THE POLYNOMIAL CHAOS METHOD WITH APPLICATIONS TO RANDOM DIFFERENTIAL EQUATIONS

by

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To my daughter and my family, who were always there giving me support and motivation.

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ABSTRACT

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The role of randomness in mathematical models is of paramount importance, with emphasis placed upon the accuracy and reliability of predictions a rational approach is the use of differential equations with random parameters to describe natural phenomena. Well known methods such as Monte Carlo methods and the method of moments have been implemented to approximate the solutions to random differential equations in the last few decades. In this work, analytic solutions to a particular Riccati type differential equation and discrete delay differential equation with random coefficients are derived, also, due to its spectral rate of convergence and simplicity, the polynomial chaos expansion method is considered to approximate the moments of the solutions. The performance of the method is exhibited and potential future applications are discussed.

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CHAPTER 1

INTRODUCTION

Numerous natural phenomena have been traditionally formulated through mathematical models based on differential equations, where the underlying parameters are represented by means of numerical values or deterministic functions. The combination of many factors such as complexity, inaccuracies and uncertainty due to natural phenomena and human behavior require the consideration of randomness in mathematical models. The value of a parameter is experimentally determined, and it is usually taken to be the mean of a set of experimental observations. In reality, this set of observations represents a distribution for the parameter values, as a consequence it is advisable to use differential equations with random variables as coefficients in the mathematical modeling of natural phenomena.

Differential equations where some or all of the coefficients are considered random variables, or that incorporate stochastic effects (usually in the form of white noise) have been increasingly used in the last few decades to deal with errors and uncertainty. They represent a growing field of great scientific interest [27],[25].

A variety of methods have been applied to approximate the moments of the solutions to random differential equations. Monte Carlo methods [23] have been used to perform simulations when random effects were involved. They are simple to implement and understand but require many realizations due to their slow convergence rate and hence tend to be computationally expensive. Other methods that have been developed and used are, for example, moment methods [15],[27] and polynomial chaos methods [38],[29]. Moment method approximations use Taylor series expan-

sions about the mean value of the input parameters. The first-order moment is the deterministic value of the output parameter obtained at the mean of the input, while evaluation of the higher-order moments requires computation of sensitivities. The drawback of this approach is that it is intrinsically limited to small perturbations; it also becomes complicated beyond second-order expansions [35],[28]. In the polynomial chaos approach a high-order representation is far easier to construct and the equations are basically the same at any order, the difference lies only in the number of terms to be considered. High-order moments are easily accessible, and the spectral convergence of the stochastic approximation guarantees that high accuracy can be obtained even with a small number of terms; see [39],[35] for computational results. An alternative approach is to add white noise terms and thus obtain a system of stochastic differential equations; see, for example, [22],[10] for applications to epidemic models.

In this thesis we use the polynomial chaos approach to study the time evolution of Riccati type differential equations and discrete-delay differential equations with random parameters. Deterministic equations for the time evolution of the polynomial chaos expansion coefficients are obtained and solved numerically. From these coefficients, means, variances and even higher-order statistics can be obtained when needed. In order to test the accuracy of our method, an analytic solution to the problems considered is required. There are few random differential equations for which an analytic solution is known [28],[39]. Mean square calculus presents major differences when compared to deterministic calculus [33], therefore finding analytic solutions to random differential equations is not an easy problem. We derive rigorous mean square solutions to a particular Riccati type differential equation and discrete delay differential equation with random coefficients (chapters 2 and 3 respectively).

The performance of the method is exhibited and potential future applications are discussed.

In order to make this thesis self-contained, in what remains of this chapter we review some important concepts, definitions and results related to the random calculus that will play an important role in the understanding and results discussed in later chapters.

1.1 Mean square calculus and random differential equations

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The $p-$th moment $E[X^n]$ of a real random variable X defined on $(\Omega, \mathcal{F}, P)$ with probability density function f is given by

$$E[X^n] = \int x^n f(x) \mathrm{d}x, \quad n = 1, 2, \ldots$$

whenever $\int |x|^n f(x) \mathrm{d}$ is finite. The first moment $E[X]$ is called the expected value, and it is often referred to as the mean or mathematical expectation. $E[X]$ can be interpreted as the statistical average of the random variable X. The quantity $V[X] = E[(X - E[X])^2]$ is called the variance of the random variable X, and measures its variability around the expected value.

Let $p \geq 1$ be a real number. A real random variable X defined on $(\Omega, \mathcal{F}, P)$ is called of order p (p-r.v.), if

$$\mathrm{E}[|X|^p] < \infty \, .$$

The space L_p of all the p-r.v's, endowed with the norm

$$\|X\|_p = (\mathrm{E}[|X|^p])^{1/p} \, ,$$

is a Banach space,

$$\lim_{n \to +\infty} \|X_n - X\|_p = 0.$$

This convergence is represented by $X_n \xrightarrow[n\to+\infty]{\text{m.p.}} X$. For $p=2$, the space $(L_2, \|X\|_2)$ is not a Banach algebra, i.e., the inequality $\|XY\|_2 \leq \|X\|_2 \|Y\|_2$ is not true in general. The space L_2 is a Hilbert space and satisfies Schwarz inequality

$$\langle X, Y\rangle = \mathrm{E}[|XY|] \leq \left(\mathrm{E}\left[X^2\right]\right)^{1/2} \left(\mathrm{E}\left[Y^2\right]\right)^{1/2} = \|X\|_2 \|Y\|_2 \,. \tag{1.1}$$

If $q > p \geq 1$, and $\{X_n : n \geq 0\}$ is a sequence in L_q, that is q-th mean convergent to $X \in L_q$, then $\{X_n : n \geq 0\}$ lies in L_p and is p-th mean convergent to $X \in L_p$,

Let T be a closed interval of the real line, and let $\{X(t) : t \in T\}$ be a stochastic process. If $\mathrm{E}[|X(t)|^p] < +\infty$ for all $t \in T$, then it is called a stochastic process of order p (p-s.p.).

Definition 1.1.1. *([37, p.55]) We say that a stochastic process $\{X(t) : t \in T\}$ defined on an interval T is almost surely (a.s.) sample path continuous or that $\{X(t) : t \in T\}$ has continuous paths with probability one (w.p. 1) if*

$$\mathrm{P}\left[\bigcup_{t\in T}\left\{\omega \in \Omega : \lim_{h\to 0} |X(t+h)(\omega) - X(t)(\omega)| \neq 0\right\}\right] = 0\,.$$

A very useful result to see that a stochastic process is a.s. continuous is the Kolmogorov's criterion:

Theorem 1.1.2. *([25, p. 12]) Assume that the s.p. $\{X(t) : t \in [0, \mathcal{T}]\}$ satisfies the following condition. For all $\mathcal{T} > 0$, there exist positive constants α, β, D such that*

$$\mathrm{E}\left[|X(t) - X(s)|^\alpha\right] \leq D|t-s|^{1+\beta} \quad 0 \leq s, t \leq \mathcal{T}.$$

Then the s.p. $\{X(t) : t \in [0, \mathcal{T}]\}$ is a.s. sample path continuous.

Example 1.1.3. *Let $X(t) = 1 + aX_0 t$ be a s.p. defined on $[0, \mathcal{T}]$, $\mathcal{T} \geq 0$, and a, X_0 independent 2-r.v.'s. Then, applying the Kolmogorov's criterion, one gets the a.s. continuity of the s.p. $X(t)$ with $D = \mathrm{E}\left[a^2\right]\mathrm{E}\left[X_0^2\right]$, $\alpha = 2$, $\beta = 1$:*

$$\mathrm{E}\left[|X(t) - X(s)|^2\right] = \mathrm{E}\left[|aX_0(t-s)|^2\right] = \mathrm{E}\left[a^2\right]\mathrm{E}\left[X_0^2\right]|t-s|^2,$$

since both expectations factors are finite, because a, X_0 are 2-r.v.'s.

Proposition 1.1.4. *([21, p. 166])If* $\sup \mathrm{E}|X_n|^r < \infty$, *then* $X_n \xrightarrow[h\to 0]{P} X$ *(P stands for convergence in probability) implies* $X_n \xrightarrow[n\to+\infty]{m.r'.} X$ *for* $r' < r$.

We say that $\{X(t) : t \in T\}$ is p-th mean continuous at $t \in T$, if

$$\|X(t+h) - X(t)\|_p \to 0 \quad \text{as} \quad n \to \infty\,. \tag{1.2}$$

Furthermore, if there exists a stochastic process $\frac{dX(t)}{dt}$ of order p, such that

$$\left\| \frac{X(t+h) - X(t)}{h} - \frac{dX(t)}{dt} \right\|_p \to 0 \quad \text{as} \quad h \to 0\,, \tag{1.3}$$

then we say that $\{X(t) : t \in T\}$ is p-th mean differentiable at $t \in T$. In the particular case when $p = 2$, $p-$th continuity is usually referred to as mean square (m.s.) continuity and $p-$th differentiability as mean square differentiability.

The mean square derivative shares some of the properties with the deterministic derivative such as linearity [27, p. 95]. If $\{X(t) : t \in T\}$ and $\{Y(t) : t \in T\}$ are m.s. differentiable at $t \in T$, the m.s. derivative of $aX(t) + bY(t)$ exists at t and

$$\frac{d}{dt}\left[X(t) + Y(t)\right] = \frac{dX(t)}{dt} + \frac{dY(t)}{dt}$$

where a and b are constants. Moreover, if an ordinary function $f(t)$ is differentiable at $t \in T$ and $\{X(t) : t \in T\}$ is m.s. differentiable at $t \in T$, the the product $f(t)X(t)$ is m.s. differentiable at t and

$$\frac{d}{dt}\left[f(t)X(t)\right] = \frac{df(t)}{dt}X(t) + f(t)\frac{dX(t)}{dt}. \tag{1.4}$$

The last property can be extended to the product of two stochastic processes under suitable conditions. The next result gives sufficient conditions for obtaining the 2-th mean derivative of a s.p. which is product of two 4-s.p.'s.

Lemma 1.1.5. *([33, p. 120]) Let* $\{W(t) : t \in T\}$ *and* $\{Z(t) : t \in T\}$ *be 4-s.p.'s having 4-th mean derivatives* $\frac{dW(t)}{dt}$ *and* $\frac{dZ(t)}{dt}$*, respectively. Then* $W(t)Z(t)$ *is 2-th mean differentiable at* $t \in T$ *and*

$$\frac{d}{dt}\left(W(t)Z(t)\right) = \frac{dW(t)}{dt}Z(t) + W(t)\frac{dZ(t)}{dt}\,. \tag{1.5}$$

We now state a chain rule for a C^1 function of a p-th mean differentiable process, which will be very useful in solving the Riccati random differential equation.

Theorem 1.1.6 (Chain Rule). *Let* f *be a real function with continuous derivative* f' *and* $\{X(t) : t \in T = [a, b]\}$ *be a p-s.p., with* $p \geq 2$*,* p *even, satisfying*

i. $X(t)$ *is p-th mean differentiable.*

ii. $X(t)$ *is path continuous (w.p. 1).*

iii. There is a $r > p$ *and a* $\delta > 0$ *such that* $\sup_{s\in[-\delta,\delta]} \mathrm{E}\left[\left|f'(x)|_{x=X(t+s)}\right|^r\right] < +\infty$.

Then, the $(p/2)$*-s.p.* $f(X(t))$ *is* $(p/2)$*-th mean differentiable and the* $(p/2)$*-th mean derivative is given by*

$$\frac{df(X(t))}{dt} = f'(x)\Big|_{x=X(t)}\frac{dX(t)}{dt}\,.$$

Proof. In [33, p.122], a proof is shown for the case of $p = 4$. This proof can be easily adapted to the case of $p \geq 2$, by using the triangle inequality of the norm $\|\cdot\|_p$ and the Proposition 1.1.4.

