ON THE EXISTENCE OF SOLUTIONS
OF DIFFERENTIAL EQUATIONS AND ZEROS
OF OPERATORS IN K-BANACH SPACES

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Introduction.

The theory of existence of solutions of differential equations in a 
Banach space employing norm as a measure is sufficiently well known [5, 6, 
8, 9]. Also utilizing this theory one can prove the existence of zeros of 
operators [2, 7, 8, 9, 11]. The advantage of using a generalized norm as 
a candidate in discussing the qualitative theory of differential equations 
is also known [1]. These thoughts naturally lead to the use of cone-valued 
norms as a measure since this approach unifies the existing theories as well 
as offers a more flexible mechanism for applications.

In this paper, we wish to work in such a general setting and conse-
quently we develop the appropriate theory of Banach spaces whose norm is 
cone-valued. Using this as a vehicle we then prove an existence theorem 
for differential equations in K-Banach spaces which is then utilized to 
prove the existence of zeros of nonlinear operators.

1. Preliminaries.

Throughout this paper $E$ will denote a real Banach space with norm, 
$||\cdot||$ and $0$ will denote the zero of $E$. A cone, $K$, is a proper subset 
of $E$ such that if $v, \omega \in K$ and $\lambda > 0$ then $v + \omega, \lambda \omega \in K$. We will assume 
$K$ is a closed cone with non-void interior $(K^0 \neq \emptyset)$ and that $u, -u \in K$ 
implies $u = 0$. The following partial orderings on $E$ are induced by $K,$

$x \leq y$ if $y - xeK$

and $x < y$ if $y - xeK^0$. 
The following lemma states some fundamental properties of such cones for a proof see [4, 12].

**Lemma 1.1.** Let \( K \) be a closed cone and \( K^o \neq 0 \). Then

1) \( x \leq y, \ y < z \) implies \( x < z \);
2) \( x \leq y, \ z < w \) implies \( x + z < y + w \);
3) \( x_n < y_n \) for each \( n \) and \( x_n \rightarrow x, \ y_n \rightarrow y \) implies \( x \leq y \);
4) \( x \in K^o, y \in E \), then there exists \( \lambda \in \mathbb{R} \) such that \( x - \lambda y \in K^o \);
5) \( x \leq y \) and \( x \neq y \) then \( y - x \in b(K) \) where \( b(K) \) denotes the boundary of \( K \).

When limits are considered in \( E \) the concepts are usually defined in terms of \( || || \), however when dealing with cones the concept of limit with respect to the cones also makes sense.

**Definition 1.2.** Let \( \{ x_n \} \) be a sequence from \( E \) and \( x \in E \) then we say

\[
\lim_{n \to \infty} x_n = x \quad \text{(limit with respect to \( K \))}
\]

if for each \( \varepsilon \in K^o \) there exists \( N(\varepsilon) \) such that \( -\varepsilon < x_n - x < \varepsilon \) for all \( n > N(\varepsilon) \).

Similarly, if \( f: [t_0, t_0 + \alpha] \to E \) then \( \lim_{t \to t_1} f(t) = x \) if for \( \varepsilon \in K^o \) there exists \( \delta > 0 \) such that \( -\varepsilon < f(t) - x < \varepsilon \) whenever \( |t - t_1| < \delta \).

**Remark 1.** Let \( r_0 \in K^o \) with \( ||r_0|| = 1 \). For each \( \varepsilon \in K^o \) there exists \( \varepsilon > 0 \) such that \( \varepsilon r_0 < \varepsilon \). Thus in considering limit we will use the equivalent notion of convergence along the \( r_0 \)-ray. That is, \( \lim_{n \to \infty} x_n = x \) if for each \( \varepsilon > 0 \) there exists \( N(\varepsilon) \) such that \( -\varepsilon r_0 < x_n - x < \varepsilon r_0 \) for all \( n > N(\varepsilon) \), and \( \lim_{t \to t_1} f(t) = x \) if for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( -\varepsilon r_0 < f(t) - x < \varepsilon r_0 \) whenever \( |t - t_1| < \delta \).
In general, the concept of limit and Limit are different. However if we deal with normal cones these concepts coincide as we show in Lemma 1.6. For later purposes we will also need the concept of regular cones and so we introduce those definitions here.

