COMMUNITY RESILIENCE IN COLLABORATIVE LEARNING

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Abstract. This paper introduces a simplified dynamical systems framework for the study of the mechanisms behind the growth of cooperative learning in large communities. We begin from the simplifying assumption that individual-based learning focuses on increasing the individual’s “fitness” while collaborative learning may result in the increase of the group’s fitness. It is not the objective of this paper to decide which form of learning is more effective but rather to identify what types of social communities of learners can be constructed via collaborative learning. The potential value of our simplified framework is inspired by the tension observed between the theories of intellectual development (individual to collective or vice versa) identified with the views of Piaget and Vygotsky. Here they are mediated by concepts and ideas from the fields of epidemiology and evolutionary biology. The community is generated from sequences of successful “contacts” between various types of individuals, which generate multiple nonlinearities in the corresponding differential equations that form the model. A bifurcation analysis of the model provides an explanation for the impact of individual learning on community intellectual development, and for the resilience of communities constructed via multilevel epidemiological contact processes, which can survive even under conditions that would not allow them to arise. This simple cooperative framework thus addresses the generalized belief that sharp community thresholds characterize separate learning cultures. Finally, we provide an example of an application of the model. The example is autobiographical as we are members of the population in this “experiment”.

1. Introduction. In recent years, collaborative learning has been at the forefront of numerous classroom-level education reform efforts. At the same time, the top-down approaches of governmental education policies have emphasized individual achievement as a primary measure of success (through high-stakes standardized testing). It is natural to ask to what extent collective and individual learning are aligned. Many studies have described how these two processes interact within a single classroom, and others have developed snapshots of individual achievement on larger scales, but a population biology perspective offers a way to describe how the two impact each other continuously over time. Recent successes in applying mathematical models to the dynamics of complex social phenomena (from eating disorders [19] and drug use [27] to church growth [21]) inspired the present study of how such a model, based on techniques from population biology and epidemiology, can embody the dynamics of the interactions between these two aspects of learning. Confrey’s examination [9, 10, 11] of the theories of intellectual development based on the views of Vygotsky and Piaget, along with our work as faculty, participants and mentors in a collaborative learning summer research experience, instigated these efforts to build a simple model of a collaborative learning culture. A framework that incorporates the impact of individuals in the development of a community of learners is developed and used to challenge standard metaphors of “cultural” conversion.

Vygotsky believed that development begins at the social level and moves towards individual internalization, while Piaget held that development proceeds from the individual to the social world. Confrey [9, 10, 11] tried to bridge these different philosophies (highly simplified by the above characterization) through “the evolutionary biology metaphor” by incorporating environmental concerns. Confrey’s evolutionary metaphor of intellectual development assumes that learning is driven by individual and social forces; that is, it assumes that learning takes place in a landscape that is not independent of the nature of individual and social (group or
community) interactions. Our model does not differentiate explicitly between these two philosophies of learning. Instead, it focuses on modeling the impact that a culture of learning (a population-level characteristic) has on a community of learners.

The objective in this paper is not to define individual or community learning as some measure of individual or group fitness but rather, the goal here is to look at the impact of the intensity of individual interactions on community structure and (in a future paper) the community impact on the individual as a feedback mechanism. The evolutionary metaphor used in this paper implicitly assumes a unit (e.g., the individual) and a level (e.g., the community). Individuals (participants) in this community are defined as those who somewhat accept the view that learning “from others and from each other” is a desirable individual trait and community goal. Here, the impact of learning on the intellectual community landscape is based on the view that learning is not a unidirectional process. Local changes may impact the structure of the overall intellectual landscape and vice versa.

Our approach uses standard evolutionary landscape metaphors (cf. [13]). The individual process is modeled as an epidemiological (contact) process. The assumption here is that individual-based learning uses local information to increase personal “fitness” while collaborative learners (who also have the same personal goal) use and provide community support to reach the same objective. An important goal of collaborative learning is therefore to increase the mean fitness of the group. It is hoped that in the process, the fitness of most individual members is also improved. Despite our positive recent experiences with collaborative learning, here we only model the impact that a social interaction structure may have on the establishment of a community of learners. In summary, we focus on the impact of mixing social structures, a purely social phenomenon, on the construction of a community of highly interactive learners. Whether this community generates benefits that surpass typical approaches to learning is not addressed at any level, although some preliminary evidence is provided in the next section.

Gladwell [18] popularized the “tipping point” concept as the sharp community threshold that characterizes alternative cultures generated by strong social (positive and negative) forces. This idea, in our setting, asserts that one must cross a tipping point to move from one culture to another. The crossing of such a threshold would then substantially alter the structure of the interactions within a learning community. Such threshold quantities, cast as reproductive numbers, are common in population biology and epidemiology; however, some models with multiple nonlinear (contact) processes exhibit a property known as a “backward bifurcation,” in which an alternative culture can persist even when the tipping point threshold has not been crossed. The nature of social interactions may therefore change the nature of the landscape away from one described by a single threshold (for an example, see [27]).

Analysis in the present study focuses on the impact that cooperative learning has on the resilience and persistence of a community of learners as they interact (the learning landscape). The paper is organized as follows: Section 2 introduces a working definition of cooperative learning and briefly reviews some of the literature on this subject; Section 3 introduces a simple model for cooperative learning; Section 4 presents some of the analysis of the model and discusses its consequences; Section 5 outlines the conclusions of our analysis and discusses some extensions; and Section 6 describes an application of the model. We close this article with some thoughts and concluding remarks.
2. Cooperative learning: A brief overview. Norwood [23] defines cooperative learning as

“...a set of instructional strategies which bring students of all performance levels together to work in small, mixed-ability learning groups...for problem solving experiences. The students in these groups are not only responsible for learning the material being taught in class, but also for helping their group members learn the material.”

The development of an environment that is conducive to, and supportive of (for a long period of time), cooperative learning may be difficult at first [28]. Here, it is assumed (for the purpose of framing this discussion) that cooperative learning, wherever and whenever it takes place, is carried out in the context of Norwood’s definition. The use of Norwood’s definition, however, is not essential for the application and development of our model. This is not a criticism of Norwood’s approach but rather an inherent limitation of our model of collaboration at the population level. Nevertheless, a generous interpretation of our analysis suggests that cooperative learning may bring students together into situations where the impact of individual learning on others is high enough that a strong culture of learning is established (community intellectual resilience). Our review of some of the literature on collaborative learning supports the view that there are learning cultures which are worth “transmitting”. In other words, we hope that the reading of the literature on collaborative learning in the context of our modeling framework may point out the potential importance that mixing structures may have on the establishment of positive learning cultures that are or are not resilient.

There is plenty of data that demonstrates the positive impact that cooperative learning can have on a community of individuals. In fact, it is known that cooperative learning promotes achievement as well as other positive affective outcomes at the elementary and middle grade levels [28]. Students also benefit from improved social skills by working in groups [25, 26]. Working together, students learn to be tactful, to manage conflicts effectively and to respect the opinions of others [2, 28]. In particular, as collaborative skills become of increasing importance in social and professional life, students’ academic experiences are of more value when they include exercises in cooperative efforts. As Brown et al. [4] point out:

“Students who are taught individually rather than collaboratively can fail to develop skills needed for collaborative work. In the collaborative conditions of the workplace, knowing how to work collaboratively is increasingly important. If people are going to learn and work in conjunction with others, they must be given the situated opportunities to develop those skills.”

Of course, the fact that we live in a global economy has re-emphasized the importance of addressing scientific, economic, ecological, health and environmental problems on larger scales. These new challenges demand multidisciplinary skills—the type of skills that can be found on a team. Collaborative work is fundamental to many of the problems faced by society today. Where will these interdisciplinary teams come from? In the U.S., these teams will have to include a larger proportion of minority students. Unfortunately, current inequities in the U.S. educational system still exclude their full participation. Furthermore, the economic inequities that support the status quo are not likely to change at a fast enough pace. For
this reason, making use of limited resources in an effective way is a desirable objective when the goal is to provide a competitive education for a large segment of the population.