Finally, let us introduce the notion of differential equation in the mean square sense [27].

$$\frac{dX(t)}{dt} = f(X(t), t), \quad t \in T = [t_0, t_e]; \quad X(0) = X_0\,. \tag{1.6}$$

Definition 1.1.7. *Consider the equation (1.6) where* X_0 *is a* 2*-r.v. and* $f : L_2 \times T \to L_2$ *is a continuous function. The s.p.* $X(t) : T \to L_2$ *is a mean square solution of equation (1.6) on* T *if*

i. $X(t)$ is m.s. continuous on T;

ii. $X(t_0) = X_0$;

iii. $f(X(t), t)$ is the m.s. derivative of $X(t)$ on T.

There are a variety of existence and uniqueness theorems regarding equation (1.6). Typical assumptions are that the function f satisfies a mean square Lipschitz condition to guarantee that the initial value problem (1.6) possess a unique solution. This type theorem has a rather limited applicability in the mean square theory of random differential equations, the main drawback is the difficulty to show that a function of a 2-s.p. is a 2-s.p. itself, moreover, the mean square Lipschitz condition is too restrictive. As discussed in [27], when the simple first order random differential equation is considered:

$$\frac{dX(t)}{dt} = AX(t), \quad t \geq 0; \quad X(0) = 1\,,$$

where A is a 2-r.v. Strand [31],[30] has shown that the m.s. Lipschitz condition is satisfied if, and only if, A is bounded almost surely, therefore the type of theorem above mentioned is not applicable to the simple case when A has a Normal or Exponential distribution. Considering these limitations, different paths should be taken when regarding the existence of solutions to equation (1.6). This discussion will be retaken in next chapter.

We are interested on approximating the moments of the solution to the problem (1.6). In the next section we describe the polynomial chaos expansions method as it is presented in [13], and [9].

1.2 Generalized Polynomial Chaos Expansions

The term Polynomial Chaos Expansions was first introduced by Wiener in [36] to approximate a $2-$r.v. by a series of polynomials in a Normal random variable. Let (Ω, F, P) be a probability space and $\xi = (\xi)_{i=1}^{\infty}$ be an infinite but countable set of independent normal standard random variables. Let

- $\hat{\Gamma}_p$ be the space of all polynomials of degree less or equal to p in the variables ξ,
- Γ_p the set of polynomials of $\hat{\Gamma}_p$ which are orthogonal to $\hat{\Gamma}_{p-1}$,
- $\tilde{\Gamma}_p$ the space generated by Γ_p:

$$\tilde{\Gamma}_p = \tilde{\Gamma}_{p-1} \bigoplus \tilde{\Gamma}_p, \Theta = \bigoplus_{i=0}^{\infty} \tilde{\Gamma}_i.$$

The subspace $\tilde{\Gamma}_p$ of Θ is called the p-th homogeneous chaos and Γ_p is called polynomial chaos of order p. In fact, the polynomial chaos of order p is the set of all possible polynomials of degree p in all possible combinations of the random variables in ξ. The polynomial chaos expansion of a 2-r.v. $\chi(\omega)$ is

$$\chi(\omega) = a_0\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}\Gamma_1(\xi_{i_1}(\omega)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2(\xi_{i_1}(\omega), \xi_{i_2}(\omega)) + \dots$$

Cameron and Martin have shown in [4] that the above series is convergent in the mean square sense. In order to ease the manipulation of the polynomial chaos expansions, the terms in the expansion above can be rearranged as

$$\chi(\omega) = \sum_{k=0}^{\infty} \chi_k \Phi_k(\xi(\omega)). \tag{1.7}$$

where the Φ_i are properly chosen polynomial basis functions of the random variable vector ξ, orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$

$$\langle \Phi_i, \Phi_j \rangle = \int \Phi_i(\xi)\Phi_j(\xi)p(\xi)\mathrm{d}\xi \tag{1.8}$$

where p is the joint probability density function of the random variables in ξ. The number of variables in ξ represents the dimension of the chaos d, since most of the applications involve a finite number of random parameters, the number d is typically a finite number. For computational purposes the polynomial chaos expansions are to be truncated up to a certain order p, so the expansion (1.7) becomes

$$\chi(\omega) \approx \sum_{k=0}^{p} \chi_k \Phi_k(\xi(\omega)). \tag{1.9}$$

where the number of terms in the expansion depends on d and p. and it is given by

$$P + 1 = \frac{(p+d)!}{p!d!}. \tag{1.10}$$

Wiener's polynomial chaos expansions are developed around normal standard random variables ξ which determines the inner product $\langle \cdot, \cdot \rangle$ for the underlying orthogonal polynomial basis $\{\Phi_i\}$. In a more recent work by Xiu and Karniadakis [38], the Askey scheme is used to generalize Wiener's polynomial chaos expansion to a common non-Gaussian measure ξ. This generalization, referred to as generalized polynomial chaos [38], can be useful to improve the convergence for non-Gaussian random variables. In this new scheme, polynomials are orthogonal in the Hilbert space corresponding to the support and density function of one of the common non-Gaussian random variables in the Askey scheme [38].

Multi-dimensional orthogonal polynomials are required to construct polynomial chaos expansions, which can be obtained as tensor products of the corresponding one-dimensional orthogonal polynomials. Let $\{\phi_i\}_{i=0}^{\infty}$ denote the one-dimensional orthogonal polynomials from the Askey scheme, assuming the random variables to be independent, the multi-dimensional generalized polynomial chaos basis $\{\Phi_i\}$ is given by

$$\Phi_i(\xi) = \prod_{k=1}^{d} \phi_{\alpha_k^i}(\xi_k), \quad \left|\alpha^i\right| = \sum_{k=1}^{d} \alpha_k^i \leq p, \quad i = 0, 1, \ldots, P. \tag{1.11}$$

In order to find approximations to the solution of (1.6),

$$\frac{dX(t)}{dt} = f(X(t), t), \quad t \in T = [t_0, t_e]; \quad X(0) = X_0\,,$$

the solution $X(t, \omega)$ is expanded by the general polynomial chaos expansion as

$$X(t, \omega) \approx \sum_{k=1}^{P} x_k(t)\Phi(\xi(\omega)). \tag{1.12}$$

By substituting the expansion (1.12) into the governing equation (1.6), the following equation is obtained

$$\frac{d}{dt}\sum_{k=1}^{P} x_k(t)\Phi(\xi(\omega)) = f(X(t),t). \tag{1.13}$$

A Galerkin projection of the above equation into each polynomial basis function Φ_i is conducted in order to ensure the error is orthogonal to the functional space spanned by the finite-dimensional basis $\{\Phi_i\}$, thus, the randomness is transferred into the basis polynomials.

$$\left\langle \frac{d}{dt}\sum_{i=1}^{P} x_i(t)\Phi_i(\xi(\omega)), \Phi_k \right\rangle = \langle f(X(t),t), \Phi_k\rangle, \quad k = 0,1,\ldots,P. \tag{1.14}$$

By using the orthogonality of the polynomial basis, a set of $P+1$ coupled deterministic ordinary differential equations for the coefficients $x_i(t)$, $i = 0,1,\ldots,P$ is obtained

$$x_k(t) = \frac{\langle f(X(t),t), \Phi_k\rangle}{\langle \Phi_k, \Phi_k\rangle}, \quad k = 0,1,\ldots,P. \tag{1.15}$$

Finally the equation (1.15) can be integrated by a suitable method (typically Runge-Kutta method).

1.3 Monte Carlo Method

The Monte Carlo method, described in [11], is a technique for approximating expected values of quantities of interest depending on the solution of differential equations with random inputs. These expected values include not only the mean but also higher moments and standard deviation. The algorithm approximates the desired expectation by a sample average of independent identically distributed (i.i.d.) realizations. When solving differential equations with random inputs, this method implies the solution of one deterministic differential equation for each realization of the input parameters. Its numerical error is approximately $O(1/M)$, where M is the

number of realizations. As a main inconvenient of this approach we indicate that it is expensive and it waste the potential possibilities that the structure of the ordinary differential equation could have: linearity, autonomy, etc.

CHAPTER 2

RANDOM DIFFERENTIAL EQUATIONS

A 2-s.p. can be expanded in terms of orthogonal polynomials of random variables [4], i.e. polynomial chaos expansions. Xiu and Karniadakis [38] have applied these expansions to solve differential equations with random parameters, resulting in the so called Wiener-Askey polynomial chaos expansions method to solve random differential equations which was described in more detail on the introduction. In this chapter a rigorous solution to a particular family of random Riccati equations is derived and then compared against approximations obtained via polynomial chaos expansions.

2.1 Riccati differential equation with random parameters

The Riccati differential equation is of special importance since it is one of the few nonlinear ordinary differential equations that can be transformed into a linear one [32]. It also appears in applications of mathematical physics, control theory and mathematical biology, see for example [1],[24]. Recently, intensive studies of differential equations with random coefficients have been undertaken and have exerted a profound influence on the analysis of a wide variety of problems in the physical, biological, social, engineering and technological sciences [27],[17],[8],[2],[29],[6], and [38]. But there are still many open questions, especially, which is the "best" method to use in approximating the solution. The answer will depend on the equation and on the number of random parameters and their distribution. In [34] the Riccati equation

with two random parameters was analyzed and an exact mean square series solution was found.

The Riccati equation is a differential equation with a second degree polynomial nonlinearity, given by:

$$\dot{X}(t) + aX^2(t) + bX(t) + c = 0, \quad X(0) = X_0, \qquad t \in [0, \mathcal{T}], \qquad \mathcal{T} > 0, \qquad (2.1)$$

where a, b, c, X_0 are random variables to be specified later. Following ideas of [33], an exact mean square solution in explicit form of (2.1) is provided by using the p-th mean calculus. Different theoretical approaches of models of type (2.1) have been investigated. In [17], a simple epidemic is modeled by using a special case of model (2.1) when only the initial condition is considered as random variable with beta distribution. In [8], a random logistic equation is solved by using the so-called sample approach. The Itô and Stratonovich calculus is used to study a population growth under random environments by using the standard white noise in [2].

2.2 Analytic solution

We begin with a particular case of (2.1) in which $b(\omega) = c(\omega) = 0$ for all $\omega \in \Omega$, and a and X_0 are independent 8-r.v's such that:

C1.1 $1 + a(\omega)X_0(\omega)t \neq 0$, for all $\omega \in \Omega$ and $t \in [0, \mathcal{T}]$, $\mathcal{T} > 0$.

