ESTIMATATION OF THE GUARANTEE TIME IN A
TWO-PARAMETER EXPONENTIAL FAILURE MODEL

by

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ABSTRACT

There are available several classical point estimators of the guarantee time (location parameter) in a two-parameter exponential failure model. For the purpose of making a pairwise comparison of the estimators, a two-fold technique is introduced which essentially examines (a) the "odds" in favor of an estimator being closer to the true value than is a competing estimator and (b) an estimator's average closeness given that it is closer to the true value as well as given that it is not. Closeness to the true value is measured through an absolute error loss function. Joint consideration of these concepts is discussed and shown to form a basis for determining which of two estimators is preferred in a given situation.

Key Words: Two-parameter exponential distribution; Pairwise comparison of estimators; Pitman-closeness efficiency; Conditional mean absolute error.
1. **INTRODUCTION**

The two-parameter exponential distribution, denoted by \( \exp(\mu, \beta) \) and with density function

\[
f(x; \mu, \beta) = \frac{1}{\beta} \exp[-(x - \mu)/\beta], \quad x \geq \mu
\]

\[(\mu \geq 0, \beta > 0)\]  

is widely used as a failure model for component reliability. Essentially, the lifetime is guaranteed during the interval \([0, \mu)\), i.e., the probability of failure during this interval is zero. On the other hand, the failure rate \( \beta^{-1} \) is constant over the interval \([\mu, \infty)\). The location parameter \( \mu \) is often called the guarantee time.

Accuracy in point estimation of \( \mu \) can be crucial. Underestimating \( \mu \) clearly devalues the true quality of the component; whereas overestimating \( \mu \) allows for the possibility of a future component to fail during the estimated interval of guarantee. When there are several point estimators of \( \mu \), a comparison based on some "average closeness to the true value of \( \mu \)" criteria should be made in order to judge which estimator might be preferred for a given situation. An arbitrary yet frequently considered approach is mean squared efficiency, i.e., the ratio of the mean squared errors of two competing estimators (see, e.g., Box and Tiao 1973, pp. 306–307). However, the appropriateness
as well as precise physical meaning of such a ratio is not particularly apparent. Furthermore, its value can be quite misleading (Dyer, Keating, and Hensley 1978).

Alternatively, we propose a two-fold pairwise comparison of estimators technique which basically examines (a) the "odds" in favor of an estimator being closer to \( \mu \) than is a competing estimator and (b) an estimator's average closeness to \( \mu \) not only when it is closer to \( \mu \) than is a competing estimator but also when it is not. This approach will provide insight and information with regard to the relative behavior of two competing estimators. In addition, it should prove to be a suitable vehicle through which a preference between estimators can be made.

2. THEORY AND METHODOLOGY

Let \( X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(r)} \) be the ordered first \( r \) \( (2 \leq r \leq n) \) failure times of a random sample of size \( n \) from \( \text{exp} (\mu, \beta) \) when both \( \mu \) and \( \beta \) are unknown. We assume \( r \) is specified, i.e., the failure times constitute a Type II censored sample. Throughout the paper, we restrict attention to the class of estimators of \( \mu \) given by

\[
M = \{X_{(1)} - \alpha(S - X_{(1)})\},
\]

where \( \alpha = \alpha(r) \) is a nonnegative real number for fixed \( r \) and

\[
ns = \sum_{i=1}^{r} X_{(i)} + (n - r)X_{(r)}
\]

is the total time on test. As we shall see, the class \( M \) contains the estimators of \( \mu \) based on the usual classical methods of point estimation theory. The error incurred by using \( \hat{\mu}_1 \) (\( \hat{\mu}_1 \in M \)) to estimate \( \mu \) is measured by an absolute error loss function:

\[
f_1 = |\hat{\mu}_1 - \mu|.
\]
2.1. Pitman-Closeness Efficiency

Let \( \hat{\mu}_1, \hat{\mu}_2 \) be two arbitrary estimators in \( M \). Write
\[
\psi(\hat{\mu}_1, \hat{\mu}_2; \omega) = \text{Prob}\{L_1 < L_2\}, \quad \omega = (\mu, \beta, r, n).
\]
Then \( \hat{\mu}_1 \) is said to be a Pitman-closer estimator of \( \mu \) than is \( \hat{\mu}_2 \) if and only if
\[
\psi(\hat{\mu}_1, \hat{\mu}_2; \omega) > 0.5 \quad \text{for all} \quad \omega \quad \text{with strict inequality for some} \quad \omega_0
\]
(Pitman 1939). The Pitman-closeness (PC) efficiency of \( \hat{\mu}_1 \) relative to \( \hat{\mu}_2 \) is defined as

\[
\text{rel.eff.}_\text{PC}(\hat{\mu}_1, \hat{\mu}_2; \omega) = \frac{\psi(\hat{\mu}_1, \hat{\mu}_2; \omega)}{\psi(\hat{\mu}_2, \hat{\mu}_1; \omega)}. \quad (2.1)
\]

