THE METHOD OF UPPER, LOWER SOLUTIONS
AND VOLterra INTEGRAL EQUATIONS

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1. INTRODUCTION

In employing the method of upper and lower solutions to dynamical systems, one is required to impose a certain monotone property on the given system [5,6,11]. When the given system does not possess such a monotonic property, stronger forms of upper and lower solutions have to be assumed in order to obtain similar results [4,6,9]. Furthermore, if the system enjoys a mixed monotone property the method of quasi-upper and lower solutions, which is recently introduced, is most useful [7]. In this paper we shall extend these ideas to Volterra integral equations.

We want to note that Volterra integral equations and inequalities which can not be reduced to differential ones were first considered in [1,10]. We shall first consider various aspects of Müller's type result and then develop monotone iterative technique to establish the existence of coupled quasi-minimal and maximal solutions.

2. METHOD OF UPPER, LOWER SOLUTIONS

Consider the Volterra integral equation

\[ x(t) = f(t) + \int_0^t K(t,s,x(s)) \, ds, \]

where \( x, f \in C[\mathbb{I}, \mathbb{R}^n], K \in C[\mathbb{I}^2 \times \mathbb{R}^n, \mathbb{R}^n] \) and \( \mathbb{I} = [0,T] \). Without further

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mention we assume that all inequalities between vectors are componentwise.

We shall first prove the following result on integral inequalities without assuming monotone condition on \( K \).

**Theorem 1.** Assume that

(i) \( v, w, z, f \in \mathcal{C}[I, \mathbb{R}^n], K \in \mathcal{C}[I^2 \times \mathbb{R}^n, \mathbb{R}^n] \)

(2) \[ v(t) \leq f(t) + \int_0^t K[t, s, z(s)] ds \leq w(t), \]

holds whenever \( v(s) \leq z(s) \leq w(s) \) for \( 0 \leq s \leq t \) and \( t, s \in I \);

(ii) the function \( K \) satisfies

(3) \[ \|K[t, s, u] - K[t, s, \bar{u}]\| \leq L\|u - \bar{u}\|, \]

for \( s, t \in I \) and \( u, \bar{u} \in \mathbb{R}^n \), where \( L > 0 \) is a constant.

Then, if \( x(t) \) is any solution of (1) such that \( v(0) \leq x(0) \leq w(0) \), we have \( v(t) \leq x(t) \leq w(t) \) for \( t \in I \).

**Proof.** We shall first assume that \( v, w \) satisfy strict inequalities in (2) and prove the conclusion for strict inequalities. If the conclusion \( v(t) < x(t) < w(t) \) for \( t \in I \) is false, whenever \( v(0) < x(0) < w(0) \), then the set

\[ Q = \bigcup_{i=1}^n \left\{ t \in I : v_i(t) > x_i(t) \text{ or } x_i(t) > w_i(t) \right\} \]

is nonempty. Let \( t_0 = \inf Q \). Certainly \( t_0 > 0 \) and \( t_0 \in Q \), then there exists an index \( j \) such that
\[ v_j(t_0) = x_j(t_0) \quad \text{or} \quad x_j(t_0) = w_j(t_0), \]

\[ v_j(s) < x_j(s) < w_j(s), \quad s \in [0,t_0), \]

\[ v_i(s) < x_i(s) < w_i(s), \quad s \in [0,t_0], \quad i \neq j. \]

Now from (2), (4) and (1) we have

\[ v_j(t_0) < f_j(t_0) + \int_0^{t_0} k_j[t_0, s, x(s)]ds = x_j(t_0), \]

which is a contradiction to the assumption that \( Q \) is nonempty, which proves the result for strict inequalities.