C1.2 There is a $r > 8$ and a $\delta > 0$ such that

$$\sup_{s \in [-\delta, \delta]} \mathrm{E}\left[\left| -\frac{1}{(1 + aX_0(t+s))^2} \right|^r\right] < +\infty.$$

The deterministic theory suggests as a m.s. candidate solution [28, p. 118] the s.p.

$$X(t) = \frac{X_0}{1 + aX_0 t}, \qquad t \in [0, \mathcal{T}], \qquad \mathcal{T} > 0. \qquad (2.2)$$

X_0 is a 8-r.v. and therefore a 4-r.v. and its 4-th mean derivative is zero. Therefore, in order to use Lemma 1.1.5, we should prove that the s.p. $\frac{1}{1+aX_0 t}$ is 4-th mean

differentiable. Let us use the chain rule given in Theorem 1.1.6 with $p = 8$. Consider the function $f(x) = 1/x$ and the s.p. $\{Z(t) : t \in [0, \mathcal{T}]\}$ defined by $Z(t) = 1 + aX_0 t$. It is clear that f is a C^1 function for $x \neq 0$ and also that $Z(t)$ is 8-th mean order, because a and X_0 are independent 8-r.v.'s. Now, $\dot{Z}(t) = aX_0$ is the 8-th mean derivative of $Z(t)$, because of $\lim_{\tau \to 0} \| \frac{Z(t+\tau) - Z(t)}{\tau} - aX_0 \|_8^8 = \lim_{\tau \to 0} \mathrm{E}\left[\left(\frac{aX_0 \tau}{\tau} - aX_0 \right)^8 \right] = 0$. By example 1.1.3, $Z(t)$ is a.s. path continuous. Thus, $f(Z(t)) = \frac{1}{1+aX_0 t} \in L_4$ due to C1.2, and by Theorem 1.1.6, its 4-th mean derivative is given by

$$\frac{df(X(t))}{dt} = -\frac{1}{(1 + aX_0 t)^2} \, aX_0.$$

So, by Lemma 3.14 of [33], it follows that the 2-th mean derivative of $X(t)$, is:

$$\dot{X}(t) = X_0 \left(-\frac{1}{(1 + aX_0 t)^2} aX_0 \right) = -a \left(\frac{X_0}{1 + aX_0 t} \right)^2 = -aX(t)^2.$$

Summarizing the following result has been established:

Theorem 2.2.1. *Consider the initial value problem given by (2.1), in which $b(\omega) = c(\omega) = 0$ for all $\omega \in \Omega$, and a and X_0 are independent 8-r.v.'s satisfying conditions C1.1-C1.2. Then the s.p. given by (2.2) is a m.s. solution of (2.1).*

Remark 2.2.2. *Condition C1.2 in Theorem 2.2.1 is trivially true for r.v.'s a and X_0 whose signs are the same, because we choose $\delta = |t|/2$, so that $t + s \in [0, \mathcal{T}]$, $r = 10$, and so $1 > \left| -\frac{1}{(1+aX_0(t+s))^2} \right|^r$.*

Now, let us consider the equation (2.1) in which a, b, c and X_0 are 8-r.v. such that

$C2.1$ $a(\omega) \neq 0$ for all ω and there exists a real number $a_0 \neq 0$ with the same sign as a such that $a_0 \leq a$

$C2.2$ $4a(\omega)c(\omega) > b^2(\omega) > 0$ for all ω

$C2.3$ There is a $r > 8$ and a $\delta > 0$ such that

$$\sup_{s \in [-\delta, \delta]} \mathrm{E}\left[\left| \sec^2 \left(\frac{1}{2} \sqrt{-b^2 + 4ac} \, (t + s) - \arctan \left(\frac{b + 2aX_0}{\sqrt{-b^2 + 4ac}} \right) \right) \right|^r \right] < \infty.$$

In this case, a m.s. candidate solution is:

$$X(t) = -\left(\frac{b + \left(\sqrt{-b^2+4ac}\right) Z(t)}{2a}\right) \tag{2.3}$$

where $Z(t) = \tan\left[\frac{1}{2}\sqrt{-b^2+4ac}\, t - \arctan\left(\frac{b+2aX_0}{\sqrt{-b^2+4ac}}\right)\right]$. To obtain the 2-th mean derivative of $X(t)$, let us use Property 3 of [28, p. 95] and Lemma 1.1.5. In order to do that, we must prove that $\mathrm{E}\left[\left(\frac{b}{2a}\right)^2\right] < \infty$, $\mathrm{E}\left[\left(\frac{\sqrt{-b^2+4ac}}{2a}\right)^4\right] < \infty$ and the s.p. $Z(t)$ is in L_4 and has a 4-th mean derivative. On account of Schwarz inequality

$$\mathrm{E}\left[\left(\frac{\sqrt{-b^2+4ac}}{2a}\right)^4\right] \leq \mathrm{E}\left[\left(\frac{1}{2a}\right)^8\right]^{\frac{1}{2}} \mathrm{E}\left[\left(\sqrt{-b^2+4ac}\right)^8\right]^{\frac{1}{2}}.$$

As $a_0 \leq a$, $\frac{1}{a} \leq \frac{1}{a_0}$, therefore $\mathrm{E}\left[\left(\frac{1}{2a}\right)^8\right]^{\frac{1}{2}}$ is bounded by $\left(\frac{1}{2a_0}\right)^4$. Now, by the triangle inequality and Schwarz inequality,

$$\begin{aligned}\mathrm{E}\left[\left(\sqrt{-b^2+4ac}\right)^8\right]^{\frac{1}{2}} &\leq \left(\| -b^2+4ac\|_4\right)^2 \\ &\leq \left(\mathrm{E}\left[(b)^8\right]^{\frac{1}{4}} + 4\mathrm{E}\left[(a)^8\right]^{\frac{1}{8}} \mathrm{E}\left[(c)^8\right]^{\frac{1}{8}}\right)^2 < \infty,\end{aligned}$$

because a, b and c are 8-r.v.'s. Hence, $\mathrm{E}\left[\left(\frac{\sqrt{-b^2+4ac}}{2a}\right)^4\right] < \infty$. Next, $\mathrm{E}\left[\left(\frac{b}{2a}\right)^2\right] \leq \mathrm{E}\left[b^4\right]^{\frac{1}{2}} \mathrm{E}\left[\left(\frac{1}{2a}\right)^4\right]^{\frac{1}{2}} \leq \mathrm{E}\left[b^4\right]^{\frac{1}{2}} \left(\frac{1}{2a_0}\right)^2 < \infty$ and $Z(t) \in L_4$ because of condition C2.3. To prove that the s.p. $Z(t)$ has 4-th mean derivative, let us set $W(t) = \frac{1}{2}\sqrt{-b^2+4ac}\, t - \arctan\left(\frac{b+2aX_0}{\sqrt{-b^2+4ac}}\right)$ and take $f(x) = \tan(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Then, we must check conditions i-iii of Theorem 1.1.6 for $p = 8$. First of all, by Schwarz inequality and the triangle inequality one gets:

$$\|W(t)\|_8 \leq \frac{1}{2}\left\|\sqrt{-b^2+4ac}\, t\right\|_8 + \left\|\arctan\left(\frac{b+2aX_0}{\sqrt{-b^2+4ac}}\right)\right\|_8 < \infty.$$

Now, it is easy to prove that the 8-th mean derivative of $W(t)$ is $\frac{1}{2}\sqrt{-b^2+4ac}$. Observe that $\mathrm{E}\left[(W(t)-W(s))^2\right] \leq \mathrm{E}\left[\left(\frac{1}{2}\sqrt{-b^2+4ac}\right)^2\right] |t-s|^2$ and the Kolmogorov's

criterion with $\alpha = 2, \beta = 1$ and $D = \mathrm{E}\left[\left(\frac{1}{2}\sqrt{-b^2+4ac}\right)^2\right] < \infty (D \in L_8)$ implies that the s.p. $W(t)$ has a.s. continuous paths. Then, by condition C2.3 and Theorem 1.1.6, the 4-th mean derivative of $Z(t) = f(W(t))$ is given by:

$$\dot{Z}(t) = \sec^2\left[\frac{1}{2}\sqrt{-b^2+4ac}\; t - \arctan\left(\frac{b+2aX_0}{\sqrt{-b^2+4ac}}\right)\right]\left(\frac{1}{2}\sqrt{-b^2+4ac}\right)$$

Therefore, from Lemma 1.1.5, it is easy to deduce that the s.p. $X(t)$ given by (2.3) is a m.s. solution of equation (2.1). Thus, the following result has been established:

Theorem 2.2.3. *Consider the equation given by (2.1), in which a, b, c and X_0 are 8-r.v.'s satisfying conditions C2.1-C2.3. Then, the s.p. given by (2.3) is a m.s. solution of (2.1).*

Remark 2.2.4. *Under the following conditions:* $0 < a_0 \leq a \leq a_1, 0 \leq b_0 \leq b \leq b_1, 0 < c_0 \leq c \leq c_1$, $0 \leq X^* \leq X_0 \leq X^{**}$ *such that:* $4a_0c_0 - b_1^2 > 0$, $\frac{2a_1X^{**}+b_1}{\sqrt{4a_0c_0-b_1^2}} \leq \tan\left(\frac{\pi}{2} - \delta^*\right)$, $\frac{1}{2}\sqrt{-b_0^2+4a_1b_1}(\mathcal{T}+\delta) \leq \frac{\pi}{2} - \delta$ *for some* $\delta, \delta^* > 0$, *then* $\left|\frac{1}{2}\sqrt{-b^2+4ac}\;(t+s) - \arctan\left(\frac{b+2aX_0}{\sqrt{-b^2+4ac}}\right)\right| \leq \frac{\pi}{2} - \delta^*$ *and so conditions C2.1-C2.3 hold true.*

2.3 Application of Generalized Polynomial Chaos

In this section we apply the technique of Generalized Polynomial Chaos (gPC) described in section 1.2 to the random initial problem given by (2.1). We proceed to expand second order stochastic process solution of (2.1), $X(t)$ and the random variables a, b, c and X_0 in terms of a polynomial orthogonal basis $\{\phi_i\}$, that is:

$$X(t) \approx \sum_{i=0}^{P} \beta_i(t)\phi_i(\zeta) \tag{2.4}$$

and

$$a \approx \sum_{i=0}^{P} a_i\phi_i(\zeta), \quad b \approx \sum_{i=0}^{P} b_i\phi_i(\zeta), \quad c \approx \sum_{i=0}^{P} c_i\phi_i(\zeta), \quad X_0 \approx \sum_{i=0}^{P} \beta_i(0)\phi_i(\zeta) \tag{2.5}$$

where a_i, b_i and c_i are real numbers and β_i are differentiable deterministic functions to be determined, $\zeta = (\zeta_1, \ldots, \zeta_M)$ is a random vector, $M \in \mathbb{N}$, ζ_i independent and identically distributed 2-r.v.'s and $\{\phi_i\}$ satisfying:

$$< \phi_i, \phi_j >=< \phi_i^2 > \delta_{ij}. \tag{2.6}$$

Here δ_{ij} is the Kronecker delta and $< \cdot, \cdot >$ denotes the inner product in the Hilbert space of the variables ζ. The total number of terms in the expansion is $P + 1$, and is determined by the dimension of the chaos M and the highest order p of the polynomials $\{\phi_i\}$; $P + 1 = \frac{(M+p)!}{M!p!}$. The coefficients a_i, for $i = 0, \ldots, P$ are computed as follows as follows:

$$a_i = \frac{< a, \phi_i(\zeta) >}{< \phi_i(\zeta), \phi_i(\zeta) >} \tag{2.7}$$

b_i, c_i and $\beta_i(0)$ are similarly computed. Now, we substitute representations (2.4), (2.5) into the equation (2.1),

$$\begin{aligned}\sum_{i=0}^{P} \dot{\beta}_i(t)\phi_i(\zeta) = &-\sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{k=0}^{P} a_i(t)\beta_j(t)\beta_k(t)\phi_i(\zeta)\phi_j(\zeta)\phi_k(\zeta) \\ &- \sum_{i=0}^{P}\sum_{j=0}^{P} b_i(t)\beta_j(t)\phi_i(\zeta)\phi_j(\zeta) - \sum_{i=0}^{P} c_i(t)\phi_i(\zeta)\end{aligned} \tag{2.8}$$