Minority students in urban school districts face dropout rates nearly twice the national average [5] and are disproportionately attending the nation’s high schools with the weakest promoting power (correlated with high dropout rates) [3]. Educational strategies, such as cooperative learning, with the capacity to raise the achievement level of groups of students, as well as individuals, may be critically important because of their cooperative learning potential for enhancing the education of all. Empirical evidence shows that students who are doing well academically and who participate in the educational enterprise as peer tutors learn more than those who do not [17]. Even the simplest forms of collaborative learning improve the intellectual growth of individuals (e.g., being exposed to high-level conversations of mathematics during group work leads to higher gains in learning [12]). Thus, cooperative learning affords increased intellectual development to the individual members as well as the community as a whole.

In summary, although collaborative learning is unlikely to be the best approach for every community, its potential may be high in large communities with limited resources or in subpopulations of communities that traditionally have supported competitive rather than collaborative environments for all its members. This situation has often reduced Latino, Native American, African American and female participation in science and mathematics.

![Flow chart for cooperative learning model](image)

**Figure 1.** Flow chart for cooperative learning model

3. **Cooperative learning model.** We consider the simple model described in Figure 1, with the parameters as listed in Table 1. The “invasion” of the dynamics of cooperation in this model is assumed to be driven by peer pressure type interactions between individuals who have been immersed in a collaborative learning environment (the core population). The population $N$ is subdivided into three classes: $S$ denotes those individuals who are in the collaborative environment but not yet active participants in it. $E$ denotes those who have learned the value of cooperative learning and participate; they do not in general actively recruit others (in $S$),
but may by example inspire them to join. \(I\) denotes the class of “leaders,” that is, the group of individuals who have taken on responsibility for the growth of the collaborative community, not only leading others (in \(S\)) to participate in collaborative learning but training participants (in \(E\)) to become future leaders. Then \(N = S + E + I\).

If we denote the average span of time an individual spends in the system (say, four years for a high school or four-year college) by \(1/\mu\), then \(\mu N\) gives the rate at which individuals enter the system, in proportion to the total core population \(N\), and \(\mu S, \mu E,\) and \(\mu I\) represent the respective group exit rates from the system. Thus the overall recruitment and departure rates are equal, so that the total population immersed in this collaborative learning environment remains constant, as is often true in school settings (the implications of relaxing this assumption will be discussed in the conclusion).

Movement between classes is governed by the three social interactions that drive the collaborative learning community. At the lowest level, participants (\(E\)) in the collaborative learning process inspire others (\(S\)) to join by their success, at a rate proportional to the number of such “contacts” between members of the \(S\) and \(E\) classes. At the same time, mentors (\(I\)) also encourage and assist members of the \(S\) class to begin participation in the collaborative learning process. We assume that members of the \(S\) class have the type of contacts conducive to these processes at a rate \(\beta_1\); that these “peer pressure” interactions with members of each class occur in proportion to the sizes of those classes, \(\frac{S}{N},\frac{E}{N},\) and \(\frac{I}{N}\) (i.e., proportionate mixing); and that \(E\) interactions are less effective than \(I\) interactions in the conversion of \(S\) individuals, by a factor of \(q\) (where \(0 \leq q \leq 1\)). Thus effective \(S - E\) interactions occur at a rate of \(q\beta_1\frac{E}{N}S\), and effective \(S - I\) interactions occur at a rate of \(\beta_1\frac{I}{N}S\). The remaining interaction, in which the mentors (\(I\)) train participants (\(E\)) to become mentors, is assumed to occur at a rate \(\beta_2\), again in proportion to the relative number of mentors available \(\frac{I}{N}\), for a total rate of \(\beta_2\frac{I}{N}E\).

As an extension to this basic model, we also consider the possibility (addressed in Section 4.2 below) that leaders may “burn out” and retire from leadership roles without leaving the community altogether. The rate \(\gamma I\) describes the loss of cooperation (energy and leadership) in the \(I\) class. The box-flow diagram describes explicitly the rates of flows between each class.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S(t))</td>
<td>population of primarily individual learners at time (t)</td>
</tr>
<tr>
<td>(E(t))</td>
<td>population of primarily cooperative individuals at time (t)</td>
</tr>
<tr>
<td>(I(t))</td>
<td>population of cooperative learning mentors at time (t)</td>
</tr>
<tr>
<td>(N)</td>
<td>total population (constant)</td>
</tr>
<tr>
<td>(\beta_1)</td>
<td>the (maximal) mentoring rate by (I) on (S)</td>
</tr>
<tr>
<td>(q)</td>
<td>mentoring ability of (E) class (on (S)) relative to the (I) class</td>
</tr>
<tr>
<td>(\beta_1 q)</td>
<td>the (maximal) peer pressure rate by (E) on (S)</td>
</tr>
<tr>
<td>(1/\mu)</td>
<td>the average time spent by an individual in the system</td>
</tr>
<tr>
<td>(\beta_2)</td>
<td>the (maximal) mentoring rate by (I) on (E)</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>the rate of loss of cooperation (energy and leadership) by (I) class</td>
</tr>
</tbody>
</table>

Table 1. Population variables and parameter definitions
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From the model described above and illustrated in Figure 1, we derive the following equations, where variables and parameters are as given in Table 1:

\[
\frac{dS}{dt} = \mu N - \beta_1 S \left( \frac{qE + I}{N} \right) - \mu S,
\]

\[
\frac{dE}{dt} = \beta_1 S \left( \frac{qE + I}{N} \right) - \beta_2 E \left( \frac{I}{N} \right) - \mu E + \gamma I,
\]

\[
\frac{dI}{dt} = \beta_2 E \left( \frac{I}{N} \right) - \mu I - \gamma I.
\]

By adding (1), (2), and (3) we see that \(\frac{dN}{dt} = 0\) which shows that the total population is constant. Hence, without loss of generality, we normalize by setting \(N = 1\), which allows us to consider a reduced model via the substitution \(S = 1 - E - I\).

In epidemiological jargon, here we are modeling a subpopulation of the general population that has been immersed in a cooperative environment. It is in this environment where the “forces” of cooperation are significant. Since it is implicitly assumed that these are high levels of contacts between the members of this (assumed) constant subpopulation, then it is identified as the core group [22]. In an educational framework, the creation of a “core” subpopulation depends strongly on administrative decisions. If however, the core population becomes highly successful and visible then the possibility that the core increases (or decreases) its size is possible. Such a concept (variable core group size) was discussed by various authors in epidemiology (cf. [20] and references therein). The role of variable core group size will be addressed in the conclusion.

4. Mathematical analysis. Since the primary interest of this study is to evaluate the long-term survival ability of cooperative learning environments, model analysis focuses here on qualitative techniques that determine all possible long-term outcomes (equilibria) and the conditions under which each occurs.

4.1. Model A: \(\gamma = 0\).

4.1.1. Cooperation-free equilibria. We begin by considering the case where there is no reversion from \(I\) to \(E\). We thus set \(\gamma = 0\) and identify equilibria of the system. From (3), we have:

\[0 = \beta_2 EI - \mu I\]

\[\Rightarrow 0 = I(\beta_2 E - \mu)\]

\[\Rightarrow I = 0 \text{ or } E = \frac{\mu}{\beta_2}.\]

Taking \(I = 0\) and \(S = 1 - E - I\) in (2), we get:

\[0 = \beta_1 (1 - E)qE - \mu E\]

\[\Rightarrow E = 0 \text{ or } E = 1 - \frac{\mu}{\beta_1 q}.\]

So our first two equilibrium points are \(X_1 = (1, 0, 0)\) and \(X_2 = \left( \frac{\mu}{\beta_1 q}, 1 - \frac{\mu}{\beta_1 q}, 0 \right)\).

