ON EXISTENCE OF EXTREMAL SOLUTIONS OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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by

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Let $X$ be a real Banach space, $K \subset X$ a cone, $I = [0,a] \subset \mathbb{R}$ and $f: I \times K \to X$ continuous. We look for conditions on $X$, $K$ and $f$ such that the IVP

\[ u' = f(t,u), \quad u(0) = u_0 \in K, \]

has a maximal solution $\bar{u}$ and a minimal solution $\underline{u}$ with respect to the partial ordering induced by $K$. Contrary to known results, [5,6], we shall not assume that $K$ has interior points, since the standard cones of many infinite dimensional spaces have empty interior. The second essential new feature is that $f$ is supposed to be defined only on $K$ and this demands that the extra conditions on $f$ are required only with respect to points in $K$, and not on the whole space.

I. DEFINITIONS AND NOTATIONS

Let $X^*$ denote the normed dual of $X$. The interior of a set $D \subset X$ is denoted by $D^0$. A closed convex subset $K \neq \{0\}$ of $X$ is said to be a cone if $\lambda K \subset K$ for every $\lambda > 0$ and $K \cap (-K) = \{0\}$. For a cone $K$, we let $K^* = \{x^* \in X^*: x^*(x) \geq 0 \text{ for all } x \in K\}$. We define a partial order "$\leq"$ by "$x \leq y$ iff $y - x \in K$".
Let $K$ be a cone, $D \subseteq X$, $f: I \times D \to X$. Then $f$ is said to be quasimonotone (w.r. to $K$) if the following condition (2) is satisfied (see [1,5,6]),

$$\tag{2} t \in I; \quad x, y \in D \quad \text{and} \quad x \leq y; \quad x^\ast \in K^\ast \quad \text{and}$$

$$x^\ast (x - y) = 0 \Rightarrow x^\ast (f(t, x) - f(t, y)) \leq 0.$$ 

An essential tool to prove existence of solutions to problem (1) in a subset $D$ of $X$ is the boundary condition

$$\tag{3} t \in I, \quad x \in \partial D = \liminf_{\lambda \to 0^+} \text{dist}(x + \lambda f(t, x), D) = 0.$$ 

See [1,5]. In case $D$ is a cone $K$, condition (3) is equivalent to

$$\tag{4} t \in I, \quad x \in \partial K, \quad x^\ast \in K^\ast \quad \text{and} \quad x^\ast (x) = 0 \Rightarrow x^\ast (f(t, x)) \geq 0$$ 

(see Example 4.1 in [1]).

Let us remark that $f: I \times K \times X$ satisfies (4) if $f$ is quasimonotone and $f(t, 0) \geq 0$ in $I$. In fact, $x \in \partial K$ and $x^\ast (x) = 0$, for some $x^\ast \in K^\ast$, imply $x^\ast (f(t, x)) = x^\ast (f(t, x) - f(t, 0)) + x^\ast (f(t, 0)) \geq 0$.

A solution $\bar{u}$ (resp. $u$) of (1) is said to be a maximal (resp. minimal) solution of (1) if $\bar{u}(t) \geq u(t)$ (resp. $u(t) \leq u(t)$) for every solution $u$ of (1) and all $t \in I$ such that both functions are defined at $t$.

Since $K \cap (-K) = \{0\}$, there is at most one maximal (resp. minimal) solution on every fixed interval $[0, a] \subseteq I$. 
II. THE FINITE DIMENSIONAL CASE

In this section we consider \( X = \mathbb{R}^n \). A simple result is

**Proposition 1.** Let \( K \subset \mathbb{R}^n \) be a cone with \( K^0 \neq \emptyset \), \( f: I \times K \to \mathbb{R}^n \) continuous, quasimonotone and such that \( f(t,0) \geq 0 \). Then

(i) Problem (1) has a (local) maximal solution.

(ii) Problem (1) has a minimal solution if \( f(t,0) \in K^0 \) in \( I \).

