EXISTENCE AND COMPARISON RESULTS FOR DIFFERENTIAL EQUATIONS OF SOBOLEV TYPE

by

A. S. Vatsala
Department of Mathematics
University of Texas at Arlington

and

R. L. Vaughn*
Department of Mathematics
Texas Christian University
Fort Worth, Texas

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1. *INTRODUCTION*

Recently a new class of differential equations, called differential equations of Sobolev type was studied in [4] in which an existence theorem of Picard type was investigated as well as a variation of constants formula. In [5], existence and comparison results for a class of Volterra integral equation of Sobolev-type were discussed.

In this paper we recall the Peano type existence result from [5] for Sobolev-differential equations and show that solutions can be extended to the entire square under consideration. This result extends results found in [1] for nonlinear Volterra integral equations. Our results include a comparison result in addition to the usual type of differential inequalities, and a differential inequality theorem such as Müller's [6]. This in turn proves the existence of extremal solutions. For special cases of the above results see [2,3,7].
2. LOCAL EXISTENCE

We consider equations of the form

\[ u'(t, x) = f(t, x, u(t, x), u(x, t)), \quad u(t_0, x) = u_0(x), \quad \frac{d}{dt}, \quad \text{(2.1)} \]

where \( u_0 \in C[J, \mathbb{R}^n] \), \( J = [t_0, t_0 + \alpha] \) and \( f \in C[J \times J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n] \).

We need the following assumptions and definitions:

\((A_1)\) \[ |f(t, x, u, v)| \leq M \quad \text{for all} \quad (t, x, u, v) \in J \times J \times \mathbb{R}^n \times \mathbb{R}^n. \]

\((A_2)\) \[ \lim_{x_1 \to x_2} \sup_{\phi \in C[J \times J \times \mathbb{R}^n]} \int_I \left| f(s, x_1, \phi(s, x_1), \phi(x_1, s)) - f(s, x_2, \phi(s, x_2), \phi(x_2, s)) \right| ds = 0. \]

\((A_3)\) \[ \lim_{x_1 \to x_2} \sup_{\psi \in C[J \times J \times \mathbb{R}^n]} \left\{ \sup_{\phi \in C[J \times J \times \mathbb{R}^n]} \int_I \left| f(s, x_1, \phi(s, x_1), \psi(x_1, s)) - f(s, x_2, \phi(s, x_2), \psi(x_2, s)) \right| ds \right\} = 0. \]

\((A_4)\) \[ |f(t, x, u, v) - f(t, x, u, \bar{v})| \leq D|u - \bar{u}|. \]

We now prove the following existence result.

**Theorem 2.1.** Suppose that \( u_0 \in C[J, \mathbb{R}^n], f \in C[J \times J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n] \) satisfying the assumptions \((A_1)\) and \((A_2)\). Then a solution to (2.1) exists on \([t_0, t_0 + \alpha]\) for some \( \alpha > 0 \).

**Proof:** Since \( u_0 \) is continuous on \( J \), \( u_0(J) \) is bounded and uniformly continuous. Thus \( \exists \ N > 0 \) so that

\[ |u_0(x) - u_0(\bar{x})| < N \quad \text{for every} \quad x, \bar{x} \in J. \]
Let \( \alpha = \min\{\alpha, \frac{N}{M} \} \) and let \( J_\alpha = [t_0, t_0 + \alpha] \).

Define \( A \subseteq C[J_0 \times J_\alpha, R^n] \) by

\[
A = \{ \phi \in C[J_\alpha \times J_\alpha, R^n] : \sup_{t, x \in J_\alpha} ||\phi(t, x) - u_0(x)|| \leq N \}.
\]

Clearly \( A \) is closed, bounded and convex.

For any \( \phi \in A \), define the function \( T\phi \) by

\[
(T\phi)(t, x) = u_0(x) + \int_{t_0}^{t} ||f(s, x, \phi(s, x), \phi(s, \cdot))|| ds.
\]

Then \( \int_{t_0}^{t} ||f(s, x, \phi(s, x), \phi(s, \cdot))|| ds \leq \alpha M \leq N \).

Thus \( TA \subseteq A \).

Also \( \int_{t_0}^{t} ||f(s, x, \phi(s, x), \phi(s, \cdot))|| ds \leq \sup_{x \in J_\alpha} ||u_0(x)|| + N \). Thus \( TA \) is uniformly bounded.

