MONOTONE METHOD FOR NONLINEAR BOUNDARY VALUE PROBLEMS ARISING IN TRANSPORT PROCESS

by

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1. INTRODUCTION

Recently, monotone iterative methods have been successfully employed to prove existence of multiple solutions and point-wise bounds on solutions of nonlinear boundary value problems for both ordinary and partial differential equations (see, [1], [3]-[6], [9]). In the transport process [10] of different types of particles in a finite rod of length (b-a) the equation governing the particle's density is given by the following linear system of equations

\[ \begin{align*}
 x' + A_0(t)x & = A_1(t)x + A_2(t)y + p(t) \\
 -y' + B_0(t)y & = B_1(t)x + B_2(t)y + q(t)
\end{align*} \quad a \leq t \leq b,
\]

where \( A_i, B_i \) \((i = 0, 1, 2)\) are \( n \times n \) matrices and \( x, y, p, q \) are \( n \)-vectors. The components \( x_1, \ldots, x_n \) of the vector \( x \) represent the \( n \) distinct type of particles moving in the forward direction along the rod while the components \( y_1, \ldots, y_n \) of \( y \) are the ones moving in the backward direction. When the ends of the rod are subjected to incident fluxes, the boundary conditions become

\[ x(a) = x_a, \quad y(b) = y_b, \]

where the vectors \( x_a, y_b \) are given. Physical reasons demand that \( A_0, B_0 \) are diagonal matrices and all the elements in the matrices \( A_i, B_i \) \((i = 0, 1, 2)\) are nonnegative functions on \([a, b]\). This specific boundary value problem has been investigated by the method of successive approximations in [2,8] and by monotone method in [7]. Because of the importance of this problem in other physical applications, we extend in this paper the monotone technique to a general class of nonlinear boundary value problem which includes the transport problem treated in [2,7] as a special case.
2. MAIN RESULTS

Consider the boundary value problem (BVP)

\[ \begin{align*}
    x'(t) &= f(t,x,y), \quad x(a) = x_a, \quad a \leq t \leq b \\
    -y'(t) &= g(t,x,y), \quad y(b) = y_b,
\end{align*} \]

(1)

where \( f, g \in C[I \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n] \), \( I = [a,b] \), \( \mathbb{R}^n \) the n-dimensional Euclidean space, and \( x_a \) and \( y_b \) are given vectors. Without further mention, we assume that all inequalities between vectors are componentwise.

**Definition 1.** A pair of functions \((u,v)\), \( u,v \in C^1[I,\mathbb{R}^n] \) is called an upper solution of (1) if

\[ \begin{align*}
    u'(t) &\geq f(t,u,v), \quad u(a) \geq x_a, \\
    -v'(t) &\geq g(t,u,v), \quad v(b) \geq y_b,
\end{align*} \]

(2)

and a lower solution if all the inequalities in (2) are reversed.

**Definition 2.** The pairs of functions \((\bar{x},\bar{y})\) and \((\underline{x},\underline{y})\) are called maximal and minimal solutions of (1) respectively, if every other solution \((x,y)\) of (1) satisfies the relations

\[ \begin{align*}
    x(t) &\leq x(t) \leq \bar{x}(t), \quad \underline{y}(t) \leq y(t) \leq \bar{y}(t),
\end{align*} \]

for all \( t \in I \).

In order to develop monotone iterative method for (1), we need the following assumptions:
(H₁) the functions $u = (v,w)$, $\overline{u} = (\overline{v},\overline{w})$, $v, w, \overline{v}, \overline{w} \in C^1[I, R^n]$ with $u(t) \leq \overline{u}(t)$ for $t \in I$ are lower and upper solutions of (1);

(H₂) For each $i$, $f_i(t,x,y)$ is monotone nondecreasing in $y$ and $g_i(t,x,y)$ is monotone nondecreasing in $x$;

(H₃) For each $i$, $f_i(t,x,y) - f_i(t,\overline{x},y) \geq -M(x_1 - \overline{x}_1)$, whenever $\overline{v}(t) \leq \overline{x} \leq x \leq \overline{v}(t)$ for $t \in I$ and $g_i(t,x,y) - g_i(t,\overline{x},\overline{y}) \geq -M(y_1 - \overline{y}_1)$, whenever $\overline{w}(t) \leq \overline{y} \leq y \leq \overline{w}(t)$ for $t \in I$, where $M \geq 0$ is a constant.