**Proof:** (i) Consider the IVP \( v' = f(t,v) + \frac{1}{p} e \), \( v(0) = u_0 + \frac{1}{p} e \) for some fixed \( e \in K^0 \) and every \( p \geq 1 \). Since \( f \) satisfies the boundary condition (4), \( f + \frac{1}{p} e \) satisfies this condition too. Therefore, there exists a solution \( v_p \) on some \([0,b] \subset I\), with \( b > 0 \) independent of \( p \) (see Theorem 4.1 in [1]). The sequence \( (v_p) \) has a uniformly convergent subsequence, the limit \( \overline{v} \) of which is a solution of (1) on \([0,b]\). If \( u \) is any solution of (1) then \( u(t) < v_p(t) \) as long as both exist (see Lemma 5.1 in [1]), and therefore \( u(t) \leq \overline{v}(t) \) for such \( t \), i.e. \( \overline{v} \) is the maximal solution.

(ii) If \( f(t,0) \in K^0 \) in \( I \), then we consider \( v' = f(t,v) - \frac{1}{p} f(t,0) \), \( v(0) = u_0 \) for \( p \geq 2 \). Since the right hand side is quasimonotone and \( f(t,0) - \frac{1}{p} f(t,0) \in K^0 \) in \( I \), Condition (4) is satisfied. If \( u \) is a solution of (1) then \( u(t) > v_p(t) \) in \((0,b]\) for some \( b > 0 \) and again \( (v_p) \) has a uniformly convergent subsequence the limit of which is the minimal solution \( v \).

q.e.d.
In general, we do not know how to prove Proposition 1 (ii) without the condition \( f(t,0) \in K^0 \) in \( I \). However, we have

**Proposition 2.** Let \( K \subset \mathbb{R}^n \) be the standard cone, i.e. \( K = \{ x : x_i > 0 \text{ for } i = 1, \ldots, n \} \), \( f : I \times K \to \mathbb{R}^n \) quasimonotone, continuous and such that \( f(t,0) \geq 0 \) in \( I \). Then (1) has a maximal and a minimal solution.

**Proof:** The existence of \( \bar{V} \) being clear from Proposition 1, we have only to prove that \( \nu \) exists. Let \( P : \mathbb{R}^n \to K \) be the metric projection, characterized by \( |x - Px| = \text{dist}(x,K) \). It is easy to see that \( Px = (\max(x_1,0), \ldots, \max(x_n,0)) \). The function \( \tilde{f}(t,x) = f(t,Px) \) defines a continuous extension of \( f \) to \( I \times \mathbb{R}^n \), and \( \tilde{f} \) is quasimonotone there. In fact, \( x \in \mathbb{R}^n \) and \( z \in K \) and \( \nu \in K = K^+ \) such that \( (\nu, z) = 0 \) imply \( (\nu, P(x+z) - Px) = 0 \), and therefore \( (\nu, \tilde{f}(t,x) + \tilde{f}(t,x+z)) \leq 0 \). Now, we fix \( \varepsilon \in K^0 \), consider \( K_\delta = \{ x \in \mathbb{R}^n : \text{dist}(x,K) < \delta \} \) for some \( \delta > 0 \) and the function \( f_\delta(t,x) = f(t,x) - \eta x - \eta^2 \varepsilon \), with \( \eta = \delta |\varepsilon|^{-1} \). We claim that \( f_\delta \) satisfies the boundary condition (3) for \( d = K_\delta \). To see this we notice first that this condition is equivalent to

\[
(5) \quad t \in I, \ x \in \partial K_\delta, \ |x^k| = 1 \text{ and } x^k(x) = \inf_{K_\delta} x^k(y) \rightarrow x^k(f_\delta(t,x)) \geq 0.
\]

(see Lemma 4.1 in [1] with \( \sup \) replaced by \( \inf \)).

Now, since the ball \( B_\delta(0) \) is contained in \( K_\delta \), we have \( x^k(x) \leq -\delta \). We also know, that \( Px \in \partial K \) and \( x - Px + y \in K_\delta \) for
every \( y \in K \), since \(|x-Px|=\delta\). Therefore, \( x^*(x) \leq x^*(x-Px+y) \) for all \( y \in K \), and this implies \( 0 = x^*(Px) = \inf_{K} x^*(y) \). Hence \( x^* \in K^* \), \( Px \in \mathcal{E}K \) and \( x^*(Px) = 0 \) and therefore \( x^*(f(t,Px)) \geq 0 \), since \( f \) satisfies the boundary condition (4). Thus, we have \( x^*(f_\delta(t,x)) \geq -\eta x^*(x) - \eta^2|e| \geq \eta(\delta-\eta|e|) = 0 \), that is, \( f_\delta \) satisfies (5). Therefore, \( v' = f_\delta(t,v), \ v(0) = u_0 - \frac{\delta}{|e|}e \) has a solution \( v_\delta \) on some \([0,b] \subset \mathcal{I} \), where \( b > 0 \) depends only on a bound for \( f \) in a neighborhood of the point \((0,u_0)\), and the range of \( v_\delta \) is in \( K_\delta \). Now, let \( u \) be any solution of (1) on \([0,\beta] \) with \( \beta \in (0,b] \), and let \( \omega(t) = u(t) - v_\delta(t) \). Then \( \omega(0) \in K^0 \) and

