NOTE ON SPECIAL RELATIVISTIC CALCULATIONS
WITH IDENTICAL LABORATORY AND ROCKET
FRAME COMPUTERS

by

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ABSTRACT

Dynamical relativistic problems invariably require the solution of nonlinear differential equations. These equations are rarely solvable in closed form and are now being solved numerically on modern digital computers. In this paper, a numerical method is developed which has the special property that computations on identical computers in both the laboratory and rocket frames yield numerical results which continue to be related by the Lorentz transformation.
1. INTRODUCTION. It is fundamental in special relativistic dynamics that events in the laboratory and rocket frames are related by the Lorentz transformation (1). However, in a relativistic framework, the analysis of even the simplest of particle motions, like that of a harmonic oscillator, requires the solution of highly nonlinear differential equations (2). Since such nonlinear problems are now being resolved approximately by high speed computer techniques, we will assume in this paper that both laboratory and rocket observers are equipped not only with identical, synchronized clocks, but also with identical digital computers. We will then describe and analyze a new numerical method for particle motions which, if used in both the lab and rocket frames, yields numerical results which continue to be related by the Lorentz transformation, thus preserving the basic relativistic relationship even between approximated events. None of the standard numerical techniques, like the Runge-Kutta, Taylor series, and predictor-corrector methods, preserves this relationship.

2. MATHEMATICAL PRELIMINARIES. For completeness, let us summarize first the mathematical concepts and results necessary for a complete understanding of the numerical method to be developed later.

Let the two rectangular, cartesian coordinate systems XYZ and X′Y′Z′ represent the laboratory and rocket frames, respectively, with the rocket frame in motion in the X direction.
at constant speed \( u \) relative to the lab frame. At the initial time \( t_0 = 0 \), assume the frames coincide. In the lab frame, let \( \Delta t > 0 \) and let the observer make observations at the distinct times \( t_k = k \Delta t, \ k = 0, 1, 2, \ldots \). Using an identical, synchronized clock, let the observer in the rocket frame make observations at the corresponding times \( t'_k \), where \( t'_k \) on the rocket clock corresponds to \( t_k \) on the lab clock. Let particle \( P \) be at \( (x_k', y_k', z_k') \) at time \( t_k \) in the lab frame and let it be at \( (x'_k, y'_k, z'_k) \) in the rocket frame at the corresponding time \( t'_k \). Then \( (x_k, y_k, z_k, t_k) \) and \( (x'_k, y'_k, z'_k, t'_k) \) are called discrete events and are related by the Lorentz transformation

\[
\begin{align*}
\text{(2.1)} \quad x'_k &= \frac{c(x_k - ut_k)}{(c^2 - u^2)^{\frac{1}{2}}} , \quad y'_k = y_k , \quad z'_k = z_k , \quad t'_k = \frac{c^2 t_k - ux_k}{c(c^2 - u^2)^{\frac{1}{2}}} ,
\end{align*}
\]

or, equivalently, by

\[
\begin{align*}
\text{(2.2)} \quad x_k &= \frac{c(x'_k + ut'_k)}{(c^2 - u^2)^{\frac{1}{2}}} , \quad y_k = y'_k , \quad z_k = z'_k , \quad t_k = \frac{c^2 t'_k + ux'_k}{c(c^2 - u^2)^{\frac{1}{2}}} ,
\end{align*}
\]

where \( c \) is the speed of light in a vacuum and where, for mathematical simplicity, we assume that

\[
\text{(2.3)} \quad |u| < c .
\]

Consider now the motion in the \( X \) direction only of a particle \( P \). At time \( t_k \) in the lab frame, let \( P \) be at \( (x_k, y_k, z_k), \ k = 0, 1, 2, \ldots \). Then \( P \)'s velocity \( v_k = v(t_k) \) and acceleration \( a_k = a(t_k) \) are defined by
(2.4) \[ v_k = \frac{\Delta x_k}{\Delta t_k} \]