We now show that \( TA \) is equicontinuous. Let \( \varepsilon > 0 \) be given and let \( t_1, x_1, t_2, x_2 \in J_\alpha \), then

\[
||T\phi(t_1, x_1) - T\phi(t_2, x_2)|| \leq ||u_0(x_1) - u_0(x_2)|| + \int_{t_1}^{t_2} ||f(s, x_2, \phi(s, x_2), \phi(s, \cdot))|| ds.
\]
\[ + \int_{t_0}^{t_1} \left| f(e, x_2, \phi(e, x_2), \psi(x, e)) - f(e, x_1, \phi(e, x_1), \psi(x, e)) \right| ds \]
\[ = I_1 + I_2 + I_3. \]

Since \( u_0(x) \) is uniformly continuous, we can choose \( \delta_1 \), so that \( |x_1 - x_2| < \delta_1 \Rightarrow I_1 < \epsilon/\delta \). Also \( I_2 < (t_2 - t_1)M \), thus if \( |t_2 - t_1| < \frac{\epsilon}{\delta M} = \delta_2 \), \( I_2 < \epsilon/\delta \). Now using \( (A_2') \) we can choose \( \delta_3 \) so that \( |x_1 - x_2| < \delta_3 \Rightarrow I_3 < \epsilon/\delta \).

Thus if \( \max\{|t_1 - t_2|, |x_1 - x_2|\} < \min\{\delta_1, \delta_2, \delta_3\} = \delta \)
\[ |\left| (T\psi)(t_1, x_1) - (T\psi)(t_2, x_2) \right| | < \epsilon. \]

Thus \( TA \) is equicontinuous, and \( \overline{TA} \) is compact.

Now let \( \{\phi_n\} \in A \) be a sequence converging to \( \psi \). Since \( f \) is continuous
\[ \int_{t_0}^{t} f(e, x, \phi_n(e, x), \phi_n(x, e)) ds = \int_{t_0}^{t} f(e, x, \psi(e, x), \psi(x, e)) ds. \]

Thus \( T\phi_n \rightarrow T\psi \) and therefore \( T \) is continuous. Now applying the Schauder fixed point Theorem, the proof is complete.

Our next result provides conditions for the extension of solutions to equation (2.1).

**Theorem 2.2.** Let \( u_0 \in C[J, \mathbb{R}^n] \), and \( f \in C[J \times J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n] \), and suppose that assumptions \((A_1)\), \((A_3)\) and \((A_4)\) hold. Then any solution,
\( u \), of (2.1) which exists on \( J_\alpha \times J_\alpha \) can be extended to \( J_\beta \times J_\beta \)
where \( \beta = \min(2\alpha,\alpha) \).

**Proof**: Let \( u \) be a solution of (1.1). Let \( \gamma = \min(\alpha/2,\alpha) \). Restrict \( u \) to \( J_{\gamma} \times J_{\gamma} \).

Consider the equation

\[
U'(t,x) = F(t,x,u(t,x),U(x,t)) , \quad U(t_0,x) = U_0(x) ,
\]

(2.2)

where \( U_0(x) = (u(t_0+\gamma,x), u_0(x+\gamma)) \),

and \( F(t,x,V,W) = (f(t+\gamma,x,u_1(x),w_1), f(t,x+\gamma,w_1,u_2)) \)

where \( U = (u_1,u_2) \), \( W = (w_1,w_2) \) with \( u_i, w_i \in \mathbb{R}^n \) for \( i = 1,2 \).

It is clear that \( U_0 \in C[J_{\gamma,R_{2n}}] \) and that \( F \in C[J_{\gamma} \times J_{\gamma} \times \mathbb{R}_{2n} \times \mathbb{R}_{2n}, \mathbb{R}_{2n}] \)
and it is easy to verify that

\[
|U_0(x) - U_0(\bar{x})| \leq \sqrt{10} N \quad \text{and} \quad |F(t,x,V,W)| \leq \sqrt{5} M \quad \text{for}
\]

\( t,x,\bar{x} \in J_{\gamma} \).

Thus there exists a solution \( U(t,x) = (u_1(t,x),u_2(t,x)) \) to (2.2) on \( J_{\beta} \times J_{\beta} \) where \( \beta = \min(\gamma,\sqrt{5} N/M) = \gamma \).