**Definition 1.3.** A cone \( K \subseteq E \) is said to be a normal cone if there exists \( L > 0 \) such that whenever \( 0 \leq x \leq y \) then \( ||x|| \leq L \cdot ||y|| \).

**Definition 1.4.** A cone \( K \subseteq E \) is said to be regular if every non-decreasing sequence \( \{x_n\} (x_1 \leq x_2 \leq x_3 \leq \ldots) \) converges with respect to the cone.

**Theorem 1.5.** Every regular cone is normal.

The proof of the above theorem can be seen in [3].

Using a normal cone we can now show the equivalence of limit and Limit.

**Lemma 1.6.** Let \( K \) be a cone with \( K^0 \neq \emptyset \) and \( \{x_n\} \) a sequence from \( E \), \( x \in E \), and \( f:[t_0, t_0+\alpha] \rightarrow E \).

1) If \( \lim_{t \to t_1} f(t) = x \) then \( \lim_{t \to t_1} f(t) = x \).
2) If \( \lim_{n \to \infty} x_n = x \) then \( \lim_{n \to \infty} x_n = x \).

Furthermore if \( K \) is a normal cone the converses of 1) and 2) hold.

**Proof.** For \( \varepsilon > 0 \), \( \varepsilon_0 \in K^0 \) and so there exists \( \gamma > 0 \) such that if \( ||y|| < \gamma \) then \( \varepsilon_0 - y \in K^0 \) that is \( y < \varepsilon_0 \). Since \( \lim_{t \to t_1} f(t) = x \) for \( \gamma > 0 \) there exists \( \delta > 0 \) such that \( ||f(t) - x|| < \gamma \) whenever \( |t - t_1| < \delta \). Thus \( f(t) - x < \varepsilon_0 \) and \( -(f(t) - x) < \varepsilon_0 \) so that \( -\varepsilon_0 < f(t) - x < \varepsilon_0 \) and 1) is proved.
Suppose $K$ is a normal cone and $\lim_{t \to t_1} f(t) = x$. For $\epsilon > 0$ there exists $\delta > 0$ such that $-\frac{\epsilon}{4(L+1)} r_0 < f(t) - x < \frac{\epsilon}{4(L+1)} r_0$ whenever $|t - t_1| < \delta$. So $0 < f(t) - x + \frac{\epsilon}{4(L+1)} r_0 < 2 \frac{\epsilon}{4(L+1)} r_0$ and using the normality of $K$ we have

$$||f(t) - x + \frac{\epsilon}{4(L+1)}|| \leq 2L \left|\frac{\epsilon}{4(L+1)} r_0\right| = 2L \left[\frac{\epsilon}{4(L+1)}\right]$$

$$||f(t) - x|| - \left|\frac{\epsilon}{4(L+1)}||\right| \leq L \frac{\epsilon}{2(L+1)}$$

$$||f(t) - x|| \leq L \frac{\epsilon}{2(L+1)} + \frac{\epsilon}{4(L+1)} < \epsilon.$$

The proof of 2) and its converse assuming normality are similar.

Since the concepts of limits are equivalent we will no longer distinguish between them and use only the notation $\lim$. The following lemma is a direct consequence of the definitions and the proof is omitted.

**LEMMA 1.7.** Let $K$ be a closed normal cone in $E$ and let $f, g: [t_0, t_0 + a] \to E$.

1) If $\lim_{t \to t_1} g(t) = A$ and $g(t) \in K$ for each $t$ then $A \in K$.

2) If $\lim_{t \to t_1} g(t) = A$ and $\lim_{t \to t_1} f(t) = B$ then $\lim_{t \to t_1} (f + g)(t)$ exists and is equal to $A + B$.

For our purposes in this paper we need a simple differential inequality.

**THEOREM 1.8.** Let $K$ be a closed cone with $K^0 \neq \emptyset$ and let $u, v \in C\left([t_0, t_0 + a], E\right)$ such that $u(t_0) < v(t_0)$ and $u' + cu < v' + cv$ for each $t \in (t_0, t_0 + a)$ where $c > 0$. Then $u(t) < v(t)$ for all $t \in [t_0, t_0 + a]$. 