We note that \(I = 0\) in both of these equilibrium points. We shall refer to \(X_1\) as the “cooperation-free” equilibrium since no one is collaborating there (\(E = I = 0\)), while \(X_2\) can be called the “mentor-free” equilibrium since at \(X_2\) there are no mentors and the cooperative community is subsisting based entirely on the example of participants \(E\).
In order for $X_2$ to have contextual meaning, we must have $\frac{\beta_1 q}{\mu} > 1$; otherwise we have a negative population. Therefore, we define the threshold quantity $R_1 = \frac{\beta_1 q}{\mu}$. $R_1$ measures the ability of members of the $E$ class to “convert” or mentor $S$-class novices into the $E$ class before leaving the community; $R_1 > 1$ means that the example set by each $E$ individual is effective enough to bring in, on average, more than one $S$ individual to the $E$ class over his/her lifetime within the learning community, in an environment where there is little established cooperation ($S \approx 1$). We call $R_1$ the low-peer pressure basic reproduction number.

We determine the stability of the equilibrium points by linearizing about each one within the reduced two-dimensional system. First we compute the general Jacobian matrix recalling the substitution $S = 1 - E - I$:

$$
\begin{pmatrix} 
\beta_1 (q - 2qE - I - qI) - \beta_2 I - \mu & \beta_1 (1 - qE - E - 2I) - \beta_2 E \\
\beta_2 I & \beta_2 E - \mu
\end{pmatrix}.
$$

Evaluated at $X_1 = (1,0,0)$, the Jacobian is:

$$
\begin{pmatrix}
\mu(R_1 - 1) & \beta_1 \\
0 & -\mu
\end{pmatrix}.
$$

Since this matrix is upper triangular, the eigenvalues are $\lambda_1 = \mu(R_1 - 1)$ and $\lambda_2 = -\mu$. Thus, applying the Routh-Hurwitz criteria, $X_1$ is locally asymptotically stable when $R_1 < 1$ and unstable otherwise.

Note that when not only $R_1 = \frac{\beta_1 q}{\mu} < 1$ ($E$-class participants are inefficient at recruiting new members), but in fact $\frac{\beta_1 q}{\mu} < 1$ ($I$-class mentors are also inefficient at recruiting new members), we can show that the cooperation-free state $X_1$ is globally asymptotically stable by using the Lyapunov function $V = E + I$:

$$
V(X_1) = 0
$$

$$
V > 0 \text{ if } E + I \neq 0
$$

$$
\frac{dV}{dt} = \beta_1 S(qE + I) - \mu(E + I) < \beta_1 \left(S - \frac{\mu}{\beta_1}\right)(E + I)
$$

$$
< \beta_1 \left(1 - \frac{\mu}{\beta_1}\right)(E + I)
$$

$$
< 0 \text{ if } \frac{\beta_1}{\mu} < 1.
$$

Similarly, the Jacobian at $X_2 = \left(\frac{1}{R_1}, 1 - \frac{1}{R_1}, 0\right)$ is:

$$
\begin{pmatrix}
\mu(1 - R_1) & \beta_1 \left(1 - \frac{1}{R_1}\right) \left[\frac{R_1}{R_1 - 1} - (q + 1) - \frac{\beta_2}{\mu}\right] \\
0 & \mu \left(\frac{\beta_2}{\mu} \left(1 - \frac{1}{R_1}\right) - 1\right)
\end{pmatrix}.
$$

This is also an upper triangular matrix; the eigenvalues are simply $\lambda_3 = \mu(1 - R_1)$ and $\lambda_4 = \mu \left(\frac{\beta_2}{\mu} \left(1 - \frac{1}{R_1}\right) - 1\right)$. We note that $\lambda_3$ is negative when $R_1 > 1$. As for $\lambda_4$, it is negative when $\frac{\beta_2}{\mu} \left(1 - \frac{1}{R_1}\right) - 1 < 0$. Otherwise, if $R_1 > 1$ but $\lambda_4 > 0$, $X_2$ is a saddle point.

Based on this analysis and the above characterization of $X_2$, we define $R_2 = \frac{\beta_2}{\mu} \left(1 - \frac{1}{R_1}\right)$. This quantity measures the ability of mentors $I$ to recruit and train new mentors from the $E$ class during their lifetimes as mentors. The raw efficiency of this process is simply $\beta_2/\mu$, the mentoring rate multiplied by the average lifetime
in the system, but it is reduced by the relative availability \( 1 - \frac{1}{\beta_1} \) of participants 
\((E)\) at the critical point \(X_2\) where active mentoring is just beginning to take hold 
in the community \((I \approx 0)\). Note that the \(X_1 - X_2\) threshold \(R_1 = 1\) corresponds to 
\(R_2 = 0\); there can be no training of new mentors \((R_2 > 0)\) unless there are already 
participants in the cooperative community \((R_1 > 1)\). We refer to \(R_2\) as the high-
peer pressure reproductive number. \(R_2 > 1\) implies that an average \(I\) individual is 
responsible for promoting more than one \(E\) individual into the \(I\) class over her/his 
lifetime, in an environment where mentoring is just arising.

Thus we have two stages of cooperation-free status, each marked by a threshold 
measure of the strength of a different type of interaction. No cooperative learning 
community can establish itself \((X_1)\) when the basic \(S - E\) interaction is weak \((R_1 < 
1)\). When participants in the collaborative process serve as effective examples \((R_1 > 
1)\) but active mentoring is weak \((R_2 < 1)\), a mentorless learning community can 
survive \((X_2)\). The next step is to determine conditions under which a healthy 
cooperative learning community (including mentors) can survive.

4.1.2. Cooperative equilibria. We use \(E_3 = \frac{\mu}{\beta_2}\) to determine the remaining equilib-
ria. In this case we must use (2) to solve for the corresponding \(S_3\) value and then 
check \(I = 1 - S - E\) for the corresponding \(I_3\) value.

\[
0 = \beta_1 S (qE + I) - \beta_2 EI - \mu E
\begin{align*}
\Rightarrow 0 &= \beta_1 S \left( q \frac{\mu}{\beta_2} + I \right) - \beta_2 \left( \frac{\mu}{\beta_2} \right) I - \mu \left( \frac{\mu}{\beta_2} \right) \\
&= \frac{\mu (\mu + I \beta_2)}{\beta_1 (q \mu + I \beta_2)}
\end{align*}
\]

(4)

Using \(E_3 = \frac{\mu}{\beta_2}\) and (4) in \(I = 1 - S - E\), we solve for \(I_3\):

\[
0 = 1 - \frac{\mu}{\beta_2} - \frac{\mu (\mu + I_3 \beta_2)}{\beta_1 (q \mu + I_3 \beta_2)} - I_3
\begin{align*}
0 &= I_3^2 + I_3 \left( \frac{\mu}{\beta_1} + (1 + q) \frac{\mu}{\beta_2} - 1 \right) + \left( \frac{\mu^2}{\beta_1 \beta_2} - \frac{q \mu}{\beta_2} + \frac{q \mu^2}{\beta_2} \right)
\end{align*}
\]

(5)

From (5), for certain conditions there may exist one or two positive \(I_3\) values for the same \(E_3\) value.