(2.5) \[ a_k = \frac{\Delta v_k}{\Delta t_k} \]

where \( \Delta \) is the usual forward difference operator defined, in general, by

\[ \Delta(f(t_k)) = f(t_{k+1}) - f(t_k) \]

By the principle of relativity (3), at the corresponding time \( t_k' \) in the rocket frame, one must define \( v_k' \) and \( a_k' \) by

(2.6) \[ v_k' = \frac{\Delta x'_k}{\Delta t'_k} \]

(2.7) \[ a_k' = \frac{\Delta v'_k}{\Delta t'_k} \]

It follows from (2.1), (2.2) and (2.4)-(2.6) that

(2.8) \[ v_k' = \frac{c^2(v_k-u)}{c^2-uv_k} \]

(2.9) \[ a_k' = \frac{c^3(a^2-u^2)^{3/2}}{(c^2-uv_k+1)(c^2-uv_k)^2} a_k \]

provided that \( \frac{v_k}{c} < 1 \), \( \frac{v_k'}{c} < 1 \).
which will be assumed in the present paper.

In the lab frame, P's mass $m(t_k)$ is defined by

$$m(t_k) = \frac{cm_0}{(c^2-v_k^2)^{\frac{1}{2}}}$$  \hspace{1cm} (2.11)$$

where $m_0$ is the usual rest mass constant. In the rocket frame, P's mass $m'(t'_k)$ is defined by

$$m'(t'_k) = \frac{cm_0}{(c^2-v'_k^2)^{\frac{1}{2}}}$$  \hspace{1cm} (2.12)$$

In the lab frame, application of a force $F_k$, $k = 0, 1, 2, \ldots$, to $P$, in the $X$ direction only, determines the motion of $P$ uniquely from given initial data $x_0$ and $v_0$. Specifically, we will assume the arithmetic, dynamical relationship

$$x'_k = \frac{c^2m(t_k)}{((c^2-v_k^2)(c^2-v_{k+1}^2))^\frac{1}{2}} \cdot \frac{AV_k}{\Delta t_k}, \quad k = 0, 1, 2, \ldots \hspace{1cm} (2.13)$$

Correspondingly, in the rocket frame, application of $F'_k$, $k = 0, 1, 2, \ldots$, to $P$, in the $X'$ direction only, determines the motion of $P$ uniquely from given initial data $x'_0$ and $v'_0$, and, by the principle of relativity, we assume

$$F'_k = \frac{c^2m'(t'_k)}{((c^2-v'_k^2)(c^2-v'_{k+1}^2))^\frac{1}{2}} \cdot \frac{AV'_k}{\Delta t'_k}, \quad k = 0, 1, 2, \ldots \hspace{1cm} (2.14)$$

With regard to (2.13) and (2.14), the following basic
Theorem is known (2):

THEOREM 1 (Symmetry). Under the Lorentz transformation, arithmetic, dynamical formula (2.13) maps into (2.14). Moreover, in the limit, dynamical difference equations (2.13) and (2.14) converge, respectively, to the dynamical differential equations

\[ F = \frac{d}{dt}(mv), \quad m = \frac{\gamma m_0}{(c^2 - v^2)^{1/2}}, \]

\[ F' = \frac{d}{dt'}(m'v'), \quad m' = \frac{\gamma m_0}{(c^2 - v'^2)^{1/2}}. \]

3. THE NUMERICAL METHOD. Let us now describe exactly how one would determine the motion of a particle \( P \) when the force acting upon it is specified.