PROOF. Suppose not then there exists $t_1 \in (t_0, t_0 + \alpha]$ such that 
$u(t_1) \leq v(t_1)$, $u(t_1) \neq v(t_1)$ and $u(t) < v(t)$ on $[t_0, t_1)$. By the 
Ascoli-Mazur Theorem [see 12] there exists a continuous linear functional, 
$\phi$, on $E$ such that $\phi(x) > 0$ for all $x \in K^0$ and $\phi(v(t_1) - u(t_1)) = 0$. 
For $h < 0$ we have 
$\phi(v(t_1 + h) - u(t_1 + h)) > \phi(u(t_1 + h))$ 
and 
$\phi(v(t_1 + h) - u(t_1 + h)) - \phi(v(t_1) - u(t_1)) \overline{h} < 0$. Thus 
$[\phi^0; v - u]_*(t_1) \leq 0$. However 
$[\phi^0; v - u]_*(t) = \phi^0(v - u)(t) \overline{v}^* = \phi(u(t) - v(t))$ 
for all $t \in (t_0, t_0 + \alpha]$ so that for $t = t_1$ we have 
$[\phi^0; v - u]_*(t_1) > c \phi(u(t_1) - v(t_1)) = c \cdot 0 = 0$ and this contradiction proves the theorem.

With this background in mind we are now ready to introduce the main 
thought of our paper. Let $X$ be a real linear space with $\theta$ again denoting 
the zero element of $X$.

DEFINITION 1.9. A function $\| | \|_K : X \to K$ where $K$ is a closed normal 
cone in $E$ with $K^0 \neq \phi$ is said to be a $K$ norm on $X$ if 
1) $\| x \|_K \geq \theta$ and $\| x \|_K = \theta$ iff $x = \theta;$ 
2) $\| \lambda x \|_K = |\lambda| \| x \|_K$ for $\lambda$ real; 
3) $\| x + y \|_K \leq \| x \|_K + \| y \|_K$.

A sequence $\{x_n\}$ in $X$ is Cauchy if for $\epsilon > 0$ there exists $N(\epsilon)$ such 
that $\| x_n - x_m \|_K < \epsilon r_0$ for all $m, n \geq N(\epsilon)$ and similarly $\{x\}_n$ converges 
to $x$ if for each $\epsilon > 0$ there exists $N(\epsilon)$ such that $\| x_n - x \|_K < \epsilon r_0$ 
for all $n \geq N(\epsilon)$. The space $X$, $\| | \|_K$ is said to be a $K$-Banach space if 
every Cauchy sequence in $X$ converges in $X$. 
REMARK 2. Concepts such as $K$-continuity, $K$-derivatives and the generalized Reimann integral for functions in $X$ are all defined in terms of $||||_K$ which deals with limits in the cone $K$.

REMARK 3. The topology for $X$ generated by $||||_K$ where $K$ is a normal cone is indeed normable in the usual sense [see page 136 in (10)]. For purely topological purposes it is immaterial whether we view $X$ as a $K$-Banach space or as an ordinary Banach space. We do, however, have more flexibility employing the cone valued norm as a measure. For example, the concept of generalized norms where $E = R^n$ and $K = R^+_+$ [see (1)] has led to many valuable results which are not obtainable when we only consider the usual norm in $R^n$.

A final preparatory lemma is needed before attacking the problems of this paper.

LEMMA 1.10. Let $q:[t_0,t_0+a] \to X$ such that $q'(t)$ exists on $(t_0,t_0+a]$. Assume that $\lim_{h \to 0} \frac{|q(t) + hq'(t)|_K - |q(t)|_K}{h}$ exists for each $t \in (t_0,t_0+a]$. If $p(t) = ||q(t)||_K$ then $p'(t)$ exists on $(t_0,t_0+a]$ and

