FIXED POINT THEOREM FOR NON-EXPANSIVE MAPPINGS ON
BANACH SPACES WITH UNIFORMLY NORMAL STRUCTURE

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In [1] Kirk proved that if $D$ is a bounded, closed, and convex subset of a reflexive Banach space that has normal structure, then every non-expansive mapping of $D$ into $D$ has a fixed point. This was also proved for a uniformly convex space by Browder [2]. In this paper we replace uniformly convex, or reflexive and normal structure, by uniformly normal structure to obtain this result.

Let $S$ be a bounded subset of the Banach space $X$ and $x \in S$

\begin{enumerate}
  \item \( \omega_x(S) = \sup \{ ||x - y|| : y \in S \} \)
  \item \( \Omega(S) = \inf \{ \omega_x(S) : x \in S \} \)
  \item \( \delta(S) = \sup \{ \omega_x(S) : x \in S \} \)
\end{enumerate}

A space $X$ is said to have uniformly normal structure if for some $h \in R$, $0 < h < 1$, every bounded closed and convex subset $S$ of $X$, $\Omega(S) < h\delta(S)$.

In [3] Edelstein shows that a space that is uniformly convex will have uniformly normal structure. An example of a Banach space that has uniformly normal structure that is not uniformly convex is the space $<R_2, ||(x,y)|| >$ with $||(x,y)|| = \max \{ |x|, |y| \}$ for $(x,y) \in R_2$. 

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Later in this paper we give an example of a Banach space that is uniformly convex in all directions but does not have uniformly normal structure.

Hence all uniformly convex spaces will have uniformly normal structure but spaces that have uniformly normal structure will not necessarily be uniformly convex. All spaces that are uniformly convex in all directions will have normal structure but not all these spaces will have uniformly normal structure. All spaces with uniformly normal structure will have normal structure.

**Lemma.** If $X$ is a Banach space with uniformly normal structure, $D$ is a non-empty, closed, bounded, and convex subset of $X$, and $f$ is a non-expansive map of $D$ into $D$, then there exist $h \in (0,1)$, independent of $D$, and $D_1$, a non-empty, closed, convex subset of $D$ such that $f : D_1 \to D$, and $\delta(D_1) \leq h \delta(D)$.

**Proof:** Since $X$ has uniformly normal structure, there exist $h \in (0,1)$, and a point $z_1 \in D$ such that $\Omega_{z_1}(D) \leq h \delta(D)$. Define $H = \{ D_\alpha \subset D : D_\alpha \text{ is non-empty, closed, convex, and } f : D_\alpha \to D_\alpha \}$ and $L = \cap \{ D_\alpha \in H : z_1 \in D_\alpha \}$.

Then $z_1 \in L$, $L$ is closed, bounded, convex subset of $D$ and $f : L \to L$. So $A = [f(L) \cup \{ z_1 \}] \subset L$ and $\overline{\text{conv}} A \subset L$, where $\overline{\text{conv}} A$ is the closure of the convex hull of $A$.

$\overline{\text{conv}} A$ is non-empty, closed, bounded, convex subset of $D$ and $f : \overline{\text{conv}} A \to \overline{\text{conv}} A$. Hence $L \subset \overline{\text{conv}} A$ and $\overline{\text{conv}} A$ is dense in $L$.

Define $D_1 = \{ x \in L : ||x - y|| \leq h \delta(D) \ \forall \ y \in L \}$. Since $z_1 \in D_1$, then $D_1$ is non-empty.
To show $D_1$ is closed let $\{x_n\}$ be a sequence in $D_1$ with $x_n \to x_0$. Consider $y \in L$ and $\varepsilon > 0$. $||x_0 - y|| \leq ||x_0 - x_n|| + ||x_n - y|| \leq ||x_0 - x_n|| + h\delta(D)$. Since $x_n \to x_0$, then $||x_0 - y|| \leq \varepsilon + h\delta(D)$. So $x_0 \in D_1$, and $D_1$ is closed.