Consider now \( \bar{w}(t) = w(t) + \varepsilon y_0 e^{nt}, \quad \bar{v}(t) = v(t) - \varepsilon y_0 e^{nt}, \) where \( y_0 = (1,1,\ldots,1), \varepsilon > 0 \) is an arbitrary small vector. Let \( p_i(t, x) = \max\{v_i(t), \min[x_i, w_i(t)]\} \) for each \( i \). Then it is clear that if \( \bar{z} \) is such that \( \bar{v}(t) \leq \bar{z}(t) \leq \bar{w}(t) \), it follows that \( z = p(t, \bar{z}) \) satisfies \( v(t) \leq z(t) \leq w(t) \) for \( t \in I \). Hence using (i) and (ii) we have

\[ \bar{w}_i(t) \geq f_i(t) + \int_0^t k_i[t, s, z(s)]ds + \varepsilon y_0 e^{nt} \]

\[ \geq f_i(t) + \int_0^t k_i[t, s, \bar{z}(s)]ds - \int_0^t \sqrt{n} \varepsilon y_0 e^{nt} ds + \varepsilon y_0 e^{nt} \]

\[ = f_i(t) + \int_0^t k_i[t, s, \bar{z}(s)]ds + cy_0 e^{nt} \left[ 1 - \frac{1}{\sqrt{n}} \right] + \frac{\varepsilon}{\sqrt{n}} y_0 \]

\[ > f_i(t) + \int_0^t k_i[t, s, \bar{z}(s)]ds, \]
for all \( z \) such that \( \overline{v}(t) \leq \overline{z}(t) \leq \overline{w}(t) \). Here we have used the fact that
\[
|\overline{z}_j - P_j(t, \overline{z})| \leq \varepsilon y_{0j} e^{\eta \mu t} \quad \text{for each } j.
\]
Similarly, we get
\[
\overline{v}_1(t) < f_1(t) + \int_0^t K_1[t, s, \overline{z}(s)] ds,
\]
for all \( z \) such that \( \overline{v}(t) \leq \overline{z}(t) \leq \overline{w}(t) \). Since \( \overline{v}(0) < x(0) < \overline{w}(0) \), we conclude that \( \overline{v}(t) < x(t) < \overline{w}(t) \) for \( t \in I \). As \( \varepsilon \) is arbitrary, the desired result follows by letting \( \varepsilon \to 0 \), which proves the theorem.

Under the assumptions of Theorem 1 we can show directly the existence of a unique solution of (1) in the sector \( \langle v, w \rangle = \{(t, z) : v(t) \leq z \leq w(t), t \in I\} \) by the method of successive approximations.

**Theorem 2.** Assume that the hypotheses (i) and (ii) of Theorem 1 hold. Then the successive approximations defined by

\[
x_0(t) = f(t) + \int_0^t K[t, s, v(s)] ds,
\]

(5)

\[
x_n(t) = f(t) + \int_0^t K[t, s, x_{n-1}(s)] ds, \quad n \geq 1,
\]

satisfies \( v(t) \leq x_n(t) \leq w(t) \) on \( I \) and converge uniformly on \( I \) to the unique solution \( x(t) \) of (1). Moreover, \( v(t) \leq x(t) \leq w(t) \) on \( I \).

**Proof.** From (2) and (5) it is easy to observe that the sequence \( \{x_n\} \) satisfies

(6)

\[
v(t) \leq x_n(t) \leq w(t), \quad t \in I.
\]
From (2), (3) and (5) one can obtain the estimate

\[ |x_n(t) - x_{n-1}(t)| \leq \frac{M(Lt)^{n-1}}{(n-1)!}, \quad t \in I, \]

where \( M \geq 0 \) is a constant. The rest of the proof for the existence of the solution of (1) can be completed by following the argument used in [2, Theorem 6.1, p. 120]. From (6) and the fact that \( \{x_n\} \) converges to the solution of (1) it is obvious that \( x(t) \) satisfies the relation \( v(t) \leq x(t) \leq w(t) \) for \( t \in I \). This completes the proof of the theorem.

If in Theorem 1 we delete the assumption that \( K \) is Lipschitzian, it is still possible to prove that there exists a solution of (1) in the sector \((v,w)\).