In the lab frame, let \( x_0 \) and \( v_0 \) be given. The positive constant \( \Delta t \) is equal to the time interval between successive observations. Thus, \( t_k = k \Delta t, \) \( k = 0, 1, 2, \ldots, \) are all known constants. Finally, assume that

\[ F_k = F(x_k, x_{k+1}, v_k, t_k), \quad k = 0, 1, 2, \ldots, \]

is a given function of the indicated variables. Then, in a recursive fashion, beginning with \( x_0 \) and \( v_0 \), one determines \( x_{k+1} \) and \( v_{k+1} \) from \( x_k \) and \( v_k \) as follows. From (2.4), \( x_{k+1} \) is given explicitly by

\[ x_{k+1} = x_k + v_k (t_{k+1} - t_k), \]

while, from (2.14) and (3.1), \( v_{k+1} \) is defined implicitly by
(3.3) \[ \frac{v_{k+1} - v_k}{t_{k+1} - t_k} \cdot \frac{c^2 m(t_k)}{((c^2 - v_k^2)(c^2 - v_{k+1}^2))^{3/2}} = F(x_k, x_{k+1}, v_k, t_k). \]

In the rocket frame, the computations are completely analogous. Beginning with \( x_0' \) and \( v_0' \), one determines \( x_{k+1}' \) and \( v_{k+1}' \) from \( x_k' \) and \( v_k' \) as follows. From (2.5), \( x_{k+1}' \) is given explicitly by

(3.4) \[ x_{k+1}' = x_k' + v_k'(t_{k+1}' - t_k'). \]

while \( v_{k+1}' \) is defined implicitly by

(3.5) \[ \frac{v_{k+1} - v_k}{t_{k+1} - t_k} \cdot \frac{c^2 m'(t_k')}{((c^2 - v_k'^2)(c^2 - v_{k+1}'^2))^{3/2}} = F'(x_k', x_{k+1}', v_k', t_k'). \]

where

(3.6) \[ F'(x_k', x_{k+1}', v_k', t_k'). \]

For relativistic harmonic oscillation, computer implementation and numerical results have been given by Greenspan (2).

Let us then explore next the theoretical basis of the general method of this section.

4. FUNDAMENTAL THEORY. In this section we will prove two fundamental theorems related to the mathematical viability and physical significance of the numerical method of Section 3. The first theorem will be a constructive existence and uniqueness theorem. The second will demonstrate that numerical calculations
on identical computers in the lab and rocket frames yield numerical results which continue to be related by the Lorentz transformation.

For simplicity, we now introduce absolute coordinates, that is, without loss of generality, set

\[ c = m_0 = 1 \]

throughout this section.

THEOREM 2 (Existence and Uniqueness). In the laboratory frame, the numerical solution generated by the method of Section 3 exists, is unique, and is given precisely by

\[ x_{k+1} = x_k + v_k(\Delta t_k) \]

\[ v_{k+1} = \frac{v_k + (\Delta t_k)(1-v_k^2)F\sqrt{1+(\Delta t_k)^2(1-v_k^2)^2F^2} - v_k^2}{1 + (\Delta t_k)^2(1-v_k^2)^2F^2} \cdot \]

Existence, uniqueness, and formulas (4.2) and (4.3), with primes inserted, are also valid in the rocket frame.

Proof. Since the algorithms in the lab and rocket frames are, essentially, the same, the proof will be given for the lab frame only. Since (4.2) is identical to (3.2), we need show only that \( v_{k+1} \) is defined uniquely by (3.3). Without loss of generality, we will do this for \( k = 0 \), in which case (3.3) reduces to

\[ \frac{v_1 - v_0}{\Delta t} \cdot \frac{c^2m(t_0)}{((c^2-v_0^2)(c^2-v_1^2))^{\frac{1}{2}}} = F \cdot \]
By (2.11) and (4.1), (4.4) reduces to

\[(4.5) \quad \frac{v_1 - v_0}{(\Delta t)(1-v_0^2)(1-v_1^2)^{\frac{1}{2}}} = F.\]

Setting

\[(4.6) \quad A = (\Delta t)(1-v_0^2)F\]

enables one to rewrite (4.5) as

\[(4.7) \quad v_1 - v_0 = A(1-v_1^2)^{\frac{1}{2}}.\]

Note that (2.3), (2.10) and (4.1) imply

\[(4.8) \quad 1-v_0^2 > 0, \quad 1-v_1^2 > 0, \quad 1-u^2 > 0,\]

so that the sign of \(A\) in (4.4) is the same as that of \(F\) whenever \(F\) is nonzero.