Consider $g \in D_1$. Let $y \in L$ and $\varepsilon > 0$. Since $\text{conv} \ A$ is dense in $L$, there exist $\sum_{i=1}^{J} \alpha_i f(y_{i}) + \alpha_{J+1} z_1$ so $\sum_{i=1}^{J+1} \alpha_i = 1$, $0 \leq \alpha_i$, $y_i \in L$

and $||\sum_{i=1}^{J} \alpha_i f(y_{i}) + \alpha_{J+1} z_1 - y|| < \varepsilon$. Then

$$||f(g) - y|| \leq ||f(g) - \sum_{i=1}^{J} \alpha_i f(y_{i}) - \alpha_{J+1} z_1|| + \varepsilon$$

$$\leq ||\sum_{i=1}^{J} f(g) - \sum_{i=1}^{J} \alpha_i f(y_{i})|| + ||\alpha_{J+1} f(g) - \alpha_{J+1} z_1|| + \varepsilon$$

$$\leq \sum_{i=1}^{J} \alpha_i ||g - y_{i}|| + \alpha_{J+1} ||f(g) - z_1|| + \varepsilon$$

$$\leq \sum_{i=1}^{J} \alpha_i h\delta(D) + \alpha_{J+1} h\delta(D) + \varepsilon = h\delta(D) + \varepsilon.$$ 

So $||f(g) - y|| \leq h\delta(D)$ for $y \in L$ and $f : D_1 \to D_1$. It follows that $D_1$ is convex and $\delta(D_1) \leq h\delta(D)$.

Hence $D_1 \subseteq D$, $D_1$ is non-empty, closed, bounded, convex, $f : D_1 \to D_1$ and $\delta(D_1) \leq h\delta(D)$.

**THEOREM.** Let $D$ be a non-empty, closed, bounded, convex subset of the Banach space $X$ that has uniformly normal structure and $f$ a non-expansive map of $D$ into $D$. Then $f$ has a fixed point in $D$. 
Proof: By the Lemma for each \( n = 0, 1, 2, \ldots \) there exist a \( D_{n+1} \), a non-empty closed bounded and convex subset of \( D_n \) with \( f : D_{n+1} \to D_{n+1} \), such that \( \delta(D_{n+1}) \leq n \delta(D) \). Let \( g_n \in D_n \) for \( n = 1, 2, 3, \ldots \). Then the sequence \( \{g_n\} \) is a Cauchy sequence in \( X \) and there exist a point \( g_0 \) such that \( g_n \to g_0 \) with \( \{g_0\} = \cap D_n \). So \( f(g_0) = g_0 \) and \( f \) has a fixed point in \( D \).

To obtain an example of a Banach space that has normal structure but does not have uniformly normal structure, consider the Banach space \( c \), the space of convergent sequences of real numbers with \( \|\{x_i\}\|_c = \sup \{ |x_i| : i \in \mathbb{N} \} \).

Define the linear, one-one, and continuous function \( g \) from \( c \) to \( l_2 \) by \( g(\{x_i\}) = \{\frac{x_i}{2^i}\} \). Now renorm \( c \), for \( \{x_i\} \in c \),

\[
\|\|\{x_i\}\|\| = \left( \|\{x_i\}\|_c^2 + \|g(\{x_i\})\|_{l_2}^2 \right)^{1/2}
\]

In [4] Zizler shows that the Banach space \( c \) with \( \|\|\cdot\|\| \) norm is uniformly convex in every direction and hence has normal structure.

Define \( B = \{ \{x_i\} \in c : 0 \leq x_i \leq 1 \text{ for } i \in \mathbb{N}, \{x_i\} + 1 \} \). Then \( B \) is closed, bounded and convex subset of \( c \).

Define \( T : B \to B \), for \( \{x_i\} \in B \) \( T(\{x_i\}) = \{0, x_1, x_2, \ldots \} \). Note that \( T \) does not have a fixed point in \( B \).

For \( \{x_i\}, \{y_i\} \in B \)

\[
\|\|\{x_i\} - \{y_i\}\|\| = \left[ \sup \{ |x_i - y_i| : i \in \mathbb{N} \}^2 + \sum_{i=1}^{\infty} \frac{(x_i - y_i)^2}{4^i} \right]^{1/2}
\]

\[
\|T(\{x_i\}) - T(\{y_i\})\| = \left[ \sup \{ |x_i - y_i| : i \in \mathbb{N} \}^2 + \frac{1}{4} \sum_{i=1}^{\infty} \frac{(x_i - y_i)^2}{4^i} \right]^{1/2}
\]
So $T$ is non-expansive on $B$.

Since $B$ is a closed bounded and convex subset of the space $\langle \sigma, ||\cdot|| \rangle$ and $T$ is a non-expansive map of $B$ into $B$ that does not have a fixed point, by our Theorem, the Banach space $\langle \sigma, ||\cdot|| \rangle$ does not have uniformly normal structure. Since the space $\langle \sigma, ||\cdot|| \rangle$ has normal structure, by a result in [1], this space is non-reflexive.
REFERENCES


