

OPTIMAL DESIGN OF BEAMS

by

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Abstract

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Beams are basic structural components that are capable of withstanding load primarily by resisting bending. Unlike Euler-Bernoulli beams, Timoshenko beams undergo both shear deformation and rotational effects, making it suitable for analyzing the behavior of thick or short beams, composite beams and beams that are subjected to high frequency excitation when their wave length becomes shorter. This thesis work focuses on optimal design of straight and tapered Timoshenko beams under static and dynamic constraints for rectangular and circular cross sections. In this work Timoshenko beam static and dynamic equations were studied. The finite element method was used for static and dynamic analysis of the beam. In finite element method to overcome the numerical problem in shear locking, cubic interpolation of displacement and an interdependent quadratic approximation of rotation has been considered.

In order to optimize the weight of the beam with static and dynamic constraints three sets of optimizations were done. The design variables are length, cross sectional width and height, with objective function as mass, static deflection constraints were used. The second optimization set was using dynamic constraints and the last set was using both static and dynamic constraints.

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Chapter 1

Introduction

Beams have been used since dim antiquity to support loads over empty space, as roof beams supported by thick columns, or as bridges thrown across water, for example. The Egyptians invented the colonnaded building that was the inspiration for the classic Greek temple. Even with the scarcity of timber in Egypt, wooden beams supported the roofs. Early bridges were beams supported at each end by the stream banks, or on piles, on which a deck was constructed for traffic. In either case, the trunk of a tree was the usual beam, trimmed and either left round or squared. Our word "beam" is, in fact, cognate with German *Baum* or Dutch *boom*. A tree makes a very satisfactory beam, indeed, and practically all beams were originally timber beams. Stone beams, as in door lintels, could be used only for very short spans and light loads, because of the brittleness of stone. Brittle materials do not make good beams.

Through the millennia, beams were designed by empirical methods, applicable only to specific cases and incapable of generalization. Galileo studied beams, and although he did not get it quite right, he showed how the subject should be approached. The theory of beams was only perfected in the late 17th century with the rise of the science of elasticity, and was shown to be a subject of great complexity for which a full and accurate solution was very difficult. This remains true even with modern computational methods, such as the method of finite elements, which produces only numbers (not designs) but very little insight, and depends on parameters that are not well

known and models that may contain errors. These methods have great value, but are not a comprehensive solution.

The theory of beams shows remarkably well the power of the approximate methods called "strength of materials methods." These methods depend on the use of statics, superposition and simplifying assumptions that turn out to be very close to the truth. They give approximate, not exact, results that are usually more than adequate for engineering work. Calculus and little differential equations are all the mathematics required for this approach, not the partial differential equations or tensor analysis that are typical tools in elasticity.

Strength of materials methods can be used for beams of arbitrary cross sections, for beams whose shape varies along the length, for loads applied in any direction at any point, distributed or concentrated. Many of these applications are discussed in the first reference, which shows the versatility of the method. The results obtained are fully adequate for engineering design. On the other hand, an accurate and rigorous quantitative solution in these varied cases would be extremely difficult and usually impossible.

Two versions of theories have been developed for analysis of beams. In Euler-Bernoulli theory, the displacement of beams is considered without shear effects. This method gives appropriate and acceptable response in thin beam in which shear effect is insignificant. However this approach, by increasing the thickness of beam and shear effect deformation, the error of response is increasing [1]. Correspondingly, the effect of shear transformation is formulated in Timoshenko Beam Theory. Therefore, this method has a better result, especially in deep/thick beams in which shear effect is impressive.

Although the rotational inertia of thick beams was investigated by Rayleigh for the first time, Timoshenko has developed this theory and formulated shear effect. Due to the complexity of the governing equations of the static and free vibrations of beams in general, numerical methods such as finite element methods have been developed profoundly. Up to now, many elements have been proposed, based on Timoshenko theory. These elements are classified into two groups which are simple and high-order elements. Some researchers used simple Two Node elements with four degrees of freedom [2-4] Thomas et al. have examined the elements proposed by other researchers [3].

The first high-order element was proposed by Kapur with eight degrees of freedom [5]. Lees and Thomas formulated a complex element by applying independent polynomial series for displacement and rotational fields [6, 7]. Also, this method has been used by Webster [8]. Rao and Gupta have examined free vibrations of rotating beams [9]. W. L. Cleghorn and B. Tabarrok [10] have proposed a finite element formulation for tapered beam elements by providing element matrices for a tapered Timoshenko beam. A cubic polynomial was employed for the deflection distribution and a linear distribution for a shear strain. A similar approach was developed by C. W. S To [11] has been employed for the free lateral vibrations of linearly tapered Timoshenko beams considering both shear deformation and rotary inertia. Leszek Majkut [12] presented a new approach to description of the Timoshenko beam free and forced vibration by a single equation. He has employed Green's function for describing the forced vibrations of the Timoshenko beams vibrational analysis.

In some methods like, isoparametric formulation, displacement and rotational fields are assumed dependently with the same order [13]. Based on Euler-Bernoulli theory, Gonclaves et al. have presented frequency equation and vibration modes for classical boundary conditions such as clamped, free, pinned and sliding supports [14]. Lee and Schultz have considered free vibration of Timoshenko beam through Psuedospectral method [15]. Starting from the early 1960's, a number of papers by Mabie and Rogers [16] presented the exact frequency equations for various tapered beams with classical boundary conditions, by using Bessel function theory.

Chapter 2

Uniform Timoshenko Beam Analysis

2.1 Static Analysis of Uniform Timoshenko Beam

The Euler-Bernoulli beam theory of beams does not include the effects of shear deformation. For short stubby beams this contribution clearly cannot be neglected, and for this reason we include both shear deformation and rotational inertia effects such that, making it suitable for describing the behavior of short beams, sandwich composite beams or beams that are subjected to high frequency excitation when the wavelength approaches the thickness of the beam. So, physically Timoshenko's theory efficiently lowers the stiffness of beam and the result is a larger deflection under a static load and lower predicted Eigen frequencies for a given set of boundary conditions. The latter effect is more noticeable for higher frequencies as the wavelength becomes shorter, and thus the distance between opposing shear forces decreases.

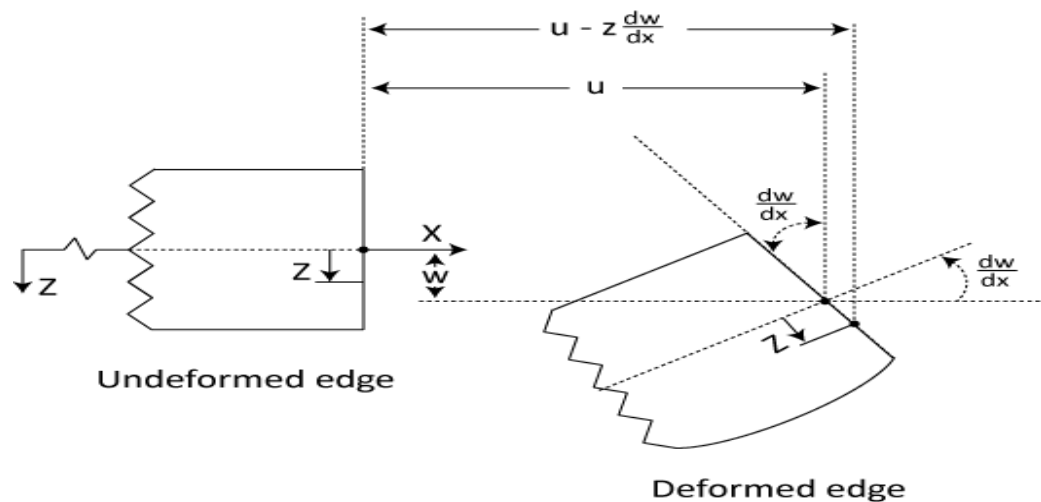


Figure 2.1 Deformation of Euler-Bernoulli Beam Element

In Timoshenko Beam Theory (TBT), we will assume that plane cross section remains plane but not necessarily normal to the longitudinal axis after deformation, i.e. transverse shear rotation $\beta(x)$ is not equal to zero. Therefore, the total rotation of a transverse plane about the Y-axis is sum of rotation due to bending and shear deformation.

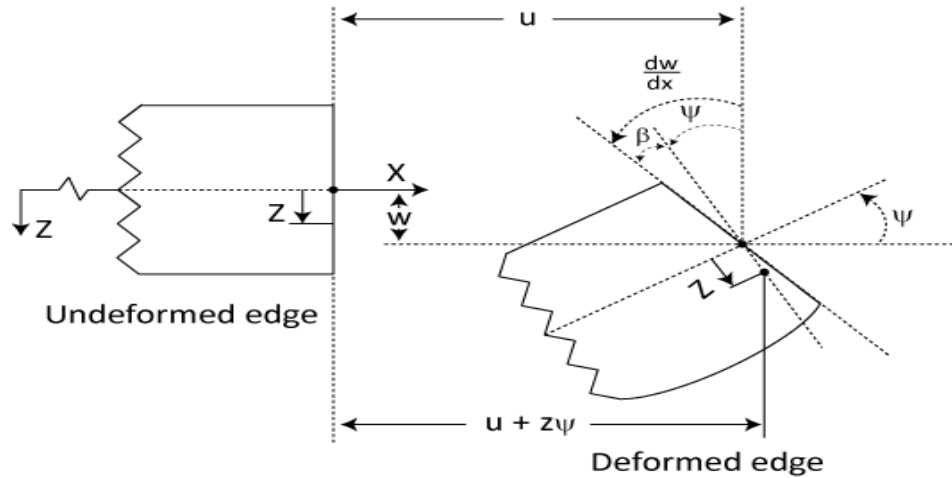


Figure 2.2 Deformation of Timoshenko Beam Element

$$\theta = \partial w / \partial x = \psi(x) + \beta(x) \quad (2.1)$$

$\theta = \partial w / \partial x =$ Total transverse rotation of plane about Y-axis

$\psi(x) =$ Rotation transverse plane about Y-axis due to bending only

$\beta(x) =$ Rotation transverse plane about Y-axis due to shear deformation

The displacement field can now be considered as the superposition of the bending and shear deformation

$$\begin{aligned}
u(x, y, z) &= -z\psi(x) \\
v(x, y, z) &= 0 \\
w(x, y, z) &= w(x)
\end{aligned}
\tag{2.2}$$

Where $u(x, y, z), v(x, y, z), w(x, y, z)$ the components of the displacement vectors in three co-ordinate directions are, $w(x)$ is the displacement of the centerline in Z-direction.

From strain-displacement relations

$$\varepsilon_{xx} = \frac{\partial u(x, y, z)}{\partial x} = -z \frac{\partial \psi}{\partial x}
\tag{2.3}$$

$$\varepsilon_{xz} = \frac{1}{2} \left[\frac{\partial u(x, y, z)}{\partial z} + \frac{\partial w(x, y, z)}{\partial x} \right] = \frac{1}{2} \left[-\psi + \frac{\partial w}{\partial x} \right]
\tag{2.4}$$

Since, the actual shear strain in the beam is not constant over the cross-section; we introduce a constant shear correction factor κ .

Where shear correction factor ‘ κ ’ is defined as,

$$k = \frac{\text{Average shear strain on a section}}{\text{Shear strain at the centroid}}$$

The significance of the shear correction factor ‘ κ ’ in multilayered plate and shell finite elements have a constant shear distribution across thickness. This causes a decrease in accuracy especially for sandwich structures. This problem is overcoming by using shear correction factor ‘ κ ’

$$\varepsilon_{xz} = \frac{1}{2} \kappa \left[-\psi + \frac{\partial w}{\partial x} \right]
\tag{2.5}$$

Now the total potential energy for the Timoshenko bam

$$\Pi = \frac{1}{2} \left[\int_0^L \int_{-h/2}^{h/2} \sigma_{xx} \varepsilon_{xx} b dx dz + \int_0^L \int_{-h/2}^{h/2} \tau_{xz} \varepsilon_{xz} b dx dz - \int_0^L q w dx \right] \quad (2.6)$$

We already know that

$$M = \int_{-h/2}^{h/2} \sigma_{xx} z b dz = -EI \frac{d\psi}{dx} \quad (2.7)$$

$$Q = \int_{-h/2}^{h/2} \tau_{xz} b dz = \kappa GA \beta(x) \quad (2.8)$$

From Eq. (2.5) and Eq. (2.8) shear force can be written as

$$Q = \int_{-h/2}^{h/2} \tau_{xz} b dz = \kappa GA \beta(x) = \kappa GA \left(-\psi + \frac{dw}{dx} \right) \quad (2.9)$$

Substitute Eq. (2.3), Eq. (2.5), Eq. (2.7) and Eq. (2.9) into Eq. (2.6)

$$\Pi = \int_0^L \left[\frac{EI}{2} \left(\frac{d\psi}{dx} \right)^2 + \frac{\kappa GA}{2} \left(-\psi + \frac{dw}{dx} \right)^2 - q w \right] dx \quad (2.10)$$

Eq. (2.10) has two functions, ψ and w

The Euler-Lagrange equations for this case are

$$\frac{d}{dx} \left(\frac{\partial F}{\partial \psi^1} \right) - \frac{\partial F}{\partial \psi} = 0 \quad (a)$$

$$\frac{d}{dx} \left(\frac{\partial F}{\partial w^1} \right) - \frac{\partial F}{\partial w} = 0 \quad (b)$$

Where
$$F = \frac{EI}{2} \left(\frac{d\psi}{dx} \right)^2 + \frac{\kappa GA}{2} \left(-\psi + \frac{dw}{dx} \right)^2 - q w$$

Now from Eq. (2.10), Eq. (a) and Eq. (b)

$$\frac{d}{dx} \left(EI \frac{d\psi}{dx} \right) + \kappa GA \left(-\psi + \frac{dw}{dx} \right) = 0 \quad (2.11)$$

$$\frac{d}{dx} \left(\kappa GA \left(\frac{dw}{dx} - \psi \right) \right) + q = 0 \quad (2.12)$$

Eq. (2.11) and Eq. (2.12) are the Governing Equations for Static Timoshenko Beam.

2.1.1 Uniform Cantilever Timoshenko Beam Analytical Solution

Let us assume that the clamped end is at $x=0$ and the free end is at $x=L$. If a point load P is applied to the free end in the positive z direction, we use right handed coordinate system, where the X -direction is positive towards right and the Z -direction is positive upwards. Following normal convention, we assume that positive forces act in the positive directions of the X & Z -directions and positive moments act in the clockwise direction. We also assume that the sign convention for bending moments is positive such that, it compresses the beam in at bottom (i.e. Lower Z) and positive shear forces rotate the beam in counter clockwise direction.

