LINEAR ALGEBRAIC COMPUTATIONAL
PROCEDURES FOR SYSTEM IDENTIFICATION PROBLEMS

by
J. Eisenfeld* and B. Soni**

Technical Report No. 59

May, 1977

* Department of Medical Computer Science, University of Texas
Health Science Center at Dallas, Texas 75235, and Department
of Mathematics, University of Texas at Arlington, Texas 76019.

** Department of Mathematics, University of Texas at Arlington,
Texas 76019.
LINEAR ALGEBRAIC COMPUTATIONAL
PROCEDURES FOR SYSTEM IDENTIFICATION PROBLEMS

by

J. Eisenfeld* and B. Soni**

Summary. An algorithm is presented for identifying exponentials sums \( I(t) = \sum_{i=1}^{N} a_i \exp(\lambda_i t) \) from discrete data \( I(t_0), I(t_1), ..., I(t_q) \). The algorithm determines the number of components, \( N \) the rate constants \( \lambda_i \) and the amplitudes \( a_i \). The method is based on the method of moments as it is applied to analyze fluorescence decay experiments. However the present method is more general. It permits greater flexibility in weighting the data and, in particular, alleviates the "cut-off error" incurred when integration is required to infinite time.

1. Introduction. We will use the following notation throughout the paper. We let \( \lambda_1, \lambda_2, ..., \lambda_N \) denote a set of distinct non-zero real numbers and we define \( X \) to be the \( N \) dimensional vector space spanned by the functions (also regarded as vectors)

\[
\phi_i(t) = \exp(\lambda_i t), \quad i = 1, 2, ..., N
\]

* Department of Medical Computer Science, University of Texas Health Science Center at Dallas, Texas 75235, and Department of Mathematics, University of Texas at Arlington, Texas 76019.

** Department of Mathematics, University of Texas at Arlington, Texas 76019.
A map \( h \) from \( X \) into the real numbers is said to be a linear functional if it satisfies the following properties: (i) \( f(x + y) = f(x) + f(y) \), \( f(\alpha x) = \alpha f(x) \) with respect to vectors \( x, y \) and scalars \( \alpha \). We use the notation \( \langle x, f \rangle \) to denote the action of a linear functional \( f \) on a vector \( x \). The dual space, denoted by \( X^\perp \), is the space of all linear functionals on \( X \). See [7]).

The role of the linear functional arises naturally from consideration of the method of moments. There one computes integrals of the form

\[
\int_0^\infty t^K x(t) dt, \quad K = 1, 2, \ldots, 2N
\]  

(1.2)

Such integrals are special cases of linear functionals. In our notation they would be expressed as \( \langle x, h_K \rangle \). There are certain problems associated with the integrals (1.2). First of all they are meaningful only when all the \( \lambda_t \)'s are negative. Second, the integrals are summed to infinite time whereas real data is collected for finite time. The resulting error is called the cutoff error. Third, the integral kernels \( t^K \) weight the data and it is not clear that this type of weighting is optimal.

Isenberg and co-workers ([3]-[6]) applied the method of moments to fluorescence decay experiments. They introduced modifications such as the moment index displacement and exponential depression to weight the data and to minimize effects of nonrandom errors ([1],[5],[6]). These modifications are obtained as special cases of the more general theory presented here. The "Isenberg method" compares favorably with other methods used in
fluorescence decay theory, including nonlinear estimation techniques. The method also applies to other types of problems [2].

In this paper we present an algorithm for identifying exponential sums

$$I(t) = \sum_{i=1}^{N} a_i \exp(\lambda_i t) \quad (1.3)$$

from discrete data

$$I(t_0), I(t_1), \ldots, I(t_Q) \quad (1.4)$$

i.e. we wish to determine the number of components $N$, the rate constants $\lambda_i$, and the amplitudes $a_i$. As in the method of moments the algorithm is based on a set of linear functionals $h_1, h_2, \ldots, h_N$ having the following properties:

(A) The set $h_1, h_2, \ldots, h_N$ are linearly independent in $X$.