\[
(6) \quad \omega'(t) = \tilde{f}(t,u(t)) - \tilde{f}(t,v_\delta(t)) + \frac{\delta}{|e|}\delta(t) + \frac{\delta^2}{|e|^2}e.
\]

Suppose that there exists a first time \( t_0 > 0 \) such that \( \omega(t_0) \in \mathcal{E}K \). Then we have \( x^*(\omega(t_0)) = 0 \) for some \( x^* \in K^* \) with \(|x^*| = 1\), and \( \phi(t) = x^*(\omega(t)) \) satisfies \( \phi'(t_0) \leq 0 \) since \( \omega(t) \in K^0 \) for \( t < t_0 \).

However (6) and the quasimonotonicity of \( \tilde{f} \) imply that

\[
\phi'(t_0) \geq \frac{\delta}{|e|} x^*(u(t)) + \frac{\delta^2}{|e|^2} x^*(e) > 0,
\]

a contradiction. Therefore, \( u(t) - v_\delta(t) \in K^0 \) in \([0,\beta] \). Now, \( (v_{\delta_n}) \) with \( \delta_n \to 0 \) has a uniformly convergent subsequence, the limit \( v \) of which is a solution of (1), and evidently the minimal solution.

q.e.d.

The essential point in this proof has been to construct a quasimonotone extension \( \tilde{f} \) of \( f \) such that \( v' = \tilde{f}(t,v) - \varepsilon e \), \( v(0) = u_0 - \varepsilon e \) has a solution lying near the cone and smaller than every solution of (1). The use of the metric projection \( P: \mathbb{R}^n + K \) for the definition of a quasimonotone extension is restricted to cones \( K \) with the following
Property \((\Theta_n')\). There exist \(m\) vectors \(e_i \in \mathbb{K}\backslash\{0\}\), for some \(1 \leq m \leq n\), such that \((e_i, e_j) = \delta_{i,j}\) for \(i, j = 1, \ldots, m\) and \(K \subset \text{span}\{e_1, \ldots, e_m\}\).

For an arbitrary cone \(K \subset \mathbb{R}^n\), it is easy to see that \((\mathbf{a} - P\mathbf{x}, P\mathbf{x}) = 0\) for all \(\mathbf{x} \in \mathbb{R}^n\). Hence, if \(K\) does not enjoy Property \((\Theta_n')\), it is easy to find \((\mathbf{x}, n, v) \in \mathbb{R}^n \times K \times K^*\) such that \((v, n) = 0\) but \((v, P(\mathbf{x} + n) - P\mathbf{x}) \neq 0\); consider for example \(K = \{\mathbf{x} \in \mathbb{R}^2: \mathbf{x}_2 \geq \frac{1}{2} |\mathbf{x}_1|\}\), where \(K^* = \{\mathbf{x} \in \mathbb{R}^2: \mathbf{x}_2 \geq 2 |\mathbf{x}_1|\}\). Therefore, we can not show that \(f(t, P\mathbf{x})\) is quasimonotone on \(I \times \mathbb{R}^n\). Suppose now, that \(K\) has \((\Theta_n')\), \(f\) is continuous, quasimonotone and \(f(t, 0) \in K\) on \(I\). Then we know that the IVP \(u' = f(t, u), u(0) = u_0 \in K\) has a local solution. Therefore, \(f(s, u_0) = u'(s) \in \text{span}\{e_1, \ldots, e_m\}\), i.e. we have already \(f: I \times K \rightarrow \text{span}\{e_1, \ldots, e_m\}\). In this space, \(K\) has nonempty interior and \(f(t, P\mathbf{x})\) defines a quasimonotone extension as before. Thus, we have