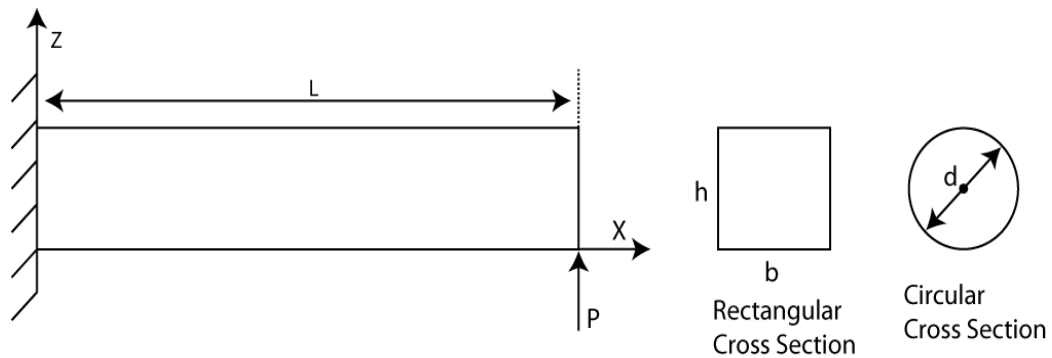


Figure 2.3 Uniform Cantilever Timoshenko Beam

From the Governing Equations of static Timoshenko Beam (For Homogeneous case with Point Load, i.e. $q=0$ in Eq. (2.12))

$$\frac{d}{dx} \left(EI \frac{d\psi}{dx} \right) = -kGA \left(\frac{dw}{dx} - \psi \right) \quad (2.13)$$

$$\frac{d}{dx} \left(kGA \left(\frac{dw}{dx} - \psi \right) \right) = 0 \quad (2.14)$$

Where Shear force is constant which is equal to applied load P,

$$Q = P = \kappa GA \left(\frac{dw}{dx} - \psi \right) \quad (2.15)$$

Now from Eq. (2.13)

$$\frac{d}{dx} \left(EI \frac{d\psi}{dx} \right) = -P \quad (2.16)$$

Integration of Eq. (2.16) and application of boundary condition $\frac{d\psi}{dx} = 0$ at $x=L$

and $\psi = 0$ at $x=0$ gives

$$\psi(x) = \frac{Px}{2EI} (2L - x) \quad (2.17)$$

From Eq. (2.15) and Eq. (2.17) we can write

$$\frac{dw}{dx} = \frac{P}{kGA} + \frac{P}{2EI} (2L - x) \quad (2.18)$$

Now first integration of Eq. (2.18) and application of boundary condition $w = 0$ at

$x = L$ gives us

$$w(x) = \frac{Px^2(6L - 2x)}{12EI} + \frac{Px}{kGA} \quad (2.19)$$

Eq. (2.17) and Eq. (2.19) are corresponding displacement and rotation equations for a cantilever beam.

2.1.2 Finite Element solution for Uniform Cantilever Timoshenko Beam

Consider an infinitesimal element of beam length δx with Young's modulus E and area moment of inertia I . The element is in static equilibrium under the forces

Total rotation of the beam is defined from Eq. (2.1)

$$dw/dx = \psi(x) + \beta(x)$$

Since applied load is a point load shear stress from governing equations static Timoshenko beam is

$$Q_x = kGA\beta(x) = C_1 \quad (2.20)$$

Static equilibrium relations are defined as

$$\begin{aligned} \frac{dM}{dx} &= Q_x \\ \frac{dQ_x}{dx} &= 0 \end{aligned} \quad (2.21)$$

*Since shear force is a constant.

Stress strain relation in bending is

$$M = EI \frac{d\psi}{dx} = EI \frac{d}{dx} \left(\frac{dw}{dx} - \beta(x) \right) \quad (2.22)$$

From Eq. (2.20) and Eq. (2.22)

$$\begin{aligned} \beta(x) &= \frac{C_1}{kGA} \\ M &= C_1 x + C_2 \end{aligned}$$

Now from Eq. (2.22) & Eq. (2.1)

$$\begin{aligned} \psi &= \frac{1}{EI} \left(C_1 \frac{x^2}{2} + C_2 x + C_3 \right) \\ w &= \frac{1}{EI} \left(C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \right) \\ \frac{dw}{dx} &= \psi(x) + \beta(x) = \frac{1}{EI} \left(C_1 \frac{x^2}{2} + C_2 x + C_3 \right) + \frac{C_1}{kGA} \end{aligned} \quad (2.23)$$

We are considering a 2-node beam element with 2 degrees of freedom at each node, i.e. we have total 4 degrees of freedom, which are $w_1, \frac{dw}{dx_1}, w_2, \frac{dw}{dx_2}$.



Figure 2.4 Two-Node Timoshenko Beam Element

Where

w_i = Deflection of cross section at node i

$\frac{dw}{dx_i}$ = Rotation of cross section due to both bending and shear deformation

These four DOF's may then be expressed in terms of constants C_j ($j=1, 2, 3, 4$)

using Eq. (2.23)

$$\begin{aligned}
w_1 = w(0) &= \frac{1}{EI} C_4 \\
\frac{dw}{dx_1} = \frac{dw}{dx}(0) &= \frac{1}{EI} (aC_1 + C_3) \\
w_2 = w(L) &= \frac{1}{EI} \left(C_1 \frac{L^3}{6} + C_2 \frac{L^2}{2} + C_3 L + C_4 \right) \\
\frac{dw}{dx_2} = \frac{dw}{dx}(L) &= \frac{1}{EI} \left[\left(\frac{L^2}{2} + a \right) C_1 \right] + C_2 L + C_3
\end{aligned} \tag{2.24}$$

In matrix form Eq. (2.24)

$$\begin{Bmatrix} w_1 \\ dw/dx_1 \\ w_2 \\ dw/dx_2 \end{Bmatrix} = \frac{1}{EI} \begin{bmatrix} 0 & 0 & 0 & 1 \\ a & 0 & 1 & 0 \\ \frac{L^3}{6} & \frac{L^2}{2} & L & 1 \\ \frac{L^2}{2} + a & L & 1 & 0 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \tag{2.25}$$

From Eq. (2.25) constant matrix $\{C_i\}$ may be written as

$$\{C_i\} = EI \begin{bmatrix} 0 & 0 & 0 & 1 \\ a & 0 & 1 & 0 \\ \frac{L^3}{6} & \frac{L^2}{2} & L & 1 \\ \frac{L^2}{2} + a & L & 1 & 0 \end{bmatrix}^{-1} \{\delta_i\} \tag{2.26}$$

$\{\delta_i\}$ = Displacement matrix

$\{C_i\}$ = Constant matrix.

Using MAT LAB®

$$\{C_i\} = EI \begin{bmatrix} \frac{12}{L^3(1+\phi)} & \frac{6}{L^2(1+\phi)} & \frac{-12}{L^3(1+\phi)} & \frac{6}{L^3(1+\phi)} \\ -6 & -(4L^2+12a) & 6 & (12a-2L^2) \\ \frac{L^2(1+\phi)}{L^3(1+\phi)} & \frac{L^3(1+\phi)}{(L^2+6a)} & \frac{L^2(1+\phi)}{L^3(1+\phi)} & \frac{L^3(1+\phi)}{L^3(1+\phi)} \\ \frac{-12a}{L^3(1+\phi)} & \frac{(L^2+6a)}{L^2(1+\phi)} & \frac{12a}{L^3(1+\phi)} & \frac{-6a}{L^2(1+\phi)} \\ 1 & 0 & 0 & 0 \end{bmatrix} \{\delta_i\} \quad (2.27)$$

$$a = \frac{EI}{kGA}$$

$$\text{Where } \phi = \frac{12a}{L^2}$$

Substitute Eq. (2.27) in Eq. (2.25) for constants C_1, C_2, C_3, C_4 which give the following equations

$$w = \frac{1}{EI} [N_{wi}] \begin{Bmatrix} w_1 \\ dw/dx_1 \\ w_2 \\ dw/dx_2 \end{Bmatrix} \quad (2.28)$$

$$[N_{wi}] = \frac{1}{1+\phi} \begin{Bmatrix} 2\xi^3 - 3\xi^2 + \phi(1-\xi) + 1 \\ L \left(\xi^3 - 2\xi^2 + \xi + \frac{\phi}{2}(\xi - \xi^2) \right) \\ -2\xi^3 + 3\xi^2 + \phi\xi \\ L \left(\xi^3 - \xi^2 + \frac{\phi}{2}(\xi^2 - \xi) \right) \end{Bmatrix} \quad (2.29)$$

$$\frac{dw}{dx} = \frac{1}{EI} [N_{\alpha}] \begin{Bmatrix} w_1 \\ dw/dx_1 \\ w_2 \\ dw/dx_2 \end{Bmatrix} \quad (2.30)$$

$$[N_{\theta i}] = \frac{1}{1+\phi} \begin{Bmatrix} \frac{6}{L}(\xi^2 - \xi) \\ (3\xi^2 - 4\xi + 1 + \phi(1 - \xi)) \\ \frac{-6}{L}(\xi^2 - \xi) \\ (3\xi^2 - 2\xi + \phi\xi) \end{Bmatrix} \quad (2.31)$$

$[N_{wi}]$ $[N_{\theta i}]$ -are displacement shape functional Matrix, rotational shape function

Matrix respectively (i=1, 2, 3, 4).

From principle of internal virtual energy

$$K = EI \int_0^L [N'_{\theta i}]^T [N'_{\theta i}] d\xi + kGA \int_0^L [N'_{wi} - N_{\theta i}]^T [N'_{wi} - N_{\theta i}] \quad (2.32)$$

Using MAT LAB

$$K = \frac{EI}{L^3(1+\phi)} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & L^2(4+\phi) & -6L & L^2(2-\phi) \\ -12 & -6L & 12 & -6L \\ 6L & L^2(2-\phi) & -6L & L^2(4+\phi) \end{bmatrix} \quad (2.33)$$

The kinetic energy T, of an element length δx of a uniform Timoshenko beam is

given as

$$T = \frac{1}{2} \int_0^L \rho A \left(\frac{dw}{dt} \right)^2 dx + \frac{1}{2} \int_0^L \rho I \left(\frac{d\psi}{dt} \right)^2 dx \quad (2.34)$$

ρ = Mass density of the material of the beam

I = Second moment of area of cross section

Therefore the mass matrix of the element has two parts, one related to transverse displacement and the other related to rotations in the form of,

Substituting the shape functions into above kinetic energy expression,

$$[M] = \rho A \int_0^L [N_w]^T [N_w] d\xi + \rho I \int_0^L [N_\theta]^T [N_\theta] d\xi \quad (2.35)$$

The first of the above equation is

$$[M_1] = \frac{\rho A L}{210(1+\phi)^2} \begin{bmatrix} (70\phi^2 + 147\phi + 78) & (35\phi^2 + 77\phi + 44)\frac{L}{4} & (35\phi^2 + 63\phi + 27) & -(35\phi^2 + 63\phi + 26)\frac{L}{4} \\ (35\phi^2 + 77\phi + 44)\frac{L}{4} & (7\phi^2 + 14\phi + 8)\frac{L^2}{4} & (35\phi^2 + 63\phi + 26)\frac{L}{4} & -(7\phi^2 + 14\phi + 6)\frac{L^2}{4} \\ (35\phi^2 + 63\phi + 27) & (35\phi^2 + 63\phi + 26)\frac{L}{4} & (70\phi^2 + 147\phi + 78) & -(35\phi^2 + 77\phi + 44)\frac{L}{4} \\ -(35\phi^2 + 63\phi + 26)\frac{L}{4} & -(7\phi^2 + 14\phi + 6)\frac{L^2}{4} & -(35\phi^2 + 77\phi + 44)\frac{L}{4} & (7\phi^2 + 14\phi + 8)\frac{L^2}{4} \end{bmatrix} \quad (2.36)$$

And the second part is

$$[M_2] = \frac{\rho I}{30(1+\phi)^2 L} \begin{bmatrix} 36 & -(15\phi - 3)L & -36 & -(15\phi - 3)L \\ -(15\phi - 3)L & (10\phi^2 + 5\phi + 4)L^2 & (15\phi - 3)L & (5\phi^2 - 5\phi - 1)L^2 \\ -36 & (15\phi - 3)L & 36 & (15\phi - 3)L \\ -(15\phi - 3)L & (5\phi^2 - 5\phi - 1)L^2 & (15\phi - 3)L & (10\phi^2 + 5\phi + 4)L^2 \end{bmatrix} \quad (2.37)$$

$$[M] = [M_1] + [M_2]$$

Rectangular Cross Section: Let us consider a cantilever beam of length L, width b, height h for which Area A_R and area moment of inertia I_R defined as

$$\begin{aligned} A_R &= b * h \\ I_R &= \frac{bh^3}{12} \end{aligned} \quad (2.38)$$

Where index R denotes Rectangular cross section

Circular Cross Section: Let us consider a cantilever beam of length L, with diameter d, for which Area A_C and area moment of inertia I_C defined as

$$A_c = \frac{\Pi d^2}{4}$$

$$I_c = \frac{\Pi d^4}{64}$$
(2.39)

Where index C denotes Circular cross section

2.2 Dynamic Analysis of Uniform Timoshenko Beam

2.2.1 Free Vibrations of a Uniform Cantilever Timoshenko Beam Analytical

Solution

Consider a beam of length L , undergoing transverse motion $w(x,t)$ caused by a load $q(x,t)$. The longitudinal coordinate is x , the flexural rigidity EI , the density ρ , the shear modulus G , and the beam cross section A . During vibration the elements of a beam perform not only translatory motion but also rotate. Hence when taking into account not only the rotary inertia but also the deflection due to shear, the slope of deflection due to shear, the slope the deflection curve $w(x,t)$ depends on the rotation $\psi(x,t)$ of the beam cross section, and one the shear i.e. on the angle of shear $\beta(x,t)$ at neutral axis.