$$p'(t) = \lim_{h \to 0} \frac{|q(t) + hq'(t)|_K - |q(t)|_K}{h}.$$

PROOF. Let $\varepsilon > 0$ and since $q'(t)$ exists we know there exists $\delta > 0$ such that $|\frac{q(t + h) - q(t)}{h} - q'(t)|_K < \varepsilon r_0$ whenever $h < 0$ and $|h| < \delta$. Using the triangular inequality and remembering $h < 0$ we have
\[
\frac{|q(t + h)|_K - |q(t) + hq'(t)|_K}{h} \leq \frac{|hq'(t) + q(t) - q(t + h)|_K}{-h} \leq \frac{|q(t + h) - q(t) - q'(t)|}{h} < e\rho_0.
\]

and
\[
\frac{|q(t + h)|_K - |q(t) + hq'(t)|_K}{h} \geq \frac{|q(t + h) - q(t) - hq'(t)|}{h} = \frac{-|q'(t) - \frac{q(t + h) - q(t)}{h}|}{h} > -e\rho_0.
\]

thus
\[
\lim_{h \to 0^-} \frac{|q(t + h)|_K - |q(t) + hq'(t)|_K}{h} = 0 \quad \text{and so}
\]

\[
p'(t) = \lim_{h \to 0^-} \frac{|q(t + h)|_K - |q(t)|_K}{h} = \lim_{h \to 0^-} \frac{|q(t + h)|_K - |q(t) + hq'(t)|_K}{h} + \lim_{h \to 0^-} \frac{|q(t) + hq'(t)|_K - |q(t)|_K}{h}
\]

\[
= \lim_{h \to 0^-} \frac{|q(t) + hq'(t)|_K - |q(t)|_K}{h}.
\]
2. Main Results.

We are now ready to pave an existence result in $K$-Banach spaces.

**Theorem 2.1.** Let $X$, $\|\cdot\|_K$ be a $K$-Banach space where $K$ is a closed regular cone in $E$ with $K^o \neq \emptyset$. Let $f: a_b(x_0) \to X$ where $a_b(x_0) = \{x \in X \mid \|x - x_0\|_K < br_0\}$ such that

1) $f$ is $K$-continuous on $a_b(x_0)$ and $\|f(x)\|_K < Mr_0$ where $M > 0$;

and 2) $\lim_{h \to 0} \frac{\|x - y + h[f(x) - f(y)]\|_K - \|x - y\|_K}{h} \leq 0$ for all $x, y \in a_b(x_0)$.

Then there exists a unique solution for the differential system $x' = f(x)$, $x(t_0) = x_0$ on $[t_0, t_0 + \alpha]$ where $\alpha = \frac{b}{M}$.

The proof of Theorem 2.1 is contained in the lemmas 2.2, 2.3, 2.4, 2.5, and 2.6 which follow.

**Lemma 2.2.** Suppose the hypothesis of Theorem 2.1. For each integer $n > 0$ there exists a function $x_n: [t_0, t_0 + \alpha] \to a_b(x_0)$ and a partition $t_0^n, t_1^n, \ldots, t_M^n$ of $[t_0, t_0 + \alpha]$ such that

1) $t_{i+1}^n - t_i^n \leq \frac{1}{n}$;

2) $\|x_n(t) - x_n(s)\|_K \leq (|t - s|)Mr_0$;

3) if $t \in (t_i^n, t_{i+1}^n]$ then $(x_n)'$ exists and equals $f(x_n(t_i^n))$;

in fact $(x_n)'(t)$ exists for all $t \in (t_0, t_0 + \alpha]$.

4) if $\|x - x_n(t_i^n)\|_K \leq M(t_{i+1}^n - t_i^n)r_0$ then $\|f(x) - f(x_n(t_i^n))\|_K \leq \frac{1}{n}r_0$.