For any $\mu = (\xi, \eta), \xi, \eta \in C[I, R^n]$ such that $\underline{u} \leq \mu \leq \overline{u}$ on $I$, consider the following linear system

$$
\begin{align*}
{x}'_1 &= f_1(t,\xi,\eta) - M(x_1 - \xi_1), \quad x_1(a) = x_a, \\
-y'_1 &= g_1(t,\xi,\eta) - M(y_1 - \eta_1), \quad y_1(b) = y_b,
\end{align*}
$$

(3)

which possesses a unique solution $u = (x,y)$ defined on $I$. For each $\mu = (\xi, \eta), \xi, \eta \in C[I, R^n]$ such that $\underline{u}(t) \leq \mu(t) \leq \overline{u}(t)$ for $t \in I$, define the mapping $A$ by

$$
A\mu = u
$$

where $u = (x,y)$ is the unique solution of (3). This mapping will be used to define the sequences that converge to the minimal and the maximal solutions of the BVP (1). For this purpose, we need the following lemma.

**Lemma 1.** Assume that the hypotheses (H₁)-(H₃) hold. Then

(i) the unique solution $u = (x,y)$ of (3) satisfies $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$ for $t \in I$;

(ii) $u \leq A\mu, \quad \overline{u} \geq A\overline{u}$;
(iii) A is monotone operator on the segment

$$\{u, u\} = \{z = (z_1, z_2), z_1, z_2 \in \mathbb{R}^n : u(t) \leq z \leq u(t), t \in I\}.$$ 

Proof. To prove (i), we shall first show $v(t) \leq x(t) \leq \bar{v}(t)$ for $t \in I$.

Define, $p_1(t) = x_1(t) - v_1(t)$. Then $p_1(a) = x_1(a) - v_1(a) \geq x_a - x_a = 0$ and

$$p_1'(t) = x_1'(t) - v_1'(t) \geq f_1(t, \xi(t), \eta(t)) - M(x_1(t) - \xi_1(t)) - f_1(t, v(t), \bar{v}(t))$$

$$\geq f_1(t, \xi(t), \bar{w}(t)) - M(x_1(t) - \xi_1(t)) - f_1(t, v(t), \bar{w}(t))$$

$$\geq -M(\xi_1(t) - v_1(t)) - M(x_1(t) - \xi_1(t)) = -M_p_1(t).$$

It then follows that $p_1(t) \geq p_1(a)e^{-M(t-a)} \geq 0$ which implies $x_1(t) \geq v_1(t)$ and hence $v(t) \leq x(t)$ for $t \in I$. Similarly, we can show that $x(t) \leq \bar{v}(t)$ for $t \in I$. This proves $v(t) \leq x(t) \leq \bar{v}(t)$ for $t \in I$. Next, we shall prove that $w(t) \leq y(t) \leq \bar{w}(t)$ for $t \in I$. We set $\phi_1(t) = y_1(t) - w_1(t)$, and note $\phi_1(b) = y_1(b) - w_1(b) \geq y_b - y_b = 0$. Also

$$\phi_1'(t) = y_1'(t) - w_1'(t) \leq -g_1(t, \xi(t), \eta(t)) + g_1(t, v(t), \bar{w}(t))$$

$$\leq -g_1(t, \xi(t), \eta(t)) + g_1(t, \xi(t), \bar{w}(t)) + M(y_1(t) - \eta_1(t))$$

$$\leq M(\eta_1(t) - w_1(t)) + M(y_1(t) - \eta_1(t)) = M\phi_1(t).$$

From this it is easy to obtain the estimate $\phi_1(t) \geq \phi_1(b)e^{-M(b-t)} \geq 0$ which implies $y_1(t) \geq w_1(t)$ and hence $w(t) \leq y(t)$ for $t \in I$. Similarly, we can obtain that $y(t) \leq \bar{w}(t)$ for $t \in I$. This proves $w(t) \leq y(t) \leq \bar{w}(t)$ for $t \in I$. Thus, we have $u(t) \leq u(t) \leq u(t)$ for $t \in I$. 

