SOME RECURSIVE DEFINITIONS OF THE
SHAPLEY VALUE AND OTHER LINEAR
VALUES OF COOPERATIVE TU GAMES

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Recursive definitions are given for the Shapley value, the Banzhaf value, the Least
Square values introduced by L. Ruiz, F. Valenciano and J. Zarzuelo, and the Semivalues
introduced by P. Dubey, A. Neyman and R. J. Weber. These definitions have been
suggested by the potential of the Shapley value, introduced by S. Hart and A. Mas-Colell,
which has also a recursive definition.

0. Introduction

Let \( N \) be a finite set of players, \( |N| = n \); a cooperative TU game in coalitional
form is a function \( v : P(N) \rightarrow R \), with \( v(\emptyset) = 0 \). It is well known that the set of all
games with the set of players \( N \), denoted below \( G(N) \), is a space of dimension \( 2^n - 1 \).
Let \( S \) be any coalition in \( v \in G(N) \) and denote by \( G(S) \) the space of games with the
set of players \( S \). If \( v \in G(N) \), then the restriction of \( v \) to \( S \) is a game in \( G(S) \). To
avoid a notation like \( v_S \), we shall denote the game \( v \) by \( (N, v) \), and its restriction to \( S \)
by \( (S, v) \). Denote by \( G^N \) the union of all spaces \( G(S) \), for all \( S \subseteq N \), \( S \neq \emptyset \). Then, a
value on \( G^N \) is a functional \( \Psi \) on \( G^N \) with values in \( R^s \) for all \( w \in G(S) \) and all
\( S \subseteq N \). In particular, for \( v \in G(N) \) the value \( \Psi \) gives \( s \)-vectors \( \Psi(S,v) \) for all
subgames of \( v \). Obviously, for \( i \in S \) we have in general \( \Psi_i(S,v) \neq \Psi_i(N,v) \) when
\( S \neq N \). This agrees with the game theoretic meaning of the value as a payoff: the win of
player \( i \) in the subgame \( (S,v) \) is, in general, different of the win of the same player in
the game \( (N,v) \), when \( S \neq N \). A value \( \Psi \) on \( G^N \) is a linear value if for any game
\( v \in G(N) \) which is a linear combination \( v = a v_1 + b v_2 \), with \( v_1, v_2 \in G(N) \) and \( a, b \in R \),
we have for all \( S \subseteq N \), \( S \neq \emptyset \), the equality \( \Psi(S,v) = a \Psi(S,v_1) + b \Psi(S,v_2) \).
We intend to give recursive definitions for the Shapley value (see [13]), the Banzhaf value
(see [1] and [10]), the Least Square values (see [12]) and the Semivalues (see [3]). As it
will be shown below, the proofs for these characterizations are using different tools, and
auxiliary results interesting by themselves.

1. A Recursive Definition of the Shapley Value.

One of the central problems of Game Theory, that of dividing fairly among the \( n \)
players the win \( v(N) \) of the grand coalition, got an early solution by the axiomatic
definition of a value introduced by L. S. Shapley (1953), (see [13], also [10] and [11]).
This definition led to the Shapley value formula for the value

\[
SH_i(T, v) = \sum_{S : i \in S \subseteq T} \frac{(s-1)!(t-s)!}{t!} [v(S) - v(S - \{i\})], \forall i \in T
\]

(1.1)

for all \( v \in G(N) \) and \( T \subseteq N \), where \( s = |S| \) and \( t = |T| \). The literature on the
Shapley value was already very rich in late 80's, as shown by the volume edited by A. E.
Roth (see [11]), when another characterization of the value has been proposed by S. Hart and A. Mas-Colell (see [9] and [11]). They introduced the concept of potential of the Shapley value that led to the proof of a new axiomatization, in which the Shapley value is the unique value satisfying the so-called reduced game axiom and standardness axiom. In this section, we give a recursive definition of the Shapley value and use the Hart/Mas-Colell potential for the proof. To make the paper self contained let us define the potential and state a major result to be used later. For any game \( v \in G(N) \), the potential of the Shapley value is the functional \( P \) recursively defined on \( G^N \) by

\[
P(S, v) = s^{-1} \left[ v(S) + \sum_{k \in S} P(S - \{k\}, v) \right], \forall S \subseteq N, |S| > 1,
\]

where \( s = |S| \). Obviously, the one-to-one correspondence \( P \) on \( G^N \), may be expressed by

\[
v(S) = \sum_{k \in S} [P(S, v) - P(S - \{k\}, v)], \forall S \subseteq N,
\]

where \( P(\emptyset, v) = 0 \). Formulas (1.2) and (1.3) will be used later, together with the following major result from [9]:

**HM-Theorem:** If \( P \) is the potential of the Shapley value defined by (1.2), or (1.3), then we have for each \( i \in S \subseteq N \),

\[P(S, v) - P(S - \{i\}, v) = SH_i(S, v), \forall S \subseteq N,
\]

and all \( v \in G(N) \).

Let \( \Psi \) be the value on \( G^N \) defined recursively by

\[
\Psi_i(\{i\}, v) = v(\{i\}), \forall i \in N,
\]

for each singleton, and

\[
\Psi_i(S, v) = s^{-1} \left[ \sum_{k \in S - \{i\}} \Psi_i(S - \{k\}, v) + v(S) - v(S - \{i\}) \right], \forall i \in S,
\]

for each subgame \( (S, v) \) with \( |S| > 1 \), where \( v \in G(N) \).

For example, for the game \( v \in G(\{1, 2, 3\}) : v(1) = 100, v(2) = 200, v(3) = 300, v(1, 2) = 400, v(1, 3) = 500, v(2, 3) = 600 \), and \( v(1, 2, 3) = 900 \), we get for the singletons

\[
\Psi_1(\{1\}, v) = 100, \quad \Psi_2(\{2\}, v) = 200, \quad \Psi_3(\{3\}, v) = 300,
\]

then for the coalitions of size two

\[
\Psi_1(\{1, 2\}, v) = 150, \quad \Psi_2(\{1, 2\}, v) = 250, \\
\Psi_1(\{1, 3\}, v) = 150, \quad \Psi_3(\{1, 3\}, v) = 350,
\]
\[ \Psi_2(\{2, 3\}, v) = 250, \quad \Psi_3(\{2, 3\}, v) = 350, \]
and for the grand coalition
\[ \Psi_1(\{1, 2, 3\}, v) = 200, \quad \Psi_2(\{1, 2, 3\}, v) = 300, \quad \Psi_3(\{1, 2, 3\}, v) = 400. \]
Note that (1.5) allows the computation of the payoffs for player \( i \) in all subgames of \( v \) without using the payoffs for the other players. Note also that (1.4) and (1.5) define uniquely a value.

