STABILITY RESULTS FOR SOLUTIONS OF REACTION DIFFUSION
SYSTEMS BY THE METHOD OF QUASISOLUTIONS*

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I. INTRODUCTION

Recently [10] the method of lower and upper solutions has been extended to systems of reaction diffusion equations which has become very useful in dealing with applications. This extension depends crucially on a certain property known as quasimonotone nondecreasing property [8] without which the results fail under natural definition of lower and upper solutions. When the quasimonotone property does not hold but a certain mixed quasimonotone property is satisfied, which is the case in several applications [7], the method of quasisolutions is more suitable [2,4,6,9]. All these results utilize monotone iterative technique. When no monotone condition holds one can also get just existence results [5] assuming Müller's type of lower and upper solutions. However in this case monotone technique fails.

In this paper, we discuss the asymptotic stability of the stationary solution of reaction-diffusion systems. We employ the method of quasisolutions and monotone technique.

II. PRELIMINARIES

We consider the reaction-diffusion system of the form

\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f_i(x,u), \quad t > 0, \quad x \in (0,1),
\]  

(2.1)

\[
B_{\mu}u(t,\mu) = \alpha_{\mu}u(t,\mu) + \beta_{\mu}(-1)^{\mu+1}\frac{\partial u}{\partial x}(t,\mu) = 0, \quad \mu = 0,1
\]  

(2.2)

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and

\[ u(0, x) = u_0(x), \quad x \in I, \quad (2.3) \]

where \( f \in C(I \times \mathbb{R}^n, \mathbb{R}^n) \), \( I = [0, 1] \), \( \alpha^i_1, \beta^i_1 \in \mathbb{R} \) with \( \alpha^0_1, \alpha^1_1 \geq 0 \), \( \beta^0_1, \beta^1_1 > 0 \) and \( i = 1, 2, \ldots, n \).

To employ the method of quasisolutions, we fix, for each \( i, 1 \leq i \leq n \), two nonnegative integers \( p_i, q_i \) such that \( p_i + q_i = n - 1 \) and split \( u \in \mathbb{R}^n \) into \( u = (u_i, [u]_{p_i}, [u]_{q_i}) \) so that (2.1) can explicitly be written as

\[
\frac{\partial u_i}{\partial t} - \frac{\partial^2 u_i}{\partial x^2} = f_i(x, u_i, [u]_{p_i}, [u]_{q_i}).
\]

In what follows the inequalities between vectors are to be understood componentwise and \( i \) always ranges from 1 to \( n \).

If \( u \in C(R^+ \times I, \mathbb{R}^n) \) has partial derivatives \( \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2} \) which are continuous in \((0, \infty) \times (0, 1)\), then we shall say that \( u \in \mathcal{C} \). The functions \( \tilde{v}, \tilde{w} \in \mathcal{C} \) with \( \tilde{v} \leq \tilde{w} \) are said to be coupled lower and upper quasisolutions respectively if

\[
\frac{\partial \tilde{v}_i}{\partial t} - \frac{\partial^2 \tilde{v}_i}{\partial x^2} \leq f_i(x, \tilde{v}_i, [\tilde{v}]_{p_i}, [\tilde{w}]_{q_i}),
\]

\[
B_\mu \tilde{v}(t, \mu) \leq 0, \quad \tilde{v}(0, x) \leq u_0(x),
\]

and

\[
\frac{\partial \tilde{w}_i}{\partial t} - \frac{\partial^2 \tilde{w}_i}{\partial x^2} \geq f_i(x, \tilde{w}_i, [\tilde{w}]_{p_i}, [\tilde{v}]_{q_i}),
\]

\[
B_\mu \tilde{w}(t, \mu) \geq 0, \quad \tilde{w}(0, x) \geq u_0(x).
\]
If equality holds in the foregoing definition, then \( \tilde{v}, \tilde{w} \) are said to be coupled quasisolutions of (2.1), (2.2), (2.3). Furthermore, we can also define coupled maximal and minimal quasisolutions of (2.1), (2.2), (2.3).

A function \( f \) is said to possess a mixed quasimonotone property (mqmp for short) if for each \( i \), \( f_i(x, u_{i_1}, [u]_{p_1}, [u]_{q_1}) \) is monotone non-decreasing in \([u]_{p_1}\) and monotone nonincreasing in \([u]_{q_1}\).

Let us consider the corresponding steady state problem

\[
- \frac{\partial^2 u_i}{\partial x^2} = f_i(x, u), \quad x \in (0, 1)
\]

(2.4)

\[
B_{\mu} u(\mu) = 0
\]

(2.5)

Analogously we can define relative to the problem (2.4), (2.5), the notion of coupled lower and upper quasisolutions, coupled minimal and maximal quasisolutions with appropriate modifications.