**Theorem 3.** Let \( v, w, z, f \in C[I, \mathbb{R}^n] \). Let \( D = \{(t,s,z): t, s \in I, 0 \leq s \leq t \) and \( v(s) \leq z \leq w(s)\} \) and \( K \in C[D, \mathbb{R}^n] \). Suppose that the assumption

\[ v(t) \leq f(t) + \int_0^t K[t,s,z(s)]ds \leq w(t), \tag{7} \]

holds whenever \( v(s) \leq z(s) \leq w(s) \) for \( 0 \leq s \leq t \) and \( t, s \in I \). Then (1) has a solution \( x(t) \) on \( I \) such that \( v(t) \leq x(t) \leq w(t) \) whenever \( v(0) \leq x(0) \leq w(0) \).

**Proof.** Consider \( P: I \times \mathbb{R}^n \to \mathbb{R}^n \) defined by \( P(t,u) = \max\{v_1(t), \min[u_1, w_1(t)]\} \). Then \( K[t,s,P(s,u)] \) defines a continuous extension of \( K \) to \( I^2 \times \mathbb{R}^n \) which is also bounded since \( K \) is bounded on \( D \). Therefore

\[ x(t) = f(t) + \int_0^t K[t,s,P(s,x(s))]ds, \tag{8} \]
has a solution \( x(t) \) on \( I \) (see, [2]). Now, we shall show that \( x(t) \) is in \( D \) and therefore a solution of (1). We note that \( v(s) \leq P(s,x(s)) \leq w(s) \), therefore,

\[
(9) \quad v(t) \leq f(t) + \int_0^t K[t,s,P(s,x(s))] ds \leq w(t),
\]

and hence, from (8) and (9) we have \( v(t) \leq x(t) \leq w(t) \) which implies \( (t,s,x(s)) \) is in \( D \). The proof of the theorem is complete.

3. MONOTONE ITERATIVE METHOD

In this section we use the monotone iterative technique to investigate the existence of coupled minimal and maximal solutions of (1). To define coupled lower and upper solutions of (1), we fix for each \( i \in \{1,2,\ldots,n\} \) two nonnegative integers \( p_i, q_i \) such that \( p_i + q_i = n \) and split \( x \in \mathbb{R}^n \) into \( x = ([x]_i^{p_i} [x]_i^{q_i}) \). Then (1) becomes

\[
(10) \quad x_i(t) = f_i(t) + \int_0^t K_i[t,s,[x(s)]_{p_i}^{q_i}] ds.
\]

The functions \( v, w \in C[I, \mathbb{R}^n] \) with \( v(t) \leq w(t) \) on \( I \) are said to be coupled lower and upper solutions of (1) if

\[
v_i(t) \leq f_i(t) + \int_0^t K_i[t,s,[v(s)]_{p_i}^{q_i}] ds,
\]

\[
w_i(t) \geq f_i(t) + \int_0^t K_i[t,s,[w(s)]_{p_i}^{q_i}] ds.
\]

The functions \( x, y \in C[I, \mathbb{R}^n] \) are said to be coupled solutions of (1) if
\[ x_i(t) = f_i(t) + \int_0^t k_i[t,s,[x(s)]_{p_i}, [y(s)]_{q_i}] ds, \]
\[ y_i(t) = f_i(t) + \int_0^t k_i[t,s,[y(s)]_{p_i}, [x(s)]_{q_i}] ds. \]

From the above definitions it is clear that one can define coupled minimal and maximal solutions of (1). In the special case when \( q_i = 0 \) for all \( i \), coupled lower and upper solutions are just upper and lower solutions of (1).

The function \( K \) is said to possess a mixed monotone property if for each \( i \), \( k_i[t,s,[x]_{p_i}, [y]_{p_i}] \) is monotone nondecreasing in \([x]_{p_i}\) and monotone nonincreasing in \([x]_{q_i}\).

We are now in a position to prove the following theorem which deals with the existence of a coupled minimal and maximal solution of (1).