By taking $< \cdot, \phi_l >$ and using the orthogonality condition given by (2.6) we obtain a set of nonlinear deterministic equations:

$$\begin{aligned}\dot{\beta}_l(t) = -\frac{1}{<\phi_l(\zeta),\phi_l(\zeta)>}\Big(&\sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{k=0}^{P} a_i(t)\beta_j(t)\beta_k(t) < \phi_i(\zeta)\phi_j(\zeta)\phi_k(\zeta), \phi_l(\zeta) > \\ &+ \sum_{i=0}^{P}\sum_{j=0}^{P} b_i(t)\beta_j(t) < \phi_i(\zeta)\phi_j(\zeta), \phi_l(\zeta) > + c_l(t) < \phi_l(\zeta), \phi_l(\zeta) >\Big),\ l = 0, \ldots, P,\end{aligned} \tag{2.9}$$

with initial conditions

$$\beta_l(0) = \frac{< X_0, \phi_l(\zeta) >}{< \phi_l(\zeta), \phi_l(\zeta) >}, \qquad l = 0, \ldots, P. \tag{2.10}$$

Once we solve for the coefficients β_l, we can compute the approximations of the mean and variance. On account of the orthogonality of $\{\phi_j(\zeta)\}$ we have

$$\mathrm{E}\left[X_P(t)\right] = \sum_{i=0}^{P} \beta_i(t)\mathrm{E}\left[\phi_i(\zeta)\right] = \beta_0(t) \tag{2.11}$$

and

$$\mathrm{E}\left[\left(X_P(t)\right)^2\right] = \sum_{i=0}^{P} \beta_i(t)^2\mathrm{E}\left[\left(\phi_i(\zeta)\right)^2\right] + \sum_{i=1}^{P}\sum_{j=0}^{i-1} \beta_i(t)\beta_j(t)\mathrm{E}\left[\phi_i(\zeta)\phi_j(\zeta)\right].$$

Thus the variance of X_P takes the form

$$\begin{aligned}\mathrm{Var}\left[X_P(t)\right] &= \mathrm{E}\left[\left(X_P(t)\right)^2\right] - \left(\mathrm{E}\left[X_P(t)\right]\right)^2 = \sum_{i=0}^{P} \beta_i(t)^2\mathrm{E}\left[\left(\phi_i(\zeta)\right)^2\right] - \beta_0(t)^2 \\ &= \sum_{i=1}^{P} \beta_i(t)^2\mathrm{E}\left[\left(\phi_i(\zeta)\right)^2\right]. \end{aligned} \tag{2.12}$$

2.4 Numerical results

In this section we provide several examples in which we consider different kind of distributions in the coefficients and initial condition of the random initial value problem (2.1).

Example 2.4.1. *Consider the equation (2.1) and assume that $b(\omega) = c(\omega) = 0$ for all $\omega \in \Omega$ and a and X_0 are independent r.v.'s such that:*

- *Case1. The two r.v.'s have Beta Distributions: $a \sim Be\,(1,3)$ and $X_0 \sim Be\,(2,2)$ with density functions:*

$$f_a(a) = \begin{cases} 3(1-a)^2 & \text{if } a \in [0,1] \\ 0, & a \notin [0,1] \end{cases}, \quad f_{X_0}(X_0) = \begin{cases} 6X_0(1-X_0) & \text{if } X_0 \in [0,1] \\ 0, & X_0 \notin [0,1], \end{cases}$$

  respectively.

- *Case 2. a as in Case 1 and* $X_0 \sim Gamma(r = 5, \theta = \frac{1}{10})$ *having density function*

$$f_{X_0}(X_0) = \begin{cases} \frac{X_0^{(r-1)} e^{-\frac{X_0}{\theta}}}{\theta^r (r-1)!} & \text{if } X_0 \in [0, \infty) \\ 0, & X_0 < 0\,, \end{cases}$$

and leading to the same mean and variance as in Case 1.

- *Case 3.* $a \sim Exp(\lambda = 4)$ *with density function:*

$$f_a(a) = \begin{cases} \lambda e^{-\lambda a} & \text{if } a \in [0, \infty) \\ 0, & a < 0, \end{cases}$$

leading to the same mean as Cases 1-2, but with a larger variance of $\mathrm{Var}[a] = 0.062$*, and* X_0 *with Beta distribution as in Case 1.*

Note that in all Cases 1-3, the r.v's are also 8-r.v's and the fact that all of them are positive implies that conditions C1.1-C1.2 are satisfied, so by Theorem 2.2.1, the m.s. solution of (2.1) in this example is given by (2.2). Therefore the first and second exact moments take the form

$$\mathrm{E}\left[(X(t))^i\right] = \int_{\mathbb{R}^2} \left(\frac{1}{1 + aX_0 t}\right)^i f_a(a) f_{X_0}(X_0) da\, dX_0, \qquad i = 1, 2. \tag{2.13}$$

From (2.11) and (2.12) we obtain the approximations of the mean and variance corresponding to the gPC. In Case 1, the Jacobi polynomials basis is employed because the r.v.'s in the equation follow Beta distributions. In Cases 2 and 3, the Laguerre polynomials are chosen since X_0 follows a Gamma distribution in Case 2 and a follows a Exponential distribution in Case 3. In all three cases the chaos dimension is taken as $M = 2$ and the order of the polynomial chaos is $p = 3$. In Figures 2.1-2.3 we show our results. The MC method with 15×10^4 simulations provides good results in Case 1, but when one of the r.v.'s is assumed to be unbounded, as in Cases 2-3, the approximations are less accurate. The gPC method gives better approximations than

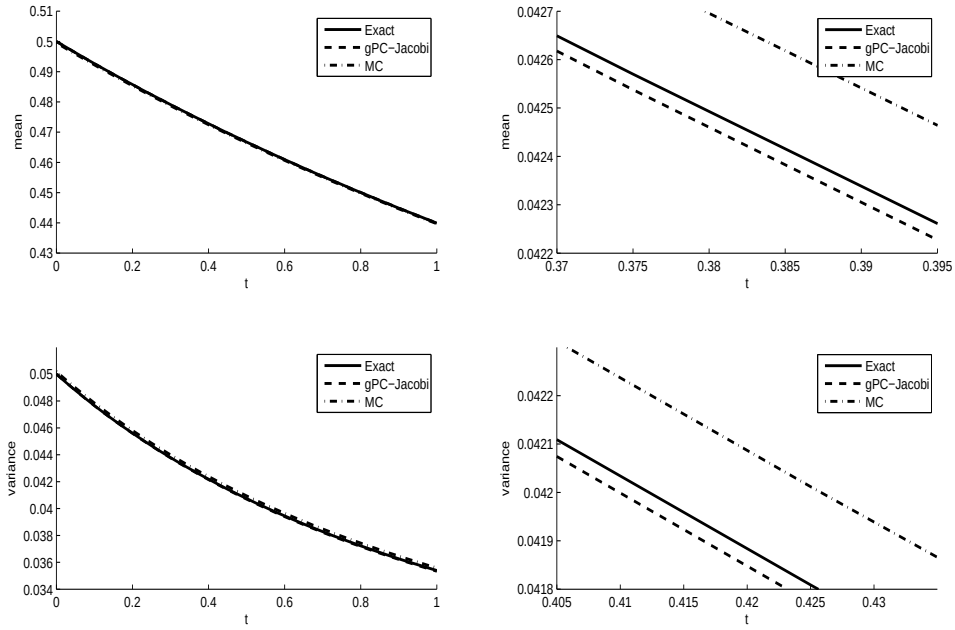

Figure 2.1. Example 2.4.1, case 1. Comparison of the exact mean and variance with its approximations obtained using the gPC with Jacobi Polynomials as the orthogonal basis and the Monte Carlo (15×10^4 simulations) methods, with $a \sim Be(1,3)$ $x_0 \sim B(\alpha = 2, \beta = 2)$.

MC method and the computational time is lower since only one large deterministic set of equations is solved to calculate the gPC expansions.

Example 2.4.2. *Consider the equation (2.1) and assume that a, b, c and* X_0 *are independent r.v.'s such that:*

- *Case 1. Both a and c have uniform distributions:* $a \sim Unif[\frac{1}{2}, \frac{2}{3}]$ *and* $c \sim Unif[\frac{2}{3}, \frac{3}{4}]$ *with density functions:*

$$f_a(a) = \begin{cases} 6 & \text{if } a \in [\frac{1}{2}, \frac{2}{3}] \\ 0, & a \notin [\frac{1}{2}, \frac{2}{3}] \end{cases}, \quad f_c(c) = \begin{cases} 12 & \text{if } c \in [\frac{2}{3}, \frac{3}{4}] \\ 0, & c \notin [\frac{2}{3}, \frac{3}{4}] \end{cases}$$

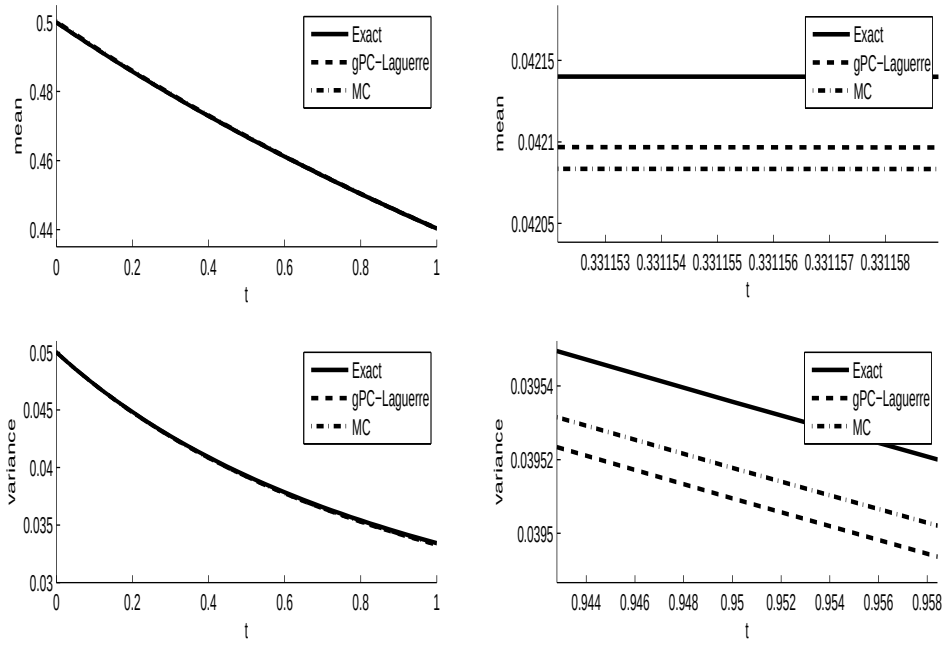

Figure 2.2. Example 2.4.1, case 2. Comparison of the exact mean and variance with its approximations obtained using the gPC with Laguerre Polynomials as the orthogonal basis and the Monte Carlo (15×10^4 simulations) methods, with $a \sim Be(1,3)$ $x_0 \sim Gamma(r=5, \theta=\frac{1}{10})$.

respectively and both b *and* X_0 *have Beta distributions:* $b \sim Be(2,2)$ *and* $X_0 \sim Be(2,5)$ *having density functions:*

$$f_b(b) = \begin{cases} 6b(1-b) & \text{if } b \in [0,1] \\ 0, & b \notin [0,1] \end{cases}, \quad f_{X_0}(X_0) = \begin{cases} 30X_0(1-X_0)^4 & \text{if } X_0 \in [0,1] \\ 0, & X_0 \notin [0,1] \end{cases}$$

respectively.