**Corollary 1.** Proposition 2 is true for every cone \(K \subset \mathbb{R}^n\) with Property \((\Theta_n')\).

The interesting question, whether Proposition 2 is true for every cone \(K\), remains open. Let us remark that following the results in [4], it is not difficult to consider the more general notion of mini-max solutions rather than maximal and minimal solutions as we have chosen to discuss.
III. EXISTENCE OF MINIMAL SOLUTIONS IN CASE $\dim X = \infty$

Consider again problem (1), where $\mathcal{f}: I \times K \times X$ is continuous, quasimonotone and $f(t,0) \in \mathcal{K}$ on $I$. Since we assume now that $\dim X = \infty$ we need extra conditions on $\mathcal{f}$ to guarantee local existence of solutions; see e.g. [1], [3], [5]. Some of these conditions are estimates involving measures of noncompactness, for instance the ball measure $\beta(B)$ for bounded $B \subset X$ which is defined by

$$\beta(B) = \inf\{r > 0: B \text{ can be covered by } \text{finitely many balls of radius } \leq r\}.$$ 

Let us mention that $\beta(B) = 0$ if $B$ is relatively compact; further properties can be found e.g. in [1], [5]. Since there are examples where the method applied in the proof to Proposition 2 works, let us state this result as

**Theorem 1.** Let $X$ be a real Banach space; $K \subset X$ a cone with $K^0 \neq \emptyset$; $I = [0,a] \subset \mathbb{R}$; $\mathcal{f}: I \times K \times X$ continuous, quasimonotone and such that $f(t,0) \in \mathcal{K}$ on $I$. Suppose also that the following hypotheses hold

(i) The metric projection $P: X \to K$ exists and satisfies $|Px-Py| \leq |x-y|$, $Px \leq P(x+\alpha)$ if $\alpha \in K$ and $x^*(P(x+\alpha)-Px) = 0$ whenever $x^* \in K^*$, $\alpha \in K$ and $x^*(\alpha) = 0$.

(ii) $\mathcal{f}$ maps bounded sets into bounded sets and $\beta(F(t,B)) \leq \omega(t,\beta(B))$ for $t \in I$ and $B \subset K$ bounded, where $\omega: I \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, monotone increasing in the second argument and such that the IVP $p' = \omega(t,\rho)$, $(0) = 0$ admits the trivial solution only.
(iii) If $X$ is not separable then $f$ is also uniformly continuous on bounded sets.

Then (1) has a minimal solution.

This theorem can be proved like Proposition 2. Since $P$ is nonexpansive, we have $\beta(PB) \geq \beta(B)$ and since $\omega$ is increasing in $\rho$, the quasimonotone extension $\tilde{f}$, defined by $\tilde{f}(t,x) = f(t,Px)$ satisfies conditions sufficient for existence of local solutions; see e.g. [1],[3], [5].

Simple examples, where condition (1) is satisfied and $K^o \neq \emptyset$, are $X = C(\Omega)$ with $\Omega \subset R^m$ compact, $X = C$, the space of all convergent sequences with the sup-norm, $X = L^\infty(\Omega)$ with $\Omega \subset R^m$ measurable, $X = l^\infty$ and the corresponding standard cones of nonnegative functions; here $Px$ is given by $(Px)(t) = \max(x(t),0)$ and $(Px)_n = \max(x_n,0)$ respectively.

Now, let $X$ be a real Banach space such that there exists a projectional scheme $\{X_n, P_n\}$, i.e. a sequence of finite dimensional subspaces $X_n$ of $X$ and a sequence of continuous linear projections $P_n : X \rightarrow X_n$ such that $P_n x \rightarrow x$ as $n \rightarrow \infty$, for every $x \in X$. Let $K \subset X$ be a cone such that $P_n K \subset K$ for every $n \geq 1$.

Example. Suppose $X$ has a (Schauder-) base $\{e_i : i \geq 1\}$. Consider $K = \{ \sum_{i \geq 1} x_i e_i : x_i \geq 0 \text{ for every } i \}$, $X_n = \text{span}(e_1, \ldots, e_n)$ and $P_n (\sum_{i \geq 1} x_i e_i) = \sum_{i \leq n} x_i e_i$. Then $\{X_n, P_n\}$ is a projectional scheme and $P_n K \subset K$ for every $n \geq 1$, but $K^o = \emptyset$.