Prob\{\( L_1 < L_2 \)\} is determined in the classical sense and thus has a relative-frequency interpretation. Consequently, \( \text{rel.eff.}_\text{PC}(\hat{\mu}_1, \hat{\mu}_2; \omega) \) gives the "odds" in favor of \( \hat{\mu}_1 \) being closer to the true value of \( \mu \) than is \( \hat{\mu}_2 \). If, for example, \( \text{rel.eff.}_\text{PC}(\hat{\mu}_1, \hat{\mu}_2; \omega) = 1.5 \), then the "odds" are 3 to 2 in favor of \( \hat{\mu}_1 \) being closer to \( \mu \) than is \( \hat{\mu}_2 \). Whenever \( \text{rel.eff.}_\text{PC}(\hat{\mu}_1, \hat{\mu}_2; \omega) > 1 \) for some \( \omega_0 \), \( \hat{\mu}_1 \) is said to be more PC efficient than \( \hat{\mu}_2 \) at \( \omega_0 \). If \( \text{rel.eff.}_\text{PC}(\hat{\mu}_1, \hat{\mu}_2; \omega) > 1 \) for all \( \omega \) with strict inequality for some \( \omega_0 \), then \( \hat{\mu}_2 \) is said to be PC inadmissible relative to \( \hat{\mu}_1 \).

The following theorem will prove useful when examining PC efficiency within the class \( M \). The proof of this and other theorems depends upon these well-known results given by Epstein and Sobel (1954): (a) \( X_1 \) and \( V = n(S - X_1) \) are independent, (b) \( (X_1, V) \) is complete sufficient for \( (\mu, \beta) \), (c) the distribution of \( X_1 \) is \( \exp(\mu, \beta/n) \), and (d) the distribution of \( V \) is gamma \( (r - 1, \beta) \). It follows that the ratio
\[
F_{2, 2(r-1)} = \frac{(r - 1)X_1 - \mu}{(S - X_1)} \quad \text{has an F-distribution with} \quad 2 \quad \text{and} \quad 2(r - 1) \quad \text{degrees of freedom}. 
\]
**Theorem 2.1:** Let \( \hat{\mu}_i = X(1) - \alpha_i(S - X(1)) \), \( i = 1,2 \), be two estimators in \( M \) such that for fixed \( r \), \( \alpha_1 > \alpha_2 > 0 \), i.e., \( \text{Prob}(\hat{\mu}_1 < \hat{\mu}_2) = 1 \). Then

\[
\psi(\hat{\mu}_1, \hat{\mu}_2; r) = \left[ \frac{2}{(\alpha_1 + \alpha_2 + 2)} \right]^{r-1} \tag{2.2}
\]

**Proof:** For fixed \( r \),

\[
\psi(\hat{\mu}_1, \hat{\mu}_2; \mu, \sigma, r, n) = \text{Prob}\{ |\hat{\mu}_1 - \mu| < |\hat{\mu}_2 - \mu| \}
\]

\[
= \text{Prob}\{ \hat{\mu}_1 + \hat{\mu}_2 > 2\mu \} = \text{Prob}\{ X(1) - \mu > (\alpha_1 + \alpha_2)(S - X(1))/2 \}
\]

\[
= \text{Prob}\{ F_{2,2(r-1)} > (\alpha_1 + \alpha_2)(r - 1)/2 \}.
\]

The result then follows since the 100\( \gamma \)th percentile of an \( F \)-distribution with 2 and 2\( (r - 1) \) degrees of freedom is \( (r - 1)[(1 - \gamma)^{-(r-1)} - 1] \). \( \blacksquare \)