Note that \( u_1'(t,x) = f(t+\gamma,x,u_1(t,x),u_2(x,t)) , u_2'(t,x) \)

\( = f(t,x+\gamma,u_2(t,x),u_1(x,t)) \), and \( u_1(x_0,s) \equiv u(x_0+t,\gamma,s) \).

Now let \( m(t) = |u_2(t,x_0) - u(t,x_0+\gamma)| \), then \( m(0) = 0 \), and by assumption (\( A_4 \)) \( m'(t) \leq \left| f(t,x_0+\gamma,u_2(t,x_0),u_1(x_0,t)) \right| \)

\( - f(t,x_0+\gamma, u(t,x_0+\gamma), u(x_0+\gamma,t)) \) \leq \( \ell m(t) \). Thus \( m(t) \equiv 0 \), and so \( u_2(t,x_0) \equiv u(t,x_0+\gamma) \).

Now consider the equation
\[ \tilde{u}'(t,x) = \tilde{f}(t,x,\tilde{u}(t,x),\tilde{u}(x, t)), \tilde{u}(t_0,x) = u_2(t_0+\gamma,x), \]  
\hspace{1cm} (2.3)

where \( \tilde{f} \) is defined by \( \tilde{f}(t,x,u,w) = f(t+\gamma,x+\gamma,u,w) \).

Using Theorem 2.1 we conclude that there exists a solution \( \tilde{u}(t,x) \) to (2.3) on \( J_\gamma \times J_\gamma \), and using (A_4), as above we find that \( \tilde{u}(t,x_0) \equiv u_1(t,x_0+\gamma) \).

Now define the function \( \bar{u}(t,x) \) on \( J_\beta \times J_\beta \) as follows:

\[
\bar{u}(t,x) = \begin{cases} 
  u(t,x) & t,x \in J_\gamma \\
  u_1(t-\gamma,x) & t \in [t_0+\gamma, t_0+\beta], \ x \in J_\gamma \\
  u_2(t,x-\gamma) & t \in [t_0+\gamma, t_0+\beta], \ x \in J_\gamma \\
  \tilde{u}(t-\gamma,x-\gamma) & t,x \in [t_0+\gamma, t_0+\beta]. 
\end{cases}
\]

We need only establish that \( \bar{u} \) is an extension of \( u \). We verify one case, suppose that \( t \in [t_0+\gamma, t_0+\beta], \ x \in J_\gamma \), then

\[
\bar{u}_1'(t,x) = u_1'(t-\gamma,x) = f(t,x,u_1(t-\gamma,x),u_2(x,t-\gamma)) \\
= f(t,x,\bar{u}(t,x),\bar{u}(x,t)).
\]

The other cases are similar. Thus \( \bar{u} \) extends the solution \( u \) to \( J_\beta \times J_\beta \).

Remark: The above theorem can be used to extend solution of (2.1) to \( J \times J \) as long as \( \alpha < +\infty \). One can easily see that if \( \alpha \geq \alpha/2 \) that the value of \( \beta \) in Theorem 2.2 is \( \alpha \). For values of \( \alpha < \alpha/2 \) one needs only to repeat the above argument a finite number of times to extend \( u \) to \( J \times J \).
Corollary 2.3. Let \( u_0 \) and \( f \) be as in Theorem 2.2. Then solutions to (2.1) can be extended to \([t_0, t_0 + \alpha] \times [t_0, t_0 + \alpha]\) as long as \( \alpha \) is finite.

3. DIFFERENTIAL INEQUALITIES

In this section, we develop the theory of differential inequalities.

Consider the following system of differential inequalities

\[
D^- u(t, x) \leq f(t, x, u(t, x), u(x, t)) \tag{3.1}
\]

\[
D^- v(t, x) \geq f(t, x, v(t, x), v(x, t)) \tag{3.2}
\]

where \( D^- v(t, x) = \lim_{h \to 0} \inf \frac{v(t+h, x) - v(t, x)}{h} \)

Definition 3.1. A function \( f(t, x, u, v) \in C([J \times J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]) \) is said to be quasimonotone nondecreasing where \( u, v \in C([J \times J, \mathbb{R}^n]) \) whenever

\[
f_j(t, x, u_j, v) \leq f_j(t, x, \bar{u}_j, v) \quad \text{where} \quad u_j \leq \bar{u}_j \quad \text{and} \quad u_j = \bar{u}_j \quad \text{for every} \quad j, \quad j = 1, 2, \ldots, n.
\]

Theorem 3.1. Let \( f \) be (i) quasimonotone nondecreasing in \( u(t, x) \) and non-decreasing in \( u(x, t) \) on \( J \times J \), then if further if one of the inequalities above is strict and \( u(t_0, x) < v(t_0, x) \), then \( u(t, x) < v(t, x) \) on \( J \times J \).