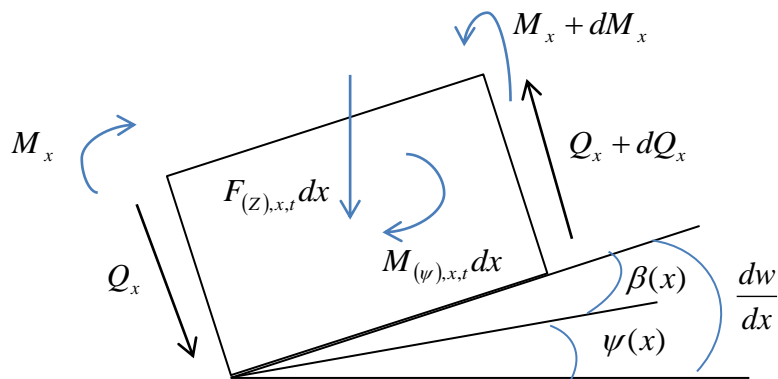


Figure 2.5 Dynamic Equilibrium of Timoshenko Beam

Another factor that affects the lateral vibration of the beam is the fact that each section of the beam rotates slightly in addition to its lateral motion when the beam deflects. The influence of the beam section rotation is taken into account through the moments of inertia, which modifies the equation of moment acting on an infinitesimal beam element

$$dM(x,t) = -I\rho \frac{\partial^2 \psi(x,t)}{\partial t^2} dx \quad (2.40)$$

By applying D' Alembert's principle, the system of coupled differential equations for transverse vibration of the Uniform Timoshenko beam with a constant cross section given by

$$\begin{aligned} -\frac{\partial Q(x,t)}{\partial x} + \rho A \frac{\partial^2 w(x,t)}{\partial t^2} &= q(x,t) \\ -\frac{\partial M(x,t)}{\partial x} + Q(x,t) - I\rho \frac{\partial^2 \psi(x,t)}{\partial t^2} &= 0 \end{aligned} \quad (2.41)$$

$$Q(x,t) = kGA\beta(x,t) = kGA \left(\frac{\partial w(x,t)}{\partial x} - \psi(x,t) \right)$$

Where $M(x,t) = EI \frac{\partial \psi}{\partial x}$

The system of above differential equations is governing equation of the Timoshenko beam vibration, where the functions are the vibration amplitude $w(x,t)$ and the angle due to pure bending $\psi(x,t)$

The Fourier method of variable separation is employed to find the functions satisfying above system of equations. It is assumed that each function $w(x,t)$ & $\psi(x,t)$

can be written in the form of a product of a function dependent on the spatial coordinate x and a function dependent on time t

$$\begin{aligned} w(x,t) &= X(x)T(t) \\ \psi(x,t) &= Y(x)T(t) \end{aligned} \quad (2.42)$$

After several simple transformations of system of equations

$$\begin{aligned} X''(x) + aX(x) - Y'(x) &= 0 \\ Y''(x) + bY(x) + cX'(x) &= 0 \\ T''(t) + \omega^2 T(t) &= 0 \end{aligned} \quad (2.43)$$

Where $a = \frac{\omega^2 \rho}{kG}$; $b = \frac{\omega^2 \rho}{E} - c$; $c = \frac{kGA}{EI}$ and ω is vibration frequency

By eliminating the function $Y(x)$ from the first two equations of system of Equations

$$Y(x) = -\frac{1}{b} \left(X'''(x) + (a+c)X'(x) \right) \quad (2.44)$$

Now substitute $Y(x)$ in system of equations to get an equation for the transverse amplitude function $X(x)$

$$X''''(x) + dX''(x) + eX(x) = 0 \quad (2.45)$$

$$\text{Where } d = a + b + c = \frac{\omega^2 \rho I \left(1 + \frac{E}{kG} \right)}{EI}; e = ab = \frac{\omega^2 \left(\omega^2 \rho^2 \frac{I}{kG} - \rho A \right)}{EI}$$

The function $Y(x)$ depends on derivatives of the vibration amplitude function $X(x)$. This equation will be used to derive the boundary conditions dependent only on the vibration amplitude function $X(x)$ and its derivatives.

The characteristic equation has the form

$$r^4 + dr^2 + e = 0$$

Replacing $r^2 = z$ in above equation

$$z^2 + dz + e = 0$$

Roots for above equation are

$$z_1 = \frac{1}{2}(-d + \sqrt{\Delta})$$

$$z_2 = \frac{1}{2}(-d - \sqrt{\Delta})$$

(2.46)

$$\Delta = d^2 - 4e = \omega^4 \rho^2 I^2 \left(1 - \frac{E}{kG}\right)^2 + 4EI\omega^2 \rho A$$

Where

It is easy to observe that $\Delta > 0 \forall \omega$

Signs of the roots z_1 & z_2 are

$$z_2 < 0 \forall \omega$$

$$z_1 > 0 \Leftrightarrow \sqrt{d^2 - 4e} > d \Rightarrow e < 0 \text{ for } \omega^2 < \frac{kGA}{\rho I}$$

$$z_1 < 0 \text{ for } \omega^2 > \frac{kGA}{\rho I}$$

Two possible solutions to system of equations come from the above discussion

$$\text{For } \omega < \sqrt{\frac{kGA}{\rho I}}$$

$$\text{The roots are } r_1 = \sqrt{z_1}; r_2 = -\sqrt{z_1}; r_3 = i\sqrt{z_2}; r_4 = -i\sqrt{z_2}$$

This gives a solution in the form

$$X(x) = C_1 e^{\sqrt{z_1}x} + C_2 e^{-\sqrt{z_1}x} + C_3 e^{i\sqrt{z_2}x} + C_4 e^{-i\sqrt{z_2}x}$$

In trigonometric and hyperbolic form

$$X(x) = P_1 \cosh \lambda_1 x + P_2 \sinh \lambda_1 x + P_3 \cos \lambda_2 x + P_4 \sin \lambda_2 x \quad (2.47)$$

Where $\lambda_1^2 = |z_1| = \frac{-d + \sqrt{\Delta}}{2}; \lambda_2^2 = |z_2| = \frac{d + \sqrt{\Delta}}{2}$

For $\omega > \sqrt{\frac{kGA}{\rho I}}$

The roots are $r_1 = i\sqrt{z_1}; r_2 = -i\sqrt{z_1}; r_3 = i\sqrt{z_2}; r_4 = -i\sqrt{z_2}$

This gives a solution in the form

$$X(x) = C_1 e^{i\sqrt{z_1}x} + C_2 e^{-i\sqrt{z_1}x} + C_3 e^{i\sqrt{z_2}x} + C_4 e^{-i\sqrt{z_2}x}$$

In trigonometric and hyperbolic form

$$X(x) = Q_1 \cos \lambda_1 x + Q_2 \sin \lambda_1 x + Q_3 \cos \lambda_2 x + Q_4 \sin \lambda_2 x \quad (2.48)$$

Where $\lambda_1^2 = |z_1| = \frac{d - \sqrt{\Delta}}{2}; \lambda_2^2 = |z_2| = \frac{d + \sqrt{\Delta}}{2}$

Boundary Conditions for Cantilever Timoshenko Beam:

Fixed/Clamped End ($x_i = 0$ or $x_i = L$) $w(x_i, t) = 0$
 $\psi(x_i, t) = 0$

After separation of variables

$$\begin{aligned} X(x_i) &= 0 \\ Y = 0 &\Leftrightarrow X'''(x_i) + (a+c)X'(x_i) = 0A \end{aligned} \quad (2.49)$$

Free End ($x_i = 0$ or $x_i = L$)

$$M(x_i, t) = EI \frac{\partial \psi(x_i, t)}{\partial x} = 0$$

$$Q(x_i, t) = kGA \left(\frac{\partial w(x_i, t)}{\partial x} - \psi(x_i, t) \right) = 0$$

After separation variables

$$X''(x_i) + aX(x_i) = 0$$

$$dX'(x_i) + X'''(x_i) = 0 \tag{2.50}$$

Cantilever Beam

Apply the boundary conditions stated above in Eq. (2.49) & Eq. (2.50).

The form of the solution for the free vibration depends on the interval to which the searched natural frequency belongs:

For frequencies $\omega < \sqrt{\frac{kGA}{\rho I}}$ the solution to Eq. (2.47) for which boundary

conditions are expressed by a matrix equation.

$$AP = 0$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & A_{22} & 0 & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$$

Where

$$\tag{2.51}$$

$$\begin{aligned}
A_{22} &= (\lambda_1^2 + a + c) * \lambda_1 \\
A_{24} &= (-\lambda_2^2 + a + c) * \lambda_2 \\
A_{31} &= (\lambda_1^2 - d) * \lambda_1 * \sinh(\lambda_1 * L) \\
A_{32} &= (\lambda_1^2 - d) * \lambda_1 * \cosh(\lambda_1 * L) \\
A_{33} &= (\lambda_1^2 + d) * \lambda_2 * \sin(\lambda_2 * L) \\
A_{34} &= (-\lambda_1^2 - d) * \lambda_2 * \cos(\lambda_2 * L) \\
A_{41} &= (\lambda_1^2 + a) * \cosh(\lambda_1 * L) \\
A_{42} &= (\lambda_1^2 + a) * \sinh(\lambda_1 * L) \\
A_{43} &= (-\lambda_2^2 + a) * \cos(\lambda_2 * L) \\
A_{44} &= (-\lambda_2^2 + a) * \sin(\lambda_2 * L) \\
P^T &= [P_1 \quad P_2 \quad P_3 \quad P_4]
\end{aligned}$$

The coefficients a, b, c, d, e are defined above. Above equation has a trivial solution at $\lambda_1 = 0$ & $\lambda_2 = 0$. This condition is possible only when $\omega = 0$, which describes the motion of the beam as a rigid body. So it is impossible for given boundary conditions. Non-trivial solutions of the main matrix are determined by equation $\det(A) = 0$. The roots of the main matrix determinate are the eigenvalues of the beam from which we can find out natural frequency of the beam.

Determinate of the matrix A has the following form

$$K_1 \sinh(\lambda_1 * L) * \sin(\lambda_2 * L) + K_2 \cosh(\lambda_1 * L) * \cos(\lambda_2 * L) = -1 \quad (2.52)$$

From observation one can tell that the above equation has infinite roots.

Since $(\lambda_1 * L) \neq 0 \Rightarrow \sinh(\lambda_1 * L) \neq 0$

Hence $\sin(\lambda_2 * L) = 0$ will satisfy the determinate at $\lambda_2 = \frac{n\pi}{L}$

Where (n = 1,2,3...)

For frequencies $\omega > \sqrt{\frac{kGA}{\rho I}}$ the solution has the form in Eq. (2.48), for which

boundary conditions are expressed by a matrix equation.

$$AQ = 0$$

$$\text{Where } A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & A_{22} & 0 & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \quad (2.53)$$

$$\begin{aligned} A_{22} &= (-\lambda_1^2 + a + c) * \lambda_1 \\ A_{24} &= (-\lambda_2^2 + a + c) * \lambda_2 \\ A_{31} &= (\lambda_1^2 + d) * \lambda_1 * \sin(\lambda_1 * L) \\ A_{32} &= (-\lambda_1^2 - d) * \lambda_1 \cos(\lambda_1 * L) \\ A_{33} &= (\lambda_2^2 + d) * \lambda_2 \sin(\lambda_2 * L) \\ A_{34} &= (-\lambda_2^2 - d) * \lambda_2 \cos(\lambda_2 * L) \\ A_{41} &= (-\lambda_1^2 + a) \cos(\lambda_1 * L) \\ A_{42} &= (-\lambda_1^2 + a) \sin(\lambda_1 * L) \\ A_{43} &= (-\lambda_2^2 + a) \cos(\lambda_2 * L) \\ A_{44} &= (-\lambda_2^2 + a) \sin(\lambda_2 * L) \end{aligned}$$

$$Q^T = [Q_1 \quad Q_2 \quad Q_3 \quad Q_4]$$

The coefficients a, b, c, d, e are defined above. Above equation has a trivial solution at $\lambda_1 = 0$ & $\lambda_2 = 0$. This condition is possible only when $\omega = 0$, which describes the motion of the beam as a rigid body. So it is impossible for given boundary conditions. Non-trivial solutions of the main matrix are determined by equation $\det(A) = 0$. The roots of the main matrix determinate are the eigenvalues of the beam from which we can find out natural frequency of the beam.

Determinate of the matrix A has the following form

$$K_1 \sin(\lambda_1 * L) * \sin(\lambda_2 * L) + K_2 \cos(\lambda_1 * L) * \cos(\lambda_2 * L) = -1 \quad (2.54)$$

From observation one can tell that the above equation has infinite roots.

$\sin(\lambda_1 * L) = 0$ Or $\sin(\lambda_2 * L) = 0$ will satisfy the determinate at respectively

$$\lambda_1 = \frac{n\pi}{L} \& \lambda_2 = \frac{n\pi}{L}$$

Where (n = 1,2,3...)

So from above discussion we can easily conclude that

$$\begin{aligned} w(x,t) &= C_1 \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n * t) \\ \psi(x,t) &= C_2 \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n * t) \end{aligned} \quad (2.55)$$

Where C_1 & C_2 are arbitrary constants

Eq. (58) is the exact solutions for the Timoshenko beam dynamic governing equations.

2.2.2 Finite Element Vibrational Analysis of Uniform Cantilever Timoshenko

Beam

Equation for FEM of, free vibrational analysis of Timoshenko beam is given as

$$MU'' + KU = 0 \quad (2.56)$$

M=Consistent Mass Matrix

K=Stiffness Matrix

$U = u * e^{-i\omega t}$, u is amplitude of vibration.

Substitute U, K, M

$$-\omega^2 Mu * e^{-i\omega t} + Ku * e^{-i\omega t} = 0$$

From above Equation we can write characteristic equation as

$$K - \lambda M = 0$$

$$\therefore \lambda = \omega^2 \tag{2.57}$$

Solving above characteristic equation we will get Eigen frequencies for the straight Timoshenko beam.

Chapter 3

Tapered Timoshenko Beam Analysis

3.1 Static Analysis of Tapered Timoshenko Beam

3.1.1 Tapered Cantilever Timoshenko Beam Analytical Solution

Let us consider a tapered Cantilever beam with length L and width b . Height of the beam linearly varying along length L , such that it follows an equation $h(x) = (h_L - h_0) \frac{x}{L} + h_0$. Where h_0 the height of the cross section is at length zero and h_L is height of the cross section at length L . As height of the beam varies along the length of the beam area and area moment of inertia of the cross section varies along the length of the beam.