2.2. **Conditional Mean Absolute Error**

In addition to the "odds" in favor of an estimator being closer to \( \mu \) than is a competing estimator, we also require a measure of its average closeness to \( \mu \) not only when it is closer but also when it is not. Again, let \( \hat{\mu}_i, \ i = 1,2 \), be two arbitrary estimators in \( M \). We write \( \text{CMAE}(\hat{\mu}_j | \hat{\mu}_1; \omega) = E(\ell_j | \ell_1 < \ell_2), \ j = 1,2, \) as the **conditional mean absolute error** (CMAE) of \( \hat{\mu}_j \) given that \( \hat{\mu}_1 \) is closer to \( \mu \). The following lemma will be needed to develop the criteria by which a pairwise comparison of the estimators in \( M \) can be made through their CMAEs.
**Lemma 2.2:** Let \((X,Y)\) be a random vector and \(h\) be a real-valued Borel measurable function. Then

\[
E[h(X,Y) | (X,Y) \in A] = E[h(X,Y) \cdot I_A(X,Y)] / \text{Prob}(X,Y \in A),
\]

where \(A\) is a Borel set in \(\mathbb{R}^2\); and \(I_A(g) = 1\) if \(g \in A\), = 0 otherwise.

The lemma follows by noting that the conditional expectation given the event \(A\) equals the unconditional expectation in which the underlying distribution is truncated outside \(A\).

**Theorem 2.3:** Let \(\hat{\mu}_i = X_{(1)} - \alpha_i (S - X_{(1)})\), \(i = 1,2\), be two estimators in \(M\) such that for fixed \(r\), \(\alpha_1 > \alpha_2 > 0\), i.e., \(\text{Prob}(\hat{\mu}_1 < \hat{\mu}_2) = 1\). Then

\[
\text{CMAE}(\hat{\mu}_j | \hat{\mu}_1; r, \beta/n). \text{Prob}(\min(\ell_1, \ell_2) = \ell_1) = \frac{\beta}{n} \left( [1 + (-1)^{j+1}] g(\alpha_1; \alpha_j) + (-1)^{j+1-1} g[(\alpha_1 + \alpha_2)/2; \alpha_j] \right) + (i-1)[-1 + \alpha_j(r-1)] \beta/n, \quad i,j = 1,2
\]

where

\[
g(x; \alpha_j) = (1 + x)^{-(r-1)} + (r - 1)(x - \alpha_j)(1 + x)^{-r}.
\]

**Proof:** We first determine an expression for \(E(\ell_j | \ell_1 < \ell_2) \cdot \text{Prob}(\ell_1 < \ell_2), \quad j = 1,2\). Let \(A = \{\ell_1 < \ell_2\}, \quad A_1 = \{\hat{\mu}_1 > \mu\}, \quad \text{and} \quad A_2 = \{\hat{\mu}_1 < \mu, \hat{\mu}_1 + \hat{\mu}_2 > 2\mu\}. \) Note that \(A = A_1 \cup A_2\) and \(A_1 \cap A_2 = \phi\). By Lemma 2.2,
\[ E(\ell_j | \ell_1 < \ell_2) \cdot \text{Prob}\{\ell_1 < \ell_2\} = E(\ell_j I_A) \]
\[ = E(\ell_j I_{A_1 \cup A_2}) = E(\ell_j I_{A_1}) + E(\ell_j I_{A_2}). \]

Since \( X = 2n(X(1) - \mu)/\beta \) has a chi-square distribution with 2 degrees of freedom, \( Y = 2n(S - X(1))/\beta \) has a chi-square distribution with \( 2(r - 1) \) degrees of freedom, \( X \) and \( Y \) are independent, then

\[ E(\ell_j I_{A_1}) = (\beta/2n) \int_0^\infty \int_0^\infty (x - \alpha_j y)^2 (x; 2) (y; 2(r - 1)) \, dx \, dy, \]

where \( \chi^2(\cdot; m) \) is the chi-square density function with \( m \) degrees of freedom. Upon evaluating the double integral we obtain

\[ E(\ell_j I_{A_1}) = (\beta/n) g(\alpha_1; \alpha_j), \]
where \( g(x; \alpha_j) \) is given by (2.5). In addition,

\[ E(\ell_j I_{A_2}) = (-1)^j (\beta/2n) \int_0^\infty \int_0^\infty (x - \alpha_j y)^2 (x; 2) \chi^2(y; 2(r - 1)) \, dx \, dy \]
\[ = (-1)^{j+1} E(\ell_j I_{A_1}) + (-1)^j (\beta/n) g(\alpha_1 + \alpha_2; 2; \alpha_j). \]

By combining terms we get (2.4) when \( i = 1 \). The case where \( i = 2 \) can be obtained in a similar way.