(B) For each $x$ in $X$, we have the relationships

$$<x', h_i> = <x', h_{i+1}>, \quad i = 1, 2, \ldots, 2N - 1 \quad (1.5)$$

A prime denotes differentiation with respect to $t$. In particular we have in mind linear functionals of the form

$$<x, h_i> = (-1)^{i-1} \int_{a}^{b} x(t) H^{(i-1)}(t) dt \quad (1.6)$$

where $H(t)$ is a "weighting function". Its choice is arbitrary except that it must have continuous derivatives $H^{(i)}$, $i = 1, 2, \ldots, 2N - 1$, on the "sampling interval" $(a, b)$ and satisfy end-point conditions

$$H^{(i)}(a) = H^{(i)}(b) = 0, \quad i = 1, 2, \ldots, 2N - 1 \quad (1.7)$$
The particular choices

\[ H_{K, \delta}(t) = t^{2N+K-1}\exp(-\delta t)/(2N + K - 1)! \]  

(1.8)

will give us the method of moments for exponential decay with index displacement of order \( K \) and exponential depression of order \( \delta \). Here \((a, b) = (0, \infty)\).

The algorithm applies more generally to systems of differential equations. This will be considered in future work.

2. Identification Process. We assume to begin with that the number of components \( N \) is known. The identification of \( N \) will be considered last. We assume that the rate constants \( \lambda_i \) in (1.3) are distinct real numbers and they correspond to nonzero amplitudes \( a_i \). We make the following observations:

(I) The function \( I^{(K)} \), \( K = 0, 1, \ldots, N - 1 \) form a linearly independent set in \( X \).

(II) The functions \( i^{(K)} \), \( K = 0, 1, \ldots, N \) form a linearly dependent set in \( X \), thus there exist constants \( c_i \) such that

\[ I^{(N)} + c_{N-1}I^{(N-1)} + \ldots + c_0I = 0 \]  

(2.1)

(III) The rate constants \( \lambda_i \) are the roots of the characteristic polynomial associated with the differential equation (2.1)

\[ P(\lambda) = \lambda^N + c_{N-1}\lambda^{N-1} + \ldots + c_0 \]  

(2.2)
Step 1: Identification of the Polynomial Coefficients $\sigma_i$. 

Operating with each $h^i_t$, $i = 1, 2, \ldots, N$ in Equation (2.1) we obtain a system of equations

$$c_0 <I, h^i_t> + c_1 <I', h^i_t> + \ldots + c_{N-1} <I^{(N-1)}, h^i_t> = -<I^N, h^i_t>$$

(2.3)

We express this system in matrix form

$$G\sigma = \pi$$

(2.4)

where

$$\sigma = (c_0, c_1, \ldots, c_{N-1})^T$$

(2.5)

$$\pi = -(<I^{(N)}, h_1>, <I^{(N)}, h_2>, \ldots, <I^{(N)}, h_N>)$$

(2.6)

and $G$ is an $N \times N$ matrix with $(i,j)$th element

$$g_{ij} = <I^{(i-1)}, h_j>$$

(2.7)

In view condition (1.5) we may conveniently express these constants in terms of

$$s_i = <I, h^i_t> = <I^{(i-1)}, h_1>$$

(2.8)

as follows

$$g_{ij} = s_{i+j-1}$$

(2.9)

$$r_i = -<I^{(N)}, h^i_t> = -s_{i+N}$$

(2.10)

Thus both $G$ and $\pi$ may be computed directly from $I(t)$. Also we have the following:

(IV) $G$ is nonsingular (in view of Appendix A and the
linear independence of both sequences $I^{(t)}$ and $h_j$)

(V) Suppose

$$A_i = (a_i \phi_i, h_i) > 0, \quad i = 1, 2, \ldots, N$$  (2.11)

(this occurs, e.g., when $a_i > 0, \quad i = 1, 2, \ldots, N$, and the function $H(t)$ in (1.6) is positive) then $G$ is positive definite (See Appendix B). Therefore $\sigma_i$ may be identified from (2.4) by means of a suitable matrix inversion algorithm.

**Step 2: Identification of the Rate Constants $\lambda_i$.**

Since the coefficients $\sigma_i$ in (2.2) have already been identified in step 1, the rate constants $\lambda_i$ are computed as the roots of $P(\lambda)$ as observed above.