**Proof.** Assume $t_0 = t_0^n, t_1^n, \ldots, t_M^n$ as well as $x_n$ have been defined inductively on $[t_0, t_i^n]$ satisfying 1), 2), 3) and 4). Choose $\delta_i$ such that $0 \leq \delta_i \leq \frac{1}{n}$ satisfying
a) \( t_i^n + \delta_i \leq t_0 + a \);

b) if \(|x - x_n(t_i^n)|_K \leq M\delta_i r_0\) then \(|f(x) - f(x(t_i^n))|_K \leq \frac{1}{n} r_0\);

and precisely one of the following holds:

c) \( \delta_i = \frac{1}{n} \) or \( t_i^n + \delta_i = t_0 + a \);

d) \( \delta_i \) could not be chosen larger satisfying a) and b). Notice d) means that for any \( \beta > 0 \) there exists \( x_\beta \in X \) such that

\[ ||x_\beta - x_n(t_i^n)||_K \leq M(\delta_i + \beta)r_0 \quad \text{and} \quad ||f(x_\beta) - f(x_n(t_i^n))||_K \leq \frac{1}{n} r_0 \]

Define \( t_{i+1}^n = t_i^n + \delta_i \) and for each \( t \in [t_i^n, t_{i+1}^n] \) define

\[ x_n(t) = x_n(t_i^n) + (t - t_i^n)f(x_n(t_i^n)). \]

Surely \( x_n(t) \in \mathcal{B}(x_0) \) and on \([t_0, t_{i+1}^n]\) we have \( x_n \) satisfying conditions 1), 2), 3), 4) of this lemma. We need to show that after some finite number of steps \( t_{i,n} = t_0 + a \). If not \( t_{i,n} + \delta \leq t_0 + a \) and \( \{x_n(t_{i,n})\}_{i=1}^\infty \) is Cauchy in \( i \) since

\[ ||x_n(t_{i+1}^n) - x_n(t_{i,n})||_K \leq |t_{i+1}^n - t_{i,n}|_K \quad \text{and} \quad \{t_{i,n}\}_{i=1}^\infty \] are Cauchy.

Since \( X \) is complete and \( \mathcal{B}(x_0) \) is closed \( x_n(t_{i,n}) \rightarrow y \in \mathcal{B}(x_0) \). Now \( f \) is \( K \)-continuous at \( y \) so there exists \( \delta < \frac{1}{n} \) such that \(|x - y|_K < \delta r_0\) implies \(|f(x) - f(y)|_K < \frac{1}{3} r_0\). For \( \delta \) sufficiently large

\[ |t_{i+1}^n - t_i^n| < \frac{\delta}{3M+1} \quad \text{and} \quad |x_n(t_{i+1}^n) - y|_K < \frac{\delta}{3} r_0. \]

However \( \delta_i = t_{i+1}^n - t_i^n < \frac{\delta}{3M+1} < \delta < \frac{1}{n} \) and \( t_{i+1}^n \neq t_0 + a \) so c) does not hold for any d) to hold. Consider d) where \( \beta = \frac{\delta}{3(M+1)} \) and we know there exists \( x_\beta \) such that

\[ ||x_\beta - x_n(t_i^n)||_K \leq M\left(\delta_i + \frac{\delta}{3(M+1)}\right)r_0 < M\frac{2\delta}{3(M+1)}r_0 < \frac{2}{3} r_0 \quad \text{and} \]

\[ ||f(x_\beta) - f(y)||_K < \frac{1}{3} r_0. \]
But 
\[ ||f(x_\beta) - f(x_n(t_n^n))||_K \leq ||f(x_\beta) - f(y)||_K + ||f(y) - f(x_n(t_n^n))||_K \]
\[ < \frac{1}{3n^0} + \frac{1}{3n^0} < \frac{1}{n^0} \]
which contradicts the choice of \( x_\beta \) since supposedly 
\[ ||f(x_\beta) - f(x_n(t_n^n))||_K \| = \frac{1}{n^0}. \]

**LEMMA 2.3.** Assume the hypothesis of Theorem 2.1 and for each \( n > 0 \) let \( x_n \) and \( t_n^n, t_n^n, \ldots, t_n^n \) be the function and partition constructed in Lemma 2.2. If \( q(t) = q_{m,n}(t) = x_n(t) - x_m(t) \) then for each \( t \) \((t, t + a)\) we have

\[ \lim_{h \to 0^-} \frac{||q(t) + ha'q(t)||_K - ||q(t)||_K}{h} \]
exists and is less than \( 2\left(\frac{1}{n} + \frac{1}{m}\right)r_0 \).

