e^{-\bar{A}x}\bar{B}(t,y)\\&\quad-H(x;t,y)\int_x^\infty d\eta\int_x^\infty ds\,e^{-\bar{A}s}\bar{B}(t,y)\Omega(s,\eta;t,y)\bar{C}(t,y)e^{-\bar{M}\eta}e^{-\bar{A}\xi}\bar{B}(t,y)=0,\end{aligned}\tag{3.51}$$

to find $H(x;t,y)$. Factoring $e^{-\bar{A}\xi}\bar{B}(t,y)$ on the right from (3.51), we obtain

$$\begin{aligned}&\Big[H(x;t,y)+\bar{C}(t,y)\,e^{-\bar{M}x}\\&\quad-\int_x^\infty d\eta\int_x^\infty ds\,H(x;t,y)\,e^{-\bar{A}s}\bar{B}(t,y)\,\Omega(s,\eta;t,y)\,\bar{C}(t,y)e^{-\bar{M}\eta}\Big]e^{-A\xi}\bar{B}(t,y)=0.\end{aligned}\tag{3.52}$$

Note that (3.52) is equivalent to

$$\begin{aligned}&H(x;t,y)+\bar{C}(t,y)e^{-\bar{M}x}\\&\quad-\int_x^\infty d\eta\int_x^\infty ds\,H(x;t,y)e^{-\bar{A}s}\bar{B}(t,y)\Omega(s,\eta;t,y)\bar{C}(t,y)e^{-\bar{M}\eta}=0.\end{aligned}\tag{3.53}$$

Factoring $H(x;t,y)$ on the left, from (3.53) we obtain

$$\begin{aligned}H(x;t,y)\Big[I-\int_x^\infty d\eta\int_x^\infty ds\,e^{-\bar{A}s}\bar{B}(t,y)\Omega(s,\eta;t,y)\bar{C}(t,y)e^{-\bar{M}\eta}\Big]&\\=-\bar{C}(t,y)e^{-\bar{M}x},&\end{aligned}\tag{3.54}$$

where I is the $p\times p$ identity matrix. When solving (3.54) it will be helpful to define the $p\times p$ matrix-valued function $\bar{\Gamma}(x;t,y)$ as

$$\bar{\Gamma}(x;t,y):=I-\int_x^\infty d\eta\int_x^\infty ds e^{-\bar{A}s}\bar{B}(t,y)\Omega(s,\eta;t,y)\bar{C}(t,y)e^{-\bar{M}\eta}.\tag{3.55}$$

Equivalently, using (3.47) we can write (3.55) as

$$\bar{\Gamma}(x;t,y) := I - \int_x^\infty d\eta \int_x^\infty ds e^{-\bar{A}s+i\bar{A}y+2i\bar{A}^2t} \bar{B}\, \Omega(s,\eta;t,y)\, \bar{C} e^{-\bar{M}\eta-i\bar{M}y+2i\bar{M}^2t}. \quad (3.56)$$

The form of $H(x;t,y)$ in (3.49) is now written explicitly in terms of the matrix quadruplet $(\bar{A},\bar{M},\bar{B},\bar{C})$ as

$$H(x;t,y) = -\bar{C}e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}\bar{\Gamma}^{-1},$$

where $\bar{\Gamma}$ is the matrix given by (3.56).

Having solved the Marchenko integral equation in (3.32) by using the separable forms of $\Omega(x,\xi;t,y)$ and $\bar{\Omega}(x,\xi;t,y)$ in (3.47) and (3.48), respectively, the solution to (3.32) is

$$K_1(x,\xi;t,y) = -\bar{C}(t,y)e^{-\bar{M}x}\bar{\Gamma}^{-1}e^{-\bar{A}\xi}\bar{B}(t,y), \quad (3.57)$$

where $\bar{\Gamma}$ is the matrix given by (3.56). We are able to use (3.27) to find $\bar{K_1}(x,\xi;t,y)$ as

$$\bar{K_1}(x,\xi;t,y) = \bar{C}(t,y)e^{-\bar{M}x}\bar{\Gamma}^{-1}\left(\int_x^\infty d\eta\, e^{-\bar{A}\eta}\bar{B}(t,y)C(t,y)e^{-M\eta}\right) e^{-A\xi}B(t,y), \quad (3.58)$$

in terms of the matrix quadruplets (A,M,B,C) and $(\bar{A},\bar{M},\bar{B},\bar{C})$.

We can find $\bar{K_2}(x,\xi;t,y)$, the solution to the Marchenko integral equation given by (3.33), in a procedure similar to the one used to find $K_1(x,\xi;t,y)$. We find the following formulation for $\bar{K_2}(x,\xi;t,y)$:

$$\bar{K_2}(x,\xi;t,y) = -C(t,y)e^{-Mx}\Gamma^{-1}e^{-A\xi}B(t,y), \quad (3.59)$$

$$K_2(x,\xi;t,y) = C(t,y)e^{-Mx}\Gamma^{-1}\left(\int_x^\infty d\eta\, e^{-A\eta}B(t,y)\bar{C}(t,y)e^{-\bar{M}\eta}\right) e^{-\bar{A}\xi}\bar{B}(t,y), \quad (3.60)$$

$$\Gamma(x;t,y) := I - \int_x^\infty d\eta \int_x^\infty ds\, e^{-As-iAy-2iA^2t} B\, \bar{\Omega}(s,\eta;t,y)\, Ce^{-M\eta+iMy-2iM^2t}. \quad (3.61)$$

It has been found in [1] that $u(x,y,t)$, $v(x,y,t)$, and $\phi(x,y,t)$ can be determined by

$$u(x,y,t) = 2K_1(x,x), \tag{3.62}$$

$$v(x,y,t) = 2\bar{K}_2(x,x;t,y), \tag{3.63}$$

$$\phi(x,y,t) = 4i\left[K_2(x,x;t,y) - \bar{K}_1(x,x;t,y)\right]_y. \tag{3.64}$$

We are now able to find $u(x,y,t)$, $v(x,y,t)$, and $\phi(x,y,t)$ explicitly using (3.62)-(3.64) and the solution to the Marchenko integral equation $K(x,\xi;t,y)$. The solution to the generalized DS II system in terms of the constant matrix quadruplets (A,M,B,C) and $(\bar{A},\bar{M},\bar{B},\bar{C})$ is

$$u(x,y,t) = -2\bar{C}e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}\bar{\Gamma}^{-1}e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{B}, \tag{3.65}$$

$$v(x,y,t) = -2Ce^{-Mx+iMy-2iM^2t}\Gamma^{-1}e^{-Ax-iAy-2iA^2t}B, \tag{3.66}$$

$$\begin{aligned}\phi(x,y,t) = 4i\Big[&Ce^{-Mx+iMy-2iM^2t}\Gamma^{-1}\int_x^\infty d\eta\left(e^{-A\eta-iAy-2iA^2t}B\bar{C}e^{-\bar{M}\eta-i\bar{M}y+2i(\bar{M})^2t}\right)\\ &\times e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{B}\\ &-\bar{C}e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}\bar{\Gamma}^{-1}\int_x^\infty d\eta\left(e^{-\bar{A}\eta+i\bar{A}y+ei\bar{A}^2t}\bar{B}Ce^{-M\eta+iMy-2iM^2t}\right)\\ &\times e^{-Ax-iAy-2iA^2t}B\Big]_y,\end{aligned} \tag{3.67}$$

where Γ and $\bar{\Gamma}$ are found in (3.61) and (3.55), respectively.

3.2 Properties of the Auxiliary Matrices Γ and $\bar{\Gamma}$

In the previous section we have obtained a solution formula for the generalized DS II system (3.1) that depends on the complex constant matrix quadruplets (A,M,B,C) and $(\bar{A},\bar{M},\bar{B},\bar{C})$ as its input. The properties of this solution formula are studied in this section. In this section we present generalized expressions for

$\Gamma(x;t,y)$ and $\bar{\Gamma}(x,y,t)$ that are no longer dependent on an integral as in (3.61) and (3.56). This section will conclude with an example.

We begin by considering the functions $\Gamma(x;t,y)$ and $\bar{\Gamma}(x;t,y)$ in terms of the matrix quadruplets (A,M,B,C) and $(\bar{A},\bar{M},\bar{B},\bar{C})$ where the matrices A, $\bar{A}$, M, and $\bar{M}$ need not commute. The $p\times p$ matrix functions $\Gamma(x;t,y)$ and $\bar{\Gamma}(x;t,y)$ from (3.61) and (3.56) can be written in the following way:

$$\Gamma(x;t,y) = I - Q(x;t,y)\,\bar{Q}(x;t,y), \tag{3.68}$$

$$\bar{\Gamma}(x;t,y) = I - \bar{Q}(x;t,y)\,Q(x;t,y), \tag{3.69}$$

where

$$Q(x;t,y) := \int_x^\infty d\eta\, e^{-A\eta-iAy-2iA^2t}B\bar{C}e^{-\bar{M}\eta-i\bar{M}y+2i(\bar{M})^2t}, \tag{3.70}$$

$$\bar{Q}(x;t,y) := \int_x^\infty d\eta\, e^{-\bar{A}\eta+i\bar{A}y+2i\bar{A}^2t}\bar{B}Ce^{-M\eta+iMy-2i(M)^2t}. \tag{3.71}$$

The integrals defining $Q(x;t,y)$ and $\bar{Q}(x;t,y)$ exist for certain choices of the matrix quadruplet (A,M,B,C) and $(\bar{A},\bar{M},\bar{B},\bar{C})$. It is known [3] that the integrals in (3.70) and (3.71) exist if the eigenvalues of the matrices A, $\bar{A}$, M, and $\bar{M}$ have positive real parts. The proof of the existence of these integrals is similar to the proof given for Proposition 4.1 in [3]. Our goal will then be to write $Q(x;t,y)$ and $\bar{Q}(x;t,y)$ in a generalized form that is not dependent on an integral and therefore requires no restriction on the quadruplets (A,M,B,C) and $(\bar{A},\bar{M},\bar{B},\bar{C})$. We begin by showing that $Q(x;t,y)$ can be written in a form independent of an integral when the eigenvalues of A, M, $\bar{A}$, and $\bar{M}$ have positive real parts. We will then redefine $Q(x;t,y)$ in a way that allows us to relax the restriction on the eigenvalues of A, M, $\bar{A}$, and $\bar{M}$. Similarly, $\bar{Q}(x;t,y)$ can be shown to have a form that is also independent of an integral.

Theorem 3.1. *Assume that the eigenvalues of A and $\bar{M}$ have positive real parts. Then, for every x, y and $t \in \mathbb{R}$ the matrix valued function $Q(x;t,y)$ in (3.70) satisfies*

$$Q(x;t,y) = e^{-Ax-iAy-2iA^2t}Pe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}, \tag{3.72}$$

where

$$P := Q(0;0,0) = \int_0^\infty ds\, e^{-As}B\bar{C}e^{-\bar{M}s}. \tag{3.73}$$

Proof. We know that the integral defining $Q(x;t,y)$ in (3.70) exists because the eigenvalues of the matrices A and $\bar{M}$ have positive real parts. Our goal is to write $Q(x;t,y)$ in a more general form that is not dependent on an integral. The integral $Q(x;t,y)$ from (3.70), can be written as

$$Q(x;t,y) := e^{-iAy-2iA^2t}\left(\int_x^\infty d\eta\, e^{-A\eta}B\bar{C}e^{-\bar{M}\eta}\right)e^{-i\bar{M}y+2i\bar{M}^2t}.$$

Using the change of variable $s = \eta - x$ we now have

$$\begin{aligned}
Q(x;t,y) &= e^{-iAy-2iA^2t}\left(\int_0^\infty ds\, e^{-A(s+x)}B\bar{C}e^{-\bar{M}(s+x)}\right)e^{-i\bar{M}y+2i\bar{M}^2t},\\
&= e^{-Ax-iAy-2iA^2t}\left(\int_0^\infty ds\, e^{-As}B\bar{C}e^{-\bar{M}s}\right)e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t},\\
&= e^{-Ax-iAy-2iA^2t}Q(0;0,0)e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}.
\end{aligned}$$

We have shown $Q(x;t,y)$ in a more general form in terms of the constant matrix quadruplet (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$ and the constant matrix $P = Q(0;0,0)$ as in (3.72).