**Step 3: Identification of the Amplitudes $a_i$.**

At this point we have already determined a set of roots $\lambda_i$ with corresponding functions (1.1). We now operate with $h_i, \quad i = 1, 2, \ldots, N$, in (1.3) to produce the system of equations

$$Qa = \sigma$$  (2.12)

where

$$a = (a_1, a_2, \ldots, a_N)^T$$  (2.13)

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)^T$$  (2.14)

and the $Q$ matrix entries

$$q_{ij} = <\phi_i, h_j>$$  (2.15)

In view of Appendix A and the linear independence of both sequences $\phi_i$ and $h_j$, the matrix $Q$ is non-
singular and \( a \) may be determined from (2.12).

**Step 0: Identification of the Number of Components, \( N \).

Suppose \( N \) is not known and we assume as \( n \) component model. The matrix \( G \) will depend on the choice of \( n \) so we now denote it by \( G_n \). In view of Appendix A we should have

\[
\det G_n \neq 0, \quad n = 1, 2, \ldots, N
\]

\[
\det G_n = 0, \quad n = N + 1, N + 2, \ldots
\]

Thus we compute the sequence \( \det G_n \) until a point \( K \) is reached such that for \( n > K \) \( \det G_n \) is recognized as zero. Then we identify \( N \) by \( N = K - 1 \).

We should point out that difficulties may arise in recognizing zeros in the sequence and therefore the \( s_i \) values should be scaled as discussed below. Further assistance occurs when the condition (2.11) is satisfied for then, in view of Appendix B, we have further

\[
\det G_n > 0, \quad n = 1, 2, \ldots, N.
\]

In practice a negative value of \( \det G_n \), which occurs because of computer errors, indicates that \( n > N \) (See Example 3.3 and Table V).

**Scaling.** Let \( A_i = a_i \angle \phi_i, h_i \), \( i = 1, 2, \ldots, N \). It is not difficult to see, in view of (2.8) and (1.3), that

\[
s_j = \sum_{i=1}^{N} A_i \lambda_j^{i-1}
\]

clearly the \( \lambda_j \)'s with larger magnitudes will tend to dominate in \( s_j \) causing stiffness in the matrix \( G \). One purpose of scaling is to attempt to
achieve, as much as possible, \( s_j \sim 1, \ j = 1,2,\ldots \). A "ball park" value of \( \lambda \) is \( \lambda^j = s_2/s_1 \). A "ball park" value for the amplitude \( A_j \) is \( s_1 \). The scaled \( s_j \) are defined by \( \tilde{s}_j = s_j/(s_1 \lambda^{j-1}) \), \( j = 1,2,\ldots \). The algorithm will then produce the scaled values \( \tilde{\lambda} \) and \( \tilde{\alpha} \). These are related to the actual values by \( \lambda_i = \tilde{\lambda} \tilde{\alpha}_i \), \( a_i = \tilde{\alpha}_i \), \( i = 1,2,\ldots,N \) where \( \tilde{\alpha} = s_1/\langle e^{\lambda t}, h_1 \rangle \). Observe that this scaling will always give \( \tilde{s}_1 = \tilde{s}_2 = 1 \). We should point out that this scaling method arises naturally from applying the algorithm (without scaling) to a one component model. We would then find that the resulting parameters are precisely \( \lambda \) and \( \alpha \) as defined above.

3. Conclusions and Examples. All computations have been performed on the IBM 370-155 computer (located at the University of Texas Regional Computer Center in Dallas, Texas) in FORTRAN IV using single precision arithmetic.

We always choose linear functionals of the form (1.6). For convenience we have chosen sampling intervals of the form \((a,b) = (0,q)\), \( q = 1,2,\ldots \). In the case of an \( N \)-component model (or for a model expected to have \( N \) components) we choose weighting functions of the form \( H(t) = [\sin(\pi t/q)]^{2N-1} \). The \( \tilde{s}_i \) numbers are then given by (See (1.6)) \( \tilde{s}_i = (-1)^{i-1} \int_0^q I(t)H(t)^{i-1}(t)dt \). All integrals are evaluated using Simpson's Rule. For convenience we choose equally
spaced data points $I(j), j = 0, 1, 2, \ldots, q$.

