SAMPLE SOLUTIONS OF STOCHASTIC BOUNDARY VALUE PROBLEMS

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ABSTRACT

We prove existence theorems for nonlinear stochastic Sturm-Liouville problems which improve results from [4]. In the simplest case this is done by means of a known result about measurable selections of multivalued maps and a new fixed point theorem for stochastic nonlinear operators which is more realistic than existing ones.

1. INTRODUCTION

We consider the boundary value problem

\[ x'' = f(t,x,x',\omega) \text{ in } J = [0,1] \]

\[ B_i(\omega)x(\cdot,\omega) = \alpha_i(\omega)x(1,\omega) + (-1)^{i+1}b_i(\omega)x'(1,\omega) = b_i(\omega) \]

for \( i = 0,1 \).

We shall always assume that we have a probability measure space \((\Omega, \mathcal{A}, P)\), the functions \( \alpha_i, b_i, B_i : \Omega \rightarrow \mathbb{R} \) are measurable, Green's function \( G(t,s,\omega) \) for

\[ x^* = 0 \text{ and } B_i(\omega)x(\cdot,\omega) = 0 \text{ for } i = 0,1 \]

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exists for every \( \omega \in \Omega \) and the function \( f: J \times R^2 \times \Omega \rightarrow R \) satisfies at least condition

\[(C_1) \quad f(\cdot, p, q, \cdot) \text{ is } (L_1 \otimes \Omega, D_1)\text{-measurable for all } (p, q) \text{ and } f(t, \cdot, \cdot, \omega) \text{ is continuous for all } (t, \omega); \text{ here } L_2 \text{ are the Lebesgue-measurable subsets of } J \text{ and } D_1 \text{ is the Borel } \sigma\text{-algebra of } R.\]

This assumption is reasonable since in typical situations one gets \( f \) from a deterministic \( g \) by replacing some coefficients etc. by stochastic processes \( z \), i.e. \( f(t, p, q, \omega) = g(t, p, q, z(t, \omega)) \), where some interesting \( z \) do not have continuous but measurable samples \( z(\cdot, \omega) \) and \( z \) is measurable in \( (t, \omega) \).

By a sample solution of (1), (2) we understand a stochastic process \( x: J \times \Omega \rightarrow R \), satisfying the boundary condition (2) on \( \Omega \), such that \( x'(\cdot, \omega) \) is a.c. (absolutely continuous) and \( x(\cdot, \omega) \) satisfies (1) \( \mu \text{-a.e. (} \mu \text{ the one-dimensional Lebesgue measure)} \), for all \( \omega \in \Omega \). Since we shall also assume some integrability conditions on \( f \), the existence of a sample solution to (1), (2) is therefore equivalent to

\[(4) \quad x(t, \omega) = \psi(t, \omega) + \int_0^1 G(t, s, \omega) f(s, x(s, \omega), x'(s, \omega), \omega) \, ds,\]

i.e. to the existence of an \( x: J \times \Omega \rightarrow R \) such that \( x(\cdot, \omega) \) is continuously differentiable, \( x(t, \cdot) \) is measurable and (4) is satisfied on \( J \times \Omega \); here \( \psi(\cdot, \omega) \in C^2(J) \) is the unique solution of \( \psi'' = 0 \) and \( B_i(\omega) \psi(\cdot, \omega) = b_i(\omega) \) for \( i = 0, 1 \). By the definition of \( \psi \) and \( G \) it is easy to see that these functions are measurable in \( \omega \). We start with the basic case where \( f(t, \cdot, \cdot, \omega) \) is bounded. Then we try to extend these results along traditional lines (upper- and lower solutions, Nagumo condition, etc.) to right hand sides \( f \) which are unbounded in \( (p, q) \).
2. THE CASE WHERE f IS BOUNDED IN (p,q).

In addition to the assumptions mentioned in the first section we now also assume that f satisfies

\[ |f(t,p,q,\omega)| \leq M(t,\omega) \text{ on } J \times \mathbb{R}^2 \times \Omega, \text{ with a measurable } M \text{ such that } \int_0^1 M(t,\omega) dt < \infty \text{ on } \Omega. \]

**Theorem 1.** Let the functions in (2) be measurable and such that Green's function for (3) exists on \( \Omega \). Suppose also that f satisfies \((C_1)\) and \((C_2)\). Then problem (1), (2) has a sample solution.

