Representation Theory of Totally Reflexive Modules
Over Non-Gorenstein Rings

by

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Abstract

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Abstract:

In the late 1960's Auslander and Bridger published *Stable Module Theory,* in which the idea of totally reflexive modules first appeared. These modules have been studied by many. However, a bulk of the information known about them is when they are over a Gorenstein ring, since in that case they are exactly the maximal Cohen-Macaulay modules. Much is already known about maximal Cohen-Macaulay modules, that is, totally reflexive modules over a Gorenstein ring. Therefore, we investigate the existence and abundance of totally reflexive modules over non-Gorenstein rings.

It is known that if there exist one non-trivial totally reflexive module over a non-Gorenstein ring, then there exists infinitely many non-trivial non-isomorphic indecomposable ones. We employ several different techniques to study the representation theory of the category of totally reflexive modules over a non-Gorenstein ring, including the classic approach of Auslander-Reiten theory. We present several of these results and conclude by giving a complete description of the totally reflexive modules over a specific family of non-Gorenstein rings.

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Chapter 1

Introduction

In order to discuss totally reflexive modules, one has to first discuss projective modules and projective resolutions. The notion of a free resolution, which is a special type of projective resolution, was given by Hilbert in 1890 [21]. However, projective modules were not introduced until 1956 by Cartan and Eilenberg in [12]. Their book was the first comprehensive source on the subject of homological algebra. Then in the late 1960s, Auslander and Bridger published *Stable Module Theory* [2] in which the idea of totally reflexive modules first appeared, though it was under the name *modules of Gorenstein dimension zero.* It was not until 2002 when Avramov and Martsinkovsky [9] first referred to them as totally reflexive modules, to better emphasize their homological properties. They were originally introduced as a generalization of a projective module, which subsequently gives a generalized notion of projection dimension. They have now become an essential object in relative (or Gorenstein) homological algebra and are used to extend the classical Ext and Tor derived functors into negative degrees, to so-called Tate (co)homology.

These modules have been studied by many, see [9] or [15], for example. However, the bulk of information known about totally reflexive modules is when they are defined over a Gorenstein ring, since in this case they are exactly the maximal Cohen-Macaulay modules. The representation theory of maximal Cohen-Macaulay modules has been well documented, see [26] or [37]. However, when there exist non-trivial totally reflexive modules over a non-Gorenstein ring, the category is often of *wild representation type.* In particular, if there exist one non-trivial totally reflexive module, then there exists infinitely many non-trivial non-isomorphic indecomposable ones [16]. In fact in [15] and [22], infinite families of non-isomorphic indecomposable totally reflexive module are constructed, both of which arise from special ring elements called exact zero divisors. These exact zero divisors play a crucial role in the construction of the filtrations in chapter 3.

For the case of totally reflexive modules over a local non-Gorenstein ring with the cube of the maximal ideal being zero there are some results, [17], [22], [38]. Over these types of rings the growth of the Betti numbers of complete resolutions is known, as well as some ring invariants. Totally reflexive modules over local non-Gorenstein ring with the cube of the maximal ideal being zero is the main focus of this thesis. Although there are results that hold for a broader class of rings. In fact, in the original definition of totally reflexive modules the only restraint on the ring was that it was two-sided Noetherian. Therefore, we make no global statement, beyond local Noetherian commutative, for our rings of study.

Since the objective of this thesis is to investigate the existence and representation theory of totally reflexive modules over non-Gorenstein rings, many different techniques are utilized. Each chapter represents a different approach, with a majority of Chapter 5 devoted to an explicit example. We begin in Chapter 2 with a general overview of definitions and concepts in homological algebra relevant to this thesis. For more background information see [3], [11], [31], and [37].

Chapter 3 is concerned with when totally reflexive modules have totally reflexive submodules. With this in mind, we define a type of filtration, and give a precise description of totally reflexive modules for which such a filtration is optimized. In addition, this allows us to describe certain aspects of a complete resolution. We conclude this chapter with an investigation of the rank of Ext^1 over a specific ring, achieving an upper bound and in a special case a lower bound. In this approach, as well as in a majority of this thesis, we study modules via a presentation matrix.

Motivated by the success of Auslander-Reiten (AR) theory in the study of modules over Artin algebras and maximal Cohen Macaulay modules over Gorenstein rings, we continue our investigation with this classical approach. Chapter 4 explores three main areas of research in AR-theory. Sections 4.2 and 4.3 are concerned with two different definitions of the AR-sequence, an important object of AR-theory, and whether the category of totally reflexive modules over non-Gorenstein rings admits them. Section 4.4 is a collection of facts discovered while investigating this question.

Chapter 5 is focused around one specific ring, whom we refer to as *Mindy*. In many ways it is "the smallest" non-Gorenstein local ring that admits non-trivial totally reflexive modules. Through techniques of Avramov [7], [8] and Eisenbud [19], we are able to completely describe the isomorphism classes of totally reflexive modules over Mindy. In addition we show that this description can be expanded to an entire class of rings.

Finally, Chapter 6 examines two methods for constructing totally reflexive modules when one is already known, the mapping cone and the tensor product of complexes. The mapping cone method discussed in Section 6.1, is of a very similar nature to the existence of the modules in Chapter 3. We conclude with an example of both construction methods.

Chapter 2

Preliminary Concepts

For this thesis all rings, R, will be commutative Noetherian. If we write $(R, \mathfrak{m}, k)$, then we mean that R is also local, with the unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. All modules considered here will be finitely generated. In addition, since we are assuming the ring R is commutative, all R-modules are automatically two sided. In the few instances when we do need to consider a module over a possible noncommutative ring, such as $\operatorname{End}(R)$, we will assume them to be left modules.

2.1 Homological Algebra

The definition of a totally reflexive module has a homological component and hence so does the study of them. We begin with an overview of the aspects of homological algebra used throughout this thesis.

2.1.1 Complexes

Definition 2.1.1.1. [31, 5.5.1] A *complex* $\mathbb{G}$ is a sequence of modules and homomorphisms, called *differentials*

$$\mathbb{G}: \qquad \cdots \longrightarrow G_{n+1} \xrightarrow{d_{n+1}} G_n \xrightarrow{d_n} G_{n-1} \longrightarrow \cdots,$$

such that the composition $d_n d_{n+1}$ is zero for all $n \in \mathbb{Z}$.

If we wish to specify the differentials without writing out the complex, we write $(\mathbb{G}, d)$. The condition $d_n d_{n+1} = 0$ is equivalent to $\operatorname{im}(d_{n+1}) \subseteq \ker(d_n)$, and therefore one can consider the quotient module $\ker(d_n)/\operatorname{im}(d_{n+1})$.

Definition 2.1.1.2. Let n be an integer. For a complex $(\mathbb{G}, d)$ its *nth homology* is

$$\mathrm{H}_n(\mathbb{G}) = \ker(d_n)/\operatorname{im}(d_{n+1}).$$

If a complex has no homology, that is, $\mathrm{H}_n(\mathbb{G}) = 0$ for all n then it is called an *exact* or *acyclic* complex. This is equivalent to saying that $\ker(d_n) = \operatorname{im}(d_{n+1})$ for all n. Therefore, the homology of a complex measures how far the complex is from being exact. If the complex

$$\alpha : \qquad 0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

is exact, we say it is a *short exact sequence*. Note that this implies that i is an injective map and p is a projection. A short exact sequence α is called *split* if there exist a map $j : C \to B$ such that $pj = 1_C$. If this holds, then $B \cong A \oplus C$.

A complex $(\mathbb{G}, d)$ over a local ring $(R, \mathfrak{m})$, is called *minimal* if we have $\operatorname{im} d_i^{\mathbb{G}} \subseteq \mathfrak{m}G_{i-1}$, for all i. Whenever possible, we will always choose complexes to be minimal.

2.1.2 Resolutions

Definition 2.1.2.1. A *projective resolution* of an R-module M is an exact sequence

$$\mathbb{P} : \qquad \cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

with P_i projective, for all $i \geq 0$.

If we omit the module M in the resolution, then it is call a *deleted projective resolution* and no information is lost by doing this since $M \cong \operatorname{coker}(P_1 \to P_0)$. If this complex is finite of length n, say

$$\mathbb{P} : \qquad 0 \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

then we say that M has *finite projective dimension* denoted $\operatorname{pd}_R(M)$. Define $\operatorname{pd}_R(M) = n$, if n is the smallest number such that M has a projective resolution of length n. If no such n exists, then $\mathrm{pd}_R(M) = \infty$. When over local rings, the length of all minimal projective resolutions of a module are equal.

Recall that an R-module F is called *free* if it is isomorphic to a direct sum of copies of the ring. This can actually be an infinite sum, however we will only consider finite ones and thus we can define $\mathrm{rank}_R(F)$ to be the number of copies of the ring comprising F. In addition, every module M is the quotient of a free module [31, 2.35]. That is, for every module M, there is a free module F and a projection $\epsilon : F \to M$.

A *free resolution* is defined in the same way as a projective resolution, except projective modules are replaced by free modules. Every module has a free resolution, and since free modules are also projective, every module has a projective resolution. A free resolution can simply be built, as we now describe. Since every module is the quotient of a free module, we start with the natural projection, ϵ, onto M and form the short exact sequence

$$0 \to K_0 \xrightarrow{i_1} F_0 \xrightarrow{\epsilon} M \to 0,$$

where i_0 the natural injection, and K_0 is the kernel of ϵ. We repeat this process with K_0 instead of M now and take d_1 to be the composition $i_0 \circ \epsilon_1$.

$$\begin{tikzcd}
 & & F_1 \arrow[rr, "d_1"] \arrow[dr, "\epsilon_1"] & & F_0 \arrow[r, "\epsilon"] & M \arrow[r] & 0 \\
 & K_1 \arrow[ur] & & K_0 \arrow[ur, "i_0"] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 & & .
\end{tikzcd}$$

This process is continued possibly indefinitely, or until one has $K_n = 0$. When over Noetherian local rings, projective modules and free modules are one in the same.

Example 2.1.2.2. Let $(R, \mathfrak{m}) = k[x, y]/(x^2, y^2)$, then

$$\cdots \longrightarrow R^4 \xrightarrow{\begin{bmatrix} x & 0 & y & 0 \\ 0 & -x & 0 & y \\ 0 & y & x & 0 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x & 0 & -y \\ 0 & y & x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} R \longrightarrow R/\mathfrak{m} \longrightarrow 0$$

is the beginning of a free resolution of the residue field $R/\mathfrak{m} = k$ of R.

Let M be a module over $(R, \mathfrak{m})$ and let

$$\mathbb{F}_M : \qquad \cdots \to F_2 \to F_1 \xrightarrow{\phi} F_0 \to M \to 0$$

be a minimal free resolution of M. Then for some $b_i \in \mathbb{N} \cup \{0\}$, we have $F_i \cong R^{b_i}$ for all i, and we can write $\mathbb{F}_M$ as

$$\cdots \to R^{b_2} \to R^{b_1} \to R^{b_0} \to M \to 0.$$

We call b_i the *ith Betti number* of M. Since $\mathbb{F}_M$ is minimal, b_0 is called the *minimal number of generators* of M, often denoted $\mu(M)$. We define a *presentation matrix* of M to be a matrix representing ϕ, say $\mathbf{\Phi}$ with respect to the standard basis. Then, we have that the $\operatorname{coker}(\mathbf{\Phi}) \cong M$. Just as with differentials of complexes, this is not unique; however from [27, 4.3], we know that $\mathbf{M_1}$ and $\mathbf{M_2}$ are both presentation matrices of an R-module M, if and only if there exist invertible matrices P, Q with entries in R such that $\mathbf{M_1} = P\mathbf{M_2}Q$.

Every module can also be imbedded as a submodule of an injective module [31, 3.38]. Therefore, we can define the dual notation of a projective resolution using injective modules instead of projective modules. One can then define the injective dimension, $\operatorname{id}_R(M)$ of a module.

2.1.3 Ext

Recall that the contravariant functor $\operatorname{Hom}_R(\_\_, N)$ is left exact. Let M and N be R-modules and suppose that $(\mathbb{F}, d)$ is a deleted free resolution of M. We obtain a complex,

$$\operatorname{Hom}_R(\mathbb{F}, N): \ 0 \longrightarrow \operatorname{Hom}_R(F_0, N) \xrightarrow{\operatorname{Hom}_R(d_1, N)} \operatorname{Hom}_R(F_1, N) \xrightarrow{\operatorname{Hom}_R(d_2, N)} \operatorname{Hom}_R(F_2, N) \longrightarrow \dots,$$

where for $f \in \operatorname{Hom}_R(F_i, N)$ we have $f \to fd_i$. There is no reason to believe that just because $\mathbb{F}$ was exact that $\operatorname{Hom}_R(\mathbb{F}, N)$ will be exact as well. The functor Ext gives us information about the homology this new complex.

Definition 2.1.3.1. Using the previous declarations,

$$\operatorname{Ext}^i_R(M, N) := \operatorname{H}^i(\operatorname{Hom}_R(\mathbb{F}, N)).$$

This definition is independent of the choice of the free resolution [31, 6.57]. We also have an isomorphism $\operatorname{Hom}_R(M, N) \cong \operatorname{Ext}^0_R(M, N)$.

Corollary 2.1.3.2 (Long Exact Sequence of Ext). *[31, 6.62] If*

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of R-modules, then for every R-module M, there is a long exact sequence of modules

$$0 \to \operatorname{Hom}_R(C, M) \to \operatorname{Hom}_R(B, M) \to \operatorname{Hom}_R(A, M) \to \operatorname{Ext}^1_R(C, M) \to \operatorname{Ext}^1_R(B, M) \to \dots$$

2.1.4 Yoneda Ext

There is another way to view the elements in $\operatorname{Ext}^i_R(M, N)$, as exact sequences starting with N and ending with M. This is called the Yoneda definition, or description, of Ext . In particular, when $i = 1$ the Yoneda definition of $\operatorname{Ext}^1_R(N, M)$ for two R-modules M, N gives a correspondence between the elements of $\operatorname{Ext}^1_R(N, M)$ and short exact sequences [18, Appendix A3],

$$0 \to M \to X \to N \to 0. \tag{2.1.4.0.1}$$

This will be the main reason for us to consider the Yoneda definition of Ext .

Let

$$\mathbb{F}_N: \qquad \cdots \longrightarrow R^{b_2} \xrightarrow{\partial_2} R^{b_1} \xrightarrow{\partial_1} R^{b_0} \to 0$$

be a (deleted) minimal free resolution of N. Suppose $[\alpha] \in \operatorname{Ext}^1_R(N, M)$, where $[\alpha]$ denotes $\alpha + \operatorname{im}\partial$. Then $\alpha \in \ker(\operatorname{Hom}_R(\partial_2, M))$, and $\alpha \in \operatorname{Hom}_R(R^{b_1}, M)$. Define the map $\iota : \operatorname{im}\partial_1 \to R^{b_0}$ to be the natural inclusion and define $\alpha'(u) := \alpha(v)$, where $\partial_1(v) = u$ for all $v \in R^{b_1}$. Using the pushout of α' and ι We can obtain the following commutative diagram,

$$\begin{tikzcd}
R^{b_1} \arrow[d] \arrow[dr, "\alpha"] & 0 \arrow[d] \\
\operatorname{im}\partial_1 \arrow[r, "\alpha'"] \arrow[d, "\iota"] & M \arrow[d, "h"] \\
R^{b_0} \arrow[r, "g"] & (M \oplus R^{b_0})/I \arrow[d] \\
& N \arrow[d] \\
& 0
\end{tikzcd}$$

where $I = (-\alpha'(r), \iota(r) | r \in \operatorname{im}\partial_1)$. The maps $g : R^{b_0} \to (M \oplus R^{b_0})/I$ and $h : M \to (M \oplus R^{b_0})/I$ are defines as inclusions composed with the natural projection. The pushout module, $(M \oplus R^{b_0})/I$, is the desired middle module X in 2.1.4.0.1.

2.1.5 Basic Theorems in Homological Algebra

We can take sequences of morphisms another step by defining a map between two complexes.

Definition 2.1.5.1. If $(\mathbb{G}, d)$ and $(\mathbb{G}', d')$ are complexes, then a *chain map* $f : \mathbb{G} \to \mathbb{G}'$ is a sequence of morphisms, $f_n : G_n \to G'_n$ for all n, making the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] & G_{n+1} \arrow[r, "d'_{n+1}"] \arrow[d, "f_{n+1}"] & G_n \arrow[r, "d'_n"] \arrow[d, "f_n"] & G_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & G'_{n+1} \arrow[r, "d_{n+1}"] & G'_n \arrow[r, "d_n"] & G'_{n-1} \arrow[r] & \cdots .
\end{tikzcd}$$

That is, $f_{n-1}d_n = d'_n f_n$ for all n.

In certain instances, given a map between two modules in different complexes, one can expand the morphism to a chain map between the complexes.

Theorem 2.1.5.2 (Comparison Theorem). *[31, 6.16] Given a homomorphism $f : A \to B$ between R-modules, consider the diagram*

$$\begin{tikzcd}
\mathbb{P} : & \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, "\check{f}_2"] & P_1 \arrow[r] \arrow[d, dashed, "\check{f}_1"] & P_0 \arrow[r] \arrow[d, dashed, "\check{f}_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\mathbb{P}' : & \arrow[r] & P'_2 \arrow[r] & P'_1 \arrow[r] & P'_0 \arrow[r] & B \arrow[r] & 0,
\end{tikzcd}$$

where the rows are complexes. If each P_n in the top row is projective, and if the bottom row is exact, then there exists a chain map $\check{f} : \mathbb{P} \to \mathbb{P}'$ making the completed diagram commute.

Theorem 2.1.5.3 (Long Exact Sequence of Homology). *[31, 6.10] If*

$$0 \to \mathbb{G}' \to \mathbb{G} \to \mathbb{G}'' \to 0$$

is an exact sequence of complexes, then there exists an exact sequence of homology

$$\cdots \to \mathrm{H}_{n+1}(\mathbb{G}'') \to \mathrm{H}_n(\mathbb{G}') \to \mathrm{H}_n(\mathbb{G}) \to \mathrm{H}_n(\mathbb{G}'') \to \mathrm{H}_{n-1}(\mathbb{G}') \to \cdots .$$

This theorem is similar to the long exact sequence of Ext given in 2.1.3.2.

Lemma 2.1.5.4 (Snake Lemma). *[31, 6.12] Given a commutative diagram with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, "g"] & A'' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0, \end{tikzcd}$$

there exist an exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0.$$

If, for a short exact sequence, a free resolution of the outer modules is known, then we can obtain a commutative diagram between all three resolutions.