- *Case 2.* a, b, c *as in Case 1 and* $X_0 \sim Gamma(r=\frac{16}{5}, \theta=\frac{5}{56})$ *having density function:*

$$f_{X_0}(X_0) = \begin{cases} \frac{X_0^{(r-1)} e^{-\frac{X_0}{\theta}}}{\theta^r \Gamma(\frac{16}{5})} & \text{if } X_0 \in [0,\infty) \\ 0, & X_0 < 0\,, \end{cases}$$

with the same mean and variance as in Case 1.

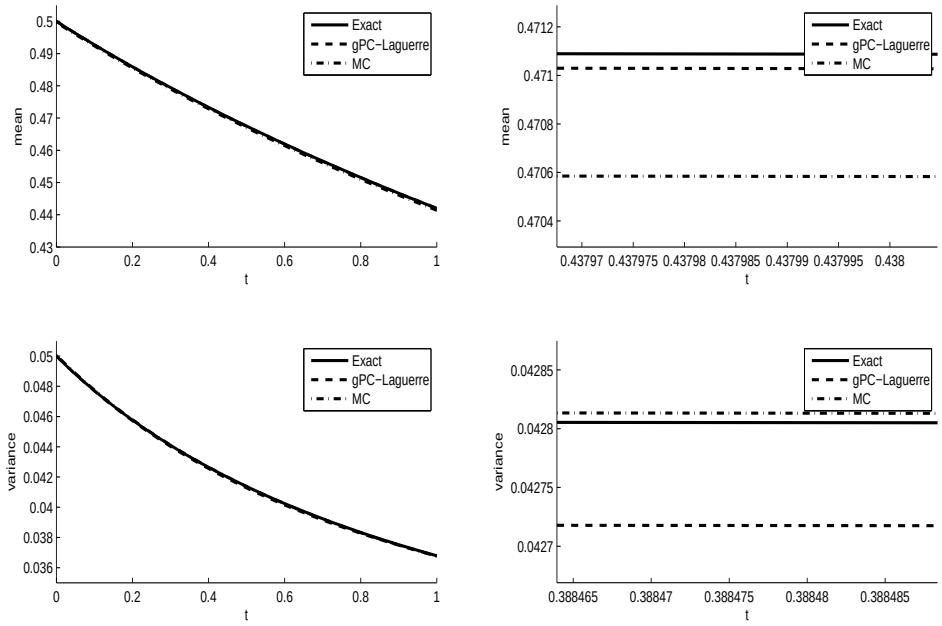

Figure 2.3. Example 2.4.1, case 3. Comparison of the exact mean and variance with its approximations obtained using the gPC with Laguerre Polynomials as the orthogonal basis and the Monte Carlo (15×10^4 simulations) methods, with $a \sim Exp(\lambda = 4)$ $x_0 \sim Be(1, 3)$.

- *Case 3.* a, b, c *as in Case 1 and* $X_0 \sim Exp(\lambda = \frac{7}{2})$*, with density function*

$$f_{X_0}(X_0) = \begin{cases} \lambda e^{-X_0 \lambda} & \text{if } X_0 \in [0, \infty) \\ 0, & X_0 < 0\,, \end{cases}$$

and leading to the same mean as in Cases 1-2, but with larger variance of $Var[X_0] = 0.0816327$.

According to Remark 2.2.4, in order to assure the existence of the exact m.s. solution in Cases 2-3 by means of Theorem 2.2.3, the r.v. X_0 must be truncated, so in both cases we select $L > 0$ sufficiently large so as to make $P[X_0 \geq L]$ arbitrarily

small, see the truncation method in [21, p. 233]. So, in Cases 2 and 3, the new density functions associate with the truncated r.v. X_0 are:

$$\hat{f}_{X_0}(X_0) = \begin{cases} \frac{X_0^{(r-1)} e^{-\frac{X_0}{\theta}}}{\int_0^L X_0^{(r-1)} e^{-\frac{X_0}{\theta}} dX_0} & \text{if } X_0 \in [0, L] \\ 0, & X_0 \in [0, L]/\mathbb{R}, \end{cases}$$

$$\hat{f}_{X_0}(X_0) = \begin{cases} \frac{e^{-X_0 \lambda}}{\int_0^L e^{-X_0 \lambda} dX_0} & \text{if } X_0 \in [0, L] \\ 0, & X_0 \in [0, L]/\mathbb{R} \end{cases}$$

respectively. Now, in all the Cases 1-3, the assumptions in Remark 2.2.4 are readily verified, and so, by Theorem 2.2.3, the m.s. solution is given by (2.3). In a similar manner as Example 2.4.1, the first and second moment can be computed by using the s.p. given by (2.3) . We have used the Jacobi polynomials basis in Case 1 and the Laguerre polynomials basis in Cases 2-3, for which no truncation on X_0 has been considered. In all cases the chaos dimension is taken as $M = 4$ and the order of the polynomial chaos is $p = 3$. Figures 2.4-2.6 show that the gPC is more accurate than the Monte Carlo method. Also we observed a reduction in computing time of about 50% for using the gPC.

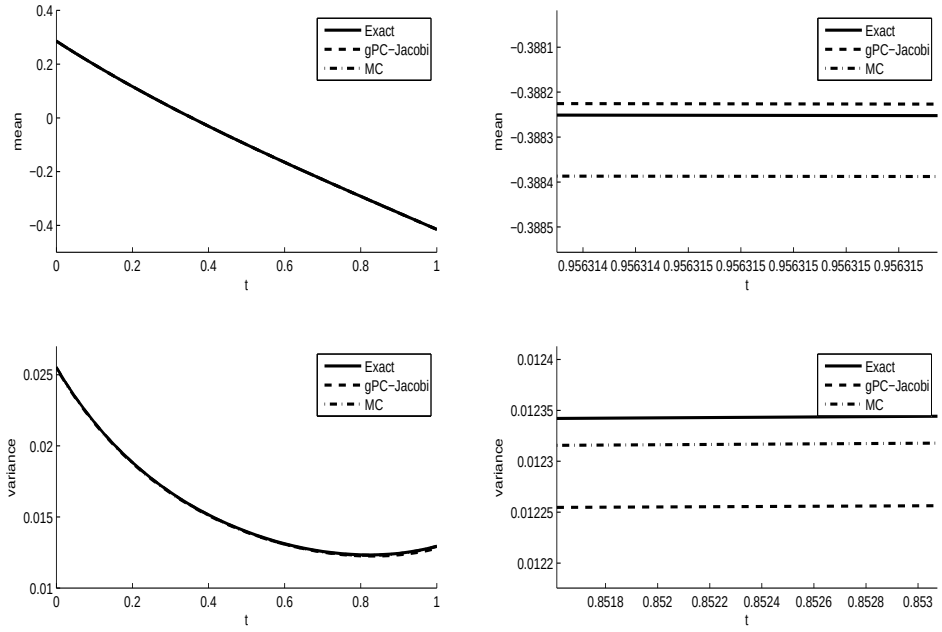

Figure 2.4. Example 2.4.2, case 1. Comparison of the exact mean and variance with its approximations obtained using the gPC with Jacobi polynomials as the orthogonal basis, and the Monte Carlo (15×10^4 simulations) methods.

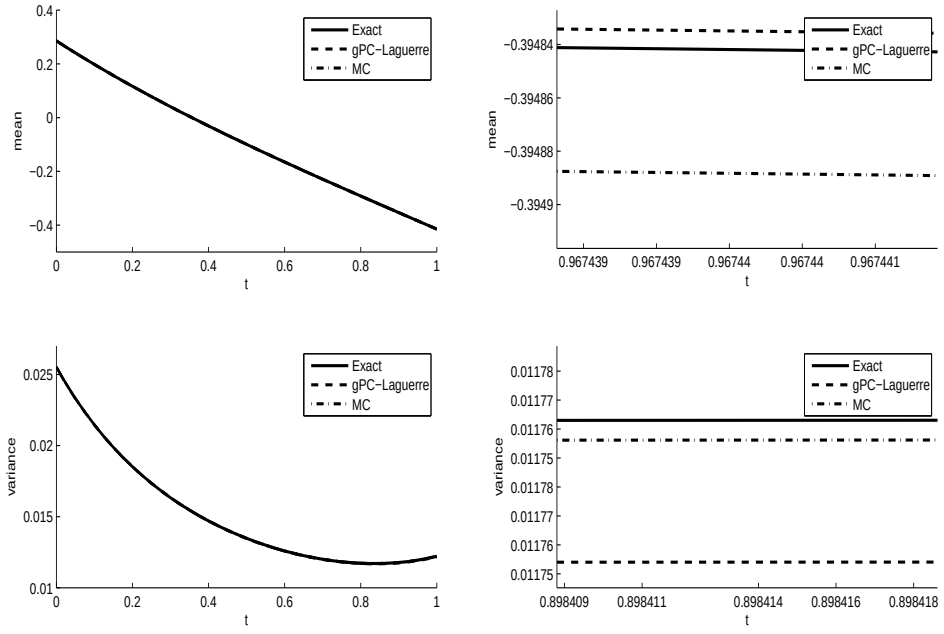

Figure 2.5. Example 2.4.2, case 2. Comparison of the exact mean and variance with its approximations obtained using the gPC with Laguerre polynomials as the orthogonal basis, and the Monte Carlo (15×10^4 simulations) methods, taking $L = 10$ and letting $P[X_0 \geq L] \simeq 3 \times 10^{-45}$.

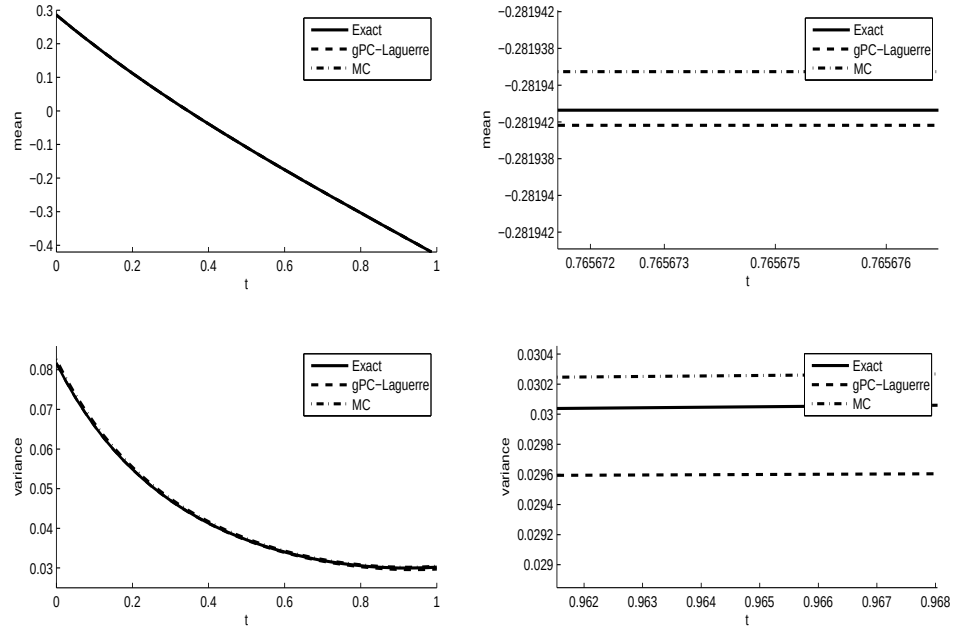

Figure 2.6. Example 2.4.2, case 3.Comparison of the exact mean and variance with its approximations obtained using the gPC with Laguerre polynomials as the orthogonal basis, and the Monte Carlo (15×10^4 simulations) methods, taking $L = 20$ and letting $P[X_0 \geq L] \simeq 3 \times 10^{-31}$.

