TWO REMARKS ON TOTALLY BALANCED GAMES

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ABSTRACT

Two results on totally balanced TU games are presented. It is first shown that the core of any subgame of a non-negative totally balanced game can be easily obtained from the maximal average value function of the game. The second result is a characterization of convex games as those games all of whose marginal games are totally balanced.

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1. Introduction.

This paper contains some simple observations on totally balanced games. Totally balancedness was defined by Shapley and Shubik (1969) as the property of having all subgames with nonempty core. These authors proved that totally balanced games coincide with market games generated by exchange economies whose traders have continuous concave utility functions. Another characterization of totally balanced games, namely, as flow games, was provided by Kalai and Zemel (1982). A flow game arises from a directed network each of whose arcs has a given capacity and belongs to a unique player; the worth of a coalition is the maximum flow that can be sent from the source to the sink by using only the arcs owned by its members. The totally balanced character of flow games is a consequence of the max-flow-min cut theorem of Ford and Fulkerson (1962), according to which the maximum source to sink flow equals the minimum capacity of a cut (i.e., of a set of arcs such that, when removed from the network, nothing can be sent from the source to the sink). Nonnegative totally balanced games are also known to be equivalent to linear production games in the sense of Owen (1975). Indeed, to any nonnegative game one can associate the linear production game in which the resources are the players, each of which owns only one unit of himself, the goods are the nonempty coalitions, each of which can be sold at a price equal to its worth, and to produce one unit of a given coalition one requires one unit of each of its members. One can easily show that the linear production game so defined is precisely the totally balanced cover of the initial game (i.e., its smallest totally balanced majorant). Note that this linear production representation of a nonnegative totally balanced game needs \( n \) resources (\( n \) being the number of players) and \( 2^n - 1 \) goods. An alternative linear production representation, requiring just one good and at most \( 2^n - 1 \) resources, can be deduced from the observation, due to Kalai and Zemel (1982), that the class of totally balanced games is the span of the additive games by the minimum operation.

Section 2 deals with nonnegative totally balanced games. For these games, a duality theory has been proposed in Martínez-Legaz (1995), relating them to a special class of convex functions. To each nontrivial nonnegative game, one associates its maximal average value (MAV) function which is convex and contains all the information on the game provided that it is totally balanced. Since totally balanced games have all subgames with nonempty core, the natural question arises how to compute these cores from the MAV function of the game. A simple answer to this question is given in Section 2. In Section 3 we consider a very special class of totally balanced games, namely, that of convex games. The main result in that section establishes that convex games are precisely those games all of whose marginal games are totally balanced.

2. Computing the core of a subgame.

An \( n \)-person TU game is a pair \( \Gamma = (N, \nu) \), where \( N \) is a finite set of players and \( \nu: 2^N \to \mathbb{R} \) is a function defined on the power set of \( N \) satisfying the condition \( \nu(\emptyset) = 0 \), called the characteristic function of the game. In this section we will only consider nontrivial nonnegative games, i.e., those whose characterization function satisfies \( \nu(S) \geq 0 \) for all \( S \in 2^N \) and it is not identically zero. As observed in Martínez-Legaz (1995), there is no loss of generality in assuming that a totally balanced game is nonnegative since one can replace the original game by another strategically equivalent
0 – normalized game, which is then totally balanced and nonnegative. For these games, the following duality theory has been developed in Martinez-Legaz (1995). Assuming that \( N = \{1, \ldots, n\} \), one defines the maximal average value (MAV) function of \( \Gamma \) as
\[
\mu: \mathbb{R}^n_+ \setminus \{0\} \rightarrow \mathbb{R}_+ \cup \{+\infty\},
\]
given by
\[
\mu(\omega) = \max_{S \subseteq N} \frac{\nu(S)}{\omega(S)}
\]
for any \( \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n_+ \setminus \{0\} \) (with the conventions \( \frac{\alpha}{0} = +\infty \) for any \( \alpha > 0 \) and \( \frac{0}{0} = 0 \)), where \( \omega(S) = \sum_{i \in S} \omega_i \). This function admits the following economic interpretation:

if the components of \( \omega \) represent the salaries demanded by the players and \( \nu(S) \) is the total amount of output produced to an employer by a set \( S' \) of players when they use his resources, then \( \mu(\omega) \) is the maximum amount of output per unit of money spent that the employer can obtain by hiring a coalition. Function \( \mu \) is convex, continuous, positively homogeneous of degree \(-1\) and finite-valued on \( \mathbb{R}^n_+ \); moreover, by Theorem 2.1 in Martinez-Legaz (1995), if \( \Gamma \) is totally balanced its characteristic function can be recovered from its MAV function by means of the following formula:
\[
\nu(S) = \min_{\omega \in \mathbb{R}^n_+ \setminus \{0\}} \mu(\omega) \omega(S) \quad \text{for any} \quad S \subseteq N
\]
(with the convention that \((+\infty) \cdot 0 = +\infty\)). It turns out that, in this case, \( \mu \) contains all the information on the game. Therefore, it is in principle possible to compute the cores of the subgames of \( \Gamma \) (which are nonempty as \( \Gamma \) is totally balanced) directly from \( \mu \). A way for doing it is suggested by the following theorem.