Lemma 2.1.5.5 (Horseshoe Lemma). *[31, 6.24] Given a diagram*

$$\begin{tikzcd} & \vdots \arrow[d] & & \vdots \arrow[d] & \\ & P_1' \arrow[d] & & P_1'' \arrow[d] & \\ & P_0' \arrow[d] & & P_0'' \arrow[d] & \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] & A'' \arrow[r] \arrow[d] & 0 \\ & 0 & & 0 & , \end{tikzcd}$$

where the columns are projective (free) resolutions and the row is exact, then there exists a projective (free) resolution of A and chain maps so that the three columns form an exact sequence of complexes.

We should note that even if the resolutions of A' and A'' are minimal there is no guarantee that the resolution of A constructed in the proof if the Horseshoe lemma will be minimal.

2.2 A Survey of Rings

The main focus of the research presented here takes place over non-Gorenstein rings. To get a sense of what types of rings these are, we will discuss the hierarchy of rings with connections to Gorenstein rings and some of their invariants.

2.2.1 Ring and Module Invariants

For a ring R, the supremum of the length of all strictly decreasing chains

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n,$$

of prime ideals of R is called the *Krull dimension* of R, denoted $\dim(R) = n$. For an R-module M, define $\dim(M) := \dim(R/\operatorname{Ann}(M))$. If not specified, dimension will always mean Krull dimension.

A module M is of *finite length* if there exists a finite chain of submodules

$$0 = M_0 \subset M_1 \subset \dots \subset M_{n-1} \subset M_n = M,$$

such that $M_i \neq M_{i+1}$ and M_{i+1}/M_i is simple for all i. If no such n exist, then M is of infinite length. Define $\operatorname{length}(M) = n$ if n is the length of the longest such chain. If $(R, \mathfrak{m}, k)$ is also an k-algebra, then we can talk about the vector space dimension of a module. It is denoted in several different ways, we will opt to use $\operatorname{rank}_k(M)$. When R is local, this is precisely the length of M. Another well used dimension is the *embedding dimension,* edim, of a ring $(R, \mathfrak{m})$ which is $\operatorname{edim}(R) = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$.

Definition 2.2.1.1. [3] For an R-module M, a sequence $\mathbf{x} = x_1, \dots, x_n$ of elements of R is called an *M-regular sequence* if

(i) x_i is a nonzero divisor of $M/(x_1, \dots, x_{i-1})$ for all $i = 1, \dots, n$, and

(ii) $M/(\mathbf{x})M \neq 0$.

If $(R, \mathfrak{m})$ is a local ring and $\mathbf{x} \subset \mathfrak{m}$, then (ii) is automatically fulfilled by Nakayama's lemma. An M-regular sequence $\mathbf{x} = x_1, \dots, x_n$ in some ideal I, is called *maximal* in I if $x_1, \dots, x_n, x_{n+1}$ is not an M-regular sequence for any $x_{n+1} \in I$.

Definition 2.2.1.2. Over a local ring, define the *depth of M* to be the length of a maximal M-regular sequence in $\mathfrak{m}$.

The depth of a module is always bounded above by its (Krull) dimension. There is a well-known formula by Auslander and Buchsbaum that states if a finitely generated R-module M has finite projection dimension then,

(2.2.1.2.1) $$\operatorname{depth}(R) = \operatorname{pd}_R(M) + \operatorname{depth}(M).$$

2.2.2 Ring Types

We start with the simplest rings, regular rings. One of the nice things about regular local rings is that they are also integral domains.

Definition 2.2.2.1. A local ring $(R, \mathfrak{m})$ is *regular* if

$$\dim(R) = \operatorname{edim}(R).$$

In general we have the inequality $\dim(R) \leq \operatorname{edim}(R)$, but having equality also implies that the maximal ideal is generated by a regular sequence of length equal to the embedding dimension. A regular ring has Krull dimension zero if and only if it is a field.

Definition 2.2.2.2. A ring R is called a *complete intersection* if there exists a regular ring Q and a regular sequence $\mathbf{x} = x_1, \dots, x_n$ in Q such that $R \cong Q/(\mathbf{x})$.

In fact, for a local complete intersection ring R and any surjective map $\phi : Q \to R$, where Q is a regular local ring, the kernel of ϕ is generated by a regular sequence in Q.

Definition 2.2.2.3. The *socle* of a local ring is the ideal which contains every element that annihilates the maximal ideal, that is,

$$\mathrm{Soc}(R) = (0 : \mathfrak{m}) \cong \mathrm{Hom}_R(k, R).$$

Definition 2.2.2.4. A local ring $(R, \mathfrak{m})$ is called a *Cohen-Macaulay ring* if

$$\mathrm{depth}(R) = \dim(R).$$

A nonzero R-module M is called a *Cohen-Macaulay module* if $\mathrm{depth}(M) = \dim(M)$. If $\mathrm{depth}(M) = \dim(M) = \dim(R)$ then the module is called a *maximal Cohen-Macaulay module,* or *MCM.*

The explicit definition of a Gorenstein ring is very involved and not very user friendly, thus the following is more often used.

Proposition 2.2.2.5. *[29, 18.1] The following are equivalent for an n-dimensional Noetherian local ring $(R, \mathfrak{m}, k)$.*

1. *R is Gorenstein.*
2. *If $n = 0$, then $\mathrm{rank}_k(\mathrm{Soc}(R)) = 1$.*
3. *R has finite injective dimension, which is equal to n.*
4. *$\mathrm{Ext}^i_R(k, R) = 0$ for $i \neq n$ and is isomorphic to k for $i = n$.*

The following is well-known.

Fact 2.2.2.6. For $(R, \mathfrak{m})$ Noetherian local, we have the implications:

$$R \text{ is regular} \Rightarrow R \text{ is a complete intersection} \Rightarrow R \text{ is Gorenstein} \Rightarrow R \text{ is a Cohen-Macaulay.}$$

The primary area of which this research lays is in Cohen-Macaulay rings that are not Gorenstein. There is another type of ring that appears in this research, called *Golod,* which overlays part of the above implications. The definition of Golod relies on certain homological invariants which we will not discuss. The only aspect of Golod rings we need is the fact that no nontrivial totally reflexive modules exist over them [9].

2.3 Totally Reflexive Modules

Totally reflexive modules, originally called modules of Gorenstein dimension zero, were defined over any ring that was two-sided Noetherian [2].

2.3.1 Two Definitions

Definition 2.3.1.1. A finitely generated R-module M is called *totally reflexive* if the following conditions are satisfied:

1. $\operatorname{Ext}^i_R(M,R) = 0$ for all $i > 0$
2. $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M,R),R) = 0$ for all $i > 0$
3. The biduality map $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M,R),R)$ is an isomorphism.

Note that the third condition is the definition for reflexive modules. It has been shown by Jorgensen and Şega in [24], that all three conditions are needed in the definition. They construct a ring R and a reflexive module M such that $\operatorname{Ext}^i_R(M,R) = 0$ for $i > 0$ but that $\operatorname{Ext}^i_R(\operatorname{Hom}_R(M,R),R) \neq 0$ for $i > 0$. The module $\operatorname{Hom}_R(M,R)$ is called the *algebraic dual* of M and is denoted by M^*. Thus condition 3 says $M \cong M^{**}$. All totally reflexive modules have a "doubly infinite" resolution of free modules, which we call a *complete resolution.* To see how this is true, consider an equivalent definition of a totally reflexive module.

Definition 2.3.1.2. An R-module M is said to be *totally reflexive* if there exists an infinite sequence of finitely generated free R-modules,

$$\mathbb{F}: \qquad \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$$

such that $M \cong \operatorname{coker}(F_1 \to F_0)$ and both $\mathbb{F}$ and $\operatorname{Hom}_R(\mathbb{F}, R)$ are exact. Such a complex is called *totally acyclic*. This totally acyclic complex is a complete resolution of M.

Fact 2.3.1.3. The two definitions for totally reflexive modules are equivalent.

Proof. Suppose M is a totally reflexive R-module and consider the free resolutions of M and its dual M^*.

$$\begin{aligned} \mathbb{F} : \qquad & \cdots \to F_1 \to F_0 \to M \to 0 \\ \mathbb{D} : \qquad & \cdots \to D_1 \to D_0 \to M^* \to 0 \end{aligned}$$

Apply $\operatorname{Hom}_R(\_\_, R)$ to $\mathbb{D}$.

$$\operatorname{Hom}_R(\mathbb{D}, R) : \qquad 0 \to \operatorname{Hom}_R(M^*, R) \to \operatorname{Hom}_R(D_0, R) \to \operatorname{Hom}_R(D_1, R) \to \cdots$$

This is the isomorphic to

$$\mathbb{D}^* : \qquad 0 \to M^{**} \to D_0^* \to D_1^* \to \cdots$$

From part (3) of definition 2.3.1.1, we have that $M \cong M^{**}$. This implies

$$\mathbb{D}^* : \qquad 0 \to M \to D_0^* \to D_1^* \to \cdots$$

which is acyclic by part (1) of definition 2.3.1.1. We can splice this complex together with the free resolution of M.

$$\mathbb{F}|\mathbb{D} : \quad \begin{tikzcd} \cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[rr] \arrow[dr] & & D_0^* \arrow[r] & D_1^* \arrow[r] & \cdots \\ & & & M \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

Since both $\mathbb{F}$ and $\mathbb{D}^*$ are exact so is $\mathbb{F}|\mathbb{D}^*$. We now have an infinite acyclic complex of free R-modules. To see that $\mathbb{F}|\mathbb{D}^*$ is in fact totally acyclic, we consider the dual of it

$$\operatorname{Hom}_R(\mathbb{F}|\mathbb{D}^*, R): \qquad \cdots \to D_1^{**} \to D_0^{**} \to F_0^* \to F_1^* \to \cdots,$$

which is isomorphic to

$$\cdots \to D_1 \to D_0 \to F_0^* \to F_1^* \to \cdots$$

This is actually $\mathbb{D}|\mathbb{F}^*$, which is exact. Therefore, definition 2.3.1.1 implies definition 2.3.1.2 of totally reflexive.

On the other hand, let

$$\mathbb{F}: \qquad \cdots \to F_1 \xrightarrow{\phi} F_0 \to F_{-1} \to \cdots$$

be a totally acyclic complex and let $M = \operatorname{coker} \phi$. Then

$$\cdots \to F_1 \to F_0 \to M \to 0$$

is acyclic, as well as its dual. Hence $\operatorname{Ext}_R^i(M, R) = 0$ for all $i > 0$. We can obtain an isomorphism of the complexes $\mathbb{F}$ and $\mathbb{F}^{**}$.

$$\begin{array}{llllll}
\mathbb{F}^{**}: & \cdots \to & F_1^{**} \to & F_0^{**} \to & F_{-1}^{**} \to & F_{-2}^{**} \to \cdots \\
 & & \downarrow\cong & \downarrow\cong & \downarrow\cong & \downarrow\cong \\
\mathbb{F}: & \cdots \to & F_1 \to & F_0 \to & F_{-1} \to & F_{-2} \to \cdots
\end{array}$$

Since $\mathbb{F}^{**}$ is exact, we have that $\operatorname{Ext}_R^i(M^*, R) = 0$, for all $i > 0$. Moreover, the isomorphisms above imply that M is reflexive, covering all three of the conditions in definition 2.3.1.1. Therefore, the two definitions of a totally reflexive module are equivalent.

$\square$

2.3.2 Properties of Totally Reflexive Modules

We begin with a collection of well-known facts about totally reflexive modules.

Remarks 2.3.2.1.

(a) Projective (free) modules are totally reflexive. We call a nonzero totally reflexive module *trivial* if it is projective (free).

(b) If M is totally reflexive, then so is M^* [13, 1.1.7].

(c) All nontrivial totally reflexive modules have infinite projective dimension.

(d) Any syzygy module in a totally acyclic complex is totally reflexive. This is most clearly seen using definition 2.3.1.2.

Totally reflexive modules have the so called "two out of three" property for short exact sequences.

Fact 2.3.2.2. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of R-modules. If A and C, or B and C are totally reflexive, then so is the third. If A and B are totally reflexive and $\operatorname{Ext}^1_R(C, R) = 0$, then C is also totally reflexive.

This fact follows directly from the long exact sequence of Ext, see 2.1.3.2. One of the simplest types nontrivial totally reflexive modules arise from a special type of ring element, first defined by Henriques and Şega [20].

Definition 2.3.2.3. For a commutative ring R, a non-unit $a \in R$ is said to be an *exact zero divisor* if there exists an $b \in R$ such that $(0 : a) = (b)$ and $(0 : b) = (a)$. If R is also local, then b is unique up to units, and we call a, and b an *exact pair of zero divisors.*

Remark 2.3.2.4. This definition of an exact zero divisor is equivalent to the existence of a free resolution of $R/(a)$ of the form

$$\mathbb{A}: \qquad \cdots \to R \xrightarrow{[b]} R \xrightarrow{[a]} R \xrightarrow{[b]} R \xrightarrow{[a]} R \to 0.$$

This complex is actually totally acyclic since $\mathbb{A} \cong \mathrm{Hom}(\mathbb{A}, R)$. Therefore, the modules $R/(a) \cong (b)$ and $R/(b) \cong (a)$ are totally reflexive.

Over Gorenstein rings, totally reflexive modules are exactly the maximal Cohen-Macaulay modules. Representation theory of maximal Cohen-Macaulay modules is a very well documented area of research. In fact, there are many books written on the subject, see [11], [26], [37], just to name a few. However, there is a much more limited amount of information on totally reflexive modules over non-Gorenstein rings.

2.3.3 Previously Known Results Over Non-Gorenstein Rings

Recall that the category of totally reflexive modules over a non-Gorenstein ring is of wild representation type if their exist one nontrivial totally reflexive module, and in [15] the authors construct infinite families of indecomposable totally reflexive modules of every admissible length. That is, for every $n \in \mathbb{N}$ there exists infinitely many nontrivial nonisomorphic indecomposable totally reflexive modules of length ne, where e is the embedding dimension of the ring, provided the residue field has characteristic zero.

Over a local ring with the cube of the maximal ideal equaling zero, there is more known about totally reflexive modules, see [17][38]. Properties about the structure of the ring are known as well. Note that the next two theorems hold for acyclic complexes and not just totally acyclic ones.

Theorem 2.3.3.1. *[17, theorem A [1]] Let* $(R, \mathfrak{m}, k)$ *be a local ring that is not Gorenstein and has* $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. *If* $\mathbb{A}$ *is a non-zero minimal acyclic complex of finitely generated free R-modules, then the ring has the following properties:*

(a) $\operatorname{Soc}(R) = \mathfrak{m}^2$,

(b) $\operatorname{edim}(R) = \operatorname{rank}_k(\operatorname{Soc}(R)) + 1$; *in particular* $\operatorname{length}(R) = 2\operatorname{edim}(R)$

In addition to giving us information about the rings, the authors of [17] discuss the complexes of acyclic modules.

Theorem 2.3.3.2. *[17, Theorem B] Let* $(R, \mathfrak{m}, k)$ *be a local ring that is not Gorenstein and has* $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. *Let* $e = \operatorname{edim}(R)$. *If* $\mathbb{A}$ *is a non-zero minimal acyclic complex of finitely generated free R-modules, then one of the following holds:*

(I) The residue field k is not a direct summand of $\operatorname{coker} \partial_i$ *for any* $i \in \mathbb{Z}$. *There is a positive integer a such that* $a = \operatorname{rank}_R A_i$, *for all* $i \in \mathbb{Z}$. *Moreover, we have* $\operatorname{length}_R(\operatorname{coker} \partial_i) = ae$ *for all* $i \in \mathbb{Z}$.

(II) There is an integer γ *such that k is a direct summand of* $\operatorname{coker} \partial_{\gamma+2}$ *and not of* $\operatorname{coker} \partial_{i+2}$ *for any* $i < \gamma$, *and a positive integer a such that*

$$a = \operatorname{rank}_R A_i \text{ for all integers } i \leq \gamma \text{ and}$$

$$\operatorname{rank}_R A_{i+1} > \operatorname{rank}_R A_i \text{ for all integers } i \geq \gamma.$$

Moreover, $\operatorname{length}_R(\operatorname{coker} \partial_{i+2}) = ae$, *for all integers* $i \geq \gamma$.

In (II) when $i \geq \gamma$, the ranks have exponential growth, [25, theorem B]. See [24, 1.4] for an example of such a complex.

[1] There are two more properties listed in [17]; however, they are beyond the scope of this thesis.

2.3.4 Gorenstein dimension

Gorenstein dimension, or G-dimension, was introduced by Auslander and Bridger in [2]; it is an analog to projective dimension. By this we mean, G-dimension plays the same role to Gorenstein rings as what finite projective dimension does for regular rings. Recall the Auslander-Buchsbaum formula 2.2.1.2.1,

$$\mathrm{pd}_R(M) = \mathrm{depth}(R) - \mathrm{depth}(M),$$

where R is local and M has finite projective dimension. If M is of finite G-dimension, then we have

$$\text{G-dim}_R(M) = \mathrm{depth}(R) - \mathrm{depth}(M).$$

Thus for an R-module M, if $\mathrm{pd}_R(M) = n < \infty$, then $\text{G-dim}_R(M) = n$. Hence the two dimensions are only possibly not equal when M has infinite projective dimension. The definition given by Auslander and Bridger for modules of Gorenstein dimension zero is proven in their paper [2, 3.8] to be equivalent the definition given in 2.3.1.1, for totally reflexive modules. Therefore, it makes sense that totally reflexive modules have also been known as *Gorenstein projective modules.* The study of Gorenstein homological algebra is of interest on its own, see [13] for example.

Chapter 3

Filtrations

Given a module M in a specific class of modules, it is of interest to know when M contains a proper submodule N belonging to the same class of modules. Such a submodule would correspond to the existence of an element in $\mathrm{Ext}^1_R(M/N, N)$. In this chapter give a theorem which explicitly states when sequences of such totally reflexive modules can be obtained over certain rings.