**Theorem 1:** The value \( \Psi \) recursively defined on \( G^N \) by (1.4) and (1.5) is the Shapley value.

**Proof:** By induction over the size of \( S \); if \( S = \{i\} \), then we have
\[ SH_i(\{i\}, v) = v(\{i\}), \]
which gives \( \Psi_i(\{i\}, v) = SH_i(\{i\}, v) \). For a fixed \( i \in N \), assuming that \( \Psi_i(S - \{k\}, v) = SH_i(S - \{k\}, v) \), \( \forall k \in S - \{i\} \), we compute the right hand side in (1.5). From the HM-Theorem, we get
\[
\sum_{k \in S - \{i\}} SH_i(S - \{k\}, v) = \sum_{k \in S - \{i\}} [P(S - \{k\}, v) - P(S - \{i, k\}, v)]. \tag{1.6}
\]
From (1.3) written for \( v(S) \) and \( v(S - \{i\}) \), we get
\[
v(S) - v(S - \{i\}) = sP(S, v) - \sum_{k \in S} P(S - \{k\}, v) - \left[(s - 1)P(S - \{i\}, v) - \sum_{k \in S - \{i\}} P(S - \{i, k\}, v) \right] = \tag{1.7}
\]
\[
= s[P(S, v) - P(S - \{i\}, v)] - \sum_{k \in S - \{i\}} [P(S - \{k\}, v) - P(S - \{i, k\}, v)].
\]
By adding up (1.6) and (1.7), we obtain
\[
\sum_{k \in S - \{i\}} SH_i(S - \{k\}, v) + v(S) - v(S - \{i\}) = s[P(S, v) - P(S - \{i\}, v)] = sSH_i(S, v),
\]
where the last equality follows again from the HM-Theorem. Hence, formula (1.5) gives \( \Psi_i(S, v) = SH_i(S, v) \) and the induction proves the theorem.

Note that we can write formula (1.5) in matrix form as follows: introduce the \( s \)-vector \( M = (M_i) \) of marginal contributions, \( M_i = v(S) - v(S - \{i\}) \), \( \forall i \in S \), then the matrix \( \overline{\Psi} = (\overline{\Psi}_{ik}) \) of order \( s \), with \( \overline{\Psi}_{ik} = \Psi_i(S - \{k\}, v) \), if \( i \neq k \) and \( \overline{\Psi}_{ik} = 0 \), if \( i = k \), for all pairs \( i, k \in S \), and the \( s \)-vector \( \overline{\Psi} = (\Psi_i(S, v)) \). Then, we rewrite (1.5) as
\[
\overline{\Psi} = s^{-1}(\overline{\Psi} \cdot e + M) \tag{1.8}
\]
where \( e \) is the \( s \)-vector with all entries equal to one. For example, for \( S = \{1, 2, 3\} \) we have
\[
\begin{pmatrix}
\Psi_1(S, u) \\
\Psi_2(S, u) \\
\Psi_3(S, u)
\end{pmatrix} = \frac{1}{3} \begin{pmatrix}
1 & \Psi_1(13, u) & \Psi_1(12, u) \\
\Psi_2(23, u) & 0 & \Psi_2(12, u) \\
\Psi_3(23, u) & \Psi_3(13, u) & 0
\end{pmatrix} \begin{pmatrix}
1 \\
3 \\
1
\end{pmatrix} + \begin{pmatrix}
v(S) - v(23) \\
v(S) - v(13) \\
v(S) - v(12)
\end{pmatrix}.
\]

We conclude that the characterization offered by Theorem 1 is allowing us to consider that (1.4) and (1.5) give a recursive definition of the Shapley value.

2. **A Recursive Definition of the Banzhaf Value.**

Another linear value has been introduced by J. F. Banzhaf III (see [1] and [10]) and it is given by the formula

\[
B(T, u) = \frac{1}{2^{T-1}} \sum_{S: i \in S \subseteq T} [v(S) - v(S - \{i\})], \forall i \in T, \tag{2.1}
\]

An axiomatic definition of the Banzhaf value may be extracted from the axiomatic definition of the Banzhaf Power Index given by P. Dubey and L. S. Shapley (1979), (see [7]). Briefly speaking, the axioms defining the Banzhaf value may be obtained from the Shapley value axioms by replacing the efficiency by a similar axiom. This fact suggests that the above given recursive definition of the Shapley value may lead to a similar definition for the Banzhaf value. To prove that such a recursive definition can be given we may use as above the potential approach, by using the potential of the Banzhaf value introduced by the author (see [3]). However, we shall use a more elegant approach here, by using the relationship between the Shapley value and the Banzhaf value proved also by the author (see [4]). Let us first describe this relationship. Let \( v \in G(N) \) and denote as above for all coalitions \( S \subseteq N \) the corresponding subgames \( (S, v) \). Consider the function

\[
\pi(S, v) = \sum_{i \in S} B_i(S, v), \forall S \subseteq N, \tag{2.2}
\]

introduced in [3]. Obviously, (2.2) defines uniquely a game \( \pi \in G(N) \) associated with \( v \), that we used to call the Power Game of \( v \), the function giving an one-to-one correspondence, as shown in [3]. The major result in [4] was:

**D-Theorem:** If \( \pi \in G(N) \) is the Power Game of \( v \in G(N) \) then we have

\[
B(S, v) = SH(S, \pi), \forall S \subseteq N, \tag{2.3}
\]

where \( B \) and \( SH \) are the Banzhaf and the Shapley value, respectively. In words, the Banzhaf value of \( v \) is the Shapley value of the Power Game of \( v \) and this is true for all subgames of \( v \). Taking into account (2.3), we shall be able to prove:

**Theorem 2:** The value \( \chi \) recursively defined by

\[
\chi_i(\{i\}, v) = v(\{i\}), i \in N, \tag{2.4}
\]

for each singleton, and
\[ \chi_i(S, v) = s^{-1} \left[ \sum_{k \in S - \{i\}} \chi_i(S - \{k\}, v) + \pi(S, v) - \pi(S - \{i\}, v) \right], \forall i \in S, \quad (2.5) \]

for each \( S \subseteq N, |S| > 1 \), where \( \pi \) is the Power Game of \( v \), is the Banzhaf value.