□

To show that $\bar{Q}(x;t,y)$ can be written as

$$\bar{Q}(x;t,y) := e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{P}e^{-Mx+iMy-2iM^2t}, \tag{3.74}$$

where $\bar{P} = \bar{Q}(0;0,0)$ is a constant matrix defined by

$$\bar{P} = \bar{Q}(0;0,0) = \int_0^\infty d\eta\, e^{-\bar{A}\eta}\bar{B}Ce^{-M\eta}$$

we use a process similar to that of the proof for Theorem 3.1. To accomplish this we use the definition for $\bar{Q}(x;t,y)$ in (3.71) and perform the change of variables $s = \eta - x$. After this change of variables we have the equivalent integral

$$\bar{Q}(x;t,y) = e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\left(\int_0^\infty ds\, e^{-\bar{A}s}\bar{B}Ce^{-Ms}\right)e^{-Mx+iMy-2iM^2t}. \tag{3.75}$$

By defining $\bar{P}$ as

$$\bar{P} := \int_0^\infty d\eta\, e^{-\bar{A}\eta}\bar{B}Ce^{-M\eta}, \tag{3.76}$$

we can rewrite the integral (3.75) as (3.74) which depends on the constant matrix $\bar{P}$.

This new formulation allows us to write $\Gamma(x;t,y)$, $\bar{\Gamma}(x;t,y)$, $Q(x;t,y)$, and $\bar{Q}(x;t,y)$ in terms of the constant matrix quadruplets (A,M,B,C) and $(\bar{A},\bar{M},\bar{B},\bar{C})$ and the constant matrices

$$P := Q(0;0,0), \tag{3.77}$$

$$\bar{P} := \bar{Q}(0;0,0). \tag{3.78}$$

Using (3.70) and (3.74) we can write $\Gamma(x;t,y)$ and $\bar{\Gamma}(x;t,y)$, repectively, as

$$\Gamma(x;t,y) = I - e^{-Ax-iAy-2iA^2t}Pe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{P}e^{-Mx+iMy-2iM^2t}, \tag{3.79}$$

$$\bar{\Gamma}(x;t,y) = I - e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{P}e^{-Mx+iMy-2iM^2t}e^{-Ax-iAy-2iA^2t}Pe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}. \tag{3.80}$$

We can now also write $K_1(x,\xi;t,y)$, $\bar{K_2}(x,\xi;t,y)$, $K_2(x,\xi;t,y)$ and $\bar{K_1}(x,\xi;t,y)$ appearing in (3.57), (3.59), (3.60) and (3.58) in terms of the matrix quadruplets

(A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$ and the constant matrices $P := Q(0; 0, 0)$ and $\bar{P} := \bar{Q}(0; 0, 0)$. Using (3.70) and (3.74) in (3.57), (3.59), (3.60), and (3.58) we obtain

$$K_1(x, \xi; t, y) = -\bar{C}e^{-\bar{M}x - i\bar{M}y + 2i\bar{M}^2 t}\bar{\Gamma}^{-1}e^{-\bar{A}\xi + i\bar{A}y + 2i\bar{A}^2 t}\bar{B}, \tag{3.81}$$

$$\bar{K}_2(x, \xi; t, y) = -Ce^{-Mx + iMy - 2iM^2 t}\Gamma^{-1}e^{-A\xi - iAy - 2iA^2 t}B, \tag{3.82}$$

$$\begin{aligned} K_2(x, \xi; t, y) = {} & Ce^{-Mx + iMy - 2iM^2 t}\Gamma^{-1}e^{-A\xi - iAy - 2iA^2 t} \\ & \times Pe^{-\bar{M}x - i\bar{M}y + 2i\bar{M}^2 t}e^{-\bar{A}\xi + i\bar{A}y + 2i\bar{A}^2 t}\bar{B}, \end{aligned} \tag{3.83}$$

$$\begin{aligned} \bar{K}_1(x, \xi; t, y) = {} & \bar{C}e^{-\bar{M}x - i\bar{M}y + 2i\bar{M}^2 t}\bar{\Gamma}^{-1}e^{-\bar{A}\xi + i\bar{A}y + 2i\bar{A}^2 t} \\ & \times \bar{P}e^{-Mx + iMy - 2iM^2 t}e^{-A\xi - iAy - 2iA^2 t}B, \end{aligned} \tag{3.84}$$

where the eigenvalues of the matrices $A, M, \bar{A}, \bar{M}$ have positive real parts.

We would like to relax the restriction on the eigenvalues of $A, M, \bar{A}, \bar{M}$ to allow for a wider range of input matrices. To do this we will redefine our auxiliary matrices P and $\bar{P}$ from (3.73) and (3.76) to be constant $p \times p$ matrices that satisfy the auxiliary matrix equations

$$\begin{aligned} B\bar{C} &= AP + P\bar{M}, \\ \bar{B}C &= \bar{A}\bar{P} + \bar{P}M. \end{aligned} \tag{3.85}$$

The auxiliary linear matrix equations (3.85) have the form of $YX - XZ = W$ where X, Y, W and Z are matrices of sizes $p \times p$, $p \times p$, $p \times p$ and $p \times p$, respectively. If $z_1, \ldots, z_p$ and $y_1, \ldots, y_p$ are the eigenvalues of Z and Y, respectively, then it is known that the equation $YX - XZ = W$ has a unique solution X if and only if $y_l - z_j \neq 0$ for any choice of l and j. The reader can reference pp. 383–387 in [9] for a proof.

By defining the constant auxiliary matrices P and $\bar{P}$ as the solutions to (3.85) we are able to choose A, M, $\bar{A}$, and $\bar{M}$ from a wider class of $p \times p$ matrices. We restrict ourselves now to matrix quadruplets (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$ such that P and $\bar{P}$ exist. Specifically, if $\{a_1, \ldots, a_p\}$, $\{m_1, \ldots, m_p\}$, $\{\bar{a}_1, \ldots, \bar{a}_p\}$, and $\{\bar{m}_1, \ldots, \bar{m}_p\}$

are eigenvalues of the matrices A, M, $\bar{A}$, and $\bar{M}$ respectively, then A, M, $\bar{A}$, and $\bar{M}$ should be chosen such that $a_l + \bar{m}_j \neq 0$ and $\bar{a}_l + m_j \neq 0$. The auxiliary matrices P and $\bar{P}$ can be found from the auxiliary equations (3.85) and used to construct the solution $u(x, y, t)$, $v(x, y, t)$, and $\phi(x, y, t)$ to (3.1) as

$$u(x,y,t) = -2\bar{C}e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}\bar{\Gamma}^{-1}e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{B}, \tag{3.86}$$

$$v(x,y,t) = -2Ce^{-Mx+iMy-2iM^2t}\Gamma^{-1}e^{-Ax-iAy-2iA^2t}B, \tag{3.87}$$

$$\begin{aligned}\phi(x,y,t) = 4i[&Ce^{-Mx+iMy-2iM^2t}\Gamma^{-1}e^{-Ax-iAy-2iA^2t}\\ &\times Pe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{B}\\ &-\bar{C}e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}\bar{\Gamma}^{-1}e^{-\bar{A}\xi+i\bar{A}y+2i\bar{A}^2t}\\ &\times \bar{P}e^{-Mx+iMy-2iM^2t}e^{-A\xi-iAy-2iA^2t}B]_y,\end{aligned} \tag{3.88}$$

where Γ and $\bar{\Gamma}$ are defined to be

$$\Gamma(x;t,y) := I - e^{-Ax-iAy-2iA^2t}Pe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{P}e^{-Mx+iMy-2iM^2t}, \tag{3.89}$$

$$\bar{\Gamma}(x;t,y) := I - e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{P}e^{-Mx+iMy-2iM^2t}e^{-Ax-iAy-2iA^2t}Pe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}. \tag{3.90}$$

Our goal will now be to show that the solution formula (3.65), (3.66), and (3.67) can be written as (3.86), (3.87), and (3.88) when the eigenvalues of the matrices A, M, $\bar{A}$, and $\bar{M}$ all have positive real parts. Alternatively, we can show that the solution formula (3.65), (3.66), and (3.67) can be written as (3.86), (3.87), and (3.88) when the eigenvalues of the matrices A, M, $\bar{A}$, and $\bar{M}$ all have negative real parts. To do this we first show that the matrix P defined in (3.73) satisfies the first line in (3.85). We will use the formulation for $Q(x;t,y)$ in (3.70) along with the equivalent formulation for $Q(x;t,y)$ in terms of (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$ in (3.72) to show that P satisfies the auxiliary equation $B\bar{C} = AP + P\bar{M}$. A similar proof can be used to show that the constant matrix $\bar{P}$ defined in (3.76) satisfies the auxiliary equation $\bar{B}C = \bar{A}\bar{P} + \bar{P}M$.

Theorem 3.2. *Assume that the eigenvalues of A and $\bar{M}$ have positive real parts. Let $a_1, \ldots, a_p$ and $\bar{m}_1, \ldots, \bar{m}_p$ be the eigenvalues of A and $\bar{M}$, respectively. The matrix P from (3.73) will satisfy the following linear matrix equation*

$$B\bar{C} = AP + P\bar{M}, \tag{3.91}$$

which will be uniquely solvable if $a_l + \bar{m}_j \neq 0$ for all $l = 1, \ldots p$ and $j = 1, \ldots p$.

Proof. Investigating the partial derivative of $Q(x;t,y)$ in (3.72) we have

$$Q_x = -e^{-Ax-iAy-2iA^2t}B\bar{C}e^{-\bar{M}-i\bar{M}y+2i\bar{m}^2t}.$$

Equivalently, we can differentiate (3.70) with respect to x

$$Q_x = -e^{-Ax-iAy-2iA^2t}APe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t} - e^{-Ax-iAy-2iA^2t}P\bar{M}e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}.$$

We can equate these expressions for Q_x

$$\begin{aligned}-e^{-Ax-iAy-2iA^2t}Pe^{-\bar{M}-i\bar{M}y+2i\bar{M}^2t} &= -e^{-Ax-iAy-2iA^2t}APe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t} \\ &\quad - e^{-Ax-iAy-2iA^2t}P\bar{M}e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}.\end{aligned}$$

Multiplying by $e^{Ax+iAy+2iA^2t}$ on the left side of the equation and $e^{\bar{M}x+i\bar{M}y-2i\bar{M}^2t}$ on the right side reveals the relation

$$B\bar{C} = AP + P\bar{M}.$$

Given a constant matrix quadruplet (A, M, B, C), we can write the auxiliary equation as

$$B\bar{C} = AP - P(-\bar{M})$$

which is uniquely solvable if $a_i + \bar{m}_j \neq 0$ for all $i = 1, \ldots p$ and $j = 1, \ldots p$. □

Similarly we can find the auxiliary equation

$$\bar{B}C = \bar{A}\bar{P} + \bar{P}M \tag{3.92}$$

by using the formulations of $\bar{Q}(x;t,y)$ in (3.71) and (3.74).

The solution found in (3.65) and (3.67) can be written as

$$u(x,y,t) = -2\bar{C}e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}\Gamma^{-1}e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{B} \tag{3.93}$$

$$v(x,y,t) = -2Ce^{-Mx+iMy-2iM^2t}\Gamma^{-1}e^{-Ax-iAy-2iA^2t}B \tag{3.94}$$

$$\begin{aligned}\phi(x,y,t) = 4i[&Ce^{-Mx+iMy-2iM^2t}\Gamma^{-1}e^{-Ax-iAy-2iA^2t}\\ &\times Pe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{B}\\ &-\bar{C}e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}\bar{\Gamma}^{-1}e^{-\bar{A}\xi+i\bar{A}y+2i\bar{A}^2t}\\ &\times \bar{P}e^{-Mx+iMy-2iM^2t}e^{-A\xi-iAy-2iA^2t}B]_y,\end{aligned} \tag{3.95}$$

where Γ and $\bar{\Gamma}$ are defined in terms of the auxiliary matrices P and $\bar{P}$ as (3.89) and (3.90), respectively.