3.1 A Saturated TR-filtration

Definition 3.1.0.1. For a totally reflexive R-module T, an *TR-filtration* of T is a chain of submodules

$$0 = T_0 \subset T_1 \subset ... \subset T_{n-1} \subset T_n = T,$$

in which each T_i is totally reflexive and T_i/T_{i-1} contains no proper nonzero totally reflexive submodules, for all $i = 1, \dots, n$. If T_i/T_{i-1} is of minimal length, then we call it a *saturated TR-filtration*

Lemma 3.1.0.2. *Let* $(R, \mathfrak{m}, k)$ *be a commutative local non-Gorenstein ring with* $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$ *and* M *an* R*-module. If a presentation matrix of* M *has a column whose entries are contained in* $\mathfrak{m}^2$*, then the first syzygy of* M *contains a copy of* k *as a direct summand.*

Proof. Let $\mathfrak{m} = (x_1, \dots, x_e)$, where e is the embedding dimension of R, and suppose M has the presentation matrix $\mathbf{M} = [c_1, \dots, c_s]$, where $c_i \in R^{b_0}$. Also, without loss of generality assume that $c_1 \subset \mathfrak{m}^2 R^{b_0}$. Let

$$\mathbb{F}: \qquad \cdots \to R^{b_2} \xrightarrow{\mathbf{N}} R^{b_1} \xrightarrow{\mathbf{M}} R^{b_0} \to 0$$

be a deleted free resolution of M. Since $\mathbf{MN} = 0$, for $1 \leq i \leq e$ we have that the elements $\begin{pmatrix} x_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, are part of a minimal generating set of the syzygies of M. without loss of generality assume that $\mathbf{N}$ has the form

$$\begin{bmatrix} x_1 & \dots & x_e & 0 & \dots & 0 \\ 0 & \dots & 0 & * & \dots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & * & \dots & * \end{bmatrix}.$$

Note that the entries in the first row, after the eth column, are all zeroes. This is true since if there was a nonzero entries there, then it would be a linear combination of the first e columns. Therefore, $\operatorname{coker} \mathbf{N} \cong k \oplus X$, for some R-module X.

$\square$

Corollary 3.1.0.3. *Let $(R, \mathfrak{m}, k)$ be a non-Gorenstein ring with $\mathfrak{m}^3 = 0$. If the nth syzygy of an R-module M contains a copy of k as a direct sum, then M is not totally reflexive.*

Proof. Assume that for some R-module X, we have $\Omega_n^R(M) \cong k \oplus X$. Now suppose that M is totally reflexive, and therefore, $\Omega_n^R(M)$ is totally reflexive as well since it is a syzygy of M. Then, for all $i > 0$, we have $\operatorname{Ext}_R^i(\Omega_n^R(M), R) = 0$ and thus $\operatorname{Ext}_R^i(k \oplus X, R) = 0$ for all $i > 0$, which implies that $\operatorname{Ext}_R^i(k, R) = 0$, for all $i > 0$. This holds if and only if R is Gorenstein. However, we assumed R not to be Gorenstein, and therefore M cannot be totally reflexive. $\square$

3.2 Filtrations and Upper Triangular Presentation Matrices

Lemma 3.2.0.4. *Let* $(R, \mathfrak{m})$ *be a non-Gorenstein ring with* $\mathfrak{m}^3 = 0$, $\mathfrak{m}^2 \neq 0$ *and embedding dimension* e. *If a totally reflexive* R*-module* T *has a presentation matrix that contains a row with only one nonzero entry, then there exists a totally reflexive submodule* $U \subset T$ *such that length* $(U) =$ *length* $(T) - e$.

Proof. Let

$$\mathbb{F}: \qquad \cdots \to F_2 \xrightarrow{\mathbf{W}} F_1 \xrightarrow{\mathbf{T}} F_0 \to F_{-1} \to \cdots$$

be a complete free resolution of T, where $\mathbf{W} = (\omega_{ij})$ is a presentation matrix of $\Omega_1^R(T)$ and let

$$\mathbf{T} = \begin{bmatrix} t_{11} & \cdots & & t_{1n} \\ \vdots & & & \vdots \\ t_{n-11} & \cdots & & t_{n-1n} \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$$

be a presentation matrix T. From [15, Theorem 5.3], we know that t_{nn} is an exact zero divisor of R. Define $(w) := (0 : t_{nn})$. Since $\mathbf{TW} = 0$, we have that $t_{nn}\omega_{nj} = 0$ for all $j = 1, \dots, n$. Thus, the ideal $(\omega_{n1}, \dots, \omega_{nn}) \subset (0 : t_{nn})$. This, with 3.1.0.2, implies that for some i we have $\omega_{ni} = w$, up to units. Therefore, every entry in the nth row of $\mathbf{W}$ is a multiple of w. We can apply column operations to $\mathbf{W}$ so that

$$\mathbf{W}' = \begin{bmatrix} w_{11} & \cdots & & w_{1n} \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & w \end{bmatrix}$$

is another presentation matrix of $\Omega_1^R(T)$.

Now consider the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & 0 \arrow[d] & \\
\mathbb{G}' : & R^{n-1} \arrow[rr, "\begin{bmatrix} w_{11} & \cdots & w_{1n-1} \\ \vdots & \ddots & \vdots \\ w_{n-11} & \cdots & w_{n-1n-1} \end{bmatrix}"] \arrow[d, "q"'] & & R^{n-1} \arrow[rr, "\begin{bmatrix} t_{11} & \cdots & t_{1n-1} \\ \vdots & \ddots & \vdots \\ t_{n-11} & \cdots & t_{n-1n-1} \end{bmatrix}"] \arrow[d, "q"'] & & R^{n-1} \arrow[r] \arrow[d, "q"'] & 0 \\
\mathbb{G} : & R^n \arrow[rr, "\mathbf{W}'"] \arrow[d, "p"'] & & R^n \arrow[rr, "\mathbf{T}"] \arrow[d, "p"'] & & R^n \arrow[r] \arrow[d, "p"'] & 0 \\
\mathbb{G}'' : & R \arrow[rr, "{[w]}"] \arrow[d] & & R \arrow[rr, "{[t_{nn}]}"] \arrow[d] & & R \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & & 0 &
\end{tikzcd}$$

where $q := \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \dots & 0 \end{bmatrix}$ and $p := \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}$. Thus we have an exact sequence of complexes

$$0 \to \mathbb{G}' \to \mathbb{G} \to \mathbb{G}'' \to 0$$

which yields the following long exact sequence.

$$\cdots \to \mathrm{H}_1(\mathbb{G}'') \to \mathrm{H}_0(\mathbb{G}') \to \mathrm{H}_0(\mathbb{G}) \to \mathrm{H}_0(\mathbb{G}'') \to 0.$$

Note that $\mathrm{H}_1(\mathbb{G}'') = 0$ since $R \xrightarrow{[w]} R \xrightarrow{[t_{nn}]} R$ is exact. Let

$$\mathbf{U} := \begin{bmatrix} t_{11} & \cdots & t_{1n-1} \\ \vdots & & \vdots \\ t_{n-11} & \cdots & t_{n-1n-1} \end{bmatrix},$$

so that $U = \operatorname{coker} \mathbf{U} \subset T$. Therefore, $\mathrm{H}_0(\mathbb{G}'') = R/(t_{nn}) \cong T/U$ and we have the short exact sequence

$$0 \to U \to T \to T/U \to 0.$$

To see that U is in fact totally reflexive, note that $T/U \cong R/(t_{nn})$ is totally reflexive since (t_{nn}) is an exact zero divisor. This, along with the fact that T is totally reflexive, implies that U is as well [13, 4.3.5].

□

Theorem 3.2.0.5. *Let* $(R, \mathfrak{m})$ *be a non-Gorenstein ring with* $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$ *that contains exact zero divisors, and suppose that* T *is a totally reflexive* R*-module. There exists a saturated TR-filtration of* T *if and only if* T *has an upper triangular presentation matrix.*

Proof. Let T be totally reflexive R-module with minimal number of generators $\mu(T) = n$ and suppose there exists a saturated TR-filtering of T. To show that T has an upper triangular presentation matrix we will use induction on $\mu(T)$. If $\mu(T) = 1$, then any presentation matrix of T would be of size 1×1 and thus trivially upper triangular.

Assuming true for $\mu(T) = n$, consider the case $\mu(T) = n + 1$, by the previous lemma there exists a totally reflexive R-module $M \subset T$ such that the sequence

$$0 \to M \to T \to T/M \to 0$$

is exact and T/M is cyclic. Define $N := T/M$ and let $\mathbb{F}$ and $\mathbb{G}$ be free resolutions of M and N respectively. Since R is local with $\mathfrak{m}^3 = 0$, we can assume that $F_i = R^n$

and $G_i = R$, for all $i \geq 0$, [17]. By applying the Horseshoe Lemma, we get the following diagram:

(3.2.0.5.1)
$$\begin{tikzcd}
0 \arrow[r] & R^n \arrow[r] \arrow[d, "d_1^M"] & R^n \oplus R \arrow[r] \arrow[d, "d_1^T"] & R \arrow[r] \arrow[d, "d_1^N"] \arrow[dl, "\sigma_1"] & 0 \\
0 \arrow[r] & R^n \arrow[r] \arrow[d, "\epsilon_M"] & R^n \oplus R \arrow[r] \arrow[d, "\epsilon_T"] & R \arrow[r] \arrow[d, "\epsilon_N"] \arrow[dl, "\sigma_0"] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $d_1^T(f,g) \mapsto d_1^M(f) + \sigma_1(g)$. Since $d_1^M(f) \in R^n$ and $\sigma_1(g) \in R^n \oplus R$, when we consider them as columns in a presentation matrix of T we have the matrix

(3.2.0.5.2)
$$\begin{bmatrix} d_1^M & f' \\ 0 & g' \end{bmatrix},$$

where $\sigma_1(g) = f' + g'$ for some $f' \in R^n$ and $g' \in R$, we see that it is upper triangular. This matrix is a presentation matrix of T. By induction, the matrix representing $d_1^M(f)$ can be taken to be upper triangular and therefore 3.2.0.5.1 is upper triangular.

Now suppose T has an upper triangular presentation matrix, say

$$\begin{bmatrix} t_{11} & \cdots & t_{1n} \\ & \ddots & \vdots \\ 0 & & t_{nn} \end{bmatrix}.$$

Again we will use induction on the minimal number of generators. If $n = 2$, then by the previous lemma 3.2.0.4 there exists a totally reflexive R-module $T_1 \cong R/(t_{11})$ which is a submodule of T. Thus, T has a saturated TR-filtration

$$0 = T_0 \subset T_1 \subset T_2 = T$$

with $T_2/T_1 \cong R/(t_{22})$. Now assume that an n-generated totally reflexive module with an upper triangular presentation matrix has a saturated TR-filtration

$$0 = T_0 \subset T_1 \subset \dots . \subset T_{n-1} \subset T_n$$

with T_i/T_{i-1} being one generated and totally reflexive for $i = 1, \dots, n$. Consider an $(n+1)$-generated totally reflexive module T, which has with an upper triangular presentation matrix. By Lemma 3.2.0.4, there exists a totally reflexive module T_n such that T_n is a submodule of T and has presentation matrix of the form

$$\begin{bmatrix} t_{11} & \cdots & t_{1n} \\ & \ddots & \vdots \\ 0 & & t_{nn} \end{bmatrix}$$

with $T/T_n \cong R/t_{n+1\,n+1}$. This, combined with the induction hypothesis, shows that T has a saturated TR-filtration.

□

Definition 3.2.0.6. An *upper triangular complex* is a complex in which every differential can simultaneously be represented by a matrix (a_{ij}) such that $a_{ij} = 0$ when $i > j$.

Corollary 3.2.0.7. *For R as in Theorem 3.2.0.5, if T is a totally reflexive R-module that has an upper triangular minimal presentation matrix, then it has an upper triangular minimal complete resolution.*

Proof. This will be done by induction on $\mu(T)$, the number of minimal generators of T. Let T be a totally reflexive R-module that has an upper triangular minimal presentation matrix with $\mu(T) = n$, and thus the matrix is of size $n \times n$. Let $\mathbb{F}$ be a free resolution of T. By [17, theorem B], we have that $\operatorname{rank}_R F_1 = n$ for all $i \in \mathbb{Z}$. If $\mu(T) = 1$, then the free resolution of T is trivially upper triangular.

Assume for $\mu(T) = n$ that $\mathbb{F}$ is an upper triangular minimal free resolution and consider the case when T has a presentation matrix of the form

$$\begin{bmatrix} t_{11} & \cdots & t_{1\,n+1} \\ & \ddots & \vdots \\ 0 & & t_{n+1\,n+1} \end{bmatrix}.$$

By theorem 3.2.0.5, there exists an saturated TR-filtration and thus the short exact sequence

$$0 \to T_n \to T \to R/(t_{n+1\,n+1}) \to 0.$$

From the Horseshoe Lemma, a diagram similar to 3.2.0.5.1 can be obtain and extended to include the first syzygies. By a similar argument to the proof of theorem 3.2.0.5, we see that the first syzygy of T has an upper triangular presentation matrix. Finally, any syzygy in a free resolution of totally reflexive module is also totally reflexive [13]. Therefore, we can apply this corollary to the nth syzygy to see that the $n+1$ the syzygy has an upper triangular presentation matrix.

□

Lemma 3.2.0.8. *Let* $(R, \mathfrak{m})$ *be a local ring with* $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. *If* $R/(a)$ *is a totally reflexive module, then* a *is an exact zero divisor.*

Proof. Let $R/(a)$ be a totally reflexive module, by 2.3.3.2 and 3.1.0.3 it has a free resolution on the form

$$\mathbb{F}: \qquad \cdots \to R \xrightarrow{b_2} R \xrightarrow{b_1} R \xrightarrow{a} R \to 0.$$

This implies that $(0 : a) = (b_1)$. Similarly, we have

$$\mathrm{Hom}_R(\mathbb{F}, R) : 0 \to \mathrm{Hom}_R(R/(a), R) \longrightarrow R \xrightarrow{b_1^*} R \xrightarrow{b_2^*} R \to \cdots,$$

which is isomorphic to

$$\mathrm{Hom}_R(\mathbb{F}, R) : 0 \to R/(a) \longrightarrow R \xrightarrow{b_1} R \xrightarrow{b_2} R \to \cdots,$$

and this implies that $(0 : b_1) = (a)$. Therefore, (a, b_1) are an exact pair of zero divisors.

□

Corollary 3.2.0.9. *For R as in Theorem 3.2.0.5, and for an R-module M that has an upper triangular presentation matrix, the non-zero entries on the main diagonal of that presentation matrix are exact zero divisors if and only if M is totally reflexive.*

Proof. Both directions of this proof rely on the existent of a saturated TR-filtration. First, suppose M is totally reflexive and has an upper triangular presentation matrix. Hence, we have a saturated TR-filtration

$$0 = M_0 \subset M_1 \subset \dots \subset M_{n-1} \subset M_n = M,$$

as well as short exact sequences

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$$

for all $i = 2, \dots, n$. If $i = 2$, then for some $u, v, \alpha \in R$ we have

$$0 \to R/(u) \to \operatorname{coker} \begin{bmatrix} u & \alpha \\ 0 & v \end{bmatrix} \to R/(v) \to 0.$$

From Lemma 3.2.0.8 both u and v are exact zero divisors. Assume this holds for $n - 1$, and consider

$$0 \to M_{n-1} \to M_n \to M_n/M_{n-1} \to 0.$$

By the induction hypothesis, M_{n-1} has a upper triangular presentation matrix with exact zero divisors on the main diagonal. A presentation matrix of M_n is obtain from presentation matrices of M_{n-1} and M_n/M_{n-1}. Moreover, if $\mathbf{M} = (a_{ij})$ is a presentation matrix of M_n, then $M_n/M_{n-1} \cong R/(a_{nn})$. Since $R/(a_{nn})$ is totally reflexive, a_{nn} is an exact zero divisor. Therefore, for all i, a_{ii} are exact zero divisors.

Now suppose M has an upper triangular presentation matrix where the non-zero entries on the main diagonal are exact zero divisors. By the inductive part of the proof of theorem 3.2.0.5, M is totally reflexive. $\square$

3.3 Filtrations and Yoneda Ext

Recall the Yoneda definition of Ext^1 from 2.1.4. Let

$$\cdots \longrightarrow R^{b_2} \xrightarrow{\partial_2} R^{b_1} \xrightarrow{\partial_1} R^{b_0} \longrightarrow N \longrightarrow 0$$

be a free resolution of N. We have the following diagram,

$$\begin{tikzcd}
R^{b_1} \arrow[d] \arrow[dr, "\alpha"] & 0 \arrow[d] \\
\operatorname{im} \partial_1 \arrow[r, "\alpha'"] \arrow[d, "\iota"] & M \arrow[d, "h"] \\
R^{b_0} \arrow[r, "g"] & (M \oplus R^{b_0})/I \arrow[d] \\
 & N \arrow[d] \\
 & 0
\end{tikzcd}$$

where $I = ((-\alpha'(r), \iota(r)) | r \in \operatorname{im} \partial_1)$ is a submodule of $M \oplus R^{b_0}$.

Let $S = k[X, Y, Z]/(X^2, Y^2, Z^2, YZ)$ and define $x = X + (X^2, Y^2, Z^2, YZ)$. Consider $\mathrm{Ext}^1_S(N, T_1)$ by the Yoneda definition for $T_1 = S/(x + by + cz)$ and $N = S/(x + dy + fz)$ where $b, c, d, f \in k$. There exists a short exact sequence of the form

$$0 \to T_1 \to T_2 \to N \to 0$$

which represents α. This will allow us to have a saturated TR-filtration of T_2,

$$0 \subset T_1 \subset T_2.$$

Let

$$\mathbb{F}_N : \qquad \cdots \longrightarrow S \xrightarrow{\partial_3} S \xrightarrow{\partial_2} S \xrightarrow{\partial_1} S \to 0$$

be a (deleted) minimal free resolution of N where ∂_i is represented by

$$\begin{cases} [x+dy+fz], & \text{if } i \text{ is odd} \\ [x-dy-fz], & \text{if } i \text{ is even.} \end{cases}$$

For $[\alpha] \in \mathrm{Ext}^1_S(N, T_1)$ where $\alpha \in \ker(\mathrm{Hom}_S(\partial_2, T_1))$, we have that $\alpha \in \mathrm{Hom}_S(S, T_1) \cong T_1$. We will identify elements of $\mathrm{Hom}_S(S, T_1)$ with T_1 under the natural isomorphism. Using the previous description we find that the pushout is $T_2 := (T_1 \oplus S)/I$. In order to find a presentation matrix for it we need to find a matrix $\mathbf{T_2}$ in which the following complex is exact

$$S^2 \xrightarrow{\mathbf{T_2}} S^2 \xrightarrow{p} T_2 = (T_1 \oplus S)/I \to 0.$$

Take e_1, e_2 to be the standard basis in S^2 and define $p(e_1) = (1,0) + I$ and $p(e_2) = (0,1) + I$. To find the image of $\mathbf{T_2}$ we just need to find the kernel of p which is shown below.

$$(x+by+cz)p(e_1) \equiv 0 \qquad \text{and} \qquad -\alpha p(e_1) + (x+dy+fz)p(e_2) \equiv 0$$

where $\alpha \in S/(x+by+cz)$. Therefore, a presentation matrix for T_2 is

$$\mathbf{T_2} = \begin{bmatrix} x+by+cz & -\tilde{\alpha} \\ 0 & x+dy+fz \end{bmatrix},$$

where $\tilde{\alpha}$ is a preimage of α in S.