Proof: If the conclusion is not true, consider the set

\[
Z(t, x) = \{(t, x) \mid u(t, x) \geq v(t, x), u(x, t) \geq v(x, t)\}
\]

which is nonempty.

Let \( Z_t \) be the projection of \( Z \) on the \( t \) axis. Let \( t_1 = \inf Z_t \).
Certainly $t_1 > t_0$. It then follows that there is an index $j$, $1 \leq j \leq n$ and for $i = 1, 2, \ldots, n$

$$u_i(s, x) < v_i(s, x) \quad \text{for } s, x \in [t_0, t_1] \times [t_0, t_0 + a]$$

$$u_i(x, s) < v_i(x, s) \quad \text{for } x, s \in [t_0, t_0 + a] \times [t_0, t_1].$$

and either

$$u_j(t_1, x) \leq v_j(t_1, x)$$

or

$$u_j(x, t_1) \leq v_j(x, t_1) \quad \forall x \in J.$$

Consequently there is an $x_1 \in J$ such that either

$$u_j(t_1, x_1) = v_j(t_1, x_1) \tag{3.3}$$

or

$$u_j(x_1, t_1) = v_j(x_1, t_1) \tag{3.4}.$$

Let $x_1$ be the minimum value of $x$ for which (3.3) or (3.4) happens.

Certainly $x_1 > t_0$.

If (3.3) happens, then

$$D_- u_j(t_1, x_1) = \liminf_{h \to 0^-} \frac{u_j(t_1 + h, x_1) - u_j(t_1, x_1)}{h}$$

$$> \liminf_{h \to 0^-} \frac{v_j(t_1 + h, x_1) - v_j(t_1, x_1)}{h} = D_- v_j(t_1, x_1).$$
But by hypothesis

\[ D_u(t_1, x_1) \leq f_j(t_1, x_1, u(t_1, x_1), u(x_1, t_1)) \]
\[ \leq f_j(t_1, x_1, v(t_1, x_1), v(x_1, t_1)) \leq D_v(t_1, x_1) \]

which leads to a contradiction.

If (3.4) happens we have \( u_j(x_1, t_1) = v_j(x_1, t_1) \). Let \( x_1 = \bar{x}, \ t_1 = \bar{t} \) ie \( u_j(\bar{x}, \bar{x}) = v_j(\bar{x}, \bar{x}) \) and \( u_j(\bar{x} + h, \bar{x}) < v_j(\bar{x} + h, \bar{x}) \),

for \( h < 0 \) by definition of \( x_1 \) and \( t_1 \). Therefore

\[ D_u(\bar{x}, \bar{x}) = \liminf_{h \to 0^-} \frac{u_j(\bar{x} + h, \bar{x}) - u_j(\bar{x}, \bar{x})}{h} \]
\[ > \liminf_{h \to 0^-} v_j(\bar{x} + h, \bar{x}) - v_j(\bar{x}, \bar{x}) = D_v(t_1, x_1) \]

but by hypothesis

\[ D_u(\bar{x}, \bar{x}) \leq f_j(\bar{x}, \bar{x}, u(\bar{x}, \bar{x}), u(\bar{x}, \bar{x})) \]
\[ \leq f_j(\bar{x}, \bar{x}, v(\bar{x}, \bar{x}), v(\bar{x}, \bar{x})) < D_v(\bar{x}, \bar{x}) \]

hence a contradiction and the theorem is complete.

**Remark 3.1.** The above theorem is true if the Dini derivative is replaced by any fixed Dini derivative. See for details [3].

If one of the inequalities (3.1), (3.2) is not assumed strict, the conclusion of theorem (3.1) fails to hold. However if \( f \) satisfies
a onesided Lipschitz condition, we get the following result.

**Theorem 3.2.** Let the assumption (i) of Theorem (3.1) hold. Suppose further that

\[
f(t, x, v_1, w_1) - f(t, x, v_2, w_2) \leq L[(v_1 - v_2) + (w_1 - w_2)] .
\]  \hspace{1cm} (3.5)

Whenever \( v_1 \geq v_2, \, w_1 \geq w_2 \). Then \( u(t_0, x) \leq v(t_0, x) \) for \( x \in J \)
implies \( u(t, x) \leq v(t, x) \) on \( J \times J \).