**Theorem 1.** Let \( \Gamma = (N, \nu) \), with \( N = \{1, \ldots, n\} \), be a nontrivial nonnegative totally balanced TU game and let \( T \subseteq N \) be such that \( \nu(T) > 0 \). For any \( x \in \mathbb{R}^n_+ \setminus \{0\} \), the following statements are equivalent:

1. \( x \) belongs to the core of the subgame \( \Gamma_T = (T, \nu|_T) \).
2. There exists \( \bar{\omega} \in \mathbb{R}^n_+ \setminus \{0\} \) such that \( x = \bar{\omega}_T \equiv (\omega_i)_{i \in T} \) and \( \mu(\bar{\omega}) = 1 \); for every \( \omega \in \mathbb{R}^n_+ \setminus \{0\} \) satisfying these conditions, \( \frac{\partial}{\partial x(T)} \) is an optimal solution of
\[
(P_T) \ \text{minimize} \ \mu(\omega) \\
\text{subject to} \ \omega(T) = 1.
\]
3. \( \bar{\omega} \in \mathbb{R}^n_+ \setminus \{0\} \) such that \( x = \bar{\omega}_T \), \( \mu(\bar{\omega}) = 1 \) and \( \frac{\partial}{\partial x(T)} \) is an optimal solution of \( (P_T) \).

**Proof.** To prove the implication (1) \( \Rightarrow \) (2), let \( x \) be a core element of \( \Gamma_T \) and take any \( \omega = (\omega_i)_{i \in T} \in \mathbb{R}^n_+ \setminus \{0\} \) such that \( \omega_T = x \) and \( \nu(S) \leq \omega(S) \) for all \( S \subseteq T \) (this condition can be achieved by giving sufficiently high values to \( \omega_i \) for \( i \notin T \)). Since we also have \( \nu(S) \leq \omega(S) \) for all \( S \subseteq T \) (as \( \omega_T = x \) is in the core of \( \Gamma_T \)), it follows that \( \mu(\omega) \leq 1 \). But we actually have \( \mu(\omega) = 1 \), as a consequence of

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\[
\mu(\omega) \geq \frac{v(T)}{\omega(T)} = \frac{v(T)}{x(T)} = 1.
\]

Let \( \omega \in \mathbb{R}_+^n \setminus \{0\} \) be any point satisfying \( x = \omega_T \) and \( \mu(\omega) = 1 \). By \( \omega_T = x \), the point \( \frac{\omega}{x(T)} \) is a feasible solution to problem \( (P_T) \). To show that it is optimal, it suffices to observe that, for each feasible \( \omega \in \mathbb{R}_+^n \setminus \{0\} \), one has

\[
\mu(\omega) \geq \frac{v(T)}{\omega(T)} = v(T) = x(T) = \mu(\omega) x(T) = \mu\left( \frac{\omega}{x(T)} \right).
\]

Implication \( (2) \Rightarrow (3) \) is obvious.

Let us now prove \( (3) \Rightarrow (1) \). Given \( \omega \) as in \( (3) \) and any \( S \subset T \), we have

\[
v(S) \leq \mu(\omega) \omega(S) = \omega(S) = x(S).
\]

Take \( \omega \in \mathbb{R}_+^n \setminus \{0\} \) such that \( \mu(\omega) \omega(T) = v(T) \) (such a point indeed exists, by the above mentioned formula to recover \( v \) from \( \mu \)). From the optimality of \( \frac{\omega}{x(T)} \), it follows that

\[
x(T) = \mu(\omega) x(T) = \mu\left( \frac{\omega}{x(T)} \right) \leq \mu\left( \frac{\omega}{\omega(T)} \right) = \mu(\omega) \omega(T) = v(T),
\]

whence \( x(T) \leq v(T) \). Since the opposite inequality also holds, we conclude that \( x \) belongs to the core of \( \Gamma_T \).

As a particular case of Theorem 1, the next result characterizes the core of the game itself.