Based on (5), we let \(x = I_3\), \(A = 1\), \(B = \left( \frac{q \mu}{\beta_2} - 1 + \frac{\mu}{\beta_1} + \frac{\mu}{\beta_2} \right)\), and \(C = 
\left( \frac{\mu^2}{\beta_1 \beta_2} - \frac{q \mu}{\beta_2} + \frac{q \mu^2}{\beta_2} \right)\). Then (5) becomes \(f(x) = Ax^2 + Bx + C = 0\). First we shall 
show that for \(R_2 > 1\), there is a unique endemic equilibrium. We observe that 
\(R_2 > 1 \Leftrightarrow f(0) < 0\) from the following:

\[
\begin{align*}
f(0) &= \frac{\mu^2}{\beta_1 \beta_2} - \frac{q \mu}{\beta_2} + \frac{q \mu^2}{\beta_2} \\
&= \frac{q \mu}{\beta_2} \left( \frac{\beta_2}{q \beta_1} - \frac{\beta_2}{\mu} + 1 \right) \\
&= \frac{q \mu^2}{\beta_2} \left( 1 - \frac{\beta_2}{\mu} \left( 1 - \frac{\mu}{q \beta_1} \right) \right) \\
&= \frac{q \mu^2}{\beta_2} (1 - R_2).
\end{align*}
\]
Next we observe that \( f(1) > 0 \).

\[
\begin{align*}
\text{det} & = 1 + \frac{q_1}{b_2} - 1 + \frac{\mu}{b_1} + \frac{\mu^2}{b_1b_2} - \frac{q_1}{b_2} + \frac{q_2^2}{b_2} \tag{7}
\end{align*}
\]

Since \( f(0) < 0 \) (when \( R_2 > 1 \)) and \( f(1) > 0 \), we know that the graph of \( f(x) \) crosses the \( x \)-axis once to the left of zero and once in \((0,1)\), proving the existence of a unique positive equilibrium point, \( X_3 = (S_3, \frac{\mu}{b_2}, I_3) \), when \( R_2 > 1 \).

To check the stability of this equilibrium point, we consider the Jacobian of the two-dimensional model evaluated at \( E_3 = \frac{\mu}{b_2} \) where \( I_3 > 0 \) (instead of solving for \( I_3 \) explicitly due to the complexity of the form of the solution). The Jacobian evaluated at \( X_3 \) is:

\[
\begin{align*}
\begin{bmatrix}
\beta_1 q - \frac{2\beta_1 q}{b_2} - I_3(\beta_1 + \beta_1 q + \beta_2) - \mu & \beta_1 - (1 + q)\frac{\beta_1}{b_2} - 2\beta_1 - \mu
\end{bmatrix}
\end{align*}
\]

with determinant and trace:

\[
\begin{align*}
\text{det} & = -\beta_2 I_3 \left[ \beta_1 \left( 1 + (1 + q)\frac{\mu}{b_2} - 2I_3 \right) - \mu \right] = \beta_1 \beta_2 I_3 (B + 2I_3), \tag{8}
\end{align*}
\]

\[
\begin{align*}
\text{trace} & = \beta_1 q - \frac{2\beta_1 q}{b_2} - I_3(\beta_1 + \beta_1 q + \beta_2) - \mu. \tag{9}
\end{align*}
\]

To satisfy the Routh-Hurwitz criteria we require the determinant \( > 0 \) and the trace \( < 0 \). We have that

\[
\text{det} > 0 \iff B + 2I_3 > 0 \iff I_3 > -B/2. \tag{10}
\]

Since \( R_2 > 1 \iff C < 0 \), and \( A = 1 > 0 \), \( I_3 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} > \frac{-B}{2A} = -B/2 \). Similarly:

\[
\text{trace} < 0 \iff I_3 > \frac{\beta_1 \beta_2 q - 2\beta_1 q\mu - \mu \beta_2}{\beta_1 \beta_2 (1 + q + \beta_2)} \tag{11}
\]

It can be shown (see Section 4.2) that this condition holds true for \( R_2 < 2 \) or \( I_3 > -B \), so this condition is also always satisfied for \( R_2 > 1 \) (when \( -4AC > 0 \) and hence \( I_3 > -B \)).

4.1.3. **Backward bifurcation.** We turn our attention to when \( R_2 < 1 \). In this case, under certain conditions there can exist two positive solutions to (5). These solutions, with appropriate stabilities, arise when the system exhibits a backward bifurcation.

Note that the condition \( R_2 < 1 \) is equivalent to:

\[
\frac{1}{\mu} < \frac{1}{\beta_1 q} + \frac{1}{\beta_2}. \tag{12}
\]

Next we consider the quadratic equation (5). With \( R_2 < 1 \iff C > 0 \), we have \( f(0) > 0 \), \( f(1) > 0 \). Hence for there to exist two positive solutions requires also (a) that the vertex be in \((0,1)\), meaning \( f'(0) = B < 0 \) and \( f'(1) > 0 \), and (b) that there are two real zero crossings, i.e., \( B^2 - 4AC > 0 \). From (a) we have

\[
\begin{align*}
f'(x) & = 2x + \frac{\mu}{\beta_1} + (1 + q)\frac{\mu}{\beta_2} - 1 \\
\Rightarrow f'(1) & = 1 + \frac{\mu}{\beta_1} + (1 + q)\frac{\mu}{\beta_2}. \tag{13}
\end{align*}
\]
which is always positive, and
\[
0 > B = \frac{\mu}{\beta_1} + (1 + q) \frac{\mu}{\beta_2} - 1
\]
\[
\frac{1}{\mu} > \frac{1}{\beta_1} + \frac{1}{\beta_2} (1 + q).
\] (14)

From (b) we have
\[
0 < \left( \frac{\mu}{\beta_1} - 1 + \frac{\mu}{\beta_2} (q + 1) \right)^2 - 4 \left( \frac{\mu^2}{\beta_1 \beta_2} - \frac{q \mu}{\beta_2} + \frac{q \mu^2}{\beta_2^2} \right)
\]
\[
0 < \frac{\mu^2}{\beta_1^2} - 2 \frac{\mu}{\beta_1} + 2 \frac{\mu^2}{\beta_1 \beta_2} (q - 1) + 1 + 2 \frac{\mu}{\beta_2} (q - 1) + \frac{\mu^2}{\beta_2^2} (q - 1)^2
\]
\[
0 < \left( \frac{\mu}{\beta_1} - 1 - \frac{\mu}{\beta_2} (1 - q) \right)^2 - 4 \frac{\mu}{\beta_2} (1 - q)
\]
\[
2 \sqrt{\frac{\mu}{\beta_2} (1 - q)} < \left| \frac{1}{\beta_1} - \frac{1}{\beta_2} (1 - q) \right|.
\] (15)

When \( B < 0 \) so that (14) holds, we have that \( \frac{\mu}{\beta_1} - 1 - \frac{\mu}{\beta_2} (1 - q) < 0 \) in (15), and we need the positive root so we multiply by \((-1)\) as follows:
\[
2 \sqrt{\frac{\mu}{\beta_2} (1 - q)} < -1 \left( \frac{\mu}{\beta_1} - 1 - \frac{\mu}{\beta_2} (1 - q) \right)
\]
\[
\frac{\mu}{\beta_1} < 1 - 2 \sqrt{\frac{\mu}{\beta_2} (1 - q)} + \frac{\mu}{\beta_2} (1 - q)
\]
\[
\frac{\mu}{\beta_1} < \left( 1 - \sqrt{\frac{\mu}{\beta_2} (1 - q)} \right)^2
\]
\[
\frac{1}{\beta_1} < 1 - \sqrt{\frac{\mu}{\beta_2} (1 - q)}
\]
\[
\left( \frac{1}{\beta_1} + \sqrt{\frac{1}{\beta_2} (1 - q)} \right)^2 < \frac{1}{\mu}.
\] (16)

Graphing inequalities (12), (14), and (16) with respect to the raw recruitment efficiencies \( r_1 = \beta_1/\mu \) and \( r_2 = \beta_2/\mu \) shows that the three boundaries intersect at \( q^2 r_1 = 1 \), \((1 - q) r_2 = 1\), and that all three can be satisfied together only for \( q R_1 < 1 \), \( R_2 < 1 \), and (16), i.e., (14) can be replaced with the simpler \( q R_1 < 1 \).

Thus, when the parameter values for conditions (12), (14) (or \( q R_1 < 1 \)), and (16) are met, we can expect the system to exhibit a backward bifurcation, meaning there are two simultaneous endemic equilibria when \( R_2 < 1 \). Note that if (12) or (14) is broken, a cooperative equilibrium is globally stable, while if (16) is broken no \( I \) class can survive. Thus we focus our interpretation on (16) in the discussion.