Proof: By induction over the size of \( S \); if \( S = \{i\} \), then we have \( B_i(\{i\}, v) = v(\{i\}) \), which gives \( \chi_i(\{i\}, v) = B_i(\{i\}, v) \). For a fixed \( i \in N \), assuming that \( \chi_i(S - \{k\}, v) = B_i(S - \{k\}, v) \), \( \forall k \in S - \{i\} \), we compute the right hand side in (2.5). From (2.3) we get via Theorem 1 that

\[
\begin{align*}
    s^{-1} \left[ \sum_{k \in S - \{i\}} B_i(S - \{k\}, v) + \pi(S, v) - \pi(S - \{i\}, v) \right] &= \\
    &= s^{-1} \left[ \sum_{k \in S - \{i\}} SH_i(S - \{k\}, \pi) + \pi(S, v) - \pi(S - \{i\}, v) \right] = \\
    &= SH_i(S, \pi) = B_i(S, v).
\end{align*}
\]

Hence, formula (2.5) gives \( \chi_i(S, v) = B_i(S, v) \) and the induction proves the theorem. By Theorem 2, the recursive definition of the Banzhaf value is

\[ B_i(\{i\}, v) = v(\{i\}), \forall i \in N, \quad (2.6) \]

for each singleton, and

\[ B_i(S, v) = s^{-1} \left[ \sum_{k \in S - \{i\}} B_i(S - \{k\}, v) + \pi(S, v) - \pi(S - \{i\}, v) \right], \forall i \in S, \quad (2.7) \]

for each \( S \subseteq N, |S| > 1 \), where \( \pi \) is the Power Game of \( v \). Notice that this does not seem to be a good recursive definition because according to (2.2) we have also the components of \( B \) in the right hand side in \( \pi \). However, this objection may be removed if the last two terms in the bracket are expressed in terms of \( v \), by means of the formula

\[ \pi(S, v) = 2^{1-s} \cdot \sum_{C \subseteq S} (2c - s)v(C), \forall S \subseteq N, \quad (2.8) \]

proved by the author (see [3]). Indeed, from (2.8) one can prove a formula for the marginal of \( i \) in the Power Game, precisely

\[ \pi(S, v) - \pi(S - \{i\}, v) = 2^{1-s} \cdot \sum_{C : i \in C \subseteq S} (2c - s)[v(C) - v(C - \{i\})] \quad (2.9) \]

which is valid for all \( S \subseteq N \), and \( i \in S \). In this way, we get from (2.9) a more attractive form of (2.7):
\[ \begin{align*}
B_i(S, v) &= s^{-1} \left\{ \sum_{k \in S - \{i\}} B_i(S - \{k\}, v) + \\
& \quad + 2^{1-s} \sum_{C : i \in C \subseteq S} (2c - s) [v(C) - v(C - \{i\})] \right\}.
\end{align*} \]

For example, for \( S = \{1, 2\} \), taking into account (2.7) we have
\[ B_1(\{1, 2\}, v) = 2^{-1} [B_1(\{1\}, v) + v(\{1, 2\}) - v(\{2\})] \]
then for \( S = \{1, 3\} \) we have
\[ B_1(\{1, 3\}, v) = 2^{-1} [B_1(\{1\}, v) + v(\{1, 3\}) - v(\{3\})] \]
and from these we get for \( S = \{1, 2, 3\} \):
\[ B_1(\{1, 2, 3\}, v) = 3^{-1} \left\{ B_1(\{1, 2\}, v) + B_1(\{1, 3\}, v) + 2^{-2} \left[ 3 [v(\{1, 2, 3\}) - v(\{2, 3\})] \right. \right. \]
\[ \left. \left. + v(\{1, 2\}) - v(\{2\}) + v(\{1, 3\}) - v(\{3\}) - v(\{1\}) \right) \right\} = \\
= 2^{-2} [v(\{1\}) + v(\{1, 2\}) - v(\{2\}) + v(\{1, 3\}) - v(\{3\}) + v(\{1, 2, 3\}) - v(\{2, 3\})] \]

3. **A Recursive Definition of the Extended Least Square Values.**

In [12], L. Ruiz, F. Valenciano and J. Zarzuelo have introduced new linear values for TU games, called Least Square values. This is a family of values defined as follows:

Consider a game \( v \in G(N) \) and any \( x \in \mathbb{R^n} \), and denote as usual the excesses of various coalitions by \( e(S, x) = v(S) - x(S) \), \( \forall S \subseteq N \); notice that if \( x \) is a preimputation, that is \( x(N) = v(N) \), then the average of excesses does not depend on \( x \), precisely
\[ \overline{e}(v) = \frac{1}{2^n - 1} \sum_{S \subseteq N} e(S, x) = \frac{1}{2^n - 1} \left[ \sum_{S \subseteq N} v(S) - 2^{n-1} v(N) \right]. \]

Let \( m(1), m(2), \ldots, m(n - 1) \) be non negative numbers, with some \( m(s) > 0 \), and consider the optimization problem: minimize
\[ \sum_{S \subseteq N} m(s) [e(S, x) - \overline{e}(v)]^2, \]
subject to
\[ x(N) = v(N). \quad (3.1) \]

Ruiz/Valenciano/Zarzuelo have shown that this optimization problem has a unique optimal solution depending on the parameters \( m(s) \) given by
\[ L S_i(N, v, m) = n^{-1} v(N) + a_i^{-1} (a_i - n^{-1} \sum_{j \in N} a_j), \forall i \in N, \quad (3.2) \]
where
\[ \alpha(n) = \sum_{s=1}^{s=n-1} m(s) \left( \begin{array}{c} n-2 \\ s-1 \end{array} \right), \quad a_i = \sum_{S:i \in S \subseteq N} m(S) v(S), \forall i \in N. \] (3.3)

This is a family of linear values on the space \( G(N) \), depending on \( n - 1 \) parameters. To be able to discuss consistency problems, they extended this definition to a value on \( G^N \), the union of spaces \( G(S), \forall S \subseteq N \), as follows. For any set of parameters \( m(s) \), denoted now \( m^n(s) = m(s), s = 1, 2, \ldots, n - 1 \), introduce the parameters \( m^{n-1}(s') \) by

\[ m^{n-1}(s') = m^n(s') + m^n(s' + 1), s' = 1, 2, \ldots, n - 2. \] (3.4)

Note that if \( m^n(s) \) are normalized, i.e. \( \alpha(n) = 1 \), then \( m^{n-1}(s') \) are also normalized. Now the parameters \( m^{n-1}(s') \) can be used to obtain the Least Square values on the vector space \( G(S) \) with |\( S \)| = \( n - 1 \), via the optimization problem similar to (3.1). This process may continue up to the two person coalitions where there is a unique parameter equal to 1 and there is a unique LS-value in the family, the Shapley value. It is enough to take the LS-values of the singletons equal to the individual worth to get the LS-values defined on \( G^N \). For an axiomatization of LS-values, the consistency relative to a reduced game of Davis/Maschler type and other properties see [12].

In [5], the author has shown other properties of LS-values and among them a new type of consistency relative to a reduced game of Hart/Mas-Colell type, which involved another family of values called Extended Least Square values: This is a family depending on \( n \) parameters which for a certain value of the new parameter reduces to the LS-values on \( G(N) \). In the following, we introduce the ELS-values by means of an optimization problem similar to (3.1) and give a recursive definition of these values.