**Proof:** Let \( \tilde{v}(t, x) = v(t, x) + \varepsilon e^{3L(t+x)} \) where \( \varepsilon > 0 \) is sufficient-
ly small vector in \( H^n \).

Then \( \tilde{v}'(t, x) = v'(t, x) + 3 \in L e^{3L(t+x)} \). That is

\[
\tilde{v}'(t, x) = f(t, x, v(t, x), v(x, t)) + 3 \in L e^{3L(t+x)} .
\]

\[ \geq f(t, x, \tilde{v}(t, x), \tilde{v}(x, t) + \varepsilon e^{3L(t+x)} .
\]

consequently we have

\[
\tilde{v}'(t, x) > f(t, x, \tilde{v}(t, x), \tilde{v}(x, t)) .
\]  \hspace{1cm} (3.6)

By Remark 3.1 we now get \( u(t, x) < \tilde{v}(t, x) \) on \( J \times J \). Taking the limit as
\( \varepsilon \to 0 \) we conclude \( u(t, x) \leq v(t, x) \) on \( J \times J \) which proves the stated
result. We now obtain bounds for solutions of (2.1) based on the
classical result of Müller's [6].

**Theorem 3.2.** Let \( v, \omega \in C^1[J \times J, H^n] \) satisfy the following inequalities
\[ v^l(t, x) < f^l(t, x, z(t, x), z'(x, t)) \quad \text{whenever} \quad v(t, x) < z(t, x) < w(t, x) \]

and \[ v^l(t, x) = z(t, x) \quad (3.7) \]

\[ w^l(t, x) > f^l(t, x, z(t, x), z'(x, t)) \quad \text{whenever} \quad v(t, x) < z(t, x) < w(t, x) \]

and \[ z(t, x) = w^l(t, x) \quad (3.8) \]

Then if \( v(t_0, x) < u(t_0, x) < w(t_0, x) \) then \( v(t, x) < u(t, x) < w(t, x) \)

where \( u(t, x) \) is the solution to (2.1).

**Proof:** If not, consider the set

\[ P(t, x) = \bigcup_{i=1}^n \{(t, x) \mid v^i(t, x) \geq u^i(t, x) \geq \omega^i(t, x) \quad \text{or} \quad v^i(x, t) \geq u^i(x, t) \geq \omega^i(x, t) \} \]

Let \( P_t \) be the projection of \( P \) on the \( t \) axis. Let \( t_1 = \inf P_t \).

Certainly \( t_1 > t_0 \) then there exists a \( j \) and an \( x_1 \) such that

\[ v^j(t_1, x) < u^j(t_1, x) < \omega^j(t_1, x) \]

\[ v^j(x, t_1) < u^j(x, t_1) < \omega^j(x, t_1) \quad \text{for} \quad i \neq j \]

and either \[ v^j(t_1, x) < u^j(t_1, x) < \omega^j(t_1, x) \quad \text{for all} \quad x \]

or

\[ v^j(x, t_1) < u^j(x, t_1) < \omega^j(x, t_1) \quad \text{for all} \quad x \]

Let \( x \) be the minimum value of \( x \) for which either \( v^j(t_1, x_1) = u^j(t_1, x_1) \)
or
\[ v_j(t_1, x_1) = u_j(t_1, x_1) \]
or
\[ u_j(t_1, x_1) = w_j(t_1, x_1) \]
or
\[ v_j(x_1, t_1) = u_j(x_1, t_1) \]
or
\[ v_j(x_1, t_1) = w_j(x_1, t_1) \].

If \( v_j(t_1, x_1) = u_j(t_1, x_1) \) then by (3.7) \( v_j'(t_1, x_1) \)
\[ < f_j(t_1, x_1, u(t_1, x_1), u(x_1, t_1)) = u_j'(t_1, x_1) \]
but \( v_j(t_1 - h, x_1) < u_j(t_1 - h, x_1) \)
hence \( v_j'(t_1, x_1) \geq u_j'(t_1, x_1) \). This implies \( u_j'(t_1, x_1) \leq v_j'(t_1, x_1) \)
\[ < u_j(t_1, x_1) \] which is absurd. Similarly we can arrive at a contradiction when \( u_j(t_1, x_1) = w_j(t_1, x_1) \).