Example 3.1 [One Component]. We consider first the case $I(t) = 0.5 \exp{(-t/30)}$. Table I presents the estimated values and relative errors for different sampling intervals. These results are typical for one component decay (including positive $\lambda$'s). The results are "good" even in the case of "small" sampling intervals. In fact accuracy tends to decrease slightly for the larger sampling intervals as indicated in Table I.

Table II presents results for different parameter values $a$ and $\lambda$ in $I(t) = a \exp{\lambda t}$. The sampling interval here is fixed at $q = 80$.

Example 3.2 [Two Components]. We consider first the case $I(t) = \exp{(-t/10)} + \exp{(-t/30)}$. As in the one component case (Table 1) we varied the sampling interval but, in contrast to the one component case, Table III shows that accuracy does improve with the size of the sampling interval.

Table IV presents results for different parameter values $a_i$ and $\lambda_i$ in $I(t) = a_1 \exp{\lambda_1 t} + a_2 \exp{\lambda_2 t}$. The sampling interval is fixed at $q = 70$. We have included one case in which we did not obtain good results. Observe that for this particular case we have $\lambda_1 < \lambda_2$ and $a_1 > a_2$. When this situation arises one has $a_1 \exp{\lambda_1 t} > a_2 \exp{\lambda_2 t}$, i.e., one component dominates another component and we can expect poor results no matter which algorithm is used.

Example 3.3 [Determination of the Number of Components]. We considered the identification of the number of compo-
ments for a single exponential $I(t) = 0.5 \exp(-t/30)$
and for a double exponential $I(t) = \exp(-t/10) + \exp(-t/30)$. Choosing the sampling interval to be (0,70) for both cases we computed the sequence of
determinants $\det G_n$ as discussed in Section 2. Table V
presents the results. For the one component case we
observe the first "zero" of the sequence when $n = 2$
and also $\det G_2$ is the first negative number in the
sequence. For the two component case we observe the
first "zero" of the sequence when $n = 3$ and $\det G_3$
is the first negative number in the sequence. In these
examples amplitudes $a_t$ and the weighting function are
all positive so that condition (2.11) is satisfied.
Thus (2.18) applies here. In view of the conditions
(2.16), (2.17) and (2.18) the occurrence of a "small"
negative value at $n = K$ suggest that the number of
components is $N = K - 1$.

Appendix A. Let $X$ be a linear space of dimension
$N$ and let $X^*$ denote the algebraic dual of $X$. Let
$\{x_1, x_2, \ldots, x_N\}$ be a set of vectors in $X$ and let
$\{g_1, g_2, \ldots, g_N\}$ be a set of linear functionals in $X^*$.
Let $G$ be the matrix with $(i,j)$th element $\langle x_i, g_j \rangle$.
Then $G$ is nonsingular if and only if both sets $\{x_i\}$
and $\{f_j\}$ are linearly independent.

Proof. Let $\{z_1, z_2, \ldots, z_N\}$ be a basis for $X$ and let
$\{g_1, g_2, \ldots, g_N\}$ be the basis for $X^*$ which is dual to
$\{z_i\}$, i.e. $\langle z_i, f_j \rangle = \delta_{i,j}$. (See [7]). Let $x_i =
\sum_{i=1}^{N} b_{ik} z_k$ and $f_j = \sum_{n=1}^{N} c_{jn} g_m$. Then $\langle x_i, f_j \rangle =
\sum_{k=1}^{L} b_{ik} c_{jk} \sum_{m=1}^{K} c_{jm} g_m$. Therefore $G = B C^T$.
Now $G$ is nonsingular if and only if both $B$ and $C$ are nonsingular which is equivalent to the condition that both $\{x_i\}$ and $\{h_j\}$ are linearly independent.

Appendix B. Let $G$ be an $N \times N$ matrix of the form described in Appendix A. Suppose further that $x_i = \sum_{m=1}^{N} a_m \lambda_i^{i-1} \phi_i$ where $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ and $\{a_1, a_2, \ldots, a_N\}$ are sequences of scalars, $\{\phi_1, \phi_2, \ldots, \phi_N\}$ is a sequence of vectors and $<\phi_i, h_j> = \lambda_i^j$ $<\phi_i, h_j>$ for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N^2 - 1$. Then:

(i) $G$ is symmetric i.e. $g_{ij} = g_{ji}$.