3.3 Globally Analytic Solutions

To guarantee that a solution will be globally analytic we will need to make restrictions on our choice of the matrix quadruplets (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}\bar{C})$. The solution formula given in (3.65)-(3.67) will provide a solution that is analytic everywhere except on the surfaces $\det[\Gamma(x;t,y)] = 0$ and $\det[\bar{\Gamma}(x;t,y)] = 0$, where $\Gamma(x;t,y)$ and $\bar{\Gamma}(x;t,y)$ are the matrix valued functions found in (3.79) and (3.80), respectively. That is, for any choice of (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$, where the auxiliary matrices P and $\bar{P}$ exist, we are guaranteed that the solutions $u(x,y,t)$, $v(x,y,t)$, and $\phi(x,y,t)$ will be valid for all $x, y, t \in \mathbb{R}$ such that the matrix functions $\Gamma(x;t,y)$ and $\bar{\Gamma}(x,y,t)$ found in (3.61) and (3.56) are invertible. In the following section we show that the solution formulas given in (3.65), (3.66), and (3.67) will provide solutions that can be expressed as a Taylor series for all $x, y, t \in \mathbb{R}$ such that $\det[\Gamma(x;t,y)] \neq 0$ and $\det[\bar{\Gamma}(x;t,y)] \neq 0$. With certain choices of the matrix quadruplets (A, M, B, C) and $(\bar{A}, \bar{M}\bar{B}, \bar{C})$ we can show that the matrix valued functions $\Gamma(x;t,y)$ and $\bar{\Gamma}(x,y,t)$ are invertible. The analyticity of $u(x,y,t)$, $v(x,y,t)$, and $\phi(x,y,t)$ in (3.65), (3.66), and (3.67) depends on the invertibility of $\Gamma(x;t,y)$ and $\bar{\Gamma}(x;t,y)$ for all x, y and t in $\mathbb{R}$.

To find globally analytic solutions to the generalized DS II system we are motivated by the solution formula to a (1+1) case of the generalized DS II system (3.1), namely the focusing NLS equation (3.2). Using the solution formula to the generalized DS II system (3.1) with the restrictions $\phi(x,y,t) \equiv 0$, $v(x,y,t) = -u^*(x,y,t)$, and suppress the y dependence of $u(x,y,t)$ we obtain a solution formula for certain solutions to the NLS equation as

$$u(x,y,t) = -2\bar{C}e^{-\bar{M}x+2i\bar{M}^2t}\bar{\Gamma}^{-1}e^{-\bar{A}x+2i\bar{A}^2t}\bar{B}, \tag{3.96}$$

where

$$\bar{\Gamma}(x;t,y) = I - e^{-\bar{A}x+2i\bar{A}^2t}\bar{P}e^{-Mx-2iM^2t}e^{-Ax-2iA^2t}Pe^{-\bar{M}x+2i\bar{M}^2t}. \tag{3.97}$$

We compare our solution formula (3.96) to the solution formula found in [3] to find some sufficiency conditions on the matrix quadruplets (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$ that will yield globally analytic solutions. We will obtain globally analytic solutions (3.96) by choosing the matrix quadruplets (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$ as

$$\begin{aligned} A &= M, \\ \bar{A} &= A^\dagger, \\ \bar{M} &= M^\dagger = A^\dagger, \\ \bar{B} &= -C^\dagger, \\ \bar{C} &= B^\dagger, \end{aligned} \tag{3.98}$$

where the eigenvalues of A have positive real parts. Substituting the sufficiency conditions (3.98) into (3.96) we obtain

$$u(x,y,t) = -2\bar{C}e^{-A^\dagger x+2i(A^\dagger)^2 t}\bar{\Gamma}^{-1}e^{-A^\dagger x+2i(\dagger)^2 t}\bar{B}, \tag{3.99}$$

where

$$\bar{\Gamma}(x;t,y) = I - e^{-A^\dagger x+2i(A^\dagger)^2 t}\bar{P}e^{-Ax-2iA^2 t}e^{-Ax-2iA^2 t}Pe^{-A^\dagger x+2i(A^\dagger)^2 t}. \tag{3.100}$$

We claim that by using the sufficiency conditions (3.98) in (3.65), (3.66), and (3.67) we can produce a globally analytic solution to the generalized DS II system. By inspecting the solution formula (3.65), (3.66), and (3.67) we see that the solution to the generalized DS II system is written as the product of exponential functions, polynomials and trigonometric functions when Γ^{-1} and $\bar{\Gamma}^{-1}$ exist. To verify the global analyticity of the solution formula (3.65), (3.66), and (3.67) we need to verify that (3.98) are sufficient conditions for the existence of the inverses of Γ and $\bar{\Gamma}$. When studying the invertibility of $\Gamma(x;t,y)$ it will be convenient to consider $\Gamma(x;t,y)$ as

$$\Gamma(x;t,y) = I - e^{-iAy}G(x;t)e^{-i\bar{M}y}e^{i\bar{A}y}\bar{G}(x;t)e^{iMy}, \tag{3.101}$$

where the matrix valued functions $G(x;t)$ and $\bar{G}(x;t)$ are defined to be

$$G(x;t) = \int_x^\infty d\eta\, e^{-A\eta-2iA^2 t}B\bar{C}\,e^{-\bar{M}\eta+2i\bar{M}^2 t}, \tag{3.102}$$

$$\bar{G}(x;t) = \int_x^\infty ds\, e^{-\bar{A}s+2i\bar{A}^2 t}\bar{B}Ce^{-Ms-2iM^2 t}. \tag{3.103}$$

It will be convenient to write $\Gamma(x;t,y)$ as

$$\Gamma(x;t,y) = I + e^{-iAy}G(x;t)e^{-i\bar{M}y}e^{i\bar{A}y}\left[-\bar{G}(x;t)\right]e^{iMy}, \tag{3.104}$$

and consider the matrix valued functions $G(x;t)$ and $-\bar{G}(x;t)$. To show that $G(x;t)$ and $-\bar{G}(x;t)$ in (3.102) and (3.103) are positive we show that $G(x;t)$ and $-G(x;t)$

can be written as the product of a matrix and its adjoint. We see that $G(x;t)$ and $-\bar{G}(x;t)$ are positive whenever we choose the matrix quadruplets (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$ such that $\bar{C} = B^\dagger$, $\bar{B} = -C^\dagger$, $\bar{A} = M^\dagger$, and $\bar{M} = A^\dagger$. Furthermore, since $G(x;t)$ and $-\bar{G}(x;t)$ are positive and self adjoint there exists positive matrix valued functions $G^{1/2}$ and $\bar{G}^{1/2}$ such that $G(x;t) = G^{1/2}G^{1/2}$ and $-\bar{G}(x;t) = \bar{G}^{1/2}\bar{G}^{1/2}$ which are self adjoint themselves.

We will first rewrite (3.68) using $G(x;t) = G^{1/2}G^{1/2}$ and $-\bar{G}(x;t) = \bar{G}^{1/2}\bar{G}^{1/2}$ as

$$\Gamma(x;t,y) = I + e^{-iAy}G^{1/2}G^{1/2}e^{-i\bar{M}y}e^{i\bar{A}y}\bar{G}^{1/2}\bar{G}^{1/2}e^{iMy}. \tag{3.105}$$

Our goal will be to show that for certain choices of $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$ the inverse of $\Gamma(x;t,y)$ will exist for all $x, y, t \in \mathbb{R}$. We will use the Sherman-Morrison-Woodbury formula

$$(W + uv^\top)^{-1} = W^{-1} - W^{-1}u(I + v^\top W^{-1}u)^{-1}v^\top W^{-1} \tag{3.106}$$

where A is an $n \times n$ matrix, and u and v are $n \times m$ matrices. Using (3.106) with

$$\begin{aligned} u &= e^{-iAy}G^{1/2} \\ v^\top &= G^{1/2}e^{-i\bar{M}y}e^{i\bar{A}y}\bar{G}^{1/2}\bar{G}^{1/2}e^{iMy} \end{aligned}$$

we have a useful expression for $\Gamma^{-1}(x;t,y)$ as

$$\begin{aligned} \Gamma^{-1}(x;t,y) =& I - e^{-iAy}G^{1/2}(I + G^{1/2}e^{-i\bar{M}y}e^{i\bar{A}y}\bar{G}^{1/2}\bar{G}^{1/2}e^{iMy}e^{-iAy}G^{1/2})^{-1} \\ & \times G^{1/2}e^{-i\bar{M}y}e^{i\bar{A}y}\bar{G}^{1/2}\bar{G}^{1/2}e^{iMy} \end{aligned} \tag{3.107}$$

We show that $\Gamma^{-1}(x;t,y)$ exists by investigating the invertibility of

$$N(x;t) := I + G^{1/2}e^{-i\bar{M}y}e^{i\bar{A}y}\bar{G}^{1/2}\bar{G}^{1/2}e^{iMy}e^{-iAy}G^{1/2}, \tag{3.108}$$

from (3.107). We show that $N(x;t)$ is invertible by expressing (3.108) as the sum of the identity matrix and a positive matrix. Using the self adjoint properties of $G(x;t)$ and $\bar{G}(x;t)$ we can express $N(x;t)$ as

$$N(x;t) = I + (G^{1/2}e^{-i\bar{M}y}e^{i\bar{A}y}\bar{G}^{1/2})(G^{1/2}e^{iA^\dagger y}e^{-iM^\dagger y}\bar{G}^{1/2})^\dagger. \tag{3.109}$$

In the expression (3.109) we see that $N(x;t)$ can be written as the sum of the $n \times n$ identity matrix and a positive matrix. Therefore, $N(x;t)$ is invertible for certain choices of the matrix quadruplet (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$. From this it follows that $\Gamma^{-1}(x;t,y)$ exists for certain choices of the matrix quadruplet (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$. Hence, for globally analytic solutions to the nonlinear PDE in (3.1) we need only choose the matrix quadruplets (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$ so that $M = A$, $\bar{A} = M^\dagger$, $\bar{M} = A^\dagger$, $\bar{B} = -C^\dagger$, and $\bar{C} = B^\dagger$. We follow this section with a detailed method to construct globally analytic solutions to the generalized Davey-Stewartson II system found in (3.1). For other choices of the matrix quadruplets (A, M, B, C) and $\bar{A}, \bar{M}, \bar{B}, \bar{C}$ we are assured to have a solution to the nonlinear PDE (3.1) with singularities found on the surfaces $\det[\Gamma(x;t,y)] = 0$ and $\det[\bar{\Gamma}(x;t,y)] = 0$.

3.3.1 Constructing the Solutions $u(x,y,t)$, $v(x,y,t)$, and $\phi(x,y,t)$ from (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$

An advantage of this method is that the solution formula found can be used without any knowledge of the method used to derive them. In this way, a large class of solutions is available to a wide audience, who need only to be familiar with linear algebra and calculus. Below is a systematic method that can be used to construct a solution $u(x,y,t)$, $v(x,y,t)$, and $\phi(x,y,t)$ to (3.1) with any given choice of constant matrix quadruplets (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$.

(i) Begin by choosing the complex valued constant quadruplets of matrices (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$, each with sizes $p \times p$, $p \times p$, $p \times 1$, and $1 \times p$, respectively, as our input. Let $a_1, \dots, a_p$ be the eigenvalues of A and $m_1, \dots, m_p$ be the eigenvalues of M. For globally analytic solutions we will choose the matrix M such that $M = A$ and $\bar{A} = M^\dagger$, $\bar{M} = A^\dagger$, $\bar{B} = -C^\dagger$, and $\bar{C} = B^\dagger$. These conditions are sufficient for a globally analytic solution. We will need to enforce a condition on the eigenvalues of A and M, namely $a_i + m_j \neq 0$ for all $i = 1, \ldots, p$ and $j = 1, \ldots, p$ to ensure the existence of the auxiliary matrices P and $\bar{P}$.