For the proof of this result we need

**Lemma 1.** Let \((\Omega, \mathcal{G})\) be a measurable space, \((X, d)\) a separable metric space and \(G : \Omega \rightarrow 2^X \sim (\omega)\) a multivalued map such that \(G(\omega)\) is complete for all \(\omega \in \Omega\) and \(\rho(x, G(\cdot))\) is measurable for every \(x \in X\) \((\rho(x, A)\) is the distance from \(x\) to \(A\)). Then \(G\) admits an \((\mathcal{G}, \mathcal{A})\)-measurable selection, i.e., an \(g : \Omega \rightarrow X\) such that \(g^{-1}(A) \in \mathcal{A}\) and \(g(\omega) \in G(\omega)\) for all \(\omega \in \Omega\) \((\mathcal{A}\) is the Borel \(\sigma\)-algebra of \((X, d)\)).

This is Theorem III.6 in [3] or Theorem 24.3 in [7] since the measurability of \(\rho(x, G(\cdot))\) for every \(x\) is equivalent to \(G^{-1}(A) \in \mathcal{A}\) for every open \(A \subseteq X\). We also need the following fixed point theorem.

**Lemma 2.** Let \(X\) be a separable Banach space, \((\Omega, \mathcal{G}, P)\) a probability measure space and \(r : \Omega \rightarrow \mathbb{R}\) measurable. Let \(F : \Omega \times X \rightarrow X\) be such that \(F(\cdot, x)\) is measurable for every \(x \in X\), \(F(\omega, \cdot)\) is continuous for every \(\omega \in \Omega\) and \(F(\omega, \cdot) : B_{r(\omega)}(o) = \{x \in X : |x| \leq r(\omega)\} \rightarrow B_{r(\omega)}(o)\) is compact (i.e., \(F(\omega, B_{r(\omega)}(o))\) compact) for every \(\omega \in \Omega\). Then there exists a measurable \(x : \Omega \rightarrow X\) such that \(x(\omega) \in B_{r(\omega)}(o)\) and \(F(\omega, x(\omega)) = x(\omega)\) on \(\Omega\).
Proof. Let \( B_0 = B_r(0) \) and \( R_0 : X \to B_0 \) the retraction of \( X \) onto \( B_0 \), defined by \( R_0 x = x \) on \( B_0 \) and \( R_0 x = r(x)/|x| \) for \( |x| > r \).

Define \( \tilde{F} : \mathcal{X} \times X \to X \) by \( \tilde{F}(\omega,x) = F(\omega,R_0 x) \) and let \( S(\omega) = \{ x \in X : \tilde{F}(\omega,x) = x \} \).

Since \( \tilde{F}(\omega,\cdot) \) is continuous and \( \tilde{F}(\omega,x) = F(\omega,R_0 x) \) is relatively compact, \( S(\omega) \) is nonempty and compact, by Schauder's fixed point theorem, \( S(\omega) = \{ x \in B_0 : F(\omega,x) = x \} \) by definition of \( \tilde{F} \). Furthermore, \( \tilde{F}(\omega,.) \) is measurable since, for \( B \in \mathcal{B} \), we have

\[
\{ \omega : \tilde{F}(\omega,x) \in B \} = (F(\omega,.)^{-1}(B) \cap r^{-1}([r,\infty))) \cup \\
U \{ (\omega : F(\omega,r(\omega)x/|x|) \in B) \cap r^{-1}(0,|x|) \},
\]

where the first intersection is obviously in \( \mathcal{F} \) while the second one is in \( \mathcal{A} \) since \( r(\cdot) \) is the pointwise limit of step functions.

We want to apply Lemma 1 to \( S(\cdot) \) and know already that \( S(\omega) \) is complete. Given \( x_0 \in X \), to show measurability of \( \rho(x_0,\cdot) \) means to show \( \{ \omega : \rho(x_0,S(\omega)) \leq \alpha \} \in \mathcal{A} \) for all \( \alpha > 0 \). For \( \alpha = 0 \) we have \( \{ \omega : x_0 \in S(\omega) \} = \tilde{F}(\omega,x_0)^{-1}(\{ x_0 \}) \in \mathcal{A} \). For \( \alpha > 0 \), \( \rho(x_0,S(\omega)) \leq \alpha \) means \( S(\omega) \cap B_\alpha(x_0) + 0 \). Since \( X \) is separable we have a countable dense subset \( \{ x_j : j \in \mathbb{N} \} \) of \( B_\alpha(x_0) \) and therefore

\[
\{ \omega : \rho(x_0,S(\omega)) \leq \alpha \} = \cap_{n \geq 1} \cup \{ \omega : |x_j - \tilde{F}(\omega,x_j)| \leq 1/n \} \in \mathcal{A},
\]

since \( \tilde{F}(\omega,x) \) is relatively compact and \( \tilde{F}(\cdot,x_j)^{-1}(B_1/n(x_j)) \in \mathcal{A} \).