3.3.1 The Rank of $\mathrm{Ext}^1_S(M, T)$

Since every element in Ext^1 can be viewed as a short exact sequence, to get a sense of how many nonequivalent sequences exist we study the rank of Ext^1.

Let $T = S/(x+by+cz)$ and $M = S/(x+dy+fz)$. If

$$\mathbb{F}_M : \cdots \to S \xrightarrow{\partial_2} S \xrightarrow{\partial_1} S \to 0$$

is a (deleted) free resolution of M, then the complex $\mathrm{Hom}_S(\mathbb{F}_M, T)$ has the form

$$0 \to T \xrightarrow{\mathrm{Hom}_S(\partial_1,T)} T \xrightarrow{\mathrm{Hom}_S(\partial_2,T)} T \xrightarrow{\mathrm{Hom}_S(\partial_3,T)} T \to \cdots$$

and $\mathrm{Ext}^1_S(M,T) = \mathrm{H}^1(\mathrm{Hom}_S(\mathbb{F}_M,T)) = \ker(\mathrm{Hom}_S(\partial_2,T))/\operatorname{im}(\mathrm{Hom}_S(\partial_1,T))$. We can compute the kernels and images of these maps:

$$\begin{aligned}
\ker(\mathrm{Hom}_S(\partial_2,T)) &= \begin{cases} (1,y,z) \cong T, & \text{if } b=-d \text{ and } c=-f \\ (y,z), & \text{otherwise} \end{cases} \\
\operatorname{im}(\mathrm{Hom}_S(\partial_1,T)) &= ((b-d)y+(c-f)z).
\end{aligned}$$

Therefore,

$$\mathrm{Ext}^1_S(M,T) = \begin{cases} T_1/((d-b)y+(f-c)z), & \text{if } b=-d \text{ and } c=-f \\ (y,z)/((d-b)y+(f-c)z), & \text{otherwise,} \end{cases}$$

and we have

$$\operatorname{rank}_k(\mathrm{Ext}^1_S(M,T)) = \begin{cases} 3, & \text{if } b=c=d=f=0 \\ 2, & \text{if } b=-d \text{ and } c=-f, \text{or if } b=d \text{ and } c=f \\ 1, & \text{otherwise.} \end{cases}$$

Suppose u and v are an exact pair of zero divisors, then for some $b,c \in k$, we have $u = x+by+cz$ and $v = x-by-cz$. This fact will be proven in Chapter **??**. From the above computations, for some $[\alpha] \in \mathrm{Ext}^1_S(S/u,S/v)$ and some S-module T, there exists the short exact sequence

$$\lambda : 0 \to S/v \to T \to S/u \to 0,$$

where T has a presentation matrix

$$\mathbf{T} = \begin{bmatrix} v & -\alpha \\ 0 & u \end{bmatrix}.$$

In this case, 1_S is a possibility for $-\alpha$ since $1 \in \ker(\operatorname{Hom}_S(\partial_2, S/v))$. However,

$$\operatorname{coker} \mathbf{T} = \operatorname{coker} \begin{bmatrix} v & 1_S \\ 0 & u \end{bmatrix} \cong S.$$

This implies that λ represents part of the complete resolution of R/v. Since this is always the case when we have an exact zero pair, we exclude it to focus on when the non-syzygy cases occurs. Hence, for S-modules $A = S/(x + by + cz)$ and $B = S/(x+dy+fz)$ we define $\Gamma_S(A, B)$ to represent the rank of the elements in $\operatorname{Ext}^1_S(A, B)$ which are not part of a complete resolution of T. Thus,

$$\Gamma_S(A, B) := \begin{cases} 2 & \text{if } b = c = d = f = 0 \text{ or } b = d \text{ and } c = f, \\ 1 & \text{otherwise.} \end{cases} \tag{3.3.1.0.1}$$

3.3.2 Bounds on $\operatorname{Ext}^1_R(T_n/T_{n-1}, T_{n-1})$

Let T_n be a totally reflexive module over $(R, \mathfrak{m}, k)$ that has an $n \times n$ upper triangular presentation matrix. Recalling that if there exist one nontrivial totally reflexive module and if k is of characteristic zero, then there are infinitely many non-isomorphic indecomposable totally reflexive modules of each admissible length [15][1.4], that is, a length that is a multiple of the embedding dimension of the ring. We investigate the rank of $\operatorname{Ext}^1_R(T_n/T_{n-1}, T_{n-1})$ to get a sense of the complexity (or simplicity) of modules of each possible length. Consider the short exact sequence

$$0 \to T_{n-1} \to T_n \to T_n/T_{n-1} \to 0.$$

From this short exact sequence we can obtain a long exact sequence of Ext

$$\cdots \to \mathrm{Ext}^1_R(T_n/T_{n-1}, T_{n-1}) \to \mathrm{Ext}^1_R(T_n/T_{n-1}, T_n) \to \mathrm{Ext}^1_R(T_n/T_{n-1}, T_n/T_{n-1}) \to \cdots.$$

We then have the inequality

$$\mathrm{rank}_k(\mathrm{Ext}^1_R(T_n/T_{n-1}, T_n)) \leq \mathrm{rank}_k(\mathrm{Ext}^1_R(T_n/T_{n-1}, T_n/T_{n-1})) + \mathrm{rank}_k(\mathrm{Ext}^1_R(T_n/T_{n-1}, T_{n-1})).$$

Now we will find an upper bound for T_i over the ring $S = k[x, y, z]/(x^2, y^2, z^2, yz)$. Since we are only interest in the nontrivial cases, we have that $1 \leq \Gamma(T_n/T_{n-1}, T_{n-1}) \leq 2$ by 3.3.1.0.1. If we consider the case $n = 2$, then we also have that $1 \leq \Gamma(T_2/T_1, T_1) \leq 2$. Therefore

$$\mathrm{rank}_k \Gamma_S(T_2/T_1, T_2) \leq 4.$$

Continuing this process we see that

$$\Gamma_S(T_n/T_{n-1}, T_n) \leq 2n \qquad \text{for } n = 2, 3 \ldots.$$

In fact, we know of cases when this rank is bounded below by zero. That is, there exist a nontrivial short exact sequence beginning in T_n and ending in T_n/T_{n-1}. To find such a case, per [15], we define the $b \times b$ upper triangular matrix

(3.3.2.0.2)
$$M_b(s, t, u, v) = \begin{bmatrix} s & u & 0 & 0 & 0 & \ldots \\ 0 & t & v & 0 & 0 & \ldots \\ 0 & 0 & s & u & 0 & \ldots \\ 0 & 0 & 0 & t & v & \\ 0 & 0 & 0 & 0 & s & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{bmatrix},$$

and consider the following theorem.

Theorem 3.3.2.1. *[15, 3.1] Let* $(R, \mathfrak{m})$ *be a local ring and assume that* s *and* t *are elements in* $\mathfrak{m} \backslash \mathfrak{m}^2$ *that form an exact pair of zero divisors. Assume further that* u *and* v *are elements in* $\mathfrak{m} \backslash \mathfrak{m}^2$ *with* $uv = 0$ *and that one of the following conditions holds:*

(a) The elements s, t, and u are linearly independent modulo $\mathfrak{m}^2$.

(b) One has $s \in (t) + \mathfrak{m}^2$ and $u, v \notin (t) + \mathfrak{m}^2$.

For every $b \in \mathbb{N}$, the R-module $M_b(s, t, u, v)$ is indecomposable, totally reflexive, and non-free. Moreover $\operatorname{coker} M_b(s, t, u, v)$ *has constant Betti numbers, equal to b.*

Over S the exact zero pairs are of the form $(x + \alpha y + \beta z, x - \alpha y - \beta z)$ for $\alpha, \beta \in k$. Now consider the choices for $u, v \in \mathfrak{m} \backslash \mathfrak{m}^2$ whose product must be zero. From part (a) of Theorem 3.3.2.1, $x + \alpha y + \beta z, x - \alpha y - \beta z$ and u must be linearly independent in $\mathfrak{m}/\mathfrak{m}^2$. For $\gamma, \eta, \lambda, \tau \in k$ let $u = \gamma y + \eta z$ and $v = \lambda y + \tau z$. Therefore, for $\gamma \neq \pm 2\alpha$ and $\eta \neq \pm 2\beta$ we have that $M_b(x+\alpha y+\beta z, x-\alpha y-\beta z, \gamma y+\eta z, \lambda y+\tau z)$ is a presentation matrix of a nontrivial indecomposable totally reflexive S-module.

Although Theorem 3.3.2.1 is useful in finding some of the indecomposable totally reflexive modules that have an upper triangular presentation matrix, it says nothing about whether the choices of s, t, u, and v will lead to non-isomorphic modules. In fact, notice that for the choices of u and v over S neither one of them contains an x term. This is because if either of them did, then one can always find an equivalent presentation matrix, and thus an isomorphic module, which does not have an x term.

3.4 Over Finite Fields

Let's consider the same ring S but over $\mathbb{Z}_2$, so $S = \mathbb{Z}_2[X, Y, Z]/(X^2, Y^2, Z^2, YZ)$. Now we can list the four one-generated totally reflexive modules:

$$(x), \quad (x + y), \quad (x + z), \quad (x + y + z),$$

all of which are non-isomorphic. From here we can construct all possible totally reflexive modules that have a 2×2 upper triangular presentation matrix, say

$$\begin{bmatrix} u & a \\ 0 & t \end{bmatrix}. \tag{3.4.0.1.1}$$

However, there may be many modules that have isomorphic presentation matrices. To discover which ones do, we start with the assumption that a does not contain an x term. The complete validation of this fact will be discussed in chapter 5. With this in mind we define

$$\mathcal{T} = \{x, x+y, x+z, x+y+z\} \qquad \text{and} \qquad \mathcal{N} = \{y, z, y+z\}.$$

We will also impose an ordering on the elements in $\mathcal{T}$ and $\mathcal{N}$ as the order listed above from smallest to largest. To order modules with a 2×2 presentation matrix we use a dictionary ordering on u, t, and then a. Through the use of the CAS Magma, we can find the isomorphism classes of all totally reflexive modules that have a 2×2 upper triangular presentation matrix. We chose the smallest presentation matrix to represent each class. There are 24 non-isomorphic indecomposable totally reflexive modules with an upper triangular presentation matrix.

In the below table, we list representatives for each isomorphism class. For a matrix in the form of 3.4.0.1.1, the options for a which represent non-isomorphic indecomposable totally reflexive modules are listed in the center of the table.

Table 3.1: Isomorphism Classes over $\mathbb{Z}_2[X, Y, Z]/(X^2, Y^2, Z^2, YZ)$

u $\downarrow$ t $\rightarrow$	x	x+y	x+z	x+y+z
x	y, z, y+z	z	y	y
x+y	z	y, z, y+z	y	y
x +z	y	y	y, z, y+z	z
x +y+z	y	y	z	y, z, y+z

Something to note about these modules when the coefficient field is $\mathbb{Z}_2$ is that it is not possible to interchange u and t, while keeping the same a, and have them be isomorphic to each other. However, this is not the case if instead we consider modules over $\mathbb{Z}_3[X, Y, Z]/(X^2, Y^2, Z^2, YZ)$. Over this ring, a module with presentation matrix $\begin{bmatrix} u & a \\ 0 & t \end{bmatrix}$ is isomorphic to one with $\begin{bmatrix} t & a \\ 0 & u \end{bmatrix}$ as a presentation matrix.

Theorem 3.2.0.5 is useful in determining how the totally reflexive submodules of a totally reflexive module are contained in one another in this special case. However, there are totally reflexive modules which do not have an upper triangular presentation matrix.

3.5 Example with No Upper Triangular Presentation Matrix

Example 3.5.0.2. Consider the S-module T with a presentation matrix of

$$\mathbf{T} := \begin{bmatrix} x & z \\ y & x \end{bmatrix},$$

where $S = k[X, Y, Z]/(X^2, Y^2, Z^2, YZ)$. This is a totally reflexive S-module. Take a free resolution of it

$$\mathbb{F}_T : \qquad \cdots \to R^2 \xrightarrow{\begin{bmatrix} x & z \\ y & x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x & -z \\ -y & x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x & z \\ y & x \end{bmatrix}} R^2 \to 0$$

and apply the functor $\operatorname{Hom}_R(\_, R)$.

$$\operatorname{Hom}_R(\mathbb{F}_T, R) : \quad 0 \to R^2 \xrightarrow{\begin{bmatrix} x & z \\ y & x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x & -z \\ -y & x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} -x & -z \\ y & x \end{bmatrix}} R^2 \to \cdots$$

Therefore, $\mathrm{Hom}_R(\mathbb{F}_T, R) \cong \mathbb{F}_T$ and hence the complex $\mathrm{Hom}_R(\mathbb{F}_T, R)$ is also exact. This implies that T is a totally reflexive R-module. In particular, this module is not isomorphic to a totally reflexive module that has an upper triangular presentation matrix. This can be done by showing that the matrix $\mathbf{T}$ is not equivalent to an upper triangular matrix.

Claim 3.5.0.3. For $\mathbf{T} = \begin{bmatrix} x & z \\ y & x \end{bmatrix}$, a totally reflexive S-module, the totally reflexive module $T = \mathrm{coker}(\mathbf{T})$ is not isomorphic to a totally reflexive module that has a upper triangular presentation matrix.

Proof. Suppose that for an upper triangular matrix $\mathbf{U}$ we have $\mathrm{coker}\,\mathbf{T} \cong \mathrm{coker}\,\mathbf{U}$, and hence $\mathbf{T}$ would be equivalent to $\mathbf{U}$. That is, there would exist two invertible matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ where $a, b, \dots, h \in k$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} T \begin{bmatrix} e & f \\ g & h \end{bmatrix} = U. \tag{3.5.0.3.1}$$

For $\alpha, \beta, \gamma, \zeta, \varphi, \psi$, and λ in k, let

$$\mathbf{U} := \begin{bmatrix} x + \alpha y + \beta z & \zeta x + \varphi y + \psi z \\ 0 & x + \gamma y + \lambda z \end{bmatrix}$$

Computing the products in (3.5.0.3.1) gives us

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & z \\ y & x \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} x+\alpha y+\beta z & \zeta x+\varphi y+\psi z \\ 0 & x+\gamma y+\lambda z \end{bmatrix}$$

$$\begin{bmatrix} (ae+bg)x+bey+agz & (af+bh)x+bfy+ahz \\ (ce+dg)x+edy+cgz & (cf-dh)x+dfy+chz \end{bmatrix} = \begin{bmatrix} x+\alpha y+\beta z & \zeta x+\varphi y+\psi z \\ 0 & x+\gamma y+\lambda z \end{bmatrix}.$$

These yield the following system of equations:

(3.5.0.3.2) $\qquad ae+bg=1, \qquad be=\alpha, \qquad ag=\beta,$

(3.5.0.3.3) $\qquad af+bh=\zeta, \qquad bf=\varphi, \qquad ah=\psi,$

(3.5.0.3.4) $\qquad ce=-df, \qquad ed=0, \qquad cg=0,$

(3.5.0.3.5) $\qquad cf+dh=1, \qquad df=\gamma, \qquad ch=\lambda.$

From (3.5.0.3.4) assume that $c=0$, and by (3.5.0.3.5) we have that $dh=1$ and $\lambda=0$. Since $d\neq 0$, we must have that $h=\frac{1}{d}$. By (3.5.0.3.4), $-dg=0$ which implies that $g=0$ and using (3.5.0.3.2), we have $a=\frac{a}{e}$ for $e\neq 0$. By (3.5.0.3.4), $ed=0$ which implies $d=0$, a contradiction to the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ being invertible. If instead we let $g=0$, then this leads to an argument similar to the above one. Therefore, $\mathbf{T}$ is not equivalent to $\mathbf{U}$ and thus T is not isomorphic to another totally reflexive module that has a upper triangular presentation matrix.

□

We will further investigate the category of totally reflexive module over the ring $S=k[X,Y,Z]/(X^2,Y^2,Z^2,YZ)$ in chapter 5.

Chapter 4

Auslander-Reiten Theory

Auslander-Reiten theory, (AR theory) which is named after its developers, was originally used to study representation theory of Artin algebras [3], but now is used as a general approach in representation theory. In particular, it has been used to study the category of maximal Cohen-Macaulay modules. Since maximal Cohen-Macaulay modules over Gorenstein rings are precisely the totally reflexive modules over those rings, we look to use similar techniques here. The two main components of AR-theory used in this chapter are the AR-translate and AR-sequences, also know as almost split sequences. The AR-translate is a composition of two functors, the transpose and the (vector space) dual.

4.1 AR-theory Preliminaries

We will begin by giving a short overview of the techniques of AR-theory used below. For a more complete view of the subject see [3]. For this chapter we will assume $(R, \mathfrak{m}, k)$ to be a finite dimensional local k-algebra.

4.1.1 The AR-translate

Definition 4.1.1.1. Suppose we have a free presentation of an R-module M

$$R^m \xrightarrow{\phi} R^n \to M \to 0,$$

and we apply the functor $\mathrm{Hom}_R(\_\_, R)$:

$$\mathrm{Hom}_R(R^n, R) \xrightarrow{\phi^*} \mathrm{Hom}_R(R^m, R) \to \operatorname{coker} \phi^* \to 0. \tag{4.1.1.1.1}$$

Define the *transpose* of M, denoted by $\mathrm{Tr}(M)$, to be coker ϕ^*. If ϕ is represented by a matrix Φ with respect to the standard basis, then ϕ^* can be represented by Φ^T with respect to the dual basis.

The transpose is well-defined up to projective summands. Therefore, the transpose of a module is zero if and only if the module is projective. Define $M \simeq N$ to mean there exist projective modules P and Q such that $M \oplus P \cong N \oplus Q$. Then we have that $\mathrm{Tr}(\mathrm{Tr}(M)) \simeq M$, for all M.

Definition 4.1.1.2. For an R-module M, define

$$\mathrm{D}(M) := \mathrm{Hom}_k(M, k).$$

This is often called the vector space dual of M. Note that $\mathrm{D}(M)$ is also an R-module. For some $f \in \mathrm{D}(M) = \mathrm{Hom}_k(M, k)$, $r \in R$, and $x \in M$ we have $rf(x) = f(rx)$. Since M is an R-module $rx \in M$, and the rest of the requirements for being a module easy follow.

Definition 4.1.1.3. The *AR-translate, $\tau(M)$,* of an R-module M is defined as the dual of the transpose,

$$\tau(M) := \mathrm{D}(\mathrm{Tr}(M)).$$

Similarly, we define $\tau^{-1}(M) = \mathrm{Tr}(\mathrm{D}(M))$

One can iterate this process. Let $\tau^1(M) = \tau(M)$, and for $n \in \mathbb{Z}$ define

$$\tau^n(M) = \begin{cases} M & \text{if } n = 0 \\ \tau(\tau^{n-1}(M)) & \text{if } n \geq 1 \\ \tau^{-1}(\tau^{n+1}(M)) & \text{if } n < 1 \end{cases}$$

The τ-*orbit* of an indecomposable module M is the collection of modules in $\{\tau^n(M)\}_{n \in \mathbb{Z}}$ [3], which is also well-defined up to projective summands.