2.3. **Weighted Average Reduction in Error**

PC efficiency and the corresponding CMAEs, when individually considered, provide insight and information with regard to the relative behavior of two competing estimators. When jointly considered, these two concepts form a rather natural basis for determining which of two estimators is preferred in a given situation. First note that

\[ E(\ell_2 - \ell_1 | \ell_1 < \ell_2) = \text{CMAE}(\hat{\mu}_2 | \hat{\mu}_1; \omega) - \text{CMAE}(\hat{\mu}_1 | \hat{\mu}_1; \omega) \]

is a measure of the average reduction in error when \( \hat{\mu}_1 \) is closer to \( \mu \) than is \( \hat{\mu}_2 \). The weighted average reduc-
tion in error (WARE) when \( \hat{\mu}_1 \) is closer to \( \mu \) than is \( \hat{\mu}_2 \) is

\[
\phi(\hat{\mu}_1, \hat{\mu}_2; \omega) = E(L_2 - L_1 | L_1 < L_2) \cdot \text{Prob}(L_1 < L_2).
\] (2.6)

Whenever \( \phi(\hat{\mu}_1, \hat{\mu}_2; \omega) > \phi(\hat{\mu}_2, \hat{\mu}_1; \omega) \) for some \( \omega_0 \), \( \hat{\mu}_1 \) is said to be WARE preferred over \( \hat{\mu}_2 \) at \( \omega_0 \). If \( \phi(\hat{\mu}_1, \hat{\mu}_2; \omega) \geq \phi(\hat{\mu}_2, \hat{\mu}_1; \omega) \) for all \( \omega \) with strict inequality for some \( \omega_0 \), then \( \hat{\mu}_2 \) is said to be WARE inadmissible relative to \( \hat{\mu}_1 \). As we shall later see, \( \hat{\mu}_1 \) being WARE preferred over \( \hat{\mu}_2 \) is equivalent to the (unconditional) mean absolute error of \( \hat{\mu}_1 \) being less than the (unconditional) mean absolute error of \( \hat{\mu}_2 \). The expression

\[
\text{DWP}(\hat{\mu}_1, \hat{\mu}_2; \omega) = \phi(\hat{\mu}_1, \hat{\mu}_2; \omega) - \phi(\hat{\mu}_2, \hat{\mu}_1; \omega)
\] (2.7)

will be called the degree of WARE preference (DWP) of \( \hat{\mu}_1 \) relative to \( \hat{\mu}_2 \).

By Theorem 2.3 and (2.7), we obtain the following

**Corollary 2.4:** The degree of WARE preference of \( \hat{\mu}_1 \) relative to \( \hat{\mu}_2 \), where \( \text{Prob}(\hat{\mu}_1 < \hat{\mu}_2) = 1 \) for fixed \( r \), is

\[
\text{DWP}(\hat{\mu}_1, \hat{\mu}_2; r, \beta/n) = [2(1 + \alpha_2)^{-(r-1)} - 2(1 + \alpha_1)^{-(r-1)} - (r - 1)(\alpha_1 - \alpha_2)] \beta/n.
\] (2.8)

3. **Comparison of Estimators**

The classical methods of statistical inference provide several point estimators of \( \mu \) based on Type II censored test data. The maximum likelihood estimator of \( \mu \) is

\[
\hat{\mu}_{\text{MLE}} = X(1),
\] (3.1)
an estimator which overestimates \( \mu \) with probability one. The minimum variance unbiased estimator of \( \mu \), as given by Epstein and Sobel (1954), is

\[
\hat{\mu}_{MVUE} = X(1) - (S - X(1))/(r - 1), \quad r \geq 2. \tag{3.2}
\]

Mann's (1969) best invariant estimator of \( \mu \) is

\[
\hat{\mu}_{BIE} = X(1) - (S - X(1))/r. \tag{3.3}
\]

All three of the above estimators are members of the class \( M \). Furthermore, it can be shown that the mean squared error of an arbitrary estimator in \( M \) is

\[
E[(\hat{\mu} - \mu)^2] = [r(r - 1)(\alpha - 1/r)^2 + (r + 1)/r]\beta^2/n^2.
\]

Thus \( \hat{\mu}_{BIE} \) is the minimum mean squared error estimator of \( \mu \) within the class \( M \).