Let us note that $P_n K \subset K$ implies $K_n := K \cap X_n = P_n K$. In connection with (1) we then consider the finite dimensional problems

$$(1_n') \quad v' = f_n(t,v), \quad v(0) = P_n u_0, \quad \text{where } f_n(t,*) = P_n f(t,*) \bigg|_{K_n}.$$
Evidently, \( f_n : I \times K_n \to X_n \) is continuous and \( f_n(t,0) \in K_n \) on \( I \).

Furthermore, \( f_n \) is quasimonotone w.r. to \( K_n \). In fact, \( x \in K_n \), \( s \in K_n \), \( x^k \in K_n \) and \( x^k(z) = 0 \) imply \( P_n x^k \in K^* \) and \( P_n x^k(s) = x^k(z) = 0 \), and therefore
\[
x^k(f_n(t,x) - f_n(t,x+z)) = P_n x^k(f(t,x) - f(t,x+z)) \leq 0.
\]

Finally, if \( u \) is any solution of (1) then \( \omega_n = P_n u \) satisfies
\[
(7) \quad \omega_n = f_n(t,\omega_n) + P_n(f(t,u(t)) - f(t,P_n u(t))) , \quad \omega_n(0) = P_n u .
\]

Now, if we look at the example, we see that \( K_n \) is the standard cone in \( R^k \). Therefore \((I_n')\) has a minimal solution \( \upsilon_n \). We also have that the defects \( \upsilon_n(t) = P_n(...\) in (7) are in \( K_n \). In fact, let \( x_0^k \in K_n^* \); then \( x^k \), defined by \( x^k(e_{i,i}) = x_0^k(e_{i,i}) \) for \( i < n \) and \( = 0 \) for \( i > n \), is in \( K^* \) and \( x^k(x - P_n x) = 0 \); we also have \( P_n x < x \) and therefore
\[
(8) \quad 0 \leq x^k(f(t,x) - f(t,P_n x)) = x_0^k(P_n f(t,x) - P_n f(t,P_n x)) ;
\]

since \( x_0^k \in K_n^* \) has been arbitrary, we have \( P_n f(t,x) \geq P_n f(t,P_n x) \).

Therefore, \( \omega_n(t) \geq \upsilon_n(t) \). In order to prove convergence of \((\upsilon_n')\), we need the following proposition which is special case of Theorem 1 in [3].

**Proposition 3.** Let \( X \) be a separable Banach space and \((x_n)\) a sequence of continuously differentiable functions \( x_n : I \to X \) such that
\[
|x_n'(t)| \leq c \text{ in } I \text{ for each } n . \text{ Then } \psi(t) = \beta\{X_n(t) : n \geq 1\} \text{ is absolutely continuous and } \psi'(t) \leq \beta\{x_n'(t) : n \geq 1\} \text{ a.e. in } I.
\]

**Theorem 2.** Let \( X \) be a real Banach space with a projectional scheme
\[
\{X_n^*, P_n^*\} ; \ K \subset X \text{ a cone, } I = [0,a] \subset R \ ; f : I \times K \to X \text{ continuous,}
\]
quasimonotone and such that \( f(t, 0) \in K \) on \( I \). Suppose also that the following hypotheses are fulfilled:

\((H_1)\) For each \( n \geq 1 \), \( P_n K \subset K \), \( K_n = P_n K \) has nonempty interior in \( X_n \) and property \( \theta_{m(n)} \), considered as a cone in \( \mathbb{R}^{m(n)} \), where \( m(n) = \dim X_n \).

\((H_2)\) \( P_n f(t, x) - P_n f(t, P_n x) \in K_n \), whenever \( t \in I \), \( x \in K \) and \( n \geq 1 \).

\((H_3)\) \( f \) maps bounded sets into bounded sets and \( \beta(f(t, B)) \leq \omega(t, \beta(B)) \) for \( t \in I \) and \( B \subset K \) bounded, where \( \omega : I \times \mathbb{R}^+ \to \mathbb{R}^+ \) is continuous and such that the IVP \( \rho' = \lambda \omega(t, \rho) \), \( \rho(0) = 0 \) admits only the trivial solution \( \rho(t) = 0 \); here \( \lambda = \sup_n |P_n| \). Then problem (1) has a minimal solution.