CHAPTER 3

RANDOM DIFFERENTIAL EQUATIONS WITH DISCRETE DELAYS

Ordinary and partial differential equations have played important roles in the mathematical modeling in biology, medicine, chemistry, physics, economics, etc., but they are generally first approximations to describe the dynamics of real systems. More realistic models often require the incorporation of the past history of the system as a consequence of delayed effects. Models that incorporate delayed effects generally include delay differential equations. Delay differential equation models have been extensively developed in the past few decades (see for instance [19],[14],[26],[7],[5], and the references therein). However, deterministic systems fail to incorporate fluctuations that occur in real life phenomena. Therefore incorporating randomness in delay differential equation models is required [16],[20],[18]. This chapter is dedicated to approximate the solutions to a family of discrete delay differential equations with random coefficients by means of the polynomial chaos expansion method.

3.1 Delay differential equations with random coefficients

Let $\{X_h(t) : t \leq 0\}$ be a stochastic process, and consider the delay differential equation with a discrete delay $\tau > 0$

$$\begin{aligned} \frac{dX(t)}{dt} &= f(X(t), X(t-\tau), t), \\ X(t) &= X_h(t) \ \text{ for } t \leq 0, \end{aligned} \tag{3.1}$$

where the randomness enters the equation through f and $X_h(t)$. We can make sense of the problem (3.1) through the definition 1.1.7, by applying the so called method

of steps. Consider the the problem (3.1) as a series of random ordinary differential equations on consecutive time intervals of length τ, the first of which is:

$$\begin{aligned} \frac{dX(t)}{dt} &= f(X(t), X_h(t), t),\ t \in (0, \tau] \\ X(0) &= X_h(0), \end{aligned}$$

from where we obtain a solution $X_1(t)$ on the interval $[0, \tau]$ and then use $X_1(t)$ as the history for the next time interval $[\tau, 2\tau]$, so we have the problem

$$\begin{aligned} \frac{dX(t)}{dt} &= f(X(t), X_1(t), t),\ t \in (\tau, 2\tau] \\ X(\tau) &= X_1(\tau). \end{aligned}$$

which has solution $X_2(t)$, defined on $[\tau, 2\tau]$. In principle, this process can continue indefinitely assuming such solutions $X_1(t), X_2(t), \ldots$ exist, so the solution to problem (3.1) is the piecewise defined process formed from $X_1(t), X_2(t), \ldots$ But, once again we don't have enough tools to ensure the existence and uniqueness of such solutions in general. In the introductory chapter we mentioned that there are existence and uniqueness theorems based on Lipschitz conditions on the function f (see for instance [27]), but as noted in [31] such conditions are too restrictive and hard to implement in general. Therefore we have to restrict (3.1) to a smaller class of equations in order to elaborate on the existence of a rigorous solution.

Let us now focus our attention to a simple problem. Consider the delay differential equation:

$$\begin{aligned} \frac{dX(t)}{dt} &= -aX(t-1), \\ X(t) &= X_0 \text{ for } t \leq 0, \end{aligned} \tag{3.2}$$

where a and X_0 are 2-r.v.'s with a prescribed distribution. In order to solve the initial value problem (3.2), we follow the same approach used in last chapter to solve (2.1). As the deterministic theory suggests, a mean square candidate for the solution is:

$$X(t) = X_0 + X_0 \sum_{k=1}^{n} \frac{(-1)^k a^k}{k!} \left[t - (k-1)\right]^k \text{ for } n-1 \leq t \leq n,\ n = 1, 2, \ldots, \quad (3.3)$$

which is obtained by applying the method of steps to the deterministic problem (i.e. a and X_0 are regarded as real constants). The reader can refer to [26] for the details on the solution of a similar equation.

Notice that the process $X(t)$ given in (3.3) is defined for all $t \geq 0$, moreover, $X(t)$ is m.s. continuous since:

$$\begin{aligned} X(n+) &= X_0 + X_0 \sum_{k=1}^{n+1} \frac{(-1)^k a^k}{k!} \left[n - (k-1)\right]^k \\ &= X_0 + \frac{(-1)^{n+1} a^{n+1}}{(n+1)!} \left[n - (n+1-1)\right]^{n+1} + X_0 \sum_{k=1}^{n} \frac{(-1)^k a^k}{k!} \left[n - (k-1)\right]^k \\ &= X_0 + 0 + X_0 \sum_{k=1}^{n} \frac{(-1)^k a^k}{k!} \left[n - (k-1)\right]^k \\ &= X(n-), \quad \text{for } n = 1, 2, \ldots. \end{aligned}$$

In order to verify that $X(t)$ is m.s. differentiable we appeal to (1.4). Notice $X(t)$ is a linear combination of the products $[t-(k-1)]^k X_0 a^n$, since $[t-(k-1)]^k$ is a

differentiable function and $X_0 a^n$ is a constant random process, it follows that $X(t)$ is m.s. differentiable, also

$$\begin{aligned}
\frac{d}{dt}X(t) &= \frac{d}{dt}X_0 + \frac{d}{dt}X_0 \sum_{k=1}^{n} \frac{(-1)^k a^k}{k!} \left[t-(k-1)\right]^k \\
&= \sum_{k=1}^{n} \frac{(-1)^k}{k!} \frac{d}{dt} \left\{ \left[t-(k-1)\right]^k X_0 a^k \right\} \\
&= \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left\{ X_0 a^k \frac{d}{dt} \left[t-(k-1)\right]^k + \left[t-(k-1)\right]^k \frac{d}{dt} X_0 a^k \right\} \\
&= \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left\{ k X_0 a^k \frac{d}{dt} \left[t-(k-1)\right]^{k-1} \right\} \\
&= -a \left(X_0 + X_0 \sum_{k=2}^{n} \frac{(-1)^{k-1}}{(k-1)!} \left\{ a^{k-1} \left[t-(k-1)\right]^{k-1} \right\} \right) \\
&= -a \left(X_0 + X_0 \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \left\{ a^k \left[t-1-(k-2)\right]^{k-1} \right\} \right) \\
&= -aX(t-1),
\end{aligned}$$

therefore $X(t)$ is the m.s. solution of (3.2).

3.2 Application of Generalized Polynomial Chaos

Now we proceed to apply the technique of Generalized Polynomial Chaos (gPC) described in section 1.2 to the random problem with delay given by (2.1). The first step is to expand the second order stochastic process $X(t)$, and the random parameters a and X_0 in terms of a polynomial orthogonal basis $\{\phi_i\}$, that is:

$$X(t) \approx \sum_{i=0}^{P} \beta_i(t)\phi_i(\zeta) \tag{3.4}$$

and

$$a \approx \sum_{i=0}^{P} a_i \phi_i(\zeta), \quad X_0 \approx \sum_{i=0}^{P} \beta_i(0)\phi_i(\zeta) \tag{3.5}$$

where a_i and $\beta_i(0)$ are real numbers and β_i are differentiable deterministic functions to be determined, $\zeta = (\zeta_1, \ldots, \zeta_M)$ is a random vector, $M \in \mathbb{N}$, ζ_i independent and

identically distributed 2-r.v.'s. The coefficients a_i and $\beta_i(0)$ for $i = 0, \ldots, P$ are computed according to (2.7).

Now, we substitute representations (3.4), (3.5) into the equation (2.1),

$$\sum_{i=0}^{P} \dot{\beta}_i(t)\phi_i(\zeta) = -\sum_{i=0}^{P}\sum_{j=0}^{P} a_i(t)\beta_j(t-1) \tag{3.6}$$

By taking $< \cdot, \phi_l >$ and using the orthogonality condition given by (2.6) we obtain a set of nonlinear deterministic delay differential equations:

$$\dot{\beta}_l(t) = -\frac{1}{<\phi_l(\zeta),\phi_l(\zeta)>}\sum_{i=0}^{P}\sum_{j=0}^{P} a_i(t)\beta_j(t-1) < \phi_i(\zeta)\phi_j(\zeta), \phi_l(\zeta) >,\ l = 0, \ldots, P,$$

with history

$$\beta_l(0) = \frac{< X_0, \phi_l(\zeta) >}{< \phi_l(\zeta), \phi_l(\zeta) >}, \qquad l = 0, \ldots, P, \qquad t \leq 0. \tag{3.7}$$

The next step is to integrate the above ordinary delay differential equations. Then the mean and variance of $X(t)$ can be computed as in (2.11) and (2.12).

3.3 Random Logistic Equation with delay

The logistic differential equation, first introduced in 1837 by the Belgian mathematician Pierre Verhulst (1804-1849), it is one of the most well-known models for population growth. The logistic equation is often used to describe the dynamics of populations under limited resources, such as food, space among others. In this section we consider a random version of the logistic differential equation with a discrete delay $\tau > 0$:

$$\begin{aligned} \frac{d}{dt}X(t) &= rX(t)(1 - KX(t-\tau)) = rX(t) - rKX(t)X(t-\tau) \\ X(t) &= X_0,\ t \leq 0 \end{aligned}$$

where r, K and X_0 are $2-$r.v.'s with a prescribed distribution. Due to the complexity of the right hand side for the case of a logistic equation like (3.8), we are unable to

derive an explicit formula for the solution like it has been done in earlier sections. According to the comment in [27, p. 142], for random differential equations with random coefficients, the m.s. solution agrees with the deterministic solution as long as they both exist. Thus we can explore the performance of the generalized polynomial expansion method for this equation regarding the deterministic solution as the m.s. solution. Let us apply the generalized polynomial chaos expansion method to equation (3.8).

Expand the second order stochastic process solution of (3.8), $X(t)$ and the random variables r, K and X_0 in terms of a polynomial orthogonal basis $\{\phi_i\}$, that is:

$$X(t) \approx \sum_{i=0}^{P} \beta_i(t)\phi_i(\zeta) \tag{3.8}$$

and

$$r \approx \sum_{i=0}^{P} r_i\phi_i(\zeta), \quad K \approx \sum_{i=0}^{P} k_i\phi_i(\zeta), \quad X_0 \approx \sum_{i=0}^{P} \beta_i(0)\phi_i(\zeta), \tag{3.9}$$

where r_i,k_i and $\beta_i(0)$ are real numbers and $\beta_i(t)$ are differentiable deterministic functions to be determined. Now, substitute representations (3.8), (3.9) into the equation (3.8),

$$\begin{aligned}\sum_{i=0}^{P} \dot{\beta}_i(t)\phi_i(\zeta) = &\sum_{i=0}^{P}\sum_{j=0}^{P} r_i(t)\beta_j(t)\phi_i(\zeta)\phi_j(\zeta) \\ &- \sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{k=0}^{P} r_i(t)k_j(t)\beta_k(t-\tau)\phi_i(\zeta)\phi_j(\zeta)\phi_k(\zeta)\end{aligned} \tag{3.10}$$

By taking $< \cdot, \phi_l >$ and using the orthogonality condition given by (2.6) we obtain a set of nonlinear deterministic delay differential equations:

$$\begin{aligned}\dot{\beta}_l(t) = \frac{1}{<\phi_l(\zeta),\phi_l(\zeta)>} &\left(\sum_{i=0}^{P}\sum_{j=0}^{P} r_i(t)\beta_j(t) < \phi_i(\zeta)\phi_j(\zeta), \phi_l(\zeta) > \right. \\ &\left. - \sum_{i=0}^{P}\sum_{j=0}^{P}\sum_{k=0}^{P} r_i(t)k_j(t)\beta_k(t-\tau) < \phi_i(\zeta)\phi_j(\zeta)\phi_k(\zeta), \phi_l(\zeta) > \right), \; l = 0, \ldots, P,\end{aligned}$$

with history

$$\beta_l(t) = \frac{< X_0, \phi_l(\zeta) >}{< \phi_l(\zeta), \phi_l(\zeta) >}, \qquad l = 0, \ldots, P, \qquad t \leq 0. \tag{3.11}$$

Once the coefficients β_l are computed, we can approximate the mean and variance of $X(t)$ according to (2.11) and (2.12).