Proposition 4.1.1.4. *If for an R-module M, we have $M \simeq \mathrm{Tr}(M)$, then $\tau^n(M) \simeq \mathrm{Tr}(\tau^{-n}(M))$, for all $n \in \mathbb{Z}$.*

Proof. If $M \simeq \mathrm{Tr}(M)$, then the proposition is trivially true for $n = 0$. Also, for $n = 1$ we have

$$\tau(M) = \mathrm{D}(\mathrm{Tr}(M)) \simeq \mathrm{D}(M) \simeq \mathrm{Tr}(\mathrm{Tr}(\mathrm{D}(M))) = \mathrm{Tr}(\tau^{-1}(M)).$$

Now assume $\tau^n(M) \simeq \mathrm{Tr}(\tau^{-n}(M))$ for $n > 0$, and consider $\tau^{n+1}(M)$

$$\begin{aligned} \tau^{n+1}(M) = \mathrm{D}(\mathrm{Tr}(\tau^n(M))) &\simeq \mathrm{D}(\mathrm{Tr}(\mathrm{Tr}(\tau^{-n}(M)))) \\ &\simeq \mathrm{D}(\tau^{-n}(M)) \\ &\simeq \mathrm{Tr}(\mathrm{Tr}(\mathrm{D}(\tau^{-n}(M)))) \\ &= \mathrm{Tr}(\tau^{-(n+1)}(M)). \end{aligned}$$

Therefore $\tau^n(M) \simeq \mathrm{Tr}(\tau^{-n}(M))$ for all $n \in \mathbb{N}$. For $n < 0$, let $i = -n$, then we have that

$$\mathrm{Tr}(\tau^{-n}(M)) = \mathrm{Tr}(\tau^{i}(M)) \simeq \mathrm{Tr}(\mathrm{Tr}(\tau^{-i}(M))) \simeq \tau^{-i}(M) = \tau^n(M).$$

□

4.1.2 The AR-sequence

There are two different, yet equivalent, descriptions of AR-sequences. They both stem from a series of papers by Auslander and Reiten [1],[4], [5], [6], which was the basis of their book [3]. The original name given to these sequences by Auslander and Reiten was *almost split sequences.* Such sequences are not split, but are as "close" to being split as possible. To give a precise definition, we first need to discuss the types of morphisms that will appear in an AR-sequence.

We say that a morphism $g : X \to Y$ *factors through* $t : B \to Y$ if there exist $s : X \to B$ such that $ts = g$. That is, the diagram

$$\begin{tikzcd} & B \arrow[d, "t"] \\ X \arrow[ur, dashed, "\exists\, s"] \arrow[r, "g"] & Y \end{tikzcd}$$

commutes. Similarly $g : X \to Y$ *factors through* $u : X \to A$ if there exists $v : A \to Y$ such that $vu = g$,

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "u"'] & Y \\ A. \arrow[ur, dashed, "\exists\, v"'] & \end{tikzcd}$$

A morphism $f : B \to C$ is called a *split epimorphism* if the identity morphism on C factors through f. Dually, a morphism $g : A \to B$ is a *split monomorphism* if the identity morphism on A factors through g. These are the types of morphisms that appear in the sequences which are "close" to being split. Thus, we define left/right almost split morphisms.

Definition 4.1.2.1. [3] A morphism $f : B \to C$ is *right almost split* if

(a) it is not a split epimorphism,

(b) any morphism $h : X \to C$, not a split epimorphism, factors through f.

Definition 4.1.2.2. [3] A morphism $g : A \to B$ is *left almost split* if

(a) it is not a split monomorphism,

(b) any morphism $h : A \to Y$, not a split monomorphism, factors through g.

These morphisms have duality.

Lemma 4.1.2.3. *[3, V 1.3] A morphism $f : B \to C$ of modules is right almost split if and only if* $\mathrm{D}(f) : \mathrm{D}(C) \to \mathrm{D}(B)$ *is left almost split.*

Another useful fact about almost split morphisms is that they give information about the indecomposability of a module.

Lemma 4.1.2.4. *[3, V 1.7] Let $f : B \to C$ be a morphism,* *(a) If f is right almost split, then C is an indecomposable module.*

(b) If f is left almost split, then B is an indecomposable module.

We now have enough terminology to define AR-sequences.

Definition 4.1.2.5. A short exact sequence

$$s: \quad 0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$

is called an *AR-sequence ending in C* if g is left almost split and f is right almost split.

From this definition it is apparent why Auslander and Reiten called these sequences almost split sequences. For all indecomposable non-projective modules , over a finite dimensional k-algebra that is local, there exist a unique AR-sequence ending in M, up to isomorphism. [3, V 1.15, 1.16]. The following proposition of Auslander and Reiten illustrates the connections between all the concepts presented thus far.

Proposition 4.1.2.6. *[3, V, 1.13] The following are equivalent for an exact sequence*

$$0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0.$$

(a) The sequence is an AR-sequence.

(b) The module A is indecomposable and f is right almost split.

(c) The module C is indecomposable and g is left almost split.

(d) $C \cong \tau^{-1}(A)$ and g is left almost split.

(e) $A \cong \tau(C)$ and f is right almost split.[1]

Given an AR-sequence one can easily find another.

Proposition 4.1.2.7. *[3, V] A short exact sequence*

$$0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$

[1]There are actually two more equivalent statements, however they involve concepts we will not be considering here.

is an AR-sequence if and only if

$$0 \longrightarrow \mathrm{D}(C) \xrightarrow{\mathrm{D}(f)} \mathrm{D}(B) \xrightarrow{\mathrm{D}(g)} \mathrm{D}(A) \longrightarrow 0$$

is an AR-sequence.

Proof. With the fact that for any module $\mathrm{D}(\mathrm{D}(M)) \cong M$, this following directly from lemma 4.1.2.3. $\square$

4.2 The AR-translate is Not Closed for Totally Reflexive Modules

If T is a totally reflexive R-module and if $\tau(T)$ were also totally reflexive, then we could employ AR-theory on the category of totally reflexive modules. However, this is not the case. To see this, we can look at a simple example.

Example 4.2.0.8. Let $S = k[X,Y,Y]/(X^2,Y^2,Y^2,YZ)$, where k is a field, and let $M = \operatorname{coker}(x)$, where the map (x) is multiplication by $x = X + (X^2,Y^2,Y^2,YZ)$. Then S is local with maximal ideal $\mathfrak{m} = (x,y,z)$. Notice that $\mathfrak{m}^3 = 0$, and thus from [17] we know that any totally reflexive module must have constant Betti numbers. Since (x) is an exact zero divisor, M is totally reflexive and has a totally acyclic complex

$$\cdots \to S \xrightarrow{x} S \xrightarrow{x} S \xrightarrow{x} S \to \cdots.$$

We can compute $\tau(M)$, and a presentation matrix of it is

$$\begin{bmatrix} z & x & y & 0 & 0 \\ 0 & 0 & -z & y & x \end{bmatrix}$$

which has a minimal free resolution

$$\cdots \to S^{23} \to S^{11} \to S^5 \to S^2 \to 0.$$

Since this minimal free resolution of $\tau(M)$ does not have constant Betti numbers, it cannot be totally reflexive. In actuality, this is true for all totally reflexive modules over $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. It is not the transpose that is the problem.

Lemma 4.2.0.9. *For any local ring R, if T is totally reflexive R-module, then* $\mathrm{Tr}(T)$ *is also a totally reflexive R-module.*

Proof. Consider the resolution from 4.1.1.1.1, the definition of the AR-transpose, which is a free presentation of $\mathrm{Tr}(M)$. By completing the exact sequence we have

$$0 \to \mathrm{Hom}_R(M,R) \to \mathrm{Hom}_R(R^n,R) \xrightarrow{\phi^*} \mathrm{Hom}_R(R^m,R) \to \operatorname{coker}\phi^* \to 0.$$

If M is totally reflexive, then $\mathrm{Hom}_R(M,R) = M^*$ is totally reflexive as well, 2.3.2.1 (b). Since this is a free presentation of $\mathrm{Tr}(M)$, M^* is the actually the second syzygy of $\mathrm{Tr}(M)$ and therefore $\mathrm{Tr}(M)$ must also be totally reflexive 2.3.2.1 (d).

□

Theorem 4.2.0.10. *Let $(R, \mathfrak{m}, k)$ be a k-algebra with $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. If R is not Gorenstein and T is a totally reflexive R-module, then $\tau(T)$ is not totally reflexive.*

Proof. By use of lemma 4.2.0.9 we will consider the case $M := \mathrm{Tr}(T)$. Assume $\mathrm{rank}_k(\mathrm{soc}(R)) = s$ and $\mu(M) = n$. Let the following be a minimal totally acyclic complex.

$$\begin{tikzcd}[column sep=small]
\mathbb{F}: & \cdots \arrow[r] & R^n \arrow[r, "\partial_1"] & R^n \arrow[rr, "\partial_0"] \arrow[dr] & & R^n \arrow[r, "\partial_{-1}"] & R^n \arrow[r] & \cdots \\
& & & & M \arrow[ur] \arrow[dr] & & & \\
& & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

We then have that $\operatorname{im}\partial_0 \cong M \cong \ker\partial_{-1}$. Since $\partial_{-1}\partial_0 = 0$ and $\partial_i(R^n) \subset \mathfrak{m}R^n$ for all i, we have that M contains the socle of R^n and hence $\mathrm{Soc}(M) \cong \mathrm{Soc}(R)$. Therefore, $\mathrm{rank}_k(\mathrm{soc}(M)) = \mu(M)s = ns$. If we assume $\mathrm{D}(M)$ is also totally reflexive, then the equality $\mathrm{rank}_k(\mathrm{soc}(\mathrm{D}(M))) = \mu(\mathrm{D}(M))s$ would also hold, and since $\mu(\mathrm{D}(M)) = \mathrm{rank}_k(\mathrm{Soc}(M))$, we have

$$\mathrm{rank}_k(\mathrm{Soc}(\mathrm{D}(M))) = \mu(\mathrm{D}(M))s = ns^2,$$

However,

$$\mathrm{rank}_k(\mathrm{Soc}(\mathrm{D}(M))) = \mu(\mathrm{D}(\mathrm{D}(M))) = \mu(M) = n,$$

which implies $n = ns^2$ or $s = 1$, but $\mathrm{rank}_k(\mathrm{soc}(R)) = 1$ if and only if R is Gorenstein. Thus $\mathrm{D}(M) = \mathrm{D}(\mathrm{Tr}(T)) = \tau(T)$ is not totally reflexive.

□

4.3 A Minimum Element of Ext^1 as the AR-sequence

Another way to view AR-sequences is to start with short exact sequences instead of morphisms.

Definition 4.3.0.11. [37, 2.1] For an indecomposable module M, define $\mathcal{S}(M)$ as

$$\mathcal{S}(M) = \{\ s:\ 0 \to N_s \to E_s \to M \to 0 \,|\, s \text{ is a nonsplit exact sequence with } N_s \text{ indecomposable}\}$$

any element in $\mathcal{S}(M)$ gives a nontrivial element of $\mathrm{Ext}^1_R(M, N_s)$.

If M is not free, then $\mathcal{S}(M)$ is nonempty. This is easy to see if we take free presentation of M and decompose $\Omega_1(M)$. In particular, we would like to consider $\mathcal{S}(M)$ when it is completely contained inside a category, such as the categories of Cohen-Macaulay modules. This is what Yoshino explores in [37]. Next we define an ordering on $\mathcal{S}(M)$.

Definition 4.3.0.12. [37, 2.3] For two elements, $s, t \in \mathcal{S}(M)$, we say that s is *bigger* than t, denoted $s > t$, if there exist morphism $f : N_s \to N_t$ making the below diagram commute,

$$\begin{tikzcd}
0 \arrow[r] & N_s \arrow[r] \arrow[d, "f"'] & E_s \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_t \arrow[r] & E_t \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

If f is an isomorphism, then we write $s \sim t$. This makes $\mathcal{S}(M)$ into a well-defined partially ordered set [37, 2.4, 2.7].

Definition 4.3.0.13. Let M be an indecomposable module, a short exact sequence $s \in \mathcal{S}(M)$ is an *AR-sequence ending in* M if s is the minimum element in $\mathcal{S}(M)$.

If the AR-sequence exist, then E_s and N_s are unique, up to isomorphisms. If for all non-free modules M in a category, there exist an AR-sequence ending in M, then we say the category *admits AR-sequences*. There are many known categories that admit AR-sequence, see [3], [37], or [26] for examples.

4.3.1 An Equivalence of Definitions for the AR-sequence

It is not obvious that these two definitions, 4.1.2.5 and 4.3.0.13, are equivalent. This is proven in [37, 2.9]. However, in those $\mathcal{S}(M)$ is defined to be inside the category of Cohen-Macaulay modules. We are interested in a slightly different set.

Definition 4.3.1.1. For an indecomposable totally reflexive module, define

$$\mathcal{S}'(M) = \{\, s : \ 0 \to N_s \to E_s \to M \to 0 \,|\, s \text{ is a nonsplit exact sequence with } N_s \\ \text{indecomposable and totally reflexive}\}\,.$$

This set is nonempty for the same reason $\mathcal{S}(M)$ is nonempty. Note that from 2.3.2.2, M and N_s both being totally reflexive implies E_s is as well.

Theorem 4.3.1.2. *Let $a \in \mathcal{S}'(M)$, then a is of the form*

$$a : \ 0 \to \tau'(M) \xrightarrow{i} E_a \xrightarrow{p} M \to 0$$

if and only if a is the minimum element in $\mathcal{S}'(M)$, where $\tau'(M)$ is defined as $\tau(M)$ except that the modules that are being factored through are totally reflexive.

Proof. Derived from the proof of Lemma 2.9 in [37].

Let a be the AR-sequence define above,

$$a:\ 0 \to \tau'(M) \xrightarrow{i} E_a \xrightarrow{p} M \to 0,$$

and suppose that t

$$t:\ 0 \to N_t \longrightarrow E_t \xrightarrow{\pi} M \to 0.$$

is the minimum element in $\mathcal{S}'(M)$. Thus $t < a$ and

$$\begin{tikzcd}
0 \arrow[r] & \tau'(M) \arrow[r, "i"] \arrow[d] & E_a \arrow[r, "p"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_t \arrow[r] & E_t \arrow[r, "\pi"] & M \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram. Since p is a right almost split morphism, there exist $\varphi : E_t \to E_a$ such that $\pi = p\varphi$. Also, since $N_t \subset E_t$, define $f : N_T \to \tau'(M)$ to be φ restricted to N_t. This gives us the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \tau'(M) \arrow[r, "i"] \arrow[d, bend right] & E_a \arrow[r, "p"] \arrow[d, bend right] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_t \arrow[r] \arrow[u, bend right, "f"'] & E_t \arrow[r, "\pi"] \arrow[u, bend right, "\varphi"'] \arrow[ur, "\pi"] & M \arrow[r] & 0.
\end{tikzcd}$$

This implies that $a < t$, but t was the minimum element in $\mathcal{S}'(M)$. Therefore $a \sim t$, and a must be the minimum element in $\mathcal{S}'(M)$. Now suppose that

$$s:\ 0 \to N_s \longrightarrow E_s \xrightarrow{p} M \to 0$$

is the minimum element in $\mathcal{S}'(M)$. For a totally reflexive module L, let $q : L \to M$ be a morphism which is not a split epimorphism. In order prove that $N_s \cong \tau'(M)$, we must show that q factors through p. In other words, the existence of a commutative triangle,

$$\begin{tikzcd}
 & L \arrow[dl, dashed, "\exists f"'] \arrow[d, "q"] \\
E_s \arrow[r, "p"] & M.
\end{tikzcd}$$

Define the sequence

$$u:\ 0 \to Q \longrightarrow E_s \oplus L \overset{\varphi}{\longrightarrow} M \to 0,$$

where $\varphi = (p, q)$ and $Q = \ker(\varphi)$. Since neither p or q is split, φ is not split [37, 1.21]. In addition, Q is totally reflexive, since E_s, L, and M are define to be totally reflexive, 2.3.2.2 . Define i to be the natural inclusion of E_s into $E_s \oplus L$, and h to be π restricted to N_s.

$$\begin{tikzcd}
s: & 0 \arrow[r] & N_s \arrow[r] \arrow[d, "h"] & E_s \arrow[r, "p"] \arrow[d, "i"] & M \arrow[r] \arrow[d, equal] & 0 \\
u: & 0 \arrow[r] & Q \arrow[r] & E_s \oplus L \arrow[r, "\varphi"] & M \arrow[r] & 0.
\end{tikzcd}$$

The sequence u is not necessary in $\mathcal{S}'(M)$, since Q may not be indecomposable. Therefore, let $Q = \sum_i Q_i$, where each Q_i is indecomposable and we have $\mathrm{Ext}^1_R(M, Q) = \sum_i \mathrm{Ext}^1_R(M, Q_i)$. Let $u = \sum_i u_i$, since u is not a split, there exist an u_i that is also not split, called it t. Note that $s > t$.

$$\begin{tikzcd}
s: & 0 \arrow[r] & N_s \arrow[r] \arrow[d, "h"] & E_s \arrow[r, "p"] \arrow[d, "i"] & M \arrow[r] \arrow[d, equal] & 0 \\
u: & 0 \arrow[r] & Q \arrow[r] \arrow[d] & E_s \oplus L \arrow[r, "\varphi"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
t: & 0 \arrow[r] & Q_t \arrow[r] & E_t \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

However, s is the minimum element in $\mathcal{S}'(M)$. Hence $s \sim t$, and there exist morphisms g and f' such that the following diagram commutes:

$$\begin{tikzcd}
u: & 0 \arrow[r] & Q \arrow[r] \arrow[d, "g"] & E_s \oplus L \arrow[r, "\varphi"] \arrow[d, "f'"] & M \arrow[r] \arrow[d, equal] & 0 \\
s: & 0 \arrow[r] & N_s \arrow[r] & E_s \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Let ι be the natural inclusion of L into $L \oplus E_s$, and define $f = f'\iota$. Then we have that the diagram

$$\begin{tikzcd}
E_s \oplus L \arrow[d, "f'"'] & L \arrow[l, "\iota"'] \arrow[d, "q"] \arrow[dl, dashed, "f"'] \\
E_s \arrow[r, "p"] & M.
\end{tikzcd}$$

commutes and hence q factors through p. Therefore, the two definitions of an AR-sequence are equivalent.