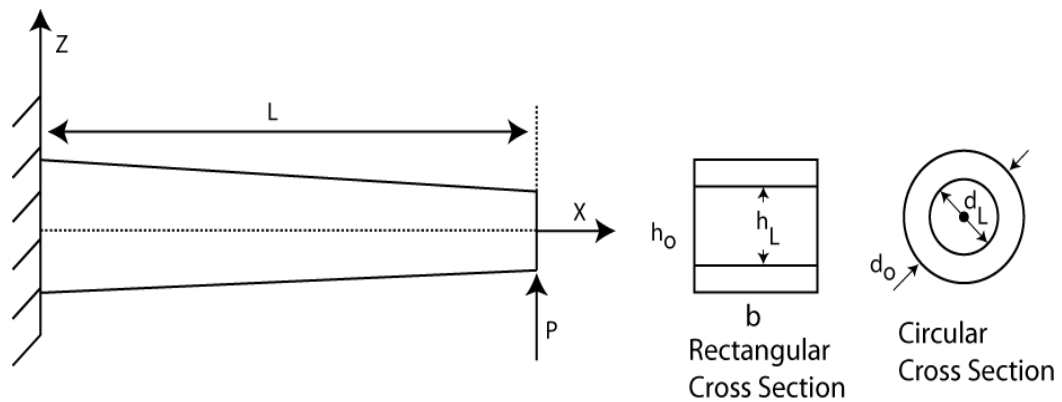


Figure 3.1 Tapered Cantilever Timoshenko Beam

Now from governing equations of Timoshenko beam (i.e. Eq. (2.11) and Eq.

(2.12))

$$\frac{d}{dx} \left(EI \frac{d\psi}{dx} \right) + \kappa GA \left(-\psi + \frac{dw}{dx} \right) = 0$$

$$\frac{d}{dx} \left(\kappa GA \left(\frac{dw}{dx} - \psi \right) \right) + q = 0$$

For the tapered beam, equations can be modified as

$$\frac{d}{dx} \left(EI(x) \frac{d\psi}{dx} \right) + \kappa GA(x) \left(-\psi + \frac{dw}{dx} \right) = 0 \quad (3.1)$$

$$\frac{d}{dx} \left(\kappa GA(x) \left(\frac{dw}{dx} - \psi \right) \right) + q = 0 \quad (3.2)$$

From above governing equations we write

$$\frac{d}{dx} \left(EI(x) \frac{d\psi}{dx} \right) = -P \quad (3.3)$$

Integrating above equation twice with respect to x gives rotation due to bending.

$$\psi(x) = \frac{1}{E} \left(-P \int \frac{x}{I(x)} dx + C_1 \int \frac{1}{I(x)} dx + C_2 \right) \quad (3.4)$$

From Eq. (62)

$$\kappa GA(x) \left(\frac{dw}{dx} - \psi \right) = P \quad (3.5)$$

Substitute Eq. (39) in above Eq.

$$\frac{dw}{dx} = \frac{1}{E} \left(-P \int \frac{x}{I(x)} dx + C_1 \int \frac{1}{I(x)} dx + C_2 \right) + \frac{P}{\kappa GA(x)} \quad (3.6)$$

Integrating above Eq. with respect to x

$$w(x) = \frac{1}{E} \left(-P \iint \frac{x}{I(x)} dx^2 + C_1 \iint \frac{1}{I(x)} dx^2 + C_2 x \right) + \frac{P}{kG} \int \frac{1}{A(x)} dx + C_3 \quad (3.7)$$

Eq. (3.4) and Eq. (3.7) are Rotation and Displacement of Tapered Timoshenko beam.

- Remember that coefficients in both equations are same.

Rectangular Cross Section

Consider a rectangular cross section tapered beam of length L, constant width b, and height varies along length L, such that it follows an equation

$$h_R(x) = (h_{RL} - h_{R0}) \frac{x}{L} + h_{R0}$$

For a rectangular cross section of tapered beam, area $A_R(x)$ and area moment of inertia $I_R(x)$ are given by

$$A_R(x) = b * h_R(x) = b * \left((h_{RL} - h_{R0}) \frac{x}{L} + h_{R0} \right) = A_{R0} * \left[\alpha_R \frac{x}{L} + 1 \right] \quad (3.8)$$

$$I_R(x) = \frac{b * h_R(x)^4}{12} = \frac{b * h_0^3}{12} \left((h_{RL} - h_{R0}) \frac{x}{L} + h_{R0} \right) = I_{R0} * \left(\alpha_R \frac{x}{L} + 1 \right)^3 \quad (3.9)$$

Substitute Eq. (69) in Eq. (64) to calculate rotation of tapered Rectangular cross section Timoshenko beam,

$$\psi_R(x) = \frac{1}{EI_{R0}} \left(-P \int \frac{x}{\left(\alpha_R \frac{x}{L} + 1 \right)^3} dx + C_1 \int \frac{1}{\left(\alpha_R \frac{x}{L} + 1 \right)^3} dx + C_2 \right) \quad (3.10)$$

$$\alpha_R = \left(\frac{h_{RL}}{h_{R0}} - 1 \right)$$

Where $I_{R0} = \frac{b^* h_{R0}^3}{12}$

Integrating the above equations and applying boundary condition $\psi = 0$ at $x=0$

and $\frac{d\psi}{dx} = 0$ at $x=L$

$$\psi_R(x) = \frac{1}{EI_{R0}} \left[\frac{PL^3(L + 2\alpha_R x)}{2\alpha_R^2(L + \alpha_R x)^2} - \frac{PL^4}{2\alpha_R(L + \alpha_R x)^2} + \frac{PL^2(\alpha_R - 1)}{2\alpha_R^2} \right] \quad (3.11)$$

Substitute Eq. (3.8) & Eq. (3.9) in Eq. (3.7)

$$w_R(x) = \frac{1}{EI_{R0}} \left(-P \iint \frac{x}{\left(\alpha_R \frac{x}{L} + 1\right)^3} dx^2 + C_1 \iint \frac{1}{\left(\alpha_R \frac{x}{L} + 1\right)^3} dx^2 + C_2 x \right) + \frac{P}{kGA_{R0}} \int \frac{1}{\left(\alpha_R \frac{x}{L} + 1\right)} dx + C_3 \quad (3.12)$$

Integrating the above equation and applying boundary condition $w=0$ at $x=0$,

$$w_R(x) = \frac{1}{EI_{R0}} \left(\left(\frac{PL^3}{\alpha_R^3} + \frac{aPL}{\alpha_R} \right) \log \left(1 + \alpha_R \frac{x}{L} \right) - \frac{PL^3 x \alpha_R (\alpha_R + 1)}{2\alpha_R^3 (L + \alpha_R x)} + \frac{PL^2 x (\alpha_R - 1)}{2\alpha_R^2} \right) \quad (3.13)$$

$$\alpha_R = \left(\frac{h_{RL}}{h_{R0}} - 1 \right)$$

$$I_{R0} = \frac{b^* h_{R0}^3}{12} \text{ \& } A_{R0} = b^* h_{R0}$$

$$a = \frac{EI_{R0}}{kGA_{R0}}$$

Where

Eq. (3.11) & Eq. (3.13) are Displacement and rotation due to bending of Rectangular cross section tapered cantilever Timoshenko beam.

Circular Cross Section

Consider a circular cross section tapered beam of length L, diameter d varies along length of the beam, such that it follows an equation $d(x) = (d_L - d_0) \frac{x}{L} + d_0$. Where d_L is diameter of the beam at length L and d_0 is diameter at length zero.

Area and area moment of inertia are given by

$$A_C(x) = \frac{\pi * d(x)^2}{4} = \frac{\pi}{4} * \left((d_L - d_0) \frac{x}{L} + d_0 \right)^2 = A_{C0} * \left(\alpha_C \frac{x}{L} + 1 \right)^2 \quad (3.14)$$

$$I_C(x) = \frac{\pi * d(x)^4}{64} = \frac{\pi}{64} * \left((d_L - d_0) \frac{x}{L} + d_0 \right)^4 = I_{C0} * \left(\alpha_C \frac{x}{L} + 1 \right)^4 \quad (3.15)$$

Substituting above equations in Eq. (3.14) & Eq. (3.15)

$$\psi_C(x) = \frac{1}{EI_{C0}} \left(-P \int \frac{x}{\left(\alpha_C \frac{x}{L} + 1 \right)^4} dx + C_1 \int \frac{1}{\left(\alpha_C \frac{x}{L} + 1 \right)^4} dx + C_2 \right) \quad (3.16)$$

$$\alpha_C = \left(\frac{d_L}{d_0} - 1 \right)$$

Where
$$I_{C0} = \frac{\pi * d_0^4}{64}$$

Integrating above equation and applying boundary condition $\psi = 0$ at $x=0$ and

$$\frac{d\psi}{dx} = 0 \text{ at } x=L$$

$$\psi_c(x) = \frac{PL^2}{6EI_{c0}\alpha_c^2} \left[\left(\frac{3\alpha_c + 1}{(\alpha_c + 1)^3} \right) - \left(\frac{3\alpha_c \left(\frac{x}{L} \right) + 1}{\left(\alpha_c \left(\frac{x}{L} \right) + 1 \right)^3} \right) \right] \quad (3.17)$$

Deflection can be derived as

$$w_c(x) = \frac{PL^2}{6EI_{c0}\alpha_c^2} \left[\left(\frac{3\alpha_c + 1}{(\alpha_c + 1)^3} \right) - \left(\frac{3\alpha_c \left(\frac{x}{L} \right) + 1}{\left(\alpha_c \left(\frac{x}{L} \right) + 1 \right)^3} \right) \right] dx + \frac{P}{kGA_{c0}} \int \frac{1}{\left(\alpha_c \left(\frac{x}{L} \right) + 1 \right)^2} dx \quad (3.18)$$

$$\alpha_c = \left(\frac{d_L}{d_0} - 1 \right)$$

Where $I_{c0} = \frac{\pi * d_0^4}{64}$ & $A_{c0} = \frac{\pi * d_0^2}{4}$

Integrating above equation and applying boundary condition $w=0$ at $x=L$

$$w_c(x) = \frac{PL^2}{6EI_{c0}\alpha_c^3} \left[\frac{L^2(3\alpha_c x + 2L)}{(\alpha_c x + L)^2} + \frac{x\alpha_c(3\alpha_c + 1) - 2L(3\alpha_c^2 + 3\alpha_c + 1)}{\alpha_c(\alpha_c + 1)^3} \right] + \frac{PL(L-x)}{kGA_{c0}(\alpha_c + 1)(\alpha_c x + L)} \quad (3.19)$$

Eq. (3.17) & Eq. (3.19) are rotation due to bending and displacement of Circular cross section tapered cantilever Timoshenko beam.

3.1.2 Finite Element Analysis of Tapered Cantilever Timoshenko Beam

Let us consider the general tapered beam element 1-2 of length L made of a homogeneous and linear elastic material of Young's modulus E . The cross section of this beam possesses a vertical axis of symmetry Y and has an area of cross section $A(x)$ and area moment of inertia $I(x)$ about the Z -axis. The element is in static equilibrium under

the forces. The shear angle β measured as positive in counter clockwise direction from the normal to the mid surface to the outer face of the beam. The element has two nodes, each possessing two degrees of freedom; they are the transverse displacement w , and total cross-section rotation dw/dx . The total cross section rotation dw/dx defined as sum of the slope ψ due to bending and the slope β due to shear distortion.

$$\frac{dw}{dx} = \psi(x) + \beta(x)$$

Let us consider a Rectangular cross section, tapered cantilever beam with linearly varying height along length L , such that it follows an equation $h(x) = (h_L - h_0) \frac{x}{L} + h_0$. Where h_0 is height of the cross section at left end of the beam i.e. at Node-1 and h_L height of cross section at right end of the beam i.e. at Node 2. As height of the beam varies along the length of the beam, Area $A(x)$ and area moment of inertia $I(x)$ of the cross section by equations Eq. (3.8) & Eq. (3.9),

$$A_R(x) = b * h_R(x) = b * \left((h_{RL} - h_{R0}) \frac{x}{L} + h_{R0} \right) = A_{R0} * \left[\alpha_R \frac{x}{L} + 1 \right]$$

$$I_R(x) = \frac{b * h_R(x)^3}{12} = \frac{b * \left((h_{RL} - h_{R0}) \frac{x}{L} + h_{R0} \right)^3}{12} = I_{R0} * \left[\alpha_R \frac{x}{L} + 1 \right]^3$$

Where $\alpha_R = \left(\frac{h_L}{h_0} - 1 \right)$

Since applied load is a point load shear stress from governing equations, static Timoshenko beam is

Static equilibrium relations are

$$\frac{dM}{dx} = Q_x$$

$$\frac{dQ_x}{dx} = 0$$

Stress strain relation in bending is

$$M = EI(x) \frac{d\psi}{dx} = EI \frac{d}{dx} \left(\frac{d\theta}{dx} - \beta(x) \right) = C_1 x + C_2$$

$$Q_x = kGA(x) \beta(x) = C_1$$

Integrating Moment equation

$$\psi(x) = \frac{1}{E} \left[C_1 \int \frac{x}{I(x)} dx + C_2 \int \frac{1}{I(x)} dx + C_3 \right]$$

Integrating above equation will give transverse displacement of the beam

$$w(x) = \frac{1}{E} \left[C_1 \iint \frac{x}{I(x)} dx^2 + C_2 \iint \frac{1}{I(x)} dx^2 + C_3 x + C_4 \right] \quad (3.20)$$

$$\frac{d\theta}{dx} = \frac{1}{E} \left[C_1 \int \frac{x}{I(x)} dx + C_2 \int \frac{1}{I(x)} dx + C_3 \right] + \frac{C_1}{kGA(x)} \quad (3.21)$$

We are considering a 2-node beam element with 2 degrees of freedom at each

node, i.e. we have 4 degrees of freedom in total, which are $w_1, \frac{dw}{dx}_1, w_2, \frac{dw}{dx}_2$.