Consider a game \( v \in G(N) \) and any \( x \in R^n \); let \( \gamma(n) \) be any positive number and notice that if \( x \) satisfies \( x(N) = \gamma(n) v(N) \), then the average of excesses does not depend on \( x \), precisely

\[ \bar{v}(v) = \frac{1}{2^{n-1}} \left( \sum_{S \subseteq N} v(S) - 2^{n-1} \gamma(n) v(N) \right) \] (3.5)

Let \( m(s), s = 1, 2, \ldots, n - 1 \), be nonnegative numbers not all zero and consider the optimization problem: minimize

\[ \sum_{S \subseteq N} m(S) [e(S, x) - \bar{v}(v)]^2, \]

subject to

\[ x(N) = \gamma(n) v(N). \] (3.6)

Now, for any values of the \( n \) parameters, the computation done by Ruiz/Valenciano/Zarzuelo may be repeated to show that (3.6) has the unique optimal solution given by
\[ ELS_i(N, v, m) = n^{-1} \gamma(n) v(N) + \alpha^{-1} (a_i - n^{-1} \sum_{j \in N} a_j), \quad \forall i \in N, \tag{3.7} \]

where \( \alpha \) and \( a_i, \forall i \in N \), are exactly the same as those shown in (3.3). A family of values on \( G(N) \) has been defined by (3.7), the \( LS \)-values being obtained for \( \gamma(n) = 1 \). Therefore (3.7) have been called Extended Least Square values. However, to define a value on \( G^N \) we followed a different procedure than the one explained above for \( LS \)-values. Before explaining this procedure we shall give a result; some notations are needed: Let the average worth of coalitions of size \( s \) be denoted by

\[ v_s = \binom{n}{s}^{-1} \sum_{S \subseteq N, |S| = s} v(S), \quad s = 1, 2, \ldots, n - 1, \tag{3.8} \]

and the average worth of coalitions of size \( s \) which do not contain the player \( i \) be denoted for each \( i \in N \) by

\[ v'_s = \binom{n-1}{s}^{-1} \sum_{S \subseteq N, |S| = s, i \notin S} v(S), \quad s = 1, 2, \ldots, n - 1, \tag{3.9} \]

These averages have been used by the author in [2] to show that the Shapley value can be expressed as

\[ SH_i(N, v) = n^{-1} v(N) + \sum_{s=1}^{s=n-1} \frac{v_s - v'_s}{s}, \quad \forall i \in N \tag{3.10} \]

Finally, introduce new nonnegative parameters by

\[ \gamma(s) = \alpha^{-1} (n - 1) \binom{n-2}{s-1} m(s), \quad s = 1, 2, \ldots, n - 1, \]

notice that the sum of these parameters makes \( n - 1 \), taking into account (3.3). Then, the proof of Theorem 1 in [5] can be used to show that we have:

**Theorem 3**: For the Extended Least Square values \( ELS(N, v, m) \) associated with parameters \( \gamma(n) \) and \( m(s), s = 1, 2, \ldots, n - 1 \), there exist parameters \( \gamma(s) \) with \( \sum_{s=1}^{n-1} \gamma(s) = n - 1 \), such that

\[ ELS_i(N, v) = n^{-1} \gamma(n) v(N) + \sum_{s=1}^{s=n-1} \gamma(s) \frac{v_s - v'_s}{s}, \quad \forall i \in N \tag{3.11} \]
In [5], the ELS-values were defined by this formula (see Def. 11). By abusing the notation we shall denote the ELS-values on $G(N)$ by $ELS(N, u, \gamma)$, where $\gamma \in R^N$ has the sum of the first $n - 1$ components equal to $n - 1$. From Theorem 3 follows:

**Corollary 4:** Let $\gamma$ be a nonnegative $n$-vector with the sum of the first $n - 1$ components equal to $n - 1$ and $\gamma(n) > 0$; then we have for all $v \in G(N)$:

$$ELS(N, u, \gamma) = SH(N, v_\gamma),$$

where $v_\gamma \in G(N)$ is defined by

$$v_\gamma(S) = \gamma(s)v(S), \ \forall S \subseteq N.$$  \hspace{1cm} (3.13)

**Proof:** This follows from (3.11) and (3.13), taking into account (3.10).

Now, we extend the ELS-values from $G(N)$ to $G^N$ as follows: for any nonnegative $n$-vector $\gamma$ with the sum of the first $n - 1$ components equal $n - 1$ and $\gamma(n) > 0$, and any $u \in G(T)$, $T \subseteq N$, $|T| \geq 2$, define on $G(T)$ the value $ELS(T, u, \gamma)$ by

$$ELS_t(T, u, \gamma) = t^{-1}\gamma(t)u(T) + \sum_{s=1}^{s=t-1} \gamma(s) \frac{u_s-u^i_s}{s}, \ \forall i \in N.$$  \hspace{1cm} (3.14)

where $u_s$ and $u^i_s$, $\forall i \in T$, $s = 1, 2, \ldots, t - 1$, are the corresponding averages per capita similar to (3.8) and (3.9).

Note that this applies to any subgame $(T, u)$ of $(N, v)$ and in this case it is a kind of restriction to $T$ because in (3.14) only the first $t$ components of $\gamma$ are used. We still have:

**Corollary 4':** Let $\gamma$ be a nonnegative $n$-vector with the sum of the first $n - 1$ components equal to $n - 1$ and $\gamma(n) > 0$; then we have for all $v \in G(N)$ and all $T \subseteq N$:

$$ELS(T, u, \gamma) = SH(T, v_\gamma),$$

where $v_\gamma \in G(N)$ is defined by (3.13).

Now we are able to give a recursive definition of ELS-values due to the following characterization:

**Theorem 5:** Let $\gamma \in R^N_+$ with the sum of the first $n - 1$ components equal to $n - 1$ and $\gamma(n) > 0$. For any $v \in G^N$, the value recursively defined by

$$\mathcal{K}_t(\{i\}, v, \gamma) = \gamma(1)v(\{i\}), \ \forall i \in N,$$  \hspace{1cm} (3.16)

for each singleton, and
\[ \mathcal{K}_i(S, v, \gamma) = s^{-1} \left[ \sum_{k \in S - \{i\}} \mathcal{K}_i(S - \{k\}, v, \gamma) + \gamma(s)v(S) - \gamma(s - 1)v(S - \{i\}) \right], \forall i \in S, \]

(3.17)

for each \( S \subseteq N, |S| > 1 \), is the Extended Least Square value.

**Proof:** By induction over the size of \( S \). If \( S = \{i\} \), then \( \mathcal{K}_i(\{i\}, v, \gamma) = ELS(\{i\}, v, \gamma) \). Assume that the statement holds for all coalitions of size \( s - 1 \) and prove it for a coalition \( S \) of size \( s \). Compute the bracket in (3.17):

\[
\sum_{k \in S - \{i\}} \mathcal{K}_i(S - \{k\}, v, \gamma) + \gamma(s)v(S) - \gamma(s - 1)v(S - \{i\}) =
\]

\[
= \sum_{k \in S - \{i\}} ELS_i(S - \{k\}, v, \gamma) + \gamma(s)v(S) - \gamma(s - 1)v(S - \{i\}) =
\]

\[
= \sum_{k \in S - \{i\}} SH_i(S - \{k\}, v, \gamma) + v_\gamma(S) - v_\gamma(S - \{i\}) = sSH_i(S, v, \gamma),
\]

where the first equality follows from the induction assumption, the second equality follows from Corollary 4 and the last one is a consequence of the recursive definition of the Shapley value. Hence, we have \( \mathcal{K}_i(S, v, \gamma) = SH_i(S, v, \gamma) = ELS_i(S, v, \gamma), \forall i \in S \). The result follows by induction.