(ii) Construct the auxiliary matrices P and $\bar{P}$ by solving the linear matrix equations

$$B\bar{C} = AP + P\bar{M},$$
$$\bar{B}C = \bar{A}\bar{P} + \bar{P}M.$$

(iii) Use the quadruplets of matrices (A, M, B, C) and $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$ and the auxiliary matrices P and $\bar{P}$ to construct the functions $\Gamma(x; t, y)$ and $\bar{\Gamma}(x; t, y)$ in the following way

$$\Gamma(x; t, y) = I - e^{-Ax-iAy-2iA^2t}Pe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{P}e^{-Mx+iMy-2iM^2t}$$
$$\bar{\Gamma}(x; t, y) = I - e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{P}e^{-Mx+iMy-2iM^2t}e^{-Ax-iAy-2iA^2t}Pe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}$$

(iv) We can then formulate the solutions $u(x, y, t)$, $v(x, y, t)$, and $\phi(x, y, t)$ to the nonlinear PDE (3.1) as

$$u(x, y, t) = -2\bar{C}e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}\bar{\Gamma}^{-1}e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{B}$$
$$v(x, y, t) = -2Ce^{-Mx+iMy-2iM^2t}\Gamma^{-1}e^{-Ax-iAy-2iA^2t}B$$
$$\begin{aligned}\phi(x, y, t) = 4i[&Ce^{-Mx+iMy-2iM^2t}\Gamma^{-1}e^{-Ax-iAy-2iA^2t} \\ &\times Pe^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}\bar{B} \\ &- \bar{C}e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}\bar{\Gamma}^{-1}e^{-\bar{A}\xi+i\bar{A}y+2i\bar{A}^2t} \\ &\times \bar{P}e^{-Mx+iMy-2iM^2t}e^{-A\xi-iAy-2iA^2t}B]_y.\end{aligned}$$

3.4 Independent Verification of the Solution Formula for the Generalized DS II System

The method used to find the solution formula in (3.93), (3.94) and (3.95) is based on the inverse scattering transform. The inverse scattering transform relies on the idea that each integrable nonlinear PDE is associated to a linear differential equation containing a spectral parameter [13]. An analysis of the associated spectral problem will provide the scattering data $S(\lambda; 0, 0)$. The inverse scattering transform then relates the time evolution of the scattering data to the solution

$$S(\lambda; 0, 0) \mapsto S(\lambda; t, y)$$

to the solution to the nonlinear PDE, $u(x, y, t)$, $v(x, y, t)$, and $\phi(x, y, t)$. In our method of obtaining exact solution formulas to integrable evolution equations, we emphasize the time evolution

$$\Omega(\eta, \xi; 0, 0) \mapsto \Omega(\eta, \xi; t, y)$$

of the Marchenko kernel. The Marchenko kernel $\Omega(\eta, \xi; t, y)$ is in general related to the scattering data $S(\lambda; t, y)$ through a Fourier transform. The relationship between the inverse scattering transform and the method outlined can be seen below.

$$\begin{array}{ccccc} u(x,0,0) & \xrightarrow{\text{direct scattering}} & S(\lambda;0,0) & \xrightarrow{\text{Fourier transform}} & \Omega(\eta,\xi;0,0) \\ \downarrow & & \downarrow & & \downarrow \\ u(x,y,t) & \xleftarrow[\text{inverse scattering}]{} & S(\lambda;t,y) & \xleftarrow[\text{inverse Fourier transform}]{} & \Omega(\eta,\xi;t,y) \end{array}$$

We now introduce some identities that will be helpful to show that the solution formula in (3.93), (3.94) and (3.95) is in fact a solution to the system of equations in (3.1). It will be convenient to adopt the following definitions:

$$\begin{aligned} E_1 &= e^{-Mx+iMy-2iM^2t}, \quad E_2 = e^{-Ax-iAy-2iA^2t}, \\ \bar{E}_1 &= e^{-\bar{M}x-i\bar{M}y+2i\bar{M}^2t}, \quad \bar{E}_2 = e^{-\bar{A}x+i\bar{A}y+2i\bar{A}^2t}. \end{aligned} \tag{3.110}$$

Theorem 3.3. *The functions* $S := E_2 P \bar{E}_1$, $\bar{S} := \bar{E}_2 \bar{P} E_1$, $\Gamma = I - S\bar{S}$, *and* $\bar{\Gamma} = I - \bar{S}S$ *satisfy the following equations:*

$$\bar{\Gamma}^{-1}\bar{S} = \bar{S}\Gamma^{-1}, \tag{3.111}$$

$$S\bar{\Gamma}^{-1} = \Gamma^{-1}S \tag{3.112}$$

$$\bar{S}\Gamma^{-1}S = \bar{\Gamma}^{-1} - I \tag{3.113}$$

$$S\bar{\Gamma}^{-1}\bar{S} = \Gamma^{-1} - I \tag{3.114}$$

Proof.

- To show that $\bar{\Gamma}^{-1}\bar{S} = \bar{S}\Gamma^{-1}$ we first consider the effect of multiplication of $\bar{S}$ on the right of $\bar{\Gamma}$ which yields

$$\bar{\Gamma}\bar{S} = (I - \bar{S}S)\bar{S} = \bar{S} - \bar{S}S\bar{S}. \tag{3.115}$$

  Similarly we consider the effect of multiplication of $\bar{S}$ on the left of Γ which yields

$$\bar{S}\Gamma = \bar{S}(I - S\bar{S}) = \bar{S} - \bar{S}S\bar{S}. \tag{3.116}$$

  Equating the equations (3.115) and (3.116) we have

$$\bar{\Gamma}\bar{S} = \bar{S}\Gamma. \tag{3.117}$$

  Multiplying the equation (3.117) on the right by Γ^{-1} and on the left by $\bar{\Gamma}^{-1}$ we obtain the identity

$$\bar{\Gamma}^{-1}\bar{S} = \bar{S}\Gamma^{-1}.$$

- The equation $S\bar{\Gamma}^{-1} = \Gamma^{-1}S$ can be shown with a proof similar to that of the proof for the equation $\bar{\Gamma}^{-1}\bar{S} = \bar{S}\Gamma^{-1}$.

- To show that $\bar{S}\Gamma^{-1}S = \bar{\Gamma} - I$ we begin by using the definition of $\bar{\Gamma}$ in $\bar{S}S\bar{\Gamma}^{-1}$ which yields

$$\bar{S}S\bar{\Gamma}^{-1} = (I - \bar{\Gamma})\bar{\Gamma}^{-1} = \bar{\Gamma}^{-1} - I.$$

Using the equation (3.112) in $\bar{S}S\bar{\Gamma}^{-1} = \bar{\Gamma}^{-1} - I$ we obtain the equation

$$\bar{S}\Gamma^{-1}S = \bar{\Gamma} - I$$

- The equation $S\bar{\Gamma}^{-1}\bar{S} = \Gamma^{-1} - I$ can be shown with a proof similar to that of the proof for the equation $\bar{S}\Gamma^{-1}S = \bar{\Gamma}^{-1} - I$.

□

Theorem 3.4. *The derivatives of the functions $S = E_2P\bar{E}_1$, $\bar{S} = \bar{E}_2\bar{P}E_1$, $\Gamma = I - S\bar{S}$, and $\bar{\Gamma} = I - \bar{S}S$ satisfy the following equations:*

$$i\Gamma_y = -A + M + A\Gamma - \Gamma M + S(\bar{A} - \bar{M})\bar{S} \tag{3.118}$$

$$i\bar{\Gamma}_y = \bar{A} - \bar{M} - \bar{A}\bar{\Gamma} + \bar{\Gamma}\bar{M} - \bar{S}(A - M)S \tag{3.119}$$

$$iS_y = -iS_x = AS + S\bar{M} \tag{3.120}$$

$$-i\bar{S}_y = \bar{A}\bar{S} + \bar{S}M \tag{3.121}$$

$$(\Gamma^{-1})_{xx} = 2\Gamma^{-1}\Gamma_x\Gamma^{-1}\Gamma_x\Gamma^{-1} - \Gamma^{-1}\Gamma_{xx}\Gamma^{-1} \tag{3.122}$$

$$\bar{\Gamma}_x = \bar{A} + \bar{M} - \bar{A}\bar{\Gamma} - \bar{\Gamma}\bar{M} + \bar{S}(M + A)S \tag{3.123}$$

$$\Gamma_x = A + M - A\Gamma - \Gamma + S(\bar{M} + \bar{A})\bar{S} \tag{3.124}$$

Proof. The equations shown in (3.118)-(3.124) can be shown directly with the use of the product rule for derivatives and the identity $\Gamma_x = -\Gamma^{-1}\Gamma_x\Gamma^{-1}$.

□

An independent verification is provided to show that the functions (3.93), (3.94), and (3.95) are indeed solutions to (3.1). To simplify the notation we will use the definitions presented in (3.110) and Theorem 3.3.

Theorem 3.5. *The solution formula (3.93), (3.94), and (3.95) can be written in terms of* E_1, E_2, $\bar{E}_1$, $\bar{E}_2$, S *and* $\bar{S}$ *from (3.110) and Theorem 3.3 as*

$$u(x, y, t) = -2\bar{C}\bar{E}_1\bar{\Gamma}^{-1}\bar{E}_2\bar{B}, \tag{3.125}$$

$$v(x; t, y) = -2CE_1\Gamma^{-1}E_2B, \tag{3.126}$$

$$\phi(x, y, t) = 4i[CE_1\Gamma^{-1}S\bar{E}_2\bar{B} - \bar{C}\bar{E}_1\bar{\Gamma}^{-1}\bar{S}E_2B]_y, \tag{3.127}$$

where $\Gamma(x; t, y)$ *and* $\bar{\Gamma}(x; t, y)$ *are defined by (3.89) and (3.90), respectively. Let* $\{a_1, \ldots, a_p\}$, $\{m_1, \ldots, m_p\}$, $\{\bar{a}_1, \ldots, \bar{a}_p\}$, *and* $\{\bar{m}_1, \ldots, \bar{m}_p\}$ *be the eigenvalues of the* $p \times p$ *matrices* A, M, $\bar{A}$, *and* $\bar{M}$. *The functions (3.125) - (3.127), in terms of the matrix quadruplets* (A, M, B, C) *and* $(\bar{A}, \bar{M}, \bar{B}, \bar{C})$, *satisfy the generalized DS II system (3.1) when* A, M, $\bar{A}$, *and* $\bar{M}$ *are chosen such that* $a_l + \bar{m}_j \neq 0$ *and* $\bar{a}_l + m_j \neq 0$ *for all* $l = 1, \ldots, p$ *and* $j = 1, \ldots, p$ *.*

Proof. We begin by taking the derivative of (3.125) with respect to x and obtain

$$u_x = 2\bar{C}\bar{E}_1[\bar{M}\bar{\Gamma}^{-1} + \bar{\Gamma}^{-1}\bar{A} + \bar{\Gamma}^{-1}\bar{\Gamma}_x\bar{\Gamma}^{-1}]\bar{E}_2\bar{B}.$$

Factoring $\bar{\Gamma}^{-1}$ from the left and right side we are able to write u_x as

$$u_x = 2\bar{C}\bar{E}_1\bar{\Gamma}^{-1}[\bar{\Gamma}\bar{M} + \bar{A}\bar{\Gamma} + \bar{\Gamma}_x]\bar{\Gamma}^{-1}\bar{E}_2\bar{B}. \tag{3.128}$$