...
Let $Au = u$, where $u = (x,y)$ is the unique solution of (3) corresponding to $u = (v,w)$. Now, we shall prove that $u \leq u$, i.e., $v \leq x$ and $w \leq y$ for $t \in I$. Setting $\phi_1(t) = x_1(t) - v_1(t)$, we have $\phi_1(a) = x_1(a) - v_1(a) \geq 0$. \[\phi_i(t) = x_i(t) - v_i(t) \geq f_i(t,v(t),w(t)) - M(x_1(t) - v_1(t)) - f_i(t,v(t),w(t)) = -M\phi_1(t),\]

which implies $\phi_1(t) \geq 0$, i.e., $x_1(t) \geq v_1(t)$ and hence $v(t) \leq x(t)$ for $t \in I$. Similarly, we can get that $w(t) \leq y(t)$ for $t \in I$. Thus, we have proved $u \leq Au$ for $t \in I$. Similar arguments show that $\overline{u} \geq A\overline{u}$, proving (ii).

To prove (iii), let $u_1 = (\xi_1,\eta_1)$, $u_2 = (\xi_2,\eta_2)$, $\xi_1,\eta_1,\xi_2,\eta_2 \in C[I,K]$ be such that $u_1, u_2 \in (\overline{v},\overline{u})$ and $u_1 \leq u_2$. Let $Au_1 = u_1$ and $Au_2 = u_2$ where $u_1 = (x_1,y_1)$ and $u_2 = (x_2,y_2)$ are the unique solutions of (3) corresponding to $u_1$ and $u_2$ respectively. Then setting $\phi_1(t) = x_{1i}(t) - x_{2i}(t)$, we have $\phi_1(a) = x_{11}(a) - x_{21}(a) = 0$ and

\[\phi_{1i}(t) = x_{1i}(t) - x_{2i}(t) = f_{1i}(t,\xi_1(t),\eta_1(t)) - M(x_{11}(t) - \xi_{11}(t)) - f_{1i}(t,\xi_2(t),\eta_2(t)) + M(x_{21}(t) - \xi_{21}(t)) \leq f_{1i}(t,\xi_1(t),\eta_1(t)) - f_{1i}(t,\xi_2(t),\eta_2(t)) - M(x_{11}(t) - \xi_{11}(t)) + M(x_{21}(t) - \xi_{21}(t)) \leq M(\xi_{21}(t) - \xi_{11}(t)) - M(x_{11}(t) - \xi_{11}(t)) + M(x_{21}(t) - \xi_{21}(t)) = -M\phi_1(t),\]

which yields $x_{1i}(t) \leq x_{2i}(t)$ i.e., $x_1(t) \leq x_2(t)$ for $t \in I$. Similarly, we can prove that $y_1(t) \leq y_2(t)$ for $t \in I$. It therefore follows that the A is monotone on the segment $(\overline{u},\overline{u})$. 

In view of Lemma 1, we can define the sequences

\[ u_n = A_{n-1}, \quad \overline{u}_n = \overline{A} \overline{u}_{n-1}, \]

where \( u_n = (v_n, w_n), \quad \overline{u}_n = (\overline{v}_n, \overline{w}_n) \) with \( u_0 = u \) and \( \overline{u}_0 = \overline{u} \). It is then easy to see that the sequences \( \{u_n\}, \{\overline{u}_n\} \) are monotone nondecreasing, nonincreasing respectively, and that \( u \leq u_n \leq \overline{u}_n \leq \overline{u} \) on \( I \). Furthermore, using the standard arguments, it follows that these sequences converge uniformly and monotonically to solutions \((x, y)\) and \((\overline{x}, \overline{y})\) of (1).