3.4 Numerical results

In this section we provide several examples in which we consider different distributions in the coefficients and history for the problems (3.2) and (3.8).

Example 3.4.1. *Consider the equation (3.8) and assume that a and X_0 are independent r.v.'s such that:*

- *Case1. The two r.v.'s have Beta Distributions $a \sim Be\,(2,5)$ and $X_0 \sim Be\,(2,2)$.*
- *Case 2. a as in Case 1 and $X_0 \sim Gamma(r = 5, \theta = \frac{1}{10})$, leading to the same mean and variance as in Case 1.*
- *Case 3. $a \sim Exp(\lambda = 4)$ leading to the same mean as Cases 1-2, but with a larger variance of $\mathrm{Var}[a] = 0.062$, and X_0 with Beta distribution as in Case 1.*

Refer to examples 2.4.1 and 2.4.2 from the previous chapter for the probability density functions of these random variables.

Note that the m.s. solution of (3.2) is given by (3.3). Therefore the first and second exact moments take the form

$$\mathrm{E}\left[(X(t))^i\right] = \int_{\mathbb{R}^2} \left(X(t) = X_0 + X_0 \sum_{k=1}^{n} \frac{(-1)^k a^k}{k!} \left[t - (k-1)\right]^k \right)^i f_a(a) f_{X_0}(X_0) da\, dX_0.$$

From (2.11) and (2.12) we obtain the approximations of the mean and variance corresponding to the gPC. In Case 1, the Jacobi polynomials basis is employed because the variable X_0 follows a Beta distribution. In Cases 2 and 3, the Laguerre polynomials are chosen since X_0 follows a Gamma distribution in Case 2 and a follows

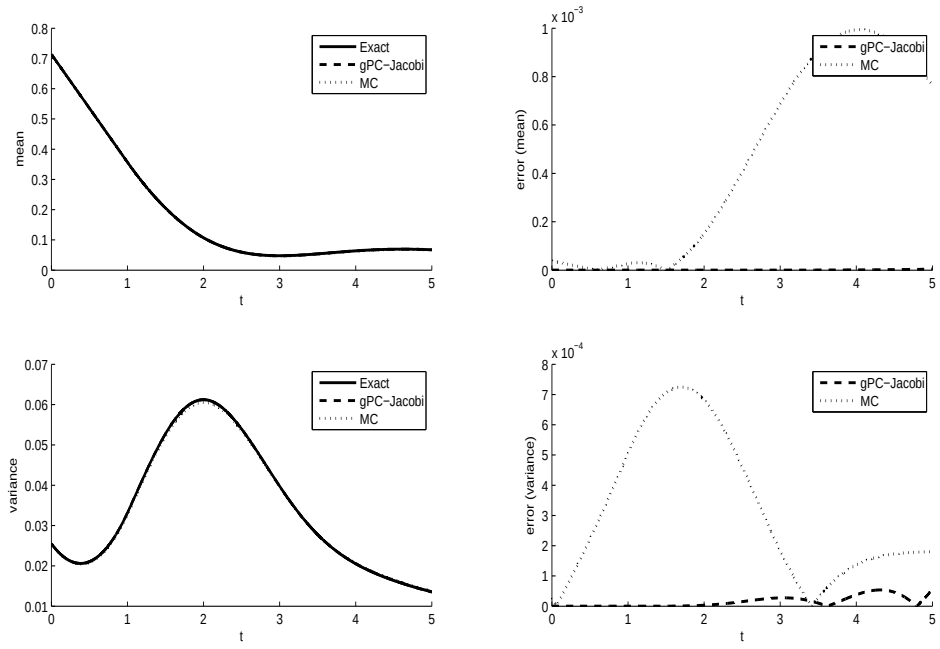

Figure 3.1. Example 3.4.1, case 1. Comparison of the exact mean and variance with its approximations obtained using the gPC with Jacobi Polynomials as the orthogonal basis and the Monte Carlo (20×10^3 simulations) methods, with $a \sim Be(2,5)$ $X_0 \sim B(\alpha = 2, \beta = 2)$.

a Exponential distribution in Case 3. In all three cases the chaos dimension is taken as $M = 2$ and the order of the polynomial chaos is $p = 3$. In Figures 3.1-3.3 we show a comparison of the gPC method and Monte Carlo simulations. Throughout the experiments conducted, dde23 matlab solver for deterministic delay differential equations has been used to integrate the deterministic problems derived from both, gPC and MC methods. Monte Carlo method with 15×10^3 simulations provides good approximations. As seen throughout examples 3 and 4 gPC provide expansions have high order of convergence for early times that eventually deteriorates, being case 3 the worst case as it can be observed that the error on approximated the variance

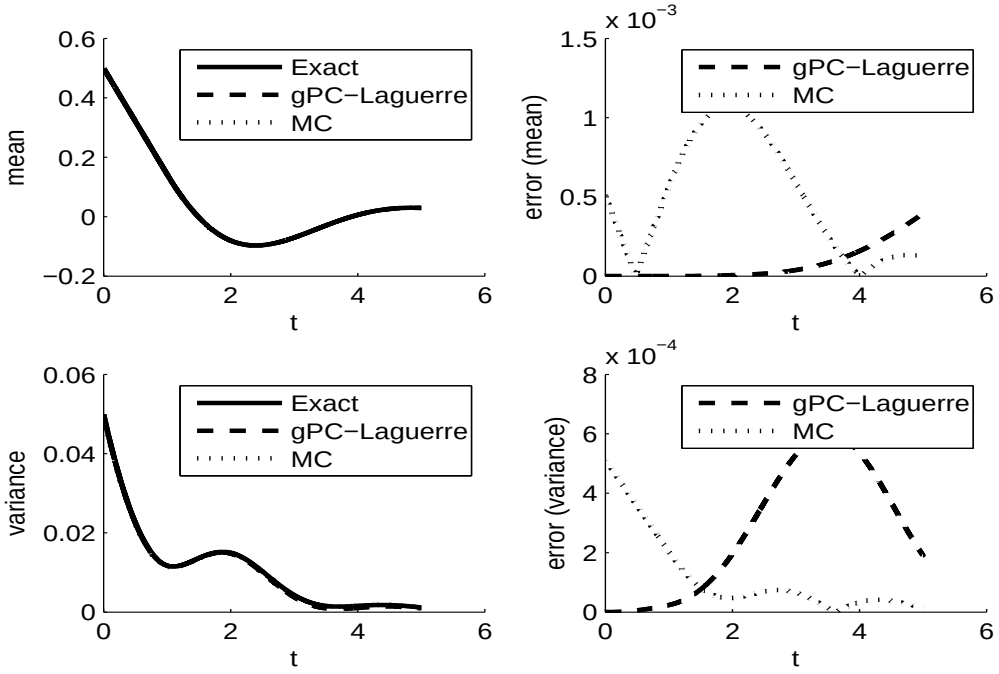

Figure 3.2. Example 3.4.1, case 2. Comparison of the exact mean and variance with its approximations obtained using the gPC with Laguerre Polynomials as the orthogonal basis and the Monte Carlo (20×10^3 simulations) methods, with $a \sim Be(2,5)$ $X_0 \sim Gamma(r = 5, \theta = \frac{1}{10})$.

blows up in time. When compared to Monte Carlo at later times, gPC still provides good approximation in cases 1 and 2.

Example 3.4.2. *Consider the equation (3.8) and assume that r, K, and X_0 are independent r.v.'s such that:*

- *Case 1. $X_0 \sim Be(2,2)$ and $K \sim Unif[\frac{1}{4}, \frac{3}{4}]$ with density function:*

$$f_K(k) = \begin{cases} 2 & \text{if } k \in [\frac{1}{4}, \frac{3}{4}] \\ 0, & k \notin [\frac{1}{4}, \frac{3}{4}] \end{cases}$$

  and $r = 1$ is a deterministic constant.

- *Case 2. $r \sim Be(2,5)$, X_0 as in Case 1 and $K = 0.5$ is deterministic.*

- *Case 3. $K \sim Exp(\lambda = 1/7)$, X_0 as in Case 1 and $r = 0.25$ is deterministic.*

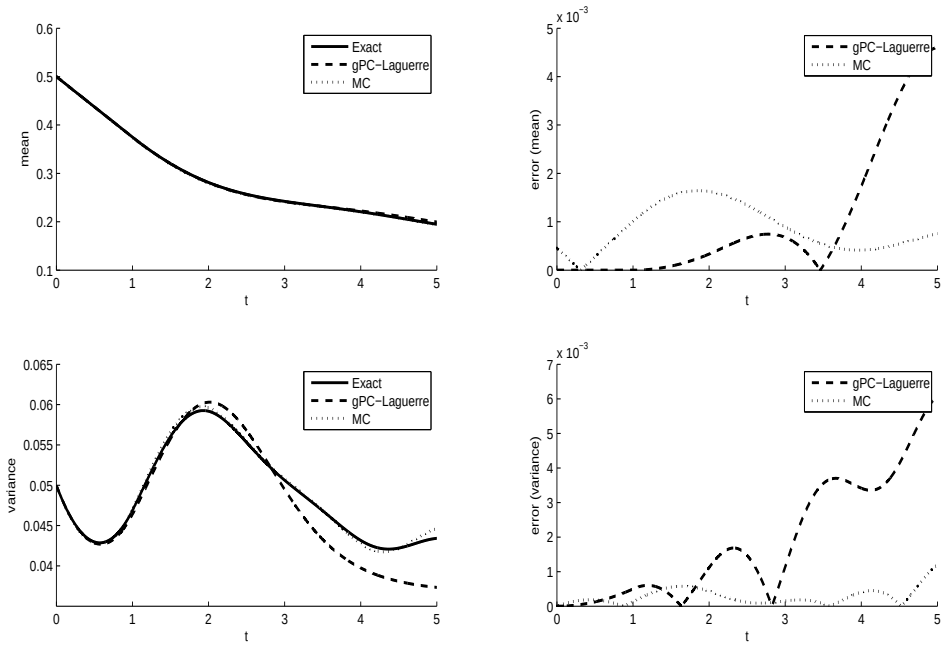

Figure 3.3. Example 3.4.1, case 3. Comparison of the exact mean and variance with its approximations obtained using the gPC with Laguerre Polynomials as the orthogonal basis and the Monte Carlo (20×10^3 simulations) methods, with $a \sim Exp(\lambda = 4)$ $X_0 \sim Be(1, 3)$.

Refer to examples 2.4.1 and 2.4.2 from the previous chapter for the probability density functions of these random variables.