**PROOF.** Let \( 0 < \alpha < 1 \) and \( h < 0 \), then

\[ ||q(t) + ha'q(t)||_K = ||\alpha[q(t) + ha'q(t)] + (1 - \alpha) q(t)||_K \leq \alpha ||q(t) + ha'q(t)||_K + (1 - \alpha) ||q(t)||_K \]
so

\[ ||q(t) + ha'q(t)||_K - ||q(t)||_K \leq \alpha \left[ ||q(t) + ha'q(t)||_K - ||q(t)||_K \right] \]
and dividing by \( ha < 0 \) we have

\[ \frac{||q(t) + ha'q(t)||_K - ||q(t)||_K}{ha} \geq \frac{||q(t) + ha'q(t)||_K - ||q(t)||_K}{h} \]

Thus \( \frac{||q(t) + ha'q(t)||_K - ||q(t)||_K}{h} \) is non-decreasing in \( h \) and since the cone is regular if we can establish an upper bound the limit will exists.

By hypothesis 2 of Theorem 2.1 there exists \( \delta > 0 \) such that \( h < 0 \) with \( |h| < \delta \) then

\[ \frac{||x_n(t) - x_m(t) + h[f(x_n(t)) - f(x_m(t))]|_K - ||x_n(t) - x_m(t)||_K}{h} \]
\[ < \left(\frac{1}{n} + \frac{1}{m}\right)r_0. \]
Since \( t \in (t_0, t_0 + \alpha) \) there exists \( i, j \) such that \( t \in \left( t_{i}^{n}, t_{i+1}^{n} \right) \) and \( t \in \left( t_{j}^{m}, t_{j+1}^{m} \right) \) and by Lemma 2.2 we have \( (x_{i}^{n})^{{'}(t)} = f(x_{i}^{n}(t)) \) and 
\( (x_{m})^{{'}(t)} = f(x_{m}(t)) \). Furthermore \( \| f(x_{i}^{n}(t)) - f(x_{i}^{n}(t_{i}^{n})) \|_{K} \leq \frac{1}{n^2 \alpha} \) and 
\( \| f(x_{m}(t)) - f(x_{m}(t_{j}^{m})) \|_{K} \leq \frac{1}{m^2 \alpha} \).

Thus for \( h < 0 \) with \( |h| < \delta \) we have

\[
\frac{\|q(t) + hq^{{'}}(t)\|_{K} - \|q(t)\|_{K}}{h} = \frac{\|x_{n}(t) - x_{m}(t) + h[f(x_{n}(t_{i}^{n})) - f(x_{m}(t_{j}^{m}))]\|_{K} - \|x_{n}(t) - x_{m}(t)\|_{K}}{h}.
\]

\[
\|x_{n}(t) - x_{m}(t) + h[f(x_{n}(t)) - f(x_{m}(t))]|_{K} - \|x_{n}(t) - x_{m}(t)\|_{K}
\]

\[
\|x_{n}(t) - x_{m}(t) + h[f(x_{n}(t_{i}^{n})) - f(x_{m}(t_{j}^{m}))]|_{K} - \|x_{n}(t) - x_{m}(t) + h[f(x_{n}(t)) - f(x_{m}(t))]|_{K}
\]

\[
\leq \left( \frac{1}{n} + \frac{1}{m} \right) r_{0} + \left| \frac{h[f(x_{n}(t_{i}^{n})) - f(x_{m}(t_{j}^{m}))] - h f(x_{n}(t)) - f(x_{m}(t))}{h} \right|_{K}
\]

\[
\leq \left( \frac{1}{n} + \frac{1}{m} \right) r_{0} + \| f(x_{n}(t_{i}^{n})) - f(x_{n}(t)) \|_{K} \| f(x_{m}(t_{j}^{m})) - f(x_{m}(t)) \|_{K}
\]

\[
\leq 2 \left( \frac{1}{n} + \frac{1}{m} \right) r_{0}.
\]

Hence we have constructed an upper bound and the lemma is proved.
**Lemma 2.4.** Assume the hypothesis of Theorem 2.1 and for each \( n > 0 \) let \( x_n \) and \( t_0^n, t_1^n, \ldots, t_n^n \) be the function and partition constructed in Lemma 2.2. Then the sequence \( \{x_n(t)\}_{n=1}^\infty \) is uniformly cauchy and thus converges uniformly to a function \( x(t) \) on \([t_0, t_0 + a]\) with \( x(t_0) = x_0 \).