When (5) has two solutions in \((0,1)\), \( R_2 < 1 \), so that the stability condition (11) that the Jacobian’s trace be negative is always satisfied. As regards the other condition (10) on the determinants, the upper solution, \( X_{3U} \), is the one that also exists when \( R_2 > 1 \), with \( I_3 = -B + \sqrt{B^2 - 4AC} > -\frac{B}{2A} \), so this equilibrium is locally asymptotically stable. The lower solution, \( X_{3L} \), however, has \( I_3 = -B + \sqrt{B^2 - 4AC} < -\frac{B}{2A} \), making it unstable.

The resulting backward bifurcation is illustrated in Figure 2. This bifurcation diagram shows that, given parameter values that meet the determined conditions:
R stable when the original model to be locally asymptotically stable. Since these conditions are similar to those obtained for the unique cooperative equilibrium points are \(Y_1\) and \(Y_2\) for a unique positive equilibrium point when \(\tilde{R}_2\) with no \(I\) class individuals is the attractor. Also important to note is that once the mentor population \(I\) is great enough to be in the basin of attraction of the stable cooperative equilibrium \(X_{3U}\), it is difficult to eliminate. No longer is it a matter of crossing a single recruitment-efficiency threshold \((R_2 = 1)\): a substantial number of mentors would have to be removed outright before the \(I\) class becomes unsustainable.

4.2. Model B: Backsliding. As an extension to the analysis of the model above, we consider the original model where there is the possibility of \(I\) class individuals returning to the \(E\) class, independent of their interactions with \(E\) class individuals. This is our burnout model in which \(\gamma > 0\) is defined as the rate at which \(I\) class individuals become less involved over time, returning to the \(E\) class. The primary question here is to what extent backsliding erodes the robustness of the learning community structure illustrated in the backward bifurcation.

Using the same analytical techniques, we determine that the first two equilibrium points are \(Y_1 = (1, 0, 0)\) and \(Y_2 = \left(\frac{1}{\beta_1}, 1 - \frac{1}{\beta_1}, 0\right)\), exactly as in the original model. \(Y_1\) is locally asymptotically stable when \(R_1 < 1\), and \(Y_2\) is locally asymptotically stable when \(R_1 > 1\) and

\[
\tilde{R}_2 = \frac{\beta_2}{\mu + \gamma} \left(1 - \frac{1}{R_1}\right) < 1.
\]

The similarities of the first two equilibria and the threshold values to those of the original model suggest that this model will also exhibit a backward bifurcation.

Solving for the cooperative equilibrium \(Y_3 = \left(\tilde{S}_3, \tilde{E}_3, \tilde{I}_3\right)\), we get \(\tilde{E}_3 = \frac{\mu + \gamma}{\beta_2}\) and the following equation for \(\tilde{I}_3:\)

\[
I^2 + I \left[\frac{\mu}{\beta_1} + \frac{(\mu + \gamma)(1 + q) - 1}{\beta_2}\right] + \frac{\mu + \gamma}{\beta_2} \left[\frac{\mu}{\beta_1} - q + \frac{q(\mu + \gamma)}{\beta_2}\right] = 0.  \tag{17}
\]

If we let \(y = \tilde{I}_3\), then we will have a quadratic equation of the form \(g(y) = \tilde{A}y^2 + \tilde{B}y + \tilde{C} = 0\). Since \(g(0) = \tilde{C} < 0 \iff \tilde{R}_2 > 1\), we conclude that we have the conditions for a unique positive equilibrium point when \(\tilde{R}_2 > 1\). Stability analysis using the Jacobian matrix indicates that \(Y_3\) is locally asymptotically stable when:

\[
\text{determinant} = \beta_1\beta_2\tilde{I}_3 \left(\tilde{B} + 2\tilde{I}_3\right) > 0, \quad  \tag{18}
\]

which is always true for \(Y_3\) when \(\tilde{R}_2 > 1\) since \(\tilde{I}_3 > -\tilde{B}/2\tilde{A}\), and

\[
\text{trace} = \beta_1 q - \frac{2\beta_1 q(\mu + \gamma)}{\beta_2} - \tilde{I}_3(\beta_1 + \beta_1 q + \beta_2) - \mu < 0, \quad  \tag{19}
\]

which is true when

\[
\tilde{I}_3 > \frac{\beta_1\beta_2 q - 2\beta_1 q(\mu + \gamma) - \mu\beta_2}{\beta_1\beta_2(1 + q) + \beta_2^2}. \quad  \tag{20}
\]

These conditions are similar to those obtained for the unique cooperative equilibrium in the original model to be locally asymptotically stable. Since \(\tilde{R}_2 < 2 \iff \tilde{I}_3 < \frac{\beta_1\beta_2 q - 2\beta_1 q(\mu + \gamma) - \mu\beta_2}{\beta_1\beta_2(1 + q) + \beta_2^2}\),
\begin{align*}
\beta_1 \beta_2 q - 2 \beta_1 q (\mu + \gamma) - \beta_2 \mu < 0, \quad (20) \text{ holds for } \tilde{R}_2 < 2. \quad (20) \text{ also holds for } \tilde{R}_2 > 1 \text{ and } I_3 > -B \text{ since}
\end{align*}

\begin{align*}
\tilde{R}_2 > 1 & \Rightarrow \beta_2 > \mu + \gamma \\
& \Rightarrow \beta_1 (1 - q) \beta_2 > \beta_1 (1 - q) (\mu + \gamma) \\
& \Rightarrow \beta_1 \beta_2 - \beta_1 (1 + q) (\mu + \gamma) - \beta_2 \mu > \beta_1 \beta_2 q - 2 \beta_1 q (\mu + \gamma) - \beta_2 \mu \\
& \Rightarrow -\tilde{B} > \frac{\beta_1 \beta_2 q - 2 \beta_1 q (\mu + \gamma) - \beta_2 \mu}{\beta_1 \beta_2} \\
& \Rightarrow -\tilde{B} > \frac{\beta_1 \beta_2 q - 2 \beta_1 q (\mu + \gamma) - \beta_2 \mu}{\beta_1 \beta_2 (1 + q) + \beta_2^2}
\end{align*}

(the last inequality follows since the right-hand side is positive for \( \tilde{R}_2 > 2 \), which is the only case left to consider).

Finally, a straightforward analysis determines the conditions under which this model also exhibits a backward bifurcation. The condition \( \tilde{R}_2 < 1 \) is equivalent to

\begin{align*}
\frac{1}{\mu} < \frac{1}{\beta_1 q} + \frac{1}{\beta_2} \frac{\mu + \gamma}{\mu}. \quad (21)
\end{align*}

Following the same strategy as with Model A, we also find conditions for multiple cooperative equilibria:

\begin{align*}
\frac{1}{\beta_1} + \frac{\mu + \gamma}{\mu} \frac{1}{\beta_2} (1 + q) < \frac{1}{\mu} \quad (22)
\end{align*}

and

\begin{align*}
\left( \sqrt{\frac{1}{\beta_1}} + \sqrt{\frac{\mu + \gamma}{\mu} \frac{1}{\beta_2} (1 - q)} \right)^2 < \frac{1}{\mu}. \quad (23)
\end{align*}

Thus, with the parameter values satisfying conditions (21), (22), and (23), we can expect the system to exhibit a backward bifurcation as in the original model.

It should be noted, however, that \( \gamma \) must be small relative to the rest of the parameters, especially \( \mu \); otherwise the conditions for a backward bifurcation are violated. In fact, by allowing backsliding, the average length of time an \( I \) individual spends in the system may be effectively reduced to the point where the high-peer pressure interactions are too weak to maintain a stable \( I \) population. As seen in Figure 3, allowing backsliding reduces the range for backward bifurcations to take place.

### 4.3. Model C: Cooperative hierarchy.

Another extension to the analysis of this model arises when we consider the case \( q = 0 \), in which all recruitment is done exclusively by mentors (\( I \)). This means that \( E \)-class participants do not assist in the recruitment of \( S \)-class individuals into the cooperative learning community.