Let \((x, y)\) be any solution of (1) such that \( v \leq x \leq \overline{v}, \quad w \leq y \leq \overline{w} \). Then, by the induction argument, it is easily seen that \( x \leq \overline{v}_n, \quad y \leq \overline{w}_n \) and \( x \geq v_n, \quad y \geq w_n \) for every \( n = 0, 1, 2, \ldots \). Hence, we have \( x \leq x \leq \overline{x} \) and \( y \leq y \leq \overline{y} \). This shows that \((\overline{x}, \overline{y})\) is a maximal solution and \((x, y)\) is a minimal solution of (1). Thus, we have proved the following theorem.

**Theorem 1.** Assume that the hypotheses \((H_1)-(H_3)\) hold. Then the sequence \((\overline{v}_n, \overline{w}_n)\) converges uniformly from above to a maximal solution \((\overline{x}, \overline{y})\) of (1) while the sequence \((v_n, w_n)\) converge uniformly from below to a minimal solution \((x, y)\) of (1). Furthermore,

\[ v \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq x \leq \overline{x} \leq \cdots \leq \overline{v}_n \leq \cdots \leq \overline{v}_2 \leq \overline{v}_1 \leq \overline{v}, \]

\[ w \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq \cdots \leq y \leq \overline{y} \leq \cdots \leq \overline{w}_n \leq \cdots \leq \overline{w}_2 \leq \overline{w}_1 \leq \overline{w}. \]

**Remark.** We note that the special nature of the BVP (1) is of immense value in developing monotone technique, because the iterative sequences \( \{u_n\}, \{\overline{u}_n\} \) are the solutions of linear initial value problems and consequently easier to compute.
Our next theorem deals with the uniqueness of the solutions of BVP (1).

**Theorem 2.** In addition to the hypotheses of Theorem 1, assume that

\begin{align}
(6) & \quad \|f(t,x,y) - f(t,z,y)\| \leq L\|x-z\|, \\
(7) & \quad \|g(t,x,y) - g(t,x,z)\| \leq N\|y-z\|,
\end{align}

where L and N are nonnegative constants. Then the maximal solution \((\bar{x},\bar{y})\) and the minimal solution \((x,y)\) obtained in Theorem 1 coincide on I, that is \(\bar{x}(t) = x(t), \bar{y}(t) = y(t)\) for \(t \in I\).

**Proof.** Since both \((\bar{x},\bar{y}), (x,y)\) are solutions of the problem (1) we have

\begin{align}
(8) & \quad \bar{x}(t) = x_a + \int_a^t f(s,\bar{x}(s),\bar{y}(s)) \, ds, \\
(9) & \quad \bar{y}(t) = y_b + \int_t^b g(s,\bar{x}(s),\bar{y}(s)) \, ds,
\end{align}

and

\begin{align}
(10) & \quad \underline{x}(t) = x_a + \int_a^t f(s,\underline{x}(s),\underline{y}(s)) \, ds, \\
(11) & \quad \underline{y}(t) = y_b + \int_t^b g(s,\underline{x}(s),\underline{y}(s)) \, ds.
\end{align}

Define \(m(t) = \|\bar{x}(t) - \bar{x}(t)\|\). Then, from (8), (10), (H2) and (6), we obtain \(m(t) \leq L \int_a^t m(s) \, ds\). Note that \(m(a) = 0\). It then follows from \(m(t) \leq c + L \int_a^t m(s) \, ds\) the estimate \(m(t) \leq c e^{L(t-a)}\) for \(t \in I\). This implies, making \(c \to 0\), \(\bar{x}(t) = \underline{x}(t)\) for \(t \in I\). Similarly, we can prove that \(\bar{y}(t) = \underline{y}(t)\) for \(t \in I\). This completes the proof of the theorem.
REFERENCES