$\square$

4.3.2 An Infinitely Decreasing Sequence

Following in Auslander's footsteps, we wish to see if the category of totally reflexive modules over non-Gorenstein rings admits AR-sequences with the above description. We construct an infinite strictly decreasing chain of short exact sequences in $\mathcal{S}'(M)$.

Example 4.3.2.1. Let $S = k[X, Y, Y]/(X^2, Y^2, Y^2, YZ)$, and consider the S-module $S/(x)$, where $x = X + (X^2, Y^2, Y^2, YZ)$. Since $(0 : x) = (x)$, we have that x is an exact zero divisor and thus by 2.3.2.4, $S/(x)$ is totally reflexive. For all integers $n \geq 2$, define $t_n \in \mathcal{S}(S/x)$ as

$$t_n : \qquad 0 \to \operatorname{coker} D_n, \to \operatorname{coker} E_n \to S/(x) \to 0$$

where D_n is an $n \times n$ matrix and E_n is an $(n+1) \times (n+1)$ matrix defined as follows:

$$D_n = \begin{bmatrix} x-y & y & 0 & 0 & \dots & 0 \\ 0 & x-y & y & 0 & \dots & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & \dots & 0 & x-y & y \\ 0 & 0 & 0 & \dots & 0 & x-y \end{bmatrix},$$

$$E_n = \begin{bmatrix} x-y & y & 0 & 0 & \dots & 0 \\ 0 & x-y & y & 0 & \dots & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & \dots & 0 & x-y & y \\ 0 & 0 & 0 & \dots & 0 & x \end{bmatrix}$$

Define

$$t_1 : \qquad 0 \to S/(x-y) \to \operatorname{coker} E_2 \to S/(x) \to 0.$$

Both $\operatorname{coker}(D_n)$ and $\operatorname{coker}(E_n)$ are totally reflexive modules since they both have a presentation matrix that is upper triangular with the entries on the main diagonal being exact zero divisors, see 3.2.0.9.

Claim 4.3.2.2. For all $n \geq 1$, $t_n > t_{n+1}$.

Proof. First, we should note that $\operatorname{coker}(D_n)$ is indecomposable [15, 3.1], and thus $t_n \in \mathcal{S}(S/(x))$. In [15] the authors construction a matrix of a more general form than E_n and prove that it is indecomposable [15, 2.6, 3.1]. Consider the diagram

$$(\star) \qquad \begin{tikzcd} t_n : & 0 \arrow[r] & \operatorname{coker}(D_n) \arrow[r, "i_n"] \arrow[d, "f_n"'] & \operatorname{coker}(E_n) \arrow[r, "p_n"] \arrow[d, "g_n"] & S/x \arrow[r] \arrow[d, equal] & 0 \\ t_{n+1} & 0 \arrow[r] & \operatorname{coker}(D_{n+1}) \arrow[r, "i_{n+1}"] & \operatorname{coker}(E_{n+1}) \arrow[r, "p_{n+1}"] & S/(x) \arrow[r] & 0. \end{tikzcd}$$

where the morphisms i_n and p_n are the natural inclusion and projection, respectively, and can be expressed as the matrices

$$i_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \dots & 0 \end{bmatrix}_{(n+1)\times n} \qquad \text{and} \qquad p_n = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}_{1\times(n+1)}.$$

The morphisms f_n and g_n are define as follows:

$$f_n = i_n \qquad g_n = \left[\begin{array}{c} \mathbf{I}_{n+1} \\ \hline 0 \dots 0\ 1 \end{array}\right]_{(n+2)\times(n+1)},$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix.

To prove the claim we shall show that the diagram $(\star)$ commutes. However, before we show commutativity, we need to make sure that g_n is a well-defined morphism. This can be done by simply showing that elements of the kernel map to zero in $\mathrm{coker}(E_{n+1})$. These elements are precisely the elements in the column space of each of the E_n matrices. The only element in the column space of E_n that could potentially not map to an element in the column space of E_{n+1} is $\begin{bmatrix} 0 & \dots & 0 & y & x \end{bmatrix}^T$, since it is the only generator in the column space of E_n that contains a nonzero entry in the $(n+2)$nd coordinate.

$$g_n\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ y \\ x \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ y \\ x \\ x \end{bmatrix}$$

Now consider the following elements in the column space of E_{n+1},

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ y \\ x-y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ y \\ x \\ x \end{bmatrix}.$$

Therefore, $g_n\left(\begin{bmatrix} 0 & \dots & 0 & y & x \end{bmatrix}^T\right)$ is zero in $\operatorname{coker} E_{n+1}$ and hence g_n is a well-defined morphism.

To show the commutativity of $(\star)$, we will start by showing that the left square commutes. Let $a = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}^T$ be a element of $\operatorname{coker} D_n$.

$$g_n(i_n(a)) \;=\; g_n\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ 0 \\ 0 \end{bmatrix}$$

$$i_{n+1}(f_n(a)) \;=\; i_{n+1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \\ 0 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the left square commutes. Similarly, let $b = \begin{bmatrix} b_1 & \dots & b_{n+1} \end{bmatrix}^T$ be in coker E_n.

$$\begin{aligned} p_{n+1}(g_n(b)) &= p_{n+1}\left(\begin{bmatrix} b_1 \\ \vdots \\ b_{n+1} \\ b_{n+1} \end{bmatrix}\right) = [b_{n+1}] \\ \mathrm{Id}(p_n(b)) &= \mathrm{Id}([b_{n+1}]) = [b_{n+1}] \end{aligned}$$

Hence $(\star)$ commutes and thus $\mathcal{S}(S/(x))$ contains an infinitely decreasing sequence of elements. Although this does not necessarily imply that no minimum element exist, just that one it is not of the form t_n.

□

Question 4.3.2.3. Does there exist an element $\alpha \in \mathcal{S}(S/(x))$ such that $\alpha < t_n$ for all n?

4.4 AR-theory for Totally Reflexive Modules

From the previous sections we know that the AR-translate does not take totally reflexive modules to totally reflexive modules. However, there are still things we can learn about totally reflexive modules using the AR-translate and AR-sequences. This section is a collection of facts obtained by employing AR-theory. The first theorem, and subsequently the prior lemmas, are true for the broader class of reflexive modules.

Lemma 4.4.0.4. *If M is a reflexive R-module with no projective summands, and there exist a short exact sequence of R-modules*

$$s: \qquad 0 \to N \to E \to M \to 0,$$

then there exist a short exact sequence

$$0 \to \mathrm{Tr}(M) \to \mathrm{Tr}(E) \to \mathrm{Tr}(N) \to 0.$$

Proof. By applying the functor $\mathrm{Hom}_R(\_\_, R)$ we obtain the left exact sequence

$$s^* : \qquad 0 \to M^* \to E^* \to N^* \to \mathrm{Ext}^1_R(M, R) \to \dots .$$

Since M is reflexive, $\mathrm{Ext}^1_R(M, R) = 0$ and thus s^* is short exact. Let $\mathbb{F}$ and $\mathbb{G}$ be minimal free resolutions of M and N, respectively. By applying $\mathrm{Hom}_R(\_\_, R)$ to free presentations of M and N we obtain

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & M^* \arrow[r] \arrow[d] & E^* \arrow[r] \arrow[d] & N^* \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & F_0 \arrow[r] \arrow[d] & F_0 \oplus G_0 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & F_1 \arrow[r] \arrow[d] & F_1 \oplus G_1 \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & 0 \\
 (\dagger) & 0 \arrow[r] & \mathrm{Tr}(M) \arrow[r] \arrow[d] & \mathrm{Tr}(E) \arrow[r] \arrow[d] & \mathrm{Tr}(N) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

where the row $(\dagger)$ is given by the snake lemma.

$\square$

The following lemma actually holds over noncommutative rings, therefore for this case we will consider one sided modules.

Lemma 4.4.0.5. *If*

$$s : \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence of left R-modules, then there exist a short exact sequence

$$\mathrm{D}(s): \qquad 0 \longrightarrow \mathrm{D}(C) \xrightarrow{\mathrm{D}(g)} \mathrm{D}(B) \xrightarrow{\mathrm{D}(f)} \mathrm{D}(A) \longrightarrow 0$$

of right R-modules.

Proof. The existence of $\mathrm{D}(s)$ is obvious when viewed k-vector spaces. All we need to show is that $\mathrm{D}(f)$ is in fact a right R-module homomorphism. That is, for $\beta \in \mathrm{D}(B)$ and $r \in R$ we have $\mathrm{D}(f)(\beta r) = (\mathrm{D}(f)(\beta))r$. By definition $\mathrm{D}(f)(\beta r) = (\beta r) \circ f$, so for $a \in A$

$$(\mathrm{D}(f(\beta r))(a) = ((\beta r) \circ f)(a) = (\beta r)(f(a)) = \beta(rf(a))$$

and since f is a left R-homomorphism we have

$$\beta(rf(a)) = \beta(f(ra)) = (\beta \circ f)(ra) = ((\beta \circ f)(r))(a) = ((\mathrm{D}(f)(\beta))r)(a).$$

Thus $\mathrm{D}(f)$ is right R-linear and so it is a right R-homomorphism.

□

Theorem 4.4.0.6. *Let*

$$s: \qquad 0 \to N \to E \to M \to 0$$

be a short exact sequence of R-modules. If M is reflexive, then

$$\tau(s): \qquad 0 \to \tau(N) \to \tau(E) \to \tau(M) \to 0$$

is exact. In particular, for the AR-sequence

$$a: \qquad 0 \to \tau(M) \to E \to M \to 0$$

we have the AR-sequence

$$\tau(a): \qquad 0 \to \tau^2(M) \to \tau(E) \to \tau(M) \to 0.$$

Proof. This follows directly from lemma 4.4.0.4 and lemma 4.4.0.5. To see that $\tau(a)$ is in fact an AR-sequence, recall that an AR-sequence is unique.

$\square$

Corollary 4.4.0.7. *Let*

$$0 \to A \to B \to C \to 0$$

be a non-split short exact sequence of R-modules. If C is reflexive module, then the following diagram commutes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tau(A) \arrow[r, "\phi"] \arrow[d, "\tau(i)"'] & L \arrow[r, "\lambda"] \arrow[d, "f"] \arrow[dl, dashed, "\alpha"'] & A \arrow[r] \arrow[d, "i"] \arrow[dl, dashed, "\gamma"'] & 0 \\
0 \arrow[r] & \tau(B) \arrow[r, "\psi"] \arrow[d, "\tau(p)"'] & M \arrow[r, "\varsigma"] \arrow[d, "g"] \arrow[dl, dashed, "\beta"'] & B \arrow[r] \arrow[d, "p"] \arrow[dl, dashed, "\delta"'] & 0 \\
0 \arrow[r] & \tau(C) \arrow[r, "\theta"] \arrow[d] & N \arrow[r, "\eta"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & .
\end{tikzcd}$$

Before we can prove this corollary, we need the following property for AR-sequences.

Proposition 4.4.0.8. *[4, 4.1] Let M be an indecomposable R-module with no projective summands and let*

$$a : \qquad 0 \to \tau(M) \xrightarrow{i} E \xrightarrow{p} M \to 0$$

be an AR-sequence, then a has the following properties.

(a) If Y is an arbitrary R-module, and $u : \tau(M) \to Y$ is not a split monomorphism, then there exist a morphism $v : E \to Y$ such that u factors through v, i.e. $vi = u$.

(b) If X is an arbitrary R-module, and $g : X \to M$ is not a split epimorphism, then there exist a morphism $h : X \to E$ such that g factors through i, i.e. $ih = g$.

Proof. [Corollary 4.4.0.7] Notice that all the horizontal sequences are AR-sequences, and thus by 4.4.0.8 there exist morphism $\alpha, \beta, \gamma, \delta$ such that

$$\tau(i) = \alpha\phi, \qquad i = \varsigma\gamma, \qquad \tau(p) = \beta\psi, \qquad p = \eta\delta.$$

Define $f : L \to M$ as $f(l) = (f_1 + f_2)(l)$, for all $l \in L$ where

$$f_1(l) = \psi(\alpha(l)) \qquad \text{and} \qquad f_2 = \gamma(\lambda(l)).$$

Similarly define $g : M \to N$ as $g(m) = (g_1 + g_2)(m)$, for all $m \in M$ where

$$g_1(m) = \theta(\beta(m)) \qquad \text{and} \qquad g_2(m) = \delta(\varsigma(m)).$$

All that is left to show is that the four inner squares commute. All four have symmetric proofs, thus we will only show that the top left one commutes. Let $x \in \tau(A)$, then

$$\begin{aligned} f(\phi(x)) &= f_1(\phi(x)) + f_2(\phi(x)) \\ &= \psi(\alpha(\phi(x))) + \gamma(\lambda(\phi(x))) \\ &= \psi(\tau(i)(x)) + \gamma(0) \\ &= \psi(\tau(i)(x)). \end{aligned}$$

Thus, this square commutes. Therefore, the diagram is commutative and by the snake lemma, the middle vertical sequence is also exact.

□

Chapter 5

Mindy

5.1 Why Mindy?

In order to gain a better understanding of how totally reflexive modules behave and which techniques of representation theory have potential applications towards them, many examples were computed. Since the category of nontrivial totally reflexive modules over non-Gorenstein rings is often of wild type, we want to perform our investigation this over the "smallest" ring possible which admits nontrivial totally reflexive modules. Hereafter, we refer to Mindy as the "smallest" non-Gorenstein local ring that admits non-trivial totally reflexive modules. The following argument justifies why Mindy is the "smallest" such non-Gorenstein ring.

We will consider Artinian local rings $(R, \mathfrak{m})$, and thus there exists a $n \in \mathbb{N}$ such that $\mathfrak{m}^n = 0$. We can immediately discard rings where $\mathfrak{m}^2 = 0$, since they are of minimal multiplicity. Indeed, non-Gorenstein rings with minimal multiplicity are Golod [8, 5.2.8], and by [9, 3.5] they only admit trivial totally reflexive modules. Thus, we want a (local) ring such that the cubic of the maximal ideal is zero. After this, we continue by considering embedding dimensions to finding the smallest length ring that has nontrivial totally reflexive modules. To do this, we will use part of a theorem by Christensen and Veliche [17, theorem A] that is relevant to our needs, see 2.3.3.1 for the full theorem.

For a local non-Gorenstein ring $(R, \mathfrak{m})$ *with* $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$*, if there exists a non-zero minimal acyclic complex* $\mathbb{A}$ *of finitely generated free*

R-modules, then $\operatorname{Soc}(R) = \mathfrak{m}^2$ *and* $\operatorname{length} R = 2e$, *where* $e = \operatorname{edim}(R)$.

A ring with an embedding dimension of one would be a hypersurface which is Gorenstein, and a ring with an embedding dimension of two would be a complete intersection [32] which is also Gorenstein. Therefore, in addition to $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$, we will consider rings with an embedding dimension of three. By the part Theorem 2.3.3.1 repeated above, we can now discard any rings of length not equal to six. However for explicitness, we will give a brief explanations of why such rings would not work.

If a ring with embedding dimension three has a length of four, then it is of minimal multiplicity, and thus Golod. If a ring has a length of five, then there are two cases to consider here. First, notice that such a ring would have $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. If the length of the socle is one, then it is Gorenstein, 2.2.2.5. Otherwise, the socle would contain an element from the maximal ideal and thus would not be equal to the square of the maximal ideal, a contradiction to 2.3.3.1 (a).

Definition 5.1.0.1. Let k be a field and let the ring $S = k[X, Y, Z]/(X^2, Y^2, Z^2, YZ)$ be called *Mindy*. Define $x = X + (X^2, Y^2, Z^2, YZ)$, $y = Y + (X^2, Y^2, Z^2, YZ)$, and $z = Z + (X^2, Y^2, Z^2, YZ)$. Mindy is a finite dimensional k algebra that is local with unique maximal ideal $\mathfrak{n} = (x, y, z)$, where $\mathfrak{n}^3 = 0 \neq \mathfrak{n}^2$. Since $\operatorname{rank}_k(\operatorname{Soc}(S)) = \operatorname{rank}_k(xy, xz) = 2$, it is not Gorenstein. From the above argument we consider this ring because it has the smallest length, $\operatorname{length}(S) = 6$, among rings that admit nontrivial totally reflexive modules, see 3.5.0.2 for an example of a totally reflexive S-module. For the remainder of this paper we will reserve $(S, \mathfrak{n})$ to be the specific ring Mindy described above.

5.2 The Cyclic Modules

One of the many ways Mindy is special is that we can easily describe the cyclic totally reflexive modules. They are exactly the ideals generated by an exact zero divisor.

Lemma 5.2.0.2. *The module S/I is totally reflexive if and only if I is an exact zero divisor, where I is a principal ideal.*

Proof. If I is an exact zero divisor, then by 2.3.2.4 the module S/I is totally reflexive. Now suppose that S/I is totally reflexive. Then we know by Theorem 2.3.3.2 that a minimal free resolution of S/I has constant Betti numbers. Since I is a principal ideal it is one generated, and thus a minimal free resolution of S/I has the form

$$\cdots \to S \xrightarrow{d_2} S \xrightarrow{d_1} S \xrightarrow{I} S \longrightarrow S/I \longrightarrow 0.$$

As we shall see in section 5.3.1 all minimal free resolutions of totally reflexive S-modules are, at most, two periodic. Hence, $d_{2i} = I$ and $d_{2i-1} = d_1$ for all $i \geq 1$. Therefore, I is an exact zero divisor.