Referring back to (2) and (3) in the original model, we note that in this case there is only one cooperation-free equilibrium point \( Z_1 = (1, 0, 0) \) which is always locally asymptotically stable. Since \( E \)-class cooperative learners do not influence others in this scenario, there is no mentor-free stage as in the original model.

As before, solving for the cooperative equilibrium yields a quadratic equation for \( I \):

\begin{align*}
h(I) = I^2 + \left( \frac{\mu}{\beta_1} + \frac{\mu}{\beta_2} - 1 \right) I + \frac{\mu^2}{\beta_1 \beta_2} = 0. \quad (24)
\end{align*}

Since \( h(0) > 0, h(1) > 0 \), the number of cooperative equilibria is either zero or two, depending on the sign of the discriminant of \( h(I) \). The conditions for the existence
of endemic equilibria, $Z_{2U}$ and $Z_{2L}$, reduce to:

$$
\left(\sqrt{\frac{1}{\beta_1}} + \sqrt{\frac{1}{\beta_2}}\right)^2 < \frac{1}{\mu},
$$

in which case one of the pair is stable and the other unstable. Since there is no analogue of $R_2$, there is no backward bifurcation point, only a supercritical saddle-node bifurcation at the boundary of the condition given above.

5. Discussion.

5.1. Cooperative learning model outcomes. We are primarily interested in the study of all possible qualitative outcomes of our cooperative learning model, that is, the intellectual landscapes in which our population under study might end up. For this reason, the model’s transient dynamics (which could be extremely fast) are ignored. We focus instead on the changes in the nature of these steady states as parameters (degree of peer pressure, etc.) are varied. The “final” state of a system like this typically depends on two non-independent factors: initial conditions and thresholds. In this section we describe first the possible end states, and then the threshold quantities involved. The next section will describe how and when the initial conditions come into play.

There exist five possible outcomes for the population over time, depending on the values of the initial conditions and thresholds described below. One of the following must occur:

1. The entire population remains a part of the least cooperative group (the “novices”). In other words, no cooperative learning culture develops; the core group population does not differ from the general population.
2. The population divides between the least cooperative and moderately cooperative groups, that is, two subcultures co-exist. Cooperation is moderately successful, but no active mentoring arises to foster it.
3. The population distributes between all three classes. Cooperation is successful and deliberately fostered; there are dramatic differences between the culture of the core and that of the general population.

4. The entire population remains in the novice group, unless a large enough initial number of cooperative individuals (including mentors) is introduced at inception, in which case the population distributes among all three classes and a cooperative learning culture takes root in a subset of the population. In other words, the establishment and survival of a collaborative culture can only occur given a strong enough initial investment in it.

5. The population distributes between the least and moderately cooperative groups but no mentoring arises, unless a large enough initial number are introduced, in which case the population distributes among all three classes. Here, cooperation can be moderately or fully successful, depending on the initial number of highly cooperative individuals introduced into the population.

In mathematical terms, the outcomes correspond to stable equilibria of $X_1$, $X_2$, $X_3$, $X_1$ vs. $X_3$, and $X_2$ vs. $X_3$, respectively. Figure 4 illustrates the relation among the outcomes and the threshold model parameters.

There are two basic threshold parameters associated with the outcomes of this model. The first is the low-peer pressure threshold value, $R_1$, which measures the average effectiveness of $E$ peer pressure on $S$ individuals, the ability of the $E$ class to establish the cooperative learning environment. $R_1$ is given by the product of the $E$-on-$S$ peer pressure rate $β_1 q$ and the average residence time $1/μ$ of an individual in the system. This interpretation assumes that the system has no $I$ individuals and that the population is composed mostly of $S$ individuals. Intuitively, if $R_1$ is greater than one then the $E$ class grows by recruiting $S$ individuals. In epidemiological terms, $E$ individuals have managed to successfully invade the $S$ population on their own. Here, all but a fraction $1/R_1$ of the population will become successful cooperative learners. A key feature of this approach is that success can never be 100% complete because some experienced learners are always leaving the system, novices are arriving, and $R_1$ is bounded (typically confined to a small range).

The second threshold parameter is the high-peer pressure threshold value, $R_2$, which measures the ability of the $I$ class to establish itself by recruiting from the $E$ class, given that the size of the latter is determined by $R_1$. $R_2$ is given by the product of the $I$-on-$E$ peer pressure times the proportion $(1 − 1/R_1)$ of the population already in the $E$ class, times the average residence time of an individual in the system. This interpretation assumes that the system has few $I$ individuals, and that the population is composed mostly of $S$ and $E$ individuals (with $R_1 > 1$). Intuitively, if $R_2$ is greater than one, then the $E$ and $I$ classes will grow by recruiting from the $S$ and $E$ populations, respectively. If, however, $R_2$ is less than one, then the $E$ and $I$ classes will shrink because on average the individuals will not replace themselves before exiting the system. In terms of our biological metaphor, $R_2 > 1$ means that $I$ individuals would then have managed to successfully invade the $S$ and $E$ populations.

The situation here, however, is not as simple as described above. Gladwell's \cite{Gladwell1999} “tipping point” view of the world is not enough to explain the cooperative dynamics of our model. Indeed, thanks in part to the third type of social interaction involved—the role of mentors in encouraging novices to join the collaborative community—initial conditions play a critical role in the outcomes, and as parameters are varied it is observed that community learning is a process that is hard to
Figure 4. Long-term outcomes of the cooperative learning model (by number), in terms of the raw recruitment efficiencies $r_1 = \beta_1/\mu$, $r_2 = \beta_2/\mu$ and $q$

undermine once established. Furthermore, our model illustrates that investing in cooperative learning is sound in the sense that it creates communities of learners that can survive in situations where they might not naturally arise (community resilience). This result contradicts the “tipping point” metaphor used by Gladwell [18] in its application of epidemic contact processes to the study of social dynamics. The limited understanding of the impact of nonlinear dynamics at multiple levels offered by Gladwell’s model results in a rather simplistic view of the role of social forces on community structure. It is worth noting that Gladwell took the “tipping point” metaphor from epidemiology. This concept was introduced, among others, by the Nobel Laureate, Sir Ronald Ross [24]. The value of this concept in epidemiology has been unprecedented [1, 14] but social contact processes differ from epidemiological processes. It is in the context of multilevel interactions that this concept breaks down, even in epidemiology (cf. [8] and references therein).

5.2. Interpretation of mathematical analysis. The typical analysis of epidemiological contact processes based on the “tipping point” concept says that for a system to “succeed,” and here success is measured by the growth and establishment of the $I$ mentor class, each $I$ individual, “on the average,” must convert more than one ($R_2 > 1$) individual into the $I$ class. This is not what happens here, as illustrated by the five model outcomes described above.

Individuals begin in the $S$ class, being part of the least contributing group. Over time, through the efforts of both the $I$ class and the $E$ class (group effort—not considered in Gladwell’s view of the world), a portion of the $S$ class becomes more proficient and cooperative, resulting in their promotion to the $E$ class. In addition to interacting with the $S$ class, $E$ class individuals also enter the $I$ class (“pressured” by $I$ individuals), where as experienced cooperative learners they take an active part in supporting and fostering the learning community.

As the mathematical analysis shows, this model exhibits a phenomenon known as a backward bifurcation, which indicates that under certain conditions two different end states coexist when $R_2 < 1$. In this “unusual” situation initial conditions determine which of the two actually occurs. That is, under some conditions, even though the high-peer pressure threshold value, $R_2$, may be less than one, high levels
of cooperation can still successfully invade and be maintained if enough I class individuals are introduced into the system. When \( R_2 > 1 \), the I class individuals are able to maintain a constant, sustainable influence on the system. While \( R_2 = 1 \) is a tipping point, in this case the initial proportion of I class individuals in the system can be as important than the threshold, or more so. In other words, there are learning conditions for which the natural tipping point has not been reached \( (R_2 < 1) \), but under which, once there has been an initial investment of resources to introduce a sufficient proportion of highly cooperative individuals, their efforts along with those of the E class are enough to maintain the new invading culture (even though \( R_2 < 1 \)). It also means that once a successful level has been reached, so that there are enough I individuals, destroying such a culture is difficult: bringing the parameters of the system down so that \( R_2 < 1 \) is not enough to eliminate the I class.