From (3.128) we are able to find

$$\begin{aligned} u_{xx} = -2\bar{C}\bar{E}_1\bar{\Gamma}^{-1}[&\bar{\Gamma}\bar{M}^2 + \bar{A}^2\bar{\Gamma}^{-1} + 2\bar{\Gamma}\bar{M}\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma} + 2\bar{\Gamma}\bar{M}\bar{\Gamma}^{-1}\bar{\Gamma}_x \\ &+ 2\bar{\Gamma}_x\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma} + 2\bar{\Gamma}_x\bar{\Gamma}^{-1}\bar{\Gamma}_x - \bar{\Gamma}_{xx}]\bar{\Gamma}^{-1}\bar{E}_2\bar{B}. \end{aligned} \tag{3.129}$$

Taking the derivative of (3.125) with respect to y we have

$$u_y = -2\bar{C}\bar{E}_1[-i\bar{M}\bar{\Gamma}^{-1} + i\bar{\Gamma}^{-1}\bar{A} - \bar{\Gamma}^{-1}\bar{\Gamma}_y\bar{\Gamma}^{-1}]\bar{E}_2\bar{B}.$$

Factoring $\bar{\Gamma}^{-1}$ from the left and right side we are able to write u_y as

$$u_y = -2\bar{C}\bar{E}_1\bar{\Gamma}^{-1}[-i\bar{\Gamma}\bar{M} + i\bar{A}\bar{\Gamma} - \bar{\Gamma}_y]\bar{\Gamma}^{-1}\bar{E}_2\bar{B}. \tag{3.130}$$

From (3.130) we are able to find

$$u_{yy} = -2\bar{C}\bar{E}_1\Gamma^{-1}[-\bar{\Gamma}\bar{M}^2 + 2\bar{\Gamma}\bar{M}\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma} + 2i\bar{\Gamma}\bar{M}\bar{\Gamma}^{-1}\bar{\Gamma}_y + \bar{A}^2\bar{\Gamma} - 2i\bar{\Gamma}_y\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma} + 2\bar{\Gamma}_y\bar{\Gamma}^{-1}\bar{\Gamma}_y - \bar{\Gamma}_{yy}]\bar{\Gamma}^{-1}\bar{E}_2\bar{B}. \tag{3.131}$$

Taking the derivative of (3.125) with respect to t and multiplying by $i = \sqrt{-1}$ we have

$$iu_t = -2\bar{C}\bar{E}_1[-2\bar{\Gamma}\bar{M}^2 - 2\bar{A}^2\bar{\Gamma} - i\bar{\Gamma}_t]\bar{E}_2\bar{B} \tag{3.132}$$

We will then use the identity

$$2\bar{\Gamma}_x\bar{\Gamma}^{-1}\bar{\Gamma}_x - 2\bar{\Gamma}_y\bar{\Gamma}^{-1}\bar{\Gamma}_y = (\bar{\Gamma}_x + i\bar{\Gamma}_y)\bar{\Gamma}^{-1}(\bar{\Gamma}_x + i\bar{\Gamma}_y) + (\bar{\Gamma}_x - i\bar{\Gamma}_y)\bar{\Gamma}^{-1}(\bar{\Gamma}_x - i\bar{\Gamma}_y) \tag{3.133}$$

to simplify (3.129) and (3.131) and obtain the term $u_{xx} - u_{yy}$ in (**??**) to be

$$u_{xx} - u_{yy} = -2\bar{C}\bar{E}_1\Gamma^{-1}[2\bar{\Gamma}\bar{M}^2 + 2\bar{A}^2\bar{\Gamma} + 2\bar{\Gamma}\bar{M}\bar{\Gamma}^{-1}(\bar{\Gamma}_x - i\bar{\Gamma}_y) + \bar{\Gamma}_{yy} - \bar{\Gamma}_{xx} + 2(\bar{\Gamma}_x + i\bar{\Gamma}_y)\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma} + (\bar{\Gamma}_x + i\bar{\Gamma}_y)\bar{\Gamma}^{-1}(\bar{\Gamma}_x + i\bar{\Gamma}_y) + (\bar{\Gamma}_x - i\bar{\Gamma}_y)\bar{\Gamma}^{-1}(\bar{\Gamma}_x - i\bar{\Gamma}_y)]\bar{\Gamma}^{-1}\bar{E}_2\bar{B}. \tag{3.134}$$

To be able to combine these terms we will want to write the derivatives of $\bar{\Gamma}$ in terms of the matrix quadruplet (A, M, B, C), the constant matrix P, and $\bar{\Gamma}$. We investigate the derivatives of $\bar{\Gamma}$ by taking the appropriate derivatives of (3.90)

$$\bar{\Gamma}_x = \bar{A}(I - \bar{\Gamma}) + (I - \bar{\Gamma})\bar{M} + \bar{E}_2\bar{P}E_1(M + A)E_2P\bar{E}_1, \tag{3.135}$$

$$i\bar{\Gamma}_y = \bar{A}(I - \bar{\Gamma}) - (I - \bar{\Gamma})\bar{M} + \bar{E}_2\bar{P}E_1(M - A)E_2P\bar{E}_1, \tag{3.136}$$

$$\bar{\Gamma}_t = -2i\bar{A}^2\bar{E}_2\bar{P}E_1E_2P\bar{E}_1 - 2i\bar{E}_2\bar{P}E_1E_2P\bar{E}_1\bar{M}^2 + 2i\bar{E}_1\bar{P}E_1(M^2 + A^2)E_2P\bar{E}_1. \tag{3.137}$$

We can use these equations to find the terms in the identity (3.166)

$$\bar{\Gamma}_x + i\bar{\Gamma}_y = 2\bar{A}(I - \bar{\Gamma}) + 2\bar{E}_2\bar{P}E_1ME_2P\bar{E}_1, \tag{3.138}$$

$$\bar{\Gamma}_x - i\bar{\Gamma}_y = 2(I - \bar{\Gamma})\bar{M} + 2\bar{E}_2\bar{P}E_1AE_2P\bar{E}_1. \tag{3.139}$$

We will also need the second derivatives of $\bar{\Gamma}$ with respect to x and y to write u_{xx} and u_{yy} from (3.1) in terms of $\bar{\Gamma}$

$$\bar{\Gamma}_{xx} = -\bar{A}^2(I - \bar{\Gamma}) - 2\bar{A}(I - \bar{\Gamma})\bar{M} - 2\bar{A}\bar{E}_2\bar{P}E_1(M + A)E_2P\bar{E}_1, \tag{3.140}$$

$$\bar{\Gamma}_{yy} = A^2(I - \bar{\Gamma}) - 2A(I - \bar{\Gamma})\bar{M} - 2\bar{A}\bar{E}_2\bar{P}E_1(M + A)E_2P\bar{E}_1 \tag{3.141}$$

$$- (I - \bar{\Gamma})\bar{M}^2 - 2\bar{E}_2\bar{P}E_1(M + A)E_2P\bar{E}_1\bar{M} - E_2\bar{P}E_1(M - A)E_2P\bar{E}_1\bar{M} \tag{3.142}$$

We substitute (3.137) into (3.132) to write iu_t in terms of $\bar{\Gamma}$

$$iu_t = -2\bar{C}\bar{E}_1\Gamma^{-1}[2\bar{E}_2\bar{P}E_1(M^2 + A^2)E_2P\bar{E}_1$$
$$- 2\bar{A}^2 - 2\bar{M}^2]\bar{\Gamma}^{-1}\bar{E}_2\bar{B}. \tag{3.143}$$
$$\tag{3.144}$$

Similarly we substitute (3.138), (3.139), (3.89), and (3.140) into (3.134) to write $u_{xx} - u_{yy}$ in terms of $\bar{\Gamma}$

$$u_{xx} - u_{yy} = 2\bar{\Gamma}\bar{M}^2 + 2\bar{A}^2\bar{\Gamma} + 2\bar{\Gamma}\bar{M}\bar{\Gamma}^{-1}[2(I - \bar{\Gamma})\bar{M} + 2\bar{E}_2\bar{P}E_1AE_2P\bar{E}_1] + 2\bar{A}^2(I - \bar{\Gamma})$$
$$+ 2(I - \bar{\Gamma})\bar{M}^2 + 4\bar{A}\bar{E}_2\bar{P}E_1ME_2P\bar{E}_1 + 4\bar{E}_2\bar{P}E_1AE_2P\bar{E}_1\bar{M}$$
$$+ 2\bar{E}_2\bar{P}E_1(M^2 + A^2)E_2P\bar{E}_1 + 4[(\bar{A}(I - \bar{\Gamma}) + \bar{E}_2\bar{P}ME_2P\bar{E}_1]\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}$$
$$+ 4[\bar{A}(I - \bar{\Gamma}) + E_2\bar{P}E_1ME_2PE_1]\bar{\Gamma}^{-1}[\bar{A}(I - \bar{\Gamma}) + \bar{E}_2\bar{P}E_1ME_2P\bar{E}_1]$$
$$+ 4[(I - \bar{\Gamma})\bar{M} + \bar{E}_2\bar{P}E_1AE_2P\bar{E}_1]\bar{\Gamma}^{-1}[(I - \bar{\Gamma})\bar{M} + \bar{E}_2\bar{P}E_1AE_2P\bar{E}_1]. \tag{3.145}$$

Adding (3.143) and (3.145) gives

$$iu_t + u_{xx} - u_{yy} = 4\bar{A}\bar{\Gamma}^{-1}\bar{A} - 4\bar{A}^2 + 4\bar{M}\bar{\Gamma}^{-1}\bar{M} + 4\bar{E}_2\bar{P}E_1M\bar{\Gamma}^{-1}E_2P\bar{E}_1\bar{A}$$
$$+ 4\bar{E}_2\bar{P}E_1M\bar{\Gamma}^{-1}ME_2P\bar{E}_1 + 4\bar{E}_2\bar{P}E_1A\bar{\Gamma}^{-1}E_2P\bar{E}_1\bar{M}$$
$$+ 4\bar{E}_2\bar{P}E_1AE_2P\bar{E}_1 \tag{3.146}$$

We now focus on the nonlinear terms of (3.1) and find $-2uvu+u\phi$ using (3.125), (3.126) and (3.127). We begin by considering the term $-2uvu$ by using (3.125) and (3.126) to obtain

$$\begin{aligned} -2uvu &= 16\bar{C}\bar{E}_1\bar{\Gamma}^{-1}\bar{E}_2\bar{B}CE_1\Gamma^{-1}E_2B\bar{C}\bar{E}_1\bar{\Gamma}^{-1}\bar{E}_2\bar{B}, \\ &= 2\bar{C}\bar{E}_1\bar{\Gamma}^{-1}[8\bar{E}_2\bar{B}CE_1\Gamma^{-1}E_2B\bar{C}\bar{E}_1]\bar{\Gamma}^{-1}\bar{E}_2\bar{B}. \end{aligned}$$

We use (3.85) to write this as

$$-2uvu = 2\bar{C}\bar{E}_1\bar{\Gamma}^{-1}[8\bar{E}_2(\bar{A}\bar{P}+\bar{P}M)E_1\Gamma^{-1}E_2(AP+P\bar{M})\bar{E}_1]\bar{\Gamma}^{-1}\bar{E}_2\bar{B}. \tag{3.147}$$

To simplify the function $\phi(x,y,t)$ we evaluate the appropriate derivatives in (3.127) as

$$\begin{aligned} \phi &= 4iC[iE_1M\Gamma^{-1}S\bar{E}_2\bar{B} - E_1\Gamma^{-1}\Gamma_y\Gamma^{-1}S\bar{E}_2 + E_1\Gamma^{-1}S_y\bar{E}_2 + iE_1\Gamma^{-1}S\bar{A}\bar{E}_2]\bar{B} \\ &\quad + 4i\bar{C}[-i\bar{E}_1\bar{M}\bar{\Gamma}^{-1}\bar{S}E_2 - \bar{E}_1\bar{\Gamma}^{-1}\bar{\Gamma}_y\bar{\Gamma}^{-1}\bar{S}E_2 + \bar{E}_1\bar{\Gamma}^{-1}\bar{S}_yE_2 - i\bar{E}_1\bar{\Gamma}^{-1}\bar{S}AE_2]B. \end{aligned} \tag{3.148}$$