Hence Lemma 1 applies.

q.e.d.

Now the proof to Theorem 1 is a direct consequence of Lemma 2:

Choose \( X = C^1(J) \), define \( F(\omega,x)(t) = \psi(t,\omega) + \int_0^1 G(t,s,\omega)f(s,x(s),x'(s),\omega)ds \) and

\[
r(\omega) = \max \{ |\psi(\cdot,\omega) + \int_0^1 G(\cdot,s,\omega)[\mathcal{N}(s,\omega)]ds|_0, |\psi'(\cdot,\omega) + \\
+ \int_0^1 G(\cdot,s,\omega)[\mathcal{N}(s,\omega)]ds|_0 \},
\]

where \( |y|_0 = \max_j |y(t)| \) for \( y \in C(J) \).
Remarks 1. Theorem 1 remains true for $\mathbb{R}^n$-valued functions, i.e. finite systems (1), (2), and it improves Theorem 2.1 in [4] which has the restrictive hypothesis that $f$ be continuous in $t$ and
\[ \int_0^1 M(t, \omega) dt \leq C \iff (P-a.e.). \]

2. Lemma 2 is much more realistic than the procedure chosen in a flood of papers on stochastic fixed point theorems, where the set in which one looks for fixed points is not allowed to vary with $\omega$; see e.g. the survey [2]. The result remains true if $F(\omega, \cdot)$:
\[ B_\omega = B_{r(\omega)}(0) \to B_\omega \] is only $\gamma$-condensing, i.e. $\gamma(F(\omega, A)) < \gamma(A)$ for all $A \subset B_\omega$ with $\gamma(A) > 0$, where $\gamma$ is either Kuratowski's or the ball-measure of noncompactness (see e.g. [9] or § 7 in [7]), since $\gamma(R_\omega(A)) \leq \gamma(A)$ for all bounded $A \subset X$.

3. Theorem 1 remains true in any Banach space $X$ (instead of $\mathbb{R}$ or $\mathbb{R}^n$; the functions $\alpha_i, B_i$ in (2) still realvalued) provided we add a condition on $f$ guaranteeing the relative compactness of
\[ \{F(\omega, x)(t) : x \in B_\omega\} \] . The natural formulation of such extra conditions is also in terms of measures of noncompactness, i.e. estimates of type
\[ \gamma(f(t, A, B, \omega)) \leq \rho(t, \gamma(A), \gamma(B), \omega) \] for bounded $A, B \subset X$
which imply
\[ \max \{\gamma(F(\omega, B_\omega)(t)), \gamma(F(\omega, B_\omega)'(t))\} = 0 \] for all $(t, \omega) \in J \times \Omega$, since "$\gamma(A) = 0$ iff $A$ is relatively compact" is one of the properties of such measures of noncompactness; a basic paper for deterministic problems (1), (2) in any Banach space is [10].

3. CONSEQUENCES OF THEOREM 1.

We first consider the case where $f$ has at most linear growth in $p$ and $q$.

Theorem 2. Let the functions in (2) be as in Theorem 1. Suppose that $f$ satisfies $(C_1)$ and
(5) \[ |f(t,p,q,\omega)| \leq k_1(t,\omega)|p| + k_2(t,\omega)|q| + k_3(t,\omega) \text{ on } J \times R^2 \times \Omega, \]

with measurable \( k_i \geq 0 \) such that (for every \( \omega \in \Omega \)) \( \int k_i(t,\omega)dt < \infty \) and the linear operator \( H(\omega) : C(J)^2 \rightarrow C(J)^2 \), defined by

\[
H(\omega)(x,y)(t) = \left( \int_0^1 |G(t,s,\omega)|p(s,x(s),y(s),\omega)|ds, \int_0^1 G_c(t,s,\omega)|p(\cdot,\cdot,\cdot,\omega)|ds \right)
\]

has spectral radius < 1; here \( p(t,p,q,\omega) = k_1(t,\omega)|p| + k_2(t,\omega)|q| \).

Then (1), (2) has a sample solution.