**Proof.** Let \( q_{m,n}(t) = x_n(t) - x_m(t) \) and recall from Lemma 2.2 that \((q_{m,n})'(t)\) exists for each \( t \in [t_0, t_1] \) and by Lemma 2.3 we know

\[
\lim_{h \to 0} \frac{||q_{m,n}(h) + h(q_{m,n})'(t)||_K - ||q(t)||_K}{h}
\]

exists. Thus by Lemma 1.10 we know \( p_{m,n}(t) = \frac{\langle q(t) \rangle}{K} \) has a left derivative for each \( t \in [t_0, t_0 + a] \) and in fact

\[
(p_{m,n})'(t) = \lim_{h \to 0} \frac{(||q_{m,n}(h) + h(q_{m,n})'(t)||_K - ||q(t)||_K)}{h}
\]

so by Lemma 2.3 \((p_{m,n})'(t) < 2\left(\frac{1}{n} + \frac{1}{m}\right)r_0^2\).

Now if we define \( v(t) = 2\left(\frac{1}{n} + \frac{1}{m}\right)r_0^2(t-t_0) + \frac{1}{n} + \frac{1}{m}r_0^2 \) for \( t \in [t_0, t_0 + a] \) then \( v_n(t) = \left[2\left(\frac{1}{n} + \frac{1}{m}\right)r_0^2(t-t_0) + \frac{1}{n} + \frac{1}{m}r_0^2 \right] \) and \( v_n(t) \) is \( C \) on \([t_0, t_0 + a] \) and \( v_n(t) > (p_{m,n})'(t) \) for all \( t \in [t_0, t_0 + a] \). By Theorem 1.8 we can conclude \( p_{m,n}(t) < v(t) \) for all \( t \in [t_0, t_0 + a] \). Hence \( ||x_n(t) - x_m(t)||_K < \left(\frac{1}{n} + \frac{1}{m}\right)r_0^2[2a + 1] \) which means \( \{x_n(t)\}_{n=1}^\infty \) is uniformly cauchy.

**Lemma 2.5.** Assume the hypothesis of Theorem 2.1 and let \( x(t) \) be the uniform limit of \( \{x_n(t)\}_{n=1}^\infty \) as assured by Lemma 2.4. Then \( x(t) \) is a solution of \( x' = f(x), \ x(t_0) = x_0 \).

**Proof.** Since each \( x_n \) is continuous and \( x_n \) converges uniformly to \( x \) we know \( x \) is continuous. Since \( (x(t)|t \in [t_0, t_0 + a]) \) is a compact subset
of $X$ and $x_n$ converge uniformly to $x$ it follows easily that $f(x_n(t))$ converges uniformly to $f(x(t))$. Let $t \in [t_0, t_0 + a] - S$ where $S$ is the union of all the partitions and $t \in (t_i^n, t_i^{n+1})$ we have

$$0 \leq ||x_n'(t) - f(x_n(t))||_K = ||f(x_n(t_i^n)) - f(x_n(t))||_K \leq \frac{1}{n} r_0 \text{ by part 4) of Lemma 2.2.}$$

Thus

$$\lim_{n \to \infty} \int_{t_0}^{t} \left[ x_n'(s) - f(x_n(s)) \right] ds = 0$$

$$x(t) = \lim_{n \to \infty} x_n(t) = \lim_{n \to \infty} \left[ x_0 + \int_{t_0}^{t} x_n'(s) ds \right] = \lim_{n \to \infty} \left[ x_0 + \int_{t_0}^{t} f(x_n(s)) ds \right] \leq \lim_{n \to \infty} \int_{t_0}^{t} \left[ x_n'(s) - f(x_n(s)) \right] ds$$

$$= x_0 + \int_{t_0}^{t} f(x(s)) ds.$$

Therefore $x$ is the desired solution of $x' = f(x), x(t_0) = x_0$.