Using (3.118)-(3.121) we can write (3.148) as

$$\begin{aligned} \phi &= 4CE_1\Gamma^{-1}[\Gamma M\Gamma^{-1}S + [A - M - A\Gamma + \Gamma M + S(\bar{M}-\bar{A})\bar{S}]\Gamma^{-1}S \\ &\quad + AS + S\bar{M} - S\bar{A}]\bar{E}_2\bar{B} \\ &\quad + 4\bar{C}\bar{E}_1\bar{\Gamma}^{-1}[-\bar{\Gamma}\bar{M}\bar{\Gamma}^{-1}\bar{S} + [\bar{A} - \bar{M} - \bar{A}\bar{\Gamma} + \bar{\Gamma}\bar{M} + \bar{S}(M-A)S]\bar{\Gamma}^{-1}\bar{S} \\ &\quad + \bar{A}\bar{S} + \bar{S}M - \bar{S}A]E_2B. \end{aligned} \tag{3.149}$$

Simplifying (3.149) we can write $\phi(x,y,t)$ as

$$\begin{aligned} \phi &= 4CE_1\Gamma^{-1}[(A-M)\Gamma^{-1}S + S(\bar{M}-\bar{A})\bar{\Gamma}^{-1}]\bar{E}_2\bar{B} \\ &\quad + 4\bar{C}\bar{E}_1\bar{\Gamma}^{-1}[(\bar{A}\bar{M})\bar{\Gamma}^{-1}\bar{S} + \bar{S}(M-A)\Gamma^{-1}]E_2B. \end{aligned} \tag{3.150}$$

Using (3.125) and (3.150) we can write ϕu as

$$\begin{aligned} \phi u = -8\bar{C}\bar{E}_1\bar{\Gamma}^{-1}\big[&[(\bar{A}-\bar{M})\bar{\Gamma}^{-1}\bar{S} + \bar{S}(M-A)\Gamma^{-1}]E_2B\bar{C}\bar{E}_1 \\ &+\bar{E}_2\bar{B}CE_1\Gamma^{-1}[(A-M)S + S(\bar{M}-\bar{A})]\big]\bar{\Gamma}^{-1}\bar{E}_2\bar{B}. \end{aligned} \tag{3.151}$$

Expanding the terms of (3.151) and using $S = E_2 P\bar{E}_1$ and $\bar{S} = \bar{E}_2\bar{P}E_1$ gives

$$\begin{aligned}
\phi u = -8\bar{C}\bar{E}_1\bar{\Gamma}^{-1}[&\bar{A}\bar{S}\Gamma^{-1}AS + \bar{A}\bar{\Gamma}^{-1}AS + \bar{A}\bar{\Gamma}^{-1}\bar{M} - \bar{A}\bar{M} \\
&- \bar{M}\bar{S}\Gamma^{-1}AS - \bar{M}\bar{\Gamma}^{-1}\bar{M} + \bar{M}^2 \\
&+ \bar{S}M\Gamma^{-1}AS + \bar{S}M\Gamma^{-1}S\bar{M} \\
&- \bar{S}A\Gamma^{-1}AS - \bar{S}A\Gamma^{-1}S\bar{M} \\
&+ \bar{A}\bar{S}\Gamma^{-1}AS - \bar{A}\bar{S}\Gamma^{-1}MS + \bar{A}\bar{\Gamma}^{-1}(\bar{M} - \bar{A}) \\
&- \bar{A}(\bar{M} - \bar{A}) + \bar{S}M\Gamma^{-1}(A - M)S + \bar{S}M\Gamma^{-1}S(\bar{M} - \bar{A}))]\bar{\Gamma}^{-1}\bar{B}
\end{aligned} \tag{3.152}$$

Combining the terms from (3.146), (3.147), and (3.152) we see that

$$iu_t + u_{xx} - u_{yy} - 2uvu + \phi u = 0.$$

We now show that (3.125) and (3.126) satisfy

$$\phi_{xx} + \phi_{yy} - 2(uv + vu)_{yy} = 0. \tag{3.153}$$

We show that the functions in (3.125), (3.126), and (3.127) satisfy (3.1) by evaluating the trace of the solutions. The scalar functions (3.125), (3.126), and (3.127) are written in terms of the product of a row vector and column vector and are equivalent to their trace values. By using the properties of the matrix trace we can change the order of the matrix products that construct the scalar functions (3.125), (3.126), and (3.127). We can use the definition

$$\alpha := E_2 B\bar{C}\bar{E}_1, \tag{3.154}$$

and the property of the matrix trace

$$\mathrm{tr}(AB) = \mathrm{tr}(BA), \tag{3.155}$$

to simplify $\mathrm{tr}(\phi)$ as

$$\mathrm{tr}(\phi) = \mathrm{tr}\left(4i[\bar{\alpha}\Gamma^{-1}S - \alpha\bar{\Gamma}^{-1}\bar{S}]_y\right). \tag{3.156}$$

Using (3.111) we can write (3.156) as

$$\mathrm{tr}(\phi) = \mathrm{tr}\left(4i[\bar{\alpha}\Gamma^{-1}S - \alpha\bar{S}\Gamma^{-1}]_y\right) = \mathrm{tr}\left(4i[\Gamma^{-1}(S\bar{\alpha} - \alpha\bar{S})]_y\right). \tag{3.157}$$

Factoring Γ^{-1} from the left and applying (**??**) to (3.157) we are able to write $\phi(x, y, t)$ in terms of Γ, A, and M as

$$\mathrm{tr}(\phi) = \left(4i[\Gamma^{-1}\left(S(\bar{A} - \bar{M})\bar{S} + (I - \Gamma)M - A(I - \Gamma)\right)]_y\right). \tag{3.158}$$

To further simplify $\phi(x, y, t)$ we use the identities (3.112), (3.113) and the property of matrix trace (3.155) to obtain

$$\begin{aligned}
\mathrm{tr}(\phi) =& \mathrm{tr}\left(4i[\Gamma^{-1}S(\bar{A} - \bar{M})\bar{S} + \Gamma^{-1}M - M - \Gamma^{-1}A + \Gamma^{-1}A\Gamma]_y\right) \\
&= \mathrm{tr}\left(4i[\bar{S}\,\Gamma^{-1}S(\bar{A} - \bar{M}) + \Gamma^{-1}M - M - \Gamma^{-1}A + A]_y\right) \\
&= \mathrm{tr}\left(4i[(\bar{\Gamma}^{-1} - I)(\bar{A} - \bar{M}) + \Gamma^{-1}M - M - \Gamma^{-1}A + A]_y\right).
\end{aligned} \tag{3.159}$$

Expanding the terms in the final line of (3.159) we obtain

$$\mathrm{tr}(\phi) = \mathrm{tr}\left(4i[\bar{\Gamma}^{-1}(\bar{A} - \bar{M}) + \Gamma^{-1}(M - A)]_y\right). \tag{3.160}$$

Consider the third equation in (3.1), the term $-2(uv + vu)_{yy}$ can be written as $-4(vu)_{yy}$ because the functions v and u are scalar functions. The term $-4vu$can be rewritten using (3.125), (3.126), the definition of α in (3.154), and the property of matrix trace in (3.155) to obtain

$$\begin{aligned}
\mathrm{tr}(-4vu) &= \mathrm{tr}\left(-16CE_1\Gamma^{-1}E_2B\bar{C}\bar{E}_1\bar{\Gamma}^{-1}\bar{E}_2\bar{B}\right) \\
&= \mathrm{tr}\left(-16\bar{E}_2\bar{B}CE_1\Gamma^{-1}E_2B\bar{C}\bar{E}_1\bar{\Gamma}^{-1}\right) \\
&= \mathrm{tr}\left(-16\bar{\alpha}\Gamma^{-1}\alpha\bar{\Gamma}^{-1}\right) \\
&= \mathrm{tr}\left(-16\Gamma^{-1}\alpha\bar{\Gamma}^{-1}\bar{\alpha}\right).
\end{aligned}$$

Our goal is to show that the scalar functions $u(x,y,t)$, $v(x,y,t)$, and $\phi(x,y,t)$ in (3.125), (3.126), and (3.127) satisfy the third equation in (3.1). After the simplification of $\phi(x,y,t)$ and $-4vu$ we can write the third equation in (3.1) as

$$[\Gamma^{-1}(M-A)+\bar{\Gamma}^{-1}(\bar{A}-\bar{M})]_{xxy}+4i[\Gamma^{-1}(M-A)+\bar{\Gamma}^{-1}(\bar{A}-\bar{M})]_{yyy}-16[\Gamma^{-1}\alpha\bar{\Gamma}^{-1}\bar{\alpha}]_{yy}=0.$$

We continue to simplify the term $-16[\Gamma^{-1}\alpha\bar{\Gamma}\bar{\alpha}]$ by using the identities (3.111) - (3.114), and the definition of α in (3.154) to obtain

$$\begin{aligned}\mathrm{tr}(-16[\Gamma^{-1}\alpha\bar{\Gamma}\bar{\alpha}]) &= \mathrm{tr}\left(-16\Gamma^{-1}(AS+S\bar{M})\bar{\Gamma}(\bar{A}\bar{S}+\bar{S}M)\right),\\ &= \mathrm{tr}\big(-16[\Gamma^{-1}AS\bar{\Gamma}^{-1}\bar{A}\bar{S}+\Gamma^{-1}AS\bar{\Gamma}^{-1}\bar{S}M+\Gamma^{-1}S\bar{M}\bar{\Gamma}^{-1}\bar{A}\bar{S}\\ &\quad+\Gamma^{-1}S\bar{M}\bar{\Gamma}^{-1}\bar{S}M]\big).\end{aligned} \tag{3.161}$$

To further simplify $-16[\Gamma^{-1}\alpha\bar{\Gamma}\bar{\alpha}]$ we use the property of matrix trace to show that

$$\mathrm{tr}(\Gamma^{-1}S\bar{M}\bar{\Gamma}^{-1}\bar{A}\bar{S})=\bar{S}\Gamma^{-1}S\bar{M}\bar{\Gamma}^{-1}\bar{A}=(\bar{\Gamma}^{-1}-I)\bar{M}\bar{\Gamma}^{-1}\bar{A}. \tag{3.162}$$

We use (3.162) in the last line of (3.161) to write $\mathrm{tr}(-16[\Gamma^{-1}\alpha\bar{\Gamma}\bar{\alpha}])$ as

$$\begin{aligned}\mathrm{tr}(-16[\Gamma^{-1}\alpha\bar{\Gamma}\bar{\alpha}]) &= \mathrm{tr}(-16[\Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S}+\Gamma^{-1}A(\Gamma^{-1}-I)M+(\bar{\Gamma}^{-1}-I)\bar{M}\bar{\Gamma}^{-1}\bar{A}\\ &\quad+\Gamma^{-1}S\bar{M}\bar{S}\Gamma^{-1}M]).\end{aligned} \tag{3.163}$$

By expanding the terms in (3.163) we obtain

$$\begin{aligned}\mathrm{tr}(-16[\Gamma^{-1}\alpha\bar{\Gamma}\bar{\alpha}]) &= \mathrm{tr}(-16[-\bar{\Gamma}^{-1}\bar{A}\bar{M}-\Gamma^{-1}AM+\Gamma^{-1}A\Gamma^{-1}M\\ &\quad+\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1}\bar{M}+\Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S}+\Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{S}]).\end{aligned} \tag{3.164}$$