Figure 3.2 Two-Node Timoshenko Beam Element with Four-Degrees of Freedom

Where w_i = Deflection of cross section at node ($i=1, 2$)

$\frac{d\theta}{dx_i}$ = Rotation of cross section due to both bending and shear deformation at

node i ($i= 1, 2$)

Applying Nodal variables to the equations

$$\begin{aligned}
 w_1 = w(0) &= \frac{1}{EI_0} \left[C_1 \left(\frac{-L^3(2\log(L)+1)}{2\alpha^3} \right) + C_2 \frac{L^2}{2\alpha^2} + C_4 \right] \\
 \frac{d\theta}{dx_1} = \frac{d\theta}{dx}(0) &= \frac{1}{EI_0} \left(C_1 \left(\frac{-L^2}{2\alpha^2} + a \right) + C_2 \left(\frac{-L}{2\alpha} \right) + C_3 \right) \\
 w_2 = w(L) &= \frac{1}{EI_0} \left(C_1 \left(\frac{-L^3(2(\alpha+1)\log(L(\alpha+1)+1)}{2\alpha^3(\alpha+1)} \right) + C_2 \left(\frac{L^2}{2\alpha^2(\alpha+1)^2} \right) + C_3 L + C_4 \right) \\
 \frac{dw}{dx_2} = \frac{dw}{dx}(L) &= \frac{1}{EI_0} \left[\left(\frac{-L^2(2\alpha+1)}{2\alpha^2(\alpha+1)^2} + \frac{a}{\alpha+1} \right) C_1 + C_2 \left(\frac{-L}{2\alpha(\alpha+1)^2} \right) + C_3 \right]
 \end{aligned}
 \tag{3.22}$$

In matrix form

$$\begin{Bmatrix} w_1 \\ d\theta/dx_1 \\ w_2 \\ d\theta/dx_2 \end{Bmatrix} = \frac{1}{EI_0} [B] \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}
 \tag{3.23}$$

From Eq. (83) constant matrix $\{C_i\}$ written as

$$\{C_i\} = EI_0 [B]^{-1} \{\delta_i\}
 \tag{3.24}$$

$\{\delta_i\}$ = Displacement matrix

$\{C_i\}$ = Constant matrix

$$B = \begin{bmatrix} \frac{-L^3(2\log(L)+1)}{2\alpha^3} & \frac{L^2}{2\alpha^2} & 0 & 1 \\ \frac{-L^2}{2\alpha^2} + a & \frac{-L}{2\alpha} & 1 & 0 \\ \frac{-L^3(2(\alpha+1)\log(L(\alpha+1)+1))}{2\alpha^3(\alpha+1)} & \frac{L^2}{2\alpha^2(\alpha+1)^2} & L & 1 \\ \frac{-L^2(2\alpha+1)}{2\alpha^2(\alpha+1)^2} + \frac{a}{\alpha+1} & \frac{-L}{2\alpha(\alpha+1)^2} & 1 & 0 \end{bmatrix} \quad (3.25)$$

Using MAT LAB inverse of above Matrix was calculated and substituted back in to Eq. (84) to get coefficients C_1, C_2, C_3, C_4 . By substituting these coefficients in Eq. (3.20) & Eq. (3.21) displacement and rotational shape functions are obtained.

$$w = \frac{1}{EI} [N_{wi}] \begin{Bmatrix} w_1 \\ dw/dx_1 \\ w_2 \\ dw/dx_2 \end{Bmatrix} \quad (3.26)$$

$$[N_{wi}] = \frac{1}{D} \begin{Bmatrix} \left(\xi^2 \left(\frac{a\alpha_R^4}{L^2} - \alpha_R^2 \right) + \xi \left(\frac{-a\alpha_R^4}{L^2} + \frac{2a\alpha_R^3}{L^3} + \alpha_R^2 - 2\alpha_R \right) + \left(2\alpha_R - \frac{2a\alpha_R^3}{L^2} \right) + (1 + \alpha_R \xi)(\alpha_R + 2) \log \left(\frac{1 + \alpha_R \xi}{1 + \alpha_R} \right) \right) \\ L \left(\xi^2 \left(\frac{a\alpha_R^3}{L^2} - \alpha_R + \log(\alpha_R + 1) \right) - \xi \left(\frac{a\alpha_R^3}{L^2} - \alpha_R - \alpha_R \log(L(\alpha_R + 1)) - 2 \log(\alpha_R + 1) \right) + \log L + (\alpha_R \xi + 1) \log(L(\alpha_R \xi + 1)) \right) \\ - \left(\xi^2 \left(\frac{a\alpha_R^4}{L^2} - \alpha_R^2 \right) + \xi \left(\frac{a\alpha_R^4}{L^2} + \frac{2a\alpha_R^3}{L^3} - \alpha_R^2 - 2\alpha_R \right) + \left(2\alpha_R - \frac{2a\alpha_R^3}{L^2} \right) + (1 + \alpha_R \xi)(\alpha_R + 2) \log \left(\frac{1 + \alpha_R \xi}{1 + \alpha_R} \right) \right) \\ - L(\alpha_R + 1) \left(\xi^2 \left(\frac{a\alpha_R^3}{L^2} - \alpha_R - (\alpha_R + 1) \log(\alpha_R + 1) \right) - \xi \left(\frac{a\alpha_R^3}{L^2} - \alpha_R - \alpha_R \log L \right) - \log L + (\alpha_R \xi + 1) \log(L(\alpha_R \xi + 1)) \right) \end{Bmatrix}$$

$$D = \left((1 + \alpha_R \xi) \left(2\alpha_R - \frac{2a\alpha_R^3}{L^2} - (\alpha_R + 2) \log(\alpha_R + 1) \right) \right) \quad (3.27)$$

$$\frac{dw}{dx} = \frac{1}{EI} [N_{\theta}] \begin{Bmatrix} w_1 \\ dw/dx_1 \\ w_2 \\ dw/dx_2 \end{Bmatrix} \quad (3.28)$$

$$[N_{\alpha}] = \frac{1}{D} \left[\begin{array}{c} \frac{1}{L(1+\alpha_R\xi)} \left(\alpha_R^2 \log(1+\alpha_R) - \alpha_R^2 + \frac{a\alpha_R^4}{L^2} \right) + \xi \left(\frac{-a\alpha_R^4}{L^2} + \frac{2a\alpha_R^3}{L^2} + \alpha_R^2 - 2\alpha_R + 2\log(1+\alpha_R) \right) + \left(\frac{-2a\alpha_R^3}{L^2} + 2\alpha - (2+\alpha_R)\log(1+\alpha_R) \right) \\ - \frac{1}{L(1+\alpha_R\xi)} \left(\alpha_R^3 \left(1 - \frac{a\alpha_R^2}{L^2} \right) (\xi - \xi^2) \right) \\ - \frac{(\alpha_R+1)}{(1+\alpha_R\xi)} \left(\xi^2 \left(\alpha_R(\alpha_R+1)\log(1+\alpha_R) - \alpha_R^2 + \frac{a\alpha_R^4}{L^2} \right) + \xi \left(\frac{a\alpha_R^4}{L^2} + \frac{2a\alpha_R^3}{L^2} - \alpha_R^2 - 2\alpha_R + 2(\alpha_R+1)\log(1+\alpha_R) \right) \right) \end{array} \right] \\ D = \left((1+\alpha_R\xi) \left(2\alpha_R - \frac{2a\alpha_R^3}{L^2} - (\alpha_R+2)\log(\alpha_R+1) \right) \right) \quad (3.29)$$

Eq. (85) & Eq. (86) are shape functions of Rectangular cross section Tapered Timoshenko Beam.

$[N_{wi}]$ $[N_{\alpha}]$ -are displacement shape functional Matrix, rotational shape function Matrix respectively (i=1, 2, 3, 4).

From principle of internal virtual energy

$$K = EI \int_0^L [N'_{\alpha}]^T [N'_{\alpha}] d\xi + kGA \int_0^L [N'_{wi} - N_{\alpha}]^T [N'_{wi} - N_{\alpha}] \quad (3.30)$$

From Kinetic Energy of the beam Mass matrix can be written as

$$M = \frac{1}{2} \left(\int \rho A(x) (w^1(x))^2 + \rho I(x) \left(\frac{dw^1}{dx} \right)^2 \right) dx \quad (3.31)$$

3.2 Dynamic Analysis of Tapered Timoshenko beam

3.2.1 Free Vibrations of a Tapered Timoshenko Beam Analytical solution

Consider the tapered cantilever beam, which was shown in Figure 3.1, with linearly varying eight $h(x)$ along length L. The dynamic governing equations for tapered Timoshenko beam are determined as,

$$\frac{d}{dx} \left(EI(x) \frac{d\psi}{dx} \right) + kGA(x) \left[\left(\frac{dw}{dx}(x,t) - \psi(x,t) \right) \right] - \rho I(x) \frac{d^2\psi}{dt^2}(x,t) = 0 \quad (3.32)$$

$$\frac{d}{dx} kgA(x) \left[\left(\frac{dw}{dx}(x,t) - \psi(x,t) \right) \right] + \rho A(x) \frac{d^2 w}{dt^2}(x,t) - q(x,t) = 0 \quad (3.33)$$

The computational results for engineering problems are frequently presented in non-dimensional form for convenience of extension to cover wider range problems.

The following non-dimensional parameters are introduced,

$$\begin{aligned} \xi &= x/L \\ \eta &= y/L \\ \mu &= G/E \\ \lambda &= \frac{L}{\sqrt{I_0/A_0}} \\ C_i &= \omega_i L^2 \sqrt{\frac{\rho A_0}{EI_0}} \end{aligned} \quad (3.34)$$

Where (ξ, η) are non-dimensional co-ordinates, μ is the elasticity ratio, λ is the slenderness ratio and C_i is the frequency parameter.

By the use of equations Eq. (3.8) and Eq. (3.9) the first derivatives of $\frac{dA}{dx}$ & $\frac{dI}{dx}$

can be given as

$$\frac{dA}{dx} = \alpha A_0 F^{\alpha-1} \frac{dF}{dx} \quad (3.35)$$

$$\frac{dI}{dx} = \alpha I_0 F^{\alpha-1} \frac{dF}{dx} \quad (3.36)$$

Where $\frac{dF}{dx} = \frac{-4(r-1)}{L} \left(\frac{2x}{L} - 1 \right)$

As of now all the three differential equations governing the vibration of the tapered Timoshenko beam can be obtained by substituting the above equations into the dynamic governing equations of tapered Timoshenko beam along with Eq. (3.35) and Eq. (3.36)

$$\frac{d\eta}{d\xi} = \psi + \beta \quad (3.37)$$

$$\frac{d^2\psi}{d\xi^2} = \frac{-\alpha}{F} \frac{dF}{d\xi} \frac{d\psi}{d\xi} - \frac{C_i^2}{\lambda^2} \psi - k\mu\lambda^3 \quad (3.38)$$

$$\frac{d\beta}{d\xi} = \frac{-C_i^2}{k\mu\lambda^2} \eta - \frac{\alpha}{F} \frac{dF}{d\xi} \beta \quad (3.39)$$

In which $F = F(\xi)$ & $\frac{dF}{d\xi}$ are defined as

$$F = -4(r-1)\xi(\xi-1)+1$$

$$\frac{dF}{d\xi} = -4(r-1)(2\xi-1)$$

The numerical methods presented by Lee and Wilson [17] and Lee et al. [18] are adopted to solve the three simultaneous differential equations. First the Runge-Kutta method is used to integrate the differential equations, subjected to boundary conditions. Second the determinant search method combined with Regula-Falsi method is used to determine the eigenvalues C_i in the differential equations.

3.2.2 Finite Element Vibrational Analysis of Tapered Timoshenko Beam

Equation for FEM of, free vibrational analysis of Timoshenko beam is given as

$$MU'' + KU = 0 \quad (3.40)$$

M=Consistent Mass Matrix

K=Stiffness Matrix

$U = u * e^{-i\omega t}$, u is amplitude of vibration.

Substitute U, K, M

$$-\omega^2 Mu * e^{-i\omega t} + Ku * e^{-i\omega t} = 0$$

From above Equation we can write characteristic equation as

$$K - \lambda M = 0$$

$$\therefore \lambda = \omega^2 \quad (3.41)$$

Solving above characteristic equation we will get Eigen frequencies for the straight Timoshenko beam.

Chapter 4

Design Optimization

4.1 Introduction to Design Optimization

Engineering consists of a number of well-established activities, including analysis, design, fabrication, sales, research and the development of systems. The processes of designing and fabricating systems have been developed over centuries. The existence of many complex systems, such as bridges, highways, automobiles, airplanes, space vehicles and others, is an excellent testimonial for this process. However, the evolution of these systems has been slow. The entire process has been both time-consuming and costly requiring substantial human and material resources. Therefore, the procedure has been to design, fabricate and use the system regardless of whether it was the best one. Improved systems were designed only after substantial investment had been recovered. These new systems perform the same or even more tasks, cost less, and more efficient.

The design of many engineering systems can be a fairly complex process. Many assumptions must be made to develop models that can be subjected to analysis by the available methods and the models must be verified by experiments. Many possibilities and factors must be considered during the problem formulation phase. The design of a system begins by analyzing various options, subsystems and their components are identified, designed and tested. This process results in a set of drawings, calculations, and reports by which the system can be fabricated. The optimum design forces the designer to identify explicitly a set of design variables, an objective function to be optimized, and the

constraint functions for the system. This rigorous formulation of the design problem helps the designer to gain a better understanding of the problem.

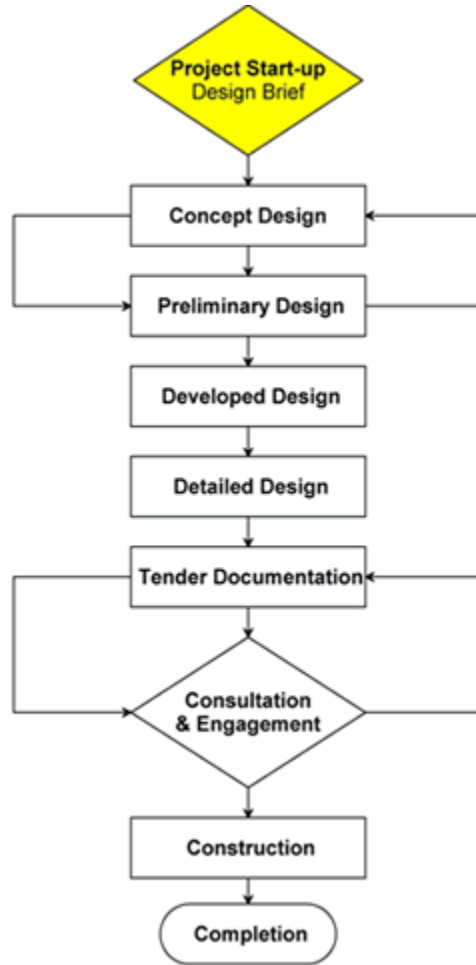
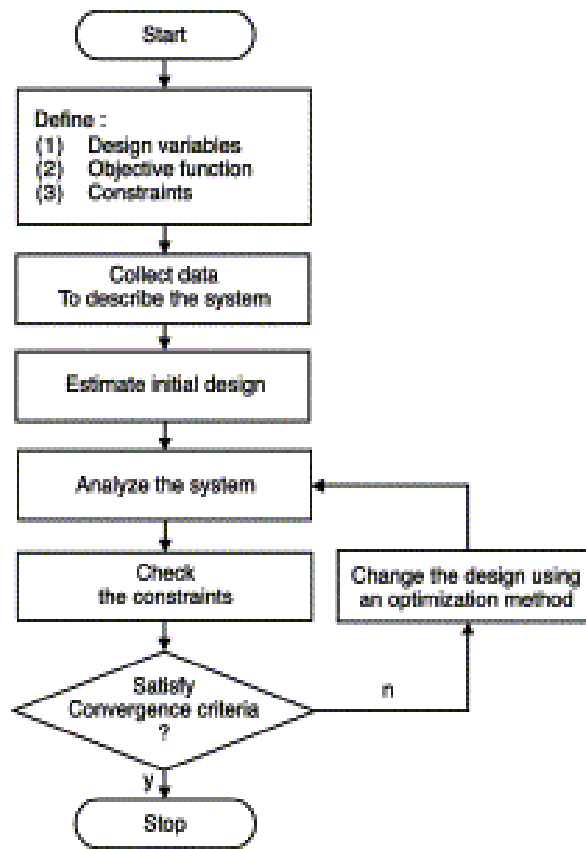


Figure 4.1 Comparison of Conventional and Optimum Design Process (A)

Conventional Design Process



Continued (B) Optimum Design Process

4.2 Optimum Design Problem Formulation

To obtain the best optimum solution to the design problem, it is important to properly formulate the design optimization problem. For most design optimization problems, we use the following

- Project/Problem Statement: The formulation process begins by developing a descriptive statement for the project. The statement describes the overall objectives and the requirements to be met.
- Data/Information Collection: To develop a mathematical formulation of the problem we need to gather material properties, performance

requirements and resource limits. In addition most problems require the capability to analyze the trial designs. Therefore, analysis procedure and analysis tools must be specified at this stage.