Proof. The reduction to the situation of Theorem 1 goes as follows. For a sample solution \( x \) of (1), (2) we have the a-priori estimate (consider (4))

\[
|x(\cdot,\omega)|, |x'(\cdot,\omega)| \leq H(\omega)(|x(\cdot,\omega)|, |x'(\cdot,\omega)|) + (M_0(\cdot,\omega), M_1(\cdot,\omega))
\]

with

\[
M_0(t,\omega) = |\psi(t,\omega)| + \int_0^1 |G(t,s,\omega)|k_3(s,\omega)ds,
\]

\[
M_1(t,\omega) = |\psi'(t,\omega)| + \int_0^1 |G_c(t,s,\omega)|k_3(s,\omega)ds,
\]

hence \( |x(t,\omega)|, |x'(t,\omega)| \leq (I-H(\omega))^{-1}(M_0(\cdot,\omega), M_1(\cdot,\omega)) \), and therefore \( \max|x(t,\omega)| \leq M_2(\omega) \) and \( \max|x'(t,\omega)| \leq M_2(\omega) \) with a measurable \( M_2(\cdot) \) since \( (I-H(\omega))^{-1} = \frac{1}{\det H(\omega)}H(\omega)^{-1} \). Now, consider \( \eta : R \times \Omega \rightarrow [0,1] \) continuous in \( \tau \) and measurable in \( \omega \) such that \( \eta(\tau,\omega) = 1 \) for \( |\tau| \leq M_2(\omega) \) and \( \eta(\tau,\omega) = 0 \) for \( |\tau| \geq M_2(\omega) + 1 \). Then \( \tilde{f} : J \times R^2 \times \Omega \rightarrow R \), defined by

\[
\tilde{f}(t,p,q,\omega) = \eta(p,\omega)\eta(q,\omega)f(t,p,q,\omega),
\]

satisfies (C_1) and (C_2). Consequently, Theorem 1 yields a sample solution \( x \) of (1), (2) with \( \tilde{f} \) instead of \( f \). Since \( \tilde{f} \) also satisfies (5) we get \( \eta(x(t,\omega),\omega) = \eta(x'(t,\omega),\omega) = 1 \) and therefore \( x \) is a sample solution of (1), (2).

q.e.d.

As another extension of Theorem 1 we consider the situation in which the behaviour of \( f \) with respect to \( x \) is controlled by upper and lower solutions and its behaviour in \( x' \) is governed by Nagumo's condition which means essentially at most quadratic growth in \( x' \).
If \( f \) is only measurable in \( t \), a reasonable definition of (sample) upper and lower solutions goes as follows.

A function \( \alpha: J \times \Omega \to \mathbb{R} \) is a lower (sample) solution of (1), (2) if \( \alpha(t, \cdot) \) is measurable for all \( t \), \( \alpha'(\cdot, \omega) \) is a.e. for every \( \omega \), \( B_1(\omega) \alpha'(\cdot, \omega) \leq b_1(\omega) \) on \( \Omega \) and for every \( \omega \in \Omega \) we have
\[
\alpha''(t, \omega) \geq f(t, \alpha(t, \omega), \alpha'(t, \omega), \omega) \mu\text{-a.e. on } J.
\]
The definition of an upper (sample) solution \( \beta \) is obtained by replacing \( \alpha \) by \( \beta \) and reversing the inequalities.

If \( \alpha \) is a lower and \( \beta \) is an upper solution for (1), (2) such that \( \alpha \leq \beta \) then we say that \( f \) satisfies a Nagumo condition w.r. to \((\alpha, \beta)\) if
\[
(6) \quad |f(t, p, q, \omega)| \leq h(|q|, \omega) \quad \text{for } (t, \omega) \in J \times \Omega, \alpha(t, \omega) \leq p \leq \beta(t, \omega), \quad q \in \mathbb{R} \]
with \( h: \mathbb{R} \times \Omega \to \mathbb{R}_+ - \{0\} \) such that \( h(\cdot, \cdot) \) is continuous, \( h(t, \cdot) \) measurable and
\[
(7) \quad \int_{\Omega} \frac{\tau d\tau}{\lambda(\omega)} \geq \max_{\Omega} \beta(t, \omega) - \min_{\Omega} \alpha(t, \omega) \quad \text{on } \Omega
\]
for some measurable \( N(\cdot) \) and \( \lambda(\omega) = \max\{|\beta(0, \omega)| - \alpha(1, \omega)|, |\beta(1, \omega)| - \alpha(0, \omega)|\} \).