**Theorem 4.** Let \( K \) possess mixed-monotone property and further assume that \( v, w \in C[I, \mathbb{R}^n] \) with \( v(t) \leq w(t) \) (denoted by \((v,w)) \) on \( I \) are coupled lower and upper solutions of (1). Then the sequences \( \{v_n\} \) and \( \{w_n\} \) defined by

\[ v_n(t) = f(t) + \int_0^t k[t,s,[v_{n-1}(s)]_{p_i}, [w_{n-1}(s)]_{q_i}] ds, \]
\[ w_n(t) = f(t) + \int_0^t k[t,s,[w_{n-1}(s)]_{p_i}, [v_{n-1}(s)]_{q_i}] ds, \]

with \( v_0 = v \) and \( w_0 = w \), converge uniformly and monotonically to coupled minimal and maximal solutions \( \alpha \) and \( \beta \) of (1) respectively,
that is, if \((x, y)\) are any coupled solutions of (1) such that \(x, y \in \langle v, w \rangle\) then

\[ v \leq v_1 \leq \ldots \leq v_n \leq x, \quad y \leq \beta \leq w_n \leq \ldots \leq w_1 \leq w \]

on \(I\). Furthermore, any solution \(x\) of (1) such that \(x \in \langle v, w \rangle\) also satisfies the inequality (13).

**Proof.** Define \(\phi_i(t) = v_{1i}(t) - v_i(t)\), then

\[
\phi_i(t) = f_i(t) + \int_0^t K_i[t, s, [v(s)]_{\alpha_i}, [w(s)]_{\beta_i}] ds - f_i(t) - \int_0^t K_i[t, s, [v(s)]_{\alpha_i}, [w(s)]_{\beta_i}] ds = 0,
\]

which implies \(v(t) \leq v_i(t)\) on \(I\). Similarly, we can show that \(w_1(t) \leq w(t)\) on \(I\). Let us assume that for some integer \(k > 0\), \(v_{k-1} \leq v_k\) and \(w_k \leq w_{k-1}\) on \(I\). Then setting \(\psi_i(t) = v_{(k+1)i}(t) - v_{ki}(t)\), and employing mixed monotone property of \(K\) we have

\[
\psi_i(t) = f_i(t) + \int_0^t K_i[t, s, [v_{k}(s)]_{\alpha_i}, [w_{k}(s)]_{\beta_i}] ds - f_i(t) - \int_0^t K_i[t, s, [v_{k-1}(s)]_{\alpha_i}, [w_{k-1}(s)]_{\beta_i}] ds
\]

\[
- f_i(t) + \int_0^t K_i[t, s, [v_{k-1}(s)]_{\alpha_i}, [w_{k-1}(s)]_{\beta_i}] ds - f_i(t) - \int_0^t K_i[t, s, [v_{k-1}(s)]_{\alpha_i}, [w_{k-1}(s)]_{\beta_i}] ds = 0,
\]

which implies \(v_k(t) \leq v_{k+1}(t)\). Similarly, we can show that \(w_{k+1}(t) \leq w_k(t)\).

Hence, it follows by induction \(v_{n-1} \leq v_n\) and \(w_n \leq w_{n-1}\) for all \(n\) on \(I\). Now, define \(w_{1i}(t) = w_{1i}(t) - v_{1i}(t)\), then using the fact that
\( v(t) \leq w(t) \) for \( t \in I \) and \( K \) has mixed-monotone property we have

\[
m_1(t) = f_1(t) + \int_{0}^{t} K_{1}[t, s, [w(s)]_{p_1}, [v(s)]_{q_1}] ds
\]

\[
-f_1(t) - \int_{0}^{t} K_{1}[t, s, [v(s)]_{p_1}, [w(s)]_{q_1}] ds
\]

\[
\geq f_1(t) + \int_{0}^{t} K_{1}[t, s, [v(s)]_{p_1}, [w(s)]_{q_1}] ds
\]

\[-f_1(t) - \int_{0}^{t} K_{1}[t, s, [v(s)]_{p_1}, [w(s)]_{q_1}] ds
\]