- **Definition of Design Variables:** In general design variables are referred to as optimization variables and are regarded as free because we can assign any value to them. Different values for design variables produce different designs. The design variables should be independent of each other as far as possible. The number of independent variables specifies the design degrees of freedom for the problem. If proper design variables are not selected for a problem, the formulation will be either incorrect or solution not possible at all.
- **Identification of Criterion to be optimized:** There can be many feasible designs for a system, and some are better than others. To compare the different designs we must have a criterion. The criterion must be a scalar function whose numerical value can be obtained once a design is specified i.e. it must be a function of the design variable

4.3 Optimization Problem

As depending on what are the design variables and objective functions determines the outcome of the design optimization. There is no unique answer in optimization problems; it is just the physics of the problem that yields the appropriate results depending on the choice of input design variables.

For this work “Fmincon” function in MATLAB was used as an optimization tool to optimize the weight/Volume of the Timoshenko beam with height as the only design variable with Eigenvalues or deflection as the constraints.

The general optimization constraint problems is stated as following,

$$\begin{array}{ll} \text{Minimize} & f(x) \quad (\text{Objective Function}) \\ \text{Subject to} & h_i(x) = e_i \quad i=1 \dots p \\ & g_i(x) \geq b_i \end{array}$$

Here in this problem Objective function is weight so for this problem objective function can be written as

$$F(x) = \rho * V \quad (4.1)$$

Where ρ =Mass Density of the material.

V=Volume of the beam.

Design Variable: Height of the beam (h)

Static Constraints: Deflection and Maximum Allowable stress in the beam.

Dynamic Constraints: Eigen Values/Natural Frequencies of the beam.

Chapter 5

Results

In this study both uniform and tapered cantilever Timoshenko beam static and vibrational analysis has been done. As parametric study, rectangular and circular cross sections were studied and for both rectangular and circular cross sections analytical and Finite Element analysis were done. In this study a beam element was considered with two nodes which have two degrees of freedom at each node i.e. deflection and total rotation. As discussed earlier in the preceding chapters 2 and 3 an unique method was developed in this study to determine the static and vibrational characteristics of the beam which is, an interdependent interpolation functions has been derived for both the deflection and total rotation from the exact governing equations of the both uniform and tapered beam for the rectangular and circular cross sections. The obtained variation is cubic approximation for deflection and for the total rotation has one order less variation than deflection i.e. quadratic variation for rotation has been derived and this variation of the deflection and rotation is well matching with the J. N. Reddy's interdependent interpolation method.

From these variations of the deflection and rotation using the finite element knowledge exact shape functions for both uniform and tapered Timoshenko beams were derived which are well in convergence with the Euler-Bernoulli shape functions. Following figures are describing the comparison of both Uniform and linearly tapered Timoshenko beam (Rectangular and Circular Cross sections) shape functions with Euler-Bernoulli shape functions.

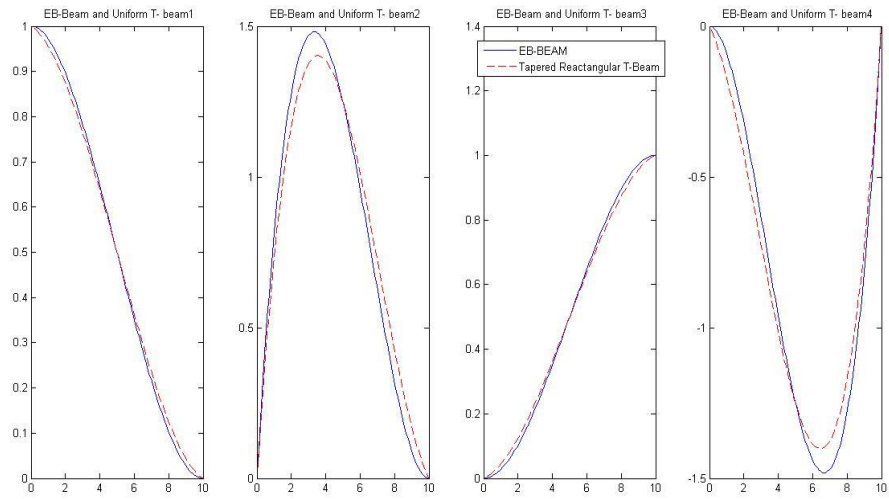


Figure 5.1 Comparison of Uniform Timoshenko Beam Shape Functions with Euler-Bernoulli Beam Shape Functions

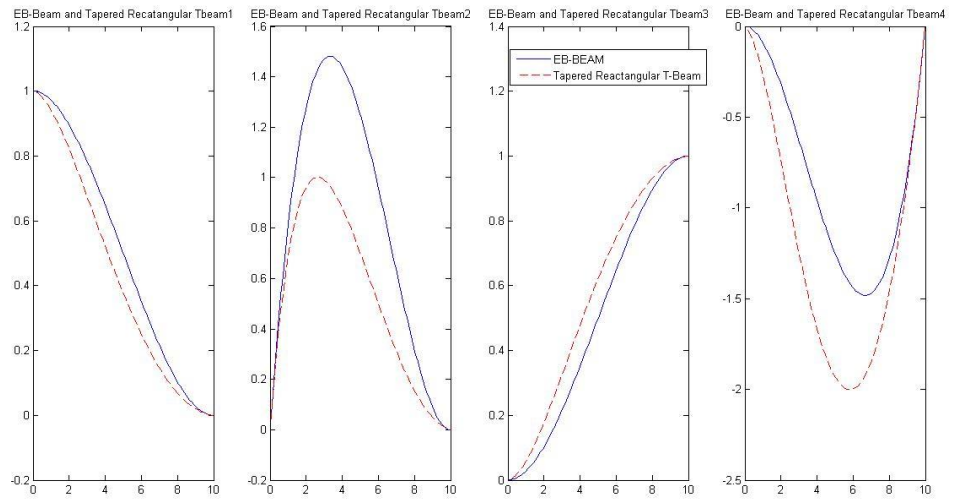


Figure 5.2 Comparison of Tapered Rectangular Timoshenko Beam Shape Functions with Euler-Bernoulli Beam Shape Functions

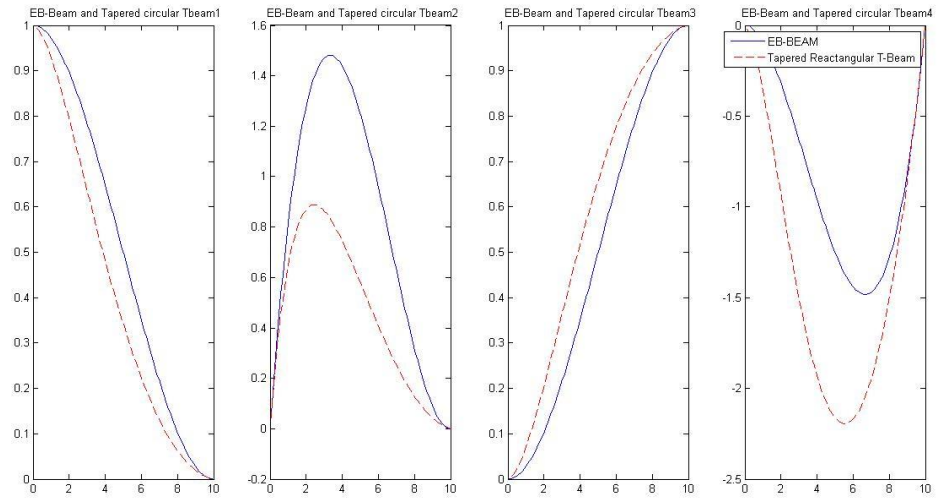


Figure 5.3 Comparison of Tapered Circular Timoshenko Beam Shape Functions with Euler-Bernoulli Beam Shape Functions