Such a function \( N(\cdot) \) exists if \( \int_{\Omega} \frac{\tau d\tau}{\lambda(\omega) h(t, \omega)} = \infty \) on \( \Omega \). To see this notice that \( H(s, \omega) := \int_{\Omega} \frac{\tau d\tau}{\lambda(\omega), h(t, \omega)} \) is strictly increasing in \( s \), denote the right hand side of (7) by \( \psi(\omega) \) and define \( N(\omega) \) by \( H(N(\omega), \omega) = \psi(\omega) + 1 \); then \( \{\omega: N(\omega) > r\} = \{\omega: \psi(\omega) + 1 < H(r, \omega)\} \in \mathcal{C} \)
for every \( r \in \mathbb{R} \) and therefore \( N(\cdot) \) is measurable. Now we have

**Theorem 3.** Let the functions in (2) be as in Theorem 1 and such that \( \alpha_i \geq 0, \beta_i \geq 0 \) and \( \alpha_i(\omega) + \beta_i(\omega) > 0 \) on \( \Omega \) for \( i = 0, 1 \).

Let \( f: J \times \mathbb{R}^2 \times \Omega \to \mathbb{R} \) satisfy condition \((C_1)\). Suppose also that there exist a lower sample solution \( \alpha \) and an upper sample solution \( \beta \) such that \( \alpha(t, \omega) \leq \beta(t, \omega) \) on \( J \times \Omega \) and \( f \) satisfies a Nagumo condition w.r. to \((\alpha, \beta)\). Then (1), (2) has a sample solution.
Proof. 1. Let $c(\omega) = 1 + \max\{N(\omega), \max_j |\alpha_j'(t,\omega)|, \max_j |\rho(t,\omega)|\}$ with $N(\cdot)$ from (7). Define $F: J \times \mathbb{R}^2 \times \Omega \to \mathbb{R}$ by

$$F(t,p,q,\omega) = f(t,\rho_1(t,p,\omega),\rho_2(q,\omega),\omega) + r(t,p,\omega)$$

where

$$\rho_1(t,p,\omega) = \max(\alpha(t,\omega), \min(p,\beta(t,\omega)))$$

$$\rho_2(q,\omega) = \max(-c(\omega), \min(q,c(\omega)))$$

$$r(t,p,\omega) = \begin{cases} \frac{p-\beta(t,\omega)}{1+p^2} & \text{if } p > \beta(t,\omega) \\ 0 & \text{if } \alpha(t,\omega) \leq p \leq \beta(t,\omega) \\ \frac{p-\alpha(t,\omega)}{1+p^2} & \text{if } p < \alpha(t,\omega) \end{cases}$$

It is easy to see that $F$ satisfies $(C_1)$ and $(C_2)$, and therefore (1), (2) with $F$ instead of $f$ has a sample solution and if we can show that there is a sample solution $x$ satisfying $\alpha(t,\omega) \leq x(t,\omega) \leq \beta(t,\omega)$ and $|x'(t,\omega)| \leq N(\omega)$ on $J \times \Omega$ then we are done.

2. Since $f$ satisfies only Carathéodory's condition, the comparison $\alpha \leq x \leq \beta$ is not directly obtainable. Therefore, we first approximate $F$ by functions $F_n$ which are measurable in $(t,\omega)$ and locally Lipschitz in $(p,q)$ such that $(C_2)$ holds with the same $M$ for all $F_n$ and $F_n(t,p,q,\omega) \to F(t,p,q,\omega)$ as $n \to \infty$, uniformly in $(p,q)$ for every $(t,\omega)$. Then we let (omitting $\omega$),

$$\alpha_n(t) = \alpha(t) - 1/n \quad \text{and} \quad \beta_n(t) = \beta(t) + 1/n,$$

$$g_n(t,p,q) = F_n(t,p,q) + \frac{p-\alpha_n(t)}{\beta_n(t)-\alpha_n(t)} [f(t,\beta,\beta')-F_n(t,\beta_n,\beta_n')]$$

$$+ \frac{\beta_n(t)-p}{\beta_n(t)-\alpha_n(t)} [f(t,\alpha,\alpha')-F_n(t,\alpha_n,\alpha_n')]$$

$$G_n(t,p,q) = g_n(t,\rho_n(t,p),q) + r_n(t,p),$$

where $\rho_n$ and $r_n$ are $\rho_1$ and $r$ with $(\alpha_n,\beta_n)$ instead of $(\alpha,\beta)$. By these definitions and Theorem 1 it is obvious that (1), (2) with $G_n$ instead of $f$ has a sample solution $x_n$ and $\alpha_n, \beta_n$ are lower and upper sample solutions, respectively, of this new problem.
If we are able to show that $\alpha_n \leq x_n \leq \beta_n$ then we can use the idea of the proof to Theorem 1 in [6] to obtain a sample solution $x$ of (1), (2) with $F$ instead of $f$ satisfying $\alpha \leq x \leq \beta$ on $J = \Omega$.