**Theorem 3.1:** \( \hat{\mu}_{MLE} \) is PC inadmissible relative to \( \hat{\mu}_{MVUE} \) as well as \( \hat{\mu}_{BIE} \). \( \hat{\mu}_{BIE} \) is more PC efficient than \( \hat{\mu}_{MVUE} \) if and only if \( r \geq 3 \).

**Proof:** By the definition of PC efficiency and Theorem 2.1,

\[
\text{rel.eff.}_{PC}(\hat{\mu}_{MVUE};\hat{\mu}_{MLE};r) = \{(1 + 1/2(r - 1))^{r-1} - 1\}^{-1}. \tag{3.4}
\]

However, \( g(r) = [1 + 1/2(r - 1)]^{r-1} \) is an increasing function of \( r \) with \( g(2) = 1.5 \) and \( \lim_{r \to \infty} g(r) = \sqrt{e} \). Thus, \( 2.0 \geq \lim_{r \to \infty} \text{rel.eff.}_{PC}(\hat{\mu}_{MVUE};\hat{\mu}_{MLE};r) > (\sqrt{e} - 1)^{-1} \approx 1.542 \) for \( r \geq 2 \), and \( \hat{\mu}_{MLE} \) is PC inadmissible relative to \( \hat{\mu}_{MVUE} \). In addition, we have

\[
\text{rel.eff.}_{PC}(\hat{\mu}_{BIE};\hat{\mu}_{MLE};r) = \{(1 + 1/2r)^{r-1} - 1\}^{-1}. \tag{3.5}
\]

Since \( h(r) = (1 + 1/2r)^{r-1} \) is an increasing function of \( r \) with \( h(2) = 1.25 \) and \( \lim_{r \to \infty} h(r) = \sqrt{e} \), then \( 4.0 \geq \lim_{r \to \infty} \text{rel.eff.}_{PC}(\hat{\mu}_{BIE};\hat{\mu}_{MLE};r) > \)}
\((\sqrt{e} - 1)^{-1} \approx 1.542\) for \(r \geq 2\). Hence \(\hat{\mu}_\text{MLE}\) is PC inadmissible relative to \(\hat{\mu}_\text{BIE}\). Finally, we have

\[
\text{rel. eff.}_{\text{PC}}(\hat{\mu}_\text{BIE}, \hat{\mu}_\text{MVUE}; r) = [1 - (2r - 1)/(2r^2 - 1)]^{-(r-1)} - 1.
\]  
(3.6)

However, \(k(r) = [1 - (2r - 1)/(2r^2 - 1)]^{r-1}\) is a decreasing function of \(r\) with \(k(2) = 4/7\), \(k(3) = 144/289\), and \(\lim_{r \to \infty} k(r) = e^{-1}\). Then

\[
\text{rel. eff.}_{\text{PC}}(\hat{\mu}_\text{BIE}, \hat{\mu}_\text{MVUE}; 2) = .75, \text{ and for } r \geq 3, 145/144 < \text{rel. eff.}_{\text{PC}}(\hat{\mu}_\text{BIE}, \hat{\mu}_\text{MVUE}; r) < e - 1 \approx 1.718.
\]

Within the class \(M\) there exists a Pitman-closest estimator of \(\mu\) in the sense that this estimator is a Pitman-closer estimator of \(\mu\) relative to all other estimators in \(M\).

**Theorem 3.2:** Within the class \(M\), the median unbiased estimator

\[
\hat{\mu}_\text{MUE} = X_{(1)} - [2(r-1)^{-1} - 1](S - X_{(1)})
\]  
(3.7)

is the Pitman-closest estimator of \(\mu\).

**Proof:** For fixed \(r\), the estimators in \(M\) are ordered. The median unbiased estimator in \(M\) (Lehmann 1959, p.83) is such that

\[
\text{Prob}(X_{(1)} - \alpha^*(S - X_{(1)}) < \mu) = .5, \text{ or, equivalently, } \text{Prob}(F_{2,2(r-1)} < \alpha^*(r - 1)) = .5. \text{ Hence, } \alpha^* = (.5)^{-1} - 1 = 2(r-1)^{-1} - 1. \text{ Since this estimator is as likely to underestimate as to overestimate } \mu, \text{ the result follows. }
\]