5.1 Uniform Timoshenko Beam Static Deflection

5.1.1 Rectangular Cantilever Timoshenko Beam Static Deflection

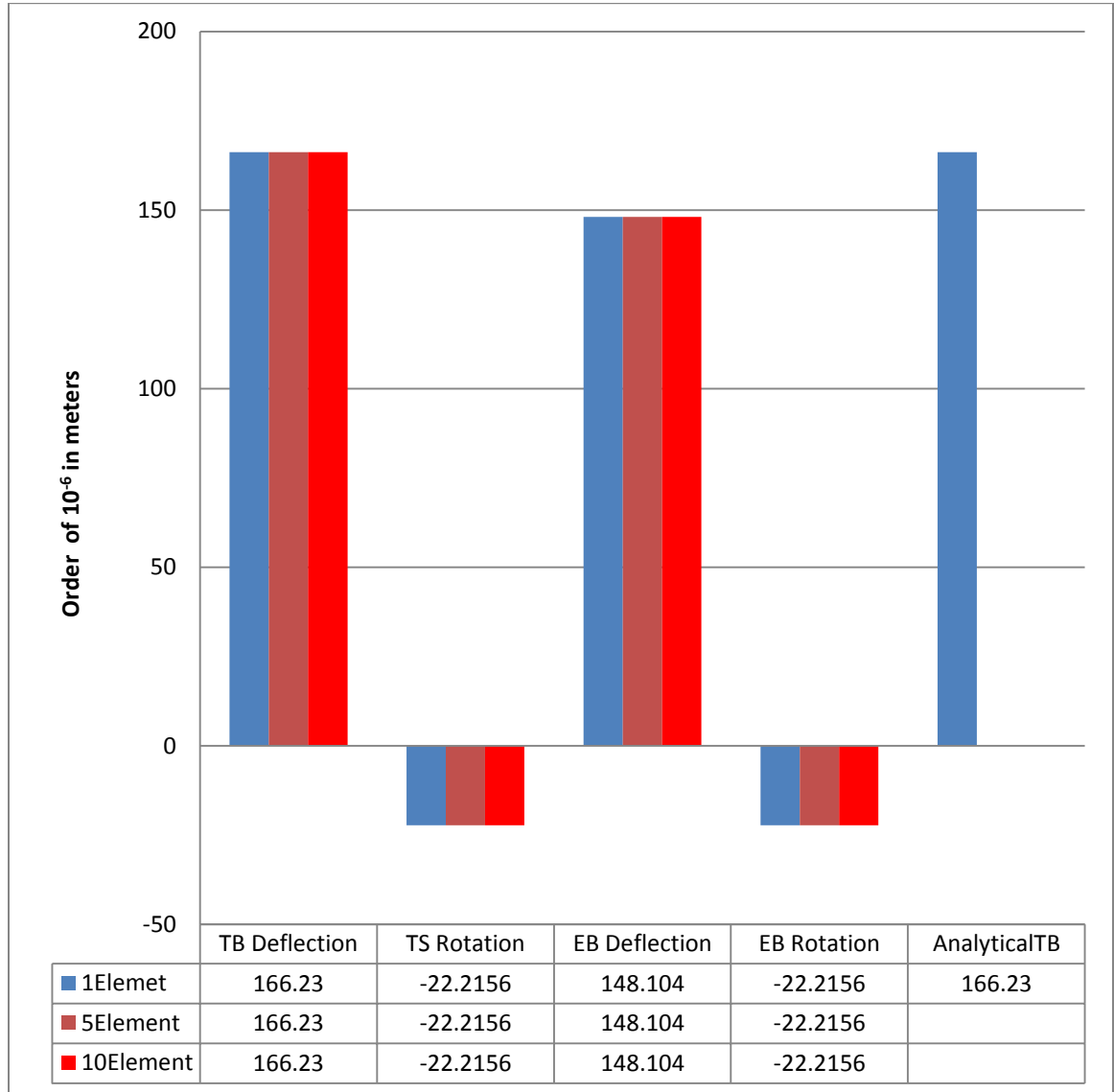


Figure 5.4 Uniform Timoshenko Beam (Rectangular Cross Section) Static Solution

5.1.2 Circular Cantilever Timoshenko Beam Static Deflection

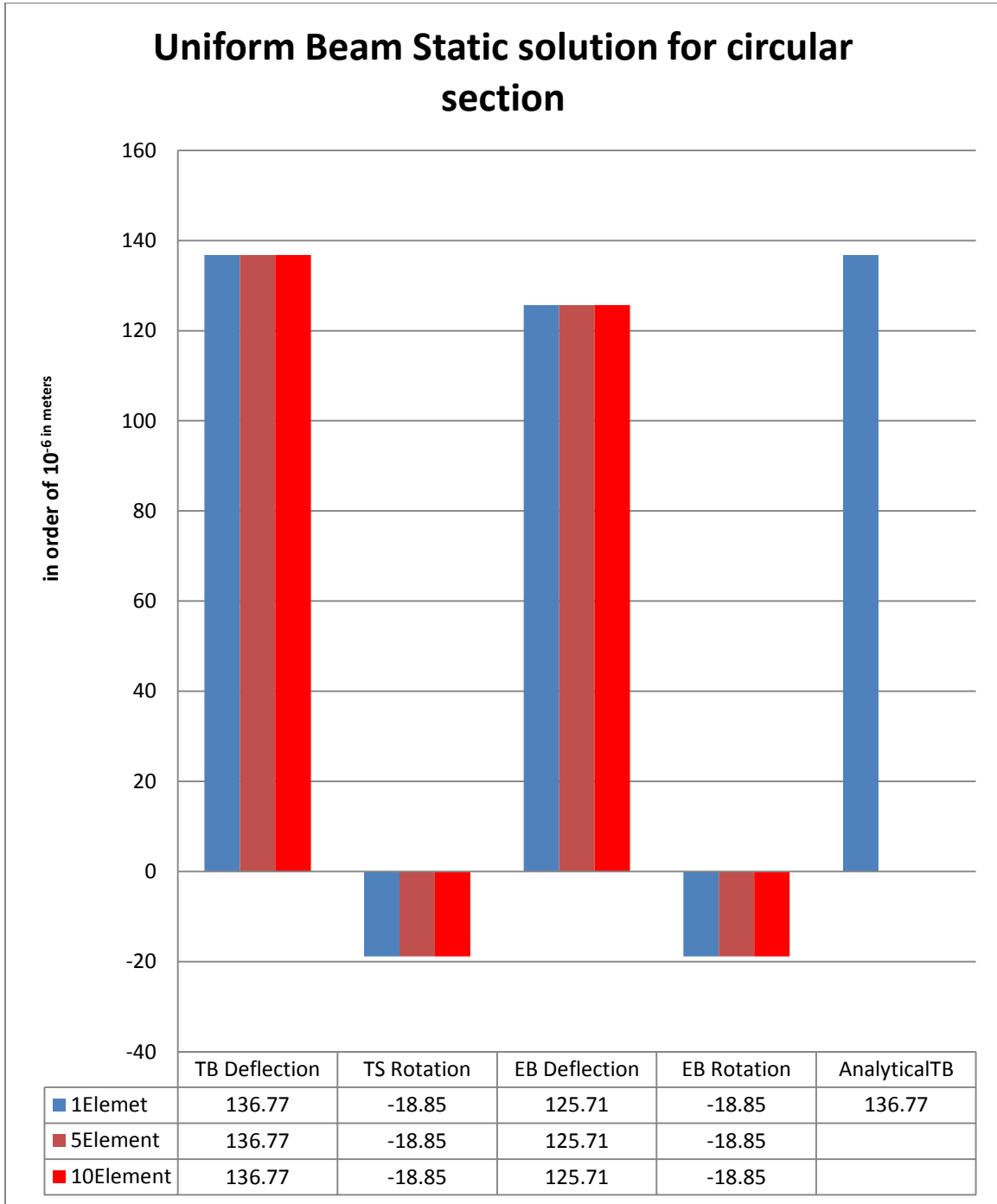


Figure 5.5 Uniform Timoshenko Beam (Circular Cross Section) Static Solution

5.2 Uniform Timoshenko Beam Dynamic Solution

5.2.1 Rectangular Cantilever Timoshenko Beam Dynamic Solution

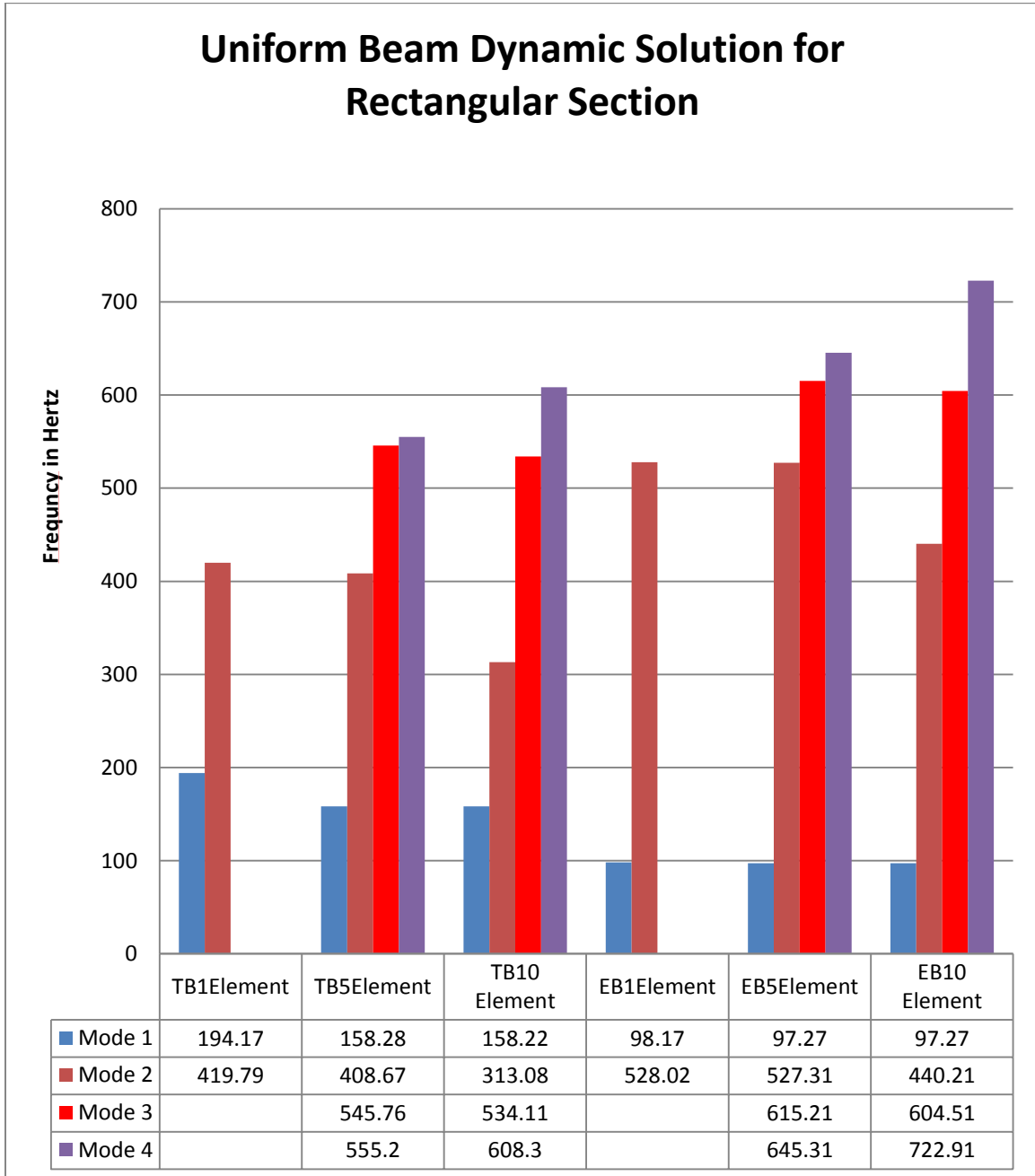


Figure 5.6 Uniform Timoshenko Beam (Rectangular Cross Section) Dynamic Solution

5.2.2 Circular Cantilever Timoshenko Beam Dynamic Solution

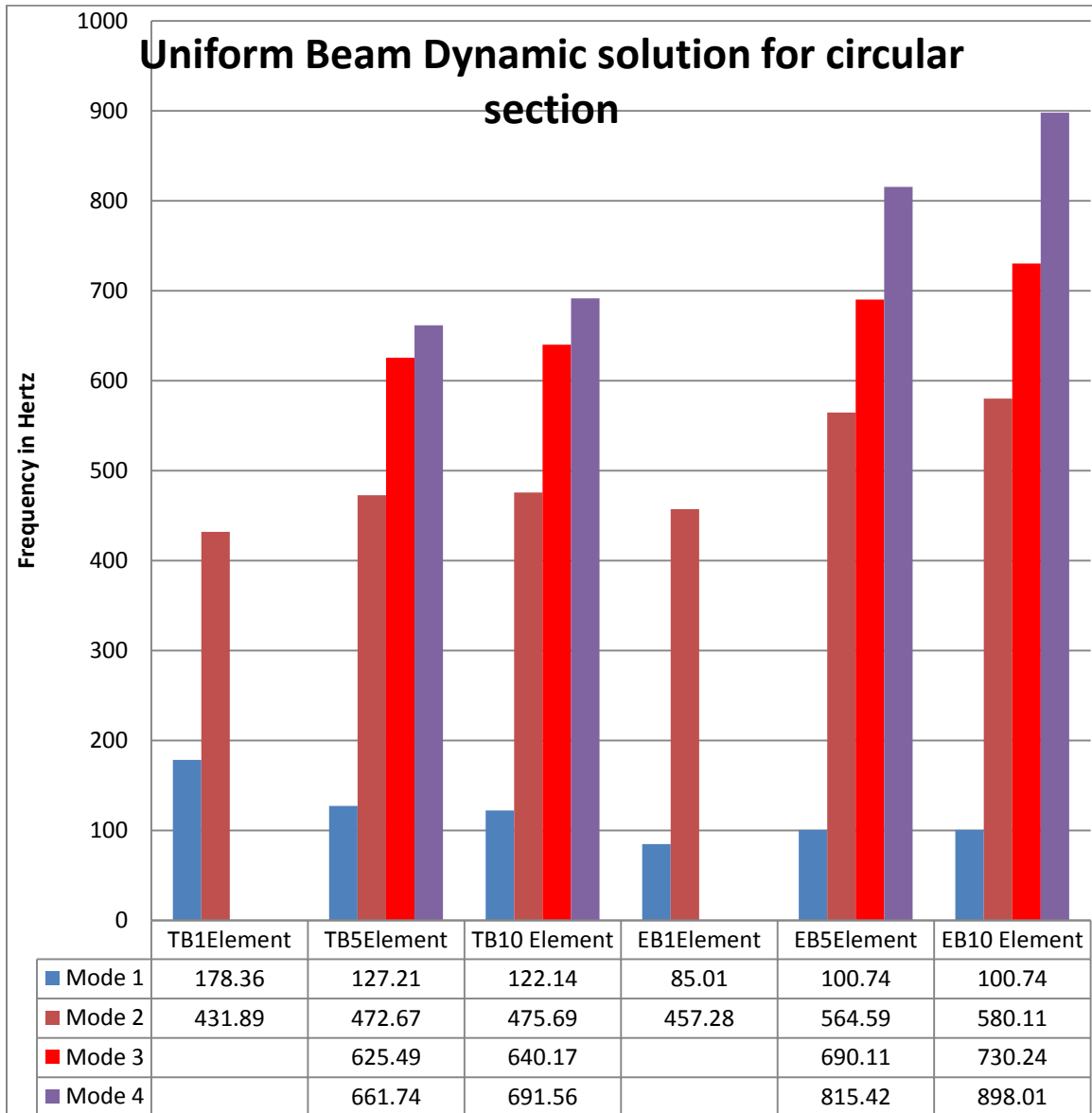


Figure 5.7 Uniform Timoshenko Beam (Circular Cross Section) Dynamic Solution

5.3 Tapered Timoshenko Beam Static Solution

5.3.1 Tapered Rectangular Cantilever Timoshenko Beam Static Deflection

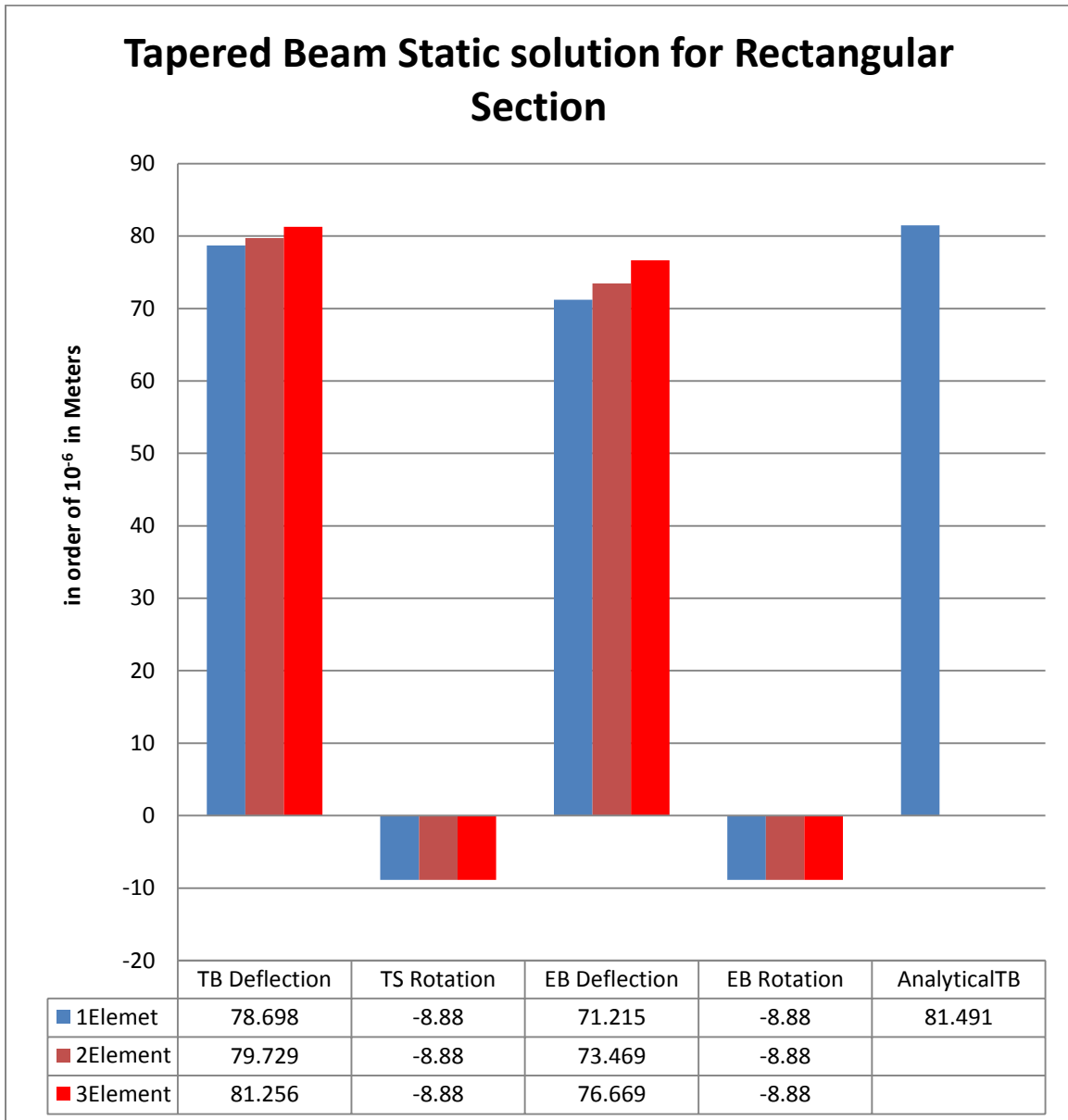


Figure 5.8 Tapered Timoshenko Beam (Rectangular Cross Section) Static Solution

5.3.2 Tapered Circular Cantilever Timoshenko Beam Static Deflection

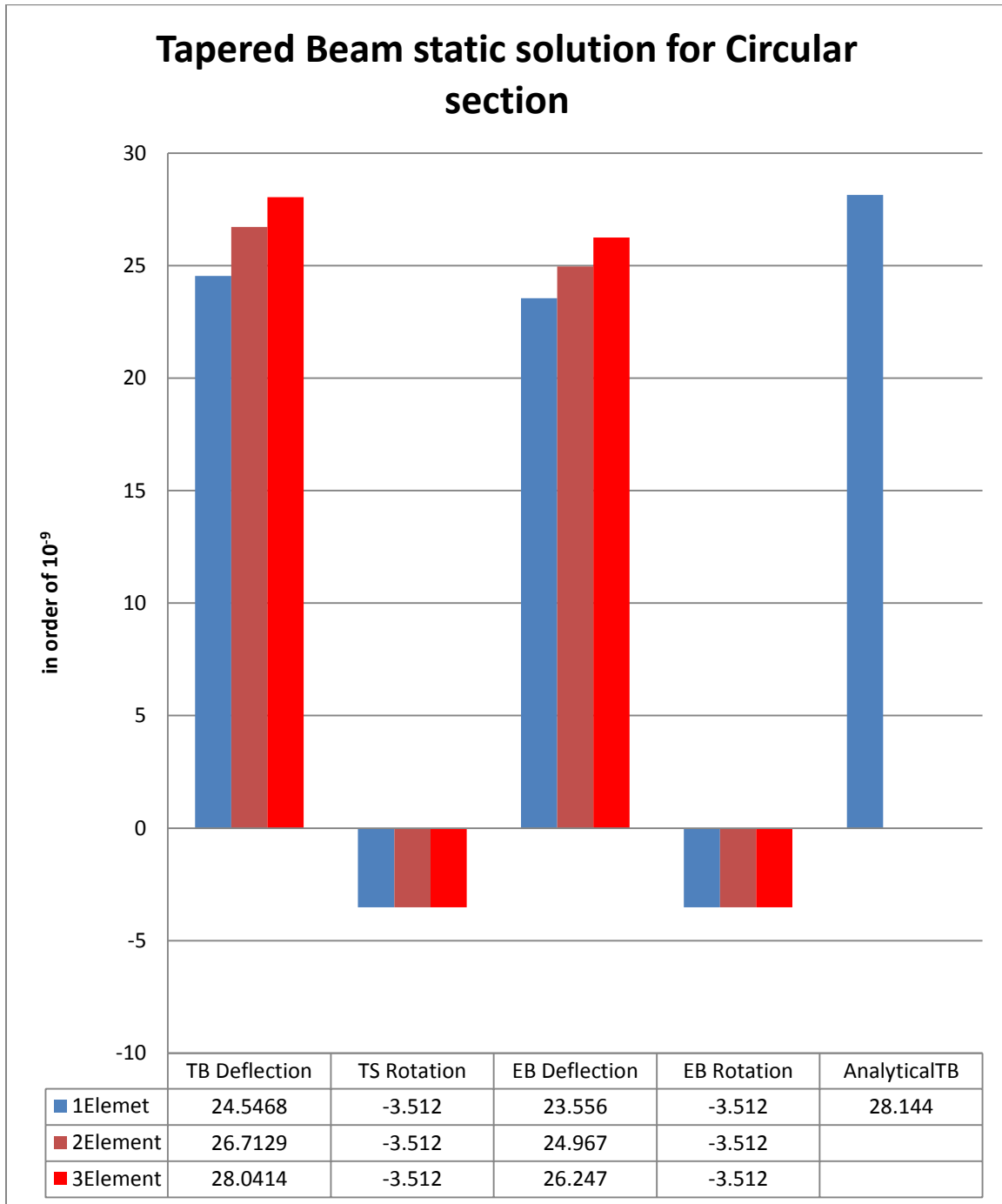


Figure 5.9 Tapered Timoshenko Beam (Circular Cross Section) Static Solution

5.4 Tapered Timoshenko Beam Dynamic Solution

5.4.1 Tapered Rectangular Cantilever Timoshenko Beam Dynamic Solution

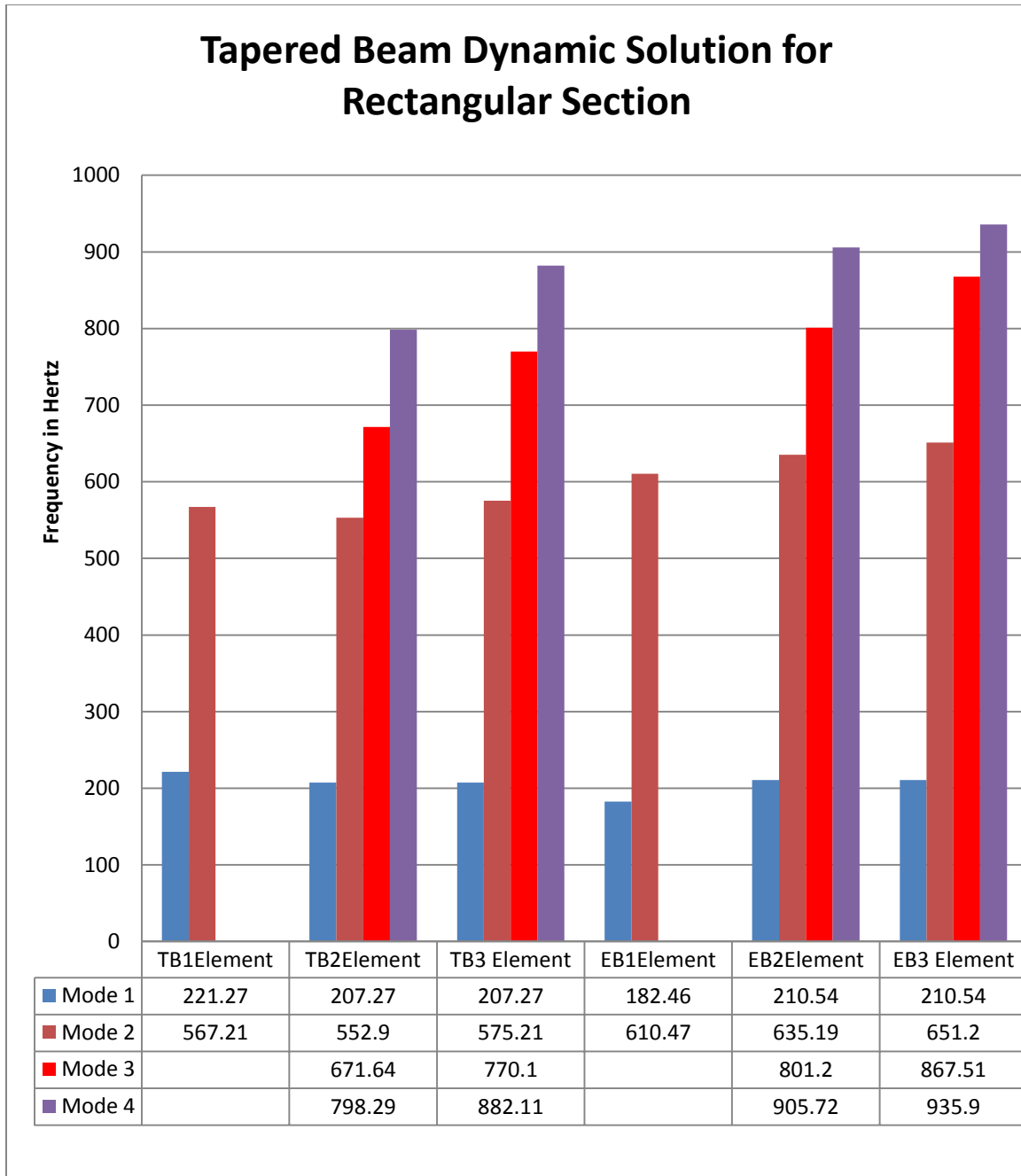


Figure 5.10 Tapered Timoshenko Beam (Rectangular Cross Section) Dynamic Solution

5.4.2 Tapered Circular Cantilever Timoshenko Beam Dynamic Solution

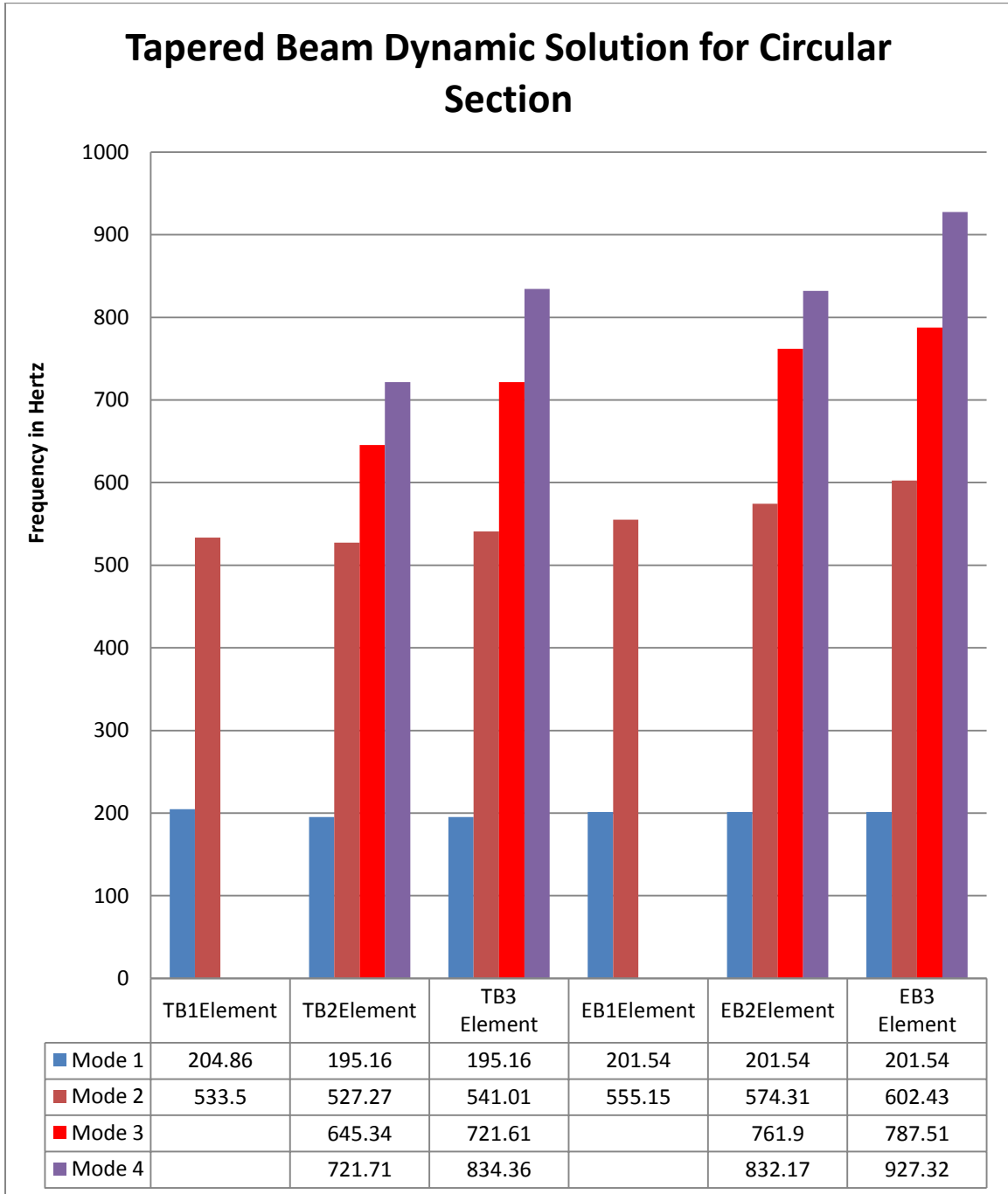


Figure 5.11 Tapered Timoshenko Beam (Circular Cross Section) Dynamic Solution

5.5 Optimization Results

Using the “Fmincon” function in MATLAB optimization tool box weight optimization of Timoshenko beam has been done with eigenvalues as constraints and height of the beam as design variable.

OBJECTIVE FUNCTION	CONSTRAINT
WEIGHT	Deflection
WEIGHT	Eigenvalues/Natural Frequencies

Table 5.1 Optimization Problem Definition

Problem 1: With the Objective function weight and Deflection ($<1.00 \cdot 10^{-6}$ m) as constraints design optimization has been done to minimize the weight with height as design variable.