In (3.4) to (3.6) we investigate the performance of bot gPC and MC methods to estimate the expected value and variance of the solution process to the random logistic equation with a discrete delay (3.8). We consider several distributions for the parameters. Throughout this example we regard the deterministic solution equation (3.8) as the exact solution. Let $\Phi(t; r; K; X_0)$ be the deterministic solution of equation (3.8) for fixed values of the parameters r, K, and X_0, then the first and second moments of $X(t)$ are calculated as follows.

$$\mathrm{E}\left[(X(t))^i\right] = \int (\Phi(t; r; K; X_0))^i \, f_r(r) f_K(K) f_{X_0}(X_0) dr dK \, dX_0, \qquad i = 1, 2. \quad (3.12)$$

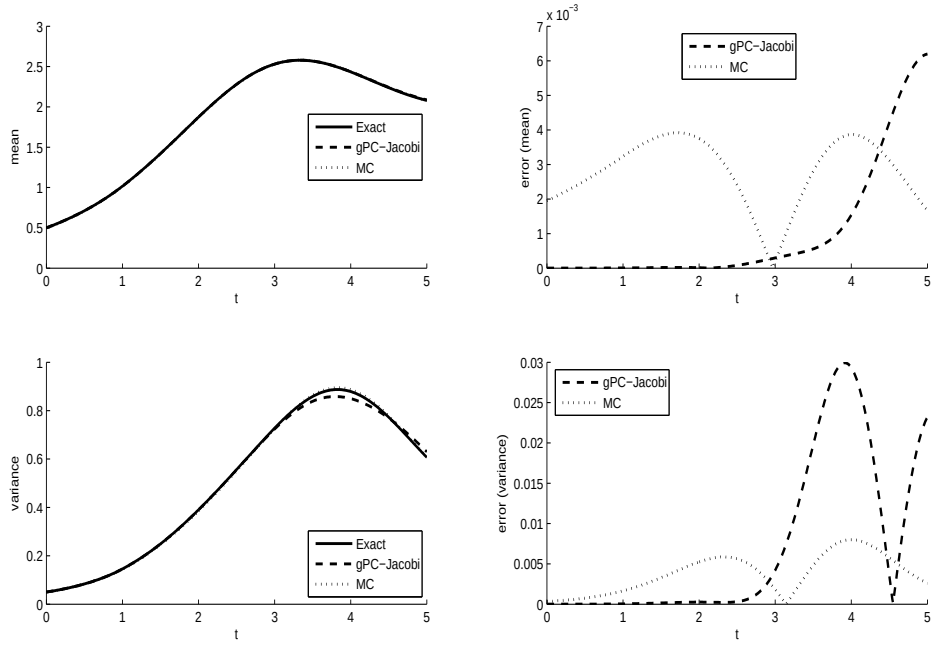

Figure 3.4. Example 3.4.2, case 1. Comparison of the exact mean and variance with its approximations obtained using the gPC with Jacobi Polynomials as the orthogonal basis and the Monte Carlo (15×10^3 simulations) methods, with $K \sim Unif(1/4, 3/4)$, $X_0 \sim Be(2, 2)$ and $r = 1$.

In our numerical experiments, we evaluate the integral above numerically, which requires the solution of the deterministic problem for every discrete value of the parameters. Then, we use the moments obtained to compare the approximations from gPCE and MC methods as shown in (3.4) to (3.6). In case 1, the expected value is better approximated by gPC up to a certain time, after which MC approximation is more accurate. In case 2 gPC seems to provide better approximations at every time, notice in this example the variance tends to zero so both methods seem to approximate well, being gPC the best one. In the last case, both the mean and the variance are increasing, in this case the expected value is well approximated by gPC but not by MC, and the variance is only approximated accurately by gPC at

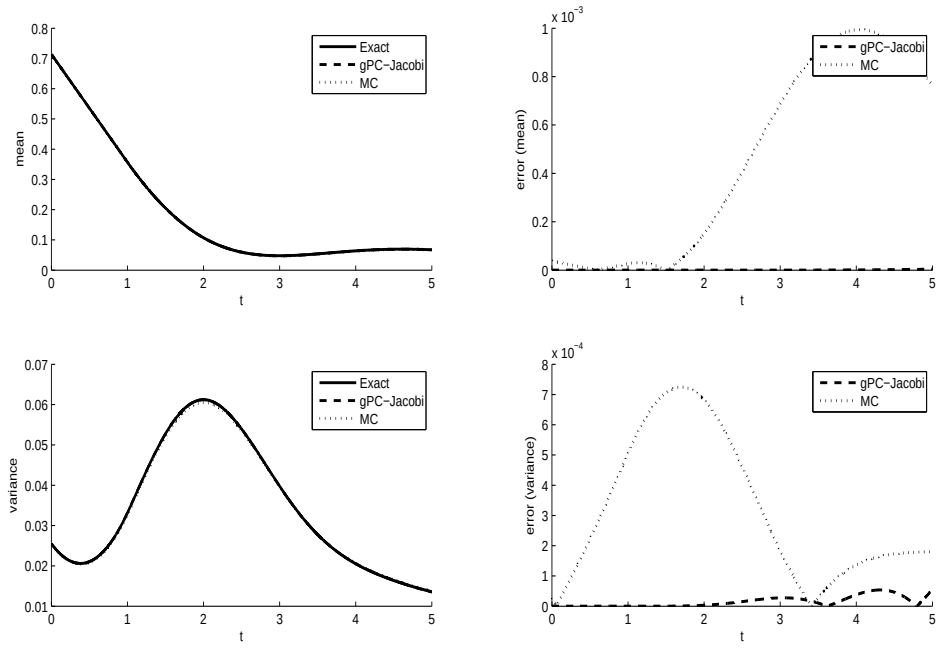

Figure 3.5. Example 3.4.2, case 2. Comparison of the exact mean and variance with its approximations obtained using the gPC with Jacobi Polynomials as the orthogonal basis and the Monte Carlo (15×10^3 simulations) methods, with $r \sim Be(2,5)$, $K = 0.5$ is a deterministic constant, adn X_0 the same as in case 1.

early times. It is known that long term integration of equations with non-linear terms becomes an challenge when approximating with polynomial chaos expansions, typically using more terms in the expansions will alleviate the impact of the non-linearity on the accuracy of the method, but this will only delay this problem. M. Gerritsma et. al. [12] have addressed this problem by introducing time-dependent polynomial chaos expansions. Generalized polynomial chaos (gPC) has non-uniform convergence and tends to break down for long-time integration. As they point out in their work, the probability density distribution of the solution evolves over time. The set of orthogonal polynomials associated with the initial distribution will therefore not be optimal at later times, thus causing the reduced efficiency of the method for

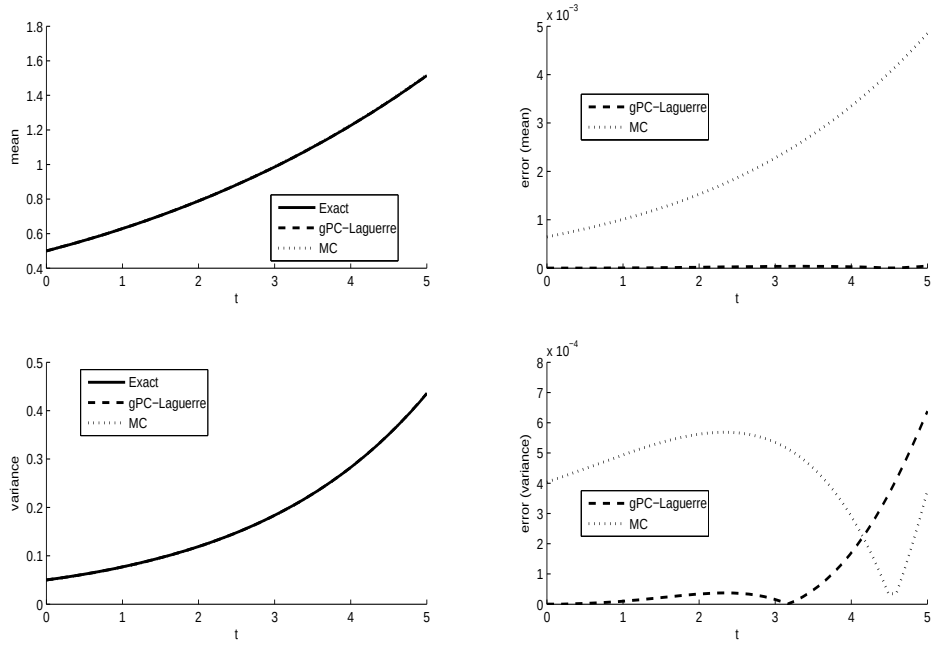

Figure 3.6. Example 3.4.2, case 3. Comparison of the exact mean and variance with its approximations obtained using the gPC with Jacobi Polynomials as the orthogonal basis and the Monte Carlo (15×10^3 simulations) methods, with $K \sim Exp(\lambda = 1/7)$, X_0 the same as in case 1 and $r = 0.25$.

long-time integration. Adaptation of the set of orthogonal polynomials with respect to the changing PDF removes the error with respect to long-time integration. In this method new stochastic variables and orthogonal polynomials are constructed as time progresses. This allows the method to use only low order polynomial approximations with high accuracy.

CHAPTER 4

CONCLUSIONS AND FUTURE WORK

In chapter 2, an analytic mean square solution for the Riccati random equations (2.1) has been derived. It was observed that under appropriate conditions on the random variables in model (2.1), the p-mean calculus is a practical tool for proving that the corresponding deterministic solution is also a mean square solution of the random Riccati equation. The chain rule of the p-mean calculus can be used in different situations, for instance, when an explicit solution of the corresponding deterministic equation is known. Having an explicit solution (exact or numerical) for random differential equations is important because it allows us to compute the main statistical functions efficiently, on the other hand having analytic solutions to random differential equations is a fundamental piece for the validation of numerical methods. Moreover, this calculus also can be useful in constructing mean square solutions (exact or numerical) of random differential equations when an analytic solution is not available [3]. Our results on the methods mentioned in the introduction of the work to model (2.1), indicate the following conclusions (these conclusions correspond to the approximations of both mean and variance functions). First, the MC method was observed to be simple to implement in both examples, but expensive to achieve a prescribed accuracy (of about 1×10^{-4}) because of the large number of simulations, specially in the second example 2.4.2 when fourth r.v.'s are simulated and the improvement of the approximations are not significant enough from 15×10^4 to 30×10^4 simulations, leading to the computing time increase of about 200%. Second, the gPC method which consists essentially of solving a coupled set of nonlinear deterministic

differential equations, provides accurate approximations for model (2.1). Even when the computational cost of this method also increased in the example 2.4.2, it is still much less than that of the MC method, besides gPG provides better approximations than the MC method as is shown in figures 2.4-2.6.

Later in chapter 3, our study is extended to random differential equations with delay, such as (3.2) and (3.8). We do not have previous knowledge of polynomial chaos method being applied to random differential equations with delays, therefore work developed in this chapter is exploratory, and it can be regarded as evidence on the applicability of the methods discussed to more general cases. An analytic m.s. solution to equation (3.2) is derived, although this is a simple problem, it proves useful when testing our numerical approximations from Monte Carlo and generalized polynomial chaos for such equations. Additionally, a logistic type delayed random differential equation has been solved by gPC methods. As observed in the examples 3.4.1 and 3.4.2 from chapter 3, generalized polynomial chaos approximations exhibit a high accuracy for early times. It is known that long term integration of the polynomial chaos equations with non-linear terms losses accuracy over time, typically using more terms in the expansions will alleviate the impact of the non-linearity on the accuracy of the method, but this will only delay this problem. M. Gerritsma et. al. [12] have addressed this problem by introducing time-dependent polynomial chaos expansions so the high nonlinearities are removed every certain period of time by recalculating the gPC expansions, therefore we do not regard this as a major drawback of the polynomial chaos method. MC method does not seem affected in the same degree by such problem as seen in examples 3 and 4, but due to its low rate of convergence and consequently high computational cost, gPC methods proves to be more efficient when solving random differential equations with delays. Further investigation of logistic type random differential equations with delays remains as future work.

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BIOGRAPHICAL STATEMENT

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