In fact, analysis shows that it can even be possible to create a sustainable cooperative learning environment involving all three classes of individuals under conditions where normally not even the intermediate E class would have been able to sustain itself \( (R_1 < 1) \). These outcomes (4 and 5 above) again depend on how many experienced cooperative learners are brought into the system at the beginning. If the I class is not able to sustain itself through recruitment from an E class sustained by S-E interactions \( (R_2 < 1) \), but it is capable of sustaining the E class by working with S individuals \( (R_1 > q) \), then the relevant condition (inequality (16)), under which a large enough infusion of experienced cooperative learners results in a stable learning environment, requires that the I class is able to interact enough with the S class, that the size of the intermediate E class grows into a pool large enough to generate a new I class before the current generation leaves the system. In other words, simultaneous cooperation at every level can sustain the community: the S-E interaction alone is not strong enough to produce an intermediate level of learners large enough to generate I individuals, but the I class' added interactions with the S class increase the size of the intermediate group just enough for new I individuals to arise. In order for this “teamwork” to succeed, however, there must already be enough I individuals in the system to foster growth; without them, that crucial added boost to the S class is missing. Therefore, the establishment of a learning community, under difficult conditions (below the threshold), can still be carried out as long as the initial support provided is sufficient to bring in enough expert learners.

To further investigate the nature of the turning point (the real “culture elimination” threshold) we consider two additional questions: (1) What happens if highly cooperative individuals become less cooperative before exiting the system, i.e., what if there is backsliding from being highly cooperative to being moderately cooperative? and (2) What happens if mid-level individuals do not positively interact with the S individuals, i.e., if only the motivated individuals do all the work?

Model B responds to the first question by incorporating the effect of the parameter \( \gamma \), defined previously as the rate at which the energy of the I class is lost, as high cooperative learners give up their leadership roles. What we find is that as the I class individuals lose their “spark”, it becomes nearly impossible to reach the turning point. The requirements for a synergistic victory over the negative invasion conditions (here defined by \( R_2 < 1 \)) are so strict that the initial investment of resources required to establish a community is likely to be too high. Therefore, successful cooperative learning communities must continually support motivated and
successful individuals in maintaining active leadership roles; otherwise even a pro-
gram that does not weaken dramatically may be lost despite its inherent resilience.
Also, providing long term resources aimed at maintaining the effectiveness of the
I class may be essential, and in the long run less costly and more effective than
having to periodically rescue a system in crisis (if the goal is to maintain a culture
of cooperation).

Model C looks at the second question by setting the E class’ contribution $q$ to
zero. The results are clear and provide information about the interactions within the
entire system. If there are not interactions among all the classes, i.e., if, in addition
to learning from the leaders in the learning community, moderately cooperative
individuals are not encouraged to work with and mentor the novice individuals, then
there is no natural turning point. In essence, the system is not a cooperative learning
environment without everyone interacting with everyone else. Even introducing high
proportions of motivated individuals will only generate a sustainable cooperative
environment if the efficiencies of both transitions are exceptionally high. Therefore,
systems dependent exclusively upon their leaders ($q = 0$) are not truly collaborative.
Hence, the value of $q$ should be considered a measure of multilevel cooperation. This
observation may help define as cooperative systems only those where two classes
interact at each level. It is our belief that such structures will always be able to
support turning point behavior and backward (subcritical) bifurcations.

6. An empirical example. Our original motivation for studying cooperative learn-
ing has been the success of the Mathematical and Theoretical Biology Institute
(MTBI)\footnote{http://mtbi.asu.edu} in which all four authors have been involved [6, 7]. MTBI is a summer
research program designed to encourage undergraduates—primarily those from un-
derrepresented backgrounds—to pursue Ph.D.’s in mathematics and other science
fields, by providing intensive sequential summer research experiences. Typically,
20–30 new students from various socioeconomic and academic backgrounds partic-
ipate each year and 5–10 more return from the previous year. The students are
housed together on a university campus throughout an eight week program com-
posed of two phases: training and research. It is within this framework that a
collaborative learning community has been established and renewed every summer.

During the program’s first four weeks, all participants attend intensive classes
in the mornings. Returning students interact with new participants by teaching
computer labs in the afternoons and helping with homework assignments in the
evenings. The first steps toward re-establishing the learning community each sum-
er begin during this phase. The amount of homework given is too much for most
students to complete in the allotted time if they work alone. Students’ initial hurdle
is thus to overcome the view that “they must be able to solve a problem on their
own before they can contribute to the group” [15]. In addition to providing help,
faculty and returning students also facilitate and encourage collaboration among
the new participants. It usually takes about a week for the majority of students to
realize not only that cooperative learning is the only way to accomplish everything
that is required, but that every individual can actually learn more through collabor-
ation. Because students react differently—and sometimes emotionally—to such
a shift in learning culture, program staff also make personal histories, perspectives,
and responses to the process itself an explicit part of regular group discussions,
in order to acknowledge their role in learning [11], clarify the program structure’s underlying intentions, and facilitate positive acculturation.

During the last four weeks of the program, participants form their own small groups and develop research projects based on topics of their own choosing, topics which frequently fall outside the applied area of expertise of the faculty. This last feature, exceptional among such programs, uproots the traditional hierarchy of intellectual authority [11], first by allowing students to set the research agenda and then by making teachers and students co-researchers [15], facing on equal ground the challenges of the endeavor (limitations on available data or expertise, difficulties with the mathematics).

In terms of the mathematical model developed in this paper, all (or nearly all) new students join the community as novices ($S$), while advanced students, who are selectively chosen, return either as experienced collaborative learners ($E$) or as mentors ($I$). The first phase of the program is designed to use the intensive classwork to prepare participants not only mathematically, but pedagogically, for the research phase, by fostering new participants’ acculturation to the cooperative learning environment (transition from $S$ to $E$), as well as returning students’ development into capable mentors (transition from $E$ to $I$). Many of the new students make the transition to collaborative learning quickly, and by the end of the classroom phase, the vast majority have generally done so. A quick cultural transformation is essential in this system in order for the research teams to function efficiently enough to complete a substantial project in the time remaining.

As with any behavioral phenomenon driven by collective influences, the true values of the parameters in the model analyzed in this paper are difficult to measure from year to year without an intensive and ongoing assessment of each participant on a daily basis. However, since (by inspection of the expressions for $R_1$, $R_2$ and inequality (16)) the outcomes are determined by three root parameters $q, r_1 = \beta_1/\mu$ and $r_2 = \beta_2/\mu$, a few simple examples serve to illustrate the robustness of the system (see Table 2). To begin with, let us assume that mentors are 3 times as efficient as other collaborative learners at fostering participation among novices: that is, let $q = 1/3$. One goal of mentors is to get most other participants to advance their collaborative abilities by the end of the second week, as many of the returning students begin to turn their attention to their own projects before the end of the first four weeks. This “training time” corresponds to $1/\beta_1$ and $1/\beta_2$, which must then be no greater than two weeks. If we take the average lifetime in the system to be the eight weeks of the summer program (in fact, for those students who later return for a second or third summer, it is longer, but here we take a conservative estimate), then the aforementioned goal of mentors fostering new students’ adaptation to cooperative learning, as well as collaborative learners’ training to become active mentors themselves, requires that $r_1$ and $r_2$ both be at least 4. As seen in the table, this combination of parameters gives $R_1 \approx 1.3$ and $R_2 = 1$. From the first threshold parameter, we see that the $E$ class will survive in this case, but normally $R_2 = 1$ is the “tipping point” for the mentor class to survive; at $R_2 = 1$ the mentor class will still normally die out (equilibrium value of 0). In this case, however, the mentors are so capable (as measured by inequality (16) holding) that even a single mentor in the initial group will ensure that the

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2 A brief caveat: Groups as small as that described here may be better described by a stochastic model which takes into account individual variation, but the simpler model applied here provides an illustration of the average dynamics of communities with this general structure.
class eventually flourishes, with the community headed toward a 2/3 participation rate (25% collaborative learners plus 42% mentors). As observed in Table 2, higher values of $r_1$ and $r_2$ (such as 5) result in a more traditional (tipping point-based) survival for the learning community.