3. We only show $\alpha_n \leq x_n$ since the inequality $x_n \leq \beta_n$ then follows similarly. Suppose on the contrary that $\alpha_n \leq x_n$ is wrong. Then there exists an $\omega$ such that $\alpha_n(\cdot, \omega) - x_n(\cdot, \omega)$ has a positive maximum, say at $t_0 \in J$. In the sequel we shall omit $\omega$.

Suppose that $t_0 \in (0, 1)$. Then we have $\alpha_n(t) \geq x_n(t) + \epsilon$ in $J_\delta = [t_0, t_0 + \delta)$ for some positive $\epsilon$ and $\delta$, $\alpha_n(t_0) = x_n(t_0)$ and $|x_n(t)| < c(\omega)$ in $J_\delta$, by the continuity of $x_n$ and the definition of $c(\omega)$. Therefore

$$\begin{cases}
\gamma'(t) \geq g_n(t, \alpha_n(t), \gamma(t)), \\
z'(t) \leq g_n(t, \alpha_n(t), z(t)) - \epsilon_1
\end{cases}$$

(u-a.e. on $J_\delta$), $\gamma(t_0) = z(t_0)$

for $\gamma(t) = x_n(t)$ and $z(t) = x_n(t)$. Hence $\gamma(t) \geq r_{**}(t)$ and $z(t) \leq r^*(t)$, where $r_{**}$ and $r^*$ are the minimal and the maximal solution of

$u' = q_n(t, \alpha_n(t), u)$, $u(t_0) = \gamma(t_0)$, respectively (see e.g. Theorem 1.10.1 in [8]). But $r_{**} = r^*$ since $g_n$ is locally Lipschitz in $q$, hence integration yields $\alpha(t) - x(t) \geq 0$ in $J_\delta$, for some $c > 0$ and therefore $\gamma(t) = z(t)$ in $J_\delta$, a contradiction to (8) since integration of (0) yields $\gamma(t) \geq z(t) + \epsilon_1(t - t_0)$ in $J_\delta$ with $\epsilon_1 > 0$. Hence $\alpha - x$ cannot attain its maximum at a $t_0 \in (0, 1)$.

Suppose next that $t_0 = 0$. Then $0 < \alpha_n(0) - x_n(0) = \alpha_n(t_0) - x_n(t_0)$ for all small $t > 0$ implies $\alpha_n(t) \leq x_n(t)$ for $t > 0$. By the first boundary condition we also have

$$\alpha_n(0) - \alpha_n(0) - \beta_n(x_n(0) - \alpha_n(0)) \geq 0$$

and therefore $\alpha_n(0) = x_n(0)$, since $\alpha_0 \geq 0$, $\beta_n \geq 0$ and $\alpha_0 - \beta_n > 0$.

Thus we may proceed as in the case $t_0 \in (0, 1)$ and likewise one sees that the maximum can also not be at $t_0 = 1$. Hence $\alpha_n(t, \omega) \leq x_n(t, \omega) \leq \beta(t, \omega)$ on $J = \Omega$.

3. We now have a sample solution $x$ satisfying $\alpha(t, \omega) \leq x(t, \omega) \leq \beta(t, \omega)$, and therefore
\[ x''(t,\omega) = f(t, x(t,\omega), a_2(x'(t,\omega),\omega)) \mu\text{-a.e. in } J \text{ for every } \omega \in \Omega. \]

Since \( c(\omega) > N(\omega) \), we may now repeat (for every \( \omega \in \Omega \)) the argument given in the proof to Theorem 1.4.1. in [1] to see that
\[ |x'(t,\omega)| \leq N(\omega) \text{ on } J \times \Omega. \]

\[ \text{q.e.d.} \]

Notice that the Nagumo condition implies boundedness of \( f \) in \( t \).
The question is whether this can be relaxed keeping the quadratic growth in \( x' \). For an interesting application of stochastic BVPs (1), (2) see [5].

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