Suppose \( v_j(x_1, t_1) = u_j(x_1, t_1) \). Let \( x_1 = \tilde{x}, \tilde{t}_1 = \tilde{x} \). Then
\[ v_j(\tilde{x}, \tilde{x}) = u_j(\tilde{x}, \tilde{x}) \] also \( v_j(\tilde{x} - h, \tilde{x}) < u_j(\tilde{x} - h, \tilde{x}) \) by the definition of \( x_1 \).
\[ u_j(\tilde{x}, \tilde{x}) \leq v_j(\tilde{x}, \tilde{x}) < f_j(\tilde{x}, \tilde{x}, u(\tilde{x}, \tilde{x}), u(\tilde{x}, \tilde{x})) = u_j(\tilde{x}, \tilde{x}) \]
because \( v(\tilde{x}, \tilde{x}) \leq u(\tilde{x}, \tilde{x}) \) and \( v_j(\tilde{x}, \tilde{x}) = u_j(\tilde{x}, \tilde{x}) \)
which leads to a contradiction. In a similar way we can arrive at a contradiction when \( v_j(x_1, t_1) = w_j(x_1, t_1) \).

**Remark 3.2.** The conclusion of the above theorem is not true if the inequalities (3.7), (3.8) are not strict. However if \( f \) satisfies a one-sided Lipschitz condition, of the following type

\[ f(t, x, (w_1(t, x), \ldots, w_n(t, x)), (w_1(x, t), \ldots, w_n(x, t))) \]
\[ - f(t, x, (w_1(t, x), \ldots, w_n(t, x)), w_1(x, t), \ldots, w_n(x, t)) \]
\[ \leq L \left[ (w_i(t, x) - v_i(t, x)) + (w_i(x, t) - v_i(x, t)) \right] \]
(3.9)
Whenever \( v_i'(t,x) \geq u_i'(t,x) \) on \( J \times J \) for \( i = 1, \ldots, n \) the conclusion is valid.

**Proof:** Consider \( v_i'(t,x) = v_i^+(t,x) - \varepsilon e^{3L_i(t+x)} \)
and \( \omega_i'(t,x) = u_i^+(t,x) + \varepsilon e^{3L_i(t+x)} \)
where \( \varepsilon > 0 \) is arbitrarily small vector

now \( \tilde{v}_i'(t,x) = v_i'(t,x) - 3L_i e^{L_i(t+x)} \)
\[ \leq f_i(t,x, a_1(t,x), \ldots, v_i(t,x), \ldots, a_n(t,x)), (a_1(x,t), \ldots, v_i(x,t), \ldots, a_n(x,t)) \]
\[ - 3 \in L_i e^{L_i(t+x)} \]
\[ \leq f_i(t,x, a_1(t,x), \ldots, \tilde{v}_i(t,x), \ldots, a_n(t,x)), (a_1(x,t), \ldots, \tilde{v}_i(x,t), \ldots, a_n(x,t)) \]
\[ - \varepsilon L_i e^{L_i(t+x)} \]

That is
\[ \tilde{v}_i(t,x) < f_i(t,x, a_1(t,x), \ldots, \tilde{v}_i(t,x), \ldots, a_n(t,x)) \]
\[ \leq f_i(t,x, a(t,x), a(x,t)) \] whenever \( \tilde{v}_i(t,x) \leq a(t,x) \leq \tilde{u}_i(t,x) \) and
\[ \tilde{v}_i(t,x) = a_i(t,x). \]

Similarly we can prove
\[ \tilde{w}_i'(t,x) > f_i(t,x, a(t,x), a(x,t)) \] whenever \( \tilde{w}_i(t,x) \leq a_i(t,x) \leq \tilde{w}_i(t,x) \) and
\[ \tilde{w}_i(t,x) = a_i(t,x). \]

Now from Theorem 3.2 it follows that
\[ \bar{V}_\epsilon(t,x) < u_\epsilon(t,x) < \bar{V}_\epsilon(t,x) \]

Taking the limit as \( \epsilon \to 0 \) it follows that

\[ v_\epsilon(t,x) \leq u_\epsilon(t,x) \leq \omega_\epsilon(t,x) \quad \text{on } J \times J. \]

Remark: The existence of extremal solutions for (2.1) can be proved by using the results of Theorem 2.1, 2.2 and by Remark 3.1. The proof follows the same lines as in [3,5].
REFERENCES