By Theorems 3.1 and 3.2 \(\hat{\mu}_\text{MVUE}\), \(\hat{\mu}_\text{BIE}\), and \(\hat{\mu}_\text{MUE}\) are each more likely to be closer to \(\mu\) than is \(\hat{\mu}_\text{MLE}\). Moreover, it can be shown by Theorem 2.3 that the average reduction in error when \(\hat{\mu}_\text{MVUE}\) (or \(\hat{\mu}_\text{BIE}\)
or \( \hat{\mu}_{MUE} \) is closer to \( \mu \) is, generally speaking, greater than when \( \hat{\mu}_{MLE} \) is closer to \( \mu \). More specifically, we prove the following

**Theorem 3.3:** \( \hat{\mu}_{MLE} \) is WARE inadmissible relative to \( \hat{\mu}_{MVUE}, \hat{\mu}_{BIE}, \) and \( \hat{\mu}_{MUE}. \) \( \hat{\mu}_{MVUE} \) is WARE inadmissible relative to \( \hat{\mu}_{BIE} \) and \( \hat{\mu}_{MUE} \).

**Proof:** For fixed \( r, \) \( f(\alpha_1;r) = 2(1 + \alpha_1)^{-r+1} + (r - 1)\alpha_1, \alpha_1 \geq 0, \) is a decreasing function of \( \alpha_1 \) if and only if \( \alpha_1 < 2^{1/r} - 1. \) Furthermore, \( f(0;r) = 2 \) and \( f[1/(r - 1);r] = 2(1 - 1/r)^{r-1} + 1 \leq 2 \) with equality only when \( r = 2. \) Since \( 2^{1/r} - 1 < 1/(r - 1), \) then \( f(\alpha_1;r) < 2 \) if and only if \( \alpha_1 < 1/(r - 1). \) But \( f(\alpha_1;r) < 2 \) is equivalent to \( DWP(\hat{\mu}_{MLE};r, \beta/n) > 0. \) Thus \( \hat{\mu}_{MLE} \) is WARE inadmissible relative to any estimator in the subclass \( M_0 = \{X(1) - \alpha(S - X(1)), 0 < \alpha < 1/(r - 1)\}, \) which includes \( \hat{\mu}_{MVUE}, \hat{\mu}_{BIE}, \) and \( \hat{\mu}_{MUE}. \)

In a similar way, it can be shown that \( \hat{\mu}_{MVUE} \) is WARE inadmissible relative to any estimator in a subclass of \( M \) which contains \( M_1 = \{X(1) - \alpha(S - X(1)), 2^{1/r} - 1 < \alpha < 1/(r - 1)\}. \) The subclass \( M_1 \) includes \( \hat{\mu}_{BIE} \) and \( \hat{\mu}_{MUE}. \)

Having eliminated \( \hat{\mu}_{MLE} \) and \( \hat{\mu}_{MVUE} \) from consideration from a WARE inadmissibility point of view, we now compare in detail \( \hat{\mu}_{BIE} \) and \( \hat{\mu}_{MUE}. \) Although \( \hat{\mu}_{BIE} \) is PC inadmissible relative to \( \hat{\mu}_{MUE}, \) i.e., the "odds" favor \( \hat{\mu}_{MUE} \) as being closer to \( \mu, \) there are certain situations in which the average reduction in error when \( \hat{\mu}_{BIE} \) is closer to \( \mu \) is considerably more than the average reduction in error when \( \hat{\mu}_{MUE} \) is closer to \( \mu. \) Table 1 compares \( \hat{\mu}_{MUE} \) and \( \hat{\mu}_{BIE} \) through PC efficiency, the corresponding CMAEs, and DWP. In conclusion, \( \hat{\mu}_{MUE} \) is WARE preferred over \( \hat{\mu}_{BIE} \) if and only if \( r > 5. \)
4. OPTIMAL ESTIMATOR OF $\mu$

By Lemma 2.2 and Theorem 2.3,

$$E(L_j) = E(L_j | L_1 < L_2) \cdot \text{Prob}(L_1 < L_2) + E(L_j | L_2 < L_1) \cdot \text{Prob}(L_2 < L_1)$$

$$= [2(1 + \alpha_j)^{-\alpha r - 1} - 1 + (r - 1)\alpha_j] \beta/n.$$  \hspace{1cm} (4.1)

The right-hand side of (4.1) may be viewed as a decomposition of $E(L_j)$, the (unconditional) mean absolute error of $\mu_j$, into two components (the CMAEs of $\mu_j$) induced by the additive constants $\text{Prob}(L_1 < L_2)$ and $\text{Prob}(L_2 < L_1)$ (the PC efficiency probabilities). This decomposition is unique only between pairs of estimators. The value of $\alpha_j$ which minimizes (4.2) for fixed $r$ is $\alpha_j = 2^{1/r} - 1$. Thus, within the class $M$, the minimum mean absolute error estimator of $\mu$ is

$$\hat{\mu}_{\text{MMAE}} = X_{(1)} - (2^{1/r} - 1)(s - X_{(1)}).$$ \hspace{1cm} (4.3)

This estimator has the following optimality property.