	Weight in Kilograms
Original Weight	13060
Optimal Weight	8734

Table 5.2 Optimal Weight of Beam with Deflection Constraints

Problem 2: With the Objective function weight and Eigenvalues (<1000 Hz) as constraints design optimization has been done to minimize the weight with height as design variable.

	Weight in Kilograms
Original Weight	13060
Optimal Weight	9072

Table 5.3 Optimized Weight of Timoshenko Beam with Eigenvalues as Constraints

Chapter 6

Conclusion and Future Work

In this thesis, detailed study of governing equations of both uniform and tapered Timoshenko beam has been studied. The numerical techniques involved to perform static and vibrational analysis of Timoshenko beam have been studied. An extensive effort has been made to derive the exact shape functions of both uniform and tapered Timoshenko beams from exact homogeneous governing equations by inter independent interpolation technique in which cubic polynomial interpolation has been obtained for the deflection and a quadratic variation obtained for deflection of the Timoshenko beam. This beam element with two nodes and four degrees of freedom successfully eliminated the numerical problem shear locking in both uniform and tapered Timoshenko beams.

Derived exact shape functions are in accordance with the Euler Bernoulli shape functions and a parametric study has been performed to understand the variations in circular and rectangular cross sections.

The finite element model for uniform Timoshenko beam is super converging with the analytical solutions whereas, for tapered beam the solution is converging for very few elements unlike the numerical techniques proposed earlier which considers large number of elements for convergence problems.

A weight optimization of Timoshenko beam with deflection and natural frequencies as constraints has been carried out in this study.

This work can be extended to various cross sections such as I-section, C-section, H-section, T-section. In this study, a linear taper in height has been considered which can be extended to breadth taper, width taper and square taper. This research can be protracted to composites with ply drop off technique considered.

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