**Example:** As \( ELS(N, v, \gamma) = LS(N, v, \gamma) \) if \( \gamma(n) = 1 \), we have the possibility of computing a component of the \( LS \)-value by using the recursive definition of the \( ELS \)-value. Consider a 3-person game and suppose that the parameters are \( \gamma(1) = 1/2, \gamma(2) = 3/2, \gamma(3) = 1 \), and we would like to compute \( LS_1(N, v, \gamma) \). We have

\[ ELS_1(\{1\}, v, \gamma) = \frac{1}{2} v(1), \]

\[ ELS_1(\{1, 2\}, v, \gamma) = \frac{1}{2} \left[ ELS_1(\{1\}, v, \gamma) + \gamma(2)v(1, 2) - \gamma(1)v(2) \right] = \]

\[ = \frac{1}{2} \left[ \frac{1}{2} v(1) + \frac{3}{2} v(1, 2) - \frac{1}{2} v(2) \right] = \frac{1}{4} \left[ 3v(1, 2) + v(1) - v(2) \right], \]

\[ ELS_1(\{1, 3\}, v, \gamma) = \frac{1}{4} \left[ 3v(1, 3) + v(1) - v(3) \right], \]

\[ LS_1(\{1, 2, 3\}, v, \gamma) = ELS_1(\{1, 2, 3\}, v, \gamma) = \]

\[ = \frac{1}{3} \left[ ELS_1(\{1, 2\}, v, \gamma) + ELS_1(\{1, 3\}, v, \gamma) + \gamma(3)v(1, 2, 3) - \gamma(2)v(2, 3) \right] = \]

\[ = \frac{1}{3} v(1, 2, 3) + \frac{1}{4} \left[ v(1, 2) + v(1, 3) - 2v(2, 3) \right] + \frac{1}{12} \left[ 2v(1) - v(2) - v(3) \right]. \]

Note that the result may be compared to the expression we get by using directly formula (3.7) with the above given \( \gamma \), or formula (3.2); from \( \gamma(1) = 1/2 \) and \( \gamma(2) = 3/2 \), we can get normalized \( m(1) = 1/4 \) and \( m(2) = 3/4 \), we have \( \alpha = 1 \), and we use one of those formulas.

Note that in [5] a potential of the \( ELS \)-value has been also defined, so that we may give a direct proof of Theorem 5 similar to the proof of Theorem 1.
4. **A Recursive Definition of the Semivalues.**

In [8], P. Dubey, A. Neyman and R. J. Weber have introduced the family of semivalues; for cooperative TU games they may be defined as follows: let $p \in \mathbb{R}^n$ be a nonnegative vector satisfying

$$\sum_{s=1}^{s=n} \left( \frac{n-1}{s-1} \right) p(s) = 1;$$

then $\rho : G(N) \rightarrow \mathbb{R}^n$ given by

$$\rho_i(N, v, p) = \sum_{S : i \in S \subseteq N} p(s)[v(S) - v(S - \{i\})], \forall i \in N,$$  \hspace{1cm} (4.2)

is a semivalue on $G(N)$. In [8], the semivalues have been defined axiomatically for a more general class of games, but for TU games the definition may be stated as above (see Lemma, p. 123). Further, to extend a semivalue to $G^N$, denote the parameters $p(s)$ by $p^n(s) = p(s), s = 1, 2, \ldots, n$, as we did in the previous section, and introduce the parameters $p^{n-1}(s')$ by

$$p^{n-1}(s') = p^n(s') + p^n(s' + 1), s' = 1, 2, \ldots, n - 1.$$  \hspace{1cm} (4.3)

Note that if (4.1) holds, then the similar normalization condition is satisfied for $p^{n-1}(s')$. Now, the parameters $p^{n-1}(s')$ were used to obtain the semivalues on the vector space $G(S)$ with $|S| = n - 1$, via a formula similar to (4.2). This process may continue up to two person coalitions; it is enough to take the semivalue of a singleton equal to the individual worth to get the family of semivalues defined on $G^N$. For the axiomatization of semivalues and the definition for more general classes of games, see [8]; note that $G^N$ has there another meaning.

To be able to prove a recursive definition of semivalues on $G^N$ we need an auxiliary result. As in the case of the Banzhaf value, the efficiency axiom does not hold for semivalues; in fact, the Banzhaf value is the semivalue corresponding to the value of parameters $p^n(s) = 2^{1-n}, s = 1, 2, \ldots, n$, as it is easily seen from (4.2); condition (4.1) obviously holds in this case. We have been able to prove Theorem 2 based upon a relationship between the Shapley value and the Banzhaf value; in the general case of a semivalue the same method of proof may be used but first we shall give a combinatorial proof. Anyway, we need the concept of Power Game. Let $\rho$ be a semivalue as defined in (4.1) and (4.2) and for $v \in G(N)$ and any $T \subseteq N$; define the number

$$\pi(T, v, p^\dagger) = \sum_{i \in T} \rho_i(T, v, p^\dagger),$$  \hspace{1cm} (4.4)

where $p^\dagger$ is obtained by successive application of rule (4.3). The functional $\pi : G(N) \rightarrow G(N)$ defined in (4.4) is generating a game, which will be called the Power
Game of \( v \). For example, if \( N = \{1, 2, 3\} \) and we take \( p^3(1) = \frac{1}{8}, \ p^3(2) = \frac{1}{4}, \ p^3(3) = \frac{3}{8} \), which obviously satisfy (4.1) and compute by (4.3) the parameters

\[
p^2(1) = \frac{3}{8}, \ p^2(2) = \frac{5}{8} \]

which again satisfy (4.1), we have for any game \( v \in G(N) \): for singletons

\[
\pi(\{1\}) = v(\{1\}), \ \pi(\{2\}) = v(\{2\}), \ \pi(\{3\}) = v(\{3\}),
\]

then, from

\[
\rho_1(\{1, 2\}, v, p^2) = \frac{3}{8} v(\{1\}) + \frac{5}{8} [v(\{1, 2\}) - v(\{2\})],
\]

we get

\[
\pi(\{1, 2\}, v, p^2) = \frac{5}{4} v(\{1, 2\}) - \frac{1}{4} [v(\{1\}) + v(\{2\})],
\]

and similarly

\[
\pi(\{1, 3\}, v, p^2) = \frac{5}{4} v(\{1, 3\}) - \frac{1}{4} [v(\{1\}) + v(\{3\})],
\]

\[
\pi(\{2, 3\}, v, p^2) = \frac{5}{4} v(\{2, 3\}) - \frac{1}{4} [v(\{2\}) + v(\{3\})];
\]

finally from \( \rho_1(\{1, 2, 3\}, v, p^3), \ \rho_2(\{1, 2, 3\}, v, p^3), \) and \( \rho_3(\{1, 2, 3\}, v, p^3) \) we get

\[
\pi(\{1, 2, 3\}, v, p^3) = \frac{9}{8} v(\{1, 2, 3\}) + [v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\})] - \frac{3}{8} [v(\{1\}) + v(\{2\}) + v(\{3\})].
\]

All expressions for \( \rho_i \) have been computed by (4.2). From this example we guess that \( \pi \) has to be an expression in the sums of the worth of coalitions of each size. We need the formula which contains also the coefficients of the sums; to write a nice expression for all coefficients we take \( p^t(t + 1) \) as an arbitrary number.