III. EXISTENCE AND UNIQUENESS

Let \( D_T = (0, T] \times (0, 1) \) where \( T > 0 \) is finite but can be arbitrarily large. Assume that \( f(x, u) \) is Hölder continuous for every bounded subset of \( I \times \mathbb{R}^n \) and \( u_0 \) is continuously differentiable in \( I \). These assumptions are needed to insure the existence of a solution for the linear reaction-diffusion problem.

For any \( v, w \in C[I, \mathbb{R}^n] \) such that \( v \leq w \) on \( I \), we define for some \( \gamma > 0 \), the sector

\[
[v, w]_\gamma = \{ u \in \mathbb{R}^n : v(x) - \gamma \leq u \leq w(x) + \gamma, \quad x \in I \}.
\]

If \( \gamma = 0 \) we shall write \([v, w]_0\). Let us list the following assumptions for convenience.
(A₁) for each \( i \) and \( x \in (0,1) \),

\[
f_i(x, \eta_1, [u]_{p_1}^{q_1}, [u]_{q_1}) - f_i(x, \eta_2, [u]_{p_1}^{q_1}, [u]_{q_1}) \geq -M_i(\eta_1 - \eta_2)
\]

whenever \( \eta_1 \geq \eta_2 \) and for some \( \gamma > 0 \), \( u, \eta_1, \eta_2 \in [v, w]_\gamma \);

(A₂) for each \( i \) and \( x \in (0,1) \),

\[
f_i(x, u_1, [u]_{p_1}^{q_1}, [\bar{u}]_{q_1}) - f_i(x, \bar{u}_1, [\bar{u}]_{p_1}^{q_1}, [u]_{q_1}) \leq M_i \sum_{j=1}^{n} (u_j - \bar{u}_j)
\]

whenever \( u \geq \bar{u} \), \( u, \bar{u} \in [v, w]_\gamma \).

We observe that whenever \( v, w \in C[I, \mathbb{R}^n] \) are coupled lower and upper quasisolutions of the problem (2.4), (2.5), it follows that \( v, w \) are also coupled lower and upper quasisolutions of the problem (2.1), (2.2), (2.3) provided that \( v(x) \leq u_0(x) \leq w(x) \) on \( I \). We then have the following result.

**Theorem 3.1.** Let \( v \leq w \) be coupled lower and upper quasisolutions for (2.4), (2.5). Let \( f \) satisfy mqmp property and (A₁) hold on \([v, w]_0 \). Then

(i) there exist monotone sequences \( \{v_n(x)\}, \{w_n(x)\} \), which converge monotonically and uniformly to coupled minimal and maximal solutions \( \alpha(x), \beta(x) \) of (2.4), (2.5), respectively such that \( v(x) \leq \alpha(x) \leq \beta(x) \leq w(x) \) on \( I \);

(ii) there exist monotone sequences \( \{v_n(t,x)\}, \{w_n(t,x)\} \) which converge monotonically and uniformly to coupled minimal and maximal solutions \( \tilde{\alpha}(t,x), \tilde{\beta}(t,x) \) of (2.1), (2.2), (2.3) respectively such that

\( v(x) \leq \tilde{\alpha}(t,x) \leq \tilde{\beta}(t,x) \leq w(x) \) for \( (t,x) \in D_T \) provided \( v(x) \leq u_0(x) \leq w(x) \) on \( I \).
(iii) If, in addition, \((A_2)\) holds on \([v,w]_0\) with \(M_1\) small enough, then there exist unique solutions \(\bar{u}(x), u(t,x)\) for the problems (2.4),(2.5) and (2.1),(2.2),(2.3) satisfying \(v(x) \leq \bar{u}(x) \leq w(x)\) on \(I\) and \(v(x) \leq u(t,x) \leq w(x)\) on \(D_T\) respectively.

**Proof.** The conclusions (i), (ii) follow directly from our recent results given in [6,9] with suitable modifications.

In order to prove (iii) it is only necessary to show that \(\beta(x) \leq \alpha(x)\) on \(I\) and \(\bar{\beta}(t,x) \leq \bar{\alpha}(t,x)\) on \(D_T\) for every \(T > 0\). We set, for each \(i\),

\[
\begin{align*}
\beta_i(x) &= \beta_i(x) - \alpha_i(x) \quad \text{and} \quad \bar{\beta}_i(t,x) = \bar{\beta}_i(t,x) - \bar{\alpha}_i(t,x).
\end{align*}
\]

It then follows because of \((A_2)\) and the fact \(\alpha(x) \leq \beta(x)\), \(\bar{\alpha}(t,x) \leq \bar{\beta}(t,x)\),

\[
\frac{\partial^2 \beta_i(x)}{\partial x^2} \geq -M_1 \sum_{j=1}^{n} m_j \quad \text{and} \quad B_{\mu} M(\mu) = 0 \quad (3.1)
\]

and

\[
\begin{align*}
\frac{\partial \bar{\beta}_i(t,x)}{\partial t} - \frac{\partial^2 \beta_i(t,x)}{\partial x^2} &\leq M_1 \sum_{j=1}^{n} m_j \\
B_{\mu} m(t,\mu) &\equiv 0, \quad m(0,x) \equiv 0
\end{align*}
\]