□

Proposition 5.2.0.3. *The ring element $ax - by - cz$ is an exact zero divisor if and only if $a \neq 0$.*

Proof. By lemma 5.2.0.2 this is equivalent to $S/(ax+by+cz)$ being a totally reflexive module. First, suppose $a \neq 0$ and consider the following exact complex which has M has a syzygy.

$$\begin{tikzcd}[column sep=small]
\mathbb{F}_M : & \cdots \arrow[r] & S \arrow[r, "{[ax+by+cz]}"] & S \arrow[rr, "{[ax-by-cz]}"] \arrow[dr] & & S \arrow[r, "{[ax+by+cz]}"] & S \arrow[r] & \cdots \\
& & & & M \arrow[ur] \arrow[dr] & & & \\
& & & 0 \arrow[ur] & & 0 & & .
\end{tikzcd}$$

This is exact by Remark 2.3.2.4. We also have that

$$\mathrm{Hom}_S(\mathbb{F}_M, S): \qquad \cdots \to S \xrightarrow{[ax-by-cz]} S \xrightarrow{[ax+by+cz]} S \xrightarrow{[ax-by-cz]} S \to \cdots$$

is also exact since it is isomorphic to $\mathbb{F}$. Therefore, $\mathbb{F}$ is a totally acyclic complex, and thus M is totally reflexive. Also, note that as ideals $(x+by+cz) \cong (x+by+cz+dxy+fxz)$

Now suppose that $M = S/(by+cz)$. Again, note that as ideals $(x+by+cz) \cong (x+by+cz+dxy+fxz)$. We can compute a deleted minimal free resolution of M.

$$\mathbb{F}_M: \qquad \cdots \to S^4 \xrightarrow{\begin{bmatrix} y & z & 0 & 0 \\ 0 & 0 & y & z \end{bmatrix}} S^2 \xrightarrow{\begin{bmatrix} y & z \end{bmatrix}} S \xrightarrow{\begin{bmatrix} by+cz \end{bmatrix}} S \to 0,$$

as well as the dual of it,

$$\mathrm{Hom}_S(\mathbb{F}_M, S): \qquad 0 \to S \xrightarrow{\begin{bmatrix} by+cz \end{bmatrix}} S \xrightarrow{\begin{bmatrix} y \\ z \end{bmatrix}} S^2 \xrightarrow{\begin{bmatrix} y & 0 \\ z & 0 \\ 0 & y \\ 0 & z \end{bmatrix}} S^4 \to \cdots.$$

If $c \neq 0$, then $y \in \ker\left(\begin{bmatrix} y \\ z \end{bmatrix}\right)$ but $y \notin \mathrm{im}\left(\begin{bmatrix} by+cz \end{bmatrix}\right)$. Also, if $b \neq 0$, then $z \in \ker\left(\begin{bmatrix} y \\ z \end{bmatrix}\right)$ but $z \notin \mathrm{im}\left(\begin{bmatrix} by+cz \end{bmatrix}\right)$. Therefore, $\mathrm{Ext}^1_S(M,S) \neq 0$, and M cannot be a totally reflexive module. Thus we have that $a \neq 0$ in order for $S/(ax+by+cz)$ to be a totally reflexive module.

$\square$

Since we must have $a \neq 0$ in order for $S/(ax + by + cz)$ to be totally reflexive we can simply divide by a to consider modules of the form $S/(x+by+cz)$. Note that $(x + by + cz)$ is an exact zero divisor with $(x - by - cz)$ as its pair. These are the only exact zero pairs in Mindy.

5.3 Description of Totally Reflexive Modules

The goal of this section is to consider a free presentation of a totally reflexive S-module T,

$$S^{b_1} \xrightarrow{\partial} S^{b_0} \to T \to 0$$

and to investigate the possibilities for ∂, as well as the other differentials in a complete resolution of T. It is already known that in a minimal presentation $b_1 = b_0$, 2.3.3.2. In fact, every free module in a complete resolution of T has the same rank. For convenience, we recall the part of this theorem from [17], previously listed 2.3.3.2, which applies to Mindy.

> *(I) The residue field k is not a direct summand of* $\operatorname{coker} \partial_i$ *for any $i \in \mathbb{Z}$, and there is a positive integer a such that $a = \operatorname{rank}_R A_i$. Moreover, for $e = \operatorname{edim}(R)$* $\operatorname{length}_R(\operatorname{coker} \partial_i) = ae$ *for all $i \in \mathbb{Z}$.*

When we combine this with the work of Avramov [7], [8] and Eisenbud [19], we can obtain more information about the structure of complete resolutions of totally reflexive modules over Mindy.

5.3.1 Periodic Resolutions

Definition 5.3.1.1. [7, 1.1] The *complexity* of an R-module M, denoted $\operatorname{cx}_R M$, is equal to d if $d-1$ is the smallest degree of a polynomial in n which bounds $b_n(M)$ from above, where $b_n(M)$ is the nth Betti number of M.

This definition, along with Theorem 2.3.3.2, shows that any totally reflexive S-module has a complexity of one. The next theorem is due to Avramov [7]. The original version in [7] has multiple conditions and groups of equivalent statements if a module satisfies one of those conditions. Below is the part of that theorem that applies to totally reflexive modules over Mindy.

Theorem 5.3.1.2. *[7, 1.6 (II)] If T is a nontrivial finitely generated totally reflexive S-module, then the following equivalent conditions hold:*

(i) $\operatorname{cx}_S T = 1$,

(ii) $b_n^S(T) = b > 0$ *for all* $n \in \mathbb{Z}$,

(iii) A minimal complete resolution of T is nonzero and periodic of at most period 2.

Therefore, any totally acyclic complex over Mindy is, at most, two-periodic. This is also made evident by the following description, 5.3.2.3.

5.3.2 Embedded Deformations

Definition 5.3.2.1. [7] A surjection of local rings $\rho : (P, \mathfrak{q}, k) \to (R, \mathfrak{m}, k)$ is called a *deformation* of R if $\ker \rho$ is generated by a P-regular sequence. A deformation is called *embedded* if $\ker \rho \subset \mathfrak{q}^2$. There is generally an abuse of language and often P is referred to as being a deformation of R.

Fact 5.3.2.2. [38, 4.2] If a ring P is an embedded deformation of a non-Gorenstein ring R that has a nontrivial totally reflexive module, and M is any R-module, then $\operatorname{G-dim}_R(M) = \operatorname{pd}_P(M) - 1$. In particular, if T is a totally reflexive R-module, then $\operatorname{pd}_P(T) = 1$.

Description 5.3.2.3. [8, 5.1.2] By finding an embedded deformation of S, we are able to completely describe the presentation matrices of totally reflexive S-modules.

Proposition 5.3.2.4. *The ring* $k[[X,Y,Z]]/(Y^2,Z^2,YZ)$ *is an embedded deformation of* S.

Proof. Let $Q = k[[X,Y,Z]]/(Y^2,Z^2,YZ)$, and $\mathfrak{p} = (X,Y,Z)$. Then, for the non-zero divisor x^2, we have $S = Q/(x^2)$. Also the natural surjection

$$\pi : Q = k[[X,Y,Z]]/(Y^2,Z^2,YZ) \to S = k[X,Y,Z]/(X^2,Y^2,Z^2,YZ)$$

has a kernel of $(x^2) \subset \mathfrak{p}^2$. Therefore, Q is an embedded deformation of S, and for the reminder of this section we will reserve Q to denote the ring defined above.

$\square$

Let T be an indecomposable totally reflexive S-module. By 5.3.2.2, it is of projective dimension one over Q. That is, for a minimal Q-free resolution of T we have that

$$0 \to Q^b \xrightarrow{\partial} Q^b \to T \to 0$$

is exact. Since x^2 annihilates T, we have $S \cong Q/(x^2)$ and thus x^2 is *null-homotopic.*

From this homotopy we can obtain the following commutative diagram, see [19].

$$\begin{tikzcd}
0 \arrow[r] & Q^b \arrow[r, "\partial"] \arrow[d, "x^2"'] & Q^b \arrow[r] \arrow[d, "x^2"] \arrow[dl, "\sigma"'] & T \arrow[r] \arrow[d, "x^2"] & 0 \\
0 \arrow[r] & Q^b \arrow[r, "\partial"] & Q^b \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

If we consider ∂ and σ as matrices, $\boldsymbol{\partial}$ and $\boldsymbol{\sigma}$, then we have that $\boldsymbol{\partial}\boldsymbol{\sigma} = x^2 I_b = \boldsymbol{\sigma}\boldsymbol{\partial}$. Thus $(\boldsymbol{\partial}, \boldsymbol{\sigma})$ is matrix factorization of x^2. Define an infinite complex of free S-modules

$$\cdots \xrightarrow{S\otimes_Q \partial} S\otimes_Q Q^b \xrightarrow{S\otimes_Q \sigma} S\otimes_Q Q^b \xrightarrow{S\otimes_Q \partial} S\otimes_Q Q^b \longrightarrow \cdots,$$

which is isomorphic to

$$\mathbb{A}: \qquad \cdots \xrightarrow{\partial^S} S^b \xrightarrow{\sigma^S} S^b \xrightarrow{\partial^S} S^b \longrightarrow \cdots,$$

where ∂^S and σ^S are ∂ and σ viewed as S-homomorphisms. The complex $\mathbb{A}$ is minimal if T does not contain any free direct sums [8, 5.1.2]. Since T is indecomposable, $\mathbb{A}$ is a minimal totally acyclic complex. Now we wish to describe a presentation matrix of such a module T, and the following will be of use in this endeavor.

Lemma 5.3.2.5. *[15, 5.1] Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m}^3 = 0$ and let*

$$\mathbb{F}: \qquad F_2 \longrightarrow F_1 \xrightarrow{\psi} F_0 \xrightarrow{\varphi} F_{-1}$$

be an exact sequence of finitely generated free R-modules, where the homomorphisms are represented by matrices with entries $\mathfrak{m}$. Let Ψ be any matrix that represents ψ. For every row Ψ_r of Ψ the following hold:

1. *The ideal $\mathfrak{r}$, generated by the entries of Ψ_r, contains $\mathfrak{m}^2$.*
2. *If $\operatorname{rank}_k \mathfrak{m}^2$ is at least 2 and $\operatorname{Hom}_R(\mathbb{F}, R)$ is exact, then Ψ_r has an entry from $\mathfrak{m}\backslash\mathfrak{m}^2$, the entries in Ψ_r from $\mathfrak{m}\backslash\mathfrak{m}^2$ generate $\mathfrak{r}$, and $\mathfrak{m}\mathfrak{r} = \mathfrak{m}^2$ holds.*

This last statement of Lemma 5.3.2.5 clearly holds for all nontrivial totally reflexive modules over non-Gorenstein rings, $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$ [17, Theorem A]. In addition, it gives us information about the entries in the presentation matrices of totally reflexive modules.

Lemma 5.3.2.6. *Let $(R, \mathfrak{m})$ be a non-Gorenstein ring with $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. If T is a totally reflexive R-module, then there exists a presentation matrix of T such that every entry in the matrix is contained in $\mathfrak{m}\backslash\mathfrak{m}^2$. That is, every entry is of a linear form.*

Proof. Recall from Lemma 4.2.0.9, that if T is a totally reflexive module, then so is $\mathrm{Tr}(T)$. Let $\Phi' = (p_{ij})$ be a presentation matrix of $\mathrm{Tr}(T)$, since $\mathrm{Tr}(T)$ is totally reflexive it is a syzygy in a totally acyclic complex. Let $\mathfrak{r}_i$ be the ideal generated by the ith row of Φ', thus $\mathfrak{r}_i = (p_{i1}, p_{i2}, \dots p_{in})$. By the previous Lemma 5.3.2.5, $\mathfrak{r}_i$ can be generated by elements of $\mathfrak{m} \backslash \mathfrak{m}^2$ for all i. Thus $\mathfrak{r}_i \cong (r_{i1}, r_{i2}, \dots r_{in})$, where $r_{ij} = p_{ij}$, if $p_{ij} \in \mathfrak{m} \backslash \mathfrak{m}^2$ otherwise $r_{ij} = 0$. Define the matrix $\Phi = (r_{ij})$. Then the row space of Φ' and Φ are isomorphic and hence so are the column spaces of $(\Phi')^T$ and $(\Phi)^T$. Since Φ is a presentation matrix of $\mathrm{Tr}(T)$, we have that $(\Phi)^T$ is a presentation matrix of T and all $r_{ji} \in \mathfrak{m} \backslash \mathfrak{m}^2$.

□

Now we are able to assume that a presentation matrix of a totally reflexive module over Mindy only has linear entries. Let

$$\mathbf{T} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

be a presentation matrix of a totally reflexive S-module T. Since $\mathrm{pd}_Q(T) = 1$, there exists no relations between the generators of $\Omega_1^Q(T)$. Thus for $\mathbf{c} = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix}^T$ in Q^n, if $\mathbf{Tc} = 0$, then $\mathbf{c} \cong 0$. That is, if for all $1 \leq i \leq n$

$$c_1 a_{i1} + c_2 a_{i2} + \dots + c_n a_{in} = 0,$$

then $c_j = 0$, for all $1 \leq j \leq n$. If $a_{ij} = \alpha y + \beta z$ where $\alpha, \beta, \in k$, then it would have the relations $y a_{ij} = 0$ and $z a_{ij} = 0$. Thus, for T to have projective dimension one over Q, there must exists at least one value j for every i such that $a_{ij} = \lambda x + \alpha' y + \beta' z$ with $\lambda \neq 0$. By use of row and column operations, one can obtain a presentation matrix of T in which there is exactly one entry in each row and in each column that contains an x term, and without lost of generality, choose a_{ii} to be that term.

Now, define the $n \times n$ matrices $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ as

$$\boldsymbol{\varphi} = xI_n + M(y,z) \qquad \boldsymbol{\psi} = xI_n - M(y,z)$$

where $M(y,z)$ is a $n \times n$ matrix whose entries are polynomials in Q of the form $\alpha y + \beta z$ for α, β in k.

Proposition 5.3.2.7. *The matrix pair* $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ *defined above is a matrix factorization of* $x^2 I_n$.

Proof. First note that $(M(y,z))^2$ is the zero matrix, since for any two polynomials $p_1 = \alpha_1 y + \beta_1 z$ and $p_2 = \alpha_2 y + \beta_2 z$ we have $p_1 p_2 = 0$ in Q. Now consider the matrix product

$$\begin{aligned} \boldsymbol{\varphi}\boldsymbol{\psi} &= (xI_n + M(y,z))\,(xI_n - M(y,z)) \\ &= x^2 I_n - xM(y,z) + M(y,z)x - (M(y,z))^2 \\ &= x^2 I_n \quad \text{since } Q \text{ is commutative} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\psi}\boldsymbol{\varphi} &= (xI_n - M(y,z))\,(xI_n + M(y,z)) \\ &= x^2 I_n - M(y,z)x + xM(y,z) - (M(y,z))^2 \\ &= x^2 I_n. \end{aligned}$$

Therefore $\boldsymbol{\varphi}\boldsymbol{\psi} = x^2 I_n = \boldsymbol{\psi}\boldsymbol{\varphi}$.

□

Therefore, the pair $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ is a matrix factorization of $x^2 I_n$, and both are presentation matrices of a totally reflexive module over S. This allows us to make the following statement.

Lemma 5.3.2.8. *A module M over Mindy is totally reflexive if and only if it has a presentation matrix of the form $xI_n + M(y,z)$, where $M(y,z)$ is an $n \times n$ matrix with* *entries of the form $\alpha y + \beta z$ for $\alpha, \beta \in k$. Moreover, $xI_n + M(y,z)$ and $xI_n - M(y,z)$ also represent the differentials in a totally acyclic complex.*

If module had a presentation matrix as stated in the above lemma , then we know that the module is totally reflexive. However, for two modules with different presentation matrices but still in this form, it is possible for them to be isomorphic. Recall that if two modules are isomorphic, then their presentation matrices are equivalent. For totally reflexive modules over Mindy we can take it a step further, to similar matrices. However, we first need the following fact.

Lemma 5.3.2.9. *If M, N are two totally reflexive S-modules, with presentation matrices $\mathbf{M} = xI_n + M(y,z)$ and $\mathbf{N} = xI_n + N(y,z)$ respectively, such that M is isomorphic to N, then there exists an invertible matrix $P \in GL_n(k)$ such that $\mathbf{M} = P\mathbf{N}P^{-1}$.*

Proof. If $M \cong N$, then there exist invertible matrices P and Q such that $P\mathbf{M}Q = \mathbf{N}$. Let $\mathbf{M} = (m_{ij})$ and $P = (p_{ij})$. Clearly, for all $i, j \leq n$, we have $p_{ij} \notin \mathfrak{n}^2$, since $\mathfrak{n}^3 = 0$. If there exist some $p_{ab} \in \mathfrak{n}/\mathfrak{n}^2$, then either $p_{ab}m_{ij} = 0$ or $p_{ab} \in \mathfrak{n}^2$. However, all totally reflexive S-modules can be represented by a matrix with entries in $\mathfrak{n}/\mathfrak{n}^2$, 5.3.2.6. Therefore, P and Q only contain entries in the field. Since $M \cong N$, we have the following:

$$\begin{aligned} P\mathbf{M}Q &= \mathbf{N} \\ PxI_n + N(y,z)Q &= xI_n + M(y,z) \\ xPQ + PM(y,z)Q &= xI_n + N(y,z). \end{aligned}$$

Since $M(y,z)$ only contains entries of the form $\alpha y + \beta z$ the product $PM(y,z)Q$ cannot have entries with an x term which implies that

$$\begin{aligned} xPQ &= xI_n \\ x(PQ - I_n) &= [0] \quad \text{the zero matrix.} \end{aligned}$$

Therefore, either $PQ = I_n$ or $PQ - I_n \subset \mathrm{Ann}(x)I_n = (x)I_n$. However, $PQ - I_n \not\subseteq (x)I_n$ since $P, Q \in GL_n(k)$. Therefore, we have that $Q = P^{-1}$.

$\square$

5.4 A Bijection of the Isomorphism Classes

Theorem 5.4.0.10. *Let $\mathcal{TR}_n(S)$ be the set of all isomorphism classes of totally reflexive S-modules minimally generated by n generators. Then there exists a bijection,*

$$\vartheta : \mathcal{TR}_n(S) \leftrightarrow Mat_n(k) \times Mat_n(k)/\sim,$$

where for $(A_1, A_2), (B_1, B_2) \in Mat_n(k) \times Mat_n(k)$ we have $(A_1, A_2) \sim (B_1, B_2)$ if there exists a $P \in GL_n(k)$ such that $P(A_1, A_2)P^{-1} := (PA_1P^{-1}, PA_2P^{-1}) = (B_1, B_2)$.

Proof. This follows directly from 5.3.2.3 and Lemma 5.3.2.9.

$\square$

This description can be expanded to a class of rings. Consider rings of the form

$$S_i = k[X, Y_1, \ldots, Y_i]/\left(X^2, (Y_1, \ldots, Y_i)^2\right)$$

for $i \geq 2$. Thus, Mindy$= S \cong S_2$.

Theorem 5.4.0.11. *Let $\mathcal{TR}_n(S_i)$ be the set of all isomorphism classes of totally reflexive modules over S_i with n minimal generators. Then there exists a bijection,*

$$\vartheta : \mathcal{TR}_n(S_i) \leftrightarrow Mat_n(k)^i/\sim,$$

where $(A_1, \ldots, A_i) \sim (B_1, \ldots, B_i)$ if $P \in GL_n(k)$ such that
$P(A_1, \ldots, A_i)P^{-1} := (PA_1P^{-1}, \ldots, PA_iP^{-1}) = (B_1, \ldots, B_i)$.