**Theorem 4.1:** Any estimator in $M$ is WARE inadmissible relative to $\hat{\mu}_{\text{MMAE}}$.

**Proof:** Let $\hat{\mu}_1$ be an arbitrary estimator in $M$ and $\hat{\mu}_j = \hat{\mu}_{\text{MMAE}}$.

Since $E(L_j) < E(L_1)$, then by (4.1)

$$0 < E(L_1) - E(L_j)$$

$$= E(L_1 | L_1 < L_2) \cdot \text{Prob}(L_1 < L_2) + E(L_1 | L_2 < L_1) \cdot \text{Prob}(L_2 < L_1)$$

$$- E(L_j | L_1 < L_2) \cdot \text{Prob}(L_1 < L_2) - E(L_j | L_2 < L_1) \cdot \text{Prob}(L_2 < L_1)$$
From the proof of Theorem 4.1, it follows that when comparing any two estimators in \( M \), the estimator with smaller mean absolute error is equivalently WARE preferred. Although \( \hat{\mu}_{MMAE} \) is WARE preferred over any estimator in \( M \), it would be of interest to examine in depth the extent of this preference for particular estimators. Tables 1 and 2 compare \( \hat{\mu}_{MMAE} \) with \( \hat{\mu}_{BIE} \) and \( \hat{\mu}_{MUE} \) through PC efficiency, the corresponding CMAEs, and the DWP.

When comparing \( \hat{\mu}_{MMAE} \) and \( \hat{\mu}_{BIE} \) for \( r < 3 \), the "odds" are approximately 2 to 1 that \( \hat{\mu}_{BIE} \) is closer to \( \mu \) than is \( \hat{\mu}_{MMAE} \). However, the average reduction in error when \( \hat{\mu}_{BIE} \) is closer to \( \mu \) is considerably less than the average reduction in error when \( \hat{\mu}_{MMAE} \) is closer to \( \mu \). A joint consideration of PC efficiency and the corresponding CMAEs (i.e., DWP) favors \( \hat{\mu}_{MMAE} \). It should also be pointed out that for all values of \( r \), \( \hat{\mu}_{BIE} \) is actually closer, on the average, to \( \mu \) when it loses (i.e., \( \hat{\mu}_{MMAE} \) is closer to \( \mu \) than is \( \hat{\mu}_{BIE} \)) than when it wins.

When \( r \geq 10 \), there is very little difference between \( \hat{\mu}_{MMAE} \) and \( \hat{\mu}_{MUE} \). Each of the estimators is about as likely to be closer to \( \mu \) as not, and the average reduction in error when each is closer is approximately the same. The DWP slightly favors \( \hat{\mu}_{MMAE} \).
1. A Comparison of Estimators of the Guarantee

**Time μ : \( \hat{\mu}_{MUE} \) versus \( \hat{\mu}_{BIE} \)**

<table>
<thead>
<tr>
<th>( r )</th>
<th>rel. eff. ( \text{rel. eff.}<em>{PC}(\hat{\mu}</em>{MUE}'; \hat{\mu}_{BIE}; r) )</th>
<th>CMAE(( \hat{\mu}_{MUE}; r )) ( \times \frac{n}{\beta} )</th>
<th>CMAE(( \hat{\mu}_{BIE}; r )) ( \times \frac{n}{\beta} )</th>
<th>DWP(( \hat{\mu}<em>{MUE}, \hat{\mu}</em>{BIE}; r )) ( \times \frac{n}{\beta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.333 (57%) (43%)</td>
<td>.893</td>
<td>1.143</td>
<td>-.167</td>
</tr>
<tr>
<td>3</td>
<td>1.127 (53%) (47%)</td>
<td>.946</td>
<td>1.059</td>
<td>-.037</td>
</tr>
<tr>
<td>4</td>
<td>1.024 (51%) (49%)</td>
<td>.988</td>
<td>1.012</td>
<td>-.006</td>
</tr>
<tr>
<td>5</td>
<td>1.037 (51%) (49%)</td>
<td>.505</td>
<td>.554</td>
<td>.008</td>
</tr>
<tr>
<td>10</td>
<td>1.172 (54%) (46%)</td>
<td>.412</td>
<td>.597</td>
<td>.028</td>
</tr>
<tr>
<td>15</td>
<td>1.223 (55%) (45%)</td>
<td>.388</td>
<td>.611</td>
<td>.033</td>
</tr>
<tr>
<td>25</td>
<td>1.265 (56%) (44%)</td>
<td>.371</td>
<td>.621</td>
<td>.037</td>
</tr>
</tbody>
</table>
### 2. A Comparison of Estimators of the Guarantee