**Proof**: (1) From the previous considerations and \((H_1)\), we know that problem \((1_n)\) has a minimal solution \( u_n \) on some interval \( I_0 = [0, b] \subset I \), where \( b > 0 \) depends only on a bound for \( |f(t, x)| \) in a neighborhood of \( (0, u_0) \), provided that \( n \) is sufficiently large, say \( n \geq n_0 \), since \( P_n u_0 + u_0 \) and \( \lambda = \sup_n |P_n| < \infty \). If \( u \) is any solution of (1) on \( I \), then \( u_n = P_n u \) satisfies (7), and by \((H_2)\) we have \( P_n u(t) \geq u_n(t) \) on \( I_0 \).

(11) Let \( \psi(t) = \beta(u_n(t) : n \geq n_0) \). Since \( u_n(0) = P_n u_0 + u_0 \), we have \( \psi(0) = 0 \). Since \( (u'_n(t)) \) is uniformly bounded on \( I_0 \), and \( u'_n(t) + 0 \) as \( n \to \infty \), Proposition 3 implies

\( \psi'(t) \leq \beta(P_n f(t, u_n(t)) ; n \geq n_0) \) a.e. on \( I_0 \). By Proposition 7.2 in [1] and \((H_3)\), we therefore have

\( \psi'(t) \leq \lambda \beta(f(t, \{u_n(t); n \geq n_0\})) \leq \lambda \omega(t, \psi(t)) \) a.e. in \( I_0 \), \( \psi(0) = 0 \).
Since $\lambda$ is a uniqueness function, this implies $\psi(t) = 0$ in $I_0$, and since $(v_n)$ is equicontinuous, we find a uniformly convergent subsequence, by the Lemma of Ascoli-Arzela, the limit $v$ of which is a solution of (1), and evidently the minimal solution of (1).

\[ q.e.d. \]

It is obvious that the whole sequence $(v_n)$ converges to $v$ since there is only one minimal solution.

IV. EXISTENCE OF MAXIMAL SOLUTIONS IN CASE $\dim \mathcal{X} = \infty$

From the proof to Proposition 1 (i) it is obvious that (1) has a maximal solution if $\mathcal{K} \neq \emptyset$ and one of the conditions sufficient for local existence of solutions, mentioned in the preceding section, is satisfied. However, if $\mathcal{K} = \emptyset$ then the problem is difficult.

Suppose again that the hypotheses of Theorem 2 except $(H_2)$ are fulfilled, let $I_0 = [0, b]$ be such that every solution of (1) exists on $I_0$, and let $S$ be the set of all solutions. Then $S$ is compact in $C_K(I_0)$. In fact, let $(u_n) \subset S$ and $\psi(t) = \beta(\{u_n(t) : n \geq 1\})$. Then $\psi(0) = 0$ and $\psi'(t) \leq \omega(t, \psi(t)) \leq \lambda \omega(t, \psi(t))$ a.e. in $I_0$, since $\lambda \geq 1$ and $\omega$ is nonnegative. Therefore $\psi(t) \equiv 0$. The compactness of $S$ implies that the defects

$$y_n(t, u) = P_n(f(t, u(t)) - f(t, P_n u(t)))$$

in (7) tend to zero as $n \to \infty$, uniformly on $I_0$, and uniformly with respect to $u \in S$. In general, this does not seem to be enough to prove existence of a maximal solution by this approach. We should find $e_n \in K_n$ such that $e_n \geq y_n(t, u)$ for all $u \in S$ and $|e_n| \to 0$ as $n \to \infty$;
then we could consider the maximal solution \( \overline{\nu}_n \) of \( \nu' = f_n(t, \nu) + \varepsilon_n \), \( \nu(0) = P_n u_0 \) and we would obtain \( \overline{u} \) in this way. But in general, we can expect only \( |\varepsilon_n| < C \sqrt{n} \delta_n \) if \( |y_n(t, u)| < \delta_n \); consider e.g. \( X = \ell^2 \).

However, in case \( X = c_0 \), the space of all sequences tending to zero (with the sup-norm), we have an upper bound \( \varepsilon_n \) with \( |\varepsilon_n| = \delta_n \), namely \( \varepsilon_n \) defined by \( \varepsilon_n^{(i)} = \delta_n \) for \( i \leq n \). Let us state this curiosity as

**Theorem 3.** Let \( X = c_0 \), \( K \subset X \) the standard cone of \( X \). Let \( f \) be continuous, quasimonotone and such that \( f(t, 0) \in K \) on \( I \). Suppose also that \( f \) satisfies (H\(_3\)) in Theorem 2. Then (1) has a maximal solution.
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