(ii) If condition (2.11) is satisfied and if the sequences $\{x_i\}$ and $\{h_j\}$ are linearly independent then $G$ is positive definite and, in particular, $\det G > 0$.

Proof. It follows from the above conditions that the $(i,j)$th entry $g_{ij} = <x_i, h_j> = \sum_{m=1}^{N} a_m \lambda_i^{i-1} \phi_m, h_j> = \sum_{m=1}^{N} a_m \lambda_i^{i-1} \phi_m, h_j> = \sum_{m=1}^{N} a_m \lambda_i^{i+j-2} \phi_m, h_1> = \sum_{m=1}^{N} A_m \lambda_i^{i+j-2}$ where $A_m = a_m <\phi_m, h_1>$ as in (2.11). Clearly $g_{ij} = g_{ji}$. In fact $G$ may be expressed in the form $G = PP^T$ where $P$ is the Vandermonde matrix

$$P = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_N \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1}
\end{bmatrix}$$
and

\[
D = \begin{bmatrix}
A_1 & & 0 \\
& A_2 & \\
& & \ddots & 0 \\
& & & \ddots & \ddots \\
0 & & & & A_N
\end{bmatrix}
\]

is a diagonal matrix. Thus \( G \) is positive if \( A_m > 0 \), \( m = 1, 2, \ldots, N \) and \( \det G = (\det P)^2A_1A_2\ldots A_N \). If in addition, the sequences \( \{x_i\} \) and \( \{h_j\} \) are linearly independent, then, in view of Appendix A, \( G \) is nonsingular. Hence also \( P \) is nonsingular and the conclusion (ii) follows.

\textbf{Acknowledgement}: The authors express their gratitude to Robert M. Dowben, M.D. and his co-workers for their useful interactions.
### Table I

<table>
<thead>
<tr>
<th>Interval</th>
<th>Estimated Values</th>
<th>Relative Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>[0, 10]</td>
<td>0.499982E00</td>
<td>-0.333262E-01</td>
</tr>
<tr>
<td>[0, 20]</td>
<td>0.499982E00</td>
<td>-0.333239E-01</td>
</tr>
<tr>
<td>[0, 30]</td>
<td>0.499999E00</td>
<td>-0.333332E-01</td>
</tr>
<tr>
<td>[0, 40]</td>
<td>0.499999E00</td>
<td>-0.333332E-01</td>
</tr>
<tr>
<td>[0, 50]</td>
<td>0.499999E00</td>
<td>-0.333332E-01</td>
</tr>
<tr>
<td>[0, 60]</td>
<td>0.499999E00</td>
<td>-0.333325E-01</td>
</tr>
<tr>
<td>[0, 70]</td>
<td>0.499999E00</td>
<td>-0.333326E-01</td>
</tr>
<tr>
<td>[0, 80]</td>
<td>0.499999E00</td>
<td>-0.333326E-01</td>
</tr>
<tr>
<td>[0, 90]</td>
<td>0.499999E00</td>
<td>-0.333326E-01</td>
</tr>
<tr>
<td>[0, 100]</td>
<td>0.499999E00</td>
<td>-0.333331E-01</td>
</tr>
</tbody>
</table>