**Lemma 2.6.** Assume the hypothesis of Theorem 2.1, then the solution of $x' = f(x), x(t_0) = x_0$ assured by Lemmas 2.2, 2.3, 2.4, and 2.5 is unique.

**Proof.** Suppose $x(t)$ and $y(t)$ are two solutions of $x' = f(x), x(t_0) = x_0$ and let $m(t) = \left| |x(t) - y(t)|_K \right|$. Using Lemma 1.10 and hypothesis 2) of Theorem 2.1 we have

$$m'(t) = \lim_{h \to 0^-} \frac{|x(t) - y(t) + h[f(x(t)) - f(y(t))]|_K - |x(t) - y(t)|_K}{h} \leq 0.$$

For any $e > 0$ consider the function $\nu(t) = er_0(t - t_0) + er_0$. Now
$m(t_0) = 0 < er_0 = v(t_0)$ and $m'(t) \leq 0 < er_0 = v'(t)$ and so by Theorem 1.8 we have

$$\theta \leq m(t) = ||x(t) - y(t)||_K < v(t) \text{ for all } t \in [t_0, t_0 + \alpha]$$

and so $x(t) = Y(t)$ for all $t \in [t_0, t_0 + \alpha]$.

Under the same hypotheses as Theorem 2.1 we can obtain global existence of solutions.

**Theorem 2.7.** Let $X, \| \cdot \|_K$ be a $K$-Banach space where $K$ is a closed regular cone in $E$ with $K^0 \neq 0$. Let $f: X \to X$ such that

1) $f$ is $K$-continuous on $X$,

and

2) $\lim_{h \to 0} \frac{||x - y + h[f(x) - f(y)]||_K - ||x - y||_K}{h} \leq \theta$

for all $x, y \in X$.

Then the differential system $x' = f(x), x(t_0) = x_0$ has a unique solution on $[t_0, \infty)$.

**Proof.** By Theorem 2.1 we know there exists a local solution of $x' = f(x), x(t_0) = x_0$. Suppose $[t_0, t_0 + \alpha)$ is the maximal interval of existence for a solution $x(t)$ of $x' = f(x), x(t_0) = x_0$. We will show $\lim_{t \to t_0 + \alpha} x(t)$ exists and is finite thus proving the solution exists globally. If we define $m(t) = ||x(t + \alpha) - x(t)||_K$ then using Lemma 1.10 and hypotheses 2) of this theorem we obtain

$$m'(t) = \lim_{h \to 0} \frac{||x(t + \alpha) - x(t) + h[f(x(t + \alpha) - f(x(t)))]||_K - ||x(t + \alpha) - x(t)||_K}{h} \leq \theta$$

For each $\epsilon > 0$ consider the function $v(t) = \left[ ||x(t_0 + \alpha) - x(t_0)||_K + e \right] (t - t_0 + 1)$
\[ m(t^*) = \| A_{t^*}(y) - A_{t^*}(w) \|_K \leq \frac{1}{2} \| y - w \|_K \] and so

\( A_{t^*} \) is a contraction mapping on \( X \) and so \( A_{t^*} \) has a fixed point \( x^* \).

We claim \( x^* \) is a fixed point of each \( A_t \). Since solutions are unique and the equation \( x' = f(x) \) is autonomous the operator \( A_t, A_{t+} \) commute and hence

\[
\| A_t(x^*) - x^* \|_K = \| A_t(A_{t^*}(x^*)) - A_{t^*}(x^*) \|_K \\
= \| A_{t^*}(A_t(x^*)) - A_{t^*}(x^*) \|_K \\
\leq \frac{1}{2} \| A_t(x^*) - x^* \|_K .
\]

Thus \( x^* \) is a fixed point of each operator \( A_t \) which means \( x(t, t_0, x^*) = x^* \) for all \( t \).

Since the constant solution is a solution of \( x' = f(x) \) we have \( 0 = x' = f(x^*) \) and hence \( T(x^*) = 0 \).

A final remark seems in order.

**Remark 4.** In view of the comments in Remark 3, if we want to work with a cone that is not normal we need to develop necessary functional analytical tools and differential inequality theory in \( K \)-Banach spaces. This we propose to consider in a later paper.
BIBLIOGRAPHY