Proof. Let $Q_i = k[[X, Y_1, \dots, Y_i]]/\left((Y_1, \dots, Y_i)^2\right)$, for $i \geq 2$. Then, Q_i is an embedded deformation of S_i. Let T be a totally reflexive module over S_i, by Theorem 2.3.3.2, $\mathrm{cx}_{Q_i} T = 1$. Hence, every totally acyclic complex is, at most, two periodic. Similar to the description 5.3.2.3, we obtain a commutitative diagram

$$\begin{tikzcd}
0 \arrow[r] & Q_i^b \arrow[r, "\partial"] \arrow[d, "x^2"'] & Q_i^b \arrow[r] \arrow[d, "x^2"] \arrow[dl, "\sigma"'] & T \arrow[r] \arrow[d, "x^2"] & 0 \\
0 \arrow[r] & Q_i^b \arrow[r, "\partial"] & Q_i^b \arrow[r] & T \arrow[r] & 0,
\end{tikzcd}$$

and Proposition 5.3.2.7 holds with the matrix pair

$$\boldsymbol{\varphi} = xI_n + M(y_1, \dots, y_i) \qquad \boldsymbol{\psi} = xI_n - M(y_1, \dots y_i).$$

$\square$

One wonders if this description can be expanded to cover other classes of rings. However, the description relays heavily on the fact that x is its own exact zero pair, and that it is the only element with this property. Another natural question that also arises here; are these types of rings the only $\mathfrak{m}^3 = 0$ ones that admit totally reflexive modules? This has been know not to be true by [15] for several years. However, one could ask the simpler question: are these types of rings the only $\mathfrak{m}^3$ ones that have exact zero pairs? Again, this is false by the same example in [15]. In fact, even imposing several restrictions on the ring $k[x_1, \dots, x_n)/I$, moreover on I, still yields to a negative answer.

Example 5.4.0.12. Let$(R, \mathfrak{m}) = k[a, b, c, d]/I$ where $I = (a^2, b^2, c^2, d^2, ac, bd, cd)$ and so we have $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. I is a homogenous ideal with monomial generators all of the same degree and thus R is very similar to S_3. However, one can easily check that no principal ideal generated by a monomial is an exact zero divisor. Although there does exist some, take the element $(a+b)+I$, whose exact zero pair is $(a-b)+I$. Also, by doing some simple computations, one can find that $T = \operatorname{coker} \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is totally reflexive.

Although we should note that Mindy is in fact the only $\mathfrak{n}^3 = 0 \neq \mathfrak{n}^2$ ring, up to isomorphisms, of the form $k[x,y,z]/I$ that contain exact zero pairs, where I is a homogenous ideal generated by monomials of the same degree.

Chapter 6

Building New Modules From Old Ones

Over a non-Gorenstein ring, if there exists one nontrivial totally reflexive module, then we know that there exists infinitely many more non-isomorphic indecomposable ones. Given that one nontrivial totally reflexive module, how do we find the others? Well, we already know that if we find a complete resolution of any totally reflexive module, then every syzygy module is also totally reflexive. However, in the case when $(R, \mathfrak{m})$ local with $\mathfrak{m}^3 = 0$, every syzygy module will be of the same size. In the case of Mindy, it is every more limiting, since any totally acyclic complex is at most two periodic. We now turn our attention more towards totally acyclic complexes, in order to investigate different methods for finding other totally reflexive modules when a nontrivial one is already known.

6.1 The Mapping Cone

The mapping cone, defined below, is an essential object in the triangulated category of totally acyclic complexes [30]. It is known that this category is closed under the mapping cone, meaning the mapping cone of two totally acyclic complexes is also a totally acyclic complex and thus its syzygies are totally reflexive modules.

Definition 6.1.0.1. [23, pg.19] Let f be a morphism between two complex $\mathbb{X} = (X_n, d_n^{\mathbb{X}})$ and $\mathbb{Y} = (Y_n, d_n^{\mathbb{Y}})$. The *mapping cone, M(f)* is the complex defined by

$$M(f)_n = X_{n-1} \oplus Y_n \quad \text{and} \quad d_n^{M(f)} = \begin{bmatrix} -d_{n-1}^{\mathbb{X}} & 0 \\ f_{n-1} & d_n^{\mathbb{Y}} \end{bmatrix}.$$

From the definition, we can clearly see what a presentation matrix of the mapping cone would look like. Therefore, we will give an example to illustrate it.

Example 6.1.0.2. Consider the following complexes over Mindy,

$$\begin{array}{llllllllll} \mathbb{X}: & \cdots \longrightarrow S^2 \xrightarrow{\begin{bmatrix} x & y \\ z & x \end{bmatrix}} S^2 \xrightarrow{\begin{bmatrix} x & -y \\ -z & x \end{bmatrix}} S^2 \xrightarrow{\begin{bmatrix} x & y \\ z & x \end{bmatrix}} S^2 \longrightarrow \cdots \\ \mathbb{Y}: & \cdots \longrightarrow S \xrightarrow{x} S \xrightarrow{x} S \xrightarrow{x} S \longrightarrow \cdots, \end{array}$$

and the chain map $f : \mathbb{X} \to \mathbb{Y}$. Defined $f_i : X_i \to Y_i$, to be the map represented by the matrix $[\ y \quad y\ ]$, for all $i \in \mathbb{Z}$. Therefore, the nth syzygy in the mapping cone of f has a presentation matrix

$$M(f)_n = \begin{bmatrix} -x & -y & 0 \\ -z & -x & 0 \\ y & y & x \end{bmatrix},$$

which is equivalent to

$$M(f)_n = \begin{bmatrix} x & y & 0 \\ z & x & 0 \\ -y & -y & x \end{bmatrix}.$$

By the work of chapter 5

$$M(f)_{n+1} = \begin{bmatrix} x & -y & 0 \\ -z & x & 0 \\ y & y & x \end{bmatrix},$$

since $M(f)_n$ and $M(f)_{n+1}$ are a matrix factorization of $x^2 I_n$.

The next statement is of a similar nature to the work done in chapter 3. In fact, we are building the same modules, just in a different context.

Proposition 6.1.0.3. *For* $(R, \mathfrak{m})$ *with* $\mathfrak{m}^3 = 0$*, let* T *be a totally reflexive* R*-module. If* T *has an upper triangular presentation matrix, then there exist complexes* $\mathbb{X}$ *and* $\mathbb{Y}$ *and a morphism* $f : \mathbb{X} \to \mathbb{Y}$ *all of which are obtained from a saturated TR-filtration of* T*, such that the mapping cone of* f *is a free resolution of* T*.*

Proof. Let $\mathbf{T}$ be an upper triangular presentation matrix of a totally reflexive R-module T, thus for some upper triangular matrix $\mathbf{A}$ of size $(n-1) \times (n-1)$ we have

$$\mathbf{T} = \left[\begin{array}{c|c} \mathbf{A} & \begin{matrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{matrix} \\ \hline 0 \ldots 0 & \alpha_n \end{array}\right]$$

where $\alpha_1, \ldots \alpha_n \in R$. Then, by 3.2.0.7, for some upper triangular matrix $\mathbf{B}$ and $\beta_1, \ldots \beta_n \in R$, a free resolution of T is of the form

$$\mathbb{F}_T : \quad \to \cdots R^n \xrightarrow{\left[\begin{array}{c|c} \mathbf{B} & \begin{matrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{matrix} \\ \hline 0 \ldots 0 & \beta_n \end{array}\right]} R^n \xrightarrow{\left[\begin{array}{c|c} \mathbf{A} & \begin{matrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{matrix} \\ \hline 0 \ldots 0 & \alpha_n \end{array}\right]} R^n \to 0.$$

Note that α_n and β_n are an exact pair od zero divisors. Define $\mathbb{X}$ to be a (deleted) free resolution of $\operatorname{coker}(\alpha_n)$, shifted by -1 and $\mathbb{Y}$ to be a (deleted) free resolution of $\operatorname{coker}\mathbf{A}$. That is,

$$\mathbb{X} \quad : \qquad \cdots \longrightarrow R \xrightarrow{\beta_n} R \xrightarrow{\alpha_n} R \xrightarrow{\beta_n} R \xrightarrow{\alpha_n} R \to 0$$

$$\mathbb{Y} \quad : \qquad \cdots \to R^{n-1} \xrightarrow{\mathbf{B}} R^{n-1} \xrightarrow{\mathbf{A}} R^{n-1} \to 0$$

Let $\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}$, and $\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}$. Now we define $f : \mathbb{X} \to \mathbb{Y}$ where $f_n : X_n \to Y_n$ is defined as $f_0 = -\alpha$, $f_1 = \beta$ and f_n is give by the Comparison Theorem 2.1.5.2 for all $n \geq 2$.

To see that f is a well defined morphism of complexes, consider the homology in the second degree of the above free resolution of $\operatorname{coker}\mathbf{B}$. Since free resolutions are exact we have

$$\left[\begin{array}{c|c} \mathbf{A} & \begin{matrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{matrix} \\ \hline 0 \ldots 0 & \alpha_n \end{array}\right] \left[\begin{array}{c|c} \mathbf{B} & \begin{matrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{matrix} \\ \hline 0 \ldots 0 & \beta_n \end{array}\right] = \left[\quad \mathbf{0} \quad \right],$$

where $[\mathbf{0}]$ is the $n \times n$ zero matrix. This lead us to the equality $\mathbf{A}\beta = \alpha\beta_n$, which is exactly what is needed to make the diagram

$$\begin{tikzcd}
\mathbb{X}: & \cdots \arrow[r] & R \arrow[r, "\alpha_n"] \arrow[d, "f_2"] & R \arrow[r, "\beta_n"] \arrow[d, "\beta"] & R \arrow[r, "\alpha_n"] \arrow[d, "-\alpha"] & R \arrow[r] \arrow[d, "f_{-1}"] & 0 \\
\mathbb{Y}: & \cdots \arrow[r] & R^{n-1} \arrow[r, "\mathbf{B}"] & R^{n-1} \arrow[r, "\mathbf{A}"] & R^{n-1} \arrow[r] & 0 &
\end{tikzcd}$$

commute. Thus, we have the mapping cone

$$M(f): \qquad \cdots \to R^n \xrightarrow{\begin{bmatrix} -\beta_n & 0 \\ \beta & \mathbf{B} \end{bmatrix}} R^n \xrightarrow{\begin{bmatrix} -\alpha_n & 0 \\ -\alpha & \mathbf{A} \end{bmatrix}} R^n \to 0,$$

and coker $\begin{bmatrix} -\alpha_n & 0 \\ -\alpha & \mathbf{A} \end{bmatrix}$ is isomorphic to T.

□

6.2 Tensor Products of Complexes

Originally the tensor product was defined to be between two modules. This notion is easily be extended to complexes. [31]

Definition 6.2.0.4. Let $(\mathbb{A}, d^A)$ and $(\mathbb{B}, d^B)$ be complexes of R-modules. The *tensor product complex,* $(\mathbb{A} \otimes_R \mathbb{B}, d^{\otimes})$ is the complex whose degree n component is defined as

$$(\mathbb{A} \otimes_R \mathbb{B})_n = \sum_{p+q=n} A_p \otimes_R B_q$$

and whose nth differential is given by

$$d_n^{\otimes} \left(d_p^A \otimes_R 1_{B_q} + (-1)^p 1_{A_p} \otimes_R d_q^B \right),$$

where $1_{\square}$ is the identity map.

One can easily see that the modules in the tensor product of complexes need not be finitely generated. However, if one of the complexes is of finite length, then it will be finitely generated. A complex $\mathbb{B}$ is said to be *bounded from the right* if $B_i = 0$ for all $i \gg 0$, and similarly from the left. A complex is *bounded* if it is bounded from both the left and the right, and hence is of finite length. It is shown in [14, 2.13] if one of the complexes, $\mathbb{B}$ is bounded on one side and $\mathbb{A}$ is an acyclic complex, then $\mathbb{A} \otimes_R \mathbb{B}$ is also acyclic.

A *perfect complex* is a complex of finitely generated free modules that is of finite length. Tensoring a perfect complex and an acyclic one also leads to an acyclic complex. Moreover, this holds for totally acyclic complexes as well.

Lemma 6.2.0.5. *[15, 2.5] Let $\mathbb{T}$ and $\mathbb{F}$ be complexes of finitely generated free R-modules. If $\mathbb{T}$ is totally acyclic and $\mathbb{F}$ is bounded from the left, then the complex $\mathbb{T} \otimes_R \mathbb{F}$ is totally acyclic.*

Since we wish all our modules to be finitely generated, we will place the further restriction on lemma 6.2.0.5 of $\mathbb{F}$ being a perfect complex. Something noteworthy of this lemma is the fact that $\mathbb{F}$ can have homology. Recall that the mapping cone construction required both complexes to be totally acyclic. This construction greatly increases the number of choices of complexes we can use. However, we can choose simple complexes which lead to complexes similar to what the mapping cone construction yields. We begin with such an example.

Example 6.2.0.6. Consider the following two complexes below both over Mindy, $S = k[x, y, z]/(x^2, y^2, z^2, yz)$.

$$\mathbb{A}: \qquad \cdots \longrightarrow S^{b_2} \xrightarrow{x-y} S^{b_1} \xrightarrow{x+y} SS^{b_0} \xrightarrow{x-y} S^{b_{-1}} \xrightarrow{x+y} S_{A_{-2}} \longrightarrow \cdots$$

$$\mathbb{P}: \qquad 0 \longrightarrow S^{c_2} \xrightarrow{z} S^{c_1} \xrightarrow{y} S^{c_0} \longrightarrow 0$$

By previous work in chapter 5, $\mathbb{A}$ is a totally acyclic complex and to see that $\mathbb{P}$ is a complex, and thus a perfect complex, note that

$$\operatorname{im}(z) \subset (y, z) = \ker(y).$$

Now we want to find a presentation matrix of the totally reflexive module $\Omega_0(\mathbb{A} \otimes_S \mathbb{P})$. First we compute the modules $(\mathbb{A} \otimes_s \mathbb{P})_0$ and $(\mathbb{A} \otimes_s \mathbb{P})_{-1}$,

$$\begin{aligned}
(\mathbb{A} \otimes_s \mathbb{P})_0 &= A_0 \otimes_S P_0 + A_{-1} \otimes_S P_1 + A_{-2} \otimes_S P_2 \\
&= S^{b_0} \otimes_S S^{c_0} + S^{b_{-1}} \otimes_S S^{c_1} + S^{b_{-2}} \otimes_S S^{c_2} \\
&\cong S^3
\end{aligned}$$

$$\begin{aligned}
(\mathbb{A} \otimes_s \mathbb{P})_{-1} &= A_{-1} \otimes_S P_0 + A_{-2} \otimes_S P_1 + A_{-3} \otimes_S P_2 \\
&= S^{b_{-1}} \otimes_S S^{c_0} + S^{b_{-2}} \otimes_S S^{c_1} + S^{b_{-3}} \otimes_S S^{c_2} \\
&\cong S^3,
\end{aligned}$$

and the differential $d_0^{\otimes}$

$$\begin{aligned}
d_0^{\otimes} &= \left(d_0^A \otimes_S 1_{P_0} + (-1)^0 1_{A_0} \otimes_S d_0^P\right) + \left(d_{-1}^A \otimes_S 1_{P_1} + (-1)^{-1} 1_{A_{-1}} \otimes_S d_1^P\right) \\
&\qquad + \left(d_{-2}^A \otimes_S 1_{P_2} + (-1)^{-2} 1_{A_{-2}} \otimes_S d_2^P\right) \\
&= \left([x-y] \otimes_S 1_{P_0} + 1_{A_0} \otimes_S [0]\right) + \left([x+y] \otimes_S 1_{P_1} - 1_{A_{-1}} \otimes_S [y]\right) \\
&\qquad + \left([x-y] \otimes_S 1_{P_2} + 1_{A_{-2}} \otimes_S [z]\right) \\
&\cong \begin{bmatrix} x-y & y & 0 \\ 0 & x+y & z \\ 0 & 0 & x-y \end{bmatrix}.
\end{aligned}$$

Thus, $\Omega_0(\mathbb{A} \otimes_S \mathbb{P}) \cong \operatorname{coker}\left(\begin{bmatrix} x-y & y & 0 \\ 0 & x+y & z \\ 0 & 0 & x-y \end{bmatrix}\right)$.

If we had chosen a $\mathbb{P}$ with a longer length, but still kept the same differentials and continued repeated them, then $\Omega_0(\mathbb{A} \otimes_S \mathbb{P})$ would have the same form as above;

the differentials of $\mathbb{A}$ down the main diagonal, and the differentials of $\mathbb{P}$ down the super diagonal. The only difference would be in the size of the matrix which represents it. Also, if the differential of $\mathbb{A}$ and $\mathbb{P}$ are chosen wisely, then the module obtain by tensoring them together would be indecomposable. By wisely, we mean that the differentials are represented by single elements which satisfies [15, 3.2], and the above example does just this.

In general, if we tensor a totally acyclic complex over R

$$\mathbb{A}: \qquad \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}^A} A_i \xrightarrow{d_i^A} A_{i-1} \longrightarrow \cdots,$$

with a perfect complex over R

$$\mathbb{P}: \qquad 0 \longrightarrow P_n \xrightarrow{d_n^P} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_1^P} P_0 \longrightarrow 0,$$

then $\Omega_i(\mathbb{A} \otimes_R \mathbb{P})$ has a block upper bidiagonal presentation matrix. In particular,

$$\Omega_i(\mathbb{A}\otimes_R\mathbb{P}) \cong \operatorname{coker} \begin{bmatrix} 1_{P_0} \otimes d_i^A & (-1)^{i-1} 1_{A_{i-1}} \otimes d_1^P & & & \\ & 1_{P_1} \otimes d_{i-1}^A & (-1)^{i-2} 1_{A_{i-2}} \otimes d_2^P & & \mathbf{0} \\ & & \ddots & \ddots & \\ & & & 1_{P_{n-1}} \otimes d_{i-(n-1)}^A & (-1)^{i-n} 1_{A_{i-n}} \otimes d_n^P \\ \mathbf{0} & & & & 1_{P_n} \otimes d_{i-n}^A \end{bmatrix}.$$

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Biographical Statement

Denise Amanda Rangel was born in Panama City, Panama in 1983 to Heidi and Frank Rangel. She is the second of three daughters. Her primary school education was received at various public schools located through out the southwest region of the United States, with her high school diploma being awarded in San Antonio, Texas in 2001. On June 21, 2014, she married Alex Tracy, a warrant officer in the United States Marine Corps.

After many changes in schools and majors, Denise received a Bachelor of Science degree in Mathematics from the University of North Carolina at Greensboro in 2008. She continued there and was awarded a Master of Arts degree in Mathematics in 2010. Following the completion of this degree she moved back to Texas and began doctoral studies at the University of Texas in Arlington. In August 2014, she earned a Ph.D. in Mathematics under the direction of David Jorgensen.

Denise's research interests lie in commutative algebra and representation theory, with particular interest in the use of homological methods.