\[= 0,
\]

which implies \( v_1(t) \leq w_1(t) \) on \( I \). By following an induction argument we have \( v_n \leq w_n \) for all \( n \) on \( I \). Thus \( \{v_n\} \) and \( \{w_n\} \) are monotone and uniformly bounded on \( I \). We now use standard arguments [2,3] to show that these sequences are equi-continuous and hence it follows that \( \{v_n\}, \{w_n\} \) converge uniformly on \( I \). Letting \( \alpha = \lim_{n \to \infty} v_n \) and \( \beta = \lim_{n \to \infty} w_n \), we find

\[
\alpha_1(t) = f_1(t) + \int_{0}^{t} K_{1}[t, s, [\alpha(s)]_{p_1}, [\beta(s)]_{q_1}] ds,
\]

\[
\beta_1(t) = f_1(t) + \int_{0}^{t} K_{1}[t, s, [\beta(s)]_{p_1}, [\alpha(s)]_{q_1}] ds.
\]

This shows that \( \alpha \) and \( \beta \) are the coupled solutions of (1).

Now we shall show that \( \alpha \) and \( \beta \) are coupled minimal and maximal solutions of (1) respectively. Let \((x, y)\) be any coupled solution of (1) such that \( x, y \in (v, w) \). Define, \( r_1(t) = x_1(t) - v_1(t) \), then using the facts that \( v \leq x, y \leq w \) and \( K \) has mixed-monotone property, we get

\[
r_1(t) = f_1(t) + \int_{0}^{t} K_{1}[t, s, [x(s)]_{p_1}, [y(s)]_{p_1}] ds
\]

\[-f_1(t) - \int_{0}^{t} K_{1}[t, s, [v(s)]_{p_1}, [w(s)]_{q_1}] ds
\]
\[ \geq f_1(t) + \int_0^t K_1[t, s, [v(s)]]_{p_1} [w(s)]_{q_1} \, ds \]
\[ -f_1(t) - \int_0^t K_1[t, s, [v(s)]]_{p_1} [w(s)]_{q_1} \, ds \]
\[ = 0, \]

which implies \( v_1(t) \leq x(t) \) on \( I \). Similarly, we can show that \( y(t) \leq w_1(t) \).

By following an induction argument, we have \( v_n \leq x, y \leq w_n \) for all \( n \) on \( I \). Hence, we have \( \alpha \leq x, y \leq \beta \) on \( I \) proving that \( \alpha \) and \( \beta \) are coupled minimal and maximal solutions of (1). Since any solution \( x \) of (1) such that \( x \in \langle v, w \rangle \) can be considered as \( (x, x) \) coupled solution of (1), we also have \( \alpha \leq x \leq \beta \) on \( I \). This completes the proof of the theorem.

**Corollary 1.** If in Theorem 4, we assume further that \( K \) satisfies the Lipschitz condition (3), then \( \alpha(t) \equiv \beta(t) \) on \( I \) and consequently, there exists a unique solution of (1) in the sector \( \langle v, w \rangle \).

**Proof.** We let \( m(t) = \| \alpha(t) - \beta(t) \| \) and obtain
\[ m(t) \leq L \int_0^t m(s) \, ds. \]

Note that \( m(0) = 0 \). It then follows from
\[ m(t) \leq \varepsilon + L \int_0^t m(s) \, ds \]
the estimate \( m(t) \leq \varepsilon e^{Lt} \) on \( I \), for all \( t > 0 \). This implies, making \( \varepsilon \to 0 \), \( m(t) \equiv 0 \) and thus \( \alpha(t) \equiv \beta(t) \).

**Remark.** In the special case \( q_1 = 0 \) for all \( i \), the coupled lower and upper solutions \( v, w \) reduce to usual lower and upper solutions and \( K[t, s, x] \) is monotone nondecreasing in \( x \). On the other hand, if \( p_i = 0 \) for all \( i \), \( v, w \) are perhaps the best possible quasi-upper and lower solutions. In either special case the results obtained are new.
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