RAY OPTICS ON SURFACES

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Abstract. Variational techniques are used to find path of light rays on surfaces. Using Fermat's principle of least time the problem is treated as a constraint optimization to obtain a system of partial differential equations.

Introduction. The time required for a beam of light to traverse a path is called the optical length of the curve. Fermat's principle states that in an optical medium, the path of light from a point \( A \) to a point \( B \) has the least optical length of all paths joining \( A \) and \( B \). Lyusternik [2, Chapter 6] gives an elementary treatment of Fermat's principle and its consequences. Goldstein [1] has a historical treatment of Fermat's principle. Weinstock [4, Chapter 5] discusses Fermat's principle and applications in geometric optics. In this article we use variational techniques to find path of light rays constraint on a given surface. By using Fermat's principle of least time we obtain a system of partial differential equations.

Discussion. First, as a review, we give differential equations for path of light rays in nonhomogeneous media. This is done in Whitham [5, pgs. 247-249]. For simplicity the proof is given in three dimensions.

Assume \( c = c(x, y, z) \) is speed of light in the medium. Let \( \sigma \) denote the total time travelled. Then

\[
\sigma = \int \frac{ds}{dt} = \int \frac{\sqrt{x'^2 + y'^2 + z'^2}}{c(x, y, z)} \, dt,
\]

where dot denotes differentiation with respect to the arc-length \( s \). Let

\[
F(x, y, z, x', y', z') = \frac{\sqrt{x'^2 + y'^2 + z'^2}}{c(x, y, z)}.
\]

Using Euler-Lagrange equations in parametric form, we obtain

\[
(1) \quad F_x - \frac{d}{dt} F_{x'} = 0
\]

\[
(2) \quad F_y - \frac{d}{dt} F_{y'} = 0
\]

\[
(3) \quad F_z - \frac{d}{dt} F_{z'} = 0
\]
Then from (1) and expression for $F(x, y, z, \dot{x}, \dot{y}, \dot{z})$ we have

$$-\frac{\partial c}{\partial x} \cdot \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} - \frac{d}{dt} \left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \cdot \frac{1}{c}\right) = 0$$

Now we use the chain rule $\frac{d}{dt} = \frac{d}{ds} \cdot \frac{ds}{dt} = \frac{d}{ds} \cdot \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$.

$$-\frac{\partial c}{\partial x} \cdot \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} - \frac{d}{ds} \left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \cdot \frac{1}{c}\right) \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = 0.$$

Note that $\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \frac{dx}{ds} \frac{ds}{dt} = \frac{dx}{ds}$. Then we obtain

$$\frac{\partial c}{\partial x} \cdot \frac{1}{c^2} + \frac{d}{ds} \left(\frac{1}{c} \frac{dx}{ds}\right) = 0.$$ (4)

Similarly for $y$ and $z$ we obtain

$$\frac{\partial c}{\partial y} \cdot \frac{1}{c^2} + \frac{d}{ds} \left(\frac{1}{c} \frac{dy}{ds}\right) = 0,$$ (5)

$$\frac{\partial c}{\partial z} \cdot \frac{1}{c^2} + \frac{d}{ds} \left(\frac{1}{c} \frac{dz}{ds}\right) = 0.$$ (6)

The general form in $n$ dimension with $c = c(x_1, \cdots, x_n)$ is given by

$$\frac{\partial c}{\partial x_i} \cdot \frac{1}{c^2} + \frac{d}{ds} \left(\frac{1}{c} \frac{dx_i}{ds}\right) = 0.$$ (7)

Note that if $c$ is constant rays are straight lines as expected.

Now we use ray optics on surfaces as a constraint optimization by minimizing $\sigma$ given by

$$\sigma = \int \frac{ds}{c(x, y, z)} = \int \frac{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}{c(x, y, z)} dt$$

subject to a surface $G(x, y, z) = 0$. We use Lagrange multiplier $\lambda(t)$. Let

$$F(x, y, z, \dot{x}, \dot{y}, \dot{z}, \lambda) = \frac{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}{c(x, y, z)} + \lambda(t)G(x, y, z).$$

See page 273 of Simmons [3] for this kind of Constraint optimization. We use Euler-Lagrange equation in parametric form. After simplification we obtain the following system of partial differential equations.
(8) \[ \frac{1}{c^2} \frac{\partial c}{\partial x} + \frac{d}{ds} \left( \frac{1}{c} \frac{dx}{ds} \right) = \lambda(t) \frac{G_x}{\sqrt{x^2 + y^2 + z^2}} , \]

(9) \[ \frac{1}{c^2} \frac{\partial c}{\partial x} + \frac{d}{ds} \left( \frac{1}{c} \frac{dy}{ds} \right) = \lambda(t) \frac{G_y}{\sqrt{x^2 + y^2 + z^2}} , \]

(10) \[ \frac{1}{c^2} \frac{\partial c}{\partial x} + \frac{d}{ds} \left( \frac{1}{c} \frac{dz}{ds} \right) = \lambda(t) \frac{G_z}{\sqrt{x^2 + y^2 + z^2}} . \]

If we assume \( c \) is constant we obtain the following equations for geodesics given in Simmons [3, p 374] as a special case

\[ \frac{d^2 x}{ds^2} \frac{1}{G_x} = \frac{d^2 y}{ds^2} \frac{1}{G_y} = \frac{d^2 z}{ds^2} \frac{1}{G_z} . \]

Also, if we let \( f = \sqrt{x^2 + y^2 + z^2} \) we can rewrite the above equations as

\[ \frac{d}{ds} \left( \frac{\frac{dx}{f}}{G_x} \right) = \frac{\frac{dx}{dt}}{G_y} = \frac{\frac{dx}{dt}}{G_z} \]

REFERENCES