Substituting the simplified forms of ϕ and $-4uv$ from (3.160) and (3.164), respectively, yields

$$\begin{aligned}0 =&\mathrm{tr}\big(\left[\Gamma^{-1}(M-A)+\bar{\Gamma}^{-1}(\bar{A}-\bar{M})\right]_{xx}\\ &+\left[\Gamma^{-1}(M-A)+\bar{\Gamma}^{-1}(\bar{A}-\bar{M})\right]_{yy}\\ &+4i[-\Gamma^{-1}AM-\bar{\Gamma}^{-1}\bar{A}\bar{M}+\Gamma^{-1}A\Gamma^{-1}M+\bar{\Gamma}\bar{A}\bar{\Gamma}^{-1}\bar{M}+\Gamma^{-1}A\Gamma^{-1}M\\ &+\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1}\bar{M}+\Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S}+\Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{S}]_{yy})\end{aligned}$$

Performing the appropriate derivatives on (3.165) we can write the third equation in (3.1) as

$$
\begin{aligned}
0 =& \mathrm{tr}([(\Gamma^{-1})_{xx} + (\Gamma^{-1})_{yy}](M - A) \\
&+ [(\bar{\Gamma}^{-1})_{xx} + (\bar{\Gamma}^{-1})_{yy}](\bar{A} - \bar{M}) \\
&+ 4i[\Gamma^{-1}\Gamma_y\Gamma^{-1}AM + \bar{\Gamma}^{-1}\bar{\Gamma}_y\bar{\Gamma}^{-1}\bar{A}\bar{M} - \Gamma^{-1}\Gamma^{-1}\Gamma^{-1}A\Gamma^{-1}M - \Gamma^{-1}A\Gamma^{-1}\Gamma_y\Gamma^{-1}M \\
&- \bar{\Gamma}^{-1}\bar{\Gamma}_y\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1}\bar{M} - \bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1}\bar{\Gamma}_y\bar{\Gamma}^{-1}\bar{M} - \Gamma^{-1}\Gamma_y\Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S} \\
&- \Gamma^{-1}A\Gamma^{-1}\Gamma_y\Gamma^{-1}S\bar{A}\bar{S} - \Gamma^{-1}\Gamma_y\Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{S} - \Gamma^{-1}M\Gamma^{-1}\Gamma_y\Gamma^{-1}S\bar{M}\bar{S} \\
&+ \Gamma^{-1}A\Gamma^{-1}S_y\bar{A}\bar{S} + \Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S}_y + \Gamma^{-1}M\Gamma^{-1}S_y\bar{M}\bar{S} + \Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{S}_y]).
\end{aligned}
\tag{3.165}
$$

We now make use of the factorization

$$
2\bar{\Gamma}_x\bar{\Gamma}^{-1}\bar{\Gamma}_x - 2\bar{\Gamma}_y\bar{\Gamma}^{-1}\bar{\Gamma}_y = (\bar{\Gamma}_x + i\bar{\Gamma}_y)\bar{\Gamma}^{-1}(\bar{\Gamma}_x + i\bar{\Gamma}_y) + (\bar{\Gamma}_x - i\bar{\Gamma}_y)\bar{\Gamma}^{-1}(\bar{\Gamma}_x - i\bar{\Gamma}_y), \tag{3.166}
$$

and the identity (3.122) to simplify the first term of (3.165). We will also use the matrix trace property to rearrange the third term in (3.165) which gives the third equation in (3.1) as

$$
\begin{aligned}
0 =& \mathrm{tr}([\Gamma^{-1}(\Gamma_x + i\Gamma_y)\Gamma^{-1}(\Gamma_x - i\Gamma_y)\Gamma^{-1} + \Gamma^{-1}(\Gamma_x - i\Gamma_y)\Gamma^{-1}(\Gamma_x + i\Gamma_y)\Gamma^{-1} \\
&- \Gamma^{-1}(\Gamma_{xx} + \Gamma_{yy})\Gamma^{-1}](M - A) \\
&+ [\bar{\Gamma}^{-1}(\bar{\Gamma}_x + i\bar{\Gamma}_y)\bar{\Gamma}^{-1}(\bar{\Gamma}_x - i\bar{\Gamma}_y)\bar{\Gamma}^{-1} + \bar{\Gamma}^{-1}(\bar{\Gamma}_x - i\bar{\Gamma}_y)\bar{\Gamma}^{-1}(\bar{\Gamma}_x + i\bar{\Gamma}_y)\bar{\Gamma}^{-1} \\
&- \bar{\Gamma}^{-1}(\bar{\Gamma}_{xx} + \bar{\Gamma}_{yy})\bar{\Gamma}^{-1}](\bar{A} - \bar{M}) \\
&+ 4i[\Gamma^{-1}AM\Gamma^{-1}\Gamma_y + \bar{\Gamma}^{-1}\bar{A}\bar{M}\bar{\Gamma}^{-1}\bar{\Gamma}_y - \Gamma^{-1}A\Gamma^{-1}M\Gamma^{-1}\Gamma_y - \Gamma^{-1}M\Gamma^{-1}A\Gamma^{-1}\Gamma_y \\
&- \bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1}\bar{M}\bar{\Gamma}^{-1}\bar{\Gamma}_y - \bar{\Gamma}^{-1}\bar{M}\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1}\bar{\Gamma}_y - \Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S}\Gamma^{-1}\Gamma_y - \Gamma^{-1}S\bar{A}\bar{S}\Gamma^{-1}A\Gamma^{-1}\Gamma_y \\
&- \Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{S}\Gamma^{-1}\Gamma_y - \Gamma^{-1}S\bar{M}\bar{S}\Gamma^{-1}M\Gamma^{-1}\Gamma_y + \Gamma^{-1}A\Gamma^{-1}S_y\bar{A}\bar{S} + \Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S}_y \\
&+ \Gamma^{-1}M\Gamma^{-1}S_y\bar{M}\bar{S} + \Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{S}_y]).
\end{aligned}
\tag{3.167}
$$

Using the identities (3.118), (3.119), (3.123), and (3.124) we will be able to simplify the first two terms in the equation (3.167). To simplify the third term in the equation (3.167) we distribute i and factor Γ^{-1}, $\bar{\Gamma}^{-1}$, $\bar{S}_y$, and S_y from the right of the appropriate terms. We then use the identities (3.118) and (3.119) to write equation (3.167) as

$$\begin{aligned}
0 =&4\mathrm{tr}([\Gamma^{-1}(M - \Gamma M + S\bar{A}\bar{S})\Gamma^{-1}(A - A\Gamma + S\bar{M}\bar{S})\Gamma^{-1} \\
&+ \Gamma^{-1}(A - A\Gamma + S\bar{M}\bar{S})\Gamma^{-1}(M - \Gamma M + S\bar{A}\bar{S})\Gamma^{-1} \\
&+ \Gamma^{-1}AM - A\Gamma M + AS\bar{A}\bar{S} + S\bar{M}\bar{S}M + S\bar{M}\bar{A}\bar{S})\Gamma^{-1}](M - A) \\
&+ 4[\bar{\Gamma}^{-1}(\bar{M} - \bar{\Gamma}\bar{M} + \bar{S}AS)\bar{\Gamma}^{-1}(\bar{A} - \bar{A}\bar{\Gamma} + \bar{S}MS)\bar{\Gamma}^{-1} \\
&+ \bar{\Gamma}^{-1}(\bar{A} - \bar{A}\bar{\Gamma} + \bar{S}MS)\bar{\Gamma}^{-1}(\bar{M} - \bar{\Gamma}\bar{M} + \bar{S}AS)\bar{\Gamma}^{-1} \\
&+ \bar{\Gamma}^{-1}(\bar{A}\bar{M} - \bar{A}\bar{\Gamma}\bar{M} + \bar{A}\bar{S}AS + \bar{S}MS\bar{M} + \bar{S}MAS)\bar{\Gamma}^{-1}](\bar{A} - \bar{M})] \\
&+ [\Gamma^{-1}AM\Gamma^{-1} - \Gamma^{-1}A\Gamma^{-1}M\Gamma^{-1} - \Gamma^{-1}M\Gamma^{-1}A\Gamma^{-1} \\
&- \Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S}\Gamma^{-1} - \Gamma^{-1}S\bar{A}\bar{S}\Gamma^{-1}A\Gamma^{-1} - \Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{S}\Gamma^{-1} \\
&- \Gamma^{-1}S\bar{M}\bar{S}\Gamma^{-1}M\Gamma^{-1}][-A + M + A\Gamma - \Gamma M + S(\bar{A} - \bar{M})\bar{S}] \\
&+ [\bar{\Gamma}^{-1}\bar{A}\bar{M}\bar{\Gamma}^{-1} - \bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1}\bar{M}\bar{\Gamma}^{-1} - \bar{\Gamma}^{-1}\bar{M}\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1}] \\
&\times [\bar{A} - \bar{M} - \bar{A}\bar{\Gamma} + \bar{\Gamma}\bar{M} - \bar{S}(A - M)S] \\
&+ [\bar{A}\bar{S}\Gamma^{-1}A\Gamma^{-1} + \bar{M}\bar{S}\Gamma^{-1}M\Gamma^{-1}][AS + S\bar{M}] \\
&- [\Gamma^{-1}A\Gamma^{-1}S\bar{A} + \Gamma^{-1}S\bar{A} + \Gamma^{-1}M\Gamma^{-1}S\bar{M}][\bar{A}\bar{S} + \bar{S}M]).
\end{aligned} \tag{3.168}$$

We can now write (3.168) as

$$\begin{aligned}
0 =&\mathrm{tr}\left(q_1(M - A) + q_2(\bar{A} - \bar{M}) + q_3(A\Gamma - \Gamma M) + q_4(\bar{\Gamma}\bar{M} - \bar{A}\bar{\Gamma})\right. \\
&\left.+q_5 S(\bar{A} - \bar{M})\bar{S} + q_6\bar{S}(M - A)S + q_7(AS + S\bar{M}) + q_8(\bar{A}\bar{S} + \bar{S}M)\right)
\end{aligned} \tag{3.169}$$