**Corollary 1.** Let \( \Gamma \) be as in Theorem 1. For any \( x \in \mathbb{R}_+^n \setminus \{0\} \), the following statements are equivalent:

1. \( x \) belongs to the core of \( \Gamma \).
2. \( \mu(x) = 1 \) and \( \frac{\omega}{x(N)} \) is an optimal solution of \( (P_N) \).

Theorem 1 shows that each point belonging to the core of a subgame \( \Gamma_T \) induces an optimal solution of the associated optimization problem \( (P_T) \). In the opposite direction, we have

**Corollary 2.** Let \( \Gamma \) and \( T \) be as in Theorem 1. For any \( \omega \in \mathbb{R}_+^n \setminus \{0\} \), the following statements are equivalent:

1. \( \omega \) is an optimal solution of \( (P_T) \).
2. \( \omega(T) = 1 \) and \( \mu(\omega) \omega_T \) belongs to the core of \( \Gamma_T \).

**Proof.** Let \( x = \mu(\omega) \omega_T \). If \( (1) \) holds, then \( \omega = \mu(\omega) \omega \) satisfies \( (3) \) of Theorem 1, whence \( (2) \) follows. Conversely, if \( (2) \) holds then \( \omega = \mu(\omega) \omega \) satisfies \( x = \omega_T \) and \( \mu(\omega) = 1 \) whence, by \( (1) \Rightarrow (2) \) in Theorem 1, we obtain \( (1) \). 

In view of Theorem 1 and Corollary 2, to compute the core of a (nontrivial) subgame \( \Gamma_T \) one can apply the following method: find all optimal solutions \( \omega \) to the problem \( (P_T) \); the elements in the core of \( \Gamma_T \) are just those of the form \( \mu(\omega) \omega_T \). Indeed, by Corollary 2, each \( \mu(\omega) \omega_T \) belongs to the core of \( \Gamma_T \). Conversely, each
element $x$ in the core of $\Gamma_T$ can be obtained in this way. To see this, take $\bar{\omega}$ as in (3) of Theorem 1. Then $\frac{\bar{\omega}}{x'(T)}$ is an optimal solution of $(P_T)$ and, as $\mu(\omega) = 1$, one has

$$x = \bar{\omega} T = \mu(\bar{\omega}) \bar{\omega} T = \mu \left( \frac{\bar{\omega}}{x(T)} \right) \frac{\bar{\omega} T}{x(T)}.$$  

One can illustrate this method by computing the core of the unanimity game $\Gamma^P = (N, \mu^P)$ associated to a nonempty coalition $P \subset N$, whose characteristic function is given by

$$\mu^P(S) = \begin{cases} 1 & \text{if } S \supset P \\ 0 & \text{otherwise.} \end{cases}$$

As shown in Martínez-Legaz (1995), the MAV function $\mu^P$ of $\Gamma^P$ is just $\mu^P(\omega) = \frac{1}{\omega(\bar{P})}$. Therefore, the minimizers of $\mu(\omega)$ under the constraint $\omega(N) = 1$ are those $\bar{\omega} \in \mathbb{R}_+^N \setminus \{0\}$ such that $\bar{\omega}(P) = 1$ and $\bar{\omega}_{N \setminus P} = 0$. Since these points satisfy $\mu^P(\bar{\omega}) = 1$, it follows that they are the core elements of $\Gamma^P$.

To summarize, our results show that the computation of the core of any subgame of the nonnegative totally balanced game reduces to the minimization of a convex function (of nonnegative variables) under one linear constraint.

3. Characterizing convex games by means of their marginals.

A very important class of totally balanced games is that of convex games. One says that $\Gamma = (N, \mu)$ is convex if for all coalitions $S, T$, one has

$$\mu(S) + \mu(T) \leq \mu(S \cup T) + \mu(S \cap T).$$

The term "convex" is due to the property of "increasing returns" enjoyed by these games. Indeed, it is well-known that $\Gamma$ is convex if and only if it satisfies

$$\mu(S \cup \{i\}) - \mu(S) \leq \mu(T \cup \{i\}) - \mu(T)$$

for each $i \in N$ and every coalitions $S, T$ such that $S \subset T \subset N \setminus \{i\}$. An example of convex games is provided by unanimity games (see Section 2).

Since totally balancedness is not a sufficient condition for a game to be convex, a natural question to ask is which additional conditions imposed on a totally balanced game ensure its convexity. The answer is given by the following theorem, which says that the required conditions are the totally balancedness of the marginal games as well. By the marginal game relative to coalition $T \subset N$, we mean the game $\Gamma'_T = (N \setminus T, \mu'_T)$ whose characteristic function is defined by $\mu'_T(S) = \mu(\bar{T} \cup S) - \mu(T)$.