### Table II

<table>
<thead>
<tr>
<th>Expected Values</th>
<th>Relative Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>0.800000E00</td>
<td>0.100000E00</td>
</tr>
<tr>
<td>1.000000E00</td>
<td>0.200000E00</td>
</tr>
<tr>
<td>1.000000E00</td>
<td>0.333333E-01</td>
</tr>
<tr>
<td>0.500000E00</td>
<td>0.333333E-01</td>
</tr>
<tr>
<td>Expected Values</td>
<td>$\alpha_1$</td>
</tr>
<tr>
<td>-----------------</td>
<td>-------------</td>
</tr>
<tr>
<td>Interval</td>
<td>0.10000E01</td>
</tr>
<tr>
<td>$[0, 0.1]$</td>
<td>0.192981E01</td>
</tr>
<tr>
<td>$[0, 0.2]$</td>
<td>0.104722E01</td>
</tr>
<tr>
<td>$[0, 0.3]$</td>
<td>0.100435E01</td>
</tr>
<tr>
<td>$[0, 0.4]$</td>
<td>0.100080E01</td>
</tr>
<tr>
<td>$[0, 0.5]$</td>
<td>0.100020E01</td>
</tr>
<tr>
<td>$[0, 0.7]$</td>
<td>0.999988E00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Estimated Values</th>
<th>$\alpha_1$</th>
<th>$\lambda_1$</th>
<th>$\alpha_2$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0, 0.7]$</td>
<td>0.929810E00</td>
<td>0.923177E00</td>
<td>0.267425E00</td>
<td>0.329267E01</td>
</tr>
<tr>
<td>$[0, 0.26]$</td>
<td>0.472231E-01</td>
<td>0.470521E-01</td>
<td>0.193936E-01</td>
<td>0.381535E-01</td>
</tr>
<tr>
<td>$[0, 0.30]$</td>
<td>0.434780E-02</td>
<td>0.436431E-02</td>
<td>0.221252E-02</td>
<td>0.304699E-02</td>
</tr>
<tr>
<td>$[0, 0.40]$</td>
<td>0.795364E-03</td>
<td>0.81867E-03</td>
<td>0.507236E-03</td>
<td>0.502467E-03</td>
</tr>
<tr>
<td>$[0, 0.50]$</td>
<td>0.197411E-03</td>
<td>0.211537E-03</td>
<td>0.164509E-03</td>
<td>0.112653E-03</td>
</tr>
<tr>
<td>$[0, 0.70]$</td>
<td>0.119209E-04</td>
<td>0.572205E-05</td>
<td>0.774860E-05</td>
<td>0.715256E-05</td>
</tr>
</tbody>
</table>

**TABLE III**
### EXPECTED VALUES

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\lambda_1$</th>
<th>$\alpha_2$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.100000E01</td>
<td>-0.100000E00</td>
<td>0.100000E01</td>
<td>-0.333333E-01</td>
</tr>
<tr>
<td>0.600000E00</td>
<td>-0.200000E00</td>
<td>0.100000E01</td>
<td>-0.333333E-01</td>
</tr>
<tr>
<td>0.100000E01</td>
<td>-0.100000E00</td>
<td>0.100000E01</td>
<td>-0.500000E-01</td>
</tr>
<tr>
<td>0.200000E00</td>
<td>-0.333333E-01</td>
<td>0.500000E-02</td>
<td>-0.156666E00</td>
</tr>
<tr>
<td>0.500000E00</td>
<td>-0.500000E-01</td>
<td>0.100000E01</td>
<td>-0.333333E-01</td>
</tr>
</tbody>
</table>

### RELATIVE ERRORS

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\lambda_1$</th>
<th>$\alpha_2$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1192094E-04</td>
<td>0.572205E-05</td>
<td>0.774860E-05</td>
<td>0.715256E-05</td>
</tr>
<tr>
<td>0.131130E-04</td>
<td>0.192523E-03</td>
<td>0.798702E-04</td>
<td>0.152886E-05</td>
</tr>
<tr>
<td>0.653227E-04</td>
<td>0.553131E-04</td>
<td>0.834465E-05</td>
<td>0.220537E-04</td>
</tr>
<tr>
<td>0.974992E00</td>
<td>0.389990E02</td>
<td>0.389991E01</td>
<td>0.800000E00</td>
</tr>
<tr>
<td>0.656894E-03</td>
<td>0.321507E-03</td>
<td>0.221431E-03</td>
<td>0.393391E-04</td>
</tr>
</tbody>
</table>

### TABLE IV

<table>
<thead>
<tr>
<th></th>
<th>One Component</th>
<th>Two Components</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\det(G_1)$</td>
<td>0.328244E01</td>
<td>0.739940E01</td>
</tr>
<tr>
<td>$\det(G_2)$</td>
<td>-0.291181E-06</td>
<td>0.243475E-01</td>
</tr>
<tr>
<td>$\det(G_3)$</td>
<td>0.339364E-15</td>
<td>-0.325250E-09</td>
</tr>
<tr>
<td>$\det(G_4)$</td>
<td>0.397607E-24</td>
<td>-0.186056E-17</td>
</tr>
</tbody>
</table>

### TABLE V
BIBLIOGRAPHY