**Lemma 6**: For any game \( v \in G(N) \), the Power Game of \( v \) defined by (4.4) for a given \( n \)-vector \( p \) of parameters is given by

\[
\pi(T, v, p^t) = \sum_{s=1}^{s=t} [sp^t(s) - (t - s)p^t(s + 1)] \sigma_s, \forall T \subseteq N,
\]

(4.5)

where \( \sigma_s \) is the sum of the worth of coalitions of size \( s \) in the subgame \( (T, v) \). Moreover, the functional \( \pi \) is an one-to-one correspondence from \( G(N) \) to \( G(N) \), if \( p^t(t) > 0 \).

**Proof**: If in (4.2) we replace \( N \) by \( T \) and \( p \) by \( p^t \) and we make the sum (4.4), then for each \( s, 1 \leq s \leq n \), any coalition \( S \) which has the size \( s \) occurs with the coefficient \( p^t(s) \) in all \( s \) components \( \rho_i \) with \( i \in S \) and with the coefficient \( -p^t(s + 1) \) in all \( t - s \) components \( \rho_i \) with \( i \in T - S \). This proves the formula.

If we return to our 3-person game above we get

\[
\pi(\{1, 2, 3\}, v, p^3) = [p^3(1) - 2p^3(2)] \sigma_1 + [2p^3(2) - p^3(3)] \sigma_2 + 3p^3(3) \sigma_3,
\]

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which for \( p^3 = \left(\frac{1}{8}, \frac{1}{4}, \frac{3}{8}\right) \) gives the number computed above.

Note that formula (4.5) is a general formula which gives among other particular expressions the formula (2.8) for the Power Game of the Banzhaf value, when we take \( p^t(s) = 2^{1-t}, s = 1, 2, \ldots, t \). Obviously, we may separate in the right hand side of (4.5) the term for \( s = t \) and move all other terms in the left hand side. We get

\[
v(T) = (tp^t(t))^{-1}\{\pi(T, v, p^t) - \sum_{s=1}^{s=t-1} [sp^t(s) - (t-s)p^t(s+1)]\sigma_s\},
\]

a recursive formula which allows the computation of \( v \) whenever the Power Game is given. Hence, the correspondence done by \( \sigma \) is an one-to-one correspondence.

Lemma 6 is allowing us to prove a recursive definition for the semivalues:

**Theorem 7**: Let \( p \in \mathbb{R}^n \) be a nonnegative vector satisfying (4.1) and define the sequence of vectors \( p^t \in \mathbb{R}^t \) by (4.3), where \( p^n = p \). For any \( u \in \mathbb{G}^N \), the value recursively defined by

\[
\rho_i(\{i\}, u, p^1) = u(\{1\}), \quad \forall i \in N, \quad (4.6)
\]

and

\[
\rho_i(T, v, p^t) = \frac{1}{t} \left[ \sum_{k \in T - \{i\}} \rho_i(T - \{k\}, v, p^{t-1}) + \pi(T, v, p^t) - \pi(T - \{i\}, v, p^{t-1}) \right] \quad (4.7)
\]

for all \( T \subseteq N \) with \( |T| > 1 \), where \( \pi \) is the Power Game of \( v \), is the semivalue with the parameters derived from \( p \in \mathbb{R}^n \) by (4.3).

**Proof**: As it is obvious that for \( |T| = 1 \) the result holds, we shall be proving that the semivalues satisfy (4.7). Notice that we may separate the parameters by writing (4.7) as

\[
\rho_i(T, v, p^t) - \frac{1}{t} \pi(T, v, p^t) = \frac{1}{t} \left[ \sum_{k \in T - \{i\}} \rho_i(T - \{k\}, v, p^{t-1}) - \pi(T - \{i\}, v, p^{t-1}) \right].
\]

We take a fixed \( i \) and we compute the coefficients of \( u(S) \) in the two sides to see that they are equal taking into account the relationship between the parameters. In the left hand side, by Lemma 6, the coefficient \( L_i(S) \) of \( u(S) \) is

\[
L_i(S) = \begin{cases} 
\frac{1}{t} [sp^t(s) - (t-s)p^t(s+1)] & \text{if } i \in S \\
-p^t(s+1) - \frac{1}{t} [sp^t(s) - (t-s)p^t(s+1)] & \text{if } i \notin S
\end{cases} \quad (4.8)
\]

because \( u(S) \) has the coefficient \( p^t(s) \) in \( \rho_i(T, v, p^t) \) if \( i \in S \) and \( -p^t(s+1) \) if \( i \notin S \). To compute the coefficient \( R_i(S) \) of \( u(S) \) in the right hand side, notice that we have to consider first the sum in the bracket. If \( i \in S \), then \( u(S) \) enters with coefficient
\( p^t(s) \) in as many terms as the number of coalitions \( T - \{k\} \) which contain \( S \), that is \( t - s \), hence the coefficient will be \((t - s)p^t-1(s)\) and if \( i \notin S \), then \( v(S) \) enters with coefficient \(-p^t-1(s+1)\) in as many terms as the number of coalitions \( T - \{k\} \) which contain \( S \cup \{i\} \), that is \( t - s - 1 \), hence the coefficient will be \(-(t - s - 1)p^t-1(s+1)\). Secondly, we have to consider \( \pi(T - \{i\}, v, p^t-1) \) to which we apply also Lemma 6. If \( i \in S \), then \( v(S) \) does not occur in \( \pi(T - \{i\}, v, p^t-1) \) while if \( i \notin S \) by the Lemma we get the coefficient equal to \(-[sp^t-1(s) - (t - s - 1)p^t-1(s+1)]\). Together, we get

\[
R_i(S) = \begin{cases} \frac{t - s}{t} p^t-1(s) & \text{if } i \in S \\ -\frac{s}{t} p^t-1(s) & \text{if } i \notin S. \end{cases} \tag{4.9}
\]

A simple algebraic computation in (4.8) gives

\[
L_i(S) = \begin{cases} \frac{t - s}{t} [p^t(s) + p^t(s+1)] & \text{if } i \in S \\ -\frac{s}{t} [p^t(s) + p^t(s+1)] & \text{if } i \notin S. \end{cases} \tag{4.10}
\]

Taking into account (4.3) written for \( t \) replacing \( n \), we see that \( L_i(S) = R_i(S) \) for all \( S \subseteq T \), and the result has been proved.