**Time \( \hat{\mu}_{\text{MMAE}} \) versus \( \hat{\mu}_{\text{BIE}} \)**

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \text{rel.eff.}<em>{PC}(\hat{\mu}</em>{\text{MMAE}}, \hat{\mu}_{\text{BIE}}; r) )</th>
<th>( \text{CMAE}(\hat{\mu}_{\text{MMAE}}; r) \times n/\beta )</th>
<th>( \text{CMAE}(\hat{\mu}_{\text{BIE}}; r) \times n/\beta )</th>
<th>( \text{DWP}(\hat{\mu}<em>{\text{MMAE}}, \hat{\mu}</em>{\text{BIE}}; r) \times n/\beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.457 (31%) (69%)</td>
<td>.389</td>
<td>.529</td>
<td>.005</td>
</tr>
<tr>
<td>3</td>
<td>.681 (41%) (59%)</td>
<td>.373</td>
<td>.562</td>
<td>.012</td>
</tr>
<tr>
<td>4</td>
<td>.814 (45%) (55%)</td>
<td>.366</td>
<td>.579</td>
<td>.017</td>
</tr>
<tr>
<td>5</td>
<td>.902 (47%) (53%)</td>
<td>.362</td>
<td>.590</td>
<td>.021</td>
</tr>
<tr>
<td>10</td>
<td>1.099 (52%) (48%)</td>
<td>.355</td>
<td>.612</td>
<td>.030</td>
</tr>
<tr>
<td>15</td>
<td>1.172 (54%) (46%)</td>
<td>.352</td>
<td>.620</td>
<td>.034</td>
</tr>
<tr>
<td>25</td>
<td>1.234 (55%) (45%)</td>
<td>.350</td>
<td>.626</td>
<td>.037</td>
</tr>
</tbody>
</table>
3. A Comparison of Estimators of the Guarantee

Time \( \mu \): \( \hat{\mu}_{\text{MMAE}} \) versus \( \hat{\mu}_{\text{MUE}} \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \text{rel.eff.}<em>{PC}(\hat{\mu}</em>{\text{MMAE}}; \hat{\mu}_{\text{MUE}}; r) )</th>
<th>( \times \frac{n}{\beta} )</th>
<th>( \text{CMAE}(\hat{\mu}_{\text{MMAE}}; r) )</th>
<th>( \times \frac{n}{\beta} )</th>
<th>( \text{CMAE}(\hat{\mu}_{\text{MUE}}; r) )</th>
<th>( \times \frac{n}{\beta} )</th>
<th>( \text{DNP}(\hat{\mu}<em>{\text{MMAE}}, \hat{\mu}</em>{\text{MUE}}; r) )</th>
<th>( \times \frac{n}{\beta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.707 (41%)</td>
<td>.343</td>
<td>1.172</td>
<td>.879</td>
<td>.172</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(59%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>.788 (44%)</td>
<td>.354</td>
<td>.734</td>
<td>.903</td>
<td>.049</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(56%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>.836 (46%)</td>
<td>.363</td>
<td>1.087</td>
<td>.609</td>
<td>.023</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(54%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>.867 (46%)</td>
<td>.368</td>
<td>1.069</td>
<td>.550</td>
<td>.013</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(54%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>.932 (48%)</td>
<td>.378</td>
<td>1.035</td>
<td>.456</td>
<td>.003</td>
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</tr>
<tr>
<td></td>
<td>(52%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>.954 (49%)</td>
<td>.381</td>
<td>1.023</td>
<td>.431</td>
<td>.001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(51%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>.972 (49%)</td>
<td>.383</td>
<td>1.014</td>
<td>.412</td>
<td>&lt;.001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(51%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
REFERENCES