The learning environment changes every year, however, with slight changes in location and personnel, and it is difficult to predict before participants arrive at the beginning of the summer how conducive the new collective personality will be toward cooperative learning. If we allow for a slight reduction in effectiveness of the mentors, lowering $r_1$ and $r_2$ to 3.5, then while $R_1$ remains above 1 (again see table), $R_2$ is now squarely below 1, meaning that according to a tipping point perspective, the mentor class should clearly die out. However, in this case inequality (16) still holds, and a large enough initial group of collaborative learners and mentors can sustain the entire community. As indicated in the table, the unstable equilibrium $X_3$, which is on the survival boundary, has about 29% of the group in the $E$ class and about 4.8% in the $I$ class. Any initial influx above this will guarantee a majority collaborative culture, with about 29% in each of the cooperative classes, and indeed our returning students and faculty mentors often make up 1/3 or more of the community, guaranteeing this initial critical core.

In practice, the potential of returning students as mentors is high, as they are selected in part for this reason. Therefore another likely scenario is that in a given year the training of new mentors from these returning students may be expected to proceed efficiently, but the readiness of the new students to switch to a primarily cooperative mode of learning may be unpredictable. The last row of Table 2 presents such a situation, in which $r_2 = 5$ but $r_1 = 2.5$. In this case even $R_1 < 1$, and a tipping point perspective would predict the collapse of the cooperative learning culture. However, once again the strength of the mentor class (measured via (16)) causes a backward bifurcation, so that with that 1/3 initial core of collaborative learners, even the mentor class survives. In this way, a returning core of dedicated mentors and experienced participants has ensured the survival of MTBI’s collaborative learning culture from one year to the next, despite the uncertainties in the thresholds that measure new arrivals’ receptiveness to it.

7. **Conclusion.** Among the challenges facing educational systems today, the search for systemic solutions is a central question. Essential for students to attain the academic standards expected of them are the basic elements of the educational process: available and sufficient materials and quality instruction. Yet many of today’s students are expected to achieve given overcrowded classrooms, lack of

<table>
<thead>
<tr>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$R_1$</th>
<th>$R_2$</th>
<th>$X_3$</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5</td>
<td>3.5</td>
<td>1.2</td>
<td>0.50</td>
<td>(29,4.8), (29,29)</td>
<td>5</td>
</tr>
<tr>
<td>4.0</td>
<td>4.0</td>
<td>1.3</td>
<td>1.0</td>
<td>(25,0), (25,42)</td>
<td>3/5</td>
</tr>
<tr>
<td>5.0</td>
<td>5.0</td>
<td>1.7</td>
<td>2.0</td>
<td>(20,56)</td>
<td>3</td>
</tr>
<tr>
<td>2.5</td>
<td>5.0</td>
<td>0.83</td>
<td>(−1)</td>
<td>(20,13), (20,20)</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 2. Outcomes predicted by Model A, as $r_1$ and $r_2$ vary. In each case $q$ is taken to be 1/3, and inequality (16) is observed to hold.
textbooks and other study guides, and at times with a teacher who does not have expertise in the given subject matter. One proposed solution is the effort underway to reduce the teacher-student ratio in several states, but the costs are high and the process slow. Moreover, the associated battle against teacher shortages does little to ensure that recruits will have developed the skills necessary to be effective educators. In this context, understanding cooperative learning as an educational strategy may provide an additional course of action.

Cooperative learning provides teachers and students an opportunity to work together, distributing academic decision making. In this way, cooperative learning promotes the deconstruction of the current pedagogical structures by encouraging teachers and students to share and even trade roles in the educational process. This benefits both students and educators by unlocking and focusing the creativity and motivation of students.

The model for cooperative learning we have considered in this study is only a simple sketch of a complex human process—community learning—and its value may lie in the insights gained from examining the role of structure and hierarchy in the learning process at the population level. Analysis of our model shows that “minimal” investments even under difficult conditions (here measured by $R_2 < 1$) in cooperative learning can establish resilient communities of learners.

As the model demonstrates, positive interactions among all the students, especially between the novice and intermediate students, are essential for the establishment of learning communities with limited resources ($R_2 < 1$). More rigid hierarchical systems ($q = 0$) preclude the second threshold described above—where the successful establishment of a cooperative learning environment depends on the proportion of motivated students initially participating, instead of on purely environmental conditions. Hence, rigid hierarchical learning environments only support fragile cooperative structures ($I$ class small, $R_2 > 1$), while less hierarchical learning environments ($q > 0$) are more resilient. In such cultures ($q > 0$), students’ individual learning and enthusiasm have an opportunity to contribute positively to the establishment of a culture of learning. The fact that the culture is built by everybody guarantees pride via ownership.

While such learning communities are able to withstand changes in some of the conditions (0 < $q < R_2 < 1$), we also find that what affects community intellectual resilience most is the loss of the highly motivated class’ cooperation and lowering the proportion of moderately cooperative individuals’ participation (critical mass threshold). Losses in the energy and leadership of the highly motivated learners causes the cooperative learning environment to be weakened against changes in the learning conditions. In other words, if the highly motivated learners interact less within the cooperative learning environment, pressures like lack of resources or the teacher’s time are more likely to break down the cooperative learning environment. Similarly, too few moderately cooperative students participating brings the cooperative learning environment closer to becoming a hierarchical learning situation which is more dependent on the learning conditions.

It is worth noticing that the simple model introduced here and its analysis have been used as a metaphor to help understand the value of cooperation in learning. A similar metaphor was used by Gladwell [18] to provide some possible mechanism for the reduction of crime in New York City. The results of this paper show that if “crime” is a multilevel cooperative activity then such reduction in crime cannot be explained by a simple sharp threshold argument.
Extensions of the above framework are under way and preliminary results indicate that the dynamics of collaborative behavior in interacting communities is rather complex and driven by multiple thresholds. It is not surprising to see the appearance of backward bifurcations at many levels. If this is the case, then there is even more good news for those who favor the spread of positive collaborative behavior. There is, however, bad news as well. The establishment of a community of learners under adverse conditions ($R_2 < 1$) depends on the existence of a core group (the subpopulation immersed in a collaborative environment). The existence of such a core group (critical mass) is essential and it may depend on administrative decisions and its ability to recruit from the general population. Models that look at conditions for the existence and survival of core subpopulations are under consideration.

In summary, inspired by our informal experiment on cooperative learning over the past decade at MTBI, we developed a model of cooperative learning where cooperation spreads as an infectious agent. There are many studies that show that cooperative learning can have a positive impact on the academic achievement of students (e.g., [16]). The introduction and the analysis of a simple mathematical model let us examine the characteristics and power of cooperative learning environments. This information helps us better understand what may make collaboration in learning an effective educational strategy (at the population level, that is, in a systemic way). We showed that under the right (sometimes rather weak) conditions a group of cooperative learners can be established. We examined its resilience to change theoretically ($0 < q < R_2 < 1$). A broad extrapolation of the discussion instigated by this model shows that the establishment of a culture of cooperative learning requires the investment of materials (both human and capital) necessary to achieve a cooperative atmosphere, and that, although maintenance is necessary, its survival (atmosphere) is not nearly as difficult as its establishment. The study of educational strategies that enhance learning at the population (community) level is essential in our efforts to facilitate the spread (invasion) of these successful approaches at large population scales.

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