Returning to our example which preceded Lemma 6, we may illustrate Theorem 7 by computing \( \rho_1(\{1, 2, 3\}, v, p^3) \); add there

\[
\rho_1(\{1\}, v, p^2) + \rho_1(\{1, 3\}, v, p^2) = \frac{3}{4} v(\{1\}) + \frac{5}{8} [v(\{1, 2\}) + v(\{1, 3\})
- v(\{2\}) - v(\{3\})]
\]

and subtract there

\[
\pi(\{1, 2, 3\}, v, p^3) - \pi(\{2, 3\}, v, p^2) = \frac{9}{8} v(\{1, 2, 3\}) + \frac{1}{8} [v(\{1, 2\}) + v(\{1, 3\}) +
+ v(\{2, 3\})] - \frac{3}{8} [v(\{1\}) + v(\{2\}) + v(\{3\})] - \frac{5}{4} v(\{2, 3\}) + \frac{1}{4} [v(\{2\}) + v(\{3\})].
\]

Now, by adding up the last two expressions and dividing by 3, as shown in (4.7), we get by Theorem 7:

\[
\rho_1(\{1, 2, 3\}, v, p^3) = \frac{4}{8} v(\{1\}) + \frac{1}{4} [v(\{1, 2\}) - v(\{2\}) + v(\{1, 3\}) - (\{3\})] +
+ \frac{2}{8} [v(\{1, 2, 3\}) - v(\{2, 3\})],
\]

that is exactly the expression we were able to get from (4.2) for our values of the parameters.

Theorem 7, which is clearly a generalization of Theorem 2, as the power game (2.2) can be obtained from the more general power game (4.4) for \( p^n(s) = 2^{1-n} \), \( s = 1, 2, \ldots, n \), suggests that some relationship between the semivalues and the Shapley
value, which would extend the relationship between the Banzhaf value and the Shapley value, may hold. Indeed, we prove:

**Theorem 8:** Let \( p \in R^n \) be a nonnegative vector satisfying (4.1) and define the sequence of vectors \( p^i \in R^t \) by (4.3), where \( p^0 = p \). Let \( \rho \) be the semivalue with these values of parameters, that is

\[
\rho_i(T, u, p^f) = \sum_{S: i \in S \subseteq T} p^f(s)[u(S) - u(S - \{i\})], \quad \forall i \in T, T \subseteq N \tag{4.11}
\]

and the Power Game of \( v \) be \( \pi \) defined by (4.4), then we have

\[
\rho(T, u, p^f) = SH(T, \pi), \quad \forall T \subseteq N. \tag{4.12}
\]

**Proof:** Let \( i \in N \) be fixed; we intend to prove (4.12) componentwise. We assume that the right hand side of (4.12) is given by the “average per capita formula”, that is

\[
SH_i(T, \pi) = t^{-1} \pi(T, u, p^f) + \sum_{h=1}^{h=t-1} \frac{\pi_{h}^i - \pi_{h}^i}{h}, \tag{4.13}
\]

where, as in (3.8) and (3.9) we denoted for \( h = 1, 2, \ldots, t - 1 \)

\[
\pi_{h}^i(T, u, p^f) = \binom{t}{h}^{-1} \sum_{Q \subseteq T \mid |Q| = h} \pi(Q, u, p^h) \tag{4.14}
\]

and

\[
\pi_{h}^i(T, u, p^f) = \binom{t-1}{h}^{-1} \sum_{Q \subseteq T \mid |Q| = h} \sum_{i \notin Q} \pi(Q, u, p^h). \tag{4.15}
\]

From (4.11) it is clear that for a fixed coalition \( S, S \subseteq T \), the coefficient of \( v(S) \) in \( \rho_i \) is \( p^f(s) \) if \( i \in S \), and \(- p^f(s+1)\) if \( i \notin S \). We shall prove by using Lemma 6 as well as (4.3), that the value of the coefficient of \( v(S) \) in the right hand side of (4.12) is the same. Obviously, with \( i \in N \) and \( S \subseteq T \subseteq N \) fixed, two cases may occur: \( i \in S \) and \( i \notin S \).

If \( i \in S \), then \( v(S) \) can not occur in any \( \pi(Q, u, p^h) \) with \( |Q| = h, \ Q \subseteq T, \ i \notin Q \), hence \( v(S) \) will not occur in \( \pi_{h}^i(T, u, p^f) \). Instead, \( v(S) \) will occur in each \( \pi(Q, u, p^h) \) with \( |Q| = h, \ Q \subseteq T \), whenever \( h \geq s \); precisely, \( v(S) \) will occur in \( \sigma_s \) once with a coefficient equal to \( sp^h(s) - (h-s)p^h(s+1) \). There are \( \binom{t-s}{h-s} \) such coalitions \( Q, S \subseteq Q \subseteq T \), hence the total coefficient of \( v(S) \) in \( \pi_{h}^i(T, u, p^f) \) will be
whenever $s \leq h \leq t$. Now, in the sum giving $SH_i(T, \pi)$ the coefficient of $\nu(S)$ for $i \in S$ will be

$$c_s = \sum_{h=s}^{h=t} \left( t \atop h \right)^{-1} \left( t-s \atop h-s \right) [sp^h(s) - (h-s)p^h(s+1)].$$

It is easy to see that based upon (4.3) we have $c_s = p^s(s)$. Indeed, take $h = s + k$ in (4.16) to get a nicer form for the factor in front of the bracket, which may have a new expression; we obtain

$$c_s = \sum_{k=0}^{k=t-s} \left( \frac{s+k}{s} \right) \left( t \atop s \right)^{-1} [sp^{s+k}(s) - kp^{s+k}(s+1)].$$

Obviously, the sum has $t - s + 1$ terms; denote by $S_k$ the partial sum of the first $k + 1$ terms. We claim that

$$S_k = \left( \frac{s+k}{s} \right) \left( t \atop s \right)^{-1} p^{s+k}(s), \quad k = 0, 1, \ldots, t-s.$$  (4.18)

From (4.17) for $k = 0$ we get that $S_0$ is the same as the number obtained from (4.18). We assume that (4.18) holds for $k < t - s$ and get

$$S_{k+1} = S_k + \left( \frac{s+k+1}{s} \right) [(s+k+1) \left( t \atop s \right)^{-1} [sp^{s+k+1}(s) - (k+1)p^{s+k+1}(s+1)].$$