**Theorem 2.** Let $\Gamma = (N, \mu)$ be a TU game. The following statements are equivalent:

1. $\Gamma$ is convex.
2. $\Gamma'_T$ is convex for every $T \subset N$.
3. $\Gamma'_T$ is totally balanced for every $T \subset N$.
4. $\Gamma'_T$ is superadditive for every $T \subset N$. 


Proof. To prove \((1) \Rightarrow (2)\), let \(T \subset N\) and \(S_1, S_2 \subset N \setminus T\). Since \(\Gamma\) is convex, we have

\[
\nu_T(S_1) + \nu_T(S_2) = \nu(T \cup S_1) + \nu(T \cup S_2) - 2\nu(T) \leq \nu(T \cup S_1 \cup S_2) + \nu(T \cup (S_1 \cap S_2)) - 2\nu(T) = \nu_T(S_1 \cup S_2) + \nu_T(S_1 \cap S_2),
\]

which shows that \(\nu_T\) is convex. Implications \((2) \Rightarrow (3) \Rightarrow (4)\) follow from the well-known facts that all convex games are totally balanced and that the latter are superadditive. So, it only remains to prove \((4) \Rightarrow (1)\); to this aim, it suffices to observe that, for each \(S_1, S_2 \subset N\), one has

\[
\nu(S_1) + \nu(S_2) = \nu_{S_1 \cap S_2}(S_1 \setminus S_2) + \nu_{S_1 \cap S_2}(S_1 \setminus S_2) + 2\nu(S_1 \cap S_2) \leq \nu_{S_1 \cap S_2}((S_1 \cup S_2) \setminus (S_1 \cap S_2)) + 2\nu(S_1 \cap S_2) = \nu(S_1 \cup S_2) + \nu(S_1 \cap S_2),
\]

where the inequality follows from the superadditivity of \(\nu_{S_1 \cap S_2}\). \(\square\)

The equivalence between statements \((1)\) and \((4)\) of the preceding theorem was implicitly used in Martínez-Legaz (1996) to prove Proposition 20 on a characterization of convex games in terms of indirect functions. Based on the equivalence \((1) \Leftrightarrow (2)\), we will next present an alternative characterization of convex games, similar to that of totally balanced games in terms of balanced sets of coalitions [cf., e.g., Shapley and Shubik (1969)]. To this aim, we need the following notion.

Definition 1. A collection \(B\) of subsets of \(P \subset N\) is marginally \(P\)-balanced if there exist positive weights \(\gamma_S\), \(S \in B' = B \setminus \left\{ \bigcap_{S \in B} S \right\}\), such that for each \(i \in P \setminus \left\{ \bigcap_{S \in B} S \right\}\) we have

\[
\sum_{S \in B' \ni i} \gamma_S = 1.
\]

Corollary 3. A TU game \(\Gamma = (N, \nu)\) is convex if and only if

\[
\nu(P) \geq \sum_{S \in B'} \gamma_S \nu(S) - \left( \sum_{S \in B'} \gamma_S - 1 \right) \nu\left( \bigcap_{S \in B} S \right)
\]

for every \(P \subset N\) and every marginally \(P\)-balanced collection \(B\) with weights \(\{\gamma_S\}_{S \in B} \).
Proof. The "only if" part follows from the totally balancedness of $\nu_T$, with $T = \bigcap S$, and the fact that the marginal $P$-balancedness of $B$ is equivalent to the balancedness of $\{S \setminus T\} \in B$ as a collection of subsets of $P$, associating to each nonempty $S \setminus T$ the weight $\gamma_S$. To prove the converse, given $S, T \subset N$ with $S \not\subseteq T$ and $T \not\subseteq S$, let $P = S \cup T$. Then $\{S, T\}$ is the marginally $P$-balanced with $\gamma_S = \gamma_T = 1$. Thus, the assumed inequality reduces to 

$$
\nu(S \cup T) \geq \nu(S) + \nu(T) - \nu(S \cap T). \quad \Box
$$

The interest of Corollary 3 lies in that it allows for an easy comparison between convex games and totally balanced games. Notice that the condition stated in Corollary 3 reduces to that of totally balancedness when restricted in collections $B$ having an empty intersection. Moreover, it admits the following interpretation. If a fraction $\gamma_S$ of coalition $S$ forms, thus yielding an output $\gamma_S \nu(S)$, the total output that $P$ can obtain is at least the sum of all these outputs minus that paid, by their extra effort, to the subcoalition consisting of those players who contributed $\sum_{S \in B'} \gamma_S$ (greater than 1) units of themselves.

This payment is the output they would be able to obtain by themselves with this extra effort. Note that, as $B$ is marginally $P$-balanced, the other players contribute exactly one unit of themselves.

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References