where

$$
\begin{aligned}
\operatorname{tr}(q_1(M-A)) =& \operatorname{tr}\big([\Gamma^{-1}M\Gamma^{-1}A\Gamma^{-1} - \Gamma^{-1}M\Gamma^{-1}A + \Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{S}\Gamma^{-1} \\
& - M\Gamma^{-1}A\Gamma^{-1} + M\Gamma^{-1}A - M\Gamma^{-1}S\bar{M}\bar{S}\Gamma^{-1} \\
& + \Gamma^{-1}S\bar{A}\bar{S}\Gamma^{-1}A\Gamma^{-1} - \Gamma^{-1}S\bar{A}\bar{S}\Gamma^{-1}A \\
& + \Gamma^{-1}S\bar{A}\bar{\Gamma}^{-1}\bar{M}\bar{S}\Gamma^{-1} - \Gamma^{-1}S\bar{A}\bar{M}\bar{A}\Gamma^{-1} \\
& + \Gamma^{-1}A\Gamma^{-1}M\Gamma^{-1} - \Gamma^{-1}AM\Gamma^{-1} + \Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S}\Gamma^{-1} \\
& + \Gamma^{-1}S\bar{M}\bar{S}\Gamma^{-1}M\Gamma^{-1} - \Gamma^{-1}S\bar{M}\bar{S}M\Gamma^{-1} \\
& + \Gamma^{-1}S\bar{M}\bar{\Gamma}^{-1}\bar{A}\bar{S}\Gamma^{-1} - \Gamma^{-1}S\bar{M}\bar{A}\bar{S}\Gamma^{-1} \\
& + \Gamma^{-1}AM\Gamma^{-1} - \Gamma^{-1}A\Gamma M\Gamma^{-1} + \Gamma^{-1}AS\bar{A}\bar{S}\Gamma^{-1} \\
& + \Gamma^{-1}S\bar{M}\bar{S}M\Gamma^{-1} + \Gamma^{-1}S\bar{M}\bar{A}\bar{S}\Gamma^{-1} \\
& + \Gamma^{-1}AM\Gamma^{-1} - \Gamma^{-1}A\Gamma M\Gamma^{-1} - \Gamma^{-1}M\Gamma^{-1}A\Gamma^{-1} \\
& + \Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S}\Gamma^{-1} - \Gamma^{-1}S\bar{A}\bar{S}\Gamma^{-1}A\Gamma^{-1} \\
& -\Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{S}\Gamma^{-1} - \Gamma^{-1}S\bar{M}\bar{S}\Gamma^{-1}M\Gamma^{-1}](M-A)\big)\,, \qquad (3.170)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}(q_2(\bar{A}-\bar{M})) =&\mathrm{tr}\left([\bar{\Gamma}^{-1}\bar{M}\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1} - \bar{\Gamma}^{-1}\bar{M}\bar{\Gamma}^{-1}\bar{A}\right. \\
&+ \bar{\Gamma}^{-1}\bar{M}\bar{S}\Gamma^{-1}M\Gamma^{-1}S - \bar{M}\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1} + \bar{M}\bar{\Gamma}^{-1}\bar{A} \\
&- \bar{M}\bar{S}\Gamma^{-1}M\Gamma^{-1}S + \bar{S}\Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{\Gamma}^{-1} \\
&- \bar{S}\Gamma^{-1}A\Gamma^{-1}S\bar{A} + \bar{S}\Gamma^{-1}A\Gamma^{-1}M\Gamma^{-1}S \\
&- \bar{S}\Gamma^{-1}AM\Gamma^{-1}S + \bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1}\bar{M}\bar{\Gamma}^{-1} \\
&- \bar{\Gamma}^{-1}\bar{A}\bar{M}\bar{\Gamma}^{-1} + \bar{\Gamma}^{-1}\bar{A}\bar{S}\Gamma^{-1}A\Gamma^{-1}S \\
&- \bar{\Gamma}^{-1}\bar{A}\bar{M}\bar{\Gamma}^{-1} + \bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}\bar{M}\bar{\Gamma}^{-1} \\
&- \bar{\Gamma}^{-1}\bar{A}\bar{S}A\Gamma^{-1}S + \bar{S}\Gamma^{-1}M\Gamma S\bar{M}\bar{\Gamma}^{-1} \\
&- \bar{S}\Gamma^{-1}MS\bar{M}\bar{\Gamma}^{-1} + \bar{S}\Gamma^{-1}M\Gamma^{-1}A\Gamma^{-1}S \\
&- \bar{S}\Gamma^{-1}MA\Gamma^{-1}S + \bar{\Gamma}^{-1}\bar{A}\bar{M}\bar{\Gamma}^{-1} \\
&- \bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}\bar{M}\bar{\Gamma}^{-1} + \bar{\Gamma}^{-1}\bar{A}\bar{S}A\Gamma^{-1}S \\
&+ \bar{S}\Gamma^{-1}MS\bar{M}\bar{\Gamma}^{-1} + \bar{S}\Gamma^{-1}MA\Gamma^{-1}S + \bar{\Gamma}^{-1}\bar{A}\bar{M}\bar{\Gamma}^{-1} \\
&+ \bar{\Gamma}^{-1}\bar{A}\bar{M}\bar{\Gamma}^{-1} - \bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1}\bar{M}\bar{\Gamma}^{-1} \\
&\left.-\bar{\Gamma}^{-1}\bar{M}\bar{\Gamma}^{-1}\bar{A}\bar{\Gamma}^{-1}](\bar{A}-\bar{M})\right),
\end{aligned} \tag{3.171}
$$

$$
\begin{aligned}
\mathrm{tr}(q_3(A\Gamma - \Gamma M)) =&\mathrm{tr}\left(\Gamma^{-1}A^2M - \Gamma^{-1}M\Gamma^{-1}A^2 - \Gamma^{-1}A\Gamma^{-1}AM\right. \\
&- \Gamma^{-1}A^2\Gamma^{-1}S\bar{A}\bar{S} - \Gamma^{-1}A\Gamma^{-1}AS\bar{A}\bar{S} \\
&- \Gamma^{-1}AM\Gamma^{-1}S\bar{M}\bar{S} - \Gamma^{-1}M\Gamma^{-1}AS\bar{M}\bar{S} \\
&- \Gamma^{-1}AM^2 + \Gamma^{-1}A\Gamma^{-1}M^2 + \Gamma^{-1}M\Gamma^{-1}AM \\
&+ \Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S}M + \Gamma AM\Gamma^{-1}S\bar{A}\bar{S} \\
&\left.+\Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{S}M + \Gamma^{-1}M^2\Gamma^{-1}S\bar{M}\bar{S}\right),
\end{aligned} \tag{3.172}
$$

$$
\begin{aligned}
\mathrm{tr}(q_4(\bar\Gamma\bar M-\bar A\bar\Gamma)) =&\mathrm{tr}\left(\bar\Gamma^{-1}\bar A\bar M^2-\bar\Gamma^{-1}\bar A^2\bar M-\bar\Gamma^{-1}\bar A\bar\Gamma^{-1}\bar M^2\right.\\
&+\bar\Gamma^{-1}\bar M\bar\Gamma^{-1}\bar A^2-\bar\Gamma^{-1}\bar M\bar\Gamma^{-1}\bar A\bar M\\
&\left.+\bar\Gamma^{-1}\bar A\bar\Gamma^{-1}\bar A\bar M\right), \qquad (3.173)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}(q_5S(\bar A-\bar M)\bar S) =&\mathrm{tr}\left(\Gamma^{-1}AM\Gamma^{-1}S\bar A\bar S-\Gamma^{-1}AM\Gamma^{-1}S\bar M\bar S\right.\\
&-\Gamma^{-1}A\Gamma^{-1}M\Gamma^{-1}S\bar A\bar S+\Gamma^{-1}A\Gamma^{-1}M\Gamma^{-1}S\bar M\bar S\\
&-\Gamma^{-1}M\Gamma^{-1}A\Gamma^{-1}S\bar A\bar S+\Gamma^{-1}M\Gamma^{-1}A\Gamma^{-1}S\bar M\bar S\\
&-\Gamma^{-1}A\Gamma^{-1}S\bar A\bar\Gamma^{-1}\bar A\bar S+\Gamma^{-1}A\Gamma^{-1}S\bar A\bar\Gamma^{-1}\bar M\bar S\\
&+\Gamma^{-1}A\Gamma^{-1}S\bar A^2\bar S-\Gamma^{-1}A\Gamma^{-1}S\bar A\bar M\bar S\\
&-bar\Gamma^{-1}\bar A\bar S\Gamma^{-1}A\Gamma^{-1}S\bar A+\bar\Gamma^{-1}\bar A\bar S\Gamma^{-1}A\Gamma^{-1}S\bar M\\
&+\bar A\bar S\Gamma^{-1}A\Gamma^{-1}S\bar A-\bar A\bar S\Gamma^{-1}A\Gamma^{-1}S\bar M\\
&-\Gamma^{-1}M\Gamma^{-1}S\bar M\bar\Gamma^{-1}\bar A\bar S+\Gamma^{-1}M\Gamma^{-1}S\bar M\bar\Gamma^{-1}\bar M\bar S\\
&+\Gamma^{-1}M\Gamma^{-1}S\bar M\bar A\bar A-\Gamma^{-1}M\Gamma^{-1}S\bar M^2\bar S\\
&-\bar\Gamma^{-1}\bar M\bar S\Gamma^{-1}M\Gamma^{-1}S\bar A+\bar\Gamma^{-1}\bar M\bar S\Gamma^{-1}M\Gamma^{-1}S\bar M\\
&\left.+\bar M\bar S\Gamma^{-1}M\Gamma^{-1}S\bar A-\bar M\bar S\Gamma^{-1}M\Gamma^{-1}S\bar M\right), \qquad (3.174)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}(q_6\bar S(M-A)S) = &\left(\Gamma^{-1}M\Gamma^{-1}S\bar A\bar M\bar S-\Gamma^{-1}A\Gamma^{-1}S\bar A\bar M\bar S\right.\\
&-\Gamma^{-1}M\Gamma^{-1}S\bar A\bar\Gamma^{-1}\bar M\bar S+\Gamma^{-1}A\Gamma^{-1}S\bar S\bar\Gamma^{-1}\bar M\bar S\\
&\left.-\Gamma^{-1}M\Gamma^{-1}S\bar M\bar\Gamma^{-1}\bar A\bar S+\Gamma^{-1}A\Gamma^{-1}S\bar M\bar\Gamma^{-1}\bar S\bar S\right), \qquad (3.175)
\end{aligned}
$$

and

$$\begin{aligned}\operatorname{tr}(q_7(AS+S\bar{M})) =& \operatorname{tr}\left(\Gamma^{-1}A\Gamma^{-1}AS\bar{A}\bar{S}+\Gamma^{-1}A\bar{\Gamma}^{-1}S\bar{M}\bar{A}\bar{S}\right.\\ &\left.+\Gamma^{-1}M\Gamma^{-1}AS\bar{M}\bar{S}+\Gamma^{-1}M\Gamma^{-1}S\bar{M}^2\bar{S},\right)\end{aligned} \tag{3.176}$$

$$\begin{aligned}\operatorname{tr}(q_8(\bar{A}\bar{S}+\bar{S}M)) =& \operatorname{tr}\left(-\Gamma^{-1}A\Gamma^{-1}S\bar{A}^2\bar{S}-\Gamma^{-1}A\Gamma^{-1}S\bar{A}\bar{S}M\right.\\ &\left.-\Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{A}\bar{S}-\Gamma^{-1}M\Gamma^{-1}S\bar{M}\bar{S}M\right).\end{aligned} \tag{3.177}$$

The remaining simplification results in the terms from equation (3.169) canceling each other. Hence, it has been shown that the solution (3.93), (3.94), and (3.95) satisfies the third equation in (3.1), which completes the proof. □

REFERENCES

[1] M. J. Ablowitz and P. A. Clarkson. *Solitons, nonlinear evolution equations and inverse scattering*, volume 149 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1991.

[2] M. J. Ablowitz and H. Segur. *Solitons and the inverse scattering transform*, volume 4 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1981.

[3] T. Aktosun, F. Demontis, and C. van der Mee. Exact solutions to the focusing nonlinear Schrödinger equation. *Inverse Problems*, 23(5):2171–2195, 2007.

[4] T. Aktosun, F. Demontis, and C. van der Mee. Exact solutions to the sine-Gordon equation. *J. Math. Phys.*, 51(12):123521, 27, 2010.

[5] T. Aktosun and C. van der Mee. Explicit solutions to the Korteweg-de Vries equation on the half line. *Inverse Problems*, 22(6):2165–2174, 2006.

[6] Tuncay Aktosun. Inverse scattering transform, KdV, and solitons. 149:1–22, 2004.

[7] Earl A. Coddington and Norman Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[8] A. Davey and K. Stewartson. On three-dimensional packets of surface waves. *Proc. Roy. Soc. London Ser. A*, 338:101–110, 1974.

[9] Harry Dym. *Linear algebra in action*. American Mathematical Society, Providence, RI, 2007.

[10] Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura. Korteweg-deVries equation and generalization. VI. Methods for exact solution. *Comm. Pure Appl. Math.*, 27:97–133, 1974.

[11] B. B. Kadomtsev and V. I. Petvishvili. On the stability of solitary waves in weakly dispersive media. *Sov. Phys. Dokl.*, 15:539–541, 1970.

[12] Alex Kasman and Michael Gekhtman. Solitons and almost-intertwining matrices. *J. Math. Phys.*, 42(8):3540–3551, 2001.

[13] Peter Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[14] Cornelia Schiebold. Explicit solution formulas for the matrix-KP. *Glasg. Math. J.*, 51(A):147–155, 2009.

[15] S.Novikov, S. V. Manakov, L.P. Pitaevskii, and V.E. Zakharov. *Theory of Solitons: The Inverse Scattering method*. Consultants Bureau, New York, NY, 1984.

[16] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Ž. Èksper. Teoret. Fiz.*, 61(1):118–134, 1971.

BIOGRAPHICAL STATEMENT

Alicia Machuca was born on May 30, 1981 in El Paso, Texas. She graduated from Tom C. Clark High School and has attended the University of Texas at San Antonio, University of North Texas and University of Texas at Arlington.