Now, if \(A = 0\), then (4.7) implies the uniqueness of \(v_1\).

Indeed, in this case, \(v_1 = v_0\). Hence, assume that

\[(4.9) \quad A \neq 0.\]

Then, squaring both sides of (4.7) and reorganizing terms implies

\[(4.10) \quad v_1^2(1+A^2) - 2v_0v_1 + v_0^2 - A^2 = 0,\]

the solutions of which are

\[(4.11) \quad v_{1,1,2} = \frac{v_0 \pm |A|\sqrt{1-v_0^2 + A^2}}{1 + A^2}, \quad v_{1,2} = \frac{v_0 - |A|\sqrt{1-v_0^2 + A^2}}{1 + A^2}.\]

Substitution of \(v_{1,1}\) into (4.7) implies

\[(4.12) \quad |A|\sqrt{1+A^2-v_0^2} - A^2v_0 = \]

\[A(1+A^2-v_0^2 + A^2v_0^2-2v_0|A|\sqrt{1+A^2-v_0^2})^{\frac{1}{2}}.\]
However,

\[(4.13) \quad |A| \sqrt{1 + A^2 - v_0^2} > A^2 v_0,\]

since (4.13) is valid if \( v_0 \leq 0 \), while, if \( v_0 > 0 \),

\[(4.14) \quad A^2(1 + A^2 - v_0^2) > A^4 v_0^2,\]

since, by (4.8).

\[(4.15) \quad A^2(1 - v_0^2) + A^4(1 - v_0^2) > 0.\]

Thus, the left side of (4.12) is positive. Since squaring both sides of (4.12) yields an identity, it follows that \( v_{1,1} \)
is a root of (4.5) if and only if \( A \), and hence \( F \), is positive.

Substitution of \( v_{1,2} \) into (4.7) implies

\[(4.16) \quad -|A| \sqrt{1 + A^2 - v_0^2} = A^2 v_0 = A(1 + A^2 - v_0^2 + A^2 v_0^2 + 2v_0 |A| \sqrt{1 + A^2 - v_0^2}^\frac{1}{2},\]

from which it follows, as above, that \( v_{1,2} \) is a root of (4.5)if and only if \( A \), and hence \( F \), is negative, from which theexistence and uniqueness follow. However, since \( v_{1,1} \) is aroot if and only if \( F > 0 \) and \( v_{1,2} \) is a root if and only if\( F < 0 \), it follows from (4.6) and (4.11) that both cases areincluded in the single formula

\[(4.17) \quad v_1 = \frac{v_0 + (\Delta t)(1-v_0^2)F \sqrt{1 + (\Delta t)^2(1-v_0^2)^2F^2 - v_0^2}}{1 + (\Delta t)^2(1-v_0^2)^2F^2}.\]

Since (4.17) is also valid for \( F = 0 \), (4.3) follows readily andthe theorem is proved.

In anticipation of the next theorem, we shall state twoauxiliary results as lemmas, the first of which is well known
and the second of which is a direct consequence of the first.

Lemma 1. If $|u| < 1$ and $|v| < 1$, then

$$|u-v| < 1-uv$$

Lemma 2. If $|v_0| < 1$, $|v_1| < 1$, and $|u| < 1$, then

$$(1-v_0v_1)(1-uv_0) + (v_1-v_0)(u-v_0) > 0$$

THEOREM 3. If the observers in the laboratory and rocket frames use identical computers in implementing the numerical method of Section 3, then their numerical results are related by the Lorentz transformation, that is, the solutions of (3.2)-(3.6) satisfy

$$(4.18) \quad x_{k+1}^r = \frac{x_{k+1} - u^r t_{k+1}}{1-u^2} , \quad k = 0,1,2,\ldots$$

and

$$(4.19) \quad v_{k+1}^r = \frac{v_{k+1}}{1-uv_{k+1}} , \quad k = 0,1,2,\ldots$$

Proof. The initial data $x_0$, $x_0^r$, $v_0$, $v_0^r$ are related by the Lorentz transformation by assumption, so that

$$(4.20) \quad x_0^r = \frac{x_0 - ut_0}{(1-u^2)^{1/2}}$$

$$(4.21) \quad v_0^r = \frac{v_0 - u}{1-uv_0}$$

To prove the theorem, we will consider only the case $k = 0$, since the proof for arbitrary $k$ follows in exactly the same way.

In the rocket frame, (3.2) has the form

$$(4.22) \quad x_{k+1}^r = x_k^r + v_k^r(t_{k+1}^r - t_k^r)$$
which, for \( k = 0 \), reduces to

\[(4.23) \quad x_1^t = x_0^t + v_0^t(t_1^t-t_0^t) .\]

From (2.1), (4.1), (4.20) and (4.21), it follows that

\[
x_1^t = \frac{x_0^t-u t_0^t}{(1-u^2)^{\frac{1}{2}}} + \frac{v_0-u}{1-uv_0} \cdot \frac{(t_1^t-t_0^t)-\sqrt{\mu}(x_1^t-x_0^t)}{(1-u^2)^{\frac{1}{2}}}
\end{align*}

\[= \frac{x_1^t-u t_1^t}{(1-u^2)^{\frac{1}{2}}} ,
\]

from which (4.18) is valid.

In the rocket frame, (3.3) has the form

\[(4.24) \quad \frac{v_{k+1}^t-v_k^t}{t_{k+1}^t-t_k^t} \cdot \frac{c_{2m^t}(t_{k+1}^t)}{(c^2-v_{k+1}^t)(c^2-v_k^t)^{\frac{1}{2}}} = F'(x_k^t,x_{k+1}^t,v_k^t,t_k^t) ,
\]

where \( F'(x_k^t,x_{k+1}^t,v_k^t,t_k^t) \) is given by (3.6). By (2.12) and (4.1), with \( k = 0 \), (4.24) becomes

\[(4.25) \quad \frac{v_1^t-v_0^t}{(t_1^t-t_0^t)(1-v_0^t)^2(1-v_1^t)^{\frac{1}{2}}} = F' .
\]

By Theorem 2, the unique solution of (4.25) for \( v_1^t \) is

\[(4.26) \quad v_1^t = \frac{v_0^t+(t_1^t-t_0^t)(1-v_0^t)^2F'\sqrt{1+(t_1^t-t_0^t)^2(1-v_0^t)^2(1-v_1^t)^2(1-v_0^t)^2}}{1+(t_1^t-t_0^t)^2(1-v_0^t)^2(1-v_1^t)^2F'}.\]

Now, by (3.6), \( F' = F \), so that substitution of (2.1), (4.5), (4.20) and (4.21) into (4.26) implies, after considerable algebraic manipulation,
(4.27) \[ v'_1 = \frac{(v_1-u)(v_0^2+1+uv_0^2v_1+uv_1-2v_0v_1-2v_0u)}{(1-u v_1)(v_0^2+1+uv_0^2v_1+uv_1-2v_0v_1-2v_0u)} \]

However, from Lemma 2,

\[ v_0^2+1+uv_0^2v_1+uv_1-2v_0v_1-2v_0u \]

\[ (1-v_0v_1)(1-uv_0) + (v_1-v_0)(u-v_0) > 0. \]

Thus, (4.27) reduces to

(4.28) \[ v'_1 = \frac{v_1-u}{1-uv_1} \]

from which (4.19) and the theorem follow readily.
REFERENCES