In $S_k$ replace $p^{s+k}(s) = p^{s+k+1}(s) + p^{s+k+1}(s+1)$ and a simple computation will give

$$S_{k+1} = \left( \frac{s+k+1}{s} \right) \left( t \atop s \right)^{-1} p^{s+k+1}(s),$$

hence our claim is correct. For $k = t - s$ we obtain the value of the entire sum, hence

$$c_s = S_{t-s} = p^s(s).$$

If $i \notin S$, then $\nu(S)$ will occur in $\pi(Q, \nu(p^h))$ with $|Q| = h$, $Q \subseteq T$, $h \geq s$, but also in $\pi(Q, \nu(p^h))$ with $|Q| = h$, $Q \subseteq T$, $i \notin Q$, $h \geq s$; therefore $\nu(S)$ will occur in $\pi_i^h(T, \nu(p^t))$ and also in $\pi_i^h(T, \nu(p^t))$. The coefficient of $\nu(S)$ in $\pi_i^h(T, \nu(p^t))$ has been computed above so that from that coefficient we have to subtract the coefficient found in $\pi_i^h(T, \nu(p^t))$. To compute the last one we use the same strategy; let us compute it. Now, $\nu(S)$ will occur in each $\pi(Q, \nu(p^h))$ with $|Q| = h$, $Q \subseteq T$, $i \notin Q$, whenever $h \geq s$; precisely, $\nu(S)$ will occur in $\sigma_s$ once with a coefficient equal to $sp^h(s) - (h-s)p^h(s+1)$. The change is that now we have $\left( \frac{t-s-1}{h-s} \right)$ coalitions
Q, S \subseteq Q \subseteq T, i \notin Q,$ hence the total coefficient in the average $\pi_h^i(T, v, p^t)$ will be (due to (4.15)) 
\[
\left(\begin{array}{c}
t - 1 \\
h
\end{array}\right)^{-1}\left(\begin{array}{c}
t - s - 1 \\
h - s
\end{array}\right)[sp^h(s) - (h - s)p^h(s + 1)],
\]
whenever $s \leq h \leq t - 1$. Notice that $h = t$ has been excluded because $T$ contains $i$. Now, in the sum giving $SH_i(T, \pi)$ the coefficient of $v(S)$ for $i \notin S$ will be
\[
c_s^i = c_S - \sum_{h=s}^{h=t-1} \left[ h \left(\begin{array}{c}
t - 1 \\
h
\end{array}\right)^{-1}\left(\begin{array}{c}
t - s - 1 \\
h - s
\end{array}\right)[sp^h(s) - (h - s)p^h(s + 1)]\right].
\]
(4.19)
To compute the sum, take $h = s + k$ in (4.19) and get a factor in front of the bracket similar to that shown in (4.17). We obtain
\[
c_s^i = c_S - \frac{t}{t-s} \sum_{k=0}^{k=t-s-1} \left(\begin{array}{c}
s + k \\
s
\end{array}\right) [(s + k)\left(\begin{array}{c}
t \\
s
\end{array}\right)]^{-1}[sp^{s+k}(s) - kp^{s+k}(s + 1)].
\]
(4.20)
Fortunately, the sum in (4.20) is almost the same as the sum in (4.17), the difference is that the last term is missing. In other words, the sum (4.20) is the partial sum $S_{t-s-1}$. Hence, from (4.18) we have
\[
c_s^i = c_S - \frac{t}{t-s} S_{t-s-1} = p^t(s) - \frac{t}{t-s} \left(\begin{array}{c}
t - 1 \\
s
\end{array}\right) \left(\begin{array}{c}
t \\
s
\end{array}\right)^{-1} p^{t-1}(s) =
\]
\[
= p^t(s) - p^{t-1}(s) = - p^t(s + 1).
\]
The result is proved.

Note that by means of Theorem 8 we can give another proof for Theorem 7, similar to the one given in Section 2 for the recursive characterization of the Banzhaf value. Indeed, we can get (4.7) as follows:
\[
\rho_i(T, v, p^t) = SH_i(T, \pi) = t^{-1} \left[ \sum_{k \in T - \{i\}} SH_i(T - \{k\}, \pi) + \pi(T, v, p^t) - \pi(T - \{i\}, v, p^{t-1}) \right]
\]
\[
= t^{-1} \left[ \sum_{k \in T - \{i\}} \rho_i(T - \{k\}, v, p^{t-1}) + \pi(T, v, p^t) - \pi(T - \{i\}, v, p^{t-1}) \right].
\]
Note also that a potential may be introduced for the semivalues similar to that introduced for the Banzhaf value in [3] and this fact may give an alternative proof of Theorem 8, similar to the proof of Theorem 1.
Remarks:
1. In [6], T. Driessen, T. Radzik and R. Wanink are characterizing the linear values possessing a generalized potential representation. As these values have to be linear, symmetric and efficient (Th. 2.7, p.5), we believe that they may be generalizations of Least Square Values which were shown in [5] to possess a potential representation.

2. The Least Square Values discussed in Section 3 were introduced in [12] as values satisfying the axioms: linearity, symmetry, coalitional monotonicity, projection and efficiency. It is easy to see from lemma 6 that a semivalue is efficient only if it is the Shapley value. Indeed, to get in (4.5) the equality $\pi(N, v, p^N) = \sigma_n = v(N)$ for all $v \in G(N)$ we have to choose $p^n(s) = \frac{(s-1)!(n-s)!}{n!}$, for $s = 1, 2, \ldots, n$. Therefore, we separated the discussions of the efficient values (the ELS-values) from that of the semivalues, in general not efficient.

3. In [7], P. Dubey, A. Neyman and R. J. Weber characterize the semivalues $\Psi^N$ on $G(N)$, where $N$ is a fixed finite set of players, as an auxiliary result needed for proving a general characterization (Th.1). We have taken this characterization as the definition (4.1), (4.2) of semivalues on $G(N)$. Then, in extending their characterization from $N$ to $N \cup \{d\}$, $d \notin N$, they have shown that to get a semivalue $\Psi^{N\cup\{d\}}$ for which the restriction to $N$ is $\Psi^N$, the parameters must satisfy $p^n(s) = p^{n+1}(s) + p^{n+1}(s + 1)$, for $s = 1, 2, \ldots, n$. Therefore, we have taken the similar formula (4.3) to be satisfied by any restriction of $\Psi^N$ to $G(T)$ with $|T| = n - 1$. Further, we proceeded in the same way for coalitions of smaller sizes, to complete the definition of semivalues on $G^N$ with that given in [7].

4. In [14], R. J. Weber is defining the values componentwise and he is imposing the axioms of a value for an arbitrary player $i$. Such a value, satisfying the linearity, dummy and monotonicity axioms is called a probabilistic value and can be expressed as

$$\Psi_i(N, v) = \sum_{S: i \in S \subseteq N} p^i_S [v(S) - v(S - \{i\})], \quad \sum_{S: i \in S \subseteq N} p^i_S = 1,$$

$$p^i_S \geq 0, \forall S \subseteq N, \text{ (Th. 5, p.106).}$$

By adding the symmetry, one gets $p^i_S = p^i_{|S|}, \forall S \subseteq N, \text{ (Th. 10, p.112).}$ Obviously, this is a component of a semivalue; the semivalues were defined in [7] by imposing the axioms: linearity, symmetry, monotonity and projection on the space of additive games (weaker than the dummy).

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