Since \(M_1\) is assumed to be small enough it is easy to check that the assumptions of the generalized maximum principle are satisfied. See [3,11]. Hence we get \(\beta(x) \leq \alpha(x)\) on \(I\) from (3.1) which proves the uniqueness of solutions of (2.4),(2.5). The uniqueness of solutions of the problem (2.1),(2.2),(2.3) follows from (3.2) as a consequence of results in [7]. The proof the theorem is therefore complete.
(2.1), (2.2), (2.3). Consequently the desired estimate (4.3) follows from Theorem 3.1.

Corollary 4.1. If the assumption of Theorem 4.1 are satisfied on $[v, w]_\gamma$ for all $\gamma > 0$, then the stationary solution $\bar{u}(x)$ is globally asymptotically (exponentially) stable.

REFERENCES


Z(1,2) = Z(1,1) + 2.0 * VZ(1,2)

80 CONTINUE

8043 K = K + 1

IF (MOD(K, KPRINT), GT, 0) GO TO 82

DO 810 I = 1, 7

C

81 WRITE (5, 81) K, I, X(1, 2), Y(1, 2), Z(1, 2), VX(1, 2), VY(1, 2), VZ(1, 2)

81 FORMAT (5X, I12, 15, 6F18.9)

810 CONTINUE

R1 = (X(1, 2) - X(2, 2))**2 + (Y(1, 2) - Y(2, 2))**2 + (Z(1, 2) - Z(2, 2))**2
R10 = (X(1, 2) - X(3, 2))**2 + (Y(1, 2) - Y(3, 2))**2 + (Z(1, 2) - Z(3, 2))**2
R80 = (X(2, 2) - X(3, 2))**2 + (Y(2, 2) - Y(3, 2))**2 + (Z(2, 2) - Z(3, 2))**2
R18 = DSQRT(R1)
R110 = DSQRT(R110)
R810 = DSQRT(R810)
COS = (-R810*R810+R18*R18+R110*R110)/(2.*R18*R110)

C

819 WRITE (5, 819) COS, R18

819 FORMAT (2X, 2F20.10)

82 IF (K, LT, 2400000) GO TO 65

IF (K, EQ, 2400000) GO TO 90
IF (K, LT, 1400000) GO TO 65
IF (K, EQ, 4600000) GO TO 91
IF (K, LT, 7200000) GO TO 65
IF (K, EQ, 7200000) GO TO 9115
IF (K, LT, 9600000) GO TO 65
IF (K, EQ, 9600000) GO TO 9116
IF (K, LT, 12000000) GO TO 65
IF (K, EQ, 12000000) GO TO 9117
IF (K, LT, 14400000) GO TO 65
IF (K, EQ, 14400000) GO TO 9118
IF (K, LT, 16800000) GO TO 65
IF (K, EQ, 16800000) GO TO 9119
IF (K, LT, 19200000) GO TO 65
IF (K, EQ, 19200000) GO TO 92

90 WRITE (21, 10) (X(1, 2), Y(1, 2), Z(1, 2), VX(1, 2), VY(1, 2),

1VZ(1, 2), I = 1, 7)

GO TO 7

91 WRITE (21, 10) (X(1, 2), Y(1, 2), Z(1, 2), VX(1, 2), VY(1, 2),

1VZ(1, 2), I = 1, 7)

GO TO 8

9115 WRITE (21, 10) (X(1, 2), Y(1, 2), Z(1, 2), VX(1, 2), VY(1, 2),

1VZ(1, 2), I = 1, 7)

GO TO 9

9116 WRITE (21, 10) (X(1, 2), Y(1, 2), Z(1, 2),

1VX(1, 2), VY(1, 2), VZ(1, 2), I = 1, 7)

GO TO 1001

9117 WRITE (21, 10) (X(1, 2), Y(1, 2), Z(1, 2),

1VX(1, 2), VY(1, 2), VZ(1, 2), I = 1, 7)

GO TO 1002

9118 WRITE (21, 10) (X(1, 2), Y(1, 2), Z(1, 2),

1VX(1, 2), VY(1, 2), VZ(1, 2), I = 1, 7)

GO TO 1003

9119 WRITE (21, 10) (X(1, 2), Y(1, 2), Z(1, 2),

1VX(1, 2), VY(1, 2), VZ(1, 2), I = 1, 17)

GO TO 1004

92 WRITE (7, 10) (X(1, 2), Y(1, 2), Z(1, 2), VX(1, 2), VY(1, 2), VZ(1, 2), I = 1, 